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CONTENTS

THE GEOMETRY OF DÜRER'S CONCHOID

HENRY E. FETTIS

The plane curve which has come to be known as "Dürer's Conchoid" was named for the 16th century architect, August Dürer, who devised it as an example of a class of curves called "conch curves" for their resemblance to sea shells [1]. Although it involves two positive parameters, a and b, their number can be reduced to one by considering either of them as unity. The curve is the one which is traced by the endpoints of a moving straightedge of length $2a$ whose midpoint remains on the x-axis of a Cartesian coordinate system at a variable distance, t, from the origin, and simultaneously passes through the point $(0, b-t)$ on the y-axis. (See Figure 1.)

From elementary considerations, the parametric equations (with t as parameter) are found to be

$$\begin{cases} x^2 + y^2 - 2xt + t^2 - a^2 = 0, \\ (y - b - x)t + t^2 + bx = 0. \end{cases} \tag{1}$$

Eliminating t from these equations gives the Cartesian equation of the curve as

$$\phi(x,y) \equiv (x^2+y^2-bx-a^2)^2 - (x+y-b)\{(x^2+y^2-a^2)(x-y+b) - 2bx^2\} = 0. \tag{2}$$

Geometrically, the curve has a number of interesting properties, some of which are given in [2, p. 159]. For example, by writing the second of equations (1) in the form

$$\frac{y}{x-t} = 1 - \frac{b}{t},$$

it can be seen that, for large values of t, the quantities y and $x-t$ both approach a constant value, α, and from the first of equations (1) we find $2\alpha^2 = a^2$, so that $\alpha = \pm a/\sqrt{2}$. Thus the curve (which consists of two branches) has horizontal asymptotes situated at equal distances above and below the x-axis. (This fact and those which follow are illustrated in Figures 2 to 4.) One branch approaches the upper asymptote from above as $x \to -\infty$ $(t \to -\infty)$ and the lower asymptote from above as $x \to \infty$ $(t \to \infty)$, while the other branch approaches, respectively, the upper and lower asymptotes from below as $x \to \infty$ and $x \to -\infty$.

The curve also has two singular points, one of which is always a nodal point, while the other may be a node, a cusp, or a conjugate (isolated) point, depending on the relative sizes of a and b. Apparently, a detailed study of this feature of the curve has not been made, although the location of the cusp and the condition for the existence of the cusp or loop are given in [2].

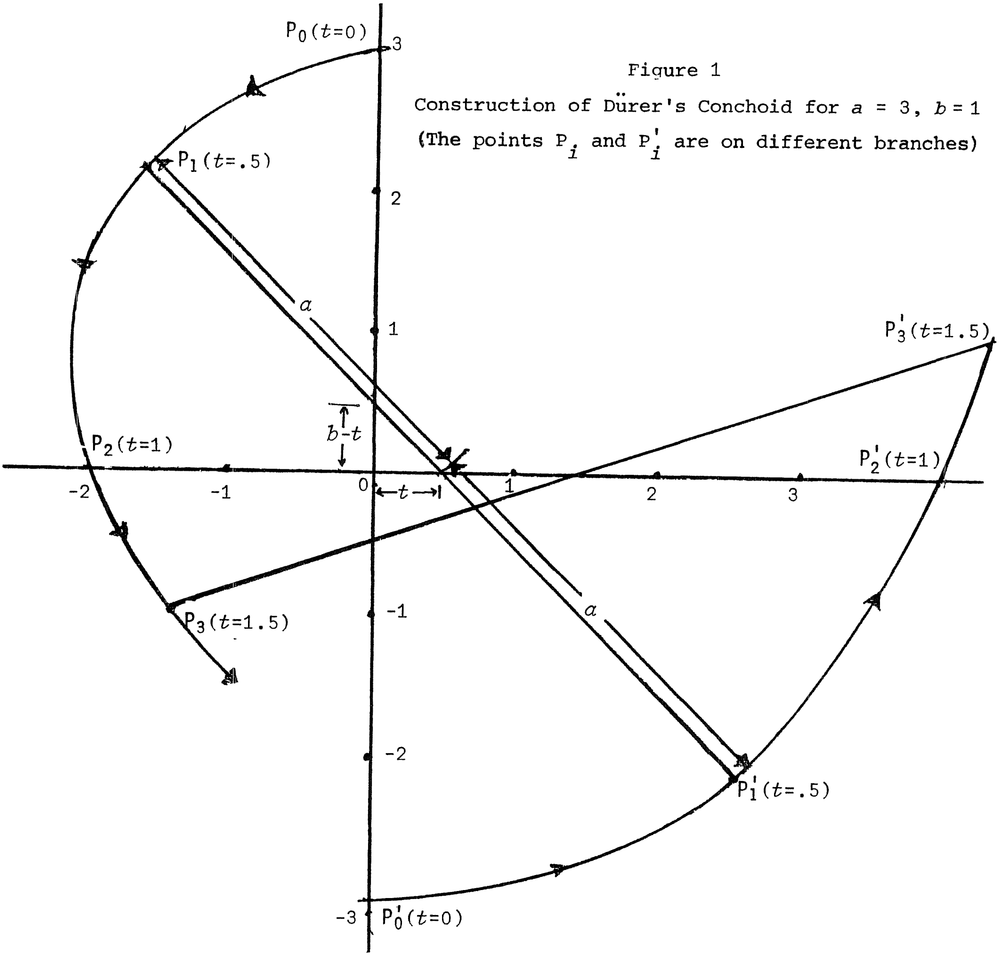

Figure 1

Construction of Dürer's Conchoid for $a = 3$, $b = 1$

(The points P_i and P'_i are on different branches)

The location and identification of the singular points of this curve from its Cartesian equation (2), by the usual criteria involving the first and second partial derivatives of $\phi(x,y)$, would prove to be a formidable task. However, by returning to the original parametric equations (1), it becomes unbelievably simple.

Eliminating t^2 between equations (1), we get

$$t = \frac{x^2 + y^2 - bx - a^2}{x + y - b},$$

and it is now evident that, to an arbitrary point (x,y) on the curve, there will correspond, in general, a unique value of t. The exception occurs when x and y are such that, simultaneously,

$$\begin{cases} x^2 + y^2 - bx - a^2 = 0, \\ x + y - b = 0. \end{cases} \tag{3}$$

The first of these equations is that of a circle with center $(b/2, 0)$ and radius $(a^2 + b^2/4)^{\frac{1}{2}}$, while the second represents a straight line with x- and y-intercepts both equal to b. The points of intersection of this circle and line determine the singular points, the abscissae of which are obtained by solving simultaneously the two equations (3), resulting in the single quadratic equation

$$2x^2 - 3bx + b^2 - a^2 = 0, \tag{4}$$

whose roots are

$$x = \frac{3b \pm \sqrt{b^2+8a^2}}{4}, \tag{5}$$

and the corresponding values of y are

$$y = \frac{b \mp \sqrt{b^2+8a^2}}{4}. \tag{6}$$

The values of t associated with these coordinates are easily found by substituting either relation (3) into the corresponding equation (1), resulting in the quadratic equation

