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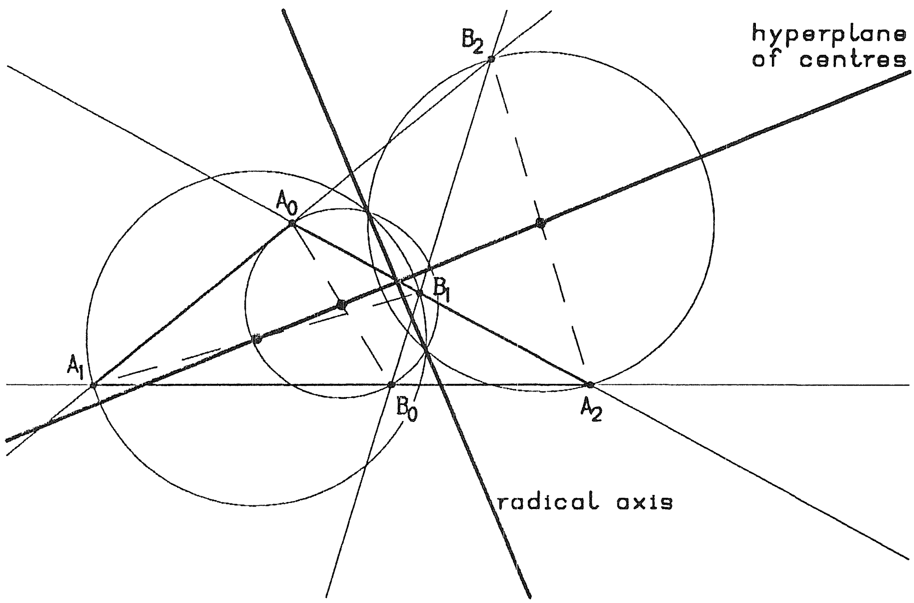
# AN $n$-DIMENSIONAL BODENMILLER THEOREM

Rudolf Fritsch

The following is an answer to Chris Fisher's query in [2]: Is there an $n$-dimensional version of Bodenmiller's Theorem? Yes, there is one, but not as Fisher conjectured. In order to make this note self-contained we repeat some facts from [2], [3], [4].




Bodenmiller's Theorem using the notation of Theorems 1 and 2

The original version of Bodenmiller's Theorem states that the three circles with the diagonals of a complete quadrilateral as diameters intersect in the same two points; more precisely, they belong to a pencil of coaxal circles. As a consequence, the midpoints of these diagonals are collinear, a fact already known to Newton and Gauss. The connection to Bodenmiller's Theorem has been exhibited by Schlömilch and Möbius; for precise references to its history see [3]. In some textbooks on geometry, e.g. [7, p. 159] it may be found in the following form:

GAUSS'S THEOREM: *If the sides (suitably extended) of triangle $A_0A_1A_2$ are cut by a straight line in three distinct points $B_0 \in A_1A_2$, $B_1 \in A_2A_0$, $B_2 \in A_0A_1$, then the midpoints of the line segments $[A_i, B_i]$, $i \in \{0,1,2\}$, are collinear.* □

This theorem has been generalized to $n$-space as a problem posed by Murray S. Klamkin:

**THEOREM 1:** *Let $\sigma = A_0A_1A_2\ldots A_n$ be an $n$-simplex and let $\mathcal{A}_i$ denote the hyperplane containing the $(n-1)$-dimensional face of $\sigma$ opposite the vertex $A_i$, $i \in \{0,1,\ldots,n\}$. If these hyperplanes are cut by a straight line in $n+1$ distinct points $B_i \in \mathcal{A}_i$, $i \in \{0,1,\ldots,n\}$, then the midpoints of the line segments $[A_i, B_i]$, $i \in \{0,1,\ldots,n\}$, lie in the same hyperplane.*

Ivan Paasche's solution [6] was translated into English for *Crux* [2]. In a comment appended to that solution, Fisher asked if this theorem can be obtained in general — as in the case $n = 2$ — from a *"Bodenmiller Theorem"*. More precisely, do the hyperspheres having the segments $[A_i, B_i]$ as diameters intersect in the same $(n-2)$-sphere? The answer is *no*, but there is a positive answer in terms of a collection of hyperspheres (defined below) that we shall call, for want of a standard terminology (cf. [5] page 34), a *system of coaxal spheres*:

**THEOREM 2:** *Under the assumptions of Theorem 1, the spheres $S_i$ having the line segments $[A_i, B_i]$ as diameters, $i \in \{0,1,\ldots,n\}$, belong to a system of coaxal spheres.*

Theorem 2 implies Theorem 1, as will be shown later on. One geometric meaning of Theorem 2 is the following: If $n$ of the $n+1$ spheres $S_i$ have two points in common, then the remaining sphere also passes through these two points.

Theorem 2 is true in a very general setting. Let $K$ be a field, $V$ a $K$-vector space with $\dim_K V = n < \infty$, and $<,>$ a nondegenerate, symmetric bilinear form

$$<,>: V \times V \longrightarrow K, \quad (\mathbf{x},\mathbf{y}) \mapsto <\mathbf{x},\mathbf{y}> = \mathbf{x}\cdot\mathbf{y}.$$

We consider a *sphere* to be a formal polynomial

$$P = P(\mathbf{x}) = \mathbf{x}^2 - \mathbf{c}\cdot\mathbf{x} + d$$

with $\mathbf{c} \in V$ and $d \in K$. For example, in the Euclidean plane a circle with centre $(c_1/2, c_2/2)$ and radius $r$ has $\mathbf{c} = (c_1, c_2)$ and $d = c_1^2/4 + c_2^2/4 - r^2$: any point $\mathbf{x} = (x, y)$ on the circle satisfies

$$x^2 + y^2 - (c_1x + c_2y) + \frac{c_1^2}{4} + \frac{c_2^2}{4} - r^2 = \left(x - \frac{c_1}{2}\right)^2 + \left(y - \frac{c_2}{2}\right)^2 - r^2 = 0.$$

Although this circle is the zero set of the function $P(\mathbf{x})$, from our general point of view distinct polynomials having the same zero sets are to be considered as distinct spheres; in particular, this applies to polynomials with empty or one-point zero sets.

The 2-set $\{\mathbf{a},\mathbf{b}\}$ is called a *diameter* of the sphere $P$ if $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $d = \mathbf{a}\cdot\mathbf{b}$. The first condition places the centre — if it exists (see the final paragraph) — at $\mathbf{c}/2$; the second condition forces $\mathbf{a}$ and $\mathbf{b}$ to satisfy the equation $P(\mathbf{x}) = 0$. Thus a sphere with diameter $\{\mathbf{a},\mathbf{b}\}$ is the locus of points $\mathbf{x}$ such that triangle $\mathbf{axb}$ has a right angle at $\mathbf{x}$:

$$0 = (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b}) = \mathbf{x}^2 - (\mathbf{a}+\mathbf{b})\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{b}.$$

A set $\mathcal{P}$ of spheres is called an *(ordinary) system of coaxal spheres* or, more briefly, a *coaxal system* if there are spheres $P_i = \mathbf{x}^2 - \mathbf{c}_i \cdot \mathbf{x} + d_i \in \mathcal{P}$, $i \in \{1, \dots, n\}$ such that the family of vectors $(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is affinely independent[1] and

$$\mathcal{P} = \left\{ P := \mathbf{x}^2 - \left( \sum_{i=1}^{n} t_i \mathbf{c}_i \right) \cdot \mathbf{x} + \sum_{i=1}^{n} t_i d_i \mid t_1, \dots, t_n \in K, \sum_{i=1}^{n} t_i = 1 \right\}.$$

(When $n = 2$ the family is usually called a *pencil of coaxal circles*, cf. [1], section 2.3; when $n = 3$ it is a *bundle of coaxal spheres*.) A family $(P_1, \dots, P_n)$ of spheres that define the system $\mathcal{P}$ is called a *set of generators* for $\mathcal{P}$; the solution set of the system of equations

$$(\mathbf{c}_i - \mathbf{c}_1) \cdot \mathbf{x} = d_i - d_1, \quad i \in \{2, \dots, n\}$$

is a straight line, called the *(radical) axis* of $\mathcal{P}$. The axis of a system is independent of the choice of generators.

Now we turn to a proof of Theorem 2. Let $\sigma$ be the given $n$-simplex — an affinely independent family $\sigma = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n) \in V^{n+1}$. For $i \in \{0, 1, \dots, n\}$ the *face* $\mathcal{A}_i$ is the hyperplane containing the vertices $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_n$. Without loss of generality we assume $\mathbf{a}_0 = 0$, implying the family $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ to be a basis of $V$. Also, let

$$\mathcal{L} = \{\tilde{\mathbf{b}} + s\mathbf{b} \mid s \in K\}$$

be a straight line cutting the faces $\mathcal{A}_i$ in $n+1$ distinct points $\mathbf{b}_i$, $i \in \{0, \dots, n\}$. Also without loss of generality, we assume $\tilde{\mathbf{b}}, \mathbf{b} \in \mathcal{A}_0$, so that

$$\tilde{\mathbf{b}} = \mathbf{b}_0 = \sum_{i=1}^{n} r_i \mathbf{a}_i, \quad \mathbf{b} = \sum_{i=1}^{n} t_i \mathbf{a}_i,$$

with

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} t_i = 1.$$

Then we have

$$\mathbf{b}_i = \mathbf{b}_0 + s_i \mathbf{b}$$

with

$$r_i + s_i \cdot t_i = 0$$

for all $i \in \{1, \dots, n\}$, all $r_i$, $s_i$, $t_i$ being different from zero.

