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CONTENTS

BOOLEAN INEQUALITIES FROM LATTICES, ARRAYS, AND POLYGONS

J.L. BRENNER

1. *An improvement from a rectangular lattice.*

In a recent article in this journal [1983: 128], we showed how to use a dissection of a triangle to obtain the following result:

1.1. *THEOREM. If $m > 1$ and $n > 1$ are integers, and if $p,q > 0$ with $p+q \le 1$, then*

(1.11) $$(1 - p^m)^n + (1 - q^n)^m > 1.$$

The proof used only Boolean arguments, such as:

1.2. If two events A and B are mutually exclusive, then $\Pr(A) + \Pr(B) \le 1$, with equality if and only if $A \vee B$ is exhaustive.

Using an idea of the editor, it is possible to sharpen (1.11). Arrange mn beads in a rectangular array of m columns and n rows, with a dogleg of max $\{m,n\}$ beads stretching from the top left to the bottom right bead. (Figure 1.3 illustrates the case $m = 5$, $n = 7$.) Color each bead randomly with exactly one of three distinct colors: using red with probability $p > 0$, using green with probability $q > 0$, or using blue with probability $r \ge 0$. The following three events are mutually exclusive, and their union is exhaustive if and only if $m = n = 1$:

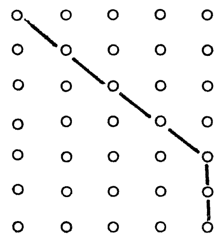

Figure 1.3

1.31. At least one row is entirely red.

1.32. At least one column is entirely green.

1.33. The dogleg is entirely blue.

It is easy to see that Pr(1.31) is the complement $1 - (1 - p^m)^n$ of the probability $(1 - p^m)^n$ that *none* of the rows be entirely red. Similarly, $\Pr(1.32) = 1 - (1-q^n)^m$, and it is clear that $\Pr(1.33) = r^{\max\{m,n\}}$. Assertion 1.2 can be extended to three mutually exclusive events, and gives

$$\{1 - (1 - p^m)^n\} + \{1 - (1 - q^n)^m\} + r^{\max\{m,n\}} \le 1.$$

Since $r = 1-p-q$, we have the following result:

1.4. *THEOREM. If $m \ge 1$ and $n \ge 1$ are integers, and if $p,q > 0$ with $p+q \le 1$, then*

$$(1 - p^m)^n + (1 - q^n)^m \ge 1 + (1-p-q)^{\max\{m,n\}},$$

with equality if and only if $m = n = 1$.

Theorem 1.4 can be extended in several ways. One extension, the proof of which requires calculus, is to let $m \geq 1$ and $n \geq 1$ be *real numbers*. Another kind of extension results if, for some given integer $k \geq 2$, we replace m,n by k positive integers n_i, and p,q by k positive real numbers p_i. It will be helpful to first state and give a (Boolean) proof of this result for $k = 3$, after which we will state without proof a far-reaching generalization which encompasses both types of extensions.

1.5. *THEOREM. If $n_1, n_2, n_3 \geq 1$ are integers, and if $p_1, p_2, p_3 > 0$ are real numbers with $p_1+p_2+p_3 \leq 1$, then*

$$(1.51)\quad (1-p_1^{l_1})^{n_1} + (1-p_2^{l_2})^{n_2} + (1-p_3^{l_3})^{n_3} \geq 2 + (1-p_1-p_2-p_3)^{\max\{n_1,n_2,n_3\}},$$

where $l_1 = n_2n_3$, $l_2 = n_3n_1$, $l_3 = n_1n_2$, with equality if and only if $n_1 = n_2 = n_3 = 1$.

Proof. Arrange $n_1n_2n_3$ beads in an $n_1 \times n_2 \times n_3$ rectangular parallelepiped with n_1 planes of dimensions $n_2 \times n_3$, n_2 planes of dimensions $n_3 \times n_1$, n_3 planes of dimensions $n_1 \times n_2$, and a "fractured" dogleg of max $\{n_1,n_2,n_3\}$ beads extending from one corner bead in a direction with equal direction cosines $1/\sqrt{3}$ until it reaches a face, proceeding then in that face in a direction with equal direction cosines $1/\sqrt{2}$ until it reaches an edge, and then remaining in that edge until it reaches the bead in the corner opposite the starting corner. (Of course, there is no "fracture" if two of the n_i are equal, and the "leg" is straight if all three of the n_i are equal.) Color each bead randomly with exactly one of four distinct colors 1,2,3,4: using color i with probability $p_i > 0$ for $i = 1,2,3$, and using color 4 with probability $r \geq 0$. The following four events are mutually exclusive, and their union is exhaustive if and only if $n_1 = n_2 = n_3 = 1$:

1.52. At least one of the n_1 planes of dimensions $n_2 \times n_3$ is entirely of color 1.

1.53. At least one of the n_2 planes of dimensions $n_3 \times n_1$ is entirely of color 2.

1.54. At least one of the n_3 planes of dimensions $n_1 \times n_2$ is entirely of color 3.

1.55. The dogleg is entirely of color 4.

It is easy to see that Pr(1.52) is the complement $1-(1-p_1^{l_1})^{n_1}$ of the probability $(1-p_1^{l_1})^{n_1}$ that *none* of the n_1 planes of dimensions $n_2 \times n_3$ be entirely of color 1, with similar results for Pr(1.53) and Pr(1.54). Obviously Pr(1.55) = $r^{\max\{n_1,n_2,n_3\}}$, so that

$$\sum_{i=1}^{3} \{1-(1-p_i^{l_i})^{n_i}\} + r^{\max\{n_1,n_2,n_3\}} \leq 1,$$

and (1.51) follows from the fact that $r = 1-p_1-p_2-p_3$; the inequality degenerates to equality if and only if $n_1 = n_2 = n_3 = 1$. □

Finally, we state the generalization announced earlier.

1.6. *THEOREM. Let* $k \geq 2$ *be a given positive integer, and suppose that, for* $i = 1,2,\ldots,k$, *we have* k *real numbers* $n_i \geq 1$ *and* k *real numbers* $p_i > 0$ *with* $S \equiv \Sigma p_i \leq 1$. *If* $N = \Pi n_i$ *and* $l_i = N/n_i$, *then*

$$\sum_{i=1}^{k} (1 - p_i^{l_i})^{n_i} \geq k - 1 + (1-S)^{\max\{n_i\}},$$

with equality if and only if all the $n_i = 1$.

The proof of this theorem in full generality requires calculus.

2. *Inequalities from a polygonal lattice.*

In the Euclidean plane, there are three regular tessellations: the rectangular one T_1 used in Section 1, the tessellation T_2 with equilateral triangles, and the dual tessellation T_3 with regular hexagons. In this section, inequalities are derived by using T_2. See Figure 2.1, in which a regular hexagon is subdivided into equilateral triangles by 45 lines, 15 (equally spaced) in each of the three prime directions. A collection of chords adjacent to and symmetrically situated with respect to a given diagonal is called *a set of central lines*. Thus, a set of central lines consists of an odd number, $2k+1$, of chords, where $0 \leq k \leq 7$.

Color each of the 169 vertices in or on the hexagon at random with exactly one of four distinct colors 1,2,3,4: using colors 1,2,3 with probabilities $p,q,r > 0$, respectively, and using color 4 with probability $s \geq 0$. A line will be said to be *of color* i if all the vertices on that line are of color i. For each k, $0 \leq k \leq 7$, the events 2.11k-2.14 given below are mutually exclusive and their union is not exhaustive.

2.11k. At least one of the $2k+1$ central vertical lines is of color 1.

2.12k. At least one of the $15-2k$ central NE-SW lines is of color 2.

2.13k. At least one of the $15-2k$ central NW-SE lines is of color 3.

2.14. Two of the main diagonals are of color 4.

For $k = 0$ and $k = 7$, nothing new emerges. For $k = 1$ (the case illustrated in Figure 2.1), the respective probabilities are:

$\Pr(2.111) = 1 - (1 - p^{14})^2(1 - p^{15})$,

$\Pr(2.121) = 1 - (1 - q^9)^2(1 - q^{10})^2\ldots(1 - q^{14})^2(1 - q^{15})$,

$\Pr(2.131) = 1 - (1 - r^9)^2(1 - r^{10})^2\ldots(1 - r^{14})^2(1 - r^{15})$,

$\Pr(2.14) \geq s^{29}$.

