Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 9, No. 3

March 1983

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin, Ottawa
Printed at Carleton University

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22 in Canada, US$20 elsewhere. Back issues: $2 each. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-8, $17 in Canada and US$15 elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, Ontario, Canada K1S 1N3.

Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Typist-compositor: Nghi Chung.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

* * *

CONTENTS

A CONSTRAINED MAXIMUM-PERIMETER TRIANGLE

M.S. KLAMKIN

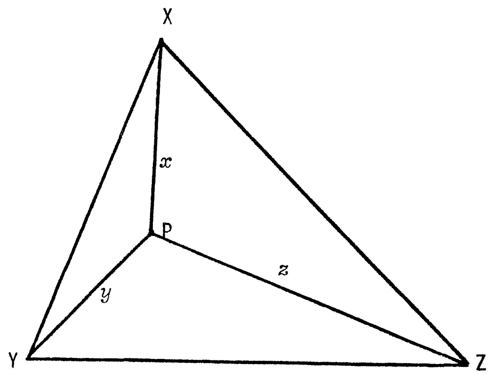
In a previous paper [1], we were given three concurrent segments of lengths x,y,z but not their relative orientation, and we determined the maximum area spanned by the three segments. For maximum area, we found that the point of concurrency P had to be the orthocenter of triangle XYZ (see figure). This led to the two-triangle inequality

$$\csc\frac{X}{2}\sin A + \csc\frac{Y}{2}\sin B + \csc\frac{Z}{2}\sin C \le \cot\frac{X}{2} + \cot\frac{Y}{2} + \cot\frac{Z}{2}, \qquad (1)$$

with equality if and only if

$$X + 2A = Y + 2B = Z + 2C = \pi.$$

(See also the solution of Crux 715 [1983: 58-62, esp. p. 61].)



Here we will determine the triangle XYZ of maximum perimeter. Again, our approach is geometric. Since X,Y,Z can be any points that are one on each of three fixed concentric circles, the possible triangles XYZ form a compact set, and thus a maximum-perimeter triangle exists. We assume that PY and PZ are positioned so that YZ is a side of that maximum-perimeter triangle, and then determine the condition on PX so that XY + XZ is as large as possible. Since the family of level curves

$$XY + XZ = \text{constant}$$

is a set of ellipses with foci at Y and Z, XY + XZ will be a maximum for the ellipse which is tangent (at X) to the circle with center P and radius x. Then, since the focal radii at X make equal angles with the normal PX to the ellipse, it follows that P lies on the internal bisector of angle YXZ. With similar results at the other vertices, we conclude that P must be the incenter of the maximum-perimeter triangle XYZ.

Let r be the inradius of the maximum-perimeter triangle XYZ, so that

$$\sin\frac{X}{2} = \frac{r}{x}, \qquad \sin\frac{Y}{2} = \frac{r}{y}, \qquad \sin\frac{Z}{2} = \frac{r}{z}. \tag{2}$$

From the triangle identity

$$\cos X + \cos Y + \cos Z = 1 + 4\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2},$$

it follows that r is the positive root of the equation

$$\left(1 - \frac{2r^2}{x^2}\right) + \left(1 - \frac{2r^2}{y^2}\right) + \left(1 - \frac{2r^2}{z^2}\right) = 1 + \frac{4r^3}{xyz},$$

which is equivalent to

$$2r^3 + \frac{y^2z^2 + z^2x^2 + x^2y^2}{xyz}\cdot r^2 - xyz = 0.$$

The positive root r of this equation having been found, the angles and the perimeter of triangle XYZ are easily determined.

Just as consideration of the maximum area led to the two-triangle inequality (1), consideration of the maximum perimeter leads to another two-triangle inequality. Let XYZ and ABC be two arbitrary triangles, and let a point P be chosen so that

$$\angle YPZ = \pi - A, \qquad \angle ZPX = \pi - B, \qquad \angle XPY = \pi - C.$$

With $PX = x$, $PY = y$, and $PZ = z$, the perimeter of triangle XYZ is

$$\sqrt{y^2+z^2+2yz\cos A} + \sqrt{z^2+x^2+2zx\cos B} + \sqrt{x^2+y^2+2xy\cos C}.$$

When x,y,z are fixed, we know that the maximum perimeter occurs just when $\pi - A = (\pi + X)/2$, or $X + 2A = \pi$, etc., and so

$$\Sigma\sqrt{y^2+z^2+2yz\cos A} \le \Sigma\sqrt{y^2+z^2+2yz\sin\frac{X}{2}}, \tag{3}$$

where the sums are cyclic with respect to x,y,z; X,Y,Z; and A,B,C; and equality occurs if and only if

$$X + 2A = Y + 2B = Z + 2C = \pi.$$

With the help of (2), we find that (3) is equivalent to the two-triangle inequality

$$\Sigma\sqrt{\sin^2\frac{Y}{2}+\sin^2\frac{Z}{2}+2\sin\frac{Y}{2}\sin\frac{Z}{2}\cos A}\cdot\sin\frac{X}{2}$$

$$\leq \Sigma\sqrt{\sin^2\frac{Y}{2}+\sin^2\frac{Z}{2}+2\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}}\cdot\sin\frac{X}{2}. \qquad (4)$$

The dual problem of finding the minimum perimeter is very easy. Here we have

$$\text{Perimeter} \geq |y-z| + |z-x| + |x-y|,$$

with equality if and only if X,Y,Z are collinear, leading to a degenerate triangle.

Inequality (1) can be extended in a similar way to simplexes, which will have to be orthocentric in order to have a maximum volume. However, the extensions of (4) even to three dimensions appear to be difficult, and we leave the following open problems:

Given the lengths, but not the relative orientations, of four concurrent segments emanating from a fixed point in space, determine the tetrahedron spanned by the four segments which maximizes (i) the total edge length, (ii) the surface area.

REFERENCE

1. M.S. Klamkin, "An Identity for Simplexes and Related Inequalities", *Simon Stevin*, 48 (1974-1975) 57-64.

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

A SYMMETRICAL-WITHIN-PANDIAGONAL PRIME MAGIC SQUARE

Composed of distinct primes, this pandiagonal sixth-order magic square has a symmetrical fourth-order magic square in its center. The computations to find it were carried out on a Model I Radio Shack TRS-80 microcomputer.

179	1259	683	677	881	311
383	839	557	743	521	947
821	911	353	167	1229	509
827	101	1163	977	419	503
761	809	587	773	491	569
1019	71	647	653	449	1151

ALLAN WM. JOHNSON JR.
Washington, D.C.

* * *

THE PUZZLE CORNER

Answer to Puzzle No. 29 [1983: 44]: Pintos (P into S).
Answer to Puzzle No. 30 [1983: 44]: Double meaning (Double mean in G).

A REMARKABLE GROUP OF SELF-COMPLEMENTARY NINE-DIGIT SQUARE ARRAYS

CHARLES W. TRIGG

Two digits are said to be *complementary* if their sum is 10. Two arrays of digits are *complementary* if their corresponding elements are complementary. For example, the complement of the magic square

$$\begin{matrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{matrix} \quad \text{is} \quad \begin{matrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{matrix}.$$

But this complement is the magic square rotated through 180°. A square array may appear in eight *equivalent* transformations (which will henceforth be considered to be indistinguishable): the square itself, its rotations through 90°, 180°, and 270°, and the mirror images of these four. Since the complement of the magic square is one of its equivalent transformations, the magic square is *self-complementary.*

In order that a nine-digit square array be self-complementary, its central element must be 5.