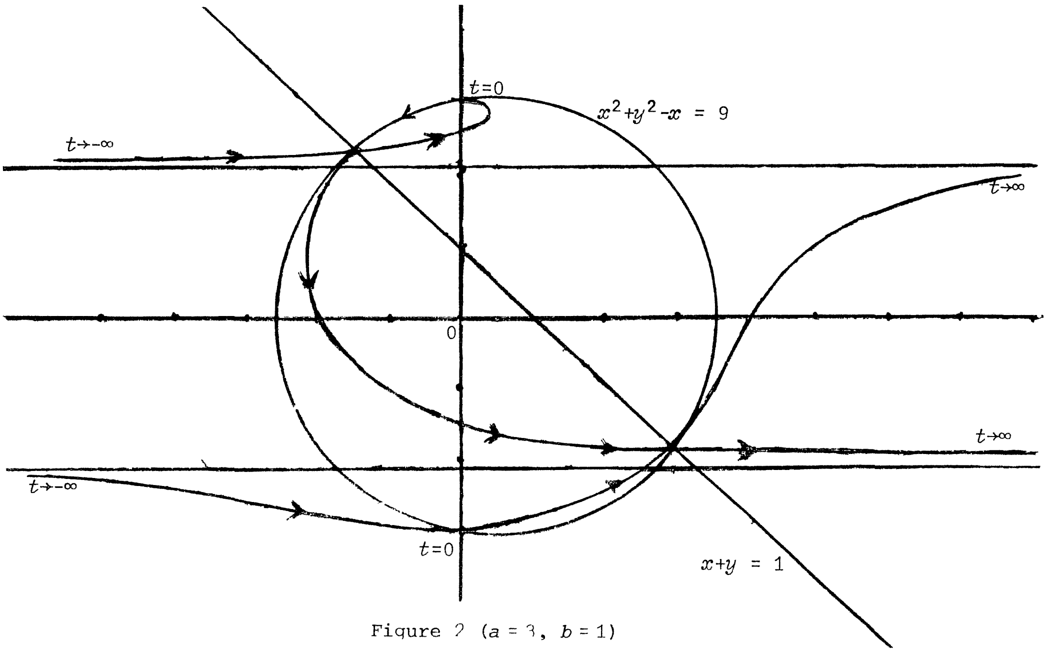

Figure 2 $(a = 3,\ b = 1)$

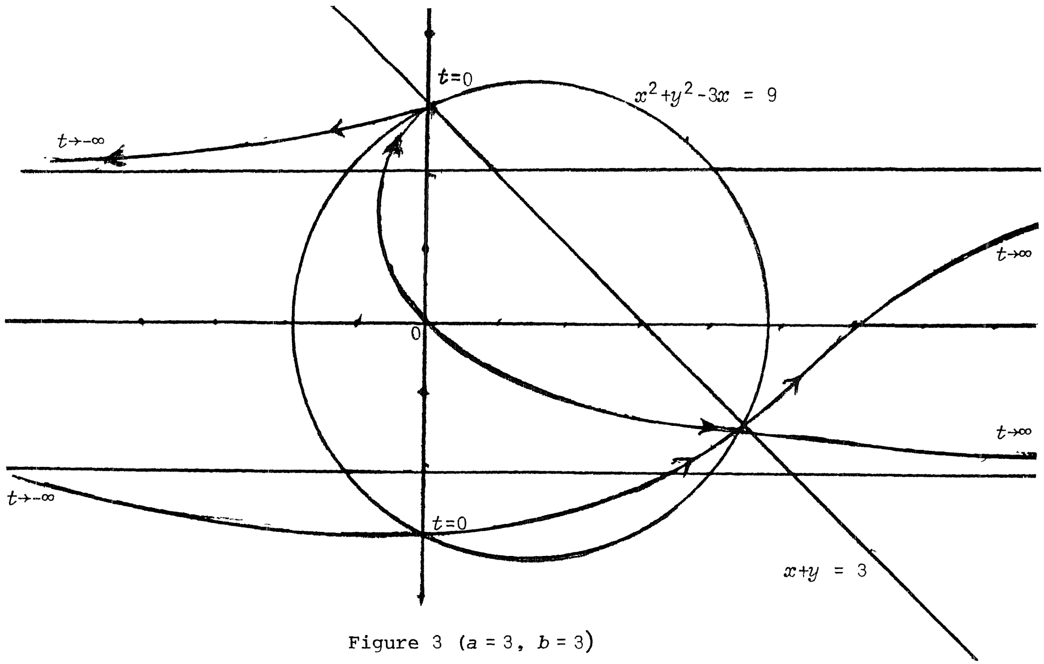

Figure 3 ($a = 3$, $b = 3$)

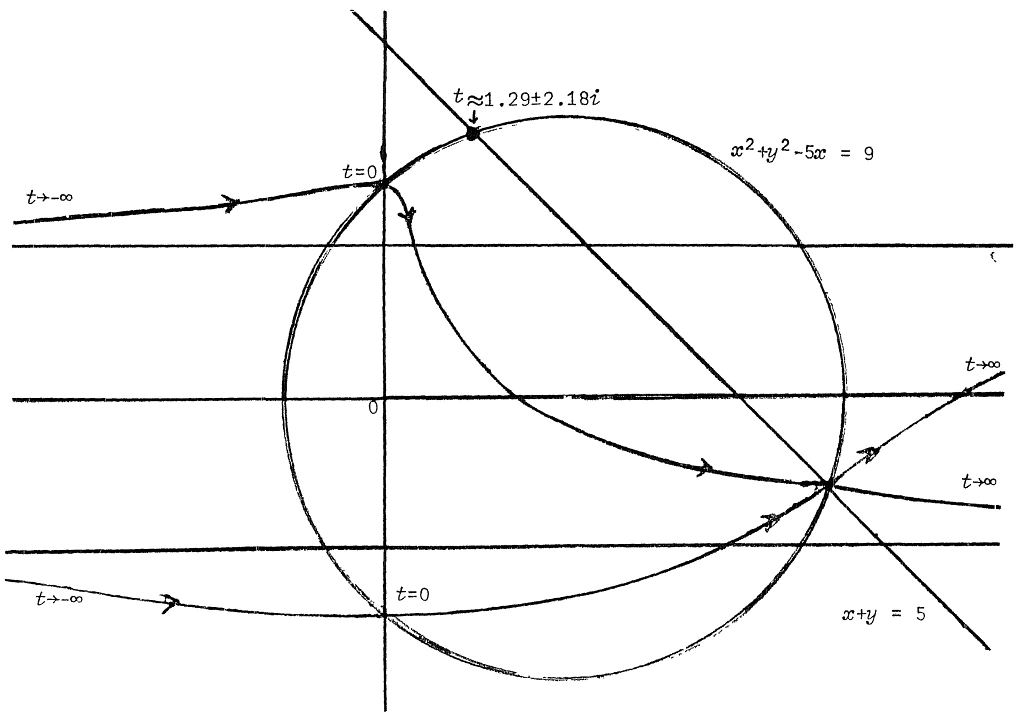

Figure 4 ($a = 3$, $b = 5$)

$$t^2 - 2xt + bx = 0,$$

with roots

$$t = x \pm \sqrt{x(x-b)}.$$

It is clear from (5) and (6) that the values of x and y are always real. However, this will not necessarily be the case for the associated values of t. (For convenience, the value of x associated with the plus sign in (5) will be designated as x_+, the other as x_-.)

It is first noted that $x_+ > 0$ and that

$$x_+ - b = \frac{2a^2}{b+\sqrt{b^2+8a^2}} > 0,$$

so that the values of t associated with this singular point are always real and distinct. Hence this point is, under all circumstances, a nodal point. On the other hand, it is seen from (4) that $x_+x_- = (b^2-a^2)/2$, and so

$$x_- \lesseqqgtr 0 \text{ according as } b \lesseqqgtr a,$$

while

$$x_- - b = \frac{-b-\sqrt{b^2+8a^2}}{4} < 0$$

in all cases. It follows that the associated values of t are

real and distinct if $b < a$,
real and equal if $b = a$,
complex conjugates if $b > a$,

and this singular point is a node, a cusp, or a conjugate point according as $b < a$, $b = a$, or $b > a$. In particular, when $b = a$, the cusp occurs at the point $(0, a)$, corresponding to $t = 0$, while the other singular point has coordinates $(3a/2, -a/2)$, with $t = \frac{1}{2}(3\pm\sqrt{3})a$.

The general shape of the curve for $a = 3$, and for each of the three cases $b = 1,3,5$, is shown in [2, p. 159]. These illustrations have been reconstructed here as Figures 2, 3, and 4.

The plane curve described here provides an excellent example of a number of curves for which the determination of the singular points from their Cartesian equation would prove formidable, whereas, with the aid of the parametric representation, the problem becomes so simple as to be almost elementary.

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* * *

PRIME POSITIONAL SEQUENCES

JACQUES P. SAUVÉ and CHARLES W. TRIGG

Tabulated below are 97 (a prime number) pairwise disjoint sequences each consisting of five or more primes less than 10^7. For any term n in any of the sequences, the next term in the same sequence is the nth prime p_n; and the first term of any sequence (which we will call the sequence *starter*) is the smallest prime that does not appear in any of the preceding sequences. We call these *prime positional sequences*. For example, the seventh sequence starts with 37, which is the smallest prime not included in any of the first six sequences, and

$$p_{37} = 157,\ p_{157} = 919,\ p_{919} = 7193,\ p_{7193} = 72727,\ p_{72727} = 919913,\ p_{919913} > 10^7.$$

The smallest prime not included in any of the 97 sequences is 727, but the sequence starting with 727 has its fifth term greater than 10^7, and is therefore omitted, as are all subsequent sequences.

