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CONTENTS

DUCCI'S FOUR-NUMBER PROBLEM: A SHORT BIBLIOGRAPHY

LEROY F. MEYERS

In a recent note, R.V. Andree [20] suggests that a computer may be useful for investigating the properties of a certain operation on k-tuples for various values of k. For each k-tuple $A = (a_1,\dots,a_k)$, let TA, the *transform* of A, be the k-tuple

$$(|a_2-a_1|,\ |a_3-a_2|,\ \dots,\ |a_k-a_{k-1}|,\ |a_1-a_k|).$$

As usual, self-composites of T are indicated by superscripts: if A is any k-tuple, then

$$T^0A = A,\ T^1A = TA,\ \text{ and }\ T^{n+1}A = T(T^nA) \text{ for } n \geq 0.$$

A k-tuple A is *desirable* just when there is a nonnegative integer n such that T^nA is the zero k-tuple $0 = (0,\dots,0)$. The *order* of a desirable k-tuple is the smallest such n.

Andree raised the question: "For which positive integers k (in particular, for $4 \leq k \leq 8$) is it true that *every* k-tuple is desirable (in particular, if the entries in the k-tuple are digits: $0, 1, \dots, 9$)?"

The question is not new, as can be seen from the bibliography appended to this note. In fact, most of Andree's questions (and many more) about the operation T are answered in what is apparently the first paper on the subject [1], wherein it is stated that the problem is due to E. Ducci.

The definition of the operation T, as given above, does not specify the nature of the entries in the k-tuple A. In most of the papers mentioned in the bibliography, these entries are required to be nonnegative integers, or they may be any real numbers. A k-tuple (in a given discussion) will be called *admissible* just when its entries are of the proper kind (for that discussion).

Many properties of the operation T are stated and proved in the literature. Most of those that I could find are listed below. In all of them, $A = (a_1,\dots,a_k)$ is an admissible k-tuple. Properties A, B, and C are obvious.

A. The cyclic permutations $(a_{j+1}, \dots, a_k, a_1, \dots, a_j)$ for $0 \leq j < k$, and the reversal $(a_k, \dots, a_1)$, are admissible k-tuples, and the m-fold repetition

$$(a_1, \dots, a_k, a_1, \dots, a_k, \dots, a_1, \dots, a_k)$$

is an admissible mk-tuple. Each of these tuples is desirable if and only if A is desirable, and in that case has the same order as A.

B. If r and s are given, and $r \neq 0$, then $(ra_1+s, \dots, ra_k+s)$ need not be admissible. However, if it is admissible, then it is desirable if and only if A is desirable, and in that case has the same order as A, unless $A = (-s/r, \ldots, -s/r) \neq 0$.

C. If $k \leq 2$, then A is desirable, and the order of A does not exceed k.

D. If k is a power of 2 and all of the entries in A are nonnegative integers, then:

D_1. A is desirable, and

D_2. the order of A does not exceed $kh-h+1$, where h is an integer, $2^h \geq a/d$, $a = \max\{a_1,\ldots,a_k\}$, and $d = \gcd\{a_1,\ldots,a_k\}$; (let h be 0 if $A = 0$);

D_3. more precisely, if $k = 4$, then the order of A does not exceed $3[q/2]+1$, where q is the smallest integer such that $a/d \leq t_q$, and the *Tribonacci number* t_q is defined so that

$$t_0 = 0, \quad t_1 = t_2 = 1, \quad \text{and} \quad t_n = t_{n-1}+t_{n-2}+t_{n-3} \text{ for } n \geq 3.$$

E. If k is not a power of 2, then A need not be desirable. In fact, $(0,0,\ldots,1)$ and its repetitions are undesirable.

F. If $k = 4$ and none of the cyclic permutations of A is a monotonic k-tuple, then the order of A does not exceed 6.

G. If $k = 4$ and $0 \leq a_1 \leq a_2 \leq a_3$ and $a_4 = a_1+a_2+a_3 > 0$, then

$$B = (a_3-a_1,\ a_3+a_1,\ a_3+2a_2+a_1,\ 3a_3+2a_2+a_1)$$

is a 4-tuple whose order (if A is desirable and not 0) is 1 more than the order of A, and whose entries satisfy inequalities and equality like those of A. (Note that $TB = 2A$.) If all entries in A are nonnegative integers, then so are the entries in B. By beginning with $(0,0,1,1)$, we see that there are desirable 4-tuples of integers of arbitrarily high order. By use of repetitions of 4-tuples, it is easily shown that there are 2^s-tuples of arbitrarily high order, where $s \geq 2$.

H. If $A = (\alpha^{k-1}, \alpha^{k-2}, \ldots, \alpha, 1)$, where $k \geq 3$ and α is the positive real number satisfying

$$\alpha^{k-1} = \alpha^{k-2} + \alpha^{k-3} + \ldots + \alpha + 1,$$

then $TA = \alpha A \neq A$, and A is undesirable. However, $\lim_{n\uparrow} T^n A = 0$. (Note that α is irrational.)

I. If all the entries in A are integers, then the sequence $(T^n A)_{n=0}^{\infty}$ is eventually periodic, and one of the k-tuples in its period consists of at most two distinct integers, one of which is 0. (If k is a power of 2, then the period is 1, and the only k-tuple in its period is 0.) The number of preliminary terms before periodicity sets in may be arbitrarily large, but in no case larger than approximately $q/(k-1)$, where q is the smallest integer for which $a/d \leq u_q$, and

$$u_0 = \ldots = u_{k-3} = 0, \quad u_{k-2} = 1, \quad u_n = u_{n-1}+u_{n-2}+\ldots+u_{n-k+1} \text{ for } n \geq k-1.$$

(Compare D_3.)

There is a short proof of D_1 and E which uses some nice ideas from the theory of polynomials. Most of this proof is in [18], and equivalent proofs using matrices are found elsewhere.

We first treat the case where each entry in A is 0 or 1. Since $|a-b| \equiv a+b \pmod 2$ whenever $\{a,b\} \subset \{0,1\}$, it is easily seen that $TA \equiv A + (a_2,\ldots,a_k,a_1) \pmod 2$.

With each k-tuple $A = (a_1,\ldots,a_k)$ we associate the polynomial

$$P_A(x) = a_1x^{k-1} + a_2x^{k-2} + \ldots + a_{k-1}x + a_k$$

in the indeterminate x. Then we have, modulo 2,

$$\begin{aligned} P_{TA}(x) &\equiv (a_1+a_2)x^{k-1} + (a_2+a_3)x^{k-2} + \ldots + (a_{k-1}+a_k)x + (a_k+a_1) \\ &\equiv (x+1)P_A(x) + a_1(x^k+1). \end{aligned}$$

If we now take remainders on division by x^k+1, we can write this as

$$P_{TA}(x) \equiv (x+1)P_A(x) \qquad (\text{mod } 2,\ x^k+1).$$

Repeating the operation then gives

$$P_{T^nA}(x) \equiv (x+1)^nP_A(x) \quad (\text{mod } 2,\ x^k+1) \quad \text{for } n \geq 0. \tag{1}$$

Now $(x+1)^2 = x^2+2x+1 \equiv x^2+1 \pmod 2$, so that an easy induction gives

$$(x+1)^{2^s} \equiv x^{2^s} + 1 \pmod 2. \tag{2}$$

Using (2) in (1) with $n := 2^s$ yields

$$P_{T^{2^s}A}(x) \equiv (x^{2^s}+1)P_A(x) \quad (\text{mod } 2,\ x^k+1).$$

If $k = 2^s$, then

$$P_{T^kA}(x) \equiv (x^k+1)P_A(x) \equiv 0 \quad (\text{mod } 2,\ x^k+1),$$

and so $T^kA = 0$.

On the other hand, if k is not a power of 2, let A be $(0,\ldots,0,1)$, so that $P_A(x) = 1$. If $T^nA = 0$, then by (1) we have $(x+1)^n \equiv 0$ (mod 2, x^k+1). Multiplication by $(x+1)^{2^s-n}$, where s is an integer such that $2^s \geq n$, then yields, from (2),

$$x^{2^s} + 1 \equiv (x+1)^{2^s} \equiv 0 \quad (\text{mod } 2,\ x^k+1). \tag{3}$$

Since k is not a power of 2, the division algorithm yields

$$2^s = kq + r, \text{ where } 0 < r < k$$

for integers q and r. Then

$$x^{2^s} + 1 \equiv (x^k)^q x^r + 1 \equiv x^r + 1 \not\equiv 0 \pmod{2,\ x^k + 1},$$

which contradicts (3). Hence (0,...,0,1) is not desirable.

We now return to the case in which the entries of A are arbitrary nonnegative integers and $k = 2^s$. Let A' be the k-tuple whose entries are the remainders on dividing the entries of A by 2. The conclusion of the first part of the proof, namely that $T^k A' = 0$, can be interpreted as stating that all entries of $T^k A$ are even, so that $\frac{1}{2}T^k A$ is an admissible k-tuple. Repetition of the argument shows that $(1/2^m)T^{mk}A$ is an admissible k-tuple for every nonnegative integer m. Let a be the maximum entry in A. Then the maximum entry in TA is not larger than a, and so the maximum entry in $T^{mk}A$ is not larger than a. But every entry in $T^{mk}A$ is divisible by 2^m. If m is chosen so that $2^m > a$, then every entry in $T^{mk}A$ must be 0, so that A is desirable, and its order does not exceed mk.