Next, we have to consider the spheres

$$P_0 = \mathbf{x}^2 - \mathbf{b}_0 \cdot \mathbf{x} \quad \text{and} \quad P_i = \mathbf{x}^2 - (\mathbf{a}_i + \mathbf{b}_i) \cdot \mathbf{x} + \mathbf{a}_i \cdot \mathbf{b}_i, \quad i \in \{1, \dots, n\};$$

we claim that the family $(P_1, \dots, P_n)$ generates a coaxal system containing $P_0$.

[1] This is the case if and only if the vectors of the family $(\mathbf{c}_2 - \mathbf{c}_1, \dots, \mathbf{c}_n - \mathbf{c}_1)$ are linearly independent.

To prove this, compute

$$(\mathbf{a}_i+\mathbf{b}_i)-(\mathbf{a}_1+\mathbf{b}_1) = \mathbf{a}_i-\mathbf{a}_1+(s_i - s_1)\mathbf{b}$$

for $i \in \{2,\dots,n\}$. In order to show the required linear independence consider the equation

$$\sum_{i=2}^{n} u_i(\mathbf{a}_i-\mathbf{a}_1+(s_i - s_1)\mathbf{b}) = 0$$

with $u_i \in K$, not all $u_i$ equal zero. As a consequence of the linear independence of the family $(\mathbf{a}_2-\mathbf{a}_1,\dots,\mathbf{a}_n-\mathbf{a}_1)$,

$$\sum_{i=2}^{n} u_i \cdot (s_i - s_1) \neq 0 .$$

Thus, we may assume

$$\sum_{i=2}^{n} u_i \cdot (s_i - s_1) = -1 ,$$

so that

$$\mathbf{b} = \sum_{i=2}^{n} u_i(\mathbf{a}_i-\mathbf{a}_1) = \sum_{i=2}^{n} u_i\mathbf{a}_i-\left(\sum_{i=2}^{n} u_i\right)\mathbf{a}_1 .$$

This implies

$$t_1 = -\sum_{i=2}^{n} u_i , \qquad t_i = u_i , \quad i \in \{2,\dots,n\} ,$$

yielding the contradiction

$$\sum_{i=1}^{n} t_i = 0 .$$

It remains to be shown that $P_0$ belongs to the coaxal system generated by the family $(P_1,\dots,P_n)$. As a matter of fact,

$$\begin{aligned}\sum_{i=1}^{n} t_i(\mathbf{a}_i+\mathbf{b}_0+s_i\mathbf{b}) &= \sum_{i=1}^{n} t_i\mathbf{a}_i+\left(\sum_{i=1}^{n} t_i\right)\mathbf{b}_0+\left(\sum_{i=1}^{n} t_i s_i\right)\mathbf{b} \\ &= \mathbf{b}+\mathbf{b}_0-\left(\sum_{i=1}^{n} r_i\right)\mathbf{b} = \mathbf{b}_0,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} t_i\mathbf{a}_i\cdot(\mathbf{b}_0+s_i\mathbf{b}) &= \left(\sum_{i=1}^{n} t_i\mathbf{a}_i\right)\cdot\mathbf{b}_0+\left(\sum_{i=1}^{n} t_i s_i\mathbf{a}_i\right)\cdot\mathbf{b} \\ &= \mathbf{b}\cdot\mathbf{b}_0-\left(\sum_{i=1}^{n} r_i\mathbf{a}_i\right)\cdot\mathbf{b} = 0 ,\end{aligned}$$

which proves Theorem 2. □

In order to derive Theorem 1 from Theorem 2 we need a further assumption. The characteristic of the underlying field $K$ must be different from 2, because otherwise we would not be able to find midpoints of line segments and centres of spheres. Under this assumption the centre of the sphere $P = \mathbf{x}^2 - \mathbf{c} \cdot \mathbf{x} + d$ is the vector $1/2 \cdot \mathbf{c}$. Given an $n$-simplex $\sigma = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n) \in V^{n+1}$ and a straight line $\mathcal{L}$ cutting the faces $\mathcal{A}_i$ in distinct points $\mathbf{b}_i$, Theorem 2 establishes that the family $(\mathbf{a}_0+\mathbf{b}_0, \mathbf{a}_1+\mathbf{b}_1, \ldots, \mathbf{a}_n+\mathbf{b}_n)$ is affinely dependent (i.e., belongs to a hyperplane). The dilatation centred at the origin with ratio $1/2$ transforms hyperplanes into hyperplanes; thus, the centres of the spheres under consideration — agreeing with the midpoints of the line segments in the classical situation — belong to a hyperplane. □

*References:*


[1] H. S. M. Coxeter and S. Greitzer, *Geometry Revisited*, Math. Assoc. of America (New Mathematical Library 19), 1975.

[2] Chris Fisher, Editor's comment to problem 1898, *Crux Mathematicorum* **20** (1994), pp. 265–266.

[3] Rudolf Fritsch, Gudermann, Bodenmiller und der Satz von Bodenmiller-Steiner, *Didaktik der Mathematik* **20** (1992), pp. 165–187.

[4] Rudolf Fritsch, Remarks on Bodenmiller's Theorem, *Journal of Geometry* **47** (1993), pp. 23–31.

[5] Roger A. Johnson, *Advanced Euclidean Geometry*, Dover Reprint, 1960.

[6] Ivan Paasche, Lösung von Aufgabe 733, *Elemente der Mathematik* **31** (1976), p. 15.

[7] Zalman Alterovich Skopets, *Geometricheskie Miniatyury*, Prosvewenie, Moscow, 1990.



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# THE SKOLIAD CORNER

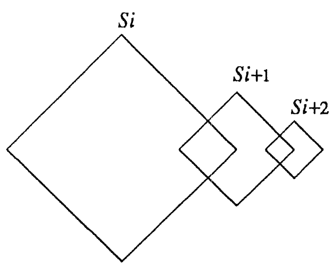
## No. 4

R. E. WOODROW

The contest we give this month is the 1995 A.I.M.E., which has been a regular feature as a pre-Olympiad contest of the Corner for many years. The American Invitational Mathematics Examination was written Thursday, March 23, 1995 and its problems are copyrighted by the Committee on the American Mathematical Competitions of the Mathematical Association of America, and they may not be reproduced without permission. The numerical solutions only will be published next issue. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588-0322.

## AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

### Thursday, March 23, 1995



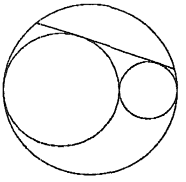

**1.** Square $S_1$ is $1 \times 1$. For $i \geq 1$, the lengths of the sides of square $S_{i+1}$ are half the lengths of the sides of square $S_i$, two adjacent sides of square $S_i$ are perpendicular bisectors of two adjacent sides of square $S_{i+1}$, and the other two sides of square $S_{i+1}$ are the perpendicular bisectors of two adjacent sides of square $S_{i+2}$. The total area enclosed by at least one of $S_1$, $S_2$, $S_3$, $S_4$, $S_5$ can be written in the form $m/n$, where $m$ and $n$ are relatively prime positive integers. Find $m - n$.

**2.** Find the last three digits of the product of the positive roots of

$$\sqrt{1995}\, x^{\log_{1995} x} = x^2.$$

**3.** Starting at (0,0), an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is left, right, up, or down, all four equally likely. Let $p$ be the probability that the object reaches $(2,2)$ in six or fewer steps. Given that $p$ can be written in the form $m/n$, where $m$ and $n$ are relatively prime positive integers, find $m + n$.

**4.** Circles of radius 3 and 6 are externally tangent to each other and are internally tangent to a circle of radius 9. The circle of radius 9 has a chord that is the common external tangent of the other two circles. Find the square of the length of this chord.



**5.** For certain real values of $a$, $b$, $c$, and $d$, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find $b$.

**6.** Let $n = 2^{31}3^{19}$. How many positive integer divisors of $n^2$ are less than $n$ but do not divide $n$?

**7.** Given that $(1 + \sin t)(1 + \cos t) = 5/4$ and

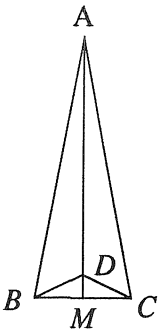
$$(1 - \sin t)(1 - \cos t) = \frac{m}{n} - \sqrt{k},$$

where $k$, $m$, and $n$ are positive integers with $m$ and $n$ relatively prime, find $k + m + n$.

**8.** For how many ordered pairs of positive integers $(x, y)$, with $y < x \le 100$, are both

$$\frac{x}{y} \quad \text{and} \quad \frac{x+1}{y+1}$$

integers?




**9.** Triangle $ABC$ is isosceles, with $AB = AC$ and altitude $AM = 11$. Suppose that there is a point $D$ on $\overline{AM}$ with $AD = 10$ and $\angle BDC = 3\angle BAC$. Then the perimeter of $\triangle ABC$ may be written in the form $a + \sqrt{b}$, where $a$ and $b$ are integers. Find $a + b$.

**10.** What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a positive composite integer?

**11.** A right rectangular prism $P$ (i.e., a rectangular parallelepiped) has sides of integral length $a$, $b$, $c$, with $a \le b \le c$. A plane parallel to one of the faces of $P$ cuts $P$ into two prisms, one of which is similar to $P$, and both of which have nonzero volume. Given that $b = 1995$, for how many ordered triples $(a, b, c)$ does such a plane exist?

**12.** Pyramid $OABCD$ has square base $ABCD$, congruent edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$, and $\angle AOB = 45°$. Let $\theta$ be the measure of the dihedral angle formed by faces $OAB$ and $OBC$. Given that $\cos\theta = m + \sqrt{n}$, where $m$ and $n$ are integers, find $m + n$.

**13.** Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find $\sum_{k=1}^{1995} \frac{1}{f(k)}$.

**14.** In a circle of radius 42, two chords of length 78 intersect at a point whose distance from the center is 18. The two chords divide the interior of the circle into four regions. Two of these regions are bordered by segments of unequal lengths, and the area of either of them can be expressed uniquely in the form $m\pi - n\sqrt{d}$, where $m$, $n$, and $d$ are positive integers and $d$ is not divisible by the square of any prime number. Find $m + n + d$.

