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CONTENTS

ON THE AREA OF A TRIANGLE IN BARYCENTRIC COORDINATES

O. BOTTEMA

We denote the area of any (oriented) triangle KLM by [KLM]. We are given a triangle ABC which will remain fixed throughout our discussion. The *homogeneous barycentric coordinates* (x,y,z) of a point P with respect to triangle ABC are defined by

$$x : y : z = [\mathrm{PBC}] : [\mathrm{PCA}] : [\mathrm{PAB}].$$

We will in this note give a few applications of the following

THEOREM. If the vertices of triangle $\mathrm{P_1P_2P_3}$ *have homogeneous barycentric coordinates* $\mathrm{P}_i = (x_i,y_i,z_i)$ *with respect to triangle* ABC, *then the area of the triangle is*

$$[\mathrm{P_1P_2P_3}] = \frac{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{\Pi(x_i+y_i+z_i)} \cdot [\mathrm{ABC}]. \tag{1}$$

It will be convenient to consider that triangle ABC has unit area. The *normalized* barycentric coordinates of point P with respect to triangle ABC are then

$$\bar{x} = [\mathrm{PBC}], \qquad \bar{y} = [\mathrm{PCA}], \qquad \bar{z} = [\mathrm{PAB}].$$

We then obviously have

$$\bar{x} + \bar{y} + \bar{z} = 1,$$

and (1) becomes

$$[\mathrm{P_1P_2P_3}] = \begin{vmatrix} \bar{x}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2 & \bar{y}_2 & \bar{z}_2 \\ \bar{x}_3 & \bar{y}_3 & \bar{z}_3 \end{vmatrix}. \tag{2}$$

We will for completeness give a proof of formula (2), which is not new and may have been known to Möbius. (Professor M.S. Klamkin informs us that a proof by trilinear coordinates can be found in S.L. Loney, *The Elements of Coordinate Geometry*, Part II, Macmillan, London, pages 39 and 57, a reference which is not available to us.)

Proof of (2). Let $\mathrm{P}_1 = (\bar{x}_1,\bar{y}_1,\bar{z}_1)$ be an arbitrary point and $\mathrm{S} = (\bar{x}_0,\bar{y}_0,0)$ a point on AB, as shown in Figure 1. We have $[\mathrm{P_1AB}] = \bar{z}_1$. Furthermore, as $\bar{x}_0 + \bar{y}_0 = 1$, we have AS : AB $= \bar{y}_0$, and so

$$[\mathrm{P_1AS}] = \bar{y}_0\bar{z}_1. \tag{3}$$

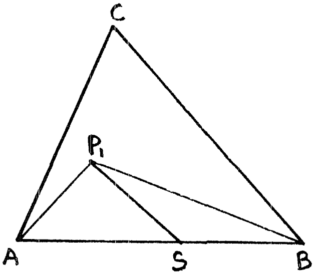


Figure 1

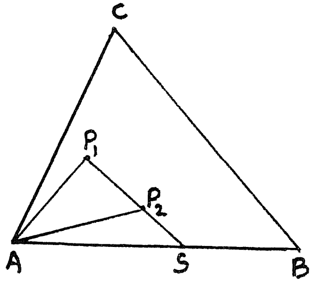


Figure 2

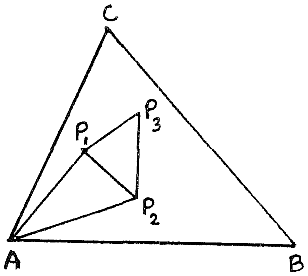


Figure 3

We now consider two points $P_1 = (\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and $P_2 = (\bar{x}_2, \bar{y}_2, \bar{z}_2)$, as shown in Figure 2. We assume that P_1P_2 is not parallel to AB, that is, $\bar{z}_1 \neq \bar{z}_2$, and meets AB in S. Then we have

$$S = (\bar{z}_2\bar{x}_1 - \bar{z}_1\bar{x}_2,\ \ \bar{z}_2\bar{y}_1 - \bar{z}_1\bar{y}_2,\ \ 0).$$

As

$$\bar{z}_2\bar{x}_1 - \bar{z}_1\bar{x}_2 + \bar{z}_2\bar{y}_1 - \bar{z}_1\bar{y}_2 = \bar{z}_2(1-\bar{z}_1) - \bar{z}_1(1-\bar{z}_2) = \bar{z}_2 - \bar{z}_1,$$

we have $S = (\bar{x}_0, \bar{y}_0, 0)$ with

$$\bar{y}_0 = \frac{\bar{z}_2\bar{y}_1 - \bar{z}_1\bar{y}_2}{\bar{z}_2 - \bar{z}_1}.$$

It now follows from (3) that

$$[P_1AS] = \frac{(\bar{z}_2\bar{y}_1 - \bar{z}_1\bar{y}_2)\bar{z}_1}{\bar{z}_2 - \bar{z}_1} \quad \text{and} \quad [P_2AS] = \frac{(\bar{z}_2\bar{y}_1 - \bar{z}_1\bar{y}_2)\bar{z}_2}{\bar{z}_2 - \bar{z}_1},$$

and therefore

$$[AP_1P_2] = \bar{y}_1\bar{z}_2 - \bar{y}_2\bar{z}_1. \tag{4}$$

For the last step, we consider three points P_1, P_2, P_3, as shown in Figure 3. With the help of (4), we obtain

$$\begin{aligned}[P_1P_2P_3] &= [AP_2P_3] + [AP_3P_1] + [AP_1P_2] \\ &= \bar{y}_2\bar{z}_3 - \bar{y}_3\bar{z}_2 + \bar{y}_3\bar{z}_1 - \bar{y}_1\bar{z}_3 + \bar{y}_1\bar{z}_2 - \bar{y}_2\bar{z}_1 \\ &= \begin{vmatrix} 1 & \bar{y}_1 & \bar{z}_1 \\ 1 & \bar{y}_2 & \bar{z}_2 \\ 1 & \bar{y}_3 & \bar{z}_3 \end{vmatrix} = \begin{vmatrix} \bar{x}_1+\bar{y}_1+\bar{z}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2+\bar{y}_2+\bar{z}_2 & \bar{y}_2 & \bar{z}_2 \\ \bar{x}_3+\bar{y}_3+\bar{z}_3 & \bar{y}_3 & \bar{z}_3 \end{vmatrix} = \begin{vmatrix} \bar{x}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2 & \bar{y}_2 & \bar{z}_2 \\ \bar{x}_3 & \bar{y}_3 & \bar{z}_3 \end{vmatrix}.\end{aligned}$$

If P_1P_2 is parallel to AB, the same argument holds with P_2, say, replaced by P_3. □

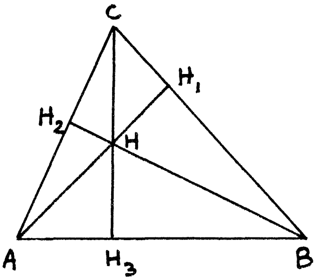


Figure 4

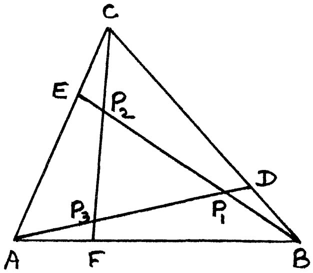


Figure 5

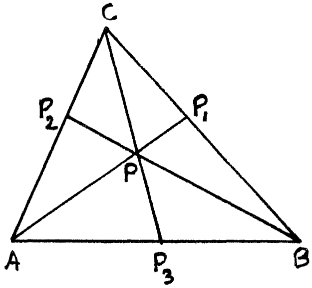


Figure 6

We are now ready to give some applications of (1).

(a) Figure 4 shows the pedal triangle $H_1H_2H_3$ of the orthocenter H of triangle ABC (with sides a,b,c and angles α,β,γ). We have $H_1 = (0,\ b\cos\gamma,\ c\cos\beta)$, etc., and we obtain from (1)

$$[H_1H_2H_3] = \frac{\begin{vmatrix} 0 & b\cos\gamma & c\cos\beta \\ a\cos\gamma & 0 & c\cos\alpha \\ a\cos\beta & b\cos\alpha & 0 \end{vmatrix}}{abc}\cdot[ABC]$$

$$= 2[ABC]\cos\alpha\cos\beta\cos\gamma.$$

(b) Referring now to Figure 5, if

$$BD : DC = r, \qquad CE : EA = s, \qquad AF : FB = t,$$

then we have

$$D = (0,1,r), \qquad E = (s,0,1), \qquad F = (1,t,0).$$

The equation of AD is $ry - z = 0$ and that of BE is $x - sz = 0$; hence $P_1 = (rs,1,r)$, and analogously $P_2 = (s,st,1)$, $P_3 = (1,t,tr)$. Since

$$\begin{vmatrix} rs & 1 & r \\ s & st & 1 \\ 1 & t & tr \end{vmatrix} = (rst-1)^2,$$

we obtain from (1)

$$[P_1P_2P_3] = \frac{(rst-1)^2}{(st+s+1)(tr+t+1)(rs+r+1)}\cdot[ABC].$$

This is *Routh's formula*, for which Klamkin recently gave various proofs in this journal [1981: 199].