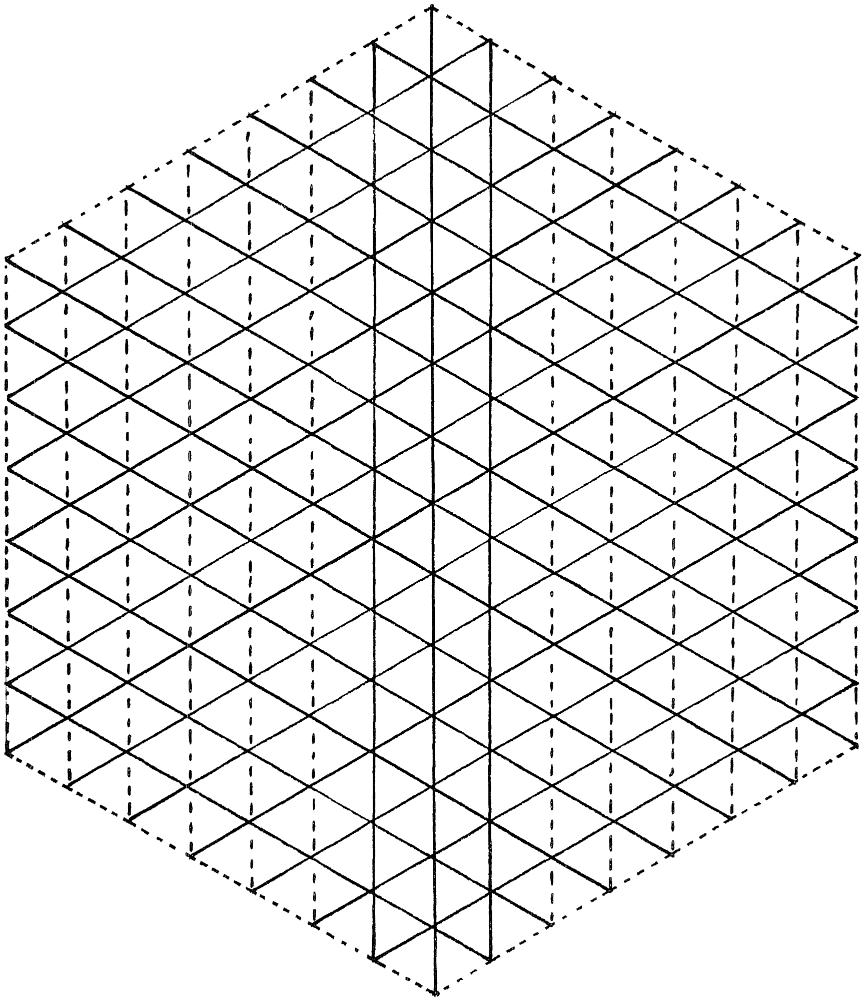

Figure 2.1

Generalizing from 15 to arbitrary odd integer $l \geq 5$, and from $k = 1$ to arbitrary integer k, $0 < k < \frac{1}{2}(l-1)$, the usual (Boolean) argument gives the following result:

2.2. *THEOREM. Let* $l \geq 5$ *be an odd integer, and let* $p,q,r > 0$, $s \geq 0$, *with* $p+q+r+s = 1$. *Then, if* k *is an integer such that* $0 < k < \frac{1}{2}(l-1)$, *the following inequality holds:*

(2.21k)
$$\begin{aligned}&(1 - p^l)(1 - p^{l-1})^2 \ldots (1 - p^{l-k})^2\\ &+ (1 - q^l)(1 - q^{l-1})^2 \ldots (1 - q^{\frac{1}{2}(l+1)+k})^2\\ &+ (1 - r^l)(1 - r^{l-1})^2 \ldots (1 - r^{\frac{1}{2}(l+1)+k})^2\\ &> 2 + s^{2l-1}.\end{aligned}$$

In particular,

(2.21l)
$$\begin{aligned}&(1 - p^l)(1 - p^{l-1})^2\\ &+ (1 - q^l)(1 - q^{l-1})^2 \ldots (1 - q^{\frac{1}{2}(l+3)})^2\\ &+ (1 - r^l)(1 - r^{l-1})^2 \ldots (1 - r^{\frac{1}{2}(l+3)})^2\\ &> 2 + s^{2l-1};\end{aligned}$$

and, if $l \equiv 1 \pmod 4$ and $a = \frac{1}{4}(l-1)$,

(2.21a)
$$f(p) + f(q) + f(r) > 2 + s^{2l-1},$$

where $f(p) = (1 - p^l)(1 - p^{l-1})^2 \ldots (1 - p^{\frac{1}{4}(3l+1)})^2$.

2.22. *COROLLARY. If $l \equiv 1 \pmod 4$ and p,q,r,s are as above, then*

(2.23)
$$(1 - p^l)^{\frac{1}{2}(l+1)} + (1 - q^l)^{\frac{1}{2}(l+1)} + (1 - r^l)^{\frac{1}{2}(l+1)} > 2 + s^{2l-1}.$$

Proof. This corollary is just a weaker form of (2.21a), since

$$1 - p^l \geq 1 - p^{l-1} \geq \ldots \geq 1 - p^{\frac{1}{4}(3l+1)},$$

and the number of factors in each term of (2.23) is the same as in $f(p)$. □

The assertion

(2.24)
$$(1 - p^l)^l + (1 - q^l)^l + (1 - r^l)^l > 2$$

is valid and is stronger than (2.23). But to obtain it a new type of diagram must be used. This is discussed in the next section.

3. *Special results obtainable from square arrays.*

In this section, straightforward applications of the formula for the sum of the probabilities of mutually exclusive events are used to obtain new inequalities.

3.1. *THEOREM. If $p_i > 0$ for $i = 1,2,3,4$ and $\Sigma p_i \leq 1$, then*

(3.2)
$$(1 - p_1^2)^2 + (1 - p_2^2)^2 + (1 - p_3^2)^2 > 2,$$

(3.3)
$$(1 - p_1^3)^3 + (1 - p_2^3)^3 + (1 - p_3^3)^3 + (1 - p_4^3)^3 > 3,$$

(3.4)
$$(1 - p_1^4)^4 + (1 - p_2^4)^4 + (1 - p_3^4)^4 > 2,$$

(3.5)
$$(1 - p_1^5)^5 + (1 - p_2^5)^5 + (1 - p_3^5)^5 + (1 - p_4^5)^5 > 3.$$

Proof of (3.2). Take four beads (numbered 1,2,3,4) and color them (with probabilities p_1, p_2, p_3) with colors c_1, c_2, c_3. The following events are mutually exclusive:

At least one of the pairs	At least one of the pairs	At least one of the pairs
1, 2;	1, 3;	1, 4;
3, 4	2, 4	2, 3
is monochromatic of color c_1.	is monochromatic of color c_2.	is monochromatic of color c_3.

The respective probabilities are

$$1 - (1 - p_1^2)^2, \quad 1 - (1 - p_2^2)^2, \quad 1 - (1 - p_3^2)^2,$$

and assertion (3.2) follows.

Proof of (3.3). Take 9 beads, four colors, and events summarized by the arrays:

1, 2, 3;	1, 4, 7;	1, 5, 9;	1, 6, 8;
4, 5, 6;	2, 5, 8;	2, 6, 7;	2, 4, 9;
7, 8, 9.	3, 6, 9.	3, 4, 8.	3, 5, 7.

The patterns for (3.4) and (3.5) should now be clear. In fact, the idea is effective for four terms (like (3.5)) when the exponents are odd, and for three terms (like (3.4)) when the exponents are even. To follow the idea, note three things: (i) the rows of the second array are the columns of the first array; (ii) the rows of the third array are the (broken) positive diagonals of the first array; (iii) the rows of the fourth array are the (broken) negative diagonals of the first array. □

The results that this method can deliver are limited. The inequality

$$(3.6) \qquad (1 - p_1^2)^2 + (1 - p_2^2)^2 + (1 - p_3^2)^2 + (1 - p_4^2)^2 > 3$$

is true, but not obtainable from a picture *with beads*. Here is the impossibility argument. The terms p_i^2 imply that the sets being colored must have two beads each. The terms $(1 - p_i^2)^2$ imply that there are two sets of two beads each. But a total of four beads can be partitioned into two such sets in only three ways.

4. *Inequalities obtainable from a (nearly) regular octagon.*

The next theorem is proved from Figure 4.1.

4.2. THEOREM. *If* $p_i > 0$ *for* $i = 1, 2, \ldots, 5$ *and* $\Sigma p_i = 1$, *then*

$$(4.3) \quad (1 - p_1^5)^3 + (1 - p_2^5)^3 + (1 - p_3^3)^2(1 - p_3^5) + (1 - p_4^3)^2(1 - p_4^5) > 3 + p_5^7.$$

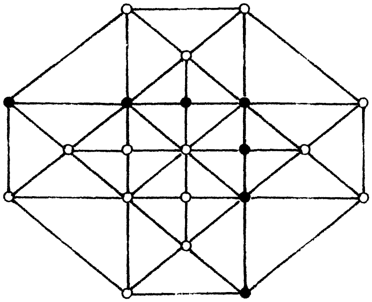

Figure 4.1

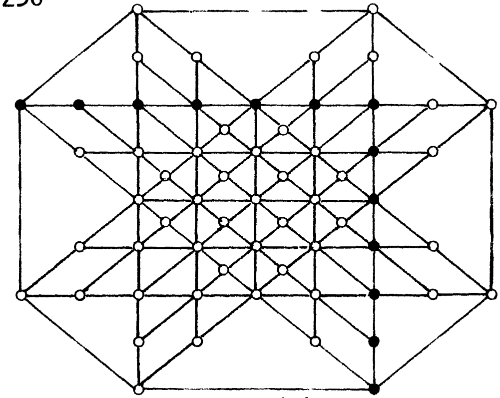

Figure 4.4

Proof. There are 21 intersection points marked with open or solid circles. Place beads at the 21 points, and color the beads (at random) with colors c_i with respective probabilities p_i $(1 \le i \le 5)$. The reader is invited to invent five mutually exclusive events and thus derive (4.3). The fifth event, with probability p_5^7, is suggested by the seven solidly marked intersection points. □

In an analogous fashion, Figure 4.4 leads to the next theorem.

THEOREM 4.5. *If* $p_i > 0$ *for* $i = 1,2,\ldots,5$ *and* $\Sigma p_i = 1$, *then*

$$(4.6) \quad (1 - p_1^7)(1 - p_1^9)^4 + (1 - p_2^7)(1 - p_2^9)^4 + (1 - p_3^5)(1 - p_3^7)^2(1 - p_3^9)^2 + (1 - p_4^5)(1 - p_4^7)^2(1 - p_4^9)^2 > 3 + p_5^{13},$$

This result (4.6) is not a corollary of either of the inequalities $\Sigma(1 - p_i^5)^5 > 3$, $\Sigma(1 - p_i^7)^7 > 3$ of Section 3.