Acknowledgments. Many thanks to Martin Gardner, who responded immediately to a request for references, which enabled the search tree to be extended. Thanks are also due to László Csirmaz, who supplied references [6] and [7].

REFERENCES

Letters in brackets at the end of a reference indicate that the similarly lettered items from the list of properties of T given above are proved or mentioned in the article or book.

1. C. Ciamberlini and A. Marengoni, "Su una interessante curiosità numerica", *Periodico di Matematiche*, (4) 7 (1937) 25-30. [D_1, E, G, H]

2. G. [Jekuthiel Ginsburg], "Curiosa 30. An interesting observation", *Scripta Mathematica*, 5 (1938) 135. [D_1] Report on [1].

3. Benedict Freedman, "The four number game", *Scripta Mathematica*, 14 (1948) 35-47. [D_1, E, G]

4. Moshe Lotan, "A problem in difference sets", *American Mathematical Monthly*, 56 (1948) 535-541. [D_1, E, F, G, H (k = 4)]

5. Boris A. Kordemskiĭ, *Matematicheskaja Smekalka*, 5th ed., Moscow, 1958, problem 345_1, pp. 302-304; solution pp. 555-557. [D_1, F]

6. D.O. Shkljarskiĭ, N.N. Chencov, and I.M. Jaglom, *Izbrannye Zadachi i Teoremy Elementarnoĭ Matematiki*, 3rd ed., part I, Moscow, 1959, problem 89: statement, p. 28; solution, pp. 177-178. [D_1, E]

7. D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book: Selected problems and theorems of elementary mathematics*, translation of [6] by John Maykovich, edited and revised by Irving Sussman, Freeman, San Francisco, 1962, problem 89: statement, p. 22; solution, pp. 166-168. [D_1, E]

8. Roland Sprague, *Recreation in Mathematics*, Blackie, 1964; Dover, 1963; item 4, pp. 5, 26-28. [I've not seen this.]

9. A. Lindgren, "Mathematical note 319", *Australian Mathematics Teacher*, 22 (1966) 61-63; answer to question proposed to the Problem Bureau of The Mathematical Association (New South Wales Branch), which appeared in *Australian Mathematics Teacher* 19 (1963) 25. [D_1, D_2, E, (F), H ($k = 4$)]

10. John M. Hammersley, "On the enfeeblement of mathematical skills by 'modern mathematics' and by similar soft intellectual trash in schools and universities", *Bulletin of the Institute of Mathematics and Its Applications*, 4 (1968) 68-85, esp. problem 6, p. 84. [Proposes D_1 (all k), E ($k = 3$), F ($k = 4$)]

11. L. Carlitz and R. Scoville, "Solution to problem 69-1: Sequences of absolute differences", *SIAM Review*, 12 (1970) 297-300; proposed by John M. Hammersley, *ibid.*, 11 (1969) 73-74. [D_1, D_3, E, F, (G), H, I]

12. Ross Honsberger, *Ingenuity in Mathematics*, New Mathematical Library, No. 23, 1970, pp. 73-74, 80-83, and 86. [D_1 ($k \in \{4,8\}$)]

13. Boris A. Kordemsky, *The Moscow Puzzles*, translation of some edition of [5] by Albert Parry, edited by Martin Gardner, Scribner's, New York, 1972, problem 342 (A): "Repeat the operation", p. 159; solution, pp. 292-293. [D_1, F]

14. Elwyn R. Berlekamp, "Design of slowly shrinking labelled squares", *Mathematics of Computation*, 29 (1975) 25-27. [$k = 4$: D_1, F, G]

15. Hans Helmut Lemmerich, "Ouadrupelfolgen", *Praxis der Mathematik* (1975), pp. 219-223. [I've not seen this.]

16. M. Burmester, R. Forcade, and E. Jacobs, "Circles of numbers", *Glasgow Mathematical Journal*, 19 (1978) 115-119. [D_1, E, F, G, H]

17. Robert Miller, "A game with n numbers", *American Mathematical Monthly*, 85 (1978) 183-185. [D_1, E, G]

18. P. Zvengrowski, "Iterated absolute differences", *Mathematics Magazine*, 52 (1979) 36-37. [D_1, D_2, E, G]

19. Michel Dumont and Jean Meeus, "The four-numbers game", *Journal of Recreational Mathematics*, 13 (1980-1981) 89-96. [D_1, E, F, G, H ($k = 4$)] An interesting sketch shows the orders of 4-tuples $(0, \alpha, \beta, 1)$, where α and β are rational and $0 \le \alpha \le \beta \le 1$.

20. Richard V. Andree, "An interesting recursive function", this journal, 8 (1982) 69; and "Postscript to ...", *ibid.*, p. 96. [Conjectures D_1, E]

21. William A. Webb, "The length of the four-number game", *Fibonacci Quarterly*, 20 (1982) 33-35. [$k = 4$: D_1, D_3, G, H]

22. Fook-Bun Wong, "Ducci processes", *Fibonacci Quarterly*, 20 (1982) 97-105. [D_1, E] Generalization in which $|a_{i+1}-a_i|$ is replaced by $g(a_i,a_{i+1})$, where g is some given function. The most interesting example is the one in which $g(a,b) = \phi(a)+\phi(b)$.

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SOME NOTABLE PALINDROMIC TRIANGULAR NUMERALS

CHARLES W. TRIGG

The numerals representing a triangular number, $T(n) = n(n+1)/2$, have different appearances in different bases, except that $T(1) = 1$ in every base. For example,

$$T(5) = 10_{\text{fifteen}} = 15_{\text{ten}} = 17_{\text{eight}} = 23_{\text{six}}.$$

Numbers palindromic in several bases.

Within the ranges studied, ten triangular numbers are palindromic in two systems of notation, and three are palindromic in three bases. They are (the n in $T(n)$ being always in base ten unless otherwise indicated):

$T(5) = 1111_{\text{two}} = 33_{\text{four}}$ $\quad$ $T(14) = 1221_{\text{four}} = 151_{\text{eight}}$

$T(6) = 10101_{\text{two}} = 111_{\text{four}}$ $\quad$ $T(18) = 333_{\text{seven}} = 171_{\text{ten}}$

$T(8) = 121_{\text{five}} = 44_{\text{eight}}$ $\quad$ $T(36) = 22122_{\text{four}} = 666_{\text{ten}}$

$T(9) = 101101_{\text{two}} = 55_{\text{eight}}$ $\quad$ $T(90) = 333333_{\text{four}} = 7777_{\text{eight}}$

$T(12) = 303_{\text{five}} = 141_{\text{seven}}$ $\quad$ $T(132) = 21112_{\text{eight}} = 8778_{\text{ten}}$

$T(10) = 313_{\text{four}} = 131_{\text{six}} = 55_{\text{ten}}$

$T(24) = 606_{\text{seven}} = 454_{\text{eight}} = 363_{\text{nine}}$

$T(25) = 101000101_{\text{two}} = 11011_{\text{four}} = 505_{\text{eight}}.$

Observe that $T(24)$ is palindromic in three consecutive bases.

In $T(5)$ and $T(90)$, both of the given representations are repdigits, and each digit is b-1, one less than the base b. In general, if two equal repdigits in bases r and s have digits $\overline{r-1}$ and $\overline{s-1}$ occurring p and q times, respectively, then

$$(r-1)[(r^p-1)/(r-1)] = (s-1)[(s^q-1)/(s-1)],$$

so $r^p = s^q$. It follows that r and s are powers of some common factor t, with $t^{pu} = t^{qv}$ and $p/q = v/u$. If the repdigits are triangular, then $t^{pu}-1 = n(n+1)/2$, so $2(t^{pu}-1)$ is the product of two consecutive integers.

Working in the decimal system, the product, P, of two consecutive integers ends in 0, 2, or 6, and $P \equiv 0, 2, 3$, or $6 \pmod 9$. If x has the form $3k$, then $P = 2(x^y-1) \equiv 7 \pmod 9$, and if x ends in 5, P ends in 8. Hence for $x < 10$ it is necessary to examine only $x = 2$ and $x = 7$, since $x = 4$ and $x = 8$ are special cases of $x = 2$. For $P < 10^8$, there are only two solutions:

$$P = 2(2^4-1) = 5 \cdot 6 \quad \text{and} \quad P = 2(2^{12}-1) = 90 \cdot 91.$$

These lead to

$$1111_{\text{two}} = 33_{\text{four}}$$

and

$$111111111111_{\text{two}} = 333333_{\text{four}} = 7777_{\text{eight}} = \overline{15}\,\overline{15}\,\overline{15}_{\text{sixteen}} = \overline{63}\,\overline{63}_{\text{sixty-four}},$$

one of the situations noted above and an expansion of the other.

Numerals palindromic in several bases.

A triangular number has the same appearance in all systems of numeration with bases greater than the number, which will be a "digit" in that system.