(c) Let $P = (x,y,z)$, and let PA,PB,PC intersect BC,CA,AB in P_1,P_2,P_3, respectively, as shown in Figure 6. Then

$$P_1 = (0,y,z), \quad P_2 = (x,0,z), \quad P_3 = (x,y,0),$$

and (1) yields

$$[P_1P_2P_3] = \frac{2xyz}{(y+z)(z+x)(x+y)} \cdot [ABC].$$

(d) For triangle ABC, let I_0 be the incenter and I_1,I_2,I_3 the excenters; r_0 the inradius and r_1,r_2,r_3 the exradii; R the circumradius and s the semiperimeter. Then

$$I_0 = (ar_0,\ br_0,\ cr_0)$$

and

$$I_1 = (-ar_1,\ br_1,\ cr_1), \quad I_2 = (ar_2,\ -br_2,\ cr_2), \quad I_3 = (ar_3,\ br_3, -cr_3).$$

Since

$$\begin{vmatrix} -ar_1 & br_1 & cr_1 \\ ar_2 & -br_2 & cr_2 \\ ar_3 & br_3 & -cr_3 \end{vmatrix} = 4abcr_1r_2r_3,$$

we obtain from (1)

$$[I_1I_2I_3] = \frac{abc[ABC]}{2(s-a)(s-b)(s-c)} = \frac{abcs}{2[ABC]} = 2Rs,$$

and in the same way

$$[I_0I_2I_3] = 2R(s-a), \quad [I_0I_3I_1] = 2R(s-b), \quad [I_0I_1I_2] = 2R(s-c).$$

Charlotte de Bourbonstraat 2, 2628 BN Delft, The Netherlands.

* * *

MATHEMATICAL CLERIHEWS

Felix Klein
Could define
Spatial loops,
Using groups.

James Clerk Maxwell
Fused his facts well:
Physics relations,
Vector equations.

Euclid and Eudoxus,
Geometric foxes,
Probed the Plane and Space
And Incommensurable Case.

G. F. B. Riemann,
Geometric demon,
Cleverly observed
That space was curved.

ALAN WAYNE
Holiday, Florida

PASSING CARS

HAYO AHLBURG

There have always been two opposite views about mathematics. Some see it as an auxiliary science applied to practical problems in daily life, from dividing the land after the floods of the Nile in ancient Egypt to the technology involved in getting to the moon. Others see it as a pure, intellectual pursuit, a glorious endeavour of the human mind: practical applications are not the primary aim and are even considered unworthy in the lofty realm of the intellect.

David Hilbert was known to hold the second view. He is said to have been asked, before giving a talk to members of a German institute of technology, not to offend them by making disparaging remarks about the relation between mathematics and technology. So he began: "Gentlemen, I have been asked not to make remarks about the relation between mathematics and technology. Of course, I couldn't make any. There is absolutely no relation between them! (Sie haben ja gar nichts miteinander zu tun!)"

While we may not share his view if expressed in that extreme way, we can surely agree that what fascinates a mathematician is the intellectual experience: here is a problem, and you have only pencil, paper, and your brains. Find the solution!

However, there are those who want to see practical applications. "What good is number theory?" As Edmund Landau said, "Man kann damit promovieren" ("You can get a Ph.D. with it"). Many a student has been "turned off" in high school by problems like the following one, taken from a Russian textbook of the twentieth century:

Problem No. 2127. The width of a postage stamp of 7 kopecks is

$$4\frac{37}{45} : \left(\frac{\frac{2}{3}+\frac{5}{6}+\frac{8}{9}+\frac{17}{18}+\frac{35}{36}}{\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{18}+\frac{1}{36}}\right)$$

of its length, this length being 9/10 of an inch. How much would you have to pay for enough stamps to cover the wall of a room 8 m wide and $5\frac{6}{7}$ m high?

The problem is taken from V. Vereshchagin, *Textbook of Arithmetic*, St. Petersburg, 1912. I obtained this gem from my Russian father-in-law who had made a note of it, being much impressed at the time by such an exercise (but not favourably).

(The width of the stamp turns out to be 7/10 of an inch. To cover 5.857×8 m^2 would require $256 \times 450 = 115200$ stamps costing 8064 rubles.)

Problems slightly closer to life, I am inclined to believe, should hold the interest of the average student more than such an exercise, although I remember how as a student I enjoyed the following problem, not so close to life either:

A man buys 123 head of fowl and pays 456 guilders. The price of the first kind of fowl is $1\frac{2}{3}$ per head, for the second kind it is $4\frac{5}{6}$ per head, and for the third kind $7\frac{8}{9}$ per head. How many of each did he buy?

The arrangement of the digits in this problem is remarkable, all the more so since, as our teacher told us, a certain Paul Chybiorz from Silesia had invented it in 1874 in about four minutes without pencil and paper! And indeed, the Diophantine system of equations

$$x + y + z = 123$$
$$1\frac{2}{3}x + 4\frac{5}{6}y + 7\frac{8}{9}z = 456$$

does have a solution in integers: $x = 64$, $y = 38$, $z = 21$.

Now, for high school students who like to turn over in their minds questions a little "closer to life", perhaps the following ideas can add a bit of interest to their math courses.

During a vacation trip on a divided superhighway, an offhand remark by my son about passing cars started a lively conversation, soon with mathematical overtones. During our lunch stop, we continued with pencil and paper; and perhaps others have thought along the same lines, proceeding further than we did. In any case, putting this down on paper may stimulate others to "put more flesh on these bones".

If on a length L of highway we find an average of A cars per kilometer[1] in each direction (admittedly a big "if"), we have $2c^* = 2LA$ cars on that stretch of highway at any given time. On a superhighway with few, if any, obstructions, drivers tend to maintain their speed within a certain range, so we will start with this approximation: let the average speed of cars in our direction be v_1, and that in the opposite direction v_2. A significant difference between v_1 and v_2 might occur, for example, if at a certain time of day many delivery trucks leave a city, all in an outgoing direction, whereas traffic in the other direction consists mainly of faster passenger cars.

If we now drive along with an average speed v, we will pass or be passed by c_1

---

[1] I use kilometers although I understand that in Canada and in the United States the metric system is just inching along[2]. Perhaps in sympathy with the old lady who complained: "I think they should have waited until all the old people were dead." (Author)

[2] The author needs to be brought up to date. The metric system was indeed inching along when, as reported in this journal five years ago [1977: 111-112], I became Chairperson of the local GO METRIC Committee ("local" meaning right here in Ottawa, deep in the heart of taxes). Since then, thanks to my efforts, the situation has undergone a twelvefold improvement: in Canada at least, the metric system is now pussyfooting along. (Edith Orr)

cars going in our direction, depending on whether $v > v_1$ or $v < v_1$. We will also meet c_2 cars of speed v_2 going the opposite way. The numbers c_1 and c_2 both depend in general on v, v_1, v_2, L, and A.

The time t_1 the average car going our way needs to cover the distance L, and the time t we need ourselves, give us the difference

$$t_1 - t = L(\frac{1}{v_1} - \frac{1}{v}).$$

The number c_1 of cars we pass (or which pass us) in our direction is thus

$$c_1 = v_1(t_1 - t)\cdot A = LA \cdot \frac{v - v_1}{v}.$$

This is always less than LA, and becomes negative when the other cars pass us $(v < v_1)$. Cars going the other way we meet according to the relations

$$t_2 + t = L(\frac{1}{v_2} + \frac{1}{v}),$$

$$c_2 = v_2(t_2 + t)\cdot A = LA \cdot \frac{v + v_2}{v}.$$

This is always positive and greater than LA. So we meet a total of

$$c = c_1 + c_2 = 2LA + LA \cdot \frac{v_2 - v_1}{v}$$

cars along our way (where faster cars passing us are counted as negative cars).

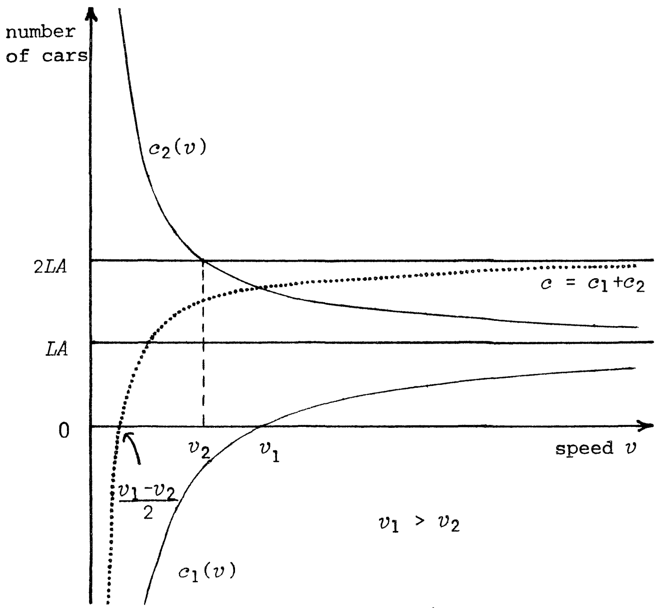


Figure 1

The graphs of $c_1(v)$ and $c_2(v)$ are branches of two hyperbolas, both with asymptote $c_1(v) \to LA$ and $c_2(v) \to LA$ as $v \to \infty$.

For $v_1 > v_2$, we have $c < 2LA$ (see Figure 1). If we drive slowly enough and $v < (v_1 - v_2)/2$, the total number c of cars met is negative, that is, more cars pass us than we meet on the other side.

For $v_1 = v_2$, the branches of hyperbolas $c_1(v)$ and $c_2(v)$ are symmetrical to $c^* = LA$, and we have $c = 2LA$. The total number of cars we meet in both directions is constant and equal to the number of cars present at any given instant along the total length of the trip, *no matter what our own speed v is.*

For $v_1 < v_2$, we have $c > 2LA$. *No matter how fast or how slow we drive ourselves*, all told we meet more cars than there are on the highway at any given instant.