2	3	5	11	31	127	709	5381	52711	648391	9737333
7	17	59	277	1787	15299	167449	2269733			
13	41	179	1063	8527	87803	1128889				
19	67	331	2221	19577	219613	3042161				
23	83	431	3001	27457	318211	4535189				
29	109	599	4397	42043	506683	7474967				
37	157	919	7193	72727	919913					
43	191	1153	9319	96797	1254739*					
47	211	1297	10631	112129	1471343					
53	241	1523	12763	137077	1828669					
61	283	1847	15823	173867	2364361					
<u>71</u>	353	2381	21179	239489	3338989					
<u>73</u>	367	2477	22093	250751	3509299					
79	401	2749	24859	285191	4030889					
89	461	3259	30133	352007	5054303					

97	509	3637	33967	401519	5823667
101	547	3943	37217	443419	6478961
103	563	4091	38833	464939	6816631
107	587	4273	40819	490643	7220981
113	617	4549	43651	527623	7807321
131	739	5623	55351	683873	
137	773	5869	57943	718807	
139	797	6113	60647	755387	
149	859	6661	66851	839483	
151	877	6823	68639	864013*	
163	967	7607	77431	985151	
167	991	7841	80071	1021271	
173	1031	8221	84347	1080923	
181	1087	8719	90023	1159901	
193	1171	9461	98519	1278779	
197	1201	9739	101701	1323503	
199	1217	9859	103069	1342907*	
223	1409	11743	125113	1656649	
227	1433	11953	127643	1693031	
229	1447	12097	129229	1715761	
233	1471	12301	131707	1751411	
239	1499	12547	134597	1793237	
251	1597	13469	145547	1950629	
257	1621	13709	148439	1993039	
263	1669	14177	153877	2071583*	
269	1723	14723	160483	2167937	
271	1741	14867	162257	2193689	
281	1823	15641	171697	2332537	
293	1913	16519	182261	2487943	
307	2027	17627	195677	2685911	
311	2063	17987	200017	2750357	
313	2081	18149	202001	2779781	
317	2099	18311	204067	2810191	
337	2269	20063	225503	3129913	
347	2341	20773	234293	3260657	
349	2351	20899	235891	3284657*	
359	2417	21529	243781	3403457	
373	2549	22811	259657	3643579	

379	2609	23431	267439	3760921*
383	2647	23801	271939	3829223
389	2683	24107	275837	3888551
397	2719	24509	280913	3965483
409	2803	25423	292489	4142053
419	2897	26371	304553	4326473
421	2909	26489	305999	4348681
433	3019	27689	321017	4578163*
439	3067	28109	326203	4658099
443	3109	28573	332099	4748047
449	3169	29153	339601	4863959
457	3229	29803	347849	4989697
463	3299	30557	357473	5138719
467	3319	30781	360293	5182717
479	3407	31667	371981	5363167
487	3469	32341	380557	5496349
491	3517	32797	386401	5587537
499	3559	33203	391711	5670851
503	3593	33569	396269	5741453
521	3733	35023	415253	6037513
523	3761	35311	418961	6095731*
541	3911	36887	439357	6415081
557	4027	38153	455849	6673993
569	4133	39239	470207	6898807
571	4153	39451	472837	6940103
577	4217	40151	481847	7081709
593	4339	41491	499403	7359427
601	4421	42293	510031	7528669
607	4463	42697	515401	7612799
613	4517	43283	522829	7730539
619	4567	43889	530773	7856939
631	4663	44879	543967	8066533
641	4759	45971	558643	8300687
643	4787	46279	562711	8365481
647	4801	46451	565069	8402833
653	4877	47297	576203	8580151
659	4933	47857	**583523**	8696917

661	4943	47963	584999	8720227
673	5021	48821	596243	8900383
677	5059	49207	601397	8982923
683	5107	49739	608459	9096533
691	5189	50591	619739	9276991
701	5281	51599	633467	9498161
719	5441	53353	657121	9878657

97 of the first 128 primes are sequence starters. One of these is even. Among the odd starters, 49 have the form $6k-1$ and 47 have the form $6k+1$. Included are 14 pairs of twin primes, underscored in the table.

Nine of the starters are palindromes: 2, 7, 101, 131, 151, 181, 313, 373, and 383. Six other palindromes appear in the sequences: 11, 191, 353, 797, 919, and the smoothly undulating 72727.

The sequence member 19577 begins with its starter, whereas their starters terminate 17, 277, 1787, 129229, and 2364361. Sequence members in which their starters (underscored) are imbedded are 127, 277, 1787, 52711, 167449, 219613, 1471343, and 2269733.

In the set of primes less than 10^7, the longest prime positional sequence has a prime number of terms, 11. In a certain sense, the last prime of that sequence, 9737333, which contains seven digits all but one of which are prime, can be said to be the primest prime less that 10^7. We observe that the two previous terms of this sequence contain the nine nonzero digits, with only the digit 1 appearing more than once. In the eleven terms of this sequence together, the frequency of occurrence of the digits

0 1 2 3 4 5 6 7 8 9

is, respectively,

1 8 3 8 1 3 1 5 2 3,

an undulating sequence. Thus there are 8 even digits and 27 odd digits, and both frequencies are cubes. The prime digits number 19, a prime, and the nonprime digits number 16, a square.

One of these prime positional sequences contains 8 terms, four have 7 terms, fourteen have 6 terms, and seventy-seven have 5 terms.

Among the primes in the sequences, three-digit repdigits occur three times at the beginning: 2221, 6661, 3338989; three times inside: 200017, 1128889, 3888551; and three times at the end: 305999, 584999, 9737333. All in all, a plethora of threes. And there are more: among the sequence starters there are 9 (a power of 3) whose digits are all powers of 3: 13, 19, 113, 131, 139, 193, 199, 311, 313; and among the sequence enders there is 1 (a power of 3) whose digits are all powers of 3: 919913. Finally, there are 8 (a 3rd power) sequence enders that contain distinct digits (these are indicated by asterisks in the table); and 1 (a power of 3) of them is a permutation of consecutive digits: 3284657.

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* * *

THE OLYMPIAD CORNER: 42

M.S. KLAMKIN

It is now official: the 24th International Mathematical Olympiad will be held in Paris, France, from July 1 to July 12, 1983. Each national delegation will consist of six students, a leader, and a deputy leader. The official languages will be English, French, German, and Russian.

*

I now give the problems posed at the 1982 University of Alberta Undergraduate Mathematics Contest. The questions were set by G. Butler, Andy Liu, and myself. I shall publish solutions in this column next month.

UNIVERSITY OF ALBERTA UNDERGRADUATE MATHEMATICS CONTEST

November 22, 1982 — Time: 3 hours

1. Find the equation of a cone with vertex at the origin and containing the intersection of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ with the sphere $x^2+y^2+z^2 = r^2$, where $a < r < c$.

2. Prove that the following two statements are equivalent:
(a) There are no positive integers a,b,c such that $a^4 - b^4 = c^2$.
(b) There are no positive integers w,x,y,z such that $w^2+x^2=y^2$ and $w^2-x^2=z^2$.
Assuming that either (a) or (b) has been proved, deduce the following special case of Fermat's Last Theorem: *There are no positive integers x,y,z such that* $x^4 + y^4 = z^4$.

3. Let f be a three times differentiable real-valued function of a real variable such that the recurrence relation $x_{n+1} = f(x_n)$ with $x_1 > 0$ always has the property that $\lim_{n\to\infty} nx_n^2 = 1$. Evaluate $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

4. Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of equations

$$\frac{1}{x_{j+1}} \sum_{i=1}^{5} a_i x_{i+j} = \frac{1}{x_{k+1}} \sum_{i=1}^{5} a_i x_{i+k}$$

for $j,k = 0,1,2,3,4$, where $x_m = x_n$ if $m \equiv n \pmod 5$.

5. An airplane flies at a constant speed relative to the wind which varies continuously with position but does not vary with time. It flies a closed path and then flies the same path in the reverse direction. Prove that the total time of flight is greater than if there were no wind.

*

I now present solutions to the problems of Practice Set 16 [1983: 14].

16-1. Solve the equation $(x^2 - 4)(x^2 - 2x) = 2$.

Solution.

The equation is equivalent to

$$(x^2-x)^2 - 2(x^2-x) + 1 = 3(x-1)^2,$$

from which

$$x^2 - x - 1 = \pm\sqrt{3}(x-1).$$

The four roots of these two quadratics are

$$x = \tfrac{1}{2}(1 + \sqrt{3} \pm \sqrt{8-2\sqrt{3}})$$

and

$$x = \tfrac{1}{2}(1 - \sqrt{3} \pm \sqrt{8+2\sqrt{3}}).$$

Alternate solution.

Since $x \neq 1$, we can set $x = 1-t$ with $t \neq 0$ and obtain the equivalent equation

$$(t - \tfrac{1}{t})^2 - 2(t - \tfrac{1}{t}) - 2 = 0,$$

from which

$$t - \frac{1}{t} = 1 \pm \sqrt{3},$$

and this leads to the same four roots as above.

16-2. Given the face angles of a trihedral angle T, determine the locus of the points of contact of its faces with its inscribed spheres.

Solution.

The locus consists of three straight lines, one in each face, and all concurrent at the vertex V of T. Since the center of any of the inscribed spheres is equidistant from the three faces of T, the centers of all the inscribed spheres lie on a line l through V which is the common intersection of the three planes which bisect the three dihedral angles of T. Then the points of contact of the spheres with any one face is obtained by projecting l orthogonally on that face.

Alternate solution.

We arrive at the same result by notinq that the set of inscribed spheres forms a homothetic family with homothetic center V, and that the points of contact on any one face are corresponding points in this homothecy.

16-3. A length L of wire is cut into two pieces which are bent into a circle and a square. Determine the minimum and the maximum of the sum of the two areas formed.

Solution.

Let L_c and A_c denote the perimeter and area, respectively, of the circle, and let L_s and A_s denote the corresponding quantities for the square. We have

$$A_c = \pi r^2, \text{ where } r = L_c/2\pi,$$

and

$$A_s = a^2, \text{ where } a = L_s/4.$$

We wish to minimize and maximize

$$A \equiv A_c + A_s = \frac{L_c^2}{4\pi} + \frac{L_s^2}{16},$$

where $L_c + L_s = L$.