There are also a few "look-alike" palindromic numerals of different magnitudes, since they are expressed in different bases. Some of these are

$$T(10_{\text{two}}) = 11_{\text{two}}, \qquad T(3_{\text{five}}) = 11_{\text{five}}, \qquad T(4_{\text{nine}}) = 11_{\text{nine}},$$

and all $T(x_b) = 11_b$ when $b = (x-1)(x+2)/2$;

$$T(11_{\text{four}}) = 33_{\text{four}}, \qquad T(10_{\text{six}}) = 33_{\text{six}},$$

and all $T(x_b) = 33_b$ when $b = (x+3)(x-2)/6$, an integer;

$$\begin{array}{llll}
T(11_{\text{six}}) & = 44_{\text{six}}, & T(10_{\text{eight}}) & = 44_{\text{eight}}, \\
T(11_{\text{eight}}) & = 55_{\text{eight}}, & T(10_{\text{ten}}) & = 55_{\text{ten}}, \\
T(12_{\text{four}}) & = 111_{\text{four}}, & T(14_{\text{nine}}) & = 111_{\text{nine}}, \\
T(101_{\text{two}}) & = 1111_{\text{two}}, & T(44_{\text{nine}}) & = 1111_{\text{nine}}, \\
T(110_{\text{two}}) & = 10101_{\text{two}}, & T(111_{\text{three}}) & = 10101_{\text{three}}, \\
T(12143_{\text{five}}) & = 102121201_{\text{five}}, & T(13401_{\text{eight}}) & = 102121201_{\text{eight}}.
\end{array}$$

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* * *

HOMAGE TO HERMAN NYON

Over the years, some of our long-term subscribers have become more than mathematical comrades, they have become personal friends. Such was Herman Nyon. We have just been informed by his wife, Beatrix Nyon, that Herman died unexpectedly on October 3, 1982, in Paramaribo, Surinam.

Since he retired in 1972, he derived much enjoyment from problem-solving and recreational mathematics, and he was generous in his moral and financial support for this journal. A few short weeks before his death, he renewed his subscription for 1983, added a generous supplement, and wrote of his intention to include *Crux* in his will. His death leaves us all greatly saddened. We extend our deepest sympathy to his wife and family.

THE OLYMPIAD CORNER: 39

M.S. KLAMKIN

We give below the problems of the 1962, 1963, and 1964 Peking Mathematics Contest (Grade 12), Paper II. The translations were provided by Andy Liu. As usual, I invite readers to send me elegant solutions.

1962

1. Evaluate

$$m! + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} + \ldots + \frac{(m+n)!}{n!}.$$

2. Six circles in a plane are such that the center of each circle is outside the other circles. Show that these six circles have empty intersection.

3. A car can carry fuel which will last a distance of a. A distance $d > a$ is to be covered with no refueling station in between. However, the car may go back and forth transporting and depositing fuel en route. What is the most economical scheme to get to the destination if $d = 4a/3$? What if $d = 23a/15$?

4. A group of children forms a circle and each child starts with an even number of pieces of candy. Each child then gives half of what he has to his right-hand neighbor. After the transaction, if a child has an odd number of pieces, he or she will receive an extra piece from an external source. Show that, after a finite number of such steps, each child will have the same number of pieces of candy.

1963

1. A polynomial $P(x)$ with integral coefficients takes on the value 2 for four distinct integral values of x. Show that $P(x)$ is never equal to 1, 3, 5, 7, or 9 for any integral value of x.

2. Nine points are randomly selected inside a square of side 1. Show that three of the points are the vertices of a triangle of area at most 1/8.

3. Given are $2n+3$ points in the plane, no three collinear and no four concyclic. Is it possible to construct a circle passing through three of the points so that exactly half of the remaining $2n$ points lie inside the circle? Justify your answer.

4. A set of 2^n objects is partitioned into a number of subsets. A move consists of transferring from one of the subsets to an equal or smaller subset a number of objects equal to the cardinality of the second subset. Prove that, irrespective

of the initial partition, all the subsets can be combined into a single set by a finite number of moves.

1964

1. ABC is a triangle in which angle A is nonacute. Let B_1DEC_1 be any inscribed square of triangle ABC with B_1 on AB, C_1 on AC, and DE on BC. Let $B_2D_1E_1C_2$ be any inscribed square of triangle AB_1C_1 with B_2 on AB_1, C_2 on AC_1, and D_1E_1 on B_1C_1. The process is continued in the same way for a finite number of steps. Prove that the sum of the areas of all the inscribed squares is strictly less than half the area of triangle ABC.

2. Let $(a_1, a_2, \ldots, a_n, \ldots)$ be a sequence of positive real numbers such that $a_n^2 \le a_n - a_{n+1}$ for all n. Show that $a_n < 1/n$ for all n.

3. A round track has n fueling stations (some possibly empty) containing a combined total of fuel sufficient for a car to travel once around the track. Prove that, irrespective of the initial distribution of fuel among the stations, it is always possible for a car with an empty tank to start from one of the stations and complete a round trip without running out of fuel on the way.

4. Show that a circle of diameter D can cover a parallelogram with perimeter $2D$. Show that the same circle can cover any planar region with perimeter $2D$.

*

We now present solutions to some problems that have appeared earlier in this column.

4. [1981: 42 ; 1982: 100] Prove that $\Sigma\{1/(i_1i_2\ldots i_k)\} = n$, where the summation is taken over all nonempty subsets $\{i_1,i_2,\ldots,i_k\}$ of $\{1,2,\ldots,n\}$.

II. *Solution by David Singmaster, Polytechnic of the South Bank, London, England.*

We have

$$\begin{aligned}\Sigma\{1/(i_1i_2\ldots i_k)\} &= (1+\tfrac{1}{1})(1+\tfrac{1}{2})\ldots(1+\tfrac{1}{n}) - 1\\ &= \frac{2}{1}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{n+1}{n} - 1\\ &= n.\end{aligned}$$

3. [1981: 43 ; 1982: 102] Let p be a prime number and n a positive integer. Prove that the following statements (a) and (b) are equivalent:

(a) None of the binomial coefficients $\binom{n}{k}$ for $k = 0,1,\ldots,n$ is divisible by p.

(b) n can be represented in the form $n = p^s q - 1$, where s and q are integers, $s \geq 0$, $0 < q < p$.

II. *Comment by David Singmaster, Polytechnic of the South Bank, London, England.*

This result follows from Corollary 14.1 given in my paper "Divisibility of binomial and multinomial coefficients by primes and prime powers", 18th Anniversary Volume of the Fibonacci Association.

1, [1982: 99] *Proposed by Jack Brennen, student, Poolesville, Maryland.*

Sum the series

$$\sum_{i=1}^{\infty} \frac{36i^2+1}{(36i^2-1)^2}.$$

I. *Solution by the proposer.*

If S is the required sum, then we have

$$2S = \sum_{i=1}^{\infty} \left\{ \frac{1}{(6i-1)^2} + \frac{1}{(6i+1)^2} \right\} = \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots \quad .$$

Now, from the well-known value of $\zeta(2)$,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},$$

we obtain successively

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24},$$

$$1 \quad + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8},$$

$$\frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \cdots = \frac{\pi^2}{72},$$

and

$$2S + 1 = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots = \frac{\pi^2}{8} - \frac{\pi^2}{72} = \frac{\pi^2}{9},$$

from which

$$S = \frac{\pi^2 - 9}{18}.$$

II. *Solution by Henry E. Fettis, Mountain View, California.*

Consider the double equality

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \frac{1}{z^2} + 2\sum_{i=1}^{\infty} \frac{i^2+z^2}{(i^2-z^2)^2}.$$

The first equality is well known (see [1] or [2]), and the second merely transforms the series into a form convenient for our purpose. With $z = 1/6$, we obtain from the above

$$4\pi^2 = \frac{\pi^2}{\sin^2(\pi/6)} = 36 + 72\sum_{i=1}^{\infty}\frac{36i^2+1}{(36i^2-1)^2},$$

from which we find the required sum to be $(\pi^2-9)/18$.

REFERENCES

1. Einar Hille, *Analytic Function Theory*, Chelsea, New York, 1973, Vol. I, p. 261.

2. V. Mangulus, *Handbook of Series for Scientists and Engineers*, Academic Press, New York, 1965, p. 78.

2. [1982: 99] *Proposed by Gregg Patruno, student, Princeton University.*

How many equations must be considered in order to maximize a given differentiable function $F(x_1, x_2, \ldots, x_n)$ in the region $0 \le x_i \le 1$, $i = 1,2,\ldots,n$, by differential calculus?

Solution by the proposer.

In addition to the 2^n endpoint equations, there are $2^k\binom{n}{k}$ $(n-k)$-dimensional cubes to be considered for $k = 0,1,\ldots,n-1$. Thus, in general the number of equations to be considered is

$$S_n = 2^n + \sum_{k=0}^{n-1} 2^k(n-k)\binom{n}{k}.$$

Now let

$$F(x) = \sum_{k=0}^{n-1}\binom{n}{k}(2x)^k = (1+2x)^n .$$

Then we have

$$S_n = 2^n + \{nF(x) - xF'(x)\}_{x=1} = 2^n + n\cdot 3^{n-1}.$$

3. [1982: 99] *Proposed by Gregg Patruno, student, Princeton University.*

A quick proof that the rationality of p, q, and $\sqrt{p}+\sqrt{q}$ implies the rationality of $\sqrt{p}$ is furnished by the identity

$$2\sqrt{p} = \frac{(\sqrt{p}+\sqrt{q})^2+p-q}{\sqrt{p}+\sqrt{q}}.$$

Prove in similar fashion that if p, q, r, and $\sqrt{p}+\sqrt{q}+\sqrt{r}$ are rational, then so is $\sqrt{p}$.