The two quartets (1,2,9,8) and (3,6,7,4) contain distinct digits. Place the digits of the first quartet, in order, cyclically at the midpoints of the sides of a square. Then place the digits of the other quartet, in order, cyclically on the corners of the square, clockwise and counterclockwise, in all possible positions. By this routine, with 5 as the central element, the following eight nine-digit arrays are generated:

$$\begin{matrix} 3 & 1 & 6 \\ 8 & 5 & 2 \\ 4 & 9 & 7 \end{matrix} \qquad \begin{matrix} 4 & 1 & 3 \\ 8 & 5 & 2 \\ 7 & 9 & 6 \end{matrix} \qquad \begin{matrix} 7 & 1 & 4 \\ 8 & 5 & 2 \\ 6 & 9 & 3 \end{matrix} \qquad \begin{matrix} 6 & 1 & 7 \\ 8 & 5 & 2 \\ 3 & 9 & 4 \end{matrix}$$

$$\begin{matrix} 3 & 1 & 4 \\ 8 & 5 & 2 \\ 6 & 9 & 7 \end{matrix} \qquad \begin{matrix} 6 & 1 & 3 \\ 8 & 5 & 2 \\ 7 & 9 & 4 \end{matrix} \qquad \begin{matrix} 7 & 1 & 6 \\ 8 & 5 & 2 \\ 4 & 9 & 3 \end{matrix} \qquad \begin{matrix} 4 & 1 & 7 \\ 8 & 5 & 2 \\ 3 & 9 & 6 \end{matrix}$$

Each of these different arrays is a 180° rotation of its complement. Consequently, the eight arrays are self-complementary. Upon interchanging the roles of the two quartets, eight more self-complementary arrays can be formed:

$$\begin{matrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{matrix} \qquad \begin{matrix} 8 & 3 & 1 \\ 4 & 5 & 6 \\ 9 & 7 & 2 \end{matrix} \qquad \begin{matrix} 9 & 3 & 8 \\ 4 & 5 & 6 \\ 2 & 7 & 1 \end{matrix} \qquad \begin{matrix} 2 & 3 & 9 \\ 4 & 5 & 6 \\ 1 & 7 & 8 \end{matrix}$$

$$\begin{matrix} 1 & 3 & 8 \\ 4 & 5 & 6 \\ 2 & 7 & 9 \end{matrix} \qquad \begin{matrix} 2 & 3 & 1 \\ 4 & 5 & 6 \\ 9 & 7 & 8 \end{matrix} \qquad \begin{matrix} 9 & 3 & 2 \\ 4 & 5 & 6 \\ 8 & 7 & 1 \end{matrix} \qquad \begin{matrix} 8 & 3 & 9 \\ 4 & 5 & 6 \\ 1 & 7 & 2 \end{matrix}$$

These same eight arrays can be obtained (in some equivalent form) by rotating the perimeter of each of the previous eight arrays through 45°.

The quartets (1,3,9,7) and (2,4,8,6) can be used to construct sixteen more self-complementary arrays; and the quartets (1,4,9,6) and (2,3,8,7) still sixteen more—a total of forty-eight self-complementary arrays.

Remarkable, yes; surprising, no! In each quartet, the alternate digits are complementary, and the routine places complementary digits in diametrically opposite positions. There are four pairs of distinct complementary digits: 1,9; 2,8; 3,7; and 4,6. Quartets can be formed from these in only six ways; and these quartets can be paired without duplicating digits only in the three sets given above. In any set, the digits can be cyclically alternated in four distinct ways; for example:

(1,2,9,8) and (3,4,7,6), (1,2,9,8) and (3,6,7,4),
(1,8,9,2) and (3,4,7,6), (1,8,9,2) and (3,6,7,4).

The routine applied to each of these four arrangements produces the same sixteen arrays.

With 5 as the central digit, the sum of the three digits in each middle row and column is 15. Then the sum of the digits in the other two rows (columns) is 45 - 15 = 30. It follows that the sum of the digits in the middle row (column) is the arithmetic mean of the sums of the digits in the outer rows (columns). That is, in every self-complementary nine-digit square array, the sums of the digits in the three rows (columns) form an arithmetic progression.

The common difference of the A.P.'s may vary from 0 to 9. The differences in the row-sum A.P.'s and the column-sum A.P.'s may be the same or different. Examples of arrays with various differences and some of the possible difference combinations are given below with the row differences, d_r, preceding the column differences, d_c:

	8	1	6	3	2	9	1	3	6	3	4	2
	3	5	7	4	5	6	8	5	2	1	5	9
	4	9	2	1	8	7	4	7	9	8	6	7
d_r, d_c =	0	,	0	1	,	7	5	,	2	6	,	3

	2	1	4	1	4	8	4	2	3	2	1	6
	3	5	7	3	5	7	1	5	9	3	5	7
	6	9	8	2	6	9	7	8	6	4	9	8
d_r, d_c =	8	,	4	2	,	9	6	,	3	6	,	6

No self-complementary nine-digit arrays exist in which $(d_r,d_c) = (0,2)$.

(0,6), (1,1), (2,3), (2,4), (2,8), (3,5), (3,7), (4,4), (4,6), (4,9), or any pairing in which $d_r+d_c \geq 14$. There are two arrays in which $(d_r,d_c) = (0,3), (0,5), (0,9), (1,8), (2,5), (2,6), (2,9)$, and $(5,6)$. The pairing (1,2) appears in three arrays, and (3,6) in four arrays.

2404 Loring Street, San Diego, California 92109.

* * *

THE PUZZLE CORNER

Puzzle No. 31: Alphametic

Though you may be at home where the buffalo roam,
Don't delude yourself thinking you're free.
You're as firmly entrenched in the stuff of this poem
AS (in soil) is the root of a TREE.

It's been said I'm a nothing—that I look like a zero...
I'll tell you right now that's as wrong as can be;
For the cube of that RUBIK (the man who's my hero)—
You might as well know—actually started with ME.

Puzzle No. 32: Alphametic

At the root of SYMMETRY, therein lies
Some logic—and yes—beauty too.
But whatever query is its disguise,
The answer, we think, must be TRUE.

Puzzle No. 33: Alphametic

The number pulled out of my hat
Was LUCAS; of 1,3,... begat.
Please don't find ME a square or a bore
But PRIME is a prime and, what's more,
A reversible one at that!

Puzzle No. 34: Alphametic

A well-known five-digit square root
Turned out to be prime, to boot.
Now that I've put mine in it,
Got its square in a minute...
'Twas TORNADO: a twister to suit.

HANS HAVERMANN, Weston, Ontario

THE OLYMPIAD CORNER: 43

M.S. KLAMKIN

I first give a new Practice Set, for which solutions will be given next month. This is followed by a more challenging set of four Russian problems from *Kvant*.

PRACTICE SET 17

17-1. The numbers 1, 2, 3, ..., n^2 are partitioned into n groups of n numbers. If s_j denotes the sum of the numbers in the jth group, determine the maximum and minimum values of $S \equiv s_1 s_2 \ldots s_n$.

Rider. Find the number of distinct partitions which yield the maximum value of S.

17-2. ABCD is a skew quadrilateral (i.e., with sides not all coplanar) such that

$$\angle ABC = \angle BCD = \angle CDA = 90^\circ.$$

Prove that $\angle DAB$ is acute.

17-3. The equation $(x-a_1)(x-a_2)\ldots(x-a_n) = 1$, where the a_i are all real, has n real roots r_i. Find the minimum number of real roots of the equation

$$(x-r_1)(x-r_2)\ldots(x-r_n) = -1.$$

*

The Russian elementary journal *Kvant* now gives some of its proposed problems in English as well as in Russian. I give below four interesting ones which appeared recently, and I shall give more from time to time. Since the solutions eventually published in *Kvant* are in Russian only, I would appreciate receiving from readers, for possible later publication here, elegant solutions in English.

K-1. Each side of a given triangle is divided into three equal parts. The six points of division are the vertices of two triangles whose intersection is a hexagon. Find the area of the hexagon in terms of the area S of the given triangle.

K-2. The sequence $(a_1, a_2, \ldots, a_n)$ is a permutation of $(1,2,\ldots,n)$.

(a) Prove that $|a_1-a_2| + |a_2-a_3| + \ldots + |a_n-a_1| \geq 2n-2$.

(b) For how many distinct permutations of $(1,2,\ldots,n)$ does equality hold in (a)?

K-3. Let $k > 2$ be a given natural number. Does there exist an infinite set E of natural numbers such that, for every finite subset A of E,

$$\sum_{a_i \in A} a_i \neq b^k$$

for any natural number b?

K-4. Some turtles are creeping in the plane in different (constant) directions, but at the same speed. Prove that the turtles will eventually be located at the vertices of a convex polygon, no matter what their initial positions were.

*

I now give an edited version of the official solutions for the 1982 University of Alberta Undergraduate Mathematics Contest [1983: 41].

UNIVERSITY OF ALBERTA UNDERGRADUATE MATHEMATICS CONTEST

November 22, 1982 — Time: 3 hours

1. Find the equation of a cone with vertex at the origin and containing the intersection of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ with the sphere $x^2+y^2+z^2 = r^2$, where $a < r < c$.

Solution.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

represents a cone with vertex at the origin, and it clearly contains the intersection of the ellipsoid and the sphere.

2. Prove that the following two statements are equivalent:

(a) There are no positive integers a,b,c such that $a^4 - b^4 = c^2$.

(b) There are no positive integers w,x,y,z such that $w^2+x^2 = y^2$ and $w^2-x^2 = z^2$.

Assuming that either (a) or (b) has been proved, deduce the following special case of Fermat's Last Theorem: *There are no positive integers x,y,z such that $x^4 + y^4 = z^4$.*

Solution.