To find the minimum, we apply Cauchy's inequality:

$$\left(\frac{L_c^2}{4\pi} + \frac{L_s^2}{16}\right)(4\pi + 16) \geq (L_c + L_s)^2 = L^2,$$

with equality if and only if

$$\frac{L_c}{4\pi} = \frac{L_s}{16} = \frac{L}{4\pi+16}.$$

Thus

$$A_{\min} = \frac{L^2}{4\pi+16},$$

and this minimum is attained when

$$L_c = \frac{4\pi L}{4\pi+16} \quad \text{and} \quad L_s = \frac{16L}{4\pi+16}.$$

To find the maximum, we note that

$$\frac{L_c^2}{4\pi} + \frac{L_s^2}{16} \le \frac{L_c^2 + L_s^2}{4\pi} \le \frac{(L_c+L_s)^2}{4\pi} = \frac{L^2}{4\pi},$$

with equality if and only if $L_s = 0$. Thus

$$A_{\max} = \frac{L^2}{4\pi}. \tag{1}$$

This result also follows from the isoperimetric theorem according to which, for all closed plane curves of given length, the circle has the maximum area.

If, as the problem states, the wire is actually cut into two pieces, then (1) is a least upper bound that is never attained.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Puzzle No. 29: Rebus (6)

$$P\overline{)S}$$

I'm fond of palominos, when they're fleet;
But slow or fast, I also like COMPLETE.

Puzzle No. 30: Rebus (6 7)

$$(G+G)/2, \quad \sqrt{G \cdot G}$$

Piece of the "Rock"
Had no musician;
"Patient" Griselda
Had no physician;
"Fair" Maid of Perth,
No ethical mission.
Words that describe
May have a CONDITION.

ALAN WAYNE, Holiday, Florida

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

811. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Some say that's how all the trouble started. But it must have been very tempting, for that Adam's APPLE was truly prime!

```
 ADAM
  ATE
 THAT
  RED
-----
APPLE
```

812. *Proposed by Dan Sokolowsky, California State University at Los Angeles.*

Let C be a given circle, and let C_i, $i = 1,2,3,4$, be circles such that

(i) C_i is tangent to C at A_i for $i = 1,2,3,4$;

(ii) C_i is tangent to C_{i+1} for $i = 1,2,3$.

Furthermore, let l be a line tangent to C at the other extremity of the diameter of C through A_1, and, for $i = 2,3,4$, let A_1A_i intersect l in P_i.

Prove that, if C, C_1, and C_4 are fixed, then the ratio of unsigned lengths P_2P_3/P_3P_4 is constant for all circles C_2 and C_3 that satisfy (i) and (ii).

813.* *Proposed by Charles W. Trigg, San Diego, California.*

The array on the right is a "staircase" of primes of the form 3_k1. When 3 is replaced by some other digit, the furthest any staircase of primes goes is 6661, since $66661 = 7 \cdot 9523$.

How much further does the 3_k1 staircase go before a composite number appears? Subsequent to that, what is the next prime in the staircase?

```
     31
    331
   3331
  33331
 333331
3333331
```

814. *Proposed by Leon Bankoff, Los Angeles, California.*

Let D denote the point on BC cut by the internal bisector of angle BAC in the Heronian triangle whose sides are AB = 14, BC = 13, CA = 15. With D as center, describe the circle touching AC in L and cutting the extension of AD in J. Show that $AJ/AL = (\sqrt{5}+1)/2$, the Golden Ratio.

815. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a,b,c, internal angle bisectors t_a, t_b,t_c, and semiperimeter s. Prove that the following inequalities hold, with equality if and only if the triangle is equilateral:

(a) $$\sqrt{3}\left(\frac{1}{at_a}+\frac{1}{bt_b}+\frac{1}{ct_c}\right) \ge \frac{4s}{abc};$$

(b) $$3\sqrt{3}\cdot\frac{\frac{1}{at_a}+\frac{1}{bt_b}+\frac{1}{ct_c}}{at_a+bt_b+ct_c} \ge 4\sqrt{\frac{2s}{(abc)^3}}.$$

816. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let a,b,c be the sides of a triangle with semiperimeter s, inradius r, and circumradius R. Prove that, with sums and product cyclic over a,b,c,

(a) $$\Pi(b+c) \le 8sR(R+2r),$$

(b) $$\Sigma bc(b+c) \le 8sR(R+r),$$

(c) $$\Sigma a^3 \le 8s(R^2-r^2).$$

817. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

(a) Suppose that to each point on the circumference of a circle we arbitrarily assign the color red or green. Three distinct points of the same color will be said to form a *monochromatic triangle*. Prove that there are monochromatic isosceles triangles.

(b)* Prove or disprove that there are monochromatic isosceles triangles if to every point on the circumference of a circle we arbitrarily assign one of k colors, where $k \ge 2$.

818. *Proposed by A.P. Guinand, Trent University, Peterborough, Ontario.*

Let ABC be a scalene triangle with circumcentre O and orthocentre H, and let P be the point where the internal bisector of angle A intersects the Euler line OH. If O, H, P only are given, construct an angle equal to angle A, using only ruler and compass.

819. *Proposed by H. Kestelman, University College, London, England.*

Let A and B be $n\times n$ Hermitian matrices. Prove that $AB - BA$ is singular if A and B have a common eigenvector. Prove that the converse is true if $n = 2$ but not if $n > 2$.

820. *Proposed by W.R. Utz, University of Missouri-Columbia.*

Let P be a polynomial with real coefficients. Devise an algorithm for summing the series

$$\sum_{n=q}^{\infty} \frac{P(n)}{n!}.$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

707. [1982: 15] *Proposed by Charles W. Trigg, San Diego, California.*

In the decimal system, how many eight-digit palindromes are the products of two consecutive integers?

I. *Composite of the (nearly identical) solutions of Bob Prielipp, University of Wisconsin-Oshkosh, and the proposer.*

Products of two consecutive integers are called *pronic numbers* [1]. Pronic numbers end in 0, 2, or 6; and they end in 06 or 56 if they end in 6. Consequently, the pronic palindromes $x(x+1)$ we are seeking must lie between

$$20000002 \text{ and } 29999992 \text{ which implies } 4472 \le x \le 5476, \tag{1}$$

or between

$$60000006 \text{ and } 60999906 \text{ which implies } 7746 \le x \le 7809, \tag{2}$$

or between

$$65000056 \text{ and } 65999956 \text{ which implies } 8062 \le x \le 8123. \tag{3}$$

An eight-digit palindrome is divisible by 11, so either x or $x+1$ is a multiple of 11. In (1), x must end in 1, 3, 6, or 8. There are 36 such integers x with $x \equiv 0 \pmod{11}$ and 37 with $x+1 \equiv 0 \pmod{11}$. In (2) and (3), x must end in 2 or 7. In these ranges, we have $x \equiv 0 \pmod{11}$ only for $x = 7777$, 8107; and $x+1 \equiv 0 \pmod{11}$ only for $x = 7787$, 8062, 8117.

We must therefore test 78 values of x. Multiplying each by the corresponding $x+1$ yields a palindrome only for $x = 5291$ and 5313. The only solutions are

$$5291 \cdot 5292 = 27999972,$$

$$5313 \cdot 5314 = 28233282.$$

II. *Comment by Milton P. Eisner, Mount Vernon College, Washington, D.C.*

If initial zeros are allowed, there are three additional solutions:

$$00000000 = 0 \cdot 1,$$

$$01233210 = 1110 \cdot 1111,$$

$$06966960 = 2639 \cdot 2640.$$

III. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Other facts of interest revealed by a computer search:

(a) There is only one eight-digit palindrome that is half a pronic number (i.e., is a triangular number):

$$35133153 = \tfrac{1}{2} \cdot 8382 \cdot 8383.$$

(b) There are no eight-digit palindromes that are squares.

(c) There are no eight-digit palindromes that are the product of three or more consecutive integers.

(d) The only eight-digit palindromes of the form $n(n+d)$, where $2 \le d \le 9$, are

$$32855823 = 5731 \cdot (5731+2), \qquad 72999927 = 8541 \cdot (8541+6),$$

$$99999999 = 9999 \cdot (9999+2), \qquad 81099018 = 9002 \cdot (9002+7),$$

$$29311392 = 5412 \cdot (5412+4), \qquad 48999984 = 6996 \cdot (6996+8),$$

$$69555596 = 8338 \cdot (8338+4), \qquad 88322388 = 9394 \cdot (9394+8),$$

$$71588517 = 8459 \cdot (8459+4), \qquad 29122192 = 5392 \cdot (5392+9).$$

IV. *Comment by the proposer.*

There are just ten pronic palindromes less than 10^{10}. These are

$$1 \cdot 2 = 2, \qquad 1621 \cdot 1622 = 2629262,$$

$$2 \cdot 3 = 6, \qquad 2457 \cdot 2458 = 6039306,$$

$$16 \cdot 17 = 272, \qquad 5291 \cdot 5292 = 27999972,$$

$$77 \cdot 78 = 6006, \qquad 5313 \cdot 5314 = 28233282,$$

$$538 \cdot 539 = 289982, \qquad 52008 \cdot 52009 = 2704884072.$$

There is only one palindrome less than 10^{10} which is the product of three or four consecutive integers (and, of course, none of any size which is the product of five or more consecutive integers). It is the reppalindrome

$$77 \cdot 78 \cdot 79 = 474474.$$

Note in particular that 77, $77 \cdot 78$, and $77 \cdot 78 \cdot 79$ are all palindromes.