Solution by the proposer.

The desired result follows from

$$\sqrt{p} = \frac{2ps^2 + \frac{1}{4}(s^2-p-q-r)^2 - (qr+rp+pq)}{s(s^2+p-q-r)},$$

where $s = \sqrt{p}+\sqrt{q}+\sqrt{r}$.

Comment by M.S.K.

It would be interesting to have a "simple" proof of the following generalization: if p_i, $i = 1,2,\ldots,n$ and $\Sigma\sqrt{p_i}$ are all rational, then so is each $\sqrt{p_i}$.

4. [1982: 99] *Proposed by Brian Hunt, student, Montgomery Blair H.S., Silver Spring, Maryland.*

If $a_i, b_i > 0$ for $i = 1,2,\ldots,n$ and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1,$$

prove that, for all $m > 1$,

$$\left\{\sum_{i=1}^{n} a_i^m\right\}^{m+1} \cdot \left\{\sum_{i=1}^{n} b_i^{-m}\right\}^{m-1} \geq 1.$$

Solution by Jeff Soesbe, student, Keystone School, San Antonio, Texas.

With all summations for $i = 1,2,\ldots,n$, we have, by the power mean inequality,

$$\frac{\Sigma a_i^m}{n} \geq \left(\frac{1}{n}\right)^m \quad \text{and} \quad \frac{\Sigma b_i^{-m}}{n} \geq \left(\frac{1}{n}\right)^{-m};$$

hence

$$(\Sigma a_i^m)^{m+1} \geq n^{1-m^2} \quad \text{and} \quad (\Sigma b_i^{-m})^{m-1} \geq n^{m^2-1}.$$

Therefore

$$(\Sigma a_i^m)^{m+1} \cdot (\Sigma b_i^{-m})^{m-1} \geq 1,$$

as required.

5. [1982: 100] *Proposed by Noam D. Elkies, student, Stuyvesant H.S., New York, N.Y.*

One solution of the Diophantine equation

$$7x^2 + 8x - 3 = y^2$$

is $(x,y) = (-3,6)$. Are there solutions for which x is positive?

Solution by the proposer.

We first note that the given equation is equivalent to

$$(3x)^2 + y^2 = (4x-1)(4x+3). \tag{1}$$

Our proof will be based on the following well-known result [1]: a positive integer N is representable as a sum of two squares if and only if, in the prime power canonical factorization of N, all primes of the form $4n+3$ occur (if at all) to an even exponent. In particular, if an odd integer N is representable as a sum of two squares, then, in any factorization $N = ab$ into two relatively prime factors, each of the factors a and b must be of the form $4n+1$. For the product of two numbers of the form $4n+1$, or of two numbers of the form $4n+3$, is of the form $4n+1$; and, because a and b are relatively prime, each prime factor of N of the form $4n+3$ must occur (if at all) an even number of times in a or an even number of times in b.

It is now immediately apparent that (1) has no solution (x,y) with $x > 0$, since the two factors on the right are relatively prime and each is of the form $4n+3$. However, there may be solutions (x,y) with $x < 0$, for then each factor in (1) would be of the form $4n+1$, and in fact the proposal gives one such solution. □

It follows from the above discussion that any equation equivalent to

$$(rx)^2 + y^2 = (4sx+1)(4tx+3),$$

where r, s, t are positive integers, has no solution in integers (x,y), either for $x \geq 0$ or $x < 0$.

REFERENCE

1. William J. LeVeque, *Topics in Number Theory*, Addison-Wesley, Reading, 1956, Vol. I, p. 126.

6. [1982: 100] *Proposed by Noam D. Elkies, Stuyvesant H.S., New York, N.Y.*

Let S_1,S_2,S_3,S_4 be four spheres in 3-space such that S_1,S_2,S_3 intersect at points A_4 and B_4; S_1,S_2,S_4 intersect at A_3 and B_3; S_1,S_3,S_4 intersect at A_2 and B_2; and, finally, S_2,S_3,S_4 intersect at A_1 and B_1. If P is any point distinct from A_i,B_i $(i = 1,2,3,4)$, prove that the circumcenters of triangles PA_1B_1, PA_2B_2, PA_3B_3, and PA_4B_4 are coplanar.

Solution by the proposer.

[This solution tacitly assumes that $P \notin A_iB_i$ for $i = 1,2,3,4$, a restriction which should have been mentioned in the proposal. (M.S.K.)]

The four spheres have a radical center which will be denoted by R. Then we have

$$RA_1 \cdot RB_1 = RA_2 \cdot RB_2 = RA_3 \cdot RB_3 = RA_4 \cdot RB_4 = k^2;$$

hence, for i = 1,2,3,4, R,A_i,B_i are collinear and the circumcircles of PA_iB_i are invariant under inversion with respect to a sphere with center R and radius k. Under the inversion, P is transformed into some other point P'. The four circumcenters must all be equidistant from P and P'; hence they must lie on the perpendicular bisecting plane of PP'. By continuity, this argument can also be used if P lies on the sphere of inversion.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Puzzle No. 23: Homonym (5; 3)

"We sailed (the skipper wrote his log)
A PRIMAL line."
"We formed (the sailor said) for grog
A FINAL line."

Puzzle No. 24: Rebus (10)

+ $\cancel{M}$

If this puzzle
Strikes you dumb,
This is what
You may become.

Puzzle No. 25: Rebus (6)

$$f(I) = I^2$$

This graph, in mathematics view.
In art they'd say, "It's ALL, won't do."

ALAN WAYNE, Holiday, Florida

Answer to Puzzle No. 22 [1982: 260]: 057 = OAK,

NUTCRACKER = 2813453764 = 53042^2 = $ACORN^2$.

* * *

MATHEMATICAL SWIFTIES

"Two lines always intersect", Tom proclaimed projectively.

M.S. KLAMKIN

"Its value is approximately 3.14159", Tom murmured piously.

EDITH ORR

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

781. *Proposed by Alan Wayne, Holiday, Florida.*

The Philosophic Father

Your brother's first to bathe, my son,
So do not look so W R Y .
You'll have to use the T U B in turn,
There's room for only O N E .

There's very little room, I guess.
Aye, there's the R U B! But try.
Use soap on cloth to R U B. Don't spurn
My counsel; answer "Y E S".

In time, you'll find it's true—
Much later, by the B Y E—
As ashes resting in an U R N,
There will be room for T W O.

Regard the preceding three patterns of three capitalized words as interdependent arithmetic additions in the decimal system. Restore the digits.

782. *Proposed by H.S.M. Coxeter, University of Toronto.*

(a) Sketch the plane cubic curve given by the parametric equations

$$x = \alpha(\beta-\gamma)^2, \quad y = \beta(\gamma-\alpha)^2, \quad z = \gamma(\alpha-\beta)^2, \quad \alpha+\beta+\gamma = 0,$$

where (x,y,z) are barycentric (or areal, or trilinear) coordinates, referred to an equilateral triangle. In what respect do its asymptotes behave differently from those of a hyperbola?

(b) Eliminate the parameters α,β,γ so as to obtain a single equation

$$x^3+y^3+z^3 + a(x^2y+x^2z+y^2z+y^2x+z^2x+z^2y) + bxyz = 0$$

for certain numbers a and b.

(c)* What equation does the curve have in terms of polar coordinates?

783. *Proposed by R.C. Lyness, Southwold, Suffolk, England.*

Let n be a fixed natural number. We are interested in finding an infinite sequence $(v_0,v_1,v_2,\ldots)$ of strictly increasing positive integers, and a finite sequence $(u_0,u_1,\ldots,u_n)$ of nonzero integers such that, for all integers $m \ge n$,

$$u_0^2v_m + u_1^2v_{m-1} + \ldots + u_n^2v_{m-n} = u_0v_m^2 + u_1v_{m-1}^2 + \ldots + u_nv_{m-n}^2. \qquad (1)$$

(a) Prove that (1) holds if

$$u_r = \text{coefficient of } x^r \text{ in } (1-x)^n$$

and

$$v_r = \text{coefficient of } x^r \text{ in } (1-x)^{-n-1}.$$

(b)* Find other sequences (u_r) and (v_r) for which (1) holds.

784. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let $F_n = a_i/b_i$, $i = 1,2,\ldots,m$, be the Farey sequence of order n, that is, the ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. (For example,

$$F_5 = (\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}),$$

with $m = 11$.) Prove that, if $P_0 = (0,0)$ and $P_i = (a_i,b_i)$, $i = 1,2,\ldots,m$, are lattice points in a Cartesian coordinate plane, then $P_0P_1\ldots P_m$ is a simple polygon of area $(m-1)/2$.

785. *Proposed by J. Chris Fisher, University of Regina.*

Suppose a closed differentiable curve has exactly one tangent line parallel to every direction. More precisely, suppose that the curve has parametrization $\vec{V}(\theta)$: $[0,2\pi] \to R^2$ for which

(i) $d\vec{V}/d\theta = (r(\theta)\cos\theta,\ r(\theta)\sin\theta)$ for some continuous real-valued function $r(\theta)$, and

(ii) $\vec{V}(\theta) = \vec{V}(\theta+\pi)$.