Assume (a). If $w^2 + x^2 = y^2$ and $w^2 - x^2 = z^2$, then

$$w^4-x^4 = (w^2+x^2)(w^2-x^2) = y^2z^2 = (yz)^2,$$

contradicting (a). So (a) $\Longrightarrow$ (b).

Now assume (b). Suppose $a^4 - b^4 = c^2$, where we assume without loss of generality that $(a,b) = 1$. Then (c,b^2,a^2) is a primitive Pythagorean triple. By the canonical representation of such triples, we have $a^2 = m^2+n^2$, while $b^2 = 2mn$ or $b^2 = m^2-n^2$ for some $m > n$. The second case is impossible by (b). In the first case, $a^2+b^2 = (m+n)^2$ and $a^2-b^2 = (m-n)^2$, again contrary to (b). So (b) ==> (a).

Finally, we deduce the special case of Fermat's Last Theorem from (a). Suppose $x^4+y^4 = z^4$. Then $z^4-y^4 = (x^2)^2$, contrary to (a).

3. Let f be a three times continuously differentiable real-valued function of a real variable such that the recurrence relation $x_{n+1} = f(x_n)$ with $x_1 > 0$ always has the property that $\lim_{n\to\infty}\sqrt{n}x_n = 1$. Evaluate $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$. [The original wording of the problem has been changed slightly.]

Solution.

With $x_1 > 0$, we have $\lim_{n\to\infty}\sqrt{n}x_n = 1$, which implies that $\lim_{n\to\infty}x_n = 0$. By the recurrence relation and the continuity of f, this is impossible unless $f(0) = 0$. With the Lagrange form of the Maclaurin formula with remainder, we now have

$$f(x_n) = f'(0)x_n + \frac{f''(0)}{2}x_n^2 + \frac{f'''(\theta x_n)}{6}x_n^3, \qquad \text{with } 0 < \theta < 1.$$

For all sufficiently large n, we therefore have

$$f(x_n) \approx f'(0)x_n \qquad \text{and} \qquad x_n \approx \frac{1}{\sqrt{n}};$$

hence

$$\frac{1}{\sqrt{n+1}} \approx \frac{f'(0)}{\sqrt{n}},$$

which is impossible unless $f'(0) = 1$. For all sufficiently large n, we now have

$$f(x_n) \approx x_n + \frac{f''(0)}{2}x_n^2,$$

and so

$$x_{n+1} - x_n \approx \frac{f''(0)}{2}x_n^2 \approx \frac{f''(0)}{2n}.$$

Hence

$$\sum_{n=N_1}^{\infty}(x_{n+1}-x_n) \approx \frac{f''(0)}{2}\sum_{n=N_1}^{\infty}\frac{1}{n},$$

where N_1 is sufficiently large. Here the left side converges (to $-x_{N_1}$) while the right side diverges unless $f''(0) = 0$. For all sufficiently large n, we now have

$$f(x_n) \approx x_n + \frac{f'''(\theta x_n)}{6}x_n^3,$$

and so, by the continuity of f''',

$$x_{n+1} - x_n \approx \frac{f'''(\theta x_n)}{6} x_n^3 \approx \frac{f'''(0)}{6n\sqrt{n}} .$$

Hence

$$\sum_{n=N_2}^{\infty} (x_{n+1} - x_n) \approx \frac{f'''(0)}{6} \sum_{n=N_2}^{\infty} \frac{1}{n\sqrt{n}},$$

where N_2 is sufficiently large. Here the left side converges to $-x_{N_2} \approx -1/\sqrt{N_2}$, while the

$$\text{right side} \approx \frac{f'''(0)}{6} \int_{N_2}^{\infty} \frac{1}{x\sqrt{x}}\, dx = \frac{f'''(0)}{6} \cdot \frac{2}{\sqrt{N_2}}.$$

We conclude that $f'''(0) = -3$.

In summary, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -3$.

4. Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of equations

$$\frac{1}{x_{j+1}} \sum_{i=1}^{5} a_i x_{i+j} = \frac{1}{x_{k+1}} \sum_{i=1}^{5} a_i x_{i+k} \qquad (1)$$

for $j,k = 0,1,2,3,4$, where $x_m = x_n$ if $m \equiv n \pmod 5$.

Solution.

If $f(j)$ denotes the left side of (1), then the system we are required to solve can be written

$$f(0) = f(1) = f(2) = f(3) = f(4). \qquad (2)$$

This is an indeterminate system of four equations in five unknowns. Let λ be the common value of the five expressions (2). Then the system is equivalent to the matrix equation

$$A \cdot X = O, \qquad x_1 x_2 x_3 x_4 x_5 \neq 0, \qquad (3)$$

where (with the usual typographical convention of using square brackets to represent a column vector)

$$X = [x_1, x_2, x_3, x_4, x_5], \qquad O = [0,0,0,0,0],$$

and

$$A = \begin{pmatrix} a_1-\lambda & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1-\lambda & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1-\lambda & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1-\lambda & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1-\lambda \end{pmatrix}.$$

Since $X \neq O$, (3) requires det $A = 0$, which is an equation of the fifth degree in λ whose roots we will now determine. For $i = 1,\ldots,5$, we multiply the ith column of A by a nonzero number v_i (to be determined) and then add the last four columns to the first. The first column of $v_1v_2v_3v_4v_5\cdot$det A now consists of

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 - \lambda v_1$$

$$a_5v_1 + a_1v_2 + a_2v_3 + a_3v_4 + a_4v_5 - \lambda v_2$$

$$a_4v_1 + a_5v_2 + a_1v_3 + a_2v_4 + a_3v_5 - \lambda v_3$$

$$a_3v_1 + a_4v_2 + a_5v_3 + a_1v_4 + a_2v_5 - \lambda v_4$$

$$a_2v_1 + a_3v_2 + a_4v_3 + a_5v_4 + a_1v_5 - \lambda v_5,$$

and det $A = 0$ when each of these five expressions vanishes. This will be found to occur when

$$\lambda = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5,$$

where $V = [v_1, v_2, v_3, v_4, v_5]$ has any of the five distinct values

$$\begin{cases} V_1 = [1, & 1, & 1, & 1, & 1], \\ V_2 = [1, & \omega, & \omega^2, & \omega^3, & \omega^4], \\ V_3 = [1, & \omega^2, & \omega^4, & \omega, & \omega^3], \\ V_4 = [1, & \omega^3, & \omega, & \omega^4, & \omega^2], \\ V_5 = [1, & \omega^4, & \omega^3, & \omega^2, & \omega], \end{cases} \tag{4}$$

where ω is a primitive fifth root of unity.

The solutions of the given system are therefore given by

$$X = kV_i, \qquad i = 1,2,3,4,5,$$

where $k \neq 0$ and the V_i are given by (4).

5. An airplane flies at a constant speed relative to the wind which varies continuously with position but does not vary with time. It flies a closed path and then flies the same path in the reverse direction. Prove that the total time of flight is greater than if there were no wind.

Solution.

Let the position of the airplane on the path be given in terms of the distance s travelled. Let $w(s)$ be the magnitude of the wind velocity and $\theta(s)$ the angle between the wind and the tangent to the path at the point s. We may assume that the constant speed of the airplane relative to the wind is 1. We resolve w into components tangential and normal to the path. The normal component $w \sin\theta$ must be equal to the normal component of the airplane. Hence the tangential component of the airplane is $(1 - w^2\sin^2\theta)^{\frac{1}{2}}$ and the resulting tangential speed is $(1 - w^2\sin^2\theta)^{\frac{1}{2}} + w\cos\theta$. The time of flight for the first traverse of the path is

$$\oint\{(1 - w^2\sin^2\theta)^{\frac{1}{2}} + w\cos\theta\}^{-1}ds. \tag{1}$$

On reversing direction, the time of flight is obtained from (1) upon replacing θ by $\theta+\pi$. Hence the total time of flight is

$$\oint(\{(1-w^2\sin^2\theta)^{\frac{1}{2}} + w\cos\theta\}^{-1} + \{(1-w^2\sin^2\theta)^{\frac{1}{2}} - w\cos\theta\}^{-1})ds.$$

By the A.M.-G.M. inequality, the integrand is at least $2/(1-w^2) > 2$ since $w \neq 0$. Alternatively, the integrand equals

$$\frac{2(1 - w^2\sin^2\theta)^{\frac{1}{2}}}{1 - w^2} \geq \frac{2(1 - w^2)^{\frac{1}{2}}}{1 - w^2} > 2.$$

Since the time of flight with no wind is $2\oint ds$, the desired result follows.