5. *Conclusion.*

In this set of two articles (the first appeared in [1983: 128]), Boolean arguments are used to derive some inequalities in which the parameters m, n, l_i, n_i are integers. No essentially analytical arguments are used. It is interesting that the same inequalities are valid when the parameters are real. This extension will be carried out in a more technical article; in it several other sharp inequalities are obtained. The article will appear in 1985 in *J. Math. Anal. & Appl.*

Note that many corollaries of the theorems of this article can be written down. One of them is:

5.1. *COROLLARY to* (3.2). *If* $a,b,c > 0$, *then*

$$(5.2) \quad \{(a+b+c)^2 - a^2\}^2 + \{(a+b+c)^2 - b^2\}^2 + \{(a+b+c)^2 - c^2\}^2 > 2(a+b+c)^4.$$

Proof. Set $p_1 = a/(a+b+c)$, $p_2 = b/(a+b+c)$, $p_3 = c/(a+b+c)$ in (3.2). □

The symbol manipulation required to prove (5.2) *directly* is rather tedious.

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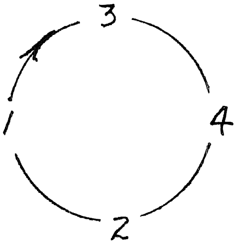
FOURTH-ORDER ADDITIVE DIGITAL BRACELETS IN BASE FIVE

CHARLES W. TRIGG

A *bracelet* is defined [1] as "one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Hence a bracelet may be regenerated by starting at any arbitrary position and applying the generating law." The distance between terms may be measured in degrees or steps. A bracelet may be cut at any arbitrary point for straight-line representation without loss of any properties.

The generating law $u_{n+2} = u_{n+1} + u_n$, where each sum is reduced modulo b (the base of the system of numeration), produces a second-order bracelet. Thus, in base five the starting pair (1, 3) leads to the four-digit bracelet shown in the adjoining figure, which in straight-line representation is



B: 1 3 4 2' 1 3.

In what follows, the degree measures of rotations and all numbers clearly identifiable from the context as magnitudes of sets are given in decimal notation (when they are not spelled out). But all sums and products of bracelet elements are reduced modulo five. Two digits with a sum of zero (modulo five) are called *complementary*.

Fourth-order additive bracelets.

The recursive formula

$$u_{n+4} = u_{n+3} + u_{n+2} + u_{n+1} + u_n$$

produces fourth-order additive digital bracelets when each sum is reduced modulo b. In base five, each of the 625 possible digit quartets appears in either the all-zero A or in one of the 312-digit bracelets H and J shown in Table 1. Bracelet H may be called the *tetranacci bracelet*, since it consists of the units' digits of the tetranacci sequence [2] in base five. The number of digits in this bracelet is $312_{\text{ten}} = 2222_{\text{five}}$.

In both H and J, the digits $90°$ apart are either all zeros or they form a cyclic permutation of B. It follows that diametrically opposite elements are complementary, and that the sum of the digits in each bracelet is zero. Furthermore, multiplication of either bracelet by 1, 3, 4, or 2 will rotate the bracelet through $0°$, $90°$, $180°$, or $270°$, respectively.

Table 1: Fourth-order bracelets

A: 0000'0000

H:	0001 1243 04133	J:	1111 4230 44143
	1302 1143 42321		2044 0313 24040
	3403 0200 24124		3244 3344 40201
	1303 2331 41404		3100 4043 13133
	4200 1343 11440		0233 3102 14241
	4201 2031 10023		1344 2332 03334
	0003 3124 02344		3333 2140 22324
	3401 3324 21413		1022 0434 12020
	4204 0100 12312		4122 4422 20103
	3404 1443 23202		4300 2024 34344
	2100 3424 33220		0144 4301 32123
	2103 1043 30014		3422 1441 04442
	0004 4312 01422		4444 1320 11412
	4203 4412 13234		3011 0242 31010
	2102 0300 31431		2311 2211 10304
	4202 3224 14101		2400 1012 42422
	1300 4212 44110		0322 2403 41314
	1304 3024 40032		4211 3223 02221
	0002 2431 03211		2222 3410 33231
	2104 2231 34142		4033 0121 43030
	1301 0400 43243		1433 1133 30402
	2101 4112 32303		1200 3031 21211
	3400 2131 22330		0411 1204 23432
	3402 4012 20041'0001		2133 4114 01113'1111

As written, in rows of thirteen elements, the sum of the digits in every column of the two bracelets is zero. In the successive 78-digit quadrants of H, the digit sums are 4, 2, 1, and 3. In J, the successive quadrant sums are 2, 1, 3, and 4. Two more appearances of the ubiquitous B.

In Table 2 are recorded the frequencies of occurrence of the five digits in the successive four quadrants of H and J. In each case, like frequencies of non-zero digits occur on diagonals slanting down toward the right.

In Table 3, the digits of each of H and J are consecutively arranged in thirteen 24-digit rows. There the sum of the digits in each column is zero.

Table 2: Frequency of digits in fourth-order bracelets

Quadrants	Digits 0	1	3	4	2
H_1	18	17	16	15	12
H_2	18	12	17	16	15
H_3	18	15	12	17	16
H_4	18	16	15	12	17

Bracelet H

Quadrants	Digits 0	1	3	4	2
J_1	13	14	21	20	10
J_2	13	10	14	21	20
J_3	13	20	10	14	21
J_4	13	21	20	10	14

Bracelet J

It follows that the sum of the digits at the vertices of any regular thirteen-gon inscribed in H or J will be zero.

The persistent B appears in the fifth row of H, and diagonally downward to the right in alternate columns of J in the form of triads of like digits interwoven with quartets of the same digits. The fifth row of H also contains two consecutive sets of four consecutive digits in increasing order of magnitude; and the twelfth row contains the five consecutive digits in decreasing order of magnitude.

The recordings of the two bracelets in three rows of 104 elements start out as

H:	0	0	0	1	1	2	4	3	0	4	1	3	3	. . .
	4	2	0	4	0	1	0	0	1	2	3	1	2	. . .
	1	3	0	0	4	2	1	2	4	4	1	1	0	. . .

and

J:	1	1	1	1	4	2	3	0	4	4	1	4	3	. . .
	4	1	2	2	4	4	2	2	2	0	1	0	3	. . .
	0	3	2	2	2	4	0	3	4	1	3	1	4	. . .

The sum of the digits in each of the columns above is zero. It follows that the sum of the digits at the vertices of any equilateral triangle inscribed in H or J will be zero. Indeed, the vertex digit sum will vanish for any inscribed regular n-gon where n is any factor of $312 = 2^3 \cdot 3 \cdot 13$.

Since all possible digit quartets appear in A, H, and J, in any clockwise matching of two of these bracelets, the sums of the corresponding digits will form a bracelet wherein the law of formation holds, so it must be a repeated A or some orientation of H or J. For example, an H (or J) added to another H (or J) that has been rotated through 180° forms a bracelet of zeros. The operation $H + J$, with the first digit of J matched with the third digit of H (from Table 1), then with the fourth digit, and so on, until it has been matched with every digit of H, produces a bracelet of 312 bracelets beginning, in order, with

$$H\ J\ J\ H\ H\ J\ J\ H\ J\ J\ J\ .\ .\ .$$

Table 3: Rearranged fourth-order bracelets

H: 0 0 0 1 1 2 4 3 0 4 1 3 3 1 3 0 2 1 1 4 3 4 2 3

2 1 3 4 0 3 0 2 0 0 2 4 1 2 4 1 3 0 3 2 3 3 1 4

1 4 0 4 4 2 0 0 1 3 4 3 1 1 4 4 0 4 2 0 1 2 0 3

1 1 0 0 2 3 0 0 0 3 3 1 2 4 0 2 3 4 4 3 4 0 1 3

3 2 4 2 1 4 1 3 4 2 0 4 0 1 0 0 1 2 3 1 2 3 4 0

4 1 4 4 3 2 3 2 0 2 2 1 0 0 3 4 2 4 3 3 2 2 0 2

1 0 3 1 0 4 3 3 0 0 1 4 0 0 0 4 4 3 1 2 0 1 4 2

2 4 2 0 3 4 4 1 2 1 3 2 3 4 2 1 0 2 0 3 0 0 3 1

4 3 1 4 2 0 2 3 2 2 4 1 4 1 0 1 1 3 0 0 4 2 1 2

4 4 1 1 0 1 3 0 4 3 0 2 4 4 0 0 3 2 0 0 0 2 2 4

3 1 0 3 2 1 1 2 1 0 4 2 2 3 1 3 4 1 4 2 1 3 0 1

0 4 0 0 4 3 2 4 3 2 1 0 1 4 1 1 2 3 2 3 0 3 3 4

0 0 2 1 3 1 2 2 3 3 0 3 4 0 2 4 0 1 2 2 0 0 4 1'0 0 0 1

J: 1 1 1 2 0 4 2 3 4 3 2 2 1 3 3 4 1 1 4 0 1 1 1 3

1 1 1 1 4 2 3 0 4 4 1 4 3 2 0 4 4 0 3 1 3 2 4 0

4 0 3 2 4 4 3 3 4 4 4 0 2 0 1 3 1 0 0 4 0 4 3 1

3 1 3 3 0 2 3 3 3 1 0 2 1 4 2 4 1 1 3 4 4 2 3 3

2 0 3 3 3 4 3 3 3 3 2 1 4 0 2 2 3 2 4 1 0 2 2 0

4 3 4 1 2 0 2 0 4 1 2 2 4 4 2 2 2 0 1 0 3 4 3 0

0 2 0 2 4 3 4 3 4 4 0 1 4 4 4 3 0 1 3 2 1 2 3 3

4 2 2 1 4 4 1 0 4 4 4 2 4 4 4 4 1 3 2 0 1 1 4 1

2 3 0 1 1 0 2 4 2 3 1 0 1 0 2 3 1 1 2 2 1 1 1 0

3 0 4 2 4 0 0 1 0 1 2 4 2 4 2 2 0 3 2 2 2 4 0 3

4 1 3 1 4 4 2 1 1 3 2 2 3 0 2 2 2 1 2 2 2 2 3 4

1 0 3 3 2 3 1 4 0 3 3 0 1 2 1 4 3 0 3 0 1 4 3 3

1 1 3 3 3 0 4 0 2 1 2 0 0 3 0 3 1 2 1 2 1 1 0 4'1 1 1 2

In order to compress the sequence of bracelets, the notation H^m has been employed to indicate that m bracelets H have been produced in succession, and J_n to indicate that n bracelets J have been produced in succession. In this notation, the entire bracelet of 312 bracelets becomes