Prove that the curve is described in the clockwise sense as θ runs from 0 to π.

786. *Proposed by O. Bottema, Delft, The Netherlands.*

Let r_1,r_2,r_3 be arbitrarily chosen positive numbers. Prove that there exists a (real) triangle whose exradii are r_1,r_2,r_3, and calculate the sides of this triangle.

787. *Proposed by J. Walter Lynch, Georgia Southern College.*

(a) Given two sides, a and b, of a triangle, what should be the length of the third side, x, in order that the area enclosed be a maximum?

(b) Given three sides, a, b, and c, of a quadrilateral, what should be the length of the fourth side, x, in order that the area enclosed be a maximum?

788. *Proposed by Meir Feder, Haifa, Israel.*

A *pandigital* integer is a (decimal) integer containing each of the ten digits exactly once.

(a) If m and n are distinct pandigital perfect squares, what is the smallest possible value of $|\sqrt{m}-\sqrt{n}|$?

(b) Find two pandigital perfect squares m and n for which this minimum value of $|\sqrt{m}-\sqrt{n}|$ is attained.

789. *Proposed by H. Kestelman, University College, London, England.*

If A and B are square matrices (any orders), then they have a common eigenvalue if and only if $AX = XB$ for some $X \neq 0$.

790. *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*

Let ABC be a triangle with sides a,b,c in the usual order, and let l_a, l_b, l_c and l'_a, l'_b, l'_c be two sets of concurrent cevians, with l_a, l_b, l_c intersecting a,b,c in L,M,N, respectively. If

$$l_a \cap l'_b = \text{P}, \qquad l_b \cap l'_c = \text{Q}, \qquad l_c \cap l'_a = \text{R},$$

prove that, independently of the choice of concurrent cevians l'_a, l'_b, l'_c, we have

$$\frac{\text{AP}}{\text{PL}}\cdot\frac{\text{BQ}}{\text{QM}}\cdot\frac{\text{CR}}{\text{RN}} = \frac{abc}{\text{BL}\cdot\text{CM}\cdot\text{AN}} \geq 8,$$

with equality occurring just when l_a, l_b, l_c are the medians of the triangle.

(This problem extends Crux 588 [1981: 306].)

762. [1982: 209] *Correction.* Part (b) should read as follows:

(b) If $a = 5$ and $3abc/4(a^3+b^3+c^3) = 5/24$, determine b and c, given that they are integers.

* * *

(continued from page 294)

identify it. In that reference, the problem, strictly equivalent to our own, was to find the common area accessible to four goats tethered at the vertices of a square garden. The editor looked into his crystal ball and found one possible source for this solution: *Mathematical Wrinkles*, by S.I. Jones, published by the S.I. Jones Co., Nashville, Tennessee, in 1930.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

90. [1975: 85 ; 1976: 34] *Proposed by Léo Sauvé, Algonquin College.*

(a) Determine, as a function of the positive integer n, the number of odd binomial coefficients in the expansion of $(a+b)^n$.

(b) Do the same for the number of odd multinomial coefficients in the expansion of $(a_1 + a_2 + \ldots + a_r)^n$.

II. *Solution of part (a) by Leroy F. Meyers, The Ohio State University.*

We will use the congruence

$$(x+1)^{2^s} \equiv x^{2^s} + 1 \pmod 2, \tag{1}$$

which is easily shown by induction to be valid for $s = 0,1,2,\ldots$.

Let n be a nonnegative integer whose binary expansion, with only the nonzero terms listed, is

$$n = 2^{s_1} + 2^{s_2} + \ldots + 2^{s_m},$$

where $0 \le s_1 < s_2 < \ldots < s_m$. Then by (1) we have

$$\begin{aligned}(x+1)^n &= (x+1)^{2^{s_1}}(x+1)^{2^{s_2}}\ldots(x+1)^{2^{s_m}}\\ &\equiv (x^{2^{s_1}}+1)(x^{2^{s_2}}+1)\ldots(x^{2^{s_m}}+1) \pmod 2.\end{aligned}$$

Now all the terms obtained by multiplying out the last line are distinct and there are exactly 2^m such terms, each with coefficient 1. This can be interpreted as saying that:

The number of odd coefficients in the expansion of $(x+1)^n$ *by the binomial theorem is* 2^m*, where* m *is the number of* 1*'s in the binary expansion of* n.

* * *

639. [1981: 146 ; 1982: 145] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

If $x + y + z = 0$, prove that

$$\frac{x^5+y^5+z^5}{5} = \frac{x^3+y^3+z^3}{3}\cdot\frac{x^2+y^2+z^2}{2}.$$

III. *Comment by M.S. Klamkin, University of Alberta.*

In a comment following the solution of this problem [1982: 148], the editor asked for a solution of the following related problem:

Let $S_r = x^r + y^r + z^r + u^r$. *Find all integer pairs* $\{m,n\}$ *such that*

$$\frac{S_{m+n}}{m+n} = \frac{S_m}{m}\cdot\frac{S_n}{n} \tag{1}$$

is meaningful and true for all quadruples (x,y,z,u) *for which* $S_1 = 0$, *having given that one solution is* $\{m,n\} = \{3,2\}$.

I show that $\{m,n\} = \{3,2\}$ is the only solution. Another solution $\{m,n\}$, if one exists, must remain valid whether or not $u = 0$, and it follows from my solution of the original problem that the only candidate is $\{m,n\} = \{5,2\}$. However, in this case (1) does not hold for the quadruple $(x,y,z,u) = (3,-1,-1,-1)$.

Editor's comment.

It would be interesting to know if there are other solutions $\{m,n\}$ valid for all quadruples (x,y,z,u) for which $S_1 = 0$ and $xyzu \neq 0$. Readers will recall that Klamkin had found that $\{m,n\} = \{3,-1\}$ was also a solution of the original problem when the restriction $xyz \neq 0$ was added.

*   *   *

665. [1981: 205 ; 1982: 221] *Proposed by Jack Garfunkel, Queens College, Flushing, N.Y.*

If A,B,C,D are the interior angles of a convex quadrilateral ABCD, prove that

$$\sqrt{2}\,\Sigma\cos\frac{A+B}{4} \le \Sigma\cot\frac{A}{2}$$

(where the four-term sum on each side is cyclic over A,B,C,D), with equality if and only if ABCD is a rectangle.

II. *Generalization by M.S. Klamkin, University of Alberta.*

The published solution of this problem depended in part on the convexity of $\cot\theta$ for acute angles θ. We use the same property, together with the convexity of $-\cos\theta$ for acute angles θ, to prove a generalization.

Let $A_1A_2\ldots A_n$ be a convex n-gon. We apply successively to each vertex A_i weights $x_r \ge 0$ and $y_r \ge 0$ such that $\Sigma x_r = \Sigma y_r = 1$ (summations throughout are for $r = 1,2,\ldots,n$) to form the weighted sums

$$\begin{aligned} B_r &= x_rA_1 + x_{r+1}A_2 + \ldots + x_{r+n-1}A_n, \\ C_r &= y_rA_1 + y_{r+1}A_2 + \ldots + y_{r+n-1}A_n, \end{aligned} \qquad r = 1,2,\ldots\, n,$$

where all subscripts are reduced modulo n. By Jensen's inequality for convex function, we have

$$\sum\cot\frac{B_r}{2} \ge n\cot\frac{(n-2)\pi}{2n} \qquad \text{and} \qquad \sum\cos\frac{C_r}{2} \le n\cos\frac{(n-2)\pi}{2n}.$$

Hence

$$\left\{n \cot \frac{(n-2)\pi}{2n}\right\}^{-1} \sum \cot \frac{B_r}{2} \geq 1 \geq \left\{n \cos \frac{(n-2)\pi}{2n}\right\}^{-1} \sum \cos \frac{C_r}{2},$$

with equality if and only if the polygon is equiangular.

The original problem is the special case for which $n = 4$, $(x_r) = (1,0,0,0)$, and $(y_r) = (\frac{1}{2},\frac{1}{2},0,0)$.

* * *

677. [1981: 240] *Proposed by E.J. Barbeau, University of Toronto.*

Prove that there do not exist four distinct square integers in arithmetic progression.

(This problem is apparently due to Fermat (1640), but the proofs mentioned in Dickson's *History of the Theory of Numbers* (Vol. II, p. 440) either have obscure references, are unclear, or appear unsatisfactory.)

The problem was solved by the proposer. Comments were received from W.J. BLUNDON, Memorial University of Newfoundland; ANDY LIU, University of Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

All the references given below were supplied by readers. The most satisfactory proof of this theorem of Fermat appears to be that of Pocklington [1], which can be found in Sierpiński [2]. It is based on the fact, also established by Pocklington, that the equation

$$x^4 - x^2y^2 + y^4 = z^2$$

has no solution in natural numbers (x,y,z) apart from the trivial one (x,x,x^2). Essentially the same proof is given in Mordell [3].