Rider. If in the problem the time-constant wind velocity is $k\vec{w}(s)$, where $k \geq 0$, prove that the time of flight increases with k.

*

Finally, I present a solution to a leftover problem from those proposed but unused at the 1981 International Mathematical Olympiad.

12. [1981: 237] Several equal spherical planets are in outer space. On the surface of each planet there is a region which is invisible from any of the remaining planets. Prove that the sum of the areas of all these regions is equal to the area of the surface of one planet. (U.S.S.R.)

Solution by J.E. Lewis, University of Alberta.

Let $S(1)$, $S(2)$, ..., $S(k)$ denote the planets. To say that the point x of the planet S is not visible from any other planet is the same as saying that

(i) there is a plane P tangent to S at x;

(ii) all of the remaining planets lie on the same side of P as does S;

(iii) the plane P is not tangent to any other planet.

Let Σ be a reference sphere, of the same radius as each of the planets. For each i, $i = 1,2,\ldots,k$, let ϕ_i be the mapping from $S(i)$ onto Σ which takes the point x of $S(i)$ to the point y of Σ where the tangent planes at x and y are parallel and where $S(i)$ and Σ both lie on the same relative side of the tangent planes. If $S(i)$ and Σ coincided, then it is clear that ϕ_i would be the identity map (for a plane can be tangent to a sphere at only one point), and this shows that the mappings ϕ_i are all congruencies.

Now let H_i denote the region of $S(i)$ that is not visible from any of the other planets, and let $\Sigma_i = \phi_i(H_i)$, so that Σ_i and H_i are congruent. It follows from (i)-(iii) above that the sets Σ_i are pairwise disjoint, and so the total measure of the invisible regions is no greater than the surface area of Σ.

Let Σ_{ij} be the subset consisting of all those points z of Σ such that the tangent plane at z is parallel to a plane that is simultaneously tangent to both $S(i)$ and $S(j)$. (Clearly, Σ_{ij} is a great circle on Σ.) Now let $\Sigma' = \Sigma - \cup(\Sigma_{ij})$. Then, for any y in Σ', there is evidently a sphere S and a point x of S such that the conditions (i)-(iii) are fulfilled. (To find S and x, let $x_i = \phi_i^{-1}(y)$, and let P_i be the plane tangent to $S(i)$ at the point x_i. Since there are only finitely many spheres $S(i)$, condition (ii) must be fulfilled for at least one of them.) This shows that $\Sigma' \subset \bigcup_i \Sigma_i$, and since the former has the same measure as the surface area of Σ, the proof is complete.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator or a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

821. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Solve the alphametic

$$\text{CRUX} = [\text{MATHEMAT}/\text{CORUM}],$$

where the brackets indicate that the remainder of the division, which is less than 500, is to be discarded.

822. *Proposed by Charles W. Trigg, San Diego, California.*

Arrange nine consecutive digits in a 3×3 array so that each of the six three-digit integers in the columns (reading downward) and rows is divisible by 7.

823. *Proposé par Olivier Lafitte, élève de Mathématiques Supérieures au Lycée Montaigne à Bordeaux, France.*

(a) Soit $\{a_1, a_2, a_3, \ldots\}$ une suite de nombres réels strictement positifs. Si

$$v_n = \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n, \qquad n = 1,2,3,\ldots,$$

montrer que $\limsup_{n\to\infty} v_n \geq e$.

(b) Trouver une suite $\{a_n\}$ pour laquelle intervient l'égalité dans (a).

824. *Proposed by J.C. Fisher and E.L. Koh, University of Regina.*

(a) *A True Story*: One problem on a recent calculus exam was to find the length of a particular curve $y = f(x)$ from $x = a$ to $x = b$. Some of our more primitive students found instead the area under the curve and, to our surprise and annoyance, came up with the same answer. What must $f(x)$ have been?

(b) If we don't want the same thing to happen on the next exam dealing with surface areas, what functions must we avoid? That is, what functions $z = f(x,y)$ will yield the same numerical answer for the volume under the surface as for the surface area, both over the same region?

825.* *Proposed by Jack Garfunkel, Flushing, N.Y.*

Of the two triangle inequalities (with sum and product cyclic over A,B,C)

$$\Sigma \tan^2\frac{A}{2} \geq 1 \qquad \text{and} \qquad 2 - 8\Pi \sin\frac{A}{2} \geq 1,$$

the first is well known and the second is equivalent to the well-known inequality $\Pi \sin(A/2) \leq 1/8$. Prove or disprove the sharper inequality

$$\Sigma \tan^2\frac{A}{2} \geq 2 - 8\Pi \sin\frac{A}{2}.$$

826.* *Proposed by Kent D. Boklan, student, Massachusetts Institute of Technology.*

It is a well-known consequence of the pigeonhole principle that, if six circles in the plane have a point in common, then one of the circles must entirely contain a radius of another.

Suppose n spherical balls have a point in common. What is the smallest value of n for which it can be said that one ball *must* entirely contain a radius of another?

827. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

For $i = 1,2,3$, let A_i be the vertices of a triangle with opposite sides

a_i, let B_i be an arbitrary point on a_i, and let M_i be the midpoint of B_jB_k. If the lines b_i are perpendicular to a_i through B_i, and if the lines m_i are perpendicular to a_i through M_i, prove that the b_i are concurrent if and only if the m_i are concurrent.

828*. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Characterize the pairs of positive integers $(2a, 2b)$, where $a < b$, for which

$$x = \frac{-a^2 + a\sqrt{a^2-ab+b^2}}{b - 2a + \sqrt{a^2-ab+b^2}}$$

is an integer. One example is $(2a, 2b) = (24, 45)$, which yields $x = 5$. (Note that $0 < a < b$ implies $x > 0$.)

829. *Proposed by O. Bottema, Delft, The Netherlands.*

Let ABC be a triangle with sides a,b,c in the usual order, centroid G, incenter I, and circumcenter O. Prove that

$$[GIO] = -\frac{(a+b+c)(b-c)(c-a)(a-b)}{48[ABC]},$$

where the square brackets denote the signed area of a triangle.

830. *Proposed by M.S. Klamkin, University of Alberta.*

Determine all real λ such that

$$|z_1\bar{z}_1(z_3-z_2) + z_2\bar{z}_2(z_1-z_3) + z_3\bar{z}_3(z_2-z_1) - iz_1\lambda| = |(z_2-z_3)(z_3-z_1)(z_1-z_2)|,$$

where z_1,z_2,z_3 are given distinct complex numbers and $z_1 \neq 0$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

260. [1977: 155; 1978: 58] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Given any triangle (other than equilateral), let P represent the projection of the incentre I on the Euler line OGNH where O, G, N, H represent respectively the circumcentre, the centroid, the centre of the nine-point circle and the orthocentre of the given triangle. Prove that P lies between G and H. In particular, prove that P coincides with N if and only if one angle of the given triangle has measure 60°.

II. *Comment by A.P. Guinand, Trent University, Peterborough, Ontario.*

This problem concerns a restriction to which the relative positions of G, H, and I are subject. In the published solution it was shown that $GH^2 > IG^2 + IH^2$, and consequently that $\angle GIH$ is obtuse (or straight).

A more direct way of describing this result is that the incentre I must lie inside the circle on diameter GH. I have called this the *critical circle* in a recent note [2] on the reconstruction of a triangle from its incentre and any two of the centres on the Euler axis. It also follows from the results of this note that, given the latter two centres, all points within the critical circle (except N) are possible as incentres I for real nonequilateral triangles.

Richard Blum pointed out in a letter that the critical circle is also the locus of points Q for which OQ = 2QN, and so another way of expressing the result of this problem is that OI > 2IN. If R and r denote the circumradius and inradius, respectively, then by Feuerbach's and Euler's Theorems [1] we have

$$OI^2 = R(R-2r) > 0 \quad \text{and} \quad IN = \tfrac{1}{2}R-r = \tfrac{1}{2}(R-2r)$$

for nonequilateral triangles. Hence

$$OI^2 - 4IN^2 = R(R-2r) - (R-2r)^2 = 2r(R-2r) > 0,$$

whence OI > 2IN, as required.

Similarly, if I_1 and r_1 denote an excentre and the corresponding exradius, we have $OI_1^2 = R(R+2r_1)$ and $I_1N = \tfrac{1}{2}R+r_1$, whence

$$OI_1^2 - 4I_1N^2 = R(R+2r_1) - (R+2r_1)^2 = -2r_1(R+2r_1) < 0,$$

and $OI_1 < 2I_1N$. Consequently, any excentre I_1 must lie outside the critical circle, and $\angle GI_1H$ is acute.