$$H^1J_2H^2J_2H^1J_3H^1J_2H^1J_1H^1J_4H^1J_1H^1J_1H^2J_3H^2J_1H^1J_1H^3J_1H^1$$

$$J_2H^2J_3H^4J_1H^2J_1H^2J_1H^2J_2H^1J_1H^2J_1H^2J_4H^1J_3H^1J_5H^4J_1H^2J_1$$

$$H^1J_3H^3J_1H^1J_2H^1J_4H^5J_1H^3J_1H^4J_2H^1J_2H^1J_1H^2J_2H^1J_2H^1J_2H^1$$

$$J_4H^3J_2H^2J_1H^1J_3H^1J_1H^1J_2H^3J_2H^1J_1H^1J_1H^4J_1H^1J_1H^2J_1H^3J_1$$

$$H^2J_2H^2\underline{J_1}H^3J_1H^2J_1H^1J_1H^1J_1H^1J_3H^1J_1H^1J_1H^4J_1H^2J_1H^2J_3H^5$$
$$J_1H^3J_6H^2J_5H^3J_1H^1J_1H^1J_1H^1J_1H^1J_3H^5J_2H^6J_3H^1J_5H^3J_2H^1J_2$$
$$H^1J_4H^1J_1H^1J_1H^3J_1H^1J_1H^1J_1H^1J_2H^1J_3.$$

The cyclic sequence of the frequencies of H is

12111 11122 13124 22212 21142 13115 34112 11132
11131 14123 22321 11114 22532 31111 56131 11131 111'12.

This is the reverse of the cyclic sequence of the frequencies of J beginning with the frequency of the second J in the fifth line of the bracelet of bracelets.

The longest palindromes imbedded in the fourth-order bracelets are in J. In order of appearance as underscored in Table 1, they are

3334333, 4442444, 2221222, and 1113111.

The leading digits, as well as the central digits, reproduce bracelet B.

Fourth-order bracelets in the decimal system have been discussed previously [3,4,5], and a discussion of second- and third-order bracelets in base five appears in [6].

REFERENCES

1. William L. Schaaf, *A Bibliography of Recreational Mathematics*, Volume 3, National Council of Teachers of Mathematics, Reston, Virginia, 1973, p. 124.

2. Mark Feinberg, "Fibonacci-Tribonacci", *Fibonacci Quarterly*, 1 (October 1963) 71-74.

3. Charles W. Trigg, "A Digital Bracelet for 1967", *Fibonacci Quarterly*, 4 (December 1967) 477-480.

4. ___________, "A Digital Bracelet for 1968", *Journal of Recreational Mathematics*, 1 (April 1968) 108-111.

5. ___________, "A Digital Bracelet for 1971". *Mathematics Teacher*, 64 (October 1971) 567-571.

6. ___________, "Additive Digital Bracelets in the Quinary System", *Crux Mathematicorum*, 9 (August-September 1983) 200-204.

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* * *

MATHEMATICAL CLERIHEWS

William Rowan Hamilton,
A toping and untrammelled son
Of the Hibernians,
Devised quaternions.

Charles Lutwidge Dodgson (Lewis Carroll)
Gave lectures in a clerk's apparel,
Was fond of logic, pun and riddle,
And also Alice Pleasance Liddell.

ALAN WAYNE, Holiday, Florida

* * *

THE OLYMPIAD CORNER: 48

M.S. KLAMKIN

Later on in this column I will give solutions to several problems proposed here earlier. But first I give two new problem sets. The first consists of the problems set at the final round of the 1982 Swedish Olympiad. The second is a set of problems proposed in the March 1983 issue of *Középiskolai Matematikai Lapok* (Hungarian Mathematical Journal for Secondary Schools). As usual, for all of these problems I solicit elegant solutions, which should be sent directly to me at the address given at the end of this column.

*

1982 SWEDISH OLYMPIAD

1. Let N be a positive integer. How many solutions to the equation

$$x^2 - [x^2] = (x - [x])^2$$

are there in the interval $1 \le x \le N$?

2. Let a,b,c be positive numbers. Prove that

$$abc \ge (b+c-a)(c+a-b)(a+b-c).$$

3. Suppose one can find a point P in the interior of the quadrilateral ABCD such that the four triangles PAB, PBC, PCD, and PDA have the same area. Show that P is on one of the diagonals AC or BD.

4. In the triangle ABC the sides are AB = 33 cm, AC = 21 cm, and BC = m cm, where m is an integer. It is possible to find a point D on AB and a point E on AC such that

$$\text{AD} = \text{DE} = \text{EC} = n \text{ cm},$$

where n is an integer. What values can m take?

5. In an orthonormal coordinate system one considers the points (x,y), where x and y are integers with $1 \le x \le 12$, $1 \le y \le 12$. Each of these 144 points is coloured red, white, or blue. Show that there is a rectangle with sides parallel to the axes and having all its vertices the same colour.

6. If $0 \le a \le 1$ and $0 \le x \le \pi$, prove that

$$(2a-1)\sin x + (1-a)\sin(1-a)x \ge 0.$$

*

FROM *KÖZÉPISKOLAI MATEMATIKAI LAPOK* (March 1983)

F. 2410. Solve the system of equations

$$\begin{aligned} x + y + z &= 5 \\ x^2 + y^2 + z^2 &= 9 \\ xy + u + vx + vy &= 0 \\ yz + u + vy + vz &= 0 \\ zx + u + vz + vx &= 0. \end{aligned}$$

F. 2411. Show that there is no party of 10 members in which the members have 9, 9, 9, 8, 8, 8, 7, 6, 4, 4 acquaintances, respectively, among themselves. (Acquaintances are supposed to be mutual.)

F. 2412. We have d cartons and one box. The cartons are numbered from 1 to d and some of them (maybe all) contain balls. We want to collect the contents of all the cartons in the box. Carton i may be emptied if it contains exactly i balls, and this is done in such a way that one ball is placed into the box and the remaining i-1 balls are placed one by one into the cartons 1, 2, ..., i-1, respectively. For which n is it possible to place n balls into an appropriate number of cartons so that they all could be collected?

F. 2413. Given is a square ABCD. Find the locus of the points P for which

$$PA + PC = \sqrt{2} \max \{PB, PD\}.$$

F. 2414. We have two regular octagons $N_1 = A_1B_1 \ldots H_1$ and $N_2 = A_2B_2 \ldots H_2$. The sides A_1B_1 and A_2B_2 lie on the same line, and also the sides D_1E_1 and D_2E_2 lie on the same line. Furthermore, G_2 coincides with C_1, and $A_2B_2 < A_1B_1$. The regular octagon N_{i+1} is in the same relation with N_i as N_2 is with N_1 ($i = 2,3,\ldots$). Assuming that all octagons N_i have been constructed, show that the sum $R_1 + R_2 + \ldots$ of the radii of their circumcircles converges to the length of the segment A_1M, where $M = A_1C_1 \cap D_1O_2$ and O_2 is the center of N_2.

F. 2415. Choose 400 different points inside a unit cube. Show that 4 of these points lie inside some sphere of radius 4/23.

P. 375. Does there exist a function $f: R \to R$ such that $\lim_{x\to\infty} f(x) = \infty$ and

$$\lim_{x\to\infty} \frac{f(x)}{\ln(\ln(\ldots(\ln x)\ldots))} = 0$$

holds for all n (where n is the number of logarithm functions in the denominator)?

P. 376. Determine the sizes of those circles whose interior points can be painted with two colours in such a way that the endpoints of all line segments of unit length have different colours.

*

9. [1981: 236; 1982: 45] Let P be a polynomial of degree n satisfying

$$P(k) = \binom{n+1}{k}^{-1}, \qquad k = 0,1,\dots,n.$$

Determine $P(n+1)$.

II. *Second solution by Noam D. Elkies, student, Columbia University.*

Since

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} P(k)$$

is the $(n+1)$st finite difference of the nth-degree polynomial P, this sum must vanish. Thus

$$\sum_{k=0}^{n} (-1)^k + (-1)^{n+1} P(n+1) = 0,$$

and it follows that

$$P(n+1) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

1. (Second Round) [1982: 12] The sequence $\{a_1, a_2, a_3, \dots\}$ is defined as follows: a_1 is an arbitrary positive integer and, for $n > 1$, $a_n = [3a_{n-1}/2]+1$. Is it possible to choose a_1 such that a_{100001} is odd and a_n is even for all $n \le 100000$?

Solution by John Morvay, Dallas, Texas.

The answer is yes. Just choose $a_1 = 2^{100000} - 2$. It then follows easily that

$$a_n = 3^{n-1} \cdot 2^{100001-n} - 2 \text{ for } n \le 100000$$

and

$$a_{100001} = 3^{100000} - 2.$$

2. [1982: 270] Let $(a_1, a_2, \dots, a_n, \dots)$ be a sequence of positive real numbers such that $a_n^2 \le a_n - a_{n+1}$ for all n. Show that $a_n < 1/n$ for all n.