**15.** Let $p$ be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Given that $p$ can be written in the form $m/n$, where $m$ and $n$ are relatively prime positive integers, find $m + n$.

* * *

Last issue we gave the problems of the Preliminary Round, 1994 Senior Mathematics Contest of the University College of the Cariboo. Below we give the answers. Winners were invited to write the Final Round, as well as to participate in an afternoon of mathematical activity and an awards dinner held at the University.

| | | | |
|---|---|---|---|
| 1. *b* | 5. *e* | 9. *a* | 13. *d* |
| 2. *d* | 6. *c* | 10. *d* | 14. *c* |
| 3. *d* | 7. *d* | 11. *d* | 15. *a* |
| 4. *a* | 8. *a* | 12. *d* | |

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That completes the Skoliad Corner for this issue. Send me your pre-Olympiad contests, as well as comments and suggestions for the future of this feature.

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# THE OLYMPIAD CORNER

## No. 164

R. E. WOODROW

*All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

The Olympiad Contest we give this number is one of several collected by Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, when he was Canadian Team Leader to the I.M.O. at Istanbul. Many thanks for collecting the materials!

# TURKISH MATHEMATICAL OLYMPIAD COMMITTEE FINAL SELECTION TEST

## April 4, 1993 — Part I

(Time: 3 hours)

**1.** Show that there is an infinite sequence of positive integers such that the first term is 16, the number of distinct positive divisors of each term is divisible by 5, and the terms of the sequence form an arithmetic progression. Of all such sequences, find the one with the smallest possible common difference between consecutive terms.

**2.** Let $M$ be the circumcenter of an acute-angled triangle $ABC$, and assume the circle $(BMA)$ intersects the segment $[BC]$ at $P$, and the segment $[AC]$ at $Q$. Show that the line $CM$ is perpendicular to the line $PQ$.

**3.** Let $(b_n)$ be sequence of positive real numbers such that

$$\text{for each } n \geq 1, \quad b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}.$$

Show that there is a natural number $K$ such that

$$\sum_{n=1}^{K} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} > \frac{1993}{1000}.$$

## April 4, 1993 — Part II

(Time: 3 hours)

**1.** Some towns are connected to each other by some roads with at most one road between any pair of towns. Let $v$ denote the number of towns, and $e$ the number of roads. Show that

(a) if $e < v-1$, then there are at least two towns such that it is impossible to travel from one to the other,

(b) if $2e > (v-1)(v-2)$, then travelling between any pair of towns is possible.

**2.** On a semicircle with diameter $AB$ and center $O$ points $E$ and $C$ are marked in such a way that $OE$ is perpendicular to $AB$, and the chord $AC$

intersects the segment $OE$ at a point $D$ which is interior to the semicircle. Find all values of the angle $\angle CAB$ such that a circle can be inscribed into the quadrilateral $OBCD$.

**3.** Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$\text{for every } x, y \in \mathbb{Q}^+, \quad f\left(x + \frac{y}{x}\right) = f(x) + \frac{f(y)}{f(x)} + 2y.$$

* * *

Last number we gave six Klamkin Quickies, promising to give the remaining six this number. Here they are for your "speedy entertainment".

## KLAMKIN QUICKIES

**7.** Determine all integral solutions of the Diophantine equation

$$(x^8 + y^8 + z^8) = 2(x^{16} + y^{16} + z^{16}).$$

**8.** Determine all the roots of the quintic equation

$$31x^5 + 165x^4 + 310x^3 + 330x^2 + 155x + 33 = 0.$$

**9.** If $F(x)$ and $G(x)$ are polynomials with integer coefficients such that $F(k)/G(k)$ is an integer for $k = 1, 2, 3 \ldots$, prove that $G(x)$ divides $F(x)$.

**10.** Given that $ABCDEF$ is a skew hexagon such that each pair of opposite sides are equal and parallel. Prove that the midpoints of the six sides are coplanar.

**11.** If $a$, $b$, $c$, $d$ are the lengths of sides of a quadrilateral, show that

$$\frac{\sqrt{a}}{(4 + \sqrt{a})}, \quad \frac{\sqrt{b}}{(4 + \sqrt{b})}, \quad \frac{\sqrt{c}}{(4 + \sqrt{c})}, \quad \frac{\sqrt{d}}{(4 + \sqrt{d})},$$

are possible lengths of sides of another quadrilateral.

**12.** Determine the maximum value of the sum of the cosines of the six dihedral angles of a tetrahedron.

* * *

Next we give Klamkin's short answers to the six Quickies used last issue. Many thanks go to Murray S. Klamkin, The University of Alberta, for originating the Corner and continuing to give it his support.

**1.** Are there any integral solutions $(x, y, z)$ of the Diophantine equation

$$(x - y - z)^3 = 27xyz$$

other than $(-a, a, a)$ or such that $xyz = 0$?

*Solution.* Let $x = u^3$, $y = v^3$, $z = w^3$, so that $u^3 - v^3 - w^3 = 3uvw$ or equivalently

$$(u - v - w)((u + v)^2 + (u + w)^2 + (v - w)^2) = 0.$$

Hence an infinite class of non-trivial solutions is given by

$$x = (v + w)^3, \quad y = v^3, \quad z = w^3.$$

Whether or not there are any other solutions is an open problem.

**2.** Does the Diophantine equation

$$(x - y - z)(x - y + z)(x + y - z) = 8xyz$$

have an infinite number of relatively prime solutions?

*Solution.* By inspection, we have the trivial solutions

$$(x, y, z) = (\pm 1, \pm 1, 0)$$

and permutations thereof. For other solutions, note that each of the three equations

$$x - y - z = 2\sqrt{yz}, \quad x - y + z = 2\sqrt{xz}, \quad x + y - z = 2\sqrt{xy}$$

is satisfied by $\sqrt{x} = \sqrt{y} + \sqrt{z}$. Consequently, we also have the infinite set of solutions

$$y = m^2, \quad z = n^2, \quad x = (m + n)^2 \quad \text{where } (m, n) = 1.$$

It is an open problem whether or not there are any other infinite sets of relatively prime solutions.

**3.** It is an easy result using calculus that if a polynomial $P(x)$ is divisible by its derivative $P'(x)$, then $P(x)$ must be of the form $a(x - r)^n$. Starting from the known result that

$$\frac{P'(x)}{P(x)} = \sum \frac{1}{x - r_i}$$

where the sum is over all the zeros $r_i$ of $P(x)$ counting multiplicities, give a non-calculus proof of the above result.

*Solution.* Since $P'(x)$ is of degree one less than that of $P(x)$,

$$\frac{P'(x)}{P(x)} = \frac{1}{a(x - r)} = \sum \left(\frac{1}{x - r_i}\right).$$

Now letting $x \to$ any $r_i$ it follows that $r = r_i$. Hence all the zeros of $P(x)$ must be the same.

**4.** Solve the simultaneous equations

$$x^2(y+z) = 1, \quad y^2(z+x) = 8, \quad z^2(x+y) = 13.$$

*Solution.* More generally we can replace the constants 1, 8, 13 by $a^3$, $b^3$, $c^3$, respectively. Then by addition of the three equations and by multiplication of the three equations, we respectively get

$$\sum x^2 y = a^3 + b^3 + c^3,$$

$$(xyz)^2\left[2xyz + \sum x^2 y\right] = (abc)^3,$$

where the sums are symmetric over $x$, $y$, $z$. Hence,

$$2t^3 + t^2(a^3 + b^3 + c^3) = (abc)^3 \tag{1}$$

where $t = xyz$. In terms of $t$, the original equations can be rewritten as

$$\frac{a^3}{tx} - \frac{1}{y} - \frac{1}{z} = 0, \quad \frac{-1}{x} + \frac{b^3}{ty} - \frac{1}{z} = 0, \quad \frac{-1}{x} - \frac{1}{y} + \frac{c^3}{tz} = 0.$$

These latter homogeneous equations are consistent since the eliminant is equation (1). Solving the last two equations for $y$ and $z$, we get

$$y = \frac{x(b_1c_1 - 1)}{c_1 + 1}, \qquad z = \frac{x(b_1c_1 - 2)}{b_1 + 1}$$

where $b_1 = b^3/t$, $c_1 = c^3/t$. On substituting back in $x^2(y+z) = a^3$, we obtain $x^3$ and then $x$, $y$, $z$.

**5.** Determine the area of a triangle of sides $a$, $b$, $c$ and semiperimeter $s$ if

$$(s-b)(s-c) = a/h, \quad (s-c)(s-a) = b/k, \quad (s-a)(s-b) = c/l,$$

where $h$, $k$, $l$, are consistent given constants.

*Solution.*

$$h = \frac{a}{(s-b)(s-c)} = \frac{1}{(s-b)} + \frac{1}{(s-c)},$$
$$k = \frac{1}{(s-c)} + \frac{1}{(s-a)},$$
$$l = \frac{1}{(s-a)} + \frac{1}{(s-b)}.$$

Hence, $h$, $k$, $l$ must satisfy the triangle inequality. Letting $2s' = h + k + l$, it follows by addition that

$$s' = \frac{1}{(s-a)} + \frac{1}{(s-b)} + \frac{1}{(s-c)}$$

and then

$$s - a = \frac{1}{(s'-h)}, \quad s - b = \frac{1}{(s'-k)}, \quad s - c = \frac{1}{(s'-l)}.$$

Adding the latter three equations, we get

$$s = \frac{1}{(s'-h)} + \frac{1}{(s'-k)} + \frac{1}{(s'-l)}.$$

Finally, the area of the triangle is given by

$$\Delta = \{s(s-a)(s-b)(s-c)\}^{1/2} = \left\{ \frac{\frac{1}{(s'-h)} + \frac{1}{(s'-k)} + \frac{1}{(s'-l)}}{(s'-h)(s'-k)(s'-l)} \right\}^{1/2}.$$

**6.** Prove that

$$3(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq xyz(x+y+z)^2$$

where $x, y, z \geq 0$.