However, not all points outside the critical circle are possible positions for excentres; for example, the circumcentre O cannot also be an excentre. More precise results for excentres will be discussed in another paper.

REFERENCES

1. Nathan Altshiller Court, *College Geometry*. Barnes & Noble, New York, 1952 pp. 85, 105.

2. A.P. Guinand, "Trianqles From Centres Out", *James Cook Mathematical Notes*, No. 30, Vol. 3, December 1982, pp. 3127-3132.

* * *

678. [1981: 240; 1982: 282] *Proposed jointly by Joe Dellinger and Ferrell Wheeler, students, Texas A & M University, College Station, Texas.*

For a given fixed integer $n \geq 2$, find the greatest common divisor of the integers in the set $\{a^n - a \mid a \in Z\}$, where Z is the set of all integers.

Editor's comment.

As we noted earlier, this problem has appeared, with two different proposers, in the August-September 1981 issue of the *American Mathematical Monthly*. A solution by Robert Breusch has just been published in the March 1983 issue of the *Monthly* (Problem E 2901, pages 212-213).

*　　　　　*　　　　　*

701. [1982: 14, 323] Late solution from J.A. McCALLUM, Medicine Hat, Alberta.

*　　　　　*　　　　　*

718. [1982: 49] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

ABC is an acute-angled triangle with circumcenter O. The lines AO, BO, CO intersect BC, CA, AB in A_1, B_1, C_1, respectively. Show that

$$OA_1 + OB_1 + OC_1 \geq \frac{3}{2}R,$$

where R is the circumradius.

I. *Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Since the triangle is acute-angled, its circumcenter O lies within it, and we may apply inequality No. 12.40 of Bottema [1]. In our notation, this states that

$$\frac{AA_1}{AO} + \frac{BB_1}{BO} + \frac{CC_1}{CO} \geq \frac{9}{2}, \tag{1}$$

where O is any interior point. When O is the circumcenter, we have

$$\frac{AA_1}{AO} = 1 + \frac{OA_1}{R}, \text{ etc.,}$$

and (1) is equivalent to the desired

$$OA_1 + OB_1 + OC_1 \geq \frac{3R}{2}. \tag{2}$$

II. *Comment by J.T. Groenman, Arnhem, The Netherlands.*

Inequalities Nos. 12.38 and 12.39 of Bottema [1] read, in our notation,

$$\frac{AO}{OA_1} + \frac{BO}{OB_1} + \frac{CO}{OC_1} \geq 6$$

and

$$\frac{AO}{OA_1} \cdot \frac{BO}{OB_1} \cdot \frac{CO}{OC_1} \geq 8,$$

where O is any interior point. When O is the circumcenter, we easily obtain from these the related inequalities

$$\frac{1}{OA_1} + \frac{1}{OB_1} + \frac{1}{OC_1} \geq \frac{6}{R} \tag{3}$$

and

$$OA_1 \cdot OB_1 \cdot OC_1 \leq \frac{R^3}{8}. \tag{4}$$

Equality holds in each case just when the triangle is equilateral. Other related inequalities can be obtained by combining (2), (3), and (4).

III. *Generalization by M.S. Klamkin, University of Alberta.*

Let $A_0A_1\ldots A_n$ be an n-simplex with an interior circumcenter O and, for $i = 0,1,\ldots,n$, let the lines A_iO intersect the face opposite to A_i in the point B_i. We show that

$$OB_0 + OB_1 + \ldots + OB_n \geq \frac{n+1}{n} \cdot R,$$

where R is the circumradius, with equality if and only if O coincides with the centroid of the n-simplex.

With the origin of vectors anywhere outside the space of the n-simplex, the barycentric representation of the circumcenter is

$$\vec{O} = x_0\vec{A}_0 + x_1\vec{A}_1 + \ldots + x_n\vec{A}_n,$$

where $x_0+x_1+\ldots+x_n = 1$ and, since O is an interior point, $0 < x_i < 1$ for $i = 0,1,\ldots,n$. Then it follows easily that

$$\vec{B}_i = \frac{\vec{O} - x_i\vec{A}_i}{1 - x_i} \quad \text{and} \quad \vec{OB}_i = \frac{x_i}{1-x_i}(\vec{O}-\vec{A}_i), \quad \text{so} \quad OB_i = \frac{x_i}{1-x_i} \cdot R.$$

Finally, from the convexity of the function $t/(1-t)$ for $0 < t < 1$, we obtain

$$\sum_{i=0}^{n} OB_i = R\sum_{i=0}^{n} \frac{x_i}{1-x_i} \geq R \cdot (n+1)\frac{1/(n+1)}{1-1/(n+1)} = \frac{n+1}{n} \cdot R,$$

with equality if and only if $x_0 = x_1 = \ldots = x_n$, that is, if and only if O coincides with the centroid of the n-simplex.

The proposed problem corresponds to the case $n = 2$.

Also solved by JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. One incorrect solution was received.

REFERENCE

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968.

* * *

719. [1982: 49] *Proposed by Noam D. Elkies, student, Stuyvesant H.S., New York, N.Y.*

Are there positive integers a,b,c such that

$$(c-a-b)^3 - 27abc = 1?$$

I. *Comment by M.S. Klamkin, University of Alberta.*

I want to explain how this problem came about. In previous Olympiad Corners, Problems J-6 and J-11 asked, in effect, for a proof that

$$\sqrt[3]{60} > 2 + \sqrt[3]{7} \quad \text{and} \quad \sqrt[3]{413} > 6 + \sqrt[3]{3}, \tag{1}$$

respectively. (For solutions, see [1980: 315] and [1981: 109].) As shown in my comment following the solution of J-6, we have

$$\begin{aligned}\sqrt[3]{c} \geq \sqrt[3]{a} + \sqrt[3]{b} &\Longleftrightarrow c-a-b \geq 3\sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{c}\\ &\Longleftrightarrow f(a,b,c) \equiv (c-a-b)^3 - 27abc \geq 0.\end{aligned} \tag{2}$$

Now equality holds in (2) for infinitely many triples (a,b,c); for example, we can take $(a,b,c) = (x^6,y^6,z^6)$ where (x,y,z) is any Pythagorean triple. But in general the inequality in (2) is strict. In the cases (1), we get

$$f(8,7,60) = 405 \quad \text{and} \quad f(216,3,413) = 75536.$$

At an Olympiad Practice Session which the proposer attended, I had asked for a proof of Problem J-6. After solving the problem, the proposer set himself the task of finding a triple (a,b,c) for which the sharpest possible strict inequality occurs in (2), that is, one for which $f(a,b,c) = 1$.

Now read on.

II. *Solution by the proposer.*

Let $p = a+c$ and $q = b+c$; then $a = p-c$, $b = q-c$, and the equation $f(a,b,c) = 1$ becomes

$$(3c-p-q)^3 - 27c(p-c)(q-c) = 1$$

which, upon expansion, is found to be equivalent to

$$9c(p^2-pq+q^2) = (p+q)^3 + 1.$$

With $r = p+q$ and $t = p-q$, this equation is equivalent to the second-degree Diophantine equation

$$9(r^2+3t^2) = 4k(r^2-r+1), \tag{3}$$

where $k = (r+1)/c$ is a positive rational parameter.

The value $k = 9/4$, for which (3) reduces to a linear equation in r, does not lead to a solution; but, for lack of a better approach, we seek values of k near $9/4$ that do. By trial and success (many trials and one success!), we find that $k = 219/97 \approx 9/4$ is satisfactory. Substituting this value in (3) yields

$$873t^2 = (r-146)^2 - 144 \cdot 146, \tag{4}$$

from which it is clear that $3 \mid (r-146)$. Moreover, since

$$c = \frac{r+1}{k} = \frac{97(r+1)}{219}$$

must be an integer, we must have $r \equiv -1 \pmod{219}$. Hence, from (4),

$$x = \frac{r-146}{3} \equiv 24 \pmod{73}$$

must satisfy

$$x^2 - 97t^2 = 16 \cdot 146. \tag{5}$$

By trying small values of t, we find that

$$34^2 - 97 \cdot 6^2 = -16 \cdot 146,$$

and the continued fraction expansion of 97 yields

$$5604^2 - 97 \cdot 569^2 = -1.$$

Using norms N, we combine these two results and obtain

$$\begin{aligned} 16 \cdot 146 &= (-1) \cdot (-16 \cdot 146) \\ &= N(5604-569\sqrt{97}) \cdot N(34+6\sqrt{97}) \\ &= N(-140622+14278\sqrt{97}) \\ &= 140622^2 - 97 \cdot 14278^2. \end{aligned}$$

Since $140622 \equiv 24 \pmod{73}$, it follows that $(x,t) = (140622, 14278)$ is a satisfactory solution of (5), which we can use to find a satisfactory triple (a,b,c). We find successively

$$r = 3x + 146 = 422012, \qquad c = \frac{97(r+1)}{219} = 186919,$$

$$p = \tfrac{1}{2}(r+t) = 218145, \qquad q = \tfrac{1}{2}(r-t) = 203867,$$

$$a = p-c = 31226, \qquad b = q-c = 16948.$$

The triple $(a,b,c) = (31226,\ 16948,\ 186919)$ satisfies $f(a,b,c) = 1$, and hence

$$\sqrt[3]{186919} > \sqrt[3]{31226} + \sqrt[3]{16948}. \tag{6}$$

It can be verified that the two sides of (6) are identical as far as the fourteenth digit after the decimal point!