The proposer noted that the problem had appeared in 1898 in the *Monthly* [4]. No complete proof was forthcoming at that time (for natural numbers) but G.B.M. Zerr found the imaginary solution

$$(6+\sqrt{-11})^2, \quad 5^2, \quad (6-\sqrt{-11})^2, \quad \left\{\sqrt{(25+\sqrt{6961})/2} - \sqrt{(25-\sqrt{6961})/2}\right\}^2,$$

with another solution obtainable by taking complex conjugates.

REFERENCES

1. H.C. Pocklington, "Some diophantine impossibilities", *Proc. Cambridge Philos. Soc.*, 17 (1914) 108-121.

2. Wacław Sierpiński, *Elementary Theory of Numbers*, Hafner, New York, 1964, pp. 74-75.

3. L.J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, pp. 20-22.

4. Problem 62 (proposed by John M. Arnold), *American Mathematical Monthly*, 5 (1898) 180.

* * *

678. [1981: 240] *Proposed jointly by Joe Dellinger and Ferrell Wheeler, students, Texas A & M University, College Station, Texas.*

For a given fixed integer $n \geq 2$, find the greatest common divisor of the integers in the set $\{a^n - a \mid a \in Z\}$, where Z is the set of all integers.

Solutions were received from FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and the proposers. Comments were submitted by W.J. BLUNDON, Memorial University of Newfoundland; and STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire. In addition, two incorrect solutions, based on a misreading of the problem, were received.

Editor's comment.

The two comments received alerted the editor to the fact that essentially the same problem appeared, with two proposers different from ours, in the August-September 1981 issue of the *American Mathematical Monthly* (No.E 2901). We received the problem on September 1, 1981, and published it immediately in the October 1981 issue of *Crux*, which was then under way. So, although the problem was published almost simultaneously in the two publications, it is clear that it must have been in the *Monthly* "pipeline" long before we received it. Our proposers must have been unaware of the impending publication of the problem in the *Monthly* when they sent it to us.

It seems only fair to let the *Monthly* carry the ball on this problem. When a solution appears in the *Monthly*, a few months hence, we will reopen the problem here and give our readers the reference.

* * *

679. [1981: 240] *Proposed by Bob Prielipp, University of Wisconsin-Oshkosh.*

The equation $x^2+y^3 = 2z^4$ has the solutions (1,1,1) and (239,1,13). Does it have infinitely many solutions (x,y,z) consisting of relatively prime positive integers?

I. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

With considerable expenditure of time on a fast computer, I found only the following six solutions with $z \leq 2770$ and $y \leq 416000$:

(1, 1, 1),	(4479, 161, 59),
(13, 7, 4),	(4445, 263, 66),
(239, 1, 13),	(490013, 11047, 944).

In view of these results, it is likely that solutions are few and far between, although I would hesitate to say that the number of solutions is finite. It also appears that the solution (239, 1, 13) is atypical, and it is likely that if an infinite family of other solutions is found, each will satisfy the relation $x \geq y \geq z$.

There are also many solutions like

$$(64,\ 16,\ 8),\quad (875,\ 25,\ 25),\quad (4174281,\ 20079,\ 1890)$$

in which x,y,z are not relatively prime. Since the given equation is not homogeneous, these do not represent multiples of more primitive solutions.

II. *First part of the solution by Kenneth M. Wilke, Topeka, Kansas.*

We first find all the solutions for which $y = 1$. The given equation then reduces to

$$x^2 = 2z^4 - 1.$$

According to Mordell [1], it has been known for two centuries that $(x, z) = (1, 1)$ and (239, 13) are solutions of this equation, and it was proved in 1942 by Ljunggren [2] that these are the only positive integer solutions. Mordell adds that "the proof is exceedingly complicated". Thus the two solutions given in the proposal are the only ones for which $y = 1$.

Editor's comment.

If the proof for $y = 1$ is "exceedingly complicated", that for $y \neq 1$ can hardly be less so. The rest of Wilke's solution was long and complicated, though not "exceedingly" so, which leads us to suspect that his solution is not complete. He found two infinite families of solutions in relatively prime integers x,y,z with $x > 0$, $y \neq 1$, $z > 0$ (thus allowing y to be negative). They are

$$\begin{cases} x = |r^4 - 24s^4|(r^8 + 816r^4s^4 + 576s^8) \\ y = -r^8 + 336r^4s^4 - 576s^8 \\ z = 6rs(r^4 + 24s^4) \end{cases} \tag{1}$$

and

$$\begin{cases} x = |3r^4 - 8s^4|(9r^8 + 816r^4s^4 + 64s^8) \\ y = -9r^8 + 336r^4s^4 - 64s^8 \\ z = 6rs(3r^4 + 8s^4) \end{cases} \tag{2}$$

where $(r, s) = 1$, r is odd, $(r, 3) = 1$ in (1), and $(s, 3) = 1$ in (2).

Of the four solutions with $y \neq 1$ found by Kierstead, only (4445, 263, 66) belongs to one of these families. So there are other families of solutions lurking in the underbrush, and finding them all will probably be an "exceedingly complicated" business.

Finally, Wilke noted that families (1) and (2) each contain infinitely many solutions with $y > 0$. In family (1), these correspond to pairs (r, s) for which

$$1.145721 \approx \sqrt[4]{168-96\sqrt{3}} < \frac{r}{s} < \sqrt[4]{168+96\sqrt{3}} \approx 4.275891,$$

and in family (2) to pairs for which

$$0.661483 \approx \sqrt[4]{\frac{168-96\sqrt{3}}{9}} < \frac{r}{s} < \sqrt[4]{\frac{168+96\sqrt{3}}{9}} \approx 2.468687.$$

REFERENCES

1. L.J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, p. 271.

2. W. Ljunggren, "Zur Theorie der Gleichung $x^2 + 1 = Dy^4$", *Avh. Norske Vid. Akad. Oslo*, No. 51 (1942).

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680. [1981: 240] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Find interesting sets of three distinct real numbers such that their product is equal to their sum.

Editor's comment.

The following remarks were culled from the solutions of several solvers. If $\{a,b,c\}$ is a satisfactory set, then so is $\{-a,-b,-c\}$; and $\{1,2,3\}$ is the only satisfactory set of natural numbers (see [1] and [2]). If $a \neq 0$, then $\{0,a,-a\}$ is a satisfactory set.

To save space, in all solutions given below it will be tacitly assumed that suitable restrictions are imposed upon the numbers concerned to make them real and distinct, as the problem requires. Most solvers submitted several solutions, but only the ones that the editor judged to be particularly "interesting" are given below.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

It is easy to show that the set $\{a,b,c\}$ satisfies $abc = a+b+c$ if and only if $ab \neq 1$ and $c = (a+b)/(ab-1)$. So the most general satisfactory set is

$$\left\{a,\ b,\ \frac{a+b}{ab-1}\right\}.$$

II. *Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Krechmar [3] gives the following satisfactory set:

$$\left\{\frac{b-c}{1+bc},\ \frac{c-a}{1+ca},\ \frac{a-b}{1+ab}\right\}.$$

III. *Solution by J.A.H. Hunter, Toronto, Ontario.*

The following set is satisfactory:

$$\left\{\frac{m+1}{\sqrt{mn-1}},\quad \frac{n+1}{\sqrt{mn-1}},\quad \sqrt{mn-1}\right\}.$$

IV. *Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

If $a+b+c = 1$, the following set is satisfactory:

$$\left\{\sqrt{\frac{a}{bc}},\quad \sqrt{\frac{b}{ca}},\quad \sqrt{\frac{c}{ab}}\right\}.$$

V. *Solution by Fred A. Miller, Elkins, West Virginia.*

The following set is satisfactory:

$$\left\{\frac{c}{b},\quad \frac{c}{a-b},\quad \frac{ac}{b^2-ab+c^2}\right\}.$$

VI. *Solution by Leon Bankoff, Los Angeles, California.*

Elementary trigonometry yields the following satisfactory sets, where A, B, C are the angles of a triangle:

$$\{\tan A,\ \tan B,\ \tan C\}$$

and

$$\left\{\cot\frac{A}{2},\ \cot\frac{B}{2},\ \cot\frac{C}{2}\right\}.$$

VII. *Solution by the proposer.*

I offer two satisfactory sets consisting of an arithmetic progression,

$$\{a+\sqrt{a^2-3},\quad a,\quad a-\sqrt{a^2-3}\},$$

and a geometric progression,

$$\left\{\frac{a(a^2-1)+a\sqrt{(a^2+1)(a^2-3)}}{2},\quad a,\quad \frac{a(a^2-1)-a\sqrt{(a^2+1)(a^2-3)}}{2}\right\}.$$

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and KENNETH M. Wilke, Topeka, Kansas.

REFERENCES

1. E.P. Starke, Solution of Problem E 2262, (proposed by G.J. Simmons and D.B. Rawlinson), *American Mathematical Monthly*, 78 (1971) 1021-1022.

2. M. Misiurewicz, "Ungelöste Probleme", *Elem. Math.*, 21 (1966) 90.

3. V.A. Krechmar, *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974, p. 25.

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681. [1981: 274] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Of all the girls that are so smart,
  There's none like pretty Sally.
She is the darling of my heart,
  And she lives in our alley.
And when our alley the sunlight dapples,
  My darling SALLY

$$\begin{array}{r} \text{SALLY} \\ \text{SELLS} \\ \underline{\text{RIPE}} \\ \text{APPLES} \end{array}.$$

(With apologies to Henry Carey (c. 1687 - 1743).)