Solution by K.S. Murray, Brooklyn, N.Y.

The defining relation shows that we must have $a_2 \le a_1(1-a_1)$. Thus

$$0 < a_1 < 1 \quad \text{and} \quad a_2 \le \tfrac{1}{4} < \tfrac{1}{2}.$$

We now use $a_{n+1} \le a_n(1-a_n)$, $n = 2,3,4,\ldots$ and proceed by induction. Assume that $a_k < 1/k$ for some $k \ge 2$. Since $x(1-x)$ is increasing in the interval $[0,\frac{1}{2}]$, we have

$$a_{k+1} \le a_k(1-a_k) \le \frac{1}{k}\left(1-\frac{1}{k}\right) = \frac{k-1}{k^2} < \frac{1}{k+1},$$

and the induction is complete.

3. [1982: 270] A round track has n fueling stations (some possibly empty) containing a combined total of fuel sufficient for a car to travel once around the track. Prove that, irrespective of the initial distribution of fuel among the stations, it is always possible for a car with an empty tank to start from one of the stations and complete a round trip without running out of fuel on the way.

Comment.

See Crux 354 [1979: 57] for a solution and for references to other appearances of this problem by various proposers. However, the version given here, from a 1964 Peking Mathematics Contest, predates all the other references known to us at this time.

1. [1983: 137] Which of $(17091982!)^2$ and $17091982^{17091982}$ is greater?

Solution by Noam D. Elkies, student, Columbia University.

The result follows by setting $n = 17091982$ in the inequality

$$(n!)^2 > n^n, \qquad n > 2, \tag{1}$$

which itself follows from

$$(n!)^2 = \prod_{j=1}^{n} j(n+1-j) > \prod_{j=1}^{n} n = n^n.$$

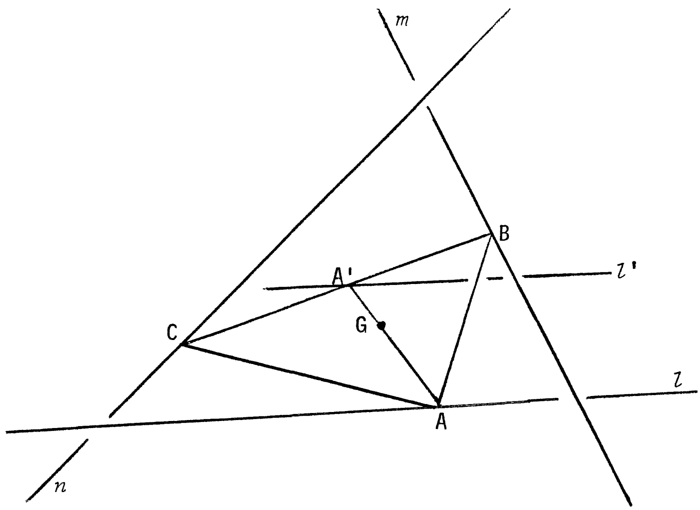
Comment by M.S.K.

Inequality (1) was also established nicely by Peter Ross (University of Santa Clara) by using the concavity of $\ln x$, and by Lones Smith (Nepean, Ontario) by means of the A.M.-G.M. inequality.

18-3. [1983: 107, 141] Show how to construct a triangle having its vertices on three given skew lines so that the centroid of the triangle coincides with a given point.

II. *Solution by Howard Eves, University of Maine.*

Denote the three mutually skew lines by l, m, n, and the given point by G, as shown in the figure. Let l' be the map of l under the homothety of center G and ratio $-\frac{1}{2}$. Let p be the plane midway between the plane through m parallel to n and the plane through n parellel to m; this plane is the locus of midpoints of all



segments joining points of m to points of n. Let p cut l' in A'. Let A'G cut l in A. Let the line of intersection of the plane determined by A' and m with the plane determined by A' and n cut m in B and n in C. Then ABC is the sought triangle.

If l' lies in p, any point on l' may serve as point A', and the problem has infinitely many solutions. If l' is parallel to p, the problem has no solution.

Comment by M.S.K.

I also received a solution equivalent to the above from Esther Szekeres (New South Wales, Australia). These two solutions were submitted in response to my request for a synthetic solution, following an analytic solution I had given earlier [1983: 141].

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

HOLLYWOOD ARITHMETIC

Item by Leonard Lyons in the September 1976 *Reader's Digest*, page 90: An actress who was offered a co-starring role with Zero Mostel declined, because Mostel is too overwhelming. "I'd be lost working with him. One plus Zero would still be Zero."

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

871. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In each of the following four (independent) cryptarithms, assign each X a different decimal digit to obtain decimal integers that make the cryptarithm arithmetically true.

(a) X•X•X•X = XX = X•X,

(b) X•X•X•X = XX simultaneously with X•X = X,

(c) X•X•X•X = XX simultaneously with X•X = X + X,

(d) X•X•X•X = XX = (X + X)•(X + X).

872. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let T be a triangle ABC with sides a,b,c and circumradius R, and let P be a point other than a vertex in the plane of T. It is known (M.S. Klamkin, "Triangle inequalities from the triangle inequality", *Elemente der Mathematik*, Vol. 34 (1979), No. 3) that there exists a triangle T_0 with sides $a\cdot$PA, $b\cdot$PB, and $c\cdot$PC. If R_0 is the circumradius of T_0, prove that

$$\text{PA}\cdot\text{PB}\cdot\text{PC} \le R\cdot R_0.$$

When does equality occur?

873. *Proposed by W.R. Utz, University of Missouri-Columbia.*

Show that the sequence $\{3n^2 + 3n + 1\}$, n an integer, contains an infinite number of squares but only one cube.

874. *Proposed by the COPS of Ottawa.*

Let S_n be the sum of the first n primes. Show that for every n there is at least one square between S_n and S_{n+1}.

875*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Can a square be dissected into three congruent nonrectangular pieces?

876. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a,b,c, and let K_a,K_b,K_c be the circles with centers A,B,C, respectively, and radii $\lambda\sqrt{bc}$, $\lambda\sqrt{ca}$, $\lambda\sqrt{ab}$, respectively, where $\lambda \geq 0$. Find the locus of the radical center of K_a, K_b, K_c as λ ranges over the nonnegative real numbers.

877. *Proposed by Charles W. Trigg, San Diego, California.*

Prove or disprove the following statement: *Every prime which is the reverse of a square integer is congruent to* 1 *modulo* 6.

Examples: $61 = 10\cdot 6 + 1$ is the reverse of 4^2, and $12391 = 2065\cdot 6 + 1$ is the reverse of 139^2.

878.* *Proposed by Kent D. Boklan, student, Massachusetts Institute of Technology.*

Let A be a real $n\times n$ matrix with nonzero entries. If A is singular (i.e., $\det A = 0$), does there always exist a real $n\times n$ matrix B such that $\det(AB+BA) \neq 0$?

879. *Proposed by Leroy F. Meyers, The Ohio State University.*

The U.S. Social Security numbers consist of 9 digits (with inital zeros permitted). How many such numbers are there which do not contain any digit three or more times consecutively?

880.* *Proposed by Clark Kimberling, University of Evansville, Indiana.*

For a given triangle ABC, what curve is formed by all the points P in three-dimensional space satisfying

$$\angle BPC = \angle CPA = \angle APB?$$

* * *

INTRODUCING MAYBE THE NEXT EDITOR OF *CRUX MATHEMATICORUM*

After the second round of crossword competition, Stanley Newman jumped up and said: "The Wicked Wasp of Twickenham! That's Alexander Pope! With a mathematics background, I'm not supposed to know such things, but I do. I picked them up from doing puzzles."

From a news item in the *New York Times*, Sunday, August 21, 1983, about the Second United States Open Crossword Puzzle Championship, which took place at the Loeb Student Center at New York University on August 20. One of the four puzzles everyone had done by mail to qualify them for the event had a mathematical clue that stumped a lot of people. It was "Kilenc plus kilenc". Paul Erdös will confirm that the correct answer is "Tizennyolc".

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

721. [1982: 77; 1983: 87] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

A propos of the editor's comment following Crux 611 [1982: 30], verify that, with decimal integers,

(a) uniquely, TRIGG is three times WRONG;

(b) independently, but also uniquely, WAYNE is seven times RIGHT.

III. *Letter to the editor by Stewart Metchette, Culver City, California.*

Per your comment to Problem 721 [1983: 89]:

I hope digitomaniacs continue their efforts, even at the risk of repeating what has been done before. For example, you quote Victor Thébault's work [1] on the 2-factor digital products: if your quote is accurate, his solution list for the 10-digit case is incomplete; if inaccurate, you have done him a disservice.

There are an additional 9 and 6 solutions (for a total of 13 and 9) for

A•BCDE = FGHIJ and AB•CDE = FGHIJ,

respectively. These are listed by Madachy [2] and attributed by him to Charles L. Baker.

You go on to say that "finding all products containing each of the digits exactly once... has already been done." In Madachy's book, he mentions that some 3-factor solutions might be possible, but knew of none.

Inspired by Madachy's comment, I ran a computer check and reported the results in [3]:

A•BC•DE = FGHI, 12 solutions;
A•BC•DE = FGHIJ, 10 solutions;
A•BC•DEF = GHIJ, 2 solutions.

There is still much work for digitomaniacs to perform. All problems requiring solutions using the nine nonzero digits can be redefined in terms of nine *distinct* digits, which need investigation.

I wish them good hunting!

Editor's comment.