*　　　　*　　　　*

720. [1982: 49] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

On the sides AB and AC of a triangle ABC as bases, similar isosceles triangles ABE and ACD are drawn outwardly. If BD = CE, prove or disprove that AB = AC.

Solution by O. Bottema, Delft, The Netherlands.

We generalize by dropping the word "isosceles" from the proposal. We consider a triangle ABC with angles α,β,γ and sides a,b,c in the usual order, and inversely similar triangles ABE and ACD drawn outwardly on bases AB and AC, respectively, with base angles θ and ϕ, as shown in Figure 1. We will show that BD = CE if and only if AB = AC *or* if triangles ABE and ACD are isosceles with vertex at A, as shown in Figure 2.

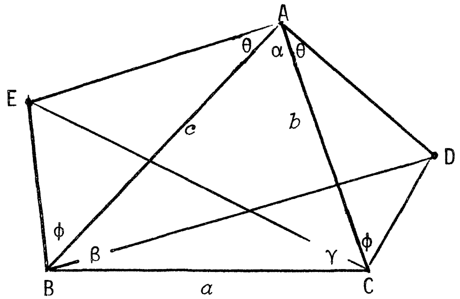


Figure 1

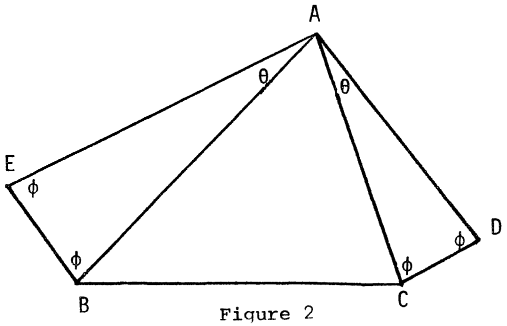


Figure 2

We have

$$\mathrm{CD} = \frac{b\sin\theta}{\sin(\theta+\phi)} \qquad \text{and} \qquad \mathrm{BE} = \frac{c\sin\theta}{\sin(\theta+\phi)},$$

from which follow

$$\mathrm{BD}^2 = a^2 + \frac{b^2\sin^2\theta}{\sin^2(\theta+\phi)} - \frac{2ab\sin\theta}{\sin(\theta+\phi)}\cdot\cos(\gamma+\phi)$$

and

$$\mathrm{CE}^2 = a^2 + \frac{c^2\sin^2\theta}{\sin^2(\theta+\phi)} - \frac{2ac\sin\theta}{\sin(\theta+\phi)}\cdot\cos(\beta+\phi).$$

From these two equations, we get

$$\begin{aligned} BD = CE \quad &\iff \quad (b^2-c^2)\sin\theta - 2a\sin(\theta+\phi)\{b\cos(\gamma+\phi) - c\cos(\beta+\phi)\} = 0 \\ &\iff \quad (b^2-c^2)\sin\theta - \sin(\theta+\phi)\cos\phi\,(2ab\cos\gamma - 2ac\cos\beta) = 0 \\ &\iff \quad (b^2-c^2)\sin\theta - \sin(\theta+\phi)\cos\phi\cdot 2(b^2-c^2) = 0 \\ &\iff \quad (b^2-c^2)\{\sin\theta - 2\sin(\theta+\phi)\cos\phi\} = 0 \\ &\iff \quad b = c \text{ or } \sin\theta - 2\sin(\theta+\phi)\cos\phi = 0. \qquad (1) \end{aligned}$$

Using elementary trigonometric identities, the second equation in (1) is easily shown to be equivalent to $\sin(\theta+2\phi) = 0$, from which follow

$$\theta + 2\phi = \pi, \qquad \pi - (\theta+\phi) = \phi, \qquad \angle ADC = \angle AEB = \phi.$$

(The proof is very easily modified to yield the same result when triangles ABE and ACD are drawn inwardly on bases AB and AC.)

In particular, if $\theta = \phi$ as in the proposed problem, we have

BD = CE $\iff$ AB = AC *or* triangles ABE and ACD are equilateral.

It is a well-known theorem that BD = CE for all triangles ABC when ABE and ACD are equilateral [1].

Also solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; W.J. BLUNDON, Memorial University of Newfoundland; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; F.G.B. MASKELL, Algonquin College, Ottawa; DAN PEDOE, University of Minnesota; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

REFERENCE

1. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library No. 19, Random House, The L.W. Singer Company, 1967, pp. 82-83 (now available from the Mathematical Association of America, Washington, D.C.).

* * *

721. [1982: 77] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

A propos of the editor's comment following Crux 611 [1982: 30], verify that, with decimal integers,

(a) uniquely, TRIGG is three times WRONG;

(b) independently, but also uniquely, WAYNE is seven times RIGHT.

I. *Solution of part (a) by Clayton W. Dodge, University of Maine of Orono.*

It is clear that G = 0 or 5. But G = 0 allows no value for N, so G = 5, and it immediately follows that N = 8. Now

$$(O, I) \text{ is one of } (0, 2), (3, 1), (6, 0), (7, 3).$$

No R can satisfy the first two possibilities, but both R = 4 and R = 9 satisfy each of the last two. Of the four possibilities for (O, I, R), only (6, 0, 4) leads to satisfactory values for W and T: W = 2 and T = 7. The unique solution is

$$3 \cdot 24685 = 74055.$$

II. *Solution of part (b) by Charles W. Trigg, San Diego, California.*

Immediately, R = 1; I = 0, 2, 3, or 4; and T ≠ 0, 5, 1, or 3. If T = 2, then E = 4, and RIGHT cannot contain a 4. Likewise, if T = 4, 6, 7, 8, or 9, then RIGHT must be devoid of 8, 2, 9, 6, or 3, respectively.

Each of the ten decimal digits appears just once, so

$$\text{RIGHT} + \text{WAYNE} = 8 \cdot \text{RIGHT} \equiv 0 \pmod 9,$$

and each of RIGHT and WAYNE is a multiple of 9.

When RIGHT is in the span 10233-11428, W = 7; in the span 11429-12857, W = 8; and in the span 12858-14285, W = 9. Furthermore, when HT = 02, 74, or 59, then N = 1. When HT = 32, 42, 92, 34, 64, 84, 26, 46, 57, 67, 97, 38, 68, 98, 39, or 49, a duplicated digit appears among H, T, N, and E. These restrictions reduce the number of possible values of RIGHT to the following twenty distinct-digit multiples of 9:

10269	10386	10458	10548	10629
10836	10962	12069	12537	12609
12654	13086	13248	13428	13527
13572	13752	13806	14058	14076

Multiplication of each of these by 7 gives products that contain duplicating digits, except in the unique case

$$7 \cdot 14076 = 98532. \quad \square$$

Obviously, WAYNE can never be WRONG. However, three WRONGs can make a RIGHT in at least three ways:

$$3 \cdot 13268 = 39804, \quad 3 \cdot 28639 = 85917, \quad 3 \cdot 28319 = 84957,$$

even though the last WRONG is a prime one. Thus it would appear that TRIGG with his three WRONGs can join WAYNE on the tree of perfection, although on a much lower branch.

Alphabetically, TRIGG is not far from being RIGHT, needing merely a one-step cyclic permutation and one replacement of a letter with the one following it in the alphabet.

Also Solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; CLAYTON W. DODGE, University of Maine at Orono (part (b) also); MEIR FEDER, Haifa, Israel; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; J.A. McCALLUM, Medicine Hat, Alberta; CHARLES W. TRIGG, San Diego, California (part (a) also); KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

Feder noted that, in the *Journal of Recreational Mathematics*, Vol. 10 (1977-78), No. 2, pp. 110-111, there is an article by Stewart Metchette containing little more than a list of the 94 computer-generated solutions of the equation

$$n \cdot \text{ABCDE} = \text{FGHIJ},$$

where n is a positive integer. For $n = 7$ there is only one solution, that given in our part (b). Astonishingly, in the very same *Journal* one year later (Vol. 11 (1978-79), No. 3, pp. 236-237) appeared an article by Steven Kahan containing little more than the same list of 94 solutions of the same equation!