Solution by Charles W. Trigg, San Diego, California.

Proceeding from the right, the columns establish the following equations:

$$Y + E = 10, \tag{1}$$

$$2L + P + 1 = E + 10k, \quad k \in \{0,1,2\}, \tag{2}$$

$$L + I + k = 10, \tag{3}$$

$$A + E + R + 1 = P + 10m, \quad m \in \{0,1,2\}, \tag{4}$$

$$2S + m = P + 10A.$$

If $A = 2$, then $S = 9$, $P = 0$, and $m = 2$. However, the maximum for $A+E+R+1$ is $2+8+7+1 < 20$, so $A = 1$ and $m \neq 2$. Hence, from (1), $E \neq 0$, 1, 5, or 9. In (4), if $A+E+1 = 10$, then $R = P$, so $E \neq 8$. In (2), if $L = 0$, then $k = 0$, and then (3) gives the impossible $I = 10$. Thus $L \neq 0$.

We now try in succession the possible values of E: 2, 3, 4, 6, and 7. The details for $E = 2$ are given below, with a star to indicate where the attempted solution fails.

A	E	Y	L	P	(k)	I	R	(m)	S	
1	2	8	3	5	(1)	6	*			
			4	3	(1)	5	9	(1)	6	(solution!)
			5	*						
			6	9	(2)	*				
			7	*						
			9	3	(2)	*				

Proceeding in like manner with $E = 3$, 4, 6, and 7, no additional solutions are found. Therefore, the unique solution is

$$\begin{array}{r} 61448 \\ 62446 \\ \underline{9532} \\ 133426 \end{array}.$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; SID KRAVITZ, Dover, New Jersey; J.A. McCALLUM, Medicine Hat, Alberta; LISA PETROCCO, Erskine College, Due West, South Carolina; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; RICHARD RHOAD, New Trier H.S., Winnetka, Illinois; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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682. [1981: 274] *Proposed by Robert C. Lyness, Southwold, Suffolk, England.*

Triangle ABC is acute-angled and Δ_1 is its orthic triangle (its vertices are the feet of the altitudes of triangle ABC). Δ_2 is the triangular hull of the three excircles of triangle ABC (that is, its sides are the external common tangents of the three pairs of excircles that are not sides of triangle ABC).

Prove that the area of triangle Δ_2 is at least 100 times the area of triangle Δ_1.

I. *Solution by George Tsintsifas, Thessaloniki, Greece.*

Let Δ = ABC, Δ_1 = DEF (with D on BC, etc.), Δ_2 = A'B'C' (with A' opposite A, etc.), and let K, K_1, K_2 be the areas of Δ, Δ_1, Δ_2, respectively, with square brackets denoting the areas of other triangles as required. (Make a figure!) We must show that

$$K_2 \geq 100K_1. \tag{1}$$

Let BC intersect A'B' and A'C' in M and N, respectively, let AC intersect A'B' in P, and let AB intersect A'C' in Q. Since

$$\angle FDB = \angle QNB = \angle A \quad \text{and} \quad \angle CDE = \angle CMP = \angle A,$$

and AB = BN, AC = CM, we have

$$[BNQ] = [CPM] = K.$$

Also, MN = $2s$, where s is the semiperimeter of Δ, so that [A'MN] = $s^2 \tan A$. With these and similar results, we get

$$K_2 = s^2(\tan A + \tan B + \tan C) - 2K$$

and (1) is equivalent to

$$s^2(\tan A + \tan B + \tan C) \geq 100K_1 + 2K. \tag{2}$$

We establish the triple in inequality

$$s^2(\tan A + \tan B + \tan C) \geq 3\sqrt{3}s^2 \geq 27K \geq 100K_1 + 2K, \tag{3}$$

from which (2) follows. The first inequality follows from

$$\tan A + \tan B + \tan C \geq 3\sqrt{3},$$

which holds for all acute-angled triangles [1, p. 26]; the second follows from

$s^2 \geq 3\sqrt{3}K$, which holds for all triangles [1, p. 42]; and the third is a consequence of the well-known inequality

$$K \geq 4K_1. \tag{4}$$

Equality holds throughout in (3) just when Δ is equilateral.

II. *Outline of the proposer's solution.*

It is easy by angle-chasing, using symmetry about each line joining a pair of excentres, to show that corresponding sides of Δ_1 and Δ_2 are parallel. It is not difficult to show that the linear enlargement is

$$\frac{1 + \cos A + \cos B + \cos C}{2\cos A \cos B \cos C} \geq 10 \tag{5}$$

for acute-angled triangles, and (1) follows, with equality just when Δ is equilateral. □

The homothetic centre of Δ_1 and Δ_2 has trilinear coordinates

$$(\tan A,\ \tan B,\ \tan C). \tag{6}$$

It is well known that $(\cos A, \cos B, \cos C)$ is the circumcentre of Δ, and that $(\sin A, \sin B, \sin C)$ is its Lemoine point. But (6) I have never seen referred to. Perhaps in view of the publication in your journal it should be christened as the *Crucial Point* of the triangle.

On the other hand, the result (like most of my amazing discoveries) may be well known to pundits of triangle geometry.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

Our first solver did not give a reference for inequality (4), but it follows from the relation

$$K_1 = 2K \cos A \cos B \cos C$$

recently established in this journal [1982: 230] and from

$$\cos A \cos B \cos C \leq \frac{1}{8},$$

which can be found in [1, p. 25]. Note that (4) holds even for nonacute-angled triangles if signed areas are used.

The proposer gave a calculus proof of inequality (5). The proof was awkward, as are most calculus proofs of triangle inequalities. The inequality is interesting in its own right. Can some reader prove it without using calculus?

Altshiller Court [2] proves that Δ_1 is also homothetic to the *tangential triangle* of Δ, formed by the tangents to the circumcircle of Δ at its vertices, but he does not give a name to their homothetic centre. If that homothetic centre is the same as that of Δ_1 and Δ_2 (can some reader prove or disprove this?), then it is doubly crucial that this point be given a name. In that case, we modestly suggest that it be called instead the *Lyness Point* of the triangle.

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

2. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 98.

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683. [1981: 274] *Proposed by Kaidy Tan, Fukien Teachers University, Foochow, China.*

Triangle ABC has AB > AC, and the internal bisector of angle A meets BC at T. Let P be any point other than T on *line* AT, and suppose lines BP,CP intersect lines AC,AB in D,E, respectively. Prove that BD > CE or BD < CE according as P lies on the same side or on the opposite side of BC as A.

Solution by Jordi Dou, Barcelona, Spain.

Our proof is based on the following elementary result. If a point P is equidistant from the sides of an angle A, and if a transversal through P cuts the sides of angle A in X and Y, then the minimum length of XY occurs when it coincides with the transversal segment MN which is perpendicular to the bisector of angle A. (This is illustrated in Figure 1.) Furthermore, if XY makes an acute angle ω with MN, then XY = $\phi(\omega)$ increases strictly from MN to ∞ as ω increases from 0 to $\alpha = (\pi - A)/2$. For

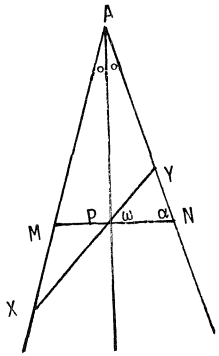

Figure 1

$$XY = \phi(\omega) = AP\cos\alpha\left\{\frac{1}{\sin(\alpha-\omega)} + \frac{1}{\sin(\alpha+\omega)}\right\}$$

and

$$\phi'(\omega) = AP\left\{\frac{\cos(\alpha-\omega)}{\sin^2(\alpha-\omega)} - \frac{\cos(\alpha+\omega)}{\sin^2(\alpha+\omega)}\right\} > 0.$$

Coming now to our problem, we assume that P is an interior point of the *segment* AT. Let B',C',D',E' be symmetric to B,C,D,E, respectively, with respect to line AT, and let MN and α be as in Figure 1. One of the segments BD or B'D' makes with

MN and acute angle (less than α) greater than the acute angle between MN and CE or C'E'. Hence

$$BD = B'D' > C'E' = CE.$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer (three solutions).

Editor's comment.

It is clear from the proposer's solutions that he intended the point P to be restricted to the *segment* AT, as in our featured solution. The proposer noted that when P is at the incentre of triangle ABC, the resulting BD > CE provides a well-known indirect proof of the Steiner-Lehmus Theorem. (See [1976: 19-24] in this journal for information about and many references to this theorem.)

It was the editor who, thinking he could espy a more general theorem, modified the proposal to allow the point P to range over the entire *line* AT. Satyanarayana and Tsintsifas showed that BD > CE continues to hold when P lies beyond A on the half-line TA. But nearly all solvers gave counterexamples to show that BD < CE does not hold for all positions of P beyond T on the half-line AT.

Let $CC_1 \parallel AB$ and $BB_1 \parallel AC$, as shown in Figure 2. Tsintsifas outlined a proof showing that $f(P) = BD - CE$ goes from 0 to $-\infty$ as P goes from T to C_1; that it goes from $-\infty$ to $+\infty$ as P goes from C_1 to B_1; and that it remains positive when P lies beyond B_1 on the half-line AT. Hence, by continuity, there is a point S between C_1 and B_1 such that $f(P)$ vanishes and BD = CE when P is at S. This point S is a remarkable point of a nonequilateral triangle whose existence has apparently been hitherto unsuspected. It would be interesting to have this point more precisely identified, and to know some of its properties and how to construct it.