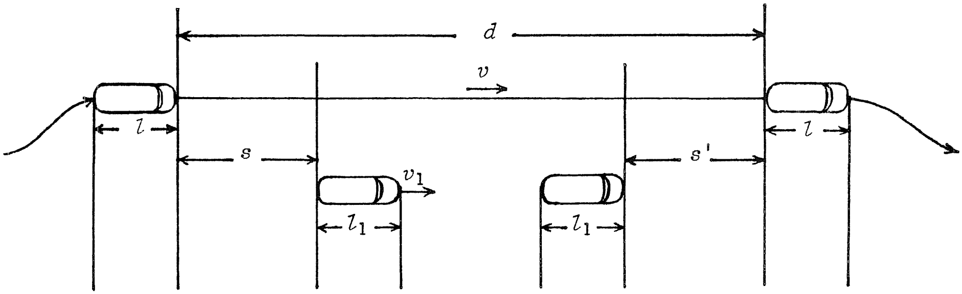


Figure 2

Let us now consider the individual event of one car passing another going in the same direction.

This begins when the faster car is still a safe distance s behind the slower car and ends when it is a safe distance s' ahead (see Figure 2). "Safe" means that at such a distance it can come from or go back into the right lane. The car lengths are l and l_1, the speeds v and $v_1 < v$, and d is the total distance cleared by the faster car while passing the slower car.

During the time T of this passing maneuver, the slower car has moved through a distance of $d - l_1 - s - s'$, while the faster car has moved ahead by $d + l$. Thus we have

$$T = \frac{d - l_1 - s - s'}{v_1} = \frac{d + l}{v},$$

from which follows

$$d = \frac{v_1 l + v(l_1 + s + s')}{v - v_1}$$

and

$$\frac{v}{v_1} = \frac{d+l}{d-l_1-s-s'}.$$

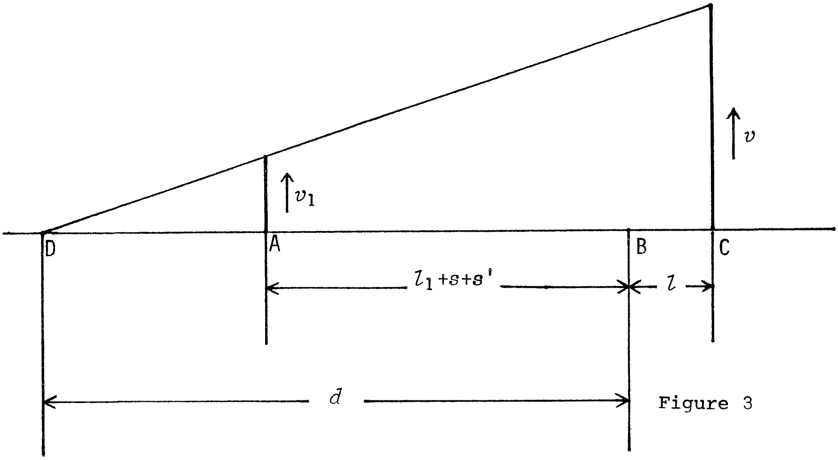


Figure 3

The equation for T leads us to Figure 3. With l, l_1, s, and s' (i.e., A,B,C) fixed on the axis of abscissas, we find the total distance d = BD, through which the faster car must stay in the passing lane, by connecting the points corresponding to speeds v_1 and v on their respective ordinates above A and C and intersecting the resulting line with the axis at D.

The figure illustrates visually what we know by instinct: the total distance traveled during the passing maneuver is vastly increased whenever the two speeds are relatively close together (v/v_1 close to 1). Obviously, an increase in the desirable safety margin $s+s'$ will also increase d.

The mathematical relations have been simplified by not taking into account acceleration, etc. Nevertheless, they give a good general picture of the situation, and of course they hold for anything from snails passing each other to rockets doing the same.

For example, we may have the following situation for two men racing each other: $l = l_1 = 0.4$ m, $v = 6$ m/sec, $v_1 = 5.4$ m/sec. With $s = s' = 0$, we already have $d \doteq 7.6$ m and $T = 1.33$ sec, while with $s = s' = 1$ m we get $d = 27.6$ m and $T = 4.67$ sec ! These figures can mean extremely tough fights between contestants at sports events.

An example for cars: $l = l_1 = 5$ m, $v = 60$ m.p.h., $v_1 = 55$ m.p.h., and $s = s' = 25$ m lead to $d = 715$ m and $T = 26.85$ sec. These "innocent" figures of nearly half a mile[1] and nearly half a minute hide quite a few accidents, whenever the available

[1] Is the metric system still inching along in Spain? (Edith Orr)

unobstructed space was less than d!

For ships and airplanes, I don't have realistic data for s, s', etc. Readers can perhaps furnish suitable values. But letting $s = s' = 0$, we have no difficulty in finding the time these vessels are actually side by side during the passing maneuver.

Apartado 35, Benidorm, Alicante, Spain.

* * *

THE OLYMPIAD CORNER: 38

M.S. KLAMKIN

Through the courtesy of Jordan B. Tabov, of the Bulgarian Academy of Sciences, I am able to give below the problems of stage IV of the XXXI Bulgarian Mathematical Olympiad. Solutions from readers would be appreciated.

May 15, 1982

1. Find all pairs of natural numbers (n,k) for which $(n+1)^k - 1 = n!$.

2. In a plane there are n circles each of unit radius. Prove that at least one of these circles contains an arc which does not intersect any of the other circles and whose length is not less than $2\pi/n$.

3. Given is a regular prism whose bases are the regular $2n$-gons $A_1A_2\ldots A_{2n}$ and $B_1B_2\ldots B_{2n}$, each with circumradius R. Prove that if the length of the edges A_iB_i varies, then the angle between the line A_1B_{n+1} and the plane through the points A_1, A_3, and B_{n+2} is maximal when $A_iB_i = 2R\cos(\pi/2n)$.

May 16, 1982

4. Let $x_1, x_2, \ldots, x_n$ be arbitrary numbers in the interval $[0,2]$. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|^2 \le n^2.$$

When is there equality?

5. Determine all values of the parameters a and b for which the polynomial

$$x^4 + (2a+1)x^3 + (a-1)^2x^2 + bx + 4$$

can be factored into a product of two polynomials $P(x)$ and $Q(x)$ of degree 2 (with leading coefficients 1) such that the equation $Q(x) = 0$ has two different roots r and s with $P(r) = s$ and $P(s) = r$.

6. Determine the set of centroids of the equilateral triangles whose vertices lie on the sides of a given square.

*

I now give the problems of the Belgian 1982 Mathematical Olympiad (15-18 yrs) and Mini-Olympiad (12-15 yrs) for which, as usual, I solicit solutions (in English or French). I obtained these problems through the courtesy of Claudine G. Festraets.

MINI-OLYMPIADE MATHÉMATIQUE BELGE 1982 - FINALE (12-15 ans)

1. Quatre verres I, II, III, IV contiennent chacun le même volume V de boisson: du vin dans le verre I, de l'eau dans les verres II, III, IV.

On verse un quart du volume de vin du verre I dans le verre II et on agite pour rendre le mélange homogène. On verse ensuite un quart du mélange obtenu du verre II dans le verre III et, après homogénéisation, on verse un quart du mélange du verre III dans le verre IV.

Que vaut le rapport des volumes de vin qui se trouvent alors dans le verre IV et dans le verre II?

2. Lequel des deux nombres

$$x = 1981(1 + 2 + 3 + 4 + \dots + 1982)$$
$$y = 1982(1 + 2 + 3 + 4 + \dots + 1981)$$

est le plus grand?

3. On choisit cinq nombres naturels a, b, c, d, e. Puis, en tournant toujours dans le même sens, on trace une ligne polygonale plane

$$P_0P_1P_2P_3P_4\dots P_{19}P_{20}$$

de telle facon que

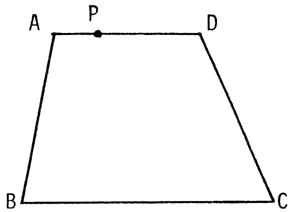
$$P_0P_1 \perp P_1P_2 \perp P_2P_3 \perp \dots \perp P_{18}P_{19} \perp P_{19}P_{20}$$

et

$$|P_0P_1| = |P_5P_6| = |P_{10}P_{11}| = |P_{15}P_{16}| = a$$
$$|P_1P_2| = |P_6P_7| = |P_{11}P_{12}| = |P_{16}P_{17}| = b$$
$$|P_2P_3| = |P_7P_8| = |P_{12}P_{13}| = |P_{17}P_{18}| = c$$
$$|P_3P_4| = |P_8P_9| = |P_{13}P_{14}| = |P_{18}P_{19}| = d$$
$$|P_4P_5| = |P_9P_{10}| = |P_{14}P_{15}| = |P_{19}P_{20}| = e.$$

Démontrer que $P_{20} = P_0$.

4. On considère un trapèze ABCD de bases AD et BC, et un point P du segment AD, comme l'indique la figure. Construire une droite passant par P et coupant le trapèze en deux parties de même aire.



OLYMPIADE MATHÉMATIQUE BELGE 1982 - FINALE (15-18 ans)

1. Deux cercles C' et C'' se coupent en deux points distincts A et B. On trace deux droites parallèles α et β passant respectivement par A et B. La droite α recoupe C' en A' et C'' en A", le point A se trouvant entre A' et A". La droite β recoupe C' en B' et C'' en B", le point B se trouvant entre B' et B".
Démontrer que le quadrilatère A'B'B"A" est un parallélogramme.

2. Quel est le plus grand nombre naturel n pour lequel le système de n inéquations

$$k < x^k < k+1 \qquad (k = 1,2,\ldots,n)$$

possède au moins une solution réelle x?