Too late, too late! For the proposer noted that, in response to a question by Victor Thébault, G.H. Biucliu and L. Tits had already given the 94 solutions on pages 205-207 of *Mathesis*, Vol. 44, 1935 B.C. (Before Computers).

Inspired by part (b) of our problem, some digitomaniac is bound to suggest finding all products containing each of the digits exactly once. Don't. It has already been done. In the *American Mathematical Monthly*, 41 (1934 B.C.) 265-266, in response to a question by W.F. Cheney, Jr., Victor Thébault found all the products containing the nine nonzero digits exactly once:

$$\begin{array}{lll} 4 \cdot 1738 = 6952 & 18 \cdot 297 = 5346 & 39 \cdot 186 = 7254 \\ 4 \cdot 1963 = 7852 & 27 \cdot 198 = 5346 & 42 \cdot 138 = 5796 \\ 12 \cdot 483 = 5796 & 28 \cdot 157 = 4396 & 48 \cdot 159 = 7632 \end{array}$$

For good measure, Thébault then gave all the products containing all ten digits exactly once:

$$\begin{array}{lll} 4 \cdot 3907 = 15628 & 7 \cdot 4093 = 28651 & 27 \cdot 594 = 16038 \\ 4 \cdot 7039 = 28156 & 7 \cdot 9304 = 65128 & 39 \cdot 402 = 15678 \\ & & 54 \cdot 297 = 16038 \end{array}$$

*   *   *

722. [1982: 77] *Proposed by Paul R. Beesack, Carleton University, Ottawa, Ontario.*

A very large prison has 10000 cells numbered from 1 to 10000 (each occupied by one prisoner), an eccentric warden, and an ingenious electronic device for opening and closing cell doors. Early one day the warden announces that all those prisoners whose cell doors are left open at the end of the day will be free to leave the prison. During the day he presses buttons which open or close cell doors as follows. First he opens all cell doors beginning with cell 1. Next, beginning with cell 2, he operates on every 2nd cell door, closing those that are open and opening those that are closed. This operation is repeated throughout the day so that at the nth step $(n = 1,2,...,10000)$, every nth cell beginning with cell n is closed if it was open, or opened if it was closed. How many prisoners are freed at the end of the day?

Solution by Sam Baethge, Southwest High School, San Antonio, Texas.

For $n = 1,2,...,10000$, a cell will be opened (if closed) or closed (if open) at step n if and only if n is a divisor of the cell number; and a cell remains open at the end of the day if and only if the cell number has an odd number of divisors, that is, if and only if the cell number is a perfect square. So 100 prisoners are freed at the end of the day, those whose cell numbers were 1^2, 2^2, ..., 100^2.

Also solved by KENT D. BOKLAN, student, Massachusetts Institute of Technology; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; JOE PETERS, Grade 11, Worsley Central School, Worsley, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer. A comment was received from JOE KONHAUSER, Macalester College, Saint Paul, Minnesota.

Editor's comment.

This journal sheepishly joins the list of publications where this problem has already appeared in some form (just as stone walls do not a prison make, so electronic door openers do not a new problem make). The list, no doubt incomplete, appears in the references, all of which were sent by readers. The proposer wrote that he first heard a variation of this problem about thirty years ago from, he thinks, Professor L.J. Mordell.

REFERENCES

1. Problem 46 (proposed by J. Lambek), *Pi Mu Epsilon Journal*, Vol. 1, No. 7 (November 1952), pp. 276-277. Solution by C.W. Trigg in Vol. 1, No. 8 (April 1953), p. 330.

2. Problem 4, *Journal of Recreational Mathematics*, Vol. 1, No. 1 (January 1968), p. 36. Solution in Vol. 1, No. 4 (October 1968), p. 240 (where two earlier references are given).

3. Ross Honsberger, *Mathematical Gems*, Mathematical Association of America, 1973, pp. 68-69.

4. Martin Gardner, "Mathematical Games", *Scientific American*, November 1974, p. 122.

5. Problem 286, (proposed by the Problem Editor of) *The Pentagon*, Vol. 35, No. 2 (Spring 1976), p. 98. Solution by Charles W. Trigg, Vol. 36, No. 2 (Spring 1977), pp. 98, 104.

* * *

723. [1982: 77] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let G be the centroid of a triangle ABC, and suppose that AG,BG,CG meet the circumcircle of the triangle again in A',B',C', respectively. Prove that

(a) $GA' + GB' + GC' \geq AG + BG + CG$;

(b) $AG/GA' + BG/GB' + CG/GC' = 3$;

(c) $GA'\cdot GB'\cdot GC' \geq AG\cdot BG\cdot CG$.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India (revised by the editor).

(a) Let a,b,c and m_a,m_b,m_c be the lengths of the sides and medians, respectively, in the usual order. From

$$4m_a^2 = 2(a^2+b^2+c^2) - 3a^2, \text{ etc.,}$$

it follows that

$$m_b \gtreqless m_c \text{ according as } b \lesseqgtr c, \text{ etc.} \tag{1}$$

If D is the midpoint of BC, then $m_a\cdot DA' = (a/2)^2$, so $DA' = a^2/4m_a$ and

$$GA' = GD + DA' = \frac{a^2+b^2+c^2}{6m_a}.$$

Thus

$$GA' - AG = \frac{a^2+b^2+c^2}{6m_a} - \frac{2}{3}m_a = \frac{(a^2-b^2) + (a^2-c^2)}{6m_a}.$$

With this and two similar results, we obtain, with sums cyclic over A,B,C and a,b,c,

$$\Sigma(GA'-AG) = \frac{1}{6}\Sigma(b^2-c^2)\left(\frac{1}{m_b}-\frac{1}{m_c}\right). \tag{2}$$

Now the right side of (2) is nonnegative by (1), and (a) follows, with equality just when the triangle is equilateral.

(b) With some results used in the solution of part (a), we have

$$\frac{AG}{GA'} = \frac{4m_a^2}{a^2+b^2+c^2} = 2 - \frac{3a^2}{a^2+b^2+c^2}, \text{ etc.}$$

Hence

$$\frac{AG}{GA'} + \frac{BG}{GB'} + \frac{CG}{GC'} = 6 - 3 = 3.$$

(c) From (b) and the A.M.-G.M. inequality, we have

$$1 = \frac{1}{3}\left(\frac{AG}{GA'} + \frac{BG}{GB'} + \frac{CG}{GC'}\right) \geq \sqrt[3]{\frac{AG}{GA'}\cdot\frac{BG}{GB'}\cdot\frac{CG}{GC'}},$$

and (c) follows, with equality just when the triangle is equilateral.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, New York; J.T. GROENMAN, Arnhem, The Netherlands (partial solution); M.S. KLAMKIN, University of Alberta; and the proposer.

Editor's comment.

Garfunkel noted that part (a) of our problem was recently proposed (but not yet solved, although the deadline for submitting solutions is past) in the *Monthly* [1]; and Klamkin found our part (b) in *Mathematics Magazine* [2], where the published solution should be compared with our own.

REFERENCES

1. Problem E 2959 (proposed by Jack Garfunkel), *American Mathematical Monthly*, 89 (1982) 498.

2. Problem 1119 (proposed by K.R.S. Sastry, solution by Thu Pham), *Mathematics Magazine*, 55 (1982) 180-182.

* * *

724. [1982: 78] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Let ABC be a triangle (with sides a,b,c) in which the angles satisfy C + A = 2B (that is, the angles are in arithmetic progression). Such a triangle has many interesting properties. Establish the following (and possibly others):

(a) $\sin(A-B) = \sin A - \sin C$.

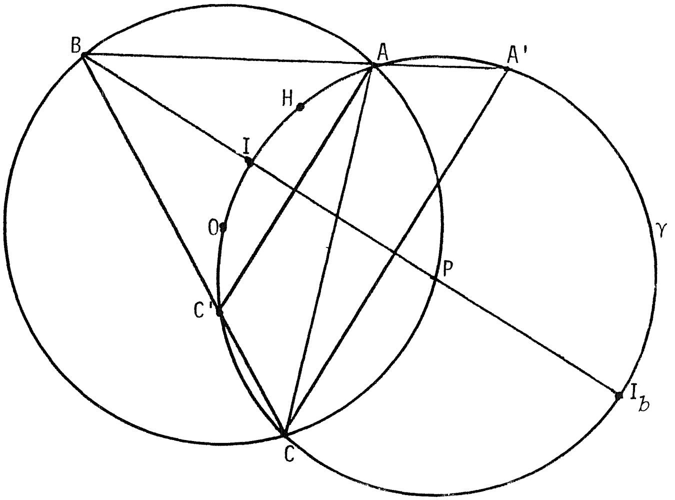
(b) $a^2 - b^2 = c(a - c)$.