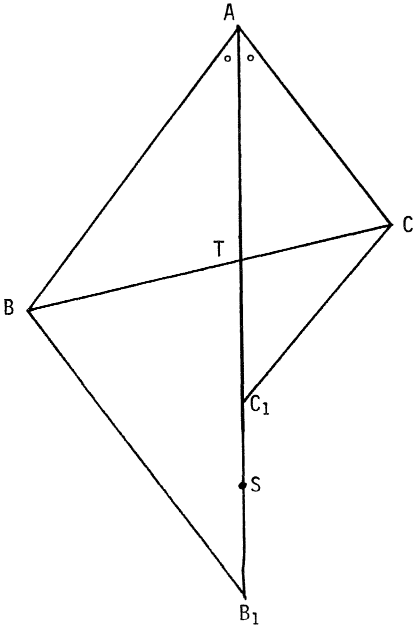

Figure 2

For certain special nonisosceles triangles ABC with AB > AC (called *pseudoisosceles triangles*), the location of the point S is known: it is the excentre I_1 opposite angle A (see the Steiner-Lehmus reference given above). These triangles are characterized by the relation

$$\sin^2\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2}.$$

One well-known example has $A = 36^\circ$, $B = 12^\circ$, $C = 132^\circ$.

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684. [1981: 275] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let O be the origin of the lattice plane, and let $M(p,q)$ be a lattice point with relatively prime positive coordinates (with $q > 1$). For $i = 1,2,\ldots,q-1$, let P_i and Q_i be the lattice points, both with ordinate i, that are respectively the left and right endpoints of the horizontal unit segment intersecting OM. Finally, let $P_iQ_i \cap OM = M_i$.

(a) Calculate

$$S_1 = \sum_{i=1}^{q-1} \overline{P_iM_i}.$$

(b) Find the minimum value of $\overline{P_iM_i}$ for $1 \le i \le q-1$.

(c) Show that $\overline{P_iM_i} + \overline{P_{q-i}M_{q-i}} = 1$, $1 \le i \le q-1$.

(d) Calculate

$$S_2 = \sum_{i=1}^{q-1} \frac{\overline{P_iM_i}}{\overline{M_iQ_i}}.$$

(e) Show that the area of a simple triangle is $\frac{1}{2}$. (A *simple triangle* is one whose vertices are lattice points and which has no other lattice point in its interior or on its perimeter.)

Solution by the proposer.

(a) Since $(p,q) = 1$, the segment OM has no lattice point in its interior. Its equation is $y = (q/p)x$; so when $y = i$ we have $x = ip/q$ and

$$\overline{P_iM_i} = \frac{ip}{q} - k_i = \frac{r_i}{q}, \tag{1}$$

where k_i is the integer $[ip/q]$ and r_i is the remainder when ip is divided by q. For $i = 1,2,\ldots,q-1$, the remainders r_i are all different and less than q. The sequence (r_i) is therefore a permutation of the sequence (i), and we have

$$S_1 = \sum_{i=1}^{q-1} \overline{P_iM_i} = \frac{1 + 2 + \ldots + (q-1)}{q} = \frac{q-1}{2}.$$

(b) Since the smallest r_i is 1, the minimum value of $\overline{P_iM_i}$ is $1/q$.

(c) With the help of (1), we get

$$\overline{P_{q-i}M_{q-i}} = \frac{(q-i)p}{q} - \left[\frac{(q-i)p}{q}\right] = p - \frac{ip}{q} - \left[p - \frac{ip}{q}\right]$$
$$= p - k_i - \frac{r_i}{q} - \left[p - k_i - \frac{r_i}{q}\right]$$
$$= p - k_i - \frac{r_i}{q} - (p - k_i - 1)$$
$$= 1 - \frac{r_i}{q},$$

from which follows

$$\overline{P_iM_i} + \overline{P_{q-i}M_{q-i}} = 1.$$

(d) Since

$$\frac{\overline{P_iM_i}}{\overline{M_iQ_i}} = \frac{r_i}{q}\Big/\left(1 - \frac{r_i}{q}\right) = \frac{r_i}{q-r_i},$$

we have

$$S_2 = \sum_{i=1}^{q-1} \frac{\overline{P_iM_i}}{\overline{M_iQ_i}} = \sum_{i=1}^{q-1} \frac{r_i}{q-r_i}$$
$$= \frac{1}{q-1} + \frac{2}{q-2} + \ldots + \frac{q-1}{1}.$$

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio.

Editor's comment.

We have omitted the proposer's proof of part (e) because this result is an immediate consequence of Pick's Theorem (see, e.g., [1]), and because, as Kierstead noted, it is equivalent to a result stated and proved in Hardy and Wright [2].

REFERENCES

1. A. Liu, "A Direct Proof of Pick's Theorem", this journal, 4 (1978) 242-244.

2. G.H. Hardy and E M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford University Press, 1979, p. 29, Theorem 34.

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685. [1981: 275] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given is a triangle ABC with internal angle bisectors t_a, t_b, t_c meeting a, b, c in U,V,W, respectively; and medians m_a, m_b, m_c meeting a, b, c in L,M,N, respectively. Let

$$m_a \cap t_b = P, \qquad m_b \cap t_c = Q, \qquad m_c \cap t_a = R.$$

Crux 588 [1981: 306] asks for a proof of the equality

$$\frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} = 8.$$

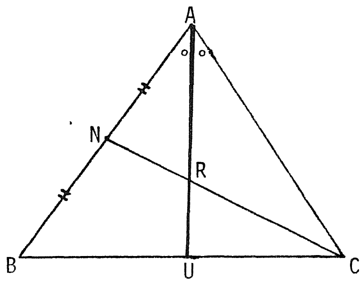
Establish here the inequality

$$\frac{AR}{RU}\cdot\frac{BP}{PV}\cdot\frac{CQ}{QW} \ge 8,$$

with equality if and only if the triangle is equilateral.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

If we apply the theorem of Menelaus to triangle ABU and transversal CRN, we get (see figure)



$$\frac{AR}{RU}\cdot\frac{UC}{CB}\cdot\frac{BN}{NA} = -1.$$

Since BN = NA, we have

$$\frac{AR}{RU} = \frac{BC}{UC} = 1 + \frac{BU}{UC} = 1 + \frac{c}{b} = \frac{b+c}{b}.$$

With this and two similar results, we now get

$$\frac{AR}{RU}\cdot\frac{BP}{PV}\cdot\frac{CQ}{QW} = \frac{b+c}{b}\cdot\frac{c+a}{c}\cdot\frac{a+b}{a} \ge 8,$$

since $(b+c)(c+a)(a+b) \ge 8abc$, with equality just when $a = b = c$ (easily shown, or see [1]).

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; JACK GARFUNKEL, Flushing, N.Y.; V.N. MURTY, Pennsylvania State University, Capitol Campus; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. One incorrect solution was received.

Editor's comment.

The present proposal points out the superficial similarity between this problem and Crux 588. See Crux 790 in this issue for still another related problem. Note that 790 is a true extension of 588 but not of the present problem; for here equality occurs if and only if the triangle is equilateral, whereas in 790 equality can occur even for nonequilateral triangles and strict inequality can occur even for equilateral triangles.

REFERENCE

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968, p. 12.

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686. [1981: 275] *Proposed by Charles W. Trigg, San Diego, California.*

Without using calculus, analytic geometry, or trigonometry, find the area of the region which is common to the four quadrants that have the vertices of a square as centers and a side of the square as a common radius.

[A solution using analytic geometry appears in *School Science and Mathematics*, 78 (April 1978) 355.]

Solution by Jordi Dou, Barcelona, Spain.

It will simplify matters if we assume that the square has sides of unit length. If certain regions have areas A, B, C, as shown in Figure 1, then we have

$$A + 4B + 4C = 1,$$
$$A + 3B + 2C = \frac{\pi}{4},$$
$$A + 2B + C = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

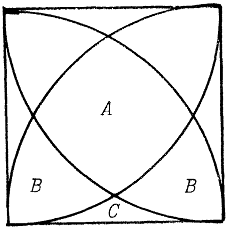

Figure 1

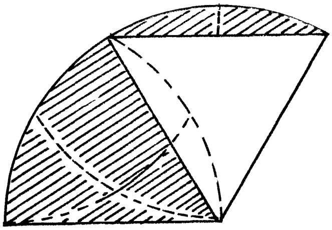

Figure 2

(Figure 2 explains the formation of the third equation.) Solving this system, we find the required value to be

$$A = 1 - \sqrt{3} + \frac{\pi}{3}.$$

For good measure, we give also

$$B = -1 + \frac{\sqrt{3}}{2} + \frac{\pi}{12}, \qquad C = 1 - \frac{\sqrt{3}}{4} - \frac{\pi}{6}.$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer. M.S. KLAMKIN, University of Alberta, sent in a comment, and one incorrect solution was received.

Editor's comment.

A photocopy of a relatively complicated solution by S. Jones, apparently taken from a book, was sent by Klamkin, who found it in his files but could not further

(continued on page 278)