*Solution.* By Cauchy's inequality

$$(x^2y + y^2z = z^2x)(zx^2 + xy^2 + yz^2) \geq (x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy})^2.$$

Hence it suffices to show that

$$\left\{\frac{(x^{3/2} + y^{3/2} + z^{3/2}}{3}\right\}^2 \geq \left\{\frac{(x+y+z)}{3}\right\}^3.$$

But this follows immediately from the power mean inequality. There is equality **iff** $x = y = z$.

* * *

To finish this number of the Corner and the solutions on file from the 1993 numbers of the Corner, we turn to readers' solutions to problems posed at the 7th Iberoamerican Mathematical Olympiad [1993: 286–287].

**1.** For each positive integer $n$, let $a_n$ be the last digit of the number

$$1 + 2 + 3 + \cdots + n.$$

Calculate $a_1 + a_2 + \cdots + a_{1991}$.

*Solutions by Seung-Jin Bang, Seoul, Korea; by Christopher J. Bradley, Clifton College, Bristol, U. K.; by Himadri Choudhury, student, Hunter High School, New York; by Tim Cross, Wolverley High School, Kidderminster, U. K.; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.*

First note that if $m \equiv n \bmod 20$, then from $m(m+1) \equiv n(n+1) \bmod 20$, we get

$$\frac{m(m+1)}{2} \equiv \frac{n(n+1)}{2} \mod 10,$$

and thus $a_m = a_n$. Now the first 20 values of $a_n$ are easily checked to be given by the following table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 3 | 6 | 0 | 5 | 1 | 8 | 6 | 5 | 5 |

| $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 6 | 8 | 1 | 5 | 0 | 6 | 3 | 1 | 0 | 0 |

Hence $\sum_{n=1}^{20} a_n = 70$. Since $1991 = 99 \times 20 + 11$ it follows that

$$\sum_{n=1}^{1991} a_n = 99 \times 70 + \sum_{n=1}^{11} a_n = 6930 + 46 = 6976.$$

**2.** Given the set of $n$ real numbers such that

$$0 < a_1 < a_2 < \cdots < a_n$$

and given the function

$$f(x) = \frac{a_1}{x+a_1} + \frac{a_2}{x+a_2} + \cdots + \frac{a_n}{x+a_n}$$

determine the sum of the lengths of all the pairwise disjoint intervals formed by all the $x$ such that $f(x) \geq 1$.

*Solutions by Seung-Jin Bang, Seoul, Korea; and by Christopher J. Bradley, Clifton College, Bristol, U. K. The solutions were similar and we give Bradley's.*

The sum of the roots of the equation

$$f(x) = \frac{a_1}{x+a_1} + \frac{a_2}{x+a_2} + \cdots + \frac{a_n}{x+a_n} = 1$$

is zero, since on multiplying up the coefficient of $x^{n-1}$ is zero. Also, from the graph of $y = f(x)$, with asymptotes at $x = -a_n, -a_{n-1}, \ldots, -a_1$ we see that $(n-1)$ roots are negative, and one is positive. So we may call the roots $-\alpha_{n-1}, -\alpha_{n-2}, \ldots, -\alpha_1$, and $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$ and where $-a_j < -\alpha_{j-1} < a_{j-1}$ ($j = 2$ to $n$), and $\alpha_0 > 0$.

The sum of the lengths of the intervals is therefore

$$(-\alpha_{n-1}+a_n)+(-\alpha_{n-2}+a_{n-1})+\cdot+(-\alpha_1+a_2)+(\alpha_0+a_1) = a_1+a_2+\cdots+a_n,$$

the intervals concerned being $(-a_n,-\alpha_{n-1}]$, $(-a_{n-1},-\alpha_{n-2}],\ldots,(-a_1,\alpha_0]$.

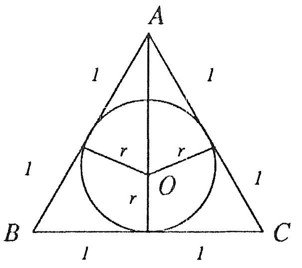
**3.** In an equilateral triangle $ABC$ (of side length 2) consider the incircle $\Gamma$.

(a) Show that for all points $P$ of $\Gamma$,

$$\overline{PA}^2+\overline{PB}^2+\overline{PC}^2=5.$$

(b) Show that for all points $P$ of $\Gamma$, it is possible to construct a triangle of sides $PA$, $PB$, $PC$, with area $\sqrt{3}/4$.

*Solutions by Christopher J. Bradley, Clifton College, Bristol, U. K.; by Himadri Choudhury, student, Hunter High School, New York; and by Tim Cross, Wolverley High School, Kidderminster, U. K. We give Cross' solution.*




Take $O$ as origin, with the $x$-axis parallel to $BC$, and $y$-axis along $OA$. Then the incircle has radius $r=1/\sqrt{3}$ and $A$, $B$, $C$ have coordinates

$$A\left(0,\frac{2}{\sqrt{3}}\right),\quad B\left(-1,\frac{-1}{\sqrt{3}}\right),\quad C\left(1,\frac{-1}{\sqrt{3}}\right).$$

A point on the incircle has coordinates parameterized by $\theta$, $0\le\theta<2\pi$ given by $P(\frac{1}{\sqrt{3}}\cos\theta,\frac{1}{\sqrt{3}}\sin\theta)$.

(a) Let $c=\cos\theta$, $s=\sin\theta$. Then

$$\begin{aligned}
&AP^2+BP^2+CP^2\\
&\quad=\left[\frac{1}{3}c^2+\frac{1}{3}(2-s)^2\right]+\left[\frac{1}{3}(c+\sqrt{3})^2+\frac{1}{3}(s+1)^2\right]\\
&\qquad\qquad\left[\frac{1}{3}(c-\sqrt{3})^2+\frac{1}{3}(s+1)^2\right]\\
&\quad=\frac{1}{3}[c^2+4-4s^2+s^2+c^2+2\sqrt{3}c+3+\\
&\qquad\qquad s^2+2s+1+c^2-2\sqrt{3}c+3+s^2+2s+1]\\
&\quad=\frac{1}{3}[3(c^2+s^2)+12]=5
\end{aligned}$$

since $c^2+s^2=1$.

(b) Now, set $x = AP = \frac{1}{\sqrt{3}}\sqrt{5-4s}$,

$$y = BP = \frac{1}{\sqrt{3}}\sqrt{5+2s+2\sqrt{3}c}, \quad \text{and} \quad z = CP = \frac{1}{\sqrt{3}}\sqrt{5+2s-2\sqrt{3}c}.$$

By reflection and rotational geometry the distances $AP$, $BP$, $CP$ will be a permutation of those obtained when $\pi/6 \le \theta \le \pi/2$, so that $1/\sqrt{3} \le x \le 1$. Also

$$y = \frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s+\frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5+4\cos\left(\theta-\frac{\pi}{6}\right)}$$

so that $\sqrt{7}/\sqrt{3} \le y \le \sqrt{3}$ and

$$z = \frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s-\frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5+4\sin\left(\theta-\frac{\pi}{3}\right)}$$

so that $\sqrt{5}/\sqrt{3} \le z \le \sqrt{7}/\sqrt{3}$. Thus

$$(x+y)_{\min} = \frac{1+\sqrt{7}}{\sqrt{3}} > z_{\max} = \frac{\sqrt{7}}{\sqrt{3}}$$

and

$$(y+z)_{\min} = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{3}} > x_{\max} = 1$$

and

$$(z+x)_{\min} = \frac{1+\sqrt{5}}{\sqrt{3}} > y_{\max} = \sqrt{3},$$

and $x$, $y$, $z$ (i.e. $AP$, $BP$, $CP$) can form the sides of a triangle.

From Heron's formula the area, $F$, of this triangle is given by

$$\begin{aligned}
F^2 &= \frac{1}{2}(x+y+z)\frac{1}{2}(-x+y+z)\cdot\frac{1}{2}(x-y+z)\cdot\frac{1}{2}(x+y-z)\\
&= \frac{1}{16}[(y+z)^2-x^2][x^2-(y-z)^2]\\
&= \frac{1}{16}[\frac{1}{3}(5+2s+2\sqrt{3}c+5+2s-2\sqrt{3}c-5+4s)\\
&\qquad +2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(c+2s-2\sqrt{3}c)^{1/2}]\\
&\quad \cdot[\frac{1}{3}(5-4s-5-2s-2\sqrt{3}c-5-2s+2\sqrt{3}c)\\
&\qquad +2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(5+2s-2\sqrt{3}c)^{1/2}]\\
&= \frac{1}{48}\left[(5+8s)+2\sqrt{(5+2s)^2-12c^2}\right]\left[-(5+8s)+2\sqrt{(5+2s)^2-12c^2}\right]\\
&= \frac{1}{48}[100+80s+16s^2-48c^2-25-80s-64s^2]\\
&= \frac{1}{48}[75-48] = \frac{9}{16}, \quad \text{since } c^2+s^2=1.
\end{aligned}$$

Thus $F = 3/4$.

[*Editor's note*: While the solution gives nice bounds for $x, y, z$ to verify that the lengths can form a triangle this also follows from the fact that $F^2$ comes out to be positive using Heron's formula.]

**4.** Let $(a_n)$ and $(b_n)$ be two sequences of integer numbers which satisfy the following conditions:

(i) $a_0 = 0$, $b_0 = 8$;

(ii) $a_{n+2} = 2a_{n+1} - a_n + 2$; $b_{n+2} = 2b_{n+1} - b_n$;

(iii) $a_n^2 + b_n^2$ is a perfect square, for all $n$.