3. Les emballages des bâtons de chocolat de la marque SUPERMATH contiennent chacun la photo d'un grand mathématicien. La collection complète comporte n photos. En supposant les photos uniformément réparties dans les emballages, combien faut-il en moyenne acheter de bâtons pour rassembler une collection complète?

4. Etant donné un nombre réel $x \geq 1$, on pose

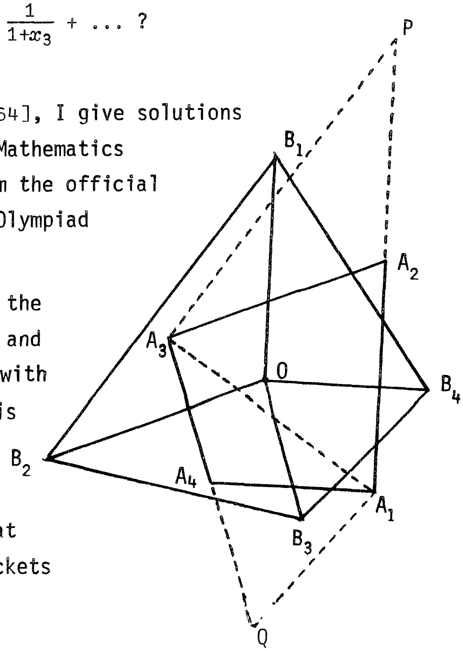
$$x_1 = x \qquad \text{et} \qquad x_{k+1} = x_k^2 + x_k \qquad (k = 1,2,3,\ldots).$$

Que vaut

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \frac{1}{1+x_3} + \ldots ?$$

*

Finally, as promised earlier [1982: 164], I give solutions for the problems set at the 1982 Canadian Mathematics Olympiad. These solutions were edited from the official ones provided by the Canadian Mathematics Olympiad Committee.



1. In the diagram [the solid lines in the adjoining figure], OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1,2,3,4$ (with $A_5 = A_1$). Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.

Solution.

Produce A_1A_2 to P and A_3A_4 to Q so that $A_2P = A_1A_2$ and $A_4Q = A_3A_4$. Then, with brackets denoting area, we have

$$[A_1A_2A_3] = [A_2PA_3] \quad \text{and} \quad [A_3A_4A_1] = [A_4QA_1].$$

Since clearly

$$\Delta OB_1B_2 \cong \Delta A_2PA_3 \quad \text{and} \quad \Delta OB_3B_4 \cong \Delta A_4QA_1,$$

we get

$$[A_1A_2A_3A_4] = [A_1A_2A_3] + [A_3A_4A_1] = [OB_1B_2] + [OB_3B_4].$$

Since we have analogously

$$[A_1A_2A_3A_4] = [OB_2B_3] + [OB_4B_1],$$

it follows that $[B_1B_2B_3B_4] = 2[A_1A_2A_3A_4]$.

Comment by M.S.K.

The above solution is simple and elegant, but some ingenuity is required to find the proper auxiliary lines upon which the solution depends. A vector solution, on the other hand, though slightly less elementary, is completely straightforward and leads to interesting related results. I give such a solution below.

Vector solution.

We take O as the origin of vectors and denote by $\vec{a}_i$ and $\vec{b}_i$ the vectors $\vec{OA}_i$ and $\vec{OB}_i$, respectively. We then have (with all subscripts reduced modulo 4)

$$\begin{aligned}
[B_1B_2B_3B_4] &= \tfrac{1}{2}|\vec{b}_1\times\vec{b}_2 + \vec{b}_2\times\vec{b}_3 + \vec{b}_3\times\vec{b}_4 + \vec{b}_4\times\vec{b}_1| \\
&= \tfrac{1}{2}|\sum_{i=1}^{4}(\vec{a}_{i+1}-\vec{a}_i)\times(\vec{a}_{i+2}-\vec{a}_{i+1})| \\
&= \tfrac{1}{2}|\sum_{\text{cyclic}}(\vec{a}_2\times\vec{a}_3 + \vec{a}_3\times\vec{a}_1 + \vec{a}_1\times\vec{a}_2)| \\
&= |\sum_{\text{cyclic}}\vec{a}_1\times\vec{a}_2| \\
&= 2[A_1A_2A_3A_4]. \quad \square
\end{aligned}$$

A similar proof would show that, if we had started with a triangle $A_1A_2A_3$ instead of a quadrilateral, then

$$[B_1B_2B_3] = 3[A_1A_2A_3],$$

and that there are no such simple results for n-gons with $n > 4$.

2. if a,b,c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,
   (i) show that a,b,c are all distinct;

(ii) show that

$$\frac{b^{1982}-c^{1982}}{b-c}+\frac{c^{1982}-a^{1982}}{c-a}+\frac{a^{1982}-b^{1982}}{a-b}$$

is an integer.

Solution.

(i) We have

$$a+b+c = 1, \qquad bc+ca+ab = -1, \qquad abc = 1. \tag{1}$$

Suppose that a,b,c are not all distinct and that, say, $b = c$. Replacing c by b and then eliminating a from the first two relations in (1), we obtain $(b-1)(3b+1) = 0$; hence

$$b = c = 1, \quad a = -1, \qquad \text{or} \qquad b = c = -\frac{1}{3}, \; a = \frac{5}{3}.$$

But in either case the third relation in (1) is not satisfied. Hence a,b,c are distinct.

(ii) Let

$$r_n = \frac{b^n-c^n}{b-c}, \qquad s_n = \frac{c^n-a^n}{c-a}, \qquad t_n = \frac{a^n-b^n}{a-b}, \qquad n \geq 1.$$

We first show that $r_{n+3} = r_{n+2} + r_{n+1} + r_n$ for $n \geq 1$. The roots b and c satisfy

$$b^3 = b^2 + b + 1 \qquad \text{and} \qquad c^3 = c^2 + c + 1;$$

hence

$$\begin{aligned} r_{n+3} &= \frac{b^{n+3}-c^{n+3}}{b-c} = \frac{b^n(b^2+b+1) - c^n(c^2+c+1)}{b-c} \\ &= \frac{b^{n+2}-c^{n+2}}{b-c} + \frac{b^{n+1}-c^{n+1}}{b-c} + \frac{b^n-c^n}{b-c} \\ &= r_{n+2} + r_{n+1} + r_n. \end{aligned}$$

Similarly, $s_{n+3} = s_{n+2} + s_{n+1} + s_n$ and $t_{n+3} = t_{n+2} + t_{n+1} + t_n$. We now show by induction that

$$r_n + s_n + t_n = \text{an integer for } n \geq 1. \tag{2}$$

It is easy to verify that

$$r_1 + s_1 + t_1 = 3,$$

$$r_2 + s_2 + t_2 = 2(a+b+c) = 2,$$

$$r_3 + s_3 + t_3 = 2(a+b+c)^2 - 3(bc+ca+ab) = 5.$$

Suppose (2) holds for $n = 1,2,\ldots,k+2$ with $k \geq 1$; then

$$r_{k+3} + s_{k+3} + t_{k+3} = (r_{k+2}+s_{k+2}+t_{k+2}) + (r_{k+1}+s_{k+1}+t_{k+1}) + (r_k+s_k+t_k)$$

is a sum of three integers by the induction assumption, and the proof of (2) is complete. In particular,

$$r_{1982} + s_{1982} + t_{1982}$$

is an integer, as required.

3. Let R^n be the n-dimensional Euclidean space. Determine the smallest number $g(n)$ of points of a set in R^n such that every point in R^n is at irrational distance from at least one point in that set.

Solution.

Clearly $g(1) > 1$, and $g(1) = 2$ follows by taking one rational point and one irrational point on the line. We now assume that $n > 1$. If we choose only two points, then there are infinitely many points at equal rational distances from both of them. Now take three points B, M, and C on a line, with BM = MC and BM^2 irrational. For any point A in R^n, we have

$$\text{AB}^2 + \text{AC}^2 - 2\text{AM}^2 = 2\text{BM}^2$$

by a well-known theorem of Apollonius. It follows that at least one of AB, AC, AM must be irrational. Hence $g(n) = 3$ for all $n > 1$.

4. Let p be a permutation of the set $S_n = \{1,2,\ldots,n\}$. An element $j \in S_n$ is called a *fixed point* of p if $p(j) = j$. Let f_n be the number of permutations having no fixed point, and g_n the number of permutations with exactly one fixed point. Show that $|f_n - g_n| = 1$.

Solution.

Let p be a permutation of S_n with exactly one fixed point j. There are n choices for j and f_{n-1} ways of defining p: $S_n-\{j\} \to S_n-\{j\}$ without additional fixed points. This proves that

$$g_n = nf_{n-1}, \qquad n \geq 2. \tag{1}$$

Now let p be a permutation of S_n with no fixed point. THen $p(1) = j$ with $j \neq 1$. There are $n-1$ choices for j. Suppose $p(j) = 1$. Then there are f_{n-2} ways of defining p: $S_n-\{1,j\} \to S_n-\{1,j\}$ without fixed points. Suppose $p(j) \neq 1$. Then there are f_{n-1} ways of defining q: $S_n-\{1\} \to S_n-\{1\}$ without fixed points. We may then define p: $S_n-\{1\} \to S_n-\{j\}$ by $p(i) = q(i)$ except that $p(i) = 1$ if $q(i) = j$. This prove that

$$f_n = (n-1)(f_{n-1} + f_{n-2}), \qquad n \geq 3. \tag{2}$$

We now prove by induction on n that

$$|f_n - g_n| = 1, \qquad n \geq 1.$$

Certainly $|f_1 - g_1| = 1$. Assume that $|f_{n-1} - g_{n-1}| = 1$ for $n \geq 2$. By (1) and (2),

$$|f_n - g_n| = |(n-1)f_{n-1} + (n-1)f_{n-2} - nf_{n-1}| = |f_{n-1} - g_{n-1}| = 1.$$

5. The altitudes of a tetrahedron ABCD are extended externally to points A', B', C', and D', respectively, where

$$AA' = k/h_a, \qquad BB' = k/h_b, \qquad CC' = k/h_c, \qquad DD' = k/h_d.$$

Here, k is a constant and h_a denotes the length of the altitude of ABCD from vertex A, etc. Prove that the centroid of the tetrahedron A'B'C'D' coincides with the centroid of ABCD.