(c) Vertices A and C, circumcentre O, incentre I, orthocentre H, and excentre I_b all lie on a circle of radius R, where R is the circumradius of the triangle. Furthermore, if this circle meets the lines AB and BC again in A' and C', respectively, then AA' = CC' = $|c-a|$.

Solution by J.T. Groenman, Arnhem, The Netherlands.

(a) Since C + A = 2B, we have B = 60° and

$$\sin(A-B) = 2\cos B\sin(A-B) = \sin A - \sin(2B-A) = \sin A - \sin C.$$



(b) From $b^2 = c^2+a^2-2ca\cos B = c^2+a^2-ca$, we obtain the desired

$$a^2 - b^2 = c(a - c).$$

(c) Let BI meet the circumcircle again in P, as shown in the figure. It is well known [1] that A, I, C, and I_b lie on a circle γ with center P and radius $2R\sin(B/2) = R$, and we must show that H and O also lie on γ. From

$$\angle AHC = 180^\circ - B = 120^\circ, \qquad \angle AOC = 2B = 120^\circ, \qquad \angle AIC = 90^\circ + \frac{B}{2} = 120^\circ,$$

we conclude that A,H,O,I,C all lie on one circle; and this circle, having the three points A,I,C in common with γ, must coincide with γ.

Now $\angle AC'C = \angle AIC = 120^\circ$ and $\angle AA'C = 180^\circ - \angle AIC = 60^\circ$; hence triangles ABC' and A'BC are both equilateral, the first having side c and the second side a. Therefore

$$AA' = CC' = |c-a|.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; G.C. GIRI, Midnapore College, West Bengal, India; GALI SALVATORE, Perkins, Québec; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec ((a) et (b) seulement); KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Here are some of the additonal properties of the configuration mentioned by solvers.

(Blundon) if C + A = 2B or, equivalently, if B = 60°, then OI = IH (Crux 739 [1982: 107] asks for a proof of the converse), and the R-r-s relationship of the triangle is $s = \sqrt{3}(R+r)$. For a proof of both properties, see the solution of Crux 260 [1978: 58].

(Wilke) [OI = IH and] $OI_b = HI_b$. For a proof, see [2].

(Dou) The center N of the nine-point circle of the triangle lies on the internal bisector of angle B (since N is the midpoint of OH).

REFERENCES

1. Roger A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, p. 185.

2. M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*, Dover, New York, 1968, pp. 75-76.

* * *

725. [1982: 78, 105] *Proposed by H. Kestelman, University College, London, England.*

An $n\times n$ matrix is called *simple* if its eigenvectors span C^n and is called *deficient* if they do not. If A and B are simple, can $A + B$ and AB both be deficient? If A is simple, show that adj A is simple; is the converse true?

(The (r,s) element of adj A is the (s,r) cofactor of A.)

Solution by the proposer.

A simple matrix which has just one eigenvalue, k, must be kI. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix};$$

then

$$A + B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad AB = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

and each of these has just one eigenvalue and is deficient. However, A and B have each two eigenvalues and are therefore simple.

Suppose the $n\times n$ matrix A is simple. To prove that adj A is simple, it is enough to show that every eigenvector of A is one of adj A. If $Av = \lambda v$ and $v \neq 0$, then

$$\lambda(\text{adj}\,A)v = (\det A)v$$

and this makes v an eigenvector of adj A if $\lambda \neq 0$. If $\lambda = 0$, there are two possibilities, (i) A has rank less than n-1, or (ii) A has rank n-1. If (i), then adj $A = 0$ and there is no more to prove. Otherwise (ii) holds and $Av = 0$; this implies that every solution of $Ax = 0$ is a scalar multiple of v. In particular, $A((\text{adj}\,A)v) = 0$

since $A(\text{adj } A) = 0$; hence $(\text{adj } A)v$ is a scalar multiple of v, i.e., v is an eigenvector of adj A.

The converse is false: if

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then A has just one eigenvalue, 0, and $A \neq 0$; hence A is deficient. But every cofactor of A is 0 and so adj A is the simple matrix 0.

Editor's comment.

We deeply regret to announce that our proposer, Professor H. Kestelman, a frequent contributor to our Problem Section, died suddenly on January 20, 1983. He collapsed in a London tube station on his way home from visiting his doctor, and was dead on arrival at hospital.

For a biographical sketch of Professor Kestelman, we can do no better than to quote from a letter dated June 24, 1982, which he sent to the editor.

"Having retired seven years ago I gave up doing analysis which had been my major interest for forty years. You know there was a famous Father William who was rebuked for standing on his head when he was old. I feel the same about analysis: beautiful but too hard when you've reached seventy five. During my last few years of teaching I gave courses on Mathematics Related to Economics and this gave me a fresh interest in convexity connected as it is with the theory of positive matrices. At first I thought I would write up what I knew about matrix theory, but this was a race against time and senility which was unfortunately lost. This explains the repeated exercises in matrix algebra which arose from a life-time's habit of an analyst of asking whether the obvious was true or false. When I was an undergraduate, matrix algebra had not reached the curriculum of a maths major (at least in England) and so I was self-taught. However, I came to it with the curiosity of an analyst, and for many years I was encouraged to enlarge my field of endeavour by physicists and chemists at UCL who demanded lectures on group representation. This made me learn groups and also more about matrices; I even learnt a little quantum mechanics on the way."

* * *

726. [1982: 78, 173] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Show that, in a regular n-simplex,

(a) the dihedral angle is Arccos $(1/n)$;

(b) the product of the altitude and the circumradius is half the square of an edge (independently of n).

Solution by M.S. Klamkin, University of Alberta.

(a) Let O and R be the circumcenter and circumradius, respectively, of the regular n-simplex $A_0A_1\ldots A_n$. For $i = 0,1,\ldots,n$, let $\vec{A}_i$ denote the vector from origin O to vertex A_i, so that $|\vec{A}_i| = R$. For all $i \neq j$ and $k \neq l$, we have

$$|\vec{A}_i - \vec{A}_j|^2 = |\vec{A}_k - \vec{A}_l|^2, \quad \text{and so} \quad \vec{A}_i\cdot\vec{A}_j = \vec{A}_k\cdot\vec{A}_l.$$

Now $\vec{A}_0 + \vec{A}_1 + \ldots + \vec{A}_n = \vec{0}$, for the centroid coincides with the circumcenter; and squaring this gives

$$(n+1)R^2 + (n+1)n\vec{A}_i\cdot\vec{A}_j = 0.$$

If $\theta_{ij} = \text{angle }(\vec{A}_i,\vec{A}_j)$, we therefore have

$$\vec{A}_i\cdot\vec{A}_j = R^2\cos\theta_{ij} = -\frac{R^2}{n} \qquad \text{and} \qquad \cos\theta_{ij} = -\frac{1}{n}.$$

Since $\vec{A}_i$ is perpendicular to the face opposite A_i, the dihedral angle corresponding to edge A_iA_j is supplementary to the angle between $-\vec{A}_i$ and $-\vec{A}_j$. Consequently, the dihedral angle is Arccos $(1/n)$, as required.

(b) As is well known, the length of the altitude or median from any vertex A_i is

$$\frac{n+1}{n}|\vec{A}_i| = \frac{n+1}{n}\cdot R.$$

Therefore, for $i \neq j$ we have

$$\tfrac{1}{2}\overline{A_iA_j}^2 = \tfrac{1}{2}(\vec{A}_i^2 + \vec{A}_j^2 - 2\vec{A}_i\cdot\vec{A}_j) = R^2(1+\frac{1}{n}) = (\frac{n+1}{n}\cdot R)R,$$

that is, the product of the altitude and circumradius is half the square of an edge (independently of n).

Also solved by STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; and the proposer.

Editor's comment.

The proposer noted that parts (a) and (b) are the results 2.8 and 3.3 on pages 40-41 of his book [1].

REFERENCE

1. Kesiraju Satyanarayana, *Dihedral Angles and In- and Ex-Elements of n-Space Simplexes*, 1979, Visalaandhra Publishing House, Vijayawada 520 004, Andhra Pradesh, India.

* * *