Determine at least two possible values for the pair $(a_{1992}, b_{1992})$.

*Solutions by Himadri Choudhury, student, Hunter High School, New York; and by Tim Cross, Wolverley High School, Kidderminster, U. K. We give the solution of Cross.*

The recurrence relation $U_{n+2} = 2U_{n+1} + U_n = 0$ has general solution $U_n = An + B$, for constants $A$ and $B$.

The recurrence relation $U_{n+2} = 2U_{n+1} + U_n = 2$ has the particular solution $U_n = n^2$. Thus $a_n = n^2 + An + B$ and $b_n = Cn + D$.

$$a_0 = 0, \quad a_1 = \alpha \quad \text{(say)} \Rightarrow B = 0, \quad A = \alpha - 1$$

$$b_0 = 8, \quad b_1 = \beta \quad \text{(say)} \Rightarrow D = 8, \quad C = \beta - 8,$$

and we have

$$a_n = n^2 + An \qquad (A = a_1 - 1)$$

$$b_n = Bn + 8 \qquad (B = b_1 - 8)$$

and

$$\begin{aligned} a_n^2 + b_n^2 &= n^4 + 2An^3 + (A^2 + B^2)n^2 + 16Bn + 64 \\ &= (n^2 + An + 8)^2 \quad \text{iff} \quad A = B = \pm 4. \end{aligned}$$

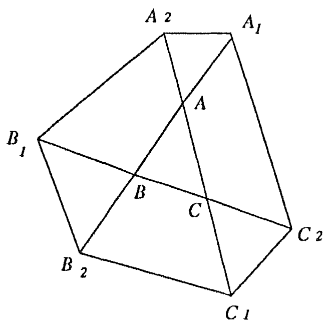
(I) $A = B = 4$ gives $a_1 = 5$, $b_1 = 12$ and $5^2 + 12^2 = 13^2$ checks, whence $a_n = n^2 + 4n$, $b_n = 4n + 8$ giving

$$(a_{1992}, b_{1992}) = (3976032, 7976).$$

(II) $A = B = -4$ gives $a_1 = -3$, $b_1 = 4$ and $(-3)^2 + 4^2 = 5^2$ checks; whence $a_n = n^2 - 4n$, $b_n = -4n + 8$ giving

$$(a_{1992}, b_{1992}) = (3960096, -7960).$$

**6.** From the triangle $T$ with vertices $A$, $B$ and $C$, the hexagon $H$ with vertices $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ is constructed, as shown in the figure. Show that the area of $H$ is at least thirteen times the area of $T$.




*Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Šefket Arslanagić, Berlin, Germany; by Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain; by Himadri Choudhury, student, Hunter High School, New York; by Tim Cross, Wolverley High School, Kidderminster, U. K.; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Choudhury.*

Note that $\Delta CC_1C_2$ and $\Delta CB_1A_2$ are both isosceles triangles and are thus similar. Also the area of $\Delta CB_1A_2 + \Delta CC_1C_2$ is equal to

$$\frac{1}{2}c^2 \sin y + \frac{1}{2}(a+b)^2 \sin y = \frac{1}{2}[c^2 + (a+b)^2]\frac{2|T|}{ab},$$

since we know that the area of $T$,

$$|T| = \frac{1}{2}ab \sin y.$$

Similarly $\Delta BB_1B_2 + \Delta BA_1C_2 = [b^2 + (a+c)^2]\frac{|T|}{ac}$ and $\Delta AA_1A_2 + \Delta AC_1B_2 = [a^2 + (b+c)^2]\frac{|T|}{bc}$. So all together

$$\begin{aligned}&\Delta CB_1A_2 + \Delta CC_1C_2 + \Delta BD_1B_2 + \Delta BA_1C_2 + \Delta AA_1A_2 + \Delta AC_1B_2\\ &\quad = |H| + 2|T| + |T|\left[\frac{c^2 + (a+b)^2}{ab} + \frac{b^2 + (a+c)^2}{ac} + \frac{a^2 + (b+c)^2}{bc}\right]\end{aligned}$$

So

$$\begin{aligned}|H| + 2|T| &= |T|\left[\frac{c^2 + a^2 + b^2}{ab} + 2 + \frac{b^2 + a^2 + c^2}{ac} + 2 + \frac{a^2 + b^2 + c^2}{bc} + 2\right]\\ &= |T|\left[(a^2 + b^2 + c^2)\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + 6\right)\right].\end{aligned}$$

Note that by the Arithmetic Mean - Geometric Mean inequality

$$(a^2 + b^2 + c^2) \geq 3(abc)^{2/3}$$

and that

$$\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \geq 3\left(\frac{1}{abc}\right)^{2/3}$$

with equality when $a = b = c$ in both cases, so we have

$$|T|\left[(a^2+b^2+c^2)\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)+6\right] \geq 15|T|$$

which implies $|H| \geq 13|T|$ as required, with equality when $\Delta ABC$ is equilateral.

[*Editor's note:* Readers should compare this problem and its solution to 1887 [ 1994: 236] for which Choudhury's solution was also used.]

* * *

That completes our space this number. Olympiad season is on us — send me your contests as well as your nice solutions to problems in the Corner.

* * * * *

# BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

*Knot Theory*, by Charles Livingston. Published by The Mathematical Association of America, 1993, Volume 24 in the Carus Mathematical Monographs. ISBN 0-88385-027-3, hardcover, 240+ pages, US $31.50. *Reviewed independently by M. E. Larsen of Copenhagen, Denmark, and U. I. Lydna of Beloretsk, Russia.*

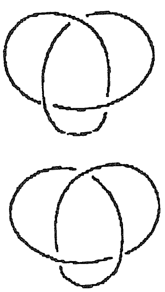


Figure 1

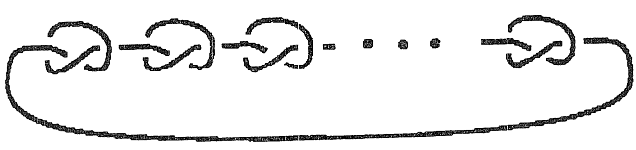


Figure 2

The physical model of a knot is simply a closed loop of string in three-dimensional space. The simplest examples are the left-handed and right-handed trefoils in Figure 1. To preclude pathological cases like Figure 2, where the number of trefoils is infinite, a knot may be formally defined as a simple closed polygonal curve in space. This does include the circle as a knot. It will be called the unknot to signify that it is an "unknotted" knot.

A link is a finite union of knots. In particular, a knot is a link with one component. Given a link, the most natural question is whether it is in fact a knot. If it is not, can one or more components be "disentangled" from the rest? If the link is in fact a knot, can it be "untwisted" into the unknot? Cutting through the link is clearly forbidden. The permissible moves can be defined more formally in terms of three basic operations known as the Reidemeister moves.

It is known that neither of the trefoils in Figure 1 can be transformed into the unknot. Moreover, they cannot be transformed into each other. In general, two knots are said to be equivalent if one can be transformed into the other. The main problem is to classify all knots into non-equivalent classes. There are infinitely many distinct classes. No two knots of the type in Figure 2 are equivalent unless they have the same number of trefoils.

Many different approaches have been used to attack this problem. The combinatorial techniques are based on the study of knot diagrams such as Figures 1 and 2. They lead to invariants such as the normalized Alexander polynomials. These polynomials are the same for two equivalent knots, but unfortunately, this can also be the case for a few pairs of non-equivalent knots.

The geometric techniques are based on the fact that every knot may be regarded as the boundary of some surface. The algebraic techniques are a subset of the field of algebraic topology. There is a construction which assigns a group to each knot. Many properties of the knot can be derived from those of the surface or the group.

All of these techniques are covered in a lucid manner in the first half of this well-written book. The treatment in the second half is more advanced. Here one finds the interaction of the three main techniques outlined above, such as different aspects of symmetry. More numerical invariants are given, including recent advances such as the Jones polynomials. Knots and links in higher dimensions are also considered.

The book keeps its feet on the ground throughout, and the abundance of appealing knot diagrams is a constant delight. There are plenty of exercises of varying degree of difficulty. This makes the book equally suitable as a classroom text or as a reference for independent study. It is very highly recommended.

* * * * *

# PROBLEMS

*Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **November 1, 1995**, *although solutions received after that date will also be considered until the time when a solution is published.*

**2015.** [1995: 53] **(Corrected)** *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*

Prove that

$$(\sin A + \sin B + \sin C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq \frac{27\sqrt{3}}{2\pi},$$

where $A$, $B$, $C$ are the angles (in radians) of a triangle.

**2031.** *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $\alpha$, $\beta$, $\gamma$ are acute angles such that

$$\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} + \frac{\sin(\beta-\gamma)}{\sin(\beta+\gamma)} + \frac{\sin(\gamma-\alpha)}{\sin(\gamma+\alpha)} = 0.$$

Prove that at least two of $\alpha$, $\beta$, $\gamma$ are equal.

**2032.** *Proposed by Tim Cross, Wolverley High School, Kidderminster, U. K.*

Prove that, for nonnegative real numbers $x$, $y$ and $z$,

$$\sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1} \geq \sqrt{6(x+y+z)}.$$

When does equality hold?

**2033.** *Proposed by K. R. S. Sastry, Dodballapur, India.*

The sides $AB$, $BC$, $CD$, $DA$ of a convex quadrilateral $ABCD$ are extended in that order to the points $P$, $Q$, $R$, $S$ such that $BP = CQ = DR = AS$. If $PQRS$ is a square, prove that $ABCD$ is also a square.

**2034**. *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Find all sequences $p_1 < p_2 < \cdots < p_n$ of distinct prime numbers such that

$$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\cdots\left(1+\frac{1}{p_n}\right)$$

is an integer.