Solution.

If any point O is taken as the origin of vectors, then

$$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 4\vec{OG},$$

where G is the centroid of ABCD, and the centroid of A'B'C'D' coincides with G if and only if

$$\vec{OA}' + \vec{OB}' + \vec{OC}' + \vec{OD}' = \vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}$$

or, equivalently,

$$\vec{AA}' + \vec{BB}' + \vec{CC}' + \vec{DD}' = \vec{0}. \tag{1}$$

The vector $\vec{AB} \times \vec{AC}$ is parallel to $\vec{DD}'$, its magnitude is $2[ABC] = 6V/h_d$, where the brackets denote area and V is the volume of ABCD, so

$$\vec{DD}' = \frac{k}{6V}(\vec{AB} \times \vec{AC}).$$

It will now be convenient to take vertex D as the origin O of vectors. Writing $\vec{a}, \vec{b}, \vec{c}$ for DA, DB, DC, respectively, we find as above that

$$\vec{AA}' = -\frac{k}{6V}(\vec{b} \times \vec{c}), \qquad \vec{BB}' = -\frac{k}{6V}(\vec{c} \times \vec{a}), \qquad \vec{CC}' = -\frac{k}{6V}(\vec{a} \times \vec{b}).$$

Hence

$$\begin{aligned} \vec{AA}' + \vec{BB}' + \vec{CC}' &= -\frac{k}{6V}(\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = -\frac{k}{6V}\{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} \\ &= -\frac{k}{6V}(\vec{AB} \times \vec{AC}) = -\vec{DD}', \end{aligned}$$

and (1) follows.

Comment by M.S.K.

This problem is a special case of the following known result: if a body is immersed in a fluid which is under constant pressure, then the sum of the forces exerted on the body by the fluid is zero. For the more sophisticated reader, this follows immediately from the integral identity

$$\int_S p\vec{N}\,dS = \int_V \nabla p\,dV$$

(derivable from the divergence theorem). Here S denotes the surface, V the volume, p is the pressure, and $\vec{N}$ is a unit vector (outwardly) normal to S. Note that the left member is the negative of the sum of all the pressure forces on S and that $\nabla p = 0$ if p is an absolute constant. Also, the pressure forces vanish if p is only constant on S.

As a rider, readers are urged to find an elementary proof of the result mentioned at the beginning of the last paragraph.

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

POSTSCRIPT TO "ON SIX CONCYCLIC POINTS"

The editor has been advised by the authors of the references given below that these references contain some of the results to be found in the article "On Six Concyclic Points", by Jan van de Craats [1982: 160], as well as several extensions and related results.

REFERENCES

1. S.N. Collings, "Cyclic Polygons and their Euler Lines", *The Mathematical Gazette*, 51 (1967) 108-114.

2. Sahib Ram Mandan, "Harmonic Chains of Equal Circles", *Scripta Mathematica*, 25 (1961) 47-64.

3. _______, "Geometric and Harmonic Chains of Equal Spheres", *Ganita*, 10 (1959) 127-140.

4. _______, "Harmonic Chains of Equal Hyperspheres", *Ganita*, 12 (1961) 15-30.

* * *

MATHEMATICAL SWIFTIES

"There's no break in that curve", Tom murmured continuously.

"All those axioms are necessary", Tom declared independently.

M.S. KLAMKIN

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

771. *Proposed by Charles W. Trigg, San Diego, California.*

The letters in the adjoining cryptarithm are in one-to-one correspondence with the ten decimal digits. What is the highest the B's can FLY with the least HUM?

$$\begin{array}{r} \text{WAX} \\ \text{HUM} \\ \underline{\text{FLY}} \\ \text{BBB} \end{array}$$

772. *Proposed by the editor.*

Find necessary and sufficient conditions on the real numbers a,b,c,d for the equation

$$z^2 + (a+bi)z + (c+di) = 0$$

to have exactly one real root.

(This is simply an exercise in proof-writing, not a great mathematical challenge. Solvers should strive for a proof that is correct, complete, concise, and linguistically as well as mathematically elegant.)

773. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Nine third-order magic squares can be combined into a ninth-order magic square. It is known (L.S. Frierson and W.S. Andrews, *Magic Squares and Cubes*, Dover, New York, 1960, p. 132) that such a ninth-order magic square can be constructed with 5 degrees of freedom, that is, with the numbers in each of the 81 cells being linear combinations of the same 5 independent variables. Prove that such a ninth-order magic square can also be constructed with 21 degrees of freedom.

774*. *Proposed by Bob Prielipp, University of Wisconsin-Oshkosh.*

Let $(G,\cdot)$ and $(G',\circ)$ both be finite groups of the same order. If, for each positive integer k, $(G,\cdot)$ and $(G',\circ)$ contain the same number of elements of order k, are the groups $(G,\cdot)$ and $(G',\circ)$ necessarily isomorphic?

775*. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

(a) For $n > 2$, let the $2n$ points $A_1, A_2, \ldots, A_{2n}$, in general position in the plane (i.e., no three collinear), be such that, for every line A_iA_j there is a line $A_kA_l \perp A_iA_j$. Prove or disprove that the $2n$ points A_i must be the vertices of a regular polygon.

(b) Conjecture and, if possible, prove an analogous result when an odd number of points are given (perhaps in 3-dimensional space).

776. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Ann watched in amazement as Sam made out the check. "I said two mugs and three plates," she reminded him.

Sam nodded. "That's right. There's no quantity discount, so it's \$4.05 for the five pieces."

"But you multiplied the two amounts instead of adding," Ann protested.

"Sure I did, lady," replied the old man. "But it made no difference to the total."

He was right! So what were the two prices?

777. *Proposed by O. Bottema, Delft, The Netherlands; and J.T. Groenman, Arnhem, The Netherlands.*

Let Q = ABCD be a convex quadrilateral with sides AB = a, BC = b, CD = c, DA = d, and area $[Q]$. The following theorem is well known: If Q has both a circumcircle and an incircle, then $[Q] = \sqrt{abcd}$.

Prove or disprove the following converse: If Q has a circumcircle and $[Q] = \sqrt{abcd}$, then there exists a circle tangent to the four lines AB, BC, CD, and DA.

778. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with incenter I, the lines AI, BI, CI meeting its circumcircle again in D, E, F, respectively. If S is the sum and P the product of the numbers

$$\frac{\mathrm{ID}}{\mathrm{AI}}, \quad \frac{\mathrm{IE}}{\mathrm{BI}}, \quad \frac{\mathrm{IF}}{\mathrm{CI}},$$

prove that $4P - S = 1$.

779. *Proposed by H. Kestelman, University College, London, England.*

Suppose $A = X + iY$, where X and Y are real square matrices. If A is invertible, show that

$$A^{-1} = \overline{A}(X^2+Y^2)^{-1}$$

if and only if X and Y commute.

If A is singular, can $X^2 + Y^2$ be invertible? If A is invertible, can $X^2 = Y^2 = 0$ (the zero matrix)?

780. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Prove that one can take a walk on Pascal's triangle, stepping from one element only to one of its nearest neighbors, in such a way that each element $\binom{m}{n}$ gets stepped on exactly $\binom{m}{n}$ times.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

638. [1981: 146; 1982: 144] A late comment was received from BIKASH K. GHOSH, Bombay, India.

* * *

651. [1981: 179; 1982: 186] *Proposed by Charles W. Trigg, San Diego, California.*

It is June, the bridal month, and LOVE is busting out all over. So THEY obey the biblical injunction to go forth and multiply, resulting paradoxically in a cryptarithmic *addition* which you are asked to investigate with averted eyes.

Find out in how many ways

$$\begin{array}{r} \text{THEY} \\ \underline{\text{MADE}} \\ \text{LOVE} \end{array}$$

and in which way their LOVE was greatest.

Editor's comment.

The published solution by Allan Wm. Johnson Jr. contained the following parenthetical sentence: (The smallest LOVE was 4596 and the average was approximately 7746.69.) Solver Johnson should not be blamed for this sentence, which contains incorrect information, because it was added by the editor, who felt that in today's world the pits of LOVE should be investigated as well as its summits.

It was Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio, who pointed out that, to arrive at the figures 4596 and 7746.69, [the editor] had thoughtlessly used only the 42 basic solutions. If all 336 solutions are considered, Kierstead found that the smallest LOVE is 4593 and that a better approximation to the average LOVE is 7745.2142857. He ended with: "And, Ms. Orr, with *that* fractional part, it has just got to be right."

To salvage some shred of dignity, the editor sheepishly observes that the digits of the more exact fractional part are those of the prime 2142857, whose rank in the sequence of primes is 158761, which is itself a prime whose rank is, oops!, 14582, not a prime. Mysterious are the ways of LOVE!

* * *

667. [1981: 205] *Proposed by Dan Sokolowsky, California State University at Los Angeles.*

A plane is determined by a line D and a point F not on D. Let C denote the conic consisting of all those points P in the plane for which PF/PQ = r, where PQ is the distance from P to D and $r > 0$ is a given real number.