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; ALAN WAYNE, Holiday, Florida; and KENNETH M. WILKE, Topeka, Kansas.

REFERENCE

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. I, p. 357.

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708. [1982: 15] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

A triangle has sides a,b,c, semiperimeter s, inradius r, and circumradius R.

(a) Prove that

$$(2a-s)(b-c)^2 + (2b-s)(c-a)^2 + (2c-s)(a-b)^2 \geq 0,$$

with equality just when the triangle is equilateral.

(b) Prove that the inequality in (a) is equivalent to each of the following:

$$3(a^3+b^3+c^3+3abc) \leq 4s(a^2+b^2+c^2),$$

$$s^2 \geq 16Rr-5r^2.$$

Solution by W.J. Blundon, Memorial University of Newfoundland.

We will first show that the three inequalities

$$(2a-s)(b-c)^2 + (2b-s)(c-a)^2 + (2c-s)(a-b)^2 \geq 0, \tag{1}$$

$$4s(a^2+b^2+c^2) \geq 3(a^3+b^3+c^3+3abc), \tag{2}$$

$$s^2 \geq 16Rr - 5r^2, \tag{3}$$

are equivalent, and then prove that one of them is (and hence all three are) valid, with equality if and only if the triangle is equilateral. In establishing equivalence, we will use the well-known relations

$$a+b+c = 2s, \qquad bc+ca+ab = s^2+4Rr+r^2, \qquad abc = 4Rrs,$$

from which follow

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(bc+ca+ab) = 2(s^2-4Rr-r^2)$$

and

$$a^3+b^3+c^3 = (a+b+c)^3 - 3(a+b+c)(bc+ca+ab) + 3abc = 2(s^3-6Rrs-3r^2s).$$

With sums cyclic over a,b,c, we have

$$\begin{aligned}\Sigma(2a-s)(b-c)^2 &= \tfrac{1}{2}\Sigma(3a-b-c)(b^2-2bc+c^2)\\ &= \tfrac{1}{2}\Sigma(3ab^2+3ac^2+b^2c+bc^2-b^3-c^3-6abc)\\ &= 2\Sigma bc(b+c) - \Sigma a^3 - 9abc\\ &= 2(\Sigma a)(\Sigma a^2) - 3\Sigma a^3 - 9abc\end{aligned}$$

$$= 4s\Sigma a^2 - 3(\Sigma a^3+3abc) \qquad (4)$$
$$= 8s(s^2-4Rr-r^2) - 6(s^3-3r^2s)$$
$$= 2s(s^2-16Rr+5r^2). \qquad (5)$$

Now (1) <==> (2) follows from (4) and, since $s > 0$, (2) <==> (3) follows from (5).

At this point, to establish the validity of (1)-(3) we could simply refer to Bottema [1] where a proof of (3) is given. We will instead establish (1) directly, because it will lead to an interesting generalization. For this purpose, we will use *Schur's Inequality* (see [2] or [3]): *If t is any real number and $x,y,z > 0$, then*

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0, \qquad (6)$$

with equality if and only if $x = y = z$.

Let $x = s-a$, $y = s-b$, $z = s-c$; then $x,y,z > 0$ and $a = y+z$, $b = z+x$, $c = x+y$. With cyclic sums again, we have

$$\begin{aligned}\Sigma(2a-s)(b-c)^2 &= \Sigma(y+z-x)(y-z)^2\\ &= \Sigma(y^3+z^3+2xyz-y^2z-yz^2-xy^2-z^2x)\\ &= 2\Sigma x^3 - 2\Sigma yz(y+z) + 6xyz\\ &= 2\Sigma x(x-y)(x-z),\end{aligned}$$

and (1) follows from (6) with $t = 1$.

More generally, for any real number t we get from (6)

$$\Sigma(s-a)^t(b-a)(c-a) \geq 0,$$

with equality if and only if $a = b = c$, and this is equivalent to (1)-(3) when $t = 1$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968, p. 50.

2. G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1952, p. 64.

3. D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York, 1970, pp. 119-121.

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709. [1982: 15] *Proposed by F.G.B. Maskell, Algonquin College, Ottawa, Ontario.*

ABC is a triangle with incentre I, and DEF is the pedal triangle of the point

I with respect to triangle ABC. Show that it is always possible to construct with straightedge and compass four circles each of which is tangent to each of the circumcircles of triangles ABC, EIF, FID, and DIE, provided that triangle ABC is not equilateral.

Solution by Jordi Dou, Barcelona, Spain.

Let α, β, γ, Ω, and ϕ be circumcircles of triangles AFE, BDF, CED, ABC, and DEF, respectively; and let Φ be the inversion with ϕ as the circle of inversion. The circles α,β,γ, which all pass through the centre of inversion I, invert into the sides of triangle DEF, and Ω inverts into ω, the nine-point circle of triangle DEF [1, p. 243, Ex. 18(a)]. By Feuerbach's Theorem [1, p. 105], ω is tangent to the four tritangent circles of triangle DEF, namely δ_0 (the incircle) and δ_1,δ_2, δ_3 (the excircles). The four required circles are therefore $\Phi(\delta_0)$, $\Phi(\delta_1)$, $\Phi(\delta_2)$, and $\Phi(\delta_3)$, each of which is tangent to α, β, γ, and Ω. It is clear that the four circles $\Phi(\delta_i)$ can be constructed with straightedge and compass. Indeed, any circle tritangent to α,β,γ can be so constructed (problem of Apollonius).

The proposer's decision to avoid equilateral triangles is not really necessary. For then $\omega = \delta_0$, so $\Phi(\delta_0)$ coincides with Ω and is thus "supertangent" to Ω.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

REFERENCE

1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952.

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710. [1982: 16] *Proposed by Gali Salvatore, Perkins, Québec.*

Let z' and z'' be the roots of the equation

$$z + \frac{1}{z} = 2(\cos\phi + i\sin\phi),$$

where $0 < \phi < \pi$.

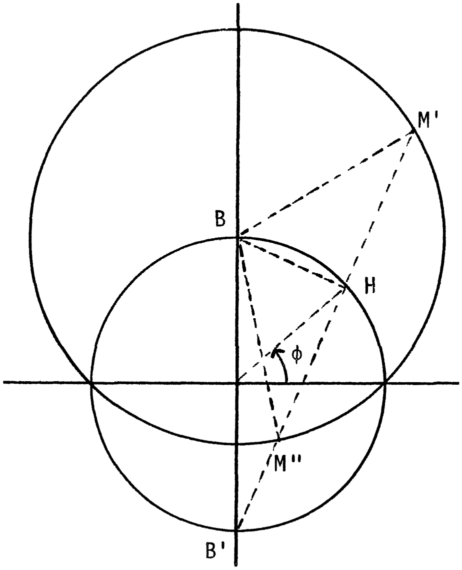
(a) Show that $z'+i$, $z''+i$ have the same argument, and that $z'-i$, $z''-i$ have the same modulus.

(b) Find the locus of the roots z',z'' in the complex plane when ϕ varies from 0 to π.

Solution by Dan Sokolowsky, California State University at Los Angeles.

(a) We will for typographical convenience write $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$. The given equation is equivalent to

$$z^2 - 2z\operatorname{cis}\phi + 1 = 0 : \qquad (1)$$



hence the complex number

$$\alpha \equiv \frac{z'+z''}{2} = \operatorname{cis}\phi$$

is the affix of the point H on the unit circle whose polar angle is ϕ (see figure). But (1) is also equivalent to

$$(z - \alpha)^2 = 2i\alpha\sin\phi;$$

hence

$$|(z-\alpha)^2| = 2\sin\phi > 0 \quad \text{and} \quad \arg(z-\alpha)^2 = \frac{\pi}{2} + \phi.$$

We can now write

$$z'-\alpha = \sqrt{2\sin\phi}\cdot\operatorname{cis}\left(\frac{\pi}{4}+\frac{\phi}{2}\right), \tag{2'}$$

$$z''-\alpha = -\sqrt{2\sin\phi}\cdot\operatorname{cis}\left(\frac{\pi}{4}+\frac{\phi}{2}\right). \tag{2''}$$

Since

$$\cos\phi = 2\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$$

and

$$1 + \sin\phi = 2\sin^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right), \tag{3}$$

we have

$$i + \alpha = \cos\phi + i(1+\sin\phi)$$
$$= 2\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\operatorname{cis}\left(\frac{\pi}{4}+\frac{\phi}{2}\right). \qquad (4)$$

Now, adding (2) and (4) and noting from (3) that

$$2\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right) > \sqrt{2\sin\phi},$$

we have

$$z'+i = \left\{2\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right) + \sqrt{2\sin\phi}\right\}\operatorname{cis}\left(\frac{\pi}{4}+\frac{\phi}{2}\right),$$
$$z''+i = \left\{2\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right) - \sqrt{2\sin\phi}\right\}\operatorname{cis}\left(\frac{\pi}{4}+\frac{\phi}{2}\right),$$

from which we see that $z'+i$ and $z''+i$ have the same argument $\pi/4 + \phi/2$.