(b) Can

$$\left(1+\frac{1}{a_1^2}\right)\left(1+\frac{1}{a_2^2}\right)\cdots\left(1+\frac{1}{a_n^2}\right)$$

be an integer where $a_1, a_2, \ldots$ are distinct integers greater than 1?

**2035**. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

If the locus of a point $E$ is an ellipse with fixed foci $F$ and $G$, prove that the locus of the incentre of triangle $EFG$ is another ellipse.

**2036**. *Proposed by Victor Oxman, Haifa University, Israel.*

You are given a circle cut out of paper, and a pair of scissors. Show how, by cutting only along folds, to cut from the circle a figure which has area between 27% and 28% of the area of the circle.

**2037**. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton.*

The lengths of the base and a slant side of an isosceles triangle are integers without common divisors. If the lengths of the angle bisectors of the triangle are all rational numbers, show that the length of the slant side is an odd perfect square.

**2038**. *Proposed by Neven Jurić, Zagreb, Croatia.*

Show that, for any positive integers $m$ and $n$, there is a positive integer $k$ so that

$$\left(\sqrt{m}+\sqrt{m-1}\right)^n = \sqrt{k}+\sqrt{k-1}.$$

**2039***. *Proposed by Dong Zhou, Fudan University, Shanghai, China, and Ji Chen, Ningbo University, China.*

Prove or disprove that

$$\frac{\sin A}{B}+\frac{\sin B}{C}+\frac{\sin C}{A} \ge \frac{9\sqrt{3}}{2\pi},$$

where $A$, $B$, $C$ are the angles (in radians) of a triangle. [Compare with *Crux* 1216 [1988: 120] and this issue!]

**2040**. *Proposed by Frederick Stern, San Jose State University, San Jose, California.*

Let $a < b$ be positive integers, and let

$$t = \frac{2^a-1}{2^b-1}.$$

What is the relative frequency of 1's (versus 0's) in the binary expansion of $t$?

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1216***. [1987: 53; 1988: 120] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi},$$

where $A, B, C$ are the angles (in radians) of a triangle.

II. *Solution by Shi-Liang Yu, Adult Technical Secondary School of Baodi, Tianjin, and Ji Chen, Ningbo University, China.*

First note that

$$\sum \frac{\sin A}{A} = \int_0^1 \sum \cos tA \, dt,$$

where the sums are cyclic over $A, B, C$. By item 2.2.33 of Chapter IX, page 160 of Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities*,

$$3\sin^2\frac{t\pi}{3} \le \sum \sin^2 tA < \sin^2 t\pi \qquad \text{for } 0 < t \le \frac{1}{2}.$$

Using

$$\sin^2\frac{t\pi}{3} = \frac{1}{2}\left(1 - \cos\frac{2t\pi}{3}\right), \qquad \text{etc.,}$$

and replacing $t$ by $t/2$, we get

$$2 + \cos t\pi < \sum \cos tA \le 3\cos\frac{t\pi}{3} \qquad \text{for } 0 < t \le 1.$$

Thus

$$2 = \int_0^1 (2 + \cos t\pi)\, dt < \int_0^1 \sum \cos tA \, dt \le \int_0^1 3\cos\frac{t\pi}{3}\, dt = \frac{9\sqrt{3}}{2\pi},$$

and the result follows.

* * * * *

**1935.** [1994: 108] *Proposed by Murray S. Klamkin, University of Alberta.*

Given an ellipse which is not a circle, prove or disprove that the locus of the midpoints of sufficiently small constant length chords is another ellipse.

*Combined solutions of Jordi Dou, Barcelona, Spain, and the proposer.*

The locus is *not* another ellipse (unless the given ellipse is a circle). Let the ellipse $\mathcal{E}$ satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let $\mathcal{L}$ be the locus of the midpoints of the chords of $\mathcal{E}$ having constant length $2k$. Note that the axes of $\mathcal{E}$ are axes of symmetry for the locus $\mathcal{L}$. If these axes meet $\mathcal{L}$ in $C$, $C'$ and $D$, $D'$, then (by plugging $(c, k)$ and $(k, d)$ into the equation for $\mathcal{E}$)

$$c^2 = \frac{a^2}{b^2}(b^2 - k^2), \quad d^2 = \frac{b^2}{a^2}(a^2 - k^2),$$

where $2c = CC'$ and $2d = DD'$. By Holditch's theorem [1] the areas $E$ and $L$ of the regions bounded by $\mathcal{E}$ and $\mathcal{L}$ satisfy

$$L = E - \pi k^2,$$

so that

$$L^2 = (\pi ab - \pi k^2)^2 = \pi^2(a^2b^2 - 2abk^2 + k^4).$$

An ellipse whose vertices are $C$, $C'$, $D$, $D'$ would surround an area of $A = \pi cd$ so that

$$A^2 = \pi^2(a^2b^2 - (a^2 + b^2)k^2 + k^4).$$

Since $L^2 - A^2 = \pi^2(a - b)^2k^2$, the area surrounded by $\mathcal{L}$ exceeds the area surrounded by this new ellipse whenever $a \neq b$ and $k \neq 0$.

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; RICHARD I. HESS, Rancho Palos Verdes, California; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; and D. J. SMEENK, Zaltbommel, The Netherlands.*

*Bradley and Smeenk showed that $\mathcal{L}$ satisfies a fourth degree equation. Penning located the problem as an "exam question" dated* 1845 [3].

*Klamkin poses a related question: can the envelope of constant-length chords of an ellipse ever be another ellipse?*

*Dou was reminded of a* Monthly *problem* [2]:

> What is the probability that the length of a chord randomly drawn in an ellipse will not exceed the length of the minor axis?

*(By "randomly drawn chords" we mean those with midpoints uniformly distributed throughout the ellipse.) The answer $b/a$ was obtained by Dou (but not in the featured solution) by observing that except for the centre, each point interior to an ellipse is the midpoint of one and only one chord; the probability follows easily from Holditch's theorem since the area swept out by the midpoints of chords having length less than $2b$ is $\pi b^2$, while the area enclosed by the ellipse is $\pi ab$.*

*References:*
[1] Arne Broman, Holditch's Theorem, *Math. Mag.* 54:3 (1981) 99–108.
[2] Frank Dapkus, Problem E 2324, *Amer. Math. Monthly* 79:10 (December 1972) 1135–1136.
[3] Question d'examen, *Nouvelle Annales de Mathematique* 4 (1845) 590–596.

* * * * *

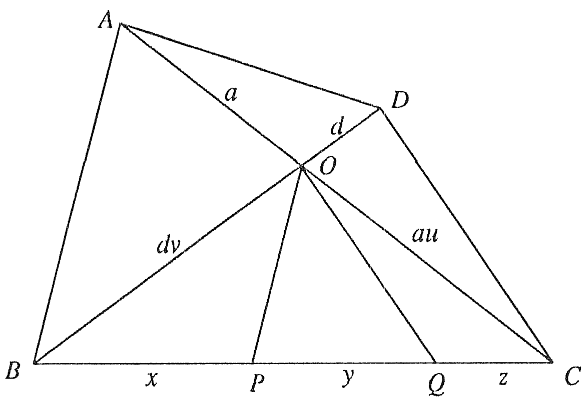
**1941.** [1994: 136] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, and $O$ is the intersection of its diagonals. Suppose that the area of the (nonconvex) pentagon $ABOCD$ is equal to the area of triangle $OBC$. Let $P$ and $Q$ be the points on $BC$ such that $OP \| AB$ and $OQ \| DC$. Prove that

$$[OAB] + [OCD] = 2[OPQ],$$

where $[XYZ]$ denotes the area of triangle $XYZ$.

*Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*




Let

$$[DOC] = u[AOD] \quad \text{and} \quad [BOA] = v[AOD];$$

then $[BCO] = uv[AOD]$, and the condition of the problem implies

$$1 + u + v = uv. \tag{1}$$

Furthermore,

$$\frac{AD}{OC} = \frac{1}{u} = \frac{BP}{PC} \quad \text{and} \quad \frac{DO}{OB} = \frac{1}{v} = \frac{CQ}{QB}.$$

Let $BP = x$, $PQ = y$, and $QC = z$. Then, since $OP \parallel AB$ and $OQ \parallel DC$,

$$\frac{x}{y+z} = \frac{1}{u} \quad \text{and} \quad \frac{z}{x+y} = \frac{1}{v},$$

so

$$z = ux - y \quad \text{and} \quad x = vz - y.$$

We may let $y = 1$, which implies $x = v(ux - 1) - 1$, so

$$x = \frac{v+1}{uv-1}.$$

Similarly

$$z = \frac{u+1}{uv-1},$$

and thus by (1)

$$x + y + z = \frac{v + 1 + uv - 1 + u + 1}{uv - 1} = \frac{2uv}{u+v}.$$

Now $[BPO] : [PQO] : [QCO] = x : y : z$, and so

$$\begin{aligned}[OPQ] &= \frac{y}{x+y+z}[BCO] = \frac{u+v}{2uv} \cdot uv[AOD] = \frac{u+v}{2}[AOD] \\ &= \frac{[DOC] + [BOA]}{2},\end{aligned}$$

which implies the result.