Given a line L in the plane, show how to determine by elementary means the intersections (if any) of L and C.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

It is clear that L and C are disjoint if $L = D$, so we assume that $L \neq D$.

If L intersects D in T, then there is a constant k such that, for every point $P \in L$, we have $PQ = kPT$; hence $P \in L \cap C$ only if

$$PF = rPQ = rkPT. \tag{1}$$

The locus of points P satisfying (1) is an Apollonian circle γ on diameter MN, where M and N divide FT internally and externally in the ratio $rk : 1$ [1]. So the intersections of L and C are the points, if any, common to L and γ.

If L is parallel to D, then $PQ = d$, a constant, for all $P \in L$; hence $P \in L \cap C$ only if

$$PF = rPQ = rd.$$

So the intersections of L and C are the points, if any, common to L and to the circle with centre F and radius rd.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer (two solutions). A comment was received from DAN PEDOE, University of Minnesota.

Editor's comment.

Pedoe found this problem in Macaulay [2].

REFERENCES

1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 15.

2. F.S. Macaulay, *Geometrical Conics*, Cambridge University Press, 1921, p. 106.

* * *

668. [1981: 205] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

For any natural number n, let $s(n)$ be the sum of the proper divisors of n (i.e., $s(n) = \sigma(n) - n$, where $\sigma(n)$ has its usual number-theoretic meaning of sum of all positive divisors of n). A set E of natural numbers is said to be *imperfectly amicable* if $s(a) = s(b)$ for all $a,b \in E$, i.e., if $s(E) = \{k\}$ for some nonnegative integer k.

An imperfectly amicable set is said to be *maximal* if none of its proper supersets is imperfectly amicable. Find all *infinite* maximal imperfectly amicable sets of natural numbers and prove there are no others.

Solution by Andy Liu, University of Alberta.

Consider the three statements:

I. $s(n) = 0$ if and only if $n = 1$.

II. $s(n) = 1$ if and only if n is prime.

III. If the positive integer $k > 1$, then the equation $s(n) = k$ has at most a finite number of solutions n.

Statement I is obvious, and very simple proofs of II and III can be found in Sierpiński [1]. An immediate consequence of these statements is that the only infinite maximal imperfectly amicable set of natural numbers is the set of all primes.

Also solved by LEROY F. MEYERS, The Ohio State University; DAVID PLOTNICK, Rockville Center, N.Y.; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; and the proposer.

Editor's comment.

Sierpiński [1] noted that $s(n) = 2$ has no solutions and that $s(n) = 3$ has precisely one solution, $n = 4$, so $\{4\}$ is one example of a finite maximal imperfectly amicable set of natural numbers. The term "imperfectly amicable" was coined in 1823 by Thomas Taylor (see Dickson [2]).

REFERENCES

1. Wacław Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1964, p. 169, Exercises 1 and 5.

2. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. I, p. 50.

* * *

669. [1981: 238] *Proposed by Charles W. Trigg, San Diego, California.*

The digits from 1 to 9 are arranged in a 3×3 array. This square array can be considered to consist of a 2×2 square and a 5-element L-shaped gnomon in four ways. If the sums of the elements in the four corner 2×2 squares are the same, the square is said to be *gnomon-magic*, and the common sum is the magic constant. [Charles W. Trigg, "Another Type of Third-Order Magic Square", *School Science and Mathematics*, 70 (May 1970) 467.] Such a square is

9	4	8
2	1	3
6	7	5

with magic constant 16. Is there a nine-digit gnomon-magic square with four odd corner digits?

Solution by the proposer.

The central digit must be odd; otherwise two of the four 2×2 corner squares have even sums and the other two have odd sums. It follows that the magic constant, M, is even. The sum of the nine distinct digits is 45, and the sum of the four even digits is 20. Hence if, in the square

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\ , \tag{1}$$

the four 2×2 corner squares are summed separately and the results added, we get

$$4M = 45 + 20 + 3e.$$

Since M is even and e is odd, the only solution is $e = 5$, $M = 20$. If a satisfactory square (1) exists, then one of its symmetries (reflections or 90° rotations) will have $a = 1$ and $b < d$. We then immediately find, in order,

$$b = 6,\quad d = 8,\quad c = 7,\quad f = 2,\quad h = 4,\quad g = 3,\quad i = 9.$$

The unique (to within symmetry) solution is thus

$$\begin{matrix} 1 & 6 & 7 \\ 8 & 5 & 2 \\ 3 & 4 & 9 \end{matrix}\ . \quad \square$$

It will be observed that on parallels to the 3-5-7 diagonal, the odd digits appear in increasing order of magnitude along alternate parallels, and the even digits appear in decreasing order of magnitude along the other parallels. Indeed, if the 2 and 8 are interchanged, the nine digits lie in order on a continuous path on three parallels. This is the only nine-digit gnomon-magic square with a central digit of 5, and the only square in which $M/e = 4$. It is the only self-complementary square. (When each digit is subtracted from 10, the square is regenerated in one of its symmetric forms.) Furthermore, each of the 3-element corner gnomons sums to 15, so rotating the perimeter through 45° generates the well-known conventional magic square

$$\begin{matrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{matrix}\ .$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; ANDY LIU, University of Alberta; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; RAM REKHA TIWARI, Radhaur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and DAVID ZAGORSKI, student, Massachusetts Institute of Technology.

670. [1981: 238] *Proposed by O. Bottema, Delft, The Netherlands.*

The points A_i, $i = 1,2,\ldots,6$, no three of which are collinear, are the vertices of a hexagon. X_0 is an arbitrary point other than A_2 on line A_1A_2. The line through X_0 parallel to A_2A_3 intersects A_3A_4 in X_1; the line through X_1 parallel to A_3A_6 intersects A_6A_1 in X_2; the line through X_2 parallel to A_5A_6 intersects A_4A_5 in X_3; and the line through X_3 parallel to A_2A_5 intersects A_1A_2 in X_4.

(a) Prove the following closure theorem: if $X_0X_1X_2X_3X_4$ is closed (that is, if X_4 coincides with X_0) for some point X_0, then it is closed for any point X_0.

(b) Show that closure takes place if and only if the six points A_i lie on a conic.

Solution by the proposer.

(a) The range of points X_4 on A_1A_2 is projective with the range X_0 because X_4 follows from X_0 after a series of (parallel) projections. If we start from A_2, then X_1, X_2, X_3, X_4 coincide with A_3, A_6, A_5, A_2, respectively. Hence A_2 is a fixed point of the projectivity. If we start from S_{12}, the point at infinity of A_1A_2, then X_1 is the point at infinity of A_3A_4, X_2 that of A_6A_1, X_3 that of A_5A_4, and X_4 coincides with S_{12}. The projectivity on A_1A_2 therefore has at least the fixed points A_2 and S_{12}. If it has a third fixed point X_0, then every point of A_1A_2 is a fixed point.

(b) If closure takes place, let X_0 (and therefore X_1) coincide with the intersection T of A_1A_2 and A_3A_4. Then TX_2 and A_3A_6 are parallel and so are TX_3 and A_2A_5. Let S be the intersection of A_2A_5 and A_3A_6. Now consider the triangles TX_2X_3 and SA_6A_5. We have

$$TX_2 \parallel SA_6, \qquad X_2X_3 \parallel A_6A_5, \qquad X_3T \parallel A_5S.$$

In view of Desargues' Theorem, the lines TS, X_2A_6, and X_3A_5 are concurrent. So if U is the intersection of A_1A_6 and A_4A_5, it follows that T, S, and U are collinear. Consider the hexagon whose vertices in order are $A_1, A_2, A_5, A_4, A_3, A_6$. The opposite sides A_1A_2 and A_4A_3 intersect at T, A_2A_5 and A_3A_6 intersect at S, and A_5A_4 and A_6A_1 intersect at U. The desired result now follows from Pascal's Theorem and its converse. □

There is an application of this problem to kinematics. Let

$$A_1A_2,\ A_2A_3,\ A_3A_4,\ A_4A_5,\ A_5A_6,\ A_6A_1,\ A_1A_4,\ A_2A_5,\ A_3A_6$$

be nine rods, hinged at their endpoints. This mechanism is in general rigid. It may be infinitesimally deformable (*wackelig*). This means that the points A_i may be given velocities such that, for any rod A_iA_j, the velocities of A_i and A_j have equal components along the line A_iA_j. Or, equivalently, if the velocity vector $\vec{v}_i$ of A_i is rotated through a right angle into $\vec{v}_i'$, then, for each rod A_iA_j, the endpoints of $\vec{v}_i'$ and $\vec{v}_j'$ are on a line parallel to A_iA_j.

If in our case closure takes place, then there is a velocity distribution for the six vertices such that for each rod the following condition is satisfied:

$$\vec{v}_1' = \vec{0}, \quad \vec{v}_4' = \vec{0}, \quad \vec{v}_2' = A_2\vec{X}_0, \quad \vec{v}_3' = A_3\vec{X}_1, \quad \vec{v}_5' = A_5\vec{X}_3, \quad \vec{v}_6' = A_6\vec{X}_2.$$

It is well known that the said mechanism is wackelig if and only if the six vertices are on a conic. (See Wunderlich, *Ebene Kinematik*, 1969, page 143.)

Also solved by JORDI DOU, Barcelona, Spain; GEORGE TSINTSIFAS, Thessaloniki, Greece; and J.T. GROENMAN, Arnhem, The Netherlands (part (a) only).