In his earlier comment, the editor meant and should have written "finding all *two-factor* products... has already been done." The italicized qualifier was unfortunately left out. The editor then went on to give Thébault's solutions, which were quoted correctly and completely from [1]. However, a more careful reading of [1] shows that Thébault wrote: "In this case [the ten-digit case] the investigation is still more complicated, and the following solutions are submitted...". Thébault then listed the 7 solutions given earlier. But he does not actually claim that these are the only solutions.

The editor, no digitomaniac, thanks reader Metchette for the clarification, and apologizes to Victor Thébault's angry shade.

REFERENCES

1. V. Thébault, Solution to Problem E 13 (proposed by W.F. Cheney, Jr.), *American Mathematical Monthly*, 41 (1934) 265-266.

2. Joseph S. Madachy, *Mathematics on Vacation*, Charles Scribner's Sons, New York, 1966, pp. 183-185.

3. Stewart Metchette, "A Note on Digital Products", *Journal of Recreational Mathematics*, 10 (1977-78) 270-271.

* * *

755. [1982: 174] *Proposed by László Csirmaz, Mathematical Institute, Hungarian Academy of Sciences.*

Find the locus of points with coordinates

$$(\cos A + \cos B + \cos C,\quad \sin A + \sin B + \sin C)$$

(a) if A,B,C are real numbers with $A + B + C = \pi$;

(b) if A,B,C are the angles of a triangle.

Solution by Jordi Dou, Barcelona, Spain.

(a) Let $A = 2\omega$ and $B = t$, so that $C = \pi-2\omega-t$. A point (x,y) lies on the locus if and only if

$$x = \cos 2\omega + \cos t + \cos(\pi-2\omega-t) = \cos 2\omega + 2\sin\omega\sin(\omega+t)$$

and

$$y = \sin 2\omega + \sin t + \sin(\pi-2\omega-t) = \sin 2\omega + 2\cos\omega\sin(\omega+t).$$

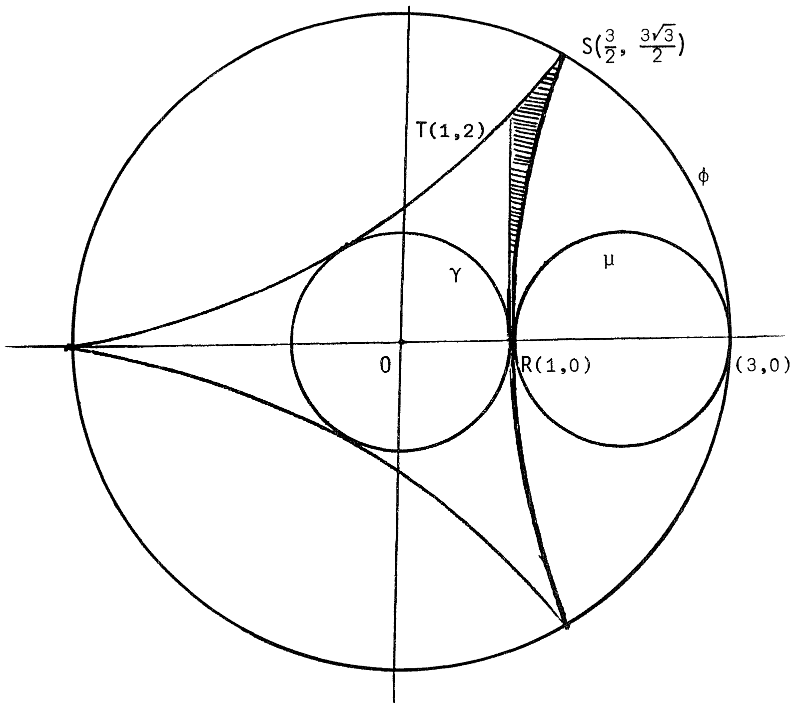
The resulting

$$\frac{y - \sin 2\omega}{x - \cos 2\omega} = \cot\omega$$

and

$$(x - \cos 2\omega)^2 + (y - \sin 2\omega)^2 = 4\sin^2(\omega+t)$$

show that, *for fixed* ω *and variable* t, the locus of (x,y) is a segment PQ of



length 4 whose midpoint ($\cos 2\omega$, $\sin 2\omega$) lies on the unit circle γ with centre at the origin O. The endpoints of PQ correspond to $t = -\omega \pm \frac{\pi}{2}$; their coordinates are

$$x_P = \cos 2\omega - 2\sin\omega, \qquad y_P = \sin 2\omega - 2\cos\omega$$

and

$$x_Q = \cos 2\omega + 2\sin\omega, \qquad y_Q = \sin 2\omega + 2\cos\omega.$$

Since $P(\omega+\pi) = Q(\omega)$, it follows that, as ω varies, P and Q describe the same curve. This curve, shown in the figure, is a 3-cusped hypocycloid, or deltoid, generated by a fixed point on a circle μ of radius 1 which rotates without slipping inside a circle ϕ of radius 3 with centre at the origin O. The interior and boundary of the deltoid constitute the desired locus. Its area is 2π and its perimeter is 16.

(b) If A,B,C are the angles of a triangle (possibly degenerate), then we have the additional restrictions

$$0 \le A=2\omega \le \pi \quad \text{and} \quad 0 \le B=t \le \pi-2\omega.$$

With these restrictions and for fixed ω, the segment PQ is either vertical or else it has a positive slope $\cot\omega$. The coordinates of the lower endpoint P are obtained by setting $t = 0$. They are

$$x_P = \cos 2\omega + 2\sin^2\omega = 1 \quad \text{and} \quad y_P = 2\sin 2\omega;$$

so, as ω varies, P describes the segment RT shown in the figure. The coordinates of the higher endpoint Q are obtained by setting $t = -\omega+\frac{\pi}{2}$. They are

$$x_Q = \cos 2\omega + 2\sin\omega \quad \text{and} \quad y_Q = \sin 2\omega + 2\cos\omega.$$

As ω varies, Q describes the arcs RS and ST of the deltoid. The required locus consists of the interior and boundary of the mixtilinear triangular lamina RST (shaded in the figure). The segment RT contains the images of all degenerate triangles, the arcs RS and ST the images of all isosceles triangles, the point S is the image of the equilateral triangle, and all scalene triangles are mapped into the interior of the lamina.

Also solved by HENRY E. FETTIS, Mountain View, California; J.T. GROENMAN, Arnhem, The Netherlands (partial solution); VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (partial solution); and the proposer.

Editor's comment.

Referring to the figure, suppose the tangent to the deltoid at T meets the arc RS in U. The proposer showed that the segment TU contains the images of all right-angled triangles, so U is the image of the isosceles right-angled triangle; and that the parts of the shaded lamina above and below TU contain the images of all acute-angled and all obtuse-angled triangles, respectively.

Suppose all triangles are labelled so that $A \le B \le C$, and suppose the tangent to the deltoid at S meets the segment RT in V. As Murty showed earlier in this journal [1982: 64-65], the points of segment VS correspond to triangles in which $B = \pi/3$; the points of the shaded lamina above and below VS correspond to the triangles with $B > \pi/3$ and $B < \pi/3$, respectively; and the point V corresponds to the degenerate triangle with angles 0, $\pi/3$, and $2\pi/3$.

If $TU \cap VS = W$, then W corresponds to the triangle with angles $\pi/6$, $\pi/3$, and $\pi/2$. Finally, R and T correspond to the degenerate triangles with angles 0, 0, π and 0, $\pi/2$, $\pi/2$, respectively.

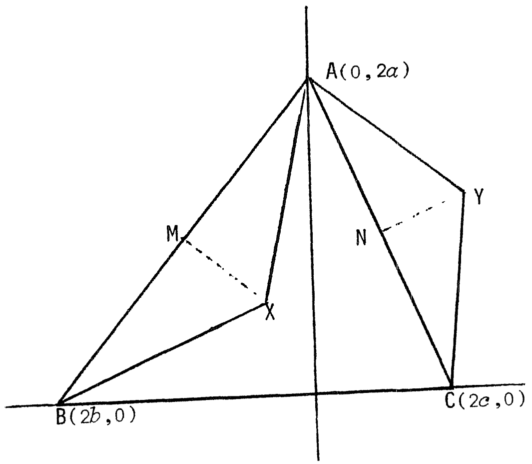
* * *

760. [1982: 175] *Proposed by Jordi Dou, Barcelona, Spain.*

Given a triangle ABC, construct with ruler and compass, on AB and AC as bases, *directly* similar isosceles triangles ABX and ACY such that BY = CX. Prove that there are exactly two such pairs of isosceles triangles.

Solution by F.G.B. Maskell, Algonquin College, Ottawa.

We introduce a rectangular coordinate system such that the vertices of the triangle have coordinates $A(0,2a)$, $B(2b,0)$, and $C(2c,0)$, as shown in the figure. The midpoints of AB and AC are then $M(b,a)$ and $N(c,a)$. It is easy to verify that ABX and ACY are directly similar isosceles triangles if and only if the coordinates of X and Y are



$$X(b+2a\rho,\ a+2b\rho)$$

and

$$Y(c+2a\rho,\ a+2c\rho),$$

where ρ is an arbitrary constant. (As a check, we find that each of the angles BAX and CAY then equals Arccos $1/\sqrt{1+4\rho^2}$.)

Now BY = CX if and only if

$$(-2b+c+2a\rho)^2 + (a+2c\rho)^2 = (-2c+b+2a\rho)^2 + (a+2b\rho)^2,$$

an equation equivalent to

$$4(b+c)\rho^2 + 16a\rho - 3(b+c) = 0.$$

If $b+c \neq 0$, this is a quadratic equation with a positive discriminant, so there are exactly two distinct satisfactory values of ρ. Those values are constructible, so are X and Y, and there are exactly two pairs of satisfactory triangles ABX and ACY.