If z', z'', and $-i$ are the affixes of the points M', M", and B', respectively, it follows from our results so far that M' and M" are both on the line B'H. Furthermore, the numbers $z'-\alpha$ and $z''-\alpha$, which are represented by the vectors $\vec{HM'}$ and $\vec{HM''}$, respectively, have the same modulus $\sqrt{2\sin\phi}$. If i is the affix of the point B, it follows that triangle BM'M" is isosceles; hence the numbers $z'-i$ and $z''-i$, which are represented by the vectors $\vec{BM'}$ and $\vec{BM''}$, respectively, have the same modulus.

These results remain valid for $\phi = 0$ and $\phi = \pi$, for then the roots z' and z'' coincide.

(b) Since

$$|\vec{BH}| = |\alpha - i| = \sqrt{\cos^2\phi + (\sin\phi - 1)^2} = \sqrt{2(1-\sin\phi)},$$

the common modulus of $z'-i$ and $z''-i$ is

$$\sqrt{|\vec{BH}|^2 + |\vec{HM'}|^2} = \sqrt{2}.$$

It follows that M' and M" describe, as ϕ increases from 0 to π, the arcs of the circle with center B and radius $\sqrt{2}$ which lie in the upper and lower half-planes, respectively.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; HENRY E. FETTIS, Mountain View, California; J.T. GROENMAN, Arnhem, The Netherlands; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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711. [1982: 47] *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

Find all the solutions of the following alphametic (in which the last word represents the sum of the preceding words), which I have worked on from time to time but never carried to completion:

A ROSE IS A ROSE IS A ROSE SO THE WARS OF THE ROSES AROSE.

I. *Solution by Allan Wm. Johnson Jr., Washington, D.C.*

In less than five minutes of wall-clock time on a Model I Radio Shack TRS-80 microcomputer, I found that this alphametic has exactly fifteen solutions in base ten, consisting of six pairs of solutions with H and I interchangeable, and three solutions in which they are not (because H = 0 and I cannot equal 0). They are listed below.

A	ROSE	IS	A	ROSE	IS	A	ROSE	SO	THE	WARS	OF	THE	ROSES	AROSE
5	3620	92	5	3620	92	5	3620	26	870	4532	61	870	36202	53620
5	3620	72	5	3620	72	5	3620	26	890	4532	61	890	36202	53620
4	2580	98	4	2580	98	4	2580	85	630	7428	51	630	25808	42580
4	2580	38	4	2580	38	4	2580	85	690	7428	51	690	25808	42580
3	1560	76	3	1560	76	3	1560	65	840	9316	52	840	15606	31560
3	1560	46	3	1560	46	3	1560	65	870	9316	52	870	15606	31560
5	3780	48	5	3780	48	5	3780	87	910	2538	76	910	37808	53780
5	3780	18	5	3780	18	5	3780	87	940	2538	76	940	37808	53780
8	6340	54	8	6340	54	8	6340	43	920	1864	37	920	63404	86340
8	6340	24	8	6340	24	8	6340	43	950	1864	37	950	63404	86340
5	3861	76	5	3861	76	5	3861	68	401	2536	89	401	38616	53861
9	7153	85	9	7153	85	9	7153	51	463	2975	10	463	71535	97153
9	7153	65	9	7153	65	9	7153	51	483	2975	10	483	71535	97153
7	5263	86	7	5263	86	7	5263	62	903	4756	21	903	52636	75263
6	4127	82	6	4127	82	6	4127	21	307	9642	15	307	41272	64127

II. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

This is the longest alphametic that I have ever seen. The only alphametic I know that comes close is the memorable

SUN		741
LOSE		5672
UNTIE	with	41982
BOTTLE	the	369952
ELISION	remarkable	2587861
NINETEEN	unique	18129221
NONENTITY	solution	161219890
EBULLIENT		234558219
INSOLUBLE		817654352 .
NEBULOSITY		1234567890

It was proposed by Steven R. Conrad in the December 1962 issue of *Recreational Mathematics Magazine* (issue 12, page 24).

It would be interesting to know if readers know of any more mammoth alphametics in the literature.

Also solved by MEIR FEDER, Haifa, Israel; and STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire. Partial solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono; HANS HAVERMANN, Weston, Ontario; ROBERT S. JOHNSON, Montréal, Québec; and HARRY L. NELSON, Livermore, California.

Editor's comment.

It was Gertrude Stein (1874-1946) who wrote "a rose is a rose is a rose", right? Wrong. This is one of the famous misquotations of literary history. The correct quotation (from *Sacred Emily*) is "Rose is a rose is a rose is a rose", and the first "Rose" is the name of a person. William Safire recently wrote [1]: "She [Gertrude Stein] did not write, 'A rose is a rose is a rose'; she wrote, 'Rose is a rose is a rose is a rose,' and the addition of the first article changes the meaninglessness. But that is neither here nor there (Miss Stein also derogated Oakland, Calif., with 'There's no there there')." Safire backtracked somewhat in a later column [2]. It was entitled "Rose Were a Rose Were a Rose". (Here the first "Rose" refers to Rose Kennedy, the mother of Senator Edward M. Kennedy and the matriarch of the Kennedy clan.)

We asked our first solver Johnson to solve our alphametic modified to include the correct Stein quotation. Again in less than five minutes of wall-clock time, he found that there are exactly eight solutions. These are given below.

ROSE	IS	A	ROSE	IS	A	ROSE	IS	A	ROSE	SO	THE	WARS	OF	THE	ROSES	AROSE
1580	48	3	1580	48	3	1580	48	3	1580	85	920	7318	56	920	15808	31580
4981	28	7	4981	28	7	4981	28	7	4981	89	601	3748	95	601	49818	74981
4382	78	6	4382	78	6	4382	78	6	4382	83	502	1648	39	502	43828	64382
2875	97	4	2875	97	4	2875	97	4	2875	78	365	1427	80	365	28757	42875
4275	97	6	4275	97	6	4275	97	6	4275	72	185	3647	20	185	42757	64275
2376	57	4	2376	57	4	2376	57	4	2376	73	196	8427	30	196	23767	42376
2839	73	4	2839	73	4	2839	73	4	2839	38	659	1423	80	659	28393	42839
1759	85	3	1759	85	3	1759	85	3	1759	57	209	6315	74	209	17595	31759

Of course, there are dissenting voices. In a book review [3], Gwendolyn MacEwen quotes the following from a book by Don Domanski [4]: "A rose is not a rose, but always a war. War in an empty house." And Noel Perrin reports [5]: "In one of Isak Dinesen's stories, a Danish nobleman sits thinking about roses. A rose *isn't* a rose, he reflects—at least..." At least she didn't say "Rose isn't a rose isn't a rose isn't a rose". Just as well: the Wars of the Roses would have to be refought, with Gertrude Stein and Isak Dinesen as the protagonists.

REFERENCES

1. *The New York Times Magazine*, December 19, 1982, p. 18.
2. *The New York Times Magazine*, February 13, 1983, p. 16.
3. *Books in Canada*, Vol. 11, No. 8 (October 1982), p. 24.
4. Don Domanski, *War in an Empty House*, House of Anansi, 1982.
5. *The New York Times Book Review*, May 2, 1982, p. 11.

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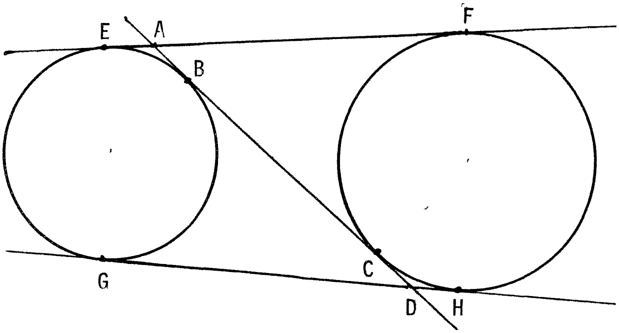
712. [1982: 47] *Proposed by Donald Aitken, Northern Alberta Institute of Technology, Edmonton, Alberta.*

Prove that AB = CD in the figure below.

Solution de Bernard Baudiffier, Collège de Sherbrooke, Sherbrooke, Québec.