* * *

671. [1981: 238] *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.*

The following alphametic is dedicated to the editors of the Problem Department in the *Two-Year College Mathematics Journal*: Erwin Just, Sam Greenspan, and Stan Friedlander. Given that one of the names is prime, solve

$$\begin{array}{r} \text{SAM} \\ \text{STAN} \\ \underline{\text{ERWIN}} \\ \text{PRIME} \end{array} .$$

Solution by Allan Wm. Johnson Jr., Washington, D.C.

There are twelve distinct answers in which at least one of the four words is prime, and in eight of these at least one name is prime. They are listed below in order of increasing PRIME.

(a)	(b)	(c)	(d)	(e)	(f)
805	983	983	814	891	891
8703	9084	9584	8719	8596	8796
16943	17564	17064	26509	30726	30526
26451	27631	27631	36042	40213	40213

(g)	(h)	(i)	(j)	(k)	(l)
908	983	983	980	905	905
9203	9281	9481	9283	9206	9306
41673	50471	50271	65143	71346	71246
51784	60735	60735	75406	81457	81457

SAM is prime in four answers: (b), (c), (h), (i).

STAN is prime in four answers: (d), (g), (h), (j).

Poor ERWIN is prime just once: in (a).

And PRIME is prime in six answers: (b), (c), (e), (f), (k), (l).

Partial solutions were received from CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The twelve answers given above were (presumably) generated by computer.

None of the other solvers found all eight answers in which (at least) one of the names is prime, least of all the proposer, who gave only one answer and did not explain how he obtained it. There would have been a unique answer if the proposer (had known and) had stated that the name of the editor-in-chief was prime, as in answer (a). (The proposer's answer was a different one.)

Sam, Stan, and Erwin deserved a better fate than to be immortalized in this poorly crafted alphametic. Proposers of alphametics are reminded that such problems should preferably have a unique answer, and that this answer should not *have* to be burped out by a computer.

* * *

672. [1981: 239] *Proposed by Jordi Dou, Barcelona, Spain.*

Given four points P,A,B,C in a plane, determine points A',B',C' on PA,PB, PC, respectively, such that

$$\frac{AA'}{PA'} = t\alpha, \qquad \frac{BB'}{PB'} = t\beta, \qquad \frac{CC'}{PC'} = t\gamma,$$

where α,β,γ are given constants, and such that the hexagon ABCA'B'C' is inscribed in a conic.

(This generalizes Crux 485 [1980: 256], which corresponds to the special case $\alpha = \beta = \gamma = 1$, $t = k/(k-1)$.)

Solution by O. Bottema, Delft, The Netherlands.

Given the points P,A,B,C and the constants α,β,γ, the points A',B',C' are determined when t is known. We introduce homogeneous barycentric point coordinates (x,y,z) with respect to triangle ABC; thus $Q = (x,y,z)$ if

$$x : y : z = [QBC] : [QCA] : [QAB],$$

where the brackets denote signed area.

The equation of the line at infinity is $x+y+z = 0$. Let $P = (p,q,r)$, where $s \equiv p+q+r \neq 0$. As $A = (1,0,0)$, a variable point on PA can be represented by (λ,q,r), with A corresponding to $\lambda_1 = \infty$, P to $\lambda_2 = p$, and the point S at infinity to $\lambda_3 = -(q+r)$. Let A' correspond to λ_0. The ratio AA'/PA' is equal to the cross ratio

$$(S\ A'\ P\ A) = (\lambda_3\ \lambda_0\ \lambda_2\ \lambda_1) = \frac{p+q+r}{p-\lambda_0} = \frac{s}{p-\lambda_0}.$$

Thus $t\alpha = s/(p-\lambda_0)$, so $\lambda_0 = p - s/t\alpha$ and we have

$$A' = (p-s/t\alpha,\ q,\ r).$$

Similarly, we find

$$B' = (p,\ q-s/t\beta,\ r), \qquad C' = (p,\ q,\ r-s/t\gamma).$$

A conic through A,B,C has an equation of the form

$$k_1yz + k_2zx + k_3xy = 0,$$

and the condition that it pass through A',B',C' gives three homogeneous linear equations in k_1,k_2,k_3 which have a nontrivial solution if and only if

$$\begin{vmatrix} qr & r(p-s/t\alpha) & (p-s/t\alpha)q \\ (q-s/t\beta)r & rp & p(q-s/t\beta) \\ q(r-s/t\gamma) & (r-s/t\gamma)p & pq \end{vmatrix} = 0.$$

If we subtract the first row from the second row and from the third row, and then multiply the first row by $t\alpha$, the second row by $t\alpha\beta/s$, and the third row by $t\alpha\gamma/s$, we obtain

$$\begin{vmatrix} qrt\alpha & rpt\alpha-rs & pqt\alpha-qs \\ -r\alpha & r\beta & -p\alpha+q\beta \\ -q\alpha & -p\alpha+r\gamma & q\gamma \end{vmatrix} = 0.$$

After some calculations, we obtain the equivalent equation

$$(p\alpha+q\beta+r\gamma)t - 2s = 0$$

from which, when $p\alpha+q\beta+r\gamma \neq 0$, we get

$$t = \frac{2(p+q+r)}{p\alpha+q\beta+r\gamma}, \tag{1}$$

and the points A',B',C' are determined. If $p\alpha+q\beta+r\gamma = 0$, then A',B',C' all coincide with P. □

When t has the value (1), the coefficients k_1,k_2,k_3 of the conic are determined from

$$k_1 : k_2 : k_3 = p(-p\alpha+q\beta+r\gamma) : q(p\alpha-q\beta+r\gamma) : r(p\alpha+q\beta-r\gamma).$$

The conic is degenerate if $k_1k_2k_3 = 0$. If, for example, $p\alpha = q\beta+r\gamma$, the equation of the conic is $x(\beta y+\gamma z) = 0$; the conic consists of the side BC and a line through A, and A' is the point $(0,q,r)$, the intersection of AP and BC.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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673. [1981: 239] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

Determine for which positive integers n the following property holds: if m is any integer satisfying

$$\frac{n(n+1)(n+2)}{6} \le m \le \frac{n(n+1)(2n+1)}{6},$$

then there exist permutations $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ of $(1,2,\ldots,n)$ such that

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = m.$$

(See Crux 563 [1981: 208].)

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

We denote the lower and upper bounds by

$$L_n = \frac{n(n+1)(n+2)}{6} \qquad \text{and} \qquad H_n = \frac{n(n+1)(2n+1)}{6}.$$

Without loss of generality, we assume that $a_i = i$ for $i = 1,2,\ldots,n$. A permutation $(b_1,\ldots,b_n)$ of $(1,\ldots,n)$ will be called an *n-representation* of a positive integer m if

$$1b_1 + 2b_2 + \ldots + nb_n = m,$$

and m will be called *n-representable* is such a permutation exists. Crux 563 [1981: 208] showed that L_n is n-representable and that no smaller integer is; and that H_n is n-representable and that no larger integer is. Consider the proposition

P_n: *if* $L_n \le m \le H_n$, *then* m *is n-representable.*

By an exhaustive enumeration of all permutations, it is seen that P_1, P_2, and P_4 are true, but that P_3 is false ($L_3 \le 12 \le H_3$, but 12 is not 3-representable). We now show that P_n is true for all $n > 4$. We do this by showing that P_k implies P_{k+1} for $k \ge 4$.

Given a k-representation of m, we form a corresponding $(k+1)$-representation of m in two ways.

First way. If $(b_1,\ldots,b_k)$ is a k-representation of m, then $(b_1,\ldots,b_k,k+1)$ is a $(k+1)$-representation of $m+(k+1)^2$. Thus, if P_k is true, then we can find a $(k+1)$-representation of all integers between

$$L_k + (k+1)^2 \qquad \text{and} \qquad H_k + (k+1)^2 = H_{k+1}.$$

Second way. If $(b_1,\ldots,b_k)$ is a k-representation of m, then $(b_1+1,\ldots,b_k+1,1)$ is a $(k+1)$-representation of $m+(k+1)(k+2)/2$. Thus, if P_k is true, then we can find a $(k+1)$-representation of all integers between

$$L_k + \frac{(k+1)(k+2)}{2} = L_{k+1} \qquad \text{and} \qquad H_k + \frac{(k+1)(k+2)}{2}.$$

These two methods give a $(k+1)$-representation of all integers between L_{k+1} and H_{K+1} (inclusive) if and only if

$$H_k + \frac{(k+1)(k+2)}{2} \geq L_k + (k+1)^2.$$

This inequality is equivalent to $k(k+1)(k-4) \geq 0$, which is true for all $k \geq 4$.

Thus P_n is true for all positive integers n except $n = 3$. □

If $L_n \leq m \leq H_n$, let $F_n(m)$ denote the *frequency* of the n-representations of m, that is, the number of distinct n-representations of m. We have prepared a table giving all frequencies $F_n(m)$ for $L_n \leq m \leq H_n$ and $3 \leq n \leq 10$. On the basis of this table, we make the following conjectures (which we have not attempted to prove, so some might well be very easy to establish):

$$F_n(L_n+i) = F_n(H_n-i) \tag{1}$$

for $i = 0,1,2,\ldots,n(n^2-1)/6$ and for all n. In particular,

$$F_n(L_n) = F_n(H_n) = 1, \qquad \text{for all } n; \tag{2}$$

$$F_n(L_n+1) = F_n(H_n-1) = n-1, \qquad \text{for all } n > 2; \tag{3}$$

$$F_n(L_n+2) = F_n(H_n-2) = \frac{(n-2)(n-3)}{2}, \quad \text{for all } n > 2. \tag{4}$$

Also solved by MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; and FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio.