If $b+c = 0$, the only acceptable value is $\rho = 0$. In this case AB = AC, and the only solution to the problem consists of the pair of degenerate triangles ABM and ACN.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. One incorrect solution was received.

* * *

761. [1982: 209] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the following independent *base eight* alphametics, each of which has a unique solution and is doubly-true in German.

```
(a) EINS      (b) EINS
    EINS          EINS
    EINS          ZWEI
    EINS          ----
    ----          VIER
    VIER
```

I. *Solution to part (a) by Kenneth M. Wilke, Topeka, Kansas.*

For ease of comprehension, we carry out all our calculations in base ten. Immediately E = 1; and since $4x \equiv 4$ or $0 \pmod 8$, N is even and $0 < 4S < 16$. Hence (S,R) = (2,0) or (3,4). If (S,R) = (2,0), then N = 4 or 6 and no valid choice of I is possible. Thus (S,R) = (3,4) and N = 2 or 6. But only N = 2 permits a valid choice for I, which is I = 5, and V = 6 follows. The unique solution is

```
1523
1523
1523
1523
----
6514.
```

II. *Solution to part (b) by the proposer.*

The alphametic is equivalent to the following system of equations, which are written in base ten:

$$2S + I = R + 8c_1, \qquad (1)$$

$$2N + c_1 = 8c_2, \qquad (2)$$

$$I + W + c_2 = 8c_3, \qquad (3)$$

$$2E + Z + c_3 = V. \qquad (4)$$

By (2), c_1 is even, so $c_1 = 0$ or 2. Also by (2), $c_2 \le 2$, so by (3) we have $c_3 = 1$. R and I have the same parity by (1). We tabulate solutions of

$$2E + Z + 1 = V; \qquad (4')$$

then for $c_2 = 0,1,2$ we enumerate solutions of

$$I + W + c_2 = 8; \qquad (3')$$

then we determine the possible R by parity with I; and finally we compute S by

$$S = \frac{R - I}{2} + 4c_1, \qquad (1')$$

remembering that, by (2), (c_2,N,c_1) is one of (0,0,0), (1,4,0), (1,3,2), or (2,7,2). This work [the details are omitted (Editor)] leads to the unique solution

```
1043
1043
2710
----
5016.
```

Also solved by the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; J.A.H. HUNTER, Toronto, Ontario (part (b) only); J.A. McCALLUM, Medicine Hat, Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas (also part (b)); ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer (also part (a)). In addition, one incorrect solution to part (a) and one to part (b) were received.

Editor's comment.

Feder noted that part (a) also has a unique solution in base ten:

$$\begin{array}{r} 1329 \\ 1329 \\ 1329 \\ \underline{1329} \\ 5316 \end{array}.$$

* * *

762. [1982: 209, 278] (Corrected) *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

ABC is a triangle with area K and sides a,b,c in the usual order. The internal bisectors of angles A,B,C meet the opposite sides in D,E,F, respectively, and the area of triangle DEF is K'.

(a) Prove that

$$\frac{3abc}{4(a^3+b^3+c^3)} \le \frac{K'}{K} \le \frac{1}{4}.$$

(b) If $a = 5$ and $3abc/4(a^3+b^3+c^3) = 5/24$, determine b and c, given that they are integers.

I. *Solution to part (a) by M.S. Klamkin, University of Alberta.*

We have AF/FB = b/a and AE/EC = c/a; hence, with brackets denoting area,

$$\frac{[\text{AFE}]}{K} = \frac{\text{AF}\cdot\text{AE}}{\text{AB}\cdot\text{AC}} = \frac{bc}{(a+b)(a+c)}.$$

With this and two similar results, we obtain

$$\frac{K'}{K} = 1 - \sum_{\text{cyclic}} \frac{bc}{(a+b)(a+c)} = \frac{2abc}{(b+c)(c+a)(a+b)},$$

and we have to show that

$$\frac{3abc}{4(a^3+b^3+c^3)} \le \frac{2abc}{(b+c)(c+a)(a+b)} \le \frac{1}{4}.$$

The first inequality will follow from

$$3(b+c)(c+a)(a+b) \le 8(a^3+b^3+c^3) \qquad (1)$$

and the second from

$$8abc \le (b+c)(c+a)(a+b). \qquad (2)$$

Now (1) and (2) are both known to hold for all nonnegative a,b,c, with equality in each case if and only if $a = b = c$. According to Bottema [1], the first is due to A. Padoa (1925) and the second to E. Cesàro (1880). □

Proofs for (1) and (2) are not given in [1]. The proof of (2) is nearly trivial ($2\sqrt{bc} \le b+c$, etc.), so for completeness we give here only a proof of (1), for which the Padoa reference is not easily accessible.

From the A.M.-G.M. inequality,

$$6abc \le 2(a^3 + b^3 + c^3); \qquad (3)$$

and from Problem 6-3 in this journal [1979: 198],

$$3(b^2c + c^2a + a^2b) \le 3(a^3 + b^3 + c^3) \qquad (4)$$

and

$$3(bc^2 + ca^2 + ab^2) \le 3(a^3 + b^3 + c^3). \qquad (5)$$

Now adding (3), (4), and (5) yields (1).

The inequality $K'/K \le \frac{1}{4}$ of part (a) was already known. It is included in the following more general result: *if* AD, BE, *and* CF *are three concurrent interior cevians of a triangle* ABC, *then the area of triangle* DEF *is a maximum if and only if the point of concurrency is the centroid of* ABC. (See the end of solution II of Crux 323 [1978: 256] and the end of the solution of Crux 585 [1981: 304].) So here we have $K'/K \le \frac{1}{4}$ with equality if and only if the incenter coincides with the centroid, that is, if and only if ABC is equilateral.

II. *Solution to part (b) by W.J. Blundon, Memorial University of Newfoundland.*

The equation we must solve for positive integers b and c is equivalent to

$$18bc = 125 + b^3 + c^3,$$

which is itself equivalent to

$$3bc = (b+c)^2 - 6(b+c) + 36 - \frac{91}{b+c+6}.$$

Since $b+c+6 \mid 91$, we have $b+c+6 = 13$ or 91, so $b+c = 7$ or 85, and the corresponding values of bc are 12 and 2250. Only the pair $(b+c,\ bc) = (7,\ 12)$ is satisfactory, and it leads to the unique solution $\{b,c\} = \{3,4\}$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundanld (part (a) also); JORDI DOU, Barcelona, Spain; G.C. GIRI, Midnapore College, West Bengal, India (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, California (part (b) only); ROBERT S. JOHNSON, Montréal, Québec (part (b) only); M.S. KLAMKIN, University of Alberta (part (b) also); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

REFERENCE

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, pp. 12-13.

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764. [1982: 209] *Proposed by Kent Boklan, student, Massachusetts Institute of Technology.*

Find all positive integer pairs $\{m,n\}$ such that

$$\frac{1}{m}+\frac{1}{n}=\frac{q}{p},$$

where p and q are consecutive primes with $p > q$.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

It is known [1] that, if p and q are positive integers, then q/p can be expressed as $1/m + 1/n$ if and only if there exist divisors p_1 and p_2 of p such that $q|p_1+p_2$. But if p is prime, its only divisors are 1 and p. Thus $q|p+1$.

Now by Bertrand's postulate, if $q > 3$ then $p+1 < 2q$, and so $q\nmid p+1$. Thus there are no solutions with $q > 3$. If $q = 2$ or 3, it is easily shown that the only solutions are

$$\frac{1}{3}+\frac{1}{3}=\frac{2}{3},\qquad \frac{1}{2}+\frac{1}{6}=\frac{2}{3},\qquad \frac{1}{2}+\frac{1}{10}=\frac{3}{5};$$

so the required pairs are $\{m,n\} = \{3,3\}$, $\{2,6\}$, and $\{2,10\}$.

Also solved by the COPS of ottawa; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; ERNEST W. FOX, South Lancaster, Ontario; J.T. GROENMAN, Arnhem, The Netherlands; KELVIN MACBETH, Queensland Institute of Technology, Brisbane, Australia and DAVID R. STONE, Georgia Southern College, Statesboro, Georgia (jointly); BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; LAWRENCE SOMER, Washington, D.C.; DAVID STEINSALTZ, Hewlett Harbor, N.Y.; RAM REKHA TIWARI, Radhaur, Bihar, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer (two solutions).

REFERENCE

1. Problem E 2875 (proposed by David Singmaster, solution by Daniel A. Rawsthorne), *American Mathematical Monthly*, 89 (1982) 501.

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765. [1982: 210] *Proposed by K.P. Shum and R.F. Turner-Smith, The Chinese University of Hong Kong.*

If n is a given positive integer, find all solutions $\theta \in [0,2\pi)$ of the equation

$$\cos^n\theta + \sin^n\theta = 1.$$

(The trivial case $n = 2$ may be omitted.)

Solution by M.S. Klamkin, University of Alberta (revised by the editor).