Les points E,F,G,H étant tels que notés sur la figure, on a

$$EF = EA + AF = AB + AC = 2AB + BC$$

et

$$GH = DH + GD = CD + BD = 2CD + BC.$$

Or la symétrie de la figure donne EF = GH; donc

$$2AB + BC = 2CD + BC \quad \text{et} \quad AB = CD.$$

Also solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; PATRICIA A. BENEDICT, Cleveland Heights H.S., Cleveland Heights, Ohio; W.J. BLUNDON, Memorial University of Newfoundland; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CECILE M. COHEN, Horace Mann School, Bronx, N.Y.; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; MILTON P. EISNER, Mount

Vernon College, Washington, D.C.; HENRY E. FETTIS, Mountain View, California; JACK GARFUNKEL, Flushing, N.Y.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; W.C. IGIPS, Danbury, Connecticut; NICK MARTIN, student, Indiana University, Bloomington, Indiana; F.G.B. MASKELL, Algonquin College, Ottawa; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DIMITRIS VATHIS, Chalcis, Greece; KENNETH S. WILLIAMS, Carleton University, Ottawa; JOHN A. WINTERINK, Albuquerque Technical-Vocational Institute, Albuquerque, New Mexico; and the proposer.

Editor's comment.

Readers will learn a bit of French and not much mathematics from the above solution, which is equivalent to more than half of the solutions received. For mathematical journals as well as for human beings, a gentle exercise once in a while is good for the circulation.

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713. [1982: 48] *Proposed jointly by Hartmut Maennel and Bernhard Leeb, West German team members, 1981 International Mathematical Olympiad.*

Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{p_i}\{\prod_{j=1}^{i} (1 - \frac{1}{p_j})\}.$$

(a) Show that the series converges if $\{p_i\}$ is the sequence of primes.

(b) Does it still converge if $\{p_i\}$ is a real sequence with each $p_i \geq 1$?

Solution by the proposers.

We will prove that the answer to part (b) is affirmative, and part (a) will follow as a special case.

The given series can be written

$$\sum_{i=1}^{\infty} \frac{1}{p_i}\cdot A_i, \qquad \text{where} \qquad A_i = \prod_{j=1}^{i}(1 - \frac{1}{p_j}). \tag{1}$$

Let $\{S_n\}$ be the sequence of partial sums of the associated series

$$\frac{1}{p_1} + \sum_{i=2}^{\infty} \frac{1}{p_i}\cdot A_{i-1}. \tag{2}$$

We show by induction that

$$S_n = 1 - A_n, \qquad n = 1,2,3,\ldots \ . \tag{3}$$

We have $S_1 = 1/p_1 = 1 - A_1$ and, assuming that (3) holds for some n,

$$S_{n+1} = S_n + \frac{1}{p_{n+1}}\cdot A_n = 1 - A_n + \frac{1}{p_{n+1}}\cdot A_n = 1 - A_{n+1}.$$

Now for all n we have

$$0 \le A_{n+1} \le A_n < 1, \qquad \text{so} \qquad S_n \le S_{n+1} \le 1\,;$$

hence the series (2) converges. Finally, from

$$\frac{1}{p_i}\cdot A_i \le \frac{1}{p_i}\cdot A_{i-1}, \qquad i = 2,3,4,\ldots,$$

it follows that (1) converges by comparison with (2).

Editor's comment.

Other would-be solvers probably gave up in despair after fruitlessly trying to prove the red herring part (a) by using deep theorems of Prime Number Theory, and *then* trying to find a counterexample for part (b). It ain't fair.

* * *

714*. [1982: 48] *Proposed by Harry D. Ruderman, Hunter College, New York, N.Y.*

Prove or disprove that for every pair (p,q) of nonnegative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer. (This problem was suggested by Problem 556 [1981: 282] proposed by Paul Erdös.)

Editor's comment.

No solution was received for this problem, which therefore remains open.

* * *

715. [1982: 48] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Let k be a real number, n an integer, and A,B,C the angles of a triangle.

(a) Prove that

$$8k(\sin nA + \sin nB + \sin nC) \le 12k^2 + 9.$$

(b) Determine for which k equality is possible in (a), and deduce that

$$|\sin nA + \sin nB + \sin nC| \le 3\sqrt{3}/2.$$

I. *Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Let

$$S = S(n) \equiv \sin nA + \sin nB + \sin nC.$$

Part (a) requires us to show that, for any real k,

$$12k^2 - 8kS + 9 = 12\{(k-\frac{S}{3})^2 + \frac{1}{9}(\frac{27}{4}-S^2)\} \geq 0. \tag{1}$$

Now (1) certainly holds for all k if $S^2 \leq 27/4$, or

$$|S(n)| \leq \frac{3\sqrt{3}}{2}, \tag{2}$$

and we proceed to establish this inequality, which is that of part (b).

It is easy to see that (2) holds if one summand of $S(n)$ is zero, or if two summands have opposite signs, for in each case $|S(n)| \leq 2 < 3\sqrt{3}/2$. We will show that it also holds if all summands have the same sign. We will need the following triple inequality, in which α,β,γ are the angles of an arbitrary triangle [1]:

$$0 < \sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}, \tag{3}$$

where each equality holds just when the triangle is equilateral.

When all summands of $S(n)$ are nonzero and have the same sign, we have

$$\begin{cases} nA = r_1\pi + A', & \text{with} \quad 0 < A' < \pi, \\ nB = r_2\pi + B', & \text{with} \quad 0 < B' < \pi, \\ nC = r_3\pi + C', & \text{with} \quad 0 < C' < \pi, \end{cases} \tag{4}$$

where r_1, r_2, r_3 are integers of the same parity (even if the summands are positive, odd if they are negative). In any case we have

$$|S(n)| = S' \equiv \sin A' + \sin B' + \sin C'.$$

Summing the relations (4), we get

$$n\pi = r\pi + (A'+B'+C'),$$

where $r = r_1+r_2+r_3$, and so

$$0 < (n-r)\pi = A' + B' + C' < 3\pi.$$

Therefore $A'+B'+C' = \pi$ or 2π. In the first case, A',B',C' are the angles of a triangle, and $S' \leq 3\sqrt{3}/2$ follows from (3). In the second case, $A'/2,B'/2,C'/2$ are the angles of a triangle, and $S' \leq 3\sqrt{3}/2$ again follows from (3). This completes the proof of (2), and (1) is established.

We have only left to determine for which k equality is possible in (1). It follows from (2) and (1) that equality is possible only when $|S| = 3\sqrt{3}/2$ and $k = S/3 = \pm\sqrt{3}/2$.

II. *Solution by M.S. Klamkin, University of Alberta (modified to use the notation of solution I when convenient).*

Inequality (1) holds if $k = 0$, so we assume that $k \neq 0$. By the A.M.-G.M. inequality, we have

$$\frac{12k^2 + 9}{8|k|} \geq \frac{3\sqrt{3}}{2}, \tag{5}$$

with equality just when $|k| = \sqrt{3}/2$. Assuming inequality (2) for the moment, we have

$$|S(n)| \leq \frac{3\sqrt{3}}{2} \leq \frac{12k^2 + 9}{8|k|}.$$

Hence (1) follows from the sharper inequality (2), and equality holds in (1) only when it holds in (5), that is, only when $k = \pm\sqrt{3}/2$. We now establish a generalization of (2).

If is known (see [2] or Crux 552 [1981: 182]) that, for arbitrary real numbers x,y,z and integer m,

$$x^2 + y^2 + z^2 \geq (-1)^{m+1}(2yz\cos mA + 2zx\cos mB + 2xy\cos mC), \tag{6}$$

with equality if and only if

$$\frac{x}{\sin mA} = \frac{y}{\sin mB} = \frac{z}{\sin mC}.$$

This is an easy consequence of the obvious inequality

$$\{x + (-1)^m(y\cos mC + z\cos mB)\}^2 + (y\sin mC - z\sin mB)^2 \geq 0. \tag{7}$$

In particular, for m even, say $m = 2n$, we have $\cos mA = 1 - 2\sin^2 nA$, etc., and (6) is equivalent to

$$(x + y + z)^2 \geq 4(yz\sin^2 nA + zx\sin^2 nB + xy\sin^2 nC). \tag{8}$$

We further particularize by assuming that x,y,z are nonnegative and setting $\sqrt{x},\sqrt{y},\sqrt{z} = p,q,r \geq 0$, respectively. Then we have, with cyclic sums,

$$\frac{(p^2+q^2+r^2)^2}{12} \geq \frac{\Sigma q^2r^2\sin^2 nA}{3} \geq \left(\frac{\Sigma qr\sin nA}{3}\right)^2, \tag{9}$$

where the first inequality comes from (8) and the second from an application of the power mean inequality. Finally, (9) yields

$$|qr\sin nA + rp\sin nB + pq\sin nC| \leq \frac{\sqrt{3}}{2}(p^2+q^2+r^2), \tag{10}$$

with equality if and only if

$$\sin nA = \sin nB = \sin nC = \pm\frac{\sqrt{3}}{2} \quad \text{and} \quad p = q = r. \tag{11}$$

This is the promised generalization of (2), which is equivalent to (2) when $p = q = r$. □

An inequality sharper than (10) is known for the special case $n = 1$. It is [3]

$$\csc\frac{X}{2}\sin A + \csc\frac{Y}{2}\sin B + \csc\frac{Z}{2}\sin C \le \cot\frac{X}{2} + \cot\frac{Y}{2} + \cot\frac{Z}{2},$$

where XYZ and ABC are arbitrary triangles, with equality if and only if

$$X + 2A = Y + 2B = Z + 2C = \pi.$$

From the obvious inequalities similar to (7),

$$\{x + (-1)^m(y\sin mC + z\sin mB)\}^2 + (y\cos mC - z\cos mB)^2 \ge 0$$

and

$$\{x + (-1)^m(y\sin mC - z\cos mB)\}^2 + (y\cos mC - z\sin mB)^2 \ge 0,$$

we obtain respectively

$$x^2 + y^2 + z^2 \ge (-1)^m(2yz\cos mA - 2zx\sin mB - 2xy\sin mC)$$

and

$$x^2 + y^2 + z^2 \ge (-1)^m(2yz\sin mA - 2zx\cos mB + 2xy\sin mC),$$

from which more inequalities analogous to (10) can be derived.