Editor's comment.

Ecker gave a simple proof of conjecture (1), and the validity of conjecture (2) follows from the solution of Crux 563 [1981: 208]. It would be interesting to have a confirmation that conjectures (3) and (4) are valid. The last word would be to have an explicit formula, in terms of n and i, for the common value of the two sides of (1).

* * *

674. [1981: 239, 276] (Corrected) *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a given triangle and let A'B'C' be its medial triangle (A' being the midpoint of BC, etc.). The incircle of the medial triangle touches its sides in R, S, T (R being on B'C', etc.).

If the points P and Q divide the perimeter of the original triangle ABC into two equal parts, prove that the midpoint of segment PQ lies on the perimeter of triangle RST.

Solution by Howard Eves, University of Maine.

Denote by D, E, F the points of contact with BC, CA, AB of the incircle of triangle ABC, and by D', E', F' the points of contact with BC, CA, AB of the escribed circles opposite the vertices A, B, C. Now D and D', E and E', F and F' are isotomic points for the sides BC, CA, AB. Also, A and D', B and E', C and F' divide the perimeter of triangle ABC into two equal parts. Because of the similarity of triangles ABC and A'B'C', we see, by the first property, that A, R, D' are collinear, B, S, E' are collinear, and C, T, F' are collinear, whence R, S, T are the midpoints of AD', BE', CF'.

We are seeking the locus of the midpoint M of PQ as P moves around the perimeter of triangle ABC. From the above, when P is at A, B, C, the point M is at R, S, T. Since AE' = D'B = $s-c$, we see that as P moves from A to E', the point Q moves congruently from D' to B. It follows, by Hjelmslev's theorem (see, e.g., Eves [1]), that M moves along RS from R to S. Similarly, as P continues from E' to C, the point Q moves from B to F', and M moves along ST from S to T. As P continues from C to D', Q moves from F' to A, and M moves along TR from T to R. As P continues on around the perimeter of triangle ABC, M once again traces the perimeter of triangle RST. □

The incircle of the medial triangle A'B'C' is known as the *Spieker circle* of triangle ABC. The Spieker circle has many interesting properties, one of the best known being that its center is the centroid of the perimeter of triangle ABC. (See, e.g., Johnson [2].)

Also solved by O. BOTTEMA, Delft, The Netherlands; JORDI DOU, Barcelona, Spain; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer. A comment was received from J.T. GROENMAN, Arnhem, The Netherlands.

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1. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, Boston, 1972, p. 311.

2. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960, pp. 226, 249.

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675. [1981: 239] *Proposed by Harry D. Ruderman, Hunter College, New York, N.Y.*

ABCD is a skew quadrilateral and P,Q,R,S are points on sides AB,BC,CD,DA, respectively. Prove that PR intersects QS if and only if

$$AP\cdot BQ\cdot CR\cdot DS = PB\cdot QC\cdot RD\cdot SA.$$

Solution by O. Bottema, Delft, The Netherlands.

The problem is correct as stated only if "PR intersects QS" is taken to mean "P,Q,R,S are coplanar", that is, only if it does not exclude the possibility that PR be parallel to QS. In this form, the problem is a well-known space version of

Menelaus' Theorem and its converse. It appears, for example, with a very simple proof, in Altshiller-Court [1], where it is called *Carnot's Theorem* (and its converse), in a form equivalent to the following:

Let ABCD *be a skew quadrilateral and* P,Q,R,S *points on sides* AB,BC,CD,DA, *respectively. Then* P,Q,R,S *are coplanar if and only if we have, both in magnitude and sign,*

$$\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RD}\cdot\frac{DS}{SA} = 1. \tag{1}$$

If the four ratios in (1) are denoted respectively by p,q,r,s, then the condition for coplanarity is $pqrs = 1$; and for PR and QS to be parallel, we have the additional condition

$$1 + p + pq + pqr = 0.$$

We state and prove a projective generalization of Carnot's Theorem.

Let ABCD *be a skew quadrilateral in projective* 3*-space. Let* V_1 *and* V_2 *be two planes and, for* $i = 1,2$, *let* V_i *intersect* AB *in* P_i, BC *in* Q_i, CD *in* R_i, *and* DA *in* S_i. *If* e_i, $i = 1,2,3,4$, *are the cross ratios*

$$e_1 = (A\ B\ P_1\ P_2),\qquad e_2 = (B\ C\ Q_1\ Q_2),\qquad e_3 = (C\ D\ R_1\ R_2),\qquad e_4 = (D\ A\ S_1\ S_2),$$

then

$$e_1e_2e_3e_4 = 1. \tag{2}$$

Proof. We introduce homogeneous projective point coordinates (x,y,z,w) with

$$A = (1,0,0,0),\qquad B = (0,1,0,0),\qquad C = (0,0,1,0),\qquad D = (0,0,0,1).$$

For $i = 1,2$, let the equation of V_i be $a_ix+b_iy+c_iz+d_iw = 0$. For AB we have $z = w = 0$; so $P_1 = (-b_1,a_1,0,0)$, $P_2 = (-b_2,a_2,0,0)$, and therefore

$$e_1 = \frac{a_1b_2}{a_2b_1}.$$

In the same way, we find

$$e_2 = \frac{b_1c_2}{b_2c_1},\qquad e_3 = \frac{c_1d_2}{c_2d_1},\qquad e_4 = \frac{d_1a_2}{d_2a_1},$$

and (2) follows. □

Carnot's Theorem is the special case when V_2 is the plane at infinity. The theorem can be generalized to spaces of n dimensions.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

REFERENCE

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Second Edition, Chelsea, New York, 1964, p. 127.

* * *

676. [1981: 240] *Proposed by William Moser, McGill University, Montréal, Québec.*

Gertie, the secretary, was so angry with her boss that she maliciously put every one of the six letters she had typed into an envelope addressed to one of the others. In fact Drab received the letter of the man to whom Gertie had mailed Crumb's letter; and the man whose letter was sent to Fatso received the letter of the man to whom Epsilon's letter was mailed. She did not mail to Axworthy the letter of the man to whom she mailed Bilk's letter. Whose letter did Crumb receive?

(I found this problem among the papers of my late brother Leo Moser (1921-1970). There was nothing to indicate prior publication.)

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

Identify each person with the first letter of his name. Define the *Gertie function*, g, as follows: if $x \in \{A,B,C,D,E,F\}$, then $g(x)$ is the person who receives the letter meant for x. The given information can then be translated into the following equations

$$g(x) \neq x, \quad g(g(C)) = D, \quad g(g(g(E))) = F, \quad g(g(B)) \neq A.$$

The Gertie function induces a permutation on the set $\{A,B,C,D,E,F\}$. Since no element maps into itself, the cycle structure of the permutation must be of one of the forms

$$(12)\ (34)\ (56), \tag{a}$$
$$(123)\ (456), \tag{b}$$
$$(12)\ (3456), \tag{c}$$
$$(123456). \tag{d}$$

Cycle structure (a) is ruled out because $g(g(C))$ would then be C, not D.

Cycle structure (b) is ruled out because $g(g(g(E)))$ would then be E, not F.

We show that cycle structure (c) is not possible. Suppose E were in the 2-cycle. Then the permutation would be of the form $(EF)\ (C4D6)$ with 4 and 6 in $\{A,B\}$. This is a contradiction because $g(g(B))$ would then equal A. Now suppose E were in the 4-cycle. This cycle would then be of the form $(E45F)$. But C and D must be two apart, and this could not happen if $\{C,D\} = \{4,5\}$ or $\{1,2\}$.

Thus the cycle structure must be of the form $(E23F56)$. Since C and D must be two apart, we must have

$$(C,D) = (3,5) \quad \text{or} \quad (6,2)$$

and the corresponding values of A and B are given by

$$(A,B) = (6,2) \quad \text{or} \quad (3,5).$$

(Note that $(A,B) = (2,6)$ or $(5,3)$ are ruled out by the condition $g(g(B)) \neq A$.)

There are therefore two possible cycles,

$$(EBCFDA) \quad \text{and} \quad (EDAFBC),$$

and in each case $g^{-1}(C) = B$, so Crumb received Bilk's letter.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; GAIL A. EISNER, McLean, Virginia; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; ANDY LIU, University of Alberta; MAHESH KUMAR SANGANERIA, Midnapore, West Bengal, India; KYOKO SASAKI (student) and STAN WAGON, Smith College, Northampton, Massachusetts (jointly); ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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The following publications of the Department of Mathematics and Statistics of Carleton University, Ottawa, may be of interest to readers of this journal:

1. *The Mathematics Calendar* for the academic year July 1982 - June 1983. This is a monthly fold-down (wall) calendar which contains monthly problems, important birth dates, general mathematical information, and much more. Price $2.00.

2. *Problems with Selected Solutions from Carleton University Annual Mathematics Competitions for High School Students 1973-1982.* This 126-page softbound book was edited by Dr. Kenneth S. Williams. Price $5.00.

These publications can be obtained by writing to

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Ottawa, Ontario
Canada K1S 5B6

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THE PUZZLE CORNER

Puzzle No. 22: Alphametic

First find the right NUTCRACKER: Deduce its square root.
If you've done it correctly, you will now have the fruit
Of an 057. Can you guess what it is?
That would be of some help if you're solving this quiz.

HANS HAVERMANN, Weston, Ontario

Answer to Puzzle No. 17 [1982: 158, 163] (Correction): ELEVEN could *also* be 202621.

Answer to Puzzle No. 20 [1982: 208]: Dissolution (*D* is solution).

Answer to Puzzle No. 21 [1982: 208]: Mi*nute*, *min*ute.