For $n = 1$, we have

$$\cos\theta + \sin\theta = \sqrt{2}\sin(\theta+\tfrac{\pi}{4}) = 1 \quad \Longleftrightarrow \quad \theta = 0 \text{ or } \tfrac{\pi}{2}.$$

For $n \ge 3$, we will use the fact that, for *every* θ,

$$\cos^n\theta + \sin^n\theta \le \cos^2\theta + \sin^2\theta \ (= 1). \tag{1}$$

Now θ is a solution of the given equation if and only if equality holds in (1), that is, if and only if

$$\cos^n\theta \ge 0 \qquad \text{and} \qquad \sin^n\theta \ge 0, \tag{2}$$

and

$$|\cos\theta| = 1 \qquad \text{or} \qquad |\sin\theta| = 1. \tag{3}$$

The values of θ satisfying (2) and (3) are easily found to be

$$\theta = \begin{cases} 0,\ \frac{\pi}{2}, & \text{for odd } n; \\ 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, & \text{for even } n \ne 2. \end{cases} \tag{4}$$

Combining the cases $n = 1$ and $n \ge 3$, we find that (4) gives all the solutions for all $n \ne 2$. □

As a generalization, we will characterize all r-dimensional *unit* vectors

$$\vec{u} = (u_1, u_2, \ldots, u_r) \quad \text{and} \quad \vec{v} = (v_1, v_2, \ldots, v_r),$$

where $r \ge 2$, whose components satisfy the equation

$$\sum_{i=1}^{r} u_i^m v_i^n = 1, \tag{5}$$

where $m \ge 0$ and $n \ge 1$ are integers. (The proposed problem corresponds to the case $r = 2$, $m = 0$.)

Case 1: $m = 0$. Here, of course, the pair $(\vec{u},\vec{v})$ satisfies (5) if and only if $\vec{v}$ satisfies

$$\sum_{i=1}^{r} v_i^n = 1. \tag{6}$$

We consider several subcases.

Case 1.1: $m = 0$, $n = 1$. It is easy to see that there is an infinite class of vectors $\vec{v}$ satisfying (6) (for which a more precise characterization depends upon the value of r), except in the case $r = 2$, where

$$v_1 + v_2 = 1 \implies (v_1 + v_2)^2 = 1 \implies v_1 v_2 = 0 \implies \vec{v} = (\pm 1, 0) \text{ or } (0, \pm 1),$$

but only (1,0) and (0,1) satisfy (6) . (This suggests an alternative proof for the case $n = 1$ in the original problem.)

Case 1.2: $m = 0$, $n = 2$. It is clear that (6) holds for all unit vectors $\vec{v}$.

Case 1.3: $m = 0$, $n \geq 3$. Here we will use the fact that, for *every* $\vec{v}$,

$$\sum_{i=1}^{r} v_i^n \leq \sum_{i=1}^{r} v_i^2 \ (= 1). \tag{7}$$

Now $\vec{v}$ satisfies (6) if and only if equality holds in (7), that is, if and only if

$$v_i^n \geq 0 \text{ for } i = 1,2,\ldots,r \tag{8}$$

and

$$|v_{i_0}| = 1 \text{ and } v_i = 0 \text{ for } i \neq i_0 \tag{9}$$

for some $i_0 \in \{1,2,\ldots,r\}$. The vectors $\vec{v}$ whose components satisfy (8) and (9) are characterized by

$$\begin{cases} v_{i_0} = \ \ 1 \text{ and } \ v_i = 0 \text{ for } i \neq i_0, \text{ if } n \text{ is odd;} \\ v_{i_0} = \pm 1 \text{ and } \ v_i = 0 \text{ for } i \neq i_0, \text{ if } n \text{ is even.} \end{cases}$$

Case 2: $m = 1$, $n = 1$. Cauchy's inequality gives, for *all* vectors $\vec{u}$ and $\vec{v}$,

$$\left(\sum_{i=1}^{r} u_i v_i\right)^2 \leq \sum_{i=1}^{r} u_i^2 \cdot \sum_{i=1}^{r} v_i^2 \ \ (= 1). \tag{10}$$

If the pair $(\vec{u},\vec{v})$ satisfies (5), then equality must hold in (10), and hence

$$u_i = kv_i \text{ and } u_i v_i = kv_i^2, \qquad i = 1,2,\ldots,r,$$

for some constant k. But (5) holds only for $k = 1$, so the only solutions are the pairs $(\vec{u},\vec{v})$ with $\vec{u} = \vec{v}$.

Case 3: $m \geq 1$, $n \geq 2$. Here we will use the fact that, for *all* vectors $\vec{u}$ and $\vec{v}$,

$$\sum_{i=1}^{r} u_i^m v_i^n \leq \sum_{i=1}^{r} v_i^2 \quad (= 1). \tag{11}$$

Now the pair $(\vec{u},\vec{v})$ satisfies (5) if and only if equality holds in (11), that is, if and only if

$$u_i^m v_i^n \geq 0 \text{ for } i = 1,2,\ldots,r \tag{12}$$

and

$$|u_{i_0}| = |v_{i_0}| = 1 \text{ and } u_i = v_i = 0 \text{ for } i \neq i_0 \tag{13}$$

for some $i_0 \in \{1,2,\ldots,r\}$. The vectors $\vec{u}$ and $\vec{v}$ whose components satisfy (12) and (13) are characterized by

$$\begin{cases} u_i = v_i = 0 \text{ for } i \neq i_0 \text{ and} \\ u_{i_0} = v_{i_0} = \pm 1, \text{ if } m \text{ and } n \text{ are both odd;} \\ u_{i_0} = 1 \text{ and } v_{i_0} = \pm 1, \text{ if } m \text{ is odd and } n \text{ is even;} \\ u_{i_0} = \pm 1 \text{ and } v_{i_0} = 1, \text{ if } m \text{ is even and } n \text{ is odd;} \\ u_{i_0} = \pm 1 \text{ and } v_{i_0} = \pm 1, \text{ if } m \text{ and } n \text{ are both even.} \end{cases}$$

Also solved by HIPPOLYTE CHARLES, Waterloo, Québec; CURTIS COOPER, Central Missouri State University; CLAYTON W. DODGE, university of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; J. WALTER LYNCH, Georgia Southern College; and the proposers. Three incorrect solutions were received.

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766. [1982: 210] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let ABC be an equilateral triangle with center O. Prove that, if P is a variable point on a fixed circle with center O, then the triangle whose sides have lengths PA, PB, PC has a constant area.

Solution by George Tsintsifas, Thessaloniki, Greece.

Let T denote triangle ABC, and let γ, R, K be its circumcircle, circumradius, and area, respectively. Now let P be any point on a circle γ' with center O and radius ρ, for some $\rho > 0$. Applying the ptolemaic inequality [1] to the quadruple of noncollinear points (P,B,C,A), we obtain, in magnitude only,

$$\text{PB}\cdot\text{CA} + \text{PC}\cdot\text{AB} \geq \text{PA}\cdot\text{BC}, \quad \text{whence} \quad \text{PB} + \text{PC} \geq \text{PA};$$

and PC + PA ≥ PB and PA + PB ≥ PC follow likewise from the quadruples (P,C,A,B) and (P,A,B,C). Each inequality is strict if P does not lie on γ (i.e., $\rho \neq R$), and equality holds if P lies on γ (in one or two of the three cases, depending on the position of P on γ).

So there *is* a triangle (possibly degenerate) with sides PA,PB,PC. If this triangle is T', we must show that its area, K', is independent of the position of P on γ'. We already know that this is true if $\rho = R$, when T' is degenerate and $K' = 0$ for all positions of P on γ. We will henceforth assume that $\rho \neq R$.

If $T_1 \equiv A_1B_1C_1$ is the pedal triangle of P with respect to T (with $PA_1 \perp BC$, etc.), then it is known [2, p. 139] that its area K_1 satisfies

$$K_1 = \frac{|R^2 - \rho^2|}{4R^2} \cdot K. \tag{1}$$

It is also known [2, p. 136] that

$$B_1C_1 = \frac{\sqrt{3}}{2}PA, \qquad C_1A_1 = \frac{\sqrt{3}}{2}PB, \qquad A_1B_1 = \frac{\sqrt{3}}{2}PC,$$

from which follows

$$K_1 = \tfrac{3}{4}K'. \tag{2}$$

Finally, from (1) and (2),

$$K' = \frac{|R^2 - \rho^2|}{3R^2} \cdot K = \frac{\sqrt{3}}{4}|R^2 - \rho^2|,$$

which is independent of the position of P on γ'.

Also solved by O. BOTTEMA, Delft, The Netherlands; JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; HENRY E. FETTIS, Mountain View, California; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles (three solutions); and the proposer. A comment was received from DAN PEDOE, University of Minnesota.

Editor's comment.

Several solutions involved lengthy calculations (some with complex numbers) resulting from the use of Heron's formula. And several solvers, trusting souls that they are, blithely set out to calculate K' without first assuring themselves that there *was* a triangle T'.

An interesting fact brought out by the above solution is that T' is similar to the pedal triangle T_1. Eves noted the following special results: if $\rho = 2R$, then $K' = K$; and if $\rho = r$, the inradius of T, then $K' = \frac{1}{4}K$. Finally, Klamkin showed that R', the circumradius of T', is not constant for all positions of P on γ', but that

$$R'_{\max} = \frac{R^2 - R\rho + \rho^2}{3|R - \rho|} \quad \text{and} \quad R'_{\min} = \frac{R^2 + R\rho + \rho^2}{3(R + \rho)}.$$

REFERENCES

1. David C. Kay, *College Geometry*, Holt, Rinehart and Winston, New York, 1969, p. 270.

2. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960.

* * *

THE PUZZLE CORNER

Puzzle No. 42: Rebus (1 4 2 8 8)

1 ≡ 2

Poor Cinna's slain
In Shakespeare's play.
Alas! It's plain—
COMPLETE, I say.

Puzzle No. 43: Phonetic Rebus (3 14)

T

At logarithms, though you're able,
You'll never find THIS in the table.

Puzzle No. 44: Enigmatic rebus (*3 *8 *6)

π

By Robert Frost, an essay you must see;
Its title, clearly pictured, is MY KEY.

Puzzle No. 45: Rebus (7 4 5)

Y = HS

The manufacturer has made his will
To put his REBUS that have shown their skill.

Puzzle No. 46: Rebus (10)

C

The THESE are used, you see,
In trigonometry.

ALAN WAYNE, Holiday, Florida

* * *