We now come back to (2), which states that

$$-\frac{3\sqrt{3}}{2} \le S(n) \le \frac{3\sqrt{3}}{2}$$

holds for every integer n and every triangle ABC, and investigate when the upper and lower bounds of $S(n)$ are attained, that is, according to (11), when

$$\sin nA = \sin nB = \sin nC = \pm\frac{\sqrt{3}}{2}. \tag{12}$$

Since $S(0) = 0$, and $S(n) = b$ if and only if $S(-n) = -b$, it suffices to consider only positive values of n. The solutions of equations (12), which are all listed in [2], are:

$$\{nA, nB, nC\} = \{r\pi + \pi/3, s\pi + \pi/3, t\pi + \pi/3\},$$

where r,s,t are nonnegative integers of the same parity such that $r+s+t = n-1$; and

$$\{nA, nB, nC\} = \{r\pi - \pi/3, s\pi - \pi/3, t\pi - \pi/3\},$$

where r,s,t are positive integers of the same parity such that $r+s+t = n+1$.

For all n, there are triangles ABC such that $S(n) = 3\sqrt{3}/2$. However, $S(n) = -3\sqrt{3}/2$ is possible if and only if $n \ge 4$. For $n = 1$ and $n = 2$, this is obvious from (3). For $n = 3$, we have

$$-2 < S(3) \leq \frac{3\sqrt{3}}{2}. \tag{13}$$

The left inequality is strict since the lower bound is attained only for the degenerate triangle with angles 0, $\pi/2$, $\pi/2$. A proof of (13) was asked in a problem set by the author for the 1981 U.S.A. Mathematical Olympiad. (For a solution, see the booklet *Math Olympiads for 1981*, obtainable by writing to Dr. Walter E. Mientka [1981: 140].) For $n = 4$, there is a unique solution for each bound. For $n = 5,6,7$, the solutions are unique only for the lower bound. For example, for $n = 5$ solutions for the upper bound are obtained from $\{r,s,t\} = \{4,0,0\}$ and $\{2,2,0\}$, but the lower bound can be attained only from $\{r,s,t\} = \{2,2,2\}$. For $n > 7$, there are multiple solutions for both bounds.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

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716. [1982: 48] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A student has been introduced to common logarithms and is wondering how their values can be calculated. He decides to obtain their binary representations (perhaps to see how a computer would do it). Help him by finding a simple algorithm to generate numbers $b_n \in \{0,1\}$ such that

$$\log_{10} x = \sum_{n=1}^{\infty} b_n \cdot 2^{-n}, \qquad 1 \leq x < 10.$$

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

We may assume from analysis that, for every x such that $1 \leq x < 10$, a binary representation

$$u = \log_{10} x = \sum_{n=1}^{\infty} b_n \cdot 2^{-n} \equiv .b_1 b_2 b_3 \ldots \tag{1}$$

exists and is unique. From this follows

$$2u = \log_{10}x^2 = b_1.b_2b_3b_4\ldots ,$$

so $b_1 = 0$ or 1 according as $x^2 < 10$ or $x^2 \geq 10$. Furthermore, having determined the first bit (binary digit) b_1, we can write

$$2u - b_1 = \log_{10}(x^2/10^{b_1}) = .b_2b_3b_4\ldots, \text{ with } 1 \leq x^2/10^{b_1} < 10, \tag{2}$$

and b_2 can be determined from (2) just as b_1 was determined from (1); and the remaining bits $b_3, b_4, b_5, \ldots$, as far as desired, can be determined successively by repeating the process. Consequently, we have proved that the following simple algorithm works:

Algorithm BCL (Binary Common Logarithm Algorithm)

Given X between 1 and 10, find the bits of $\log_{10}X$.

Step 1. [SQUARE] $Y \leftarrow X^2$.

Step 2. [GET NEXT BIT] Next bit in binary representation is $B \leftarrow 0$ if $Y < 10$, 1 if $Y \geq 10$.

Step 3. [REDUCE] $X \leftarrow Y/10^B$. Go to Step 1. □

I compared this algorithm against the algorithm actually used by the Common Run Time System of the VMS operating system on a VAX-11/780 computer and I found that, while algorithm BCL at first seems very fast, in fact it requires one multiplication for each bit desired; this is relatively slow. The algorithm my computer used was considerably faster.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; YVES GALIPEAU, Collège de Sherbrooke, Sherbrooke, Québec; M.S. KLAMKIN, University of Alberta; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

Ahlburg's solution ended with the following, presumably built up from *bits* and pieces:

$$\log_{10}7 = .84509\ 80400\ 14256\ 83071\ 22162\ 58596\ 63619\ 34835\ 72396\ 32396\ \ldots \quad .$$

All solvers except one (about whom more later) ended up with essentially the same algorithm. So this algorithm is without a doubt the best answer to our problem, in that it is the *simplest*; but it is not the *fastest*, as noted by our featured solver. No solver was willing or able to disclose any of the faster algorithms used in big computers.

The purpose of this problem seems to be to describe how a computer could find the bits of $\log_{10}x$, which He (*it* no longer seems adequate for a modern computer) would then convert into decimal $\log_{10}x$ for the benefit of lowly mortals. Our exceptional solver had an even simpler algorithm: he found the bits of $\log_{10}x$

from the decimal $\log_{10}x$, which he assumed to be known. It all seems pretty pointless: if decimal $\log_{10}x$ is already known, who (except maybe a computer) cares about the bits?

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717. [1982: 48] *Proposed jointly by J.T. Groenman, Arnhem, The Netherlands; and D.J. Smeenk, Zaltbommel, The Netherlands.*

Let P be any point in the plane of (but not on a side of) a triangle ABC. If H_a, H_b, H_c are the orthocenters of triangles PBC, PCA, PAB, respectively, prove that $[ABC] = [H_aH_bH_c]$, where the brackets denote the area of a triangle.

Solution by George Tsintsifas, Thessaloniki, Greece.

The desired result is a consequence of the following lemma, which will be proved below.

LEMMA. If ABCDEF *is a hexagon (not necessarily convex) such that* $AB \parallel DE$, $BC \parallel EF$, *and* $CD \parallel FA$, *then* $[ACE] = [BDF]$, *where signed areas are used.*

For in the given problem $AH_cBH_aCH_b$ is a hexagon with

$$AH_c \parallel H_aC \ (\perp PB), \qquad H_cB \parallel CH_b \ (\perp PA), \qquad BH_a \parallel H_bA \ (\perp PC);$$

hence, from the lemma, $[ABC] = [H_cH_aH_b] = [H_aH_bH_c]$.

Proof of the lemma. Let $BE \cap CF = R$, $CF \cap DA = S$, and $AD \cap BE = T$ (make a figure!). We have

$$BC \parallel EF \implies [CER] = [BRF],$$
$$CD \parallel FA \implies [CSA] = [FSD],$$
$$AB \parallel DE \implies [ATE] = [DTB].$$

If we sum these three equalities and then add [TSR] to each side, we obtain [ACE] = [BDF].

Also solved by O. BOTTEMA, Delft, The Netherlands; JORDI DOU, Barcelona, Spain; DAN PEDOE, University of Minnesota; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposers. A comment was received from M.S. KLAMKIN, University of Alberta.

Editor's comment.

The problem is not new. The proposers were perfectly aware of this, but they submitted the problem anyway with their own solution, as they had every right to do, because they (and the editor) felt that it was interesting and not well known. But they loyally gave their source. They found the problem (as well as the lemma used in our solution) proposed by S. Fournière (from Reims, France) in the April 1915 issue of the *Journal de Mathématiques élémentaires*.

Klamkin found the problem in Casey's *Trigonometry*, page 146, where it is credited to J. Neuberg. Since the book was published in the 1880s, the priority for the problem goes to Neuberg. In an earlier article [1981: 102-105, esp. p. 104], Klamkin had given two more recent (1960 and 1961) references for the lemma.