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; JORDI DOU, Barcelona, Spain; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.*

* * * * *

**1943.** [1994: 136] *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
In triangle $ABC$, the median $AD$ is the geometric mean of $AB$ and $AC$. Prove that

$$1 + \cos A = \sqrt{2}\,|\cos B - \cos C|.$$

*Solution by D. J. Smeenk, Zaltbommel, The Netherlands.*
Let $a$, $b$, $c$ be the sides and $\alpha$, $\beta$, $\gamma$ be the angles of the triangle. We are given that

$$bc = (AD)^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4},$$

or

$$a^2 = 2(b-c)^2. \qquad (1)$$

Without loss of generality we may assume $b > c$, so $\beta > \gamma$ and $\cos\beta < \cos\gamma$. Then from (1) it follows that $a = \sqrt{2}(b - c)$ and [from the law of sines]

$$\sin\alpha = \sqrt{2}(\sin\beta - \sin\gamma). \qquad (2)$$

We are to show

$$1 + \cos\alpha = \sqrt{2}(\cos\gamma - \cos\beta). \qquad (3)$$

From (2) and (3) it suffices to prove

$$\frac{\sin\alpha}{1+\cos\alpha} = \frac{\sin\beta - \sin\gamma}{\cos\gamma - \cos\beta},$$

or

$$\tan\frac{\alpha}{2} = \frac{2\sin[(\beta-\gamma)/2]\cos[(\beta+\gamma)/2]}{2\sin[(\beta-\gamma)/2]\sin[(\beta+\gamma)/2]} = \cot\frac{\beta+\gamma}{2},$$

and that holds.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; BILL CORRELL JR., student, Denison University, Granville, Ohio; TIM CROSS, Wolverley High School, Kidderminster, U. K.; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; BOB PRIELIPP, University of Wisconsin-Oshkosh; J.-B. ROMERO MÁRQUEZ, Universidad de Valladolid, Spain; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castello, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; P. E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

*Murray S. Klamkin, University of Alberta, notes that the result* **fails** *for the degenerate triangle $a = 0$, $b = c = 1$! Readers may like to discover where the proof goes wrong for this case.*

* * * * *

**1944.** [1994: 137] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton.*

Find the smallest positive integer $n$ so that

$$(n+1)^{2000} > (2n+1)^{1999}.$$

*Solution by Christopher J. Bradley, Clifton College, Bristol, U. K.*

Expansion by the binomial theorem and division by $n^{1999}$ gives

$$n + 2000 + \frac{2000\cdot 1999}{2n} + \cdots$$
$$> 2^{1999} + \frac{2^{1998}\cdot 1999}{n} + 2^{1997}\left(\frac{1999\cdot 1998}{2n^2}\right) + \cdots.$$

Since $n$ is going to be of the order $2^{1999}$ the terms other than the first two on either side cannot affect the issue, leaving $n$ to be the first integer greater than $2^{1999} + 999.5 - 2000$, that is,

$$n = 2^{1999} - 1000.$$

*Also solved by BILL CORRELL JR., student, Denison University, Granville, Ohio; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, Ohio State University, Columbus; P. PENNING, Delft, The Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's; and the proposer. There were two incorrect solutions and one incomplete solution sent in.*

*Actually, the proposer's original question was to find, for each given $m \geq 1$, the least integer $n = n(m)$ such that $(n+1)^{2m+2} > (2n+1)^{2m+1}$. (The present problem is the special case when $m = 999$.) He showed that $n(m) = 2^{m+1} - m - 1$. Both Correll and Penning proved that the least $n = n(k)$ such that $(n+1)^{k+1} > (2n+1)^k$ is given by $n(k) = 2^k - \lceil (k+1)/2 \rceil$. Clearly this implies the proposer's result.*

* * * * *

**1945.** [1994: 137] *Proposed by Murray S. Klamkin, University of Alberta.*

Let $A_1A_2 \ldots A_n$ be a convex $n$-gon.

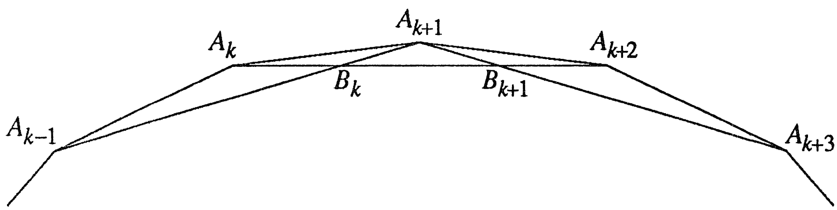
(a) Prove that

$$A_1A_2 + A_2A_3 + \cdots + A_nA_1 \leq A_1A_3 + A_2A_4 + \cdots + A_nA_2.$$

(b)* Prove or disprove that

$$2\cos\left(\frac{\pi}{n}\right)(A_1A_2 + A_2A_3 + \cdots + A_nA_1) \geq A_1A_3 + A_2A_4 + \cdots + A_nA_2.$$

*Solution by Sibylle Schwarz (student) and Johannes Waldmann, Friedrich-Schiller-Universität, Jena, Germany.*




(a) Let $n > 3$, and take all indices modulo $n$. For each $k$, define point $B_k$ as the intersection of $A_kA_{k+2}$ and $A_{k-1}A_{k+1}$. Then $A_k, B_k, B_{k+1}, A_{k+2}$ are on the line segment $A_kA_{k+2}$, and exactly in that order because of convexity of the $n$-gon. Thus

$$\begin{aligned}\sum A_kA_{k+2} &\geq \sum (A_kB_k + B_{k+1}A_{k+2}) \\ &= \sum (A_kB_k + B_kA_{k+1}) \quad \text{by shifting of indices} \\ &\geq \sum A_kA_{k+1} \quad \text{by the triangle inequality.}\end{aligned}$$

(b) This inequality is false.

Again let $n > 3$, and take all indices modulo $n$. Trivially,

$$\begin{aligned}\sum A_kA_{k+2} &\le \sum (A_kA_{k+1} + A_{k+1}A_{k+2}) \quad \text{by the triangle inequality}\\ &= \sum (A_kA_{k+1} + A_kA_{k+1}) \quad \text{by shifting of indices}\\ &= 2 \cdot \sum A_kA_{k+1}.\end{aligned}$$

However, the constant "2" cannot be lowered. Fix two points $X$ and $Y$ with distance 1. Consider the (degenerate) $n$-gon $\mathcal{A} = A_1A_2 \ldots A_n$ where

$$A_1 \equiv A_2 \equiv X, \qquad A_3 \equiv \cdots \equiv A_n \equiv Y.$$

($\mathcal{A}$ is degenerate, but it could easily be approximated by a sequence of proper convex $n$-gons.) Then $\sum A_kA_{k+2} = 4$ and $\sum A_kA_{k+1} = 2$.

*Both parts also solved by TOSHIO SEIMIYA, Kawasaki, Japan. Part* (a) *only solved by LEROY F. MEYERS, The Ohio State University, Columbus; and the proposer.*

*Seimiya's counterexample to part* (b) *is any nonsquare rectangle.*

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**1946.** [1994: 137] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

In a television commercial some months ago, a pizza restaurant announced a special sale on two pizzas, in which each pizza could independently contain up to five of the toppings the restaurant had available (no topping at all is also an option). In the commercial, a small boy declared that there were a total of 1048576 different possibilities for the two pizzas one could order. How many toppings are available at the restaurant?

*Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

If there are $n$ toppings available at the restaurant then there are

$$k = \binom{n}{5} + \binom{n}{4} + \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$$

ways to have a single pizza, since you could have up to five toppings (no toppings is a valid, although boring, choice). For two pizzas, you could have two of the same pizza ($k$ ways) or two different pizzas ($\binom{k}{2} = k(k-1)/2$ ways). Since the total number of possible combinations of the two is given as 1048576, the following must be true:

$$\frac{k(k-1)}{2} + k = 1048576.$$

But this equation doesn't have any positive integer solutions. It would seem that the restaurant in question considers getting a pizza with bacon and cheese and a pizza with mushrooms, cheese and green pepper different from getting a pizza with mushrooms, cheese and green pepper and a pizza with bacon and cheese; i.e., the order in which the order is placed matters. This seems silly to me, but so be it. In this case the equation that must be solved is $k^2 = 1048576 = 2^{20}$, so $k = 1024 = 2^{10}$. Now to find $n$ we must solve

$$\binom{n}{5} + \binom{n}{4} + \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0} = 2^{10}. \qquad (1)$$

By the symmetry of Pascal's triangle we get

$$\binom{n}{0} = \binom{n}{n}, \ \binom{n}{1} = \binom{n}{n-1}, \ \ldots, \ \binom{n}{5} = \binom{n}{n-5},$$

so our equation (1) can also be written as

$$\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \binom{n}{n-3} + \binom{n}{n-4} + \binom{n}{n-5} = 2^{10}.$$

If we add these two together we get

$$\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{n-5} + \binom{n}{5} + \binom{n}{4} + \cdots + \binom{n}{0} = 2 \cdot 2^{10} = 2^{11},$$

which is satisfied if $n = 11$, since

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Thus the restaurant has **11** different toppings.

If the order in which the order is placed doesn't matter, then the number of two-pizza orders is

$$\frac{k(k+1)}{2} = \frac{1024 \cdot 1025}{2} = 524800,$$

*not* 1048576.

*Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; THE BOOKERY PROBLEM GROUP, Walla Walla, Washington; BILL CORRELL JR., student, Denison University, Granville, Ohio; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; GREG TSENG, student, Thomas Jefferson High School for Science and Technology, Alexandria, Virginia; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

*Many solvers agree with Godin that the number of choices for the two pizzas should have been* 524800. *In fact this issue has since been raised in letters to the editor in* Mathematics Teacher, *September* 1994, *pages* 389 *and* 474, *and more recently in the March* 1995 College Math. Journal, *pages* 141–143. *One of the letters in* Mathematics Teacher *is from the pizza company responsible for the TV ad, claiming that they had* **intended** *the two pizzas to be considered as an ordered pair. (Yeah, sure... .) It seems that "ordering two pizzas" from this restaurant has unexpected mathematical meaning!*

*Geretschläger ends his solution with the groaner: "It is wonderful that* Crux *has taken to running pizza problems, as these are things you can really sink your teeth into".*

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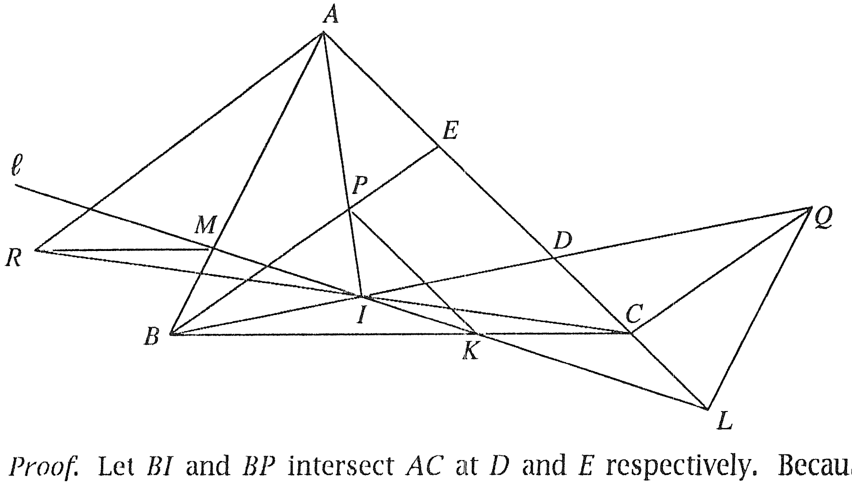
**1947.** [1994: 137] *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

Triangle $ABC$ has incenter $I$ and centroid $G$. The line $IG$ intersects $BC$, $CA$, $AB$ in $K$, $L$, $M$ respectively. The line through $K$ parallel to $CA$ intersects the internal bisector of $\angle BAC$ in $P$. The line through $L$ parallel to $AB$ intersects the internal bisector of $\angle CBA$ in $Q$. The line through $M$ parallel to $BC$ intersects the internal bisector of $\angle ACB$ in $R$. Show that $BP$, $CQ$ and $AR$ are parallel.

*Solution by Toshio Seimiya, Kawasaki, Japan.*

We shall prove the following generalization:

$ABC$ is a triangle, and a line $\ell$ intersects $BC$, $CA$ and $AB$ at $K$, $L$ and $M$ respectively. $I$ is a point on $\ell$ other than $K$, $L$ or $M$. Let $P$, $Q$ and $R$ be points on $AI$, $BI$ and $CI$ respectively, such that $KP \parallel CA$, $LQ \parallel AB$ and $MR \parallel BC$. Then $BP$, $CQ$ and $AR$ are parallel.




*Proof.* Let $BI$ and $BP$ intersect $AC$ at $D$ and $E$ respectively. Because $PK \parallel AC$ we get

$$\frac{EB}{BP} = \frac{CB}{BK} \quad \text{and} \quad \frac{PI}{IA} = \frac{KI}{IL}\,. \tag{1}$$

As $B$, $I$ and $D$ lie on a line, we have by Menelaus' theorem applied to $\Delta AEP$,

$$\frac{AD}{DE}\cdot\frac{EB}{BP}\cdot\frac{PI}{IA}=1.$$

From (1) this becomes

$$\frac{AD}{DE}\cdot\frac{CB}{BK}\cdot\frac{KI}{IL}=1. \tag{2}$$

As $B$, $I$ and $D$ lie on a line, we have by Menelaus' theorem applied to $\Delta KCL$,

$$\frac{LD}{DC}\cdot\frac{CB}{BK}\cdot\frac{KI}{IL}=1. \tag{3}$$

From (2) and (3) we have $AD/DE = LD/DC$, i.e.,

$$\frac{AD}{LD}=\frac{DE}{DC}. \tag{4}$$

Because $AB\|LQ$ we get

$$\frac{AD}{LD}=\frac{BD}{QD}. \tag{5}$$

From (4) and (5) we have

$$\frac{DE}{DC}=\frac{BD}{QD},$$

and therefore $BE\|CQ$, i.e. $BP\|CQ$. Similarly we can prove $AR\|CQ$.

*Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; P. PENNING, Delft, The Netherlands; and the proposer. All these solutions used coordinates.*

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**1948.** [1994: 137] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Are there any nonconstant differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(f(x)))=f(x)\geq 0$$

for all $x\in\mathbb{R}$?

*Solution by Chris Wildhagen, Rotterdam, The Netherlands.*
Applying $f$ to both sides of the functional equation

$$f(f(f(x)))=f(x)\geq 0 \tag{1}$$

gives $g(g(x)) = g(x)$, where $g(x) = f(f(x))$ for all $x\in\mathbb{R}$. Of course $g$ is also a differentiable function on $\mathbb{R}$ and $g(x)\geq 0$ for all $x\in\mathbb{R}$. Then the range $T = g(\mathbb{R})$ of $g$ is an interval contained in $[0,+\infty)$. Let $a$ be the infimum of $T$. Since $g(t) = t$ for all $t\in T$ and $g$ is continuous, it follows that $g(a) = a$. Assuming that $T$ has more than one element, choose $\delta > 0$ such that $(a, a+\delta)\subseteq T$. Then $x\in(a-\delta, a)$ implies $g(x)\geq g(a)$ $(= a)$, hence

$$\frac{g(x) - g(a)}{x - a} \le 0.$$

Therefore

$$g'_L(a) = \lim_{x \to a^-} \frac{g(x) - g(a)}{x - a} \le 0. \tag{2}$$

For $x \in (a, a + \delta)$ we have

$$\frac{g(x) - g(a)}{x - a} = 1,$$

hence

$$g'_R(a) = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = 1. \tag{3}$$

(2) and (3) are contradictory since $g$ is differentiable at $a$. We are led to the conclusion that $T$ is a single point, i.e., $g$ is a constant function, say $g(x) = c$ for all $x \in \mathbb{R}$. This gives, using (1), that $f(c) = f(x)$ for all $x \in \mathbb{R}$, showing that $f$ is a constant function. Thus there is no nonconstant differentiable function satisfying (1).

*Also solved by DUANE BROLINE, Eastern Illinois University, Charleston; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, Ohio State University, Columbus; MATHEMATICAL PROBLEM SOLVING LAB, University of Arizona, Tucson; DAVID C. VELLA, Skidmore College, Saratoga Springs, New York; and the proposer. There were two incorrect solutions submitted.*

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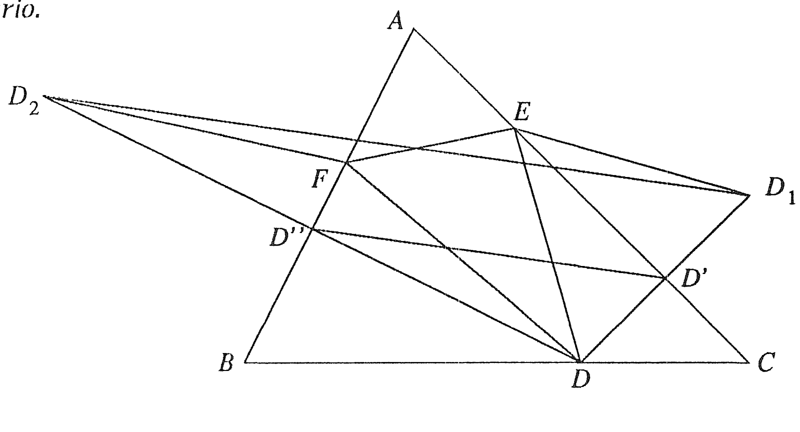
**1949.** [1994: 137] *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let $D$, $E$, $F$ be points on the sides $BC$, $CA$, $AB$ respectively of triangle $ABC$, and let $R$ be the circumradius of $ABC$. Prove that

$$\left(\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}\right)(DE + EF + FD) \ge \frac{AB + BC + CA}{R}.$$

*Solution by Cyrus Hsia, student, Woburn Collegiate, Scarborough, Ontario.*



Let

$$s = \frac{DE + EF + FD}{2} .$$

Reflect $D$ in $AC$ to get $D_1$, with $DD_1$ intersecting $AC$ at $D'$. Reflect $D$ in $AB$ to get $D_2$, with $DD_2$ intersecting $AB$ at $D''$. Then

$$DE + EF + FD = D_2F + FE + ED_1 \geq D_2D_1 = 2D''D',$$

so

$$s \geq D''D'. \qquad (1)$$

Now $AD'DD''$ is concyclic with $AD$ as diameter, so

$$\sin A = \frac{D''D'}{AD} . \qquad (2)$$

Putting (1) and (2) together,

$$\frac{s}{AD} \geq \frac{D''D'}{AD} = \sin A.$$

Likewise,

$$\frac{s}{BE} \geq \sin B \quad \text{and} \quad \frac{s}{CF} = \sin C.$$

Finally, adding these inequalities,

$$\left(\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}\right) s \geq \sin A + \sin B + \sin C = \frac{AB + BC + CA}{2R} ,$$

and the required inequality follows.

*Also solved by WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer. Pompe's solution was in fact virtually identical to Hsia's.*

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**1950.** [1994: 138] *Proposed by Svetlozar Doitchev, Stara Zagora, Bulgaria.*

The equation

$$2 \cdot 3^x + 1 = 7 \cdot 5^y,$$

where $x$ and $y$ are nonnegative integers, has $x = 1$, $y = 0$ as one solution. Find all other solutions.

*Solution by Richard K. Guy, University of Calgary, and John Selfridge, Northern Illinois University, DeKalb.*

The result can be found in a fairly famous paper [3] which deserves to be even more famous. It also appears as a special case in [2]; see the entry 4375 in Col. 7 opposite $N = 1$ in Table 4 on p. 188. For our solution, it is essential to know a little elementary number theory: there's more than enough in §2.8 of [4]. It is also very useful to have access to [1] and the extension and updates circulated by Wagstaff. The solution appears to demand formidable calculations, but there are ways to circumvent these, some of which will be indicated below. See the Remark on Calculation on p. 100 of [4].

If $y \geq 0$, then $x \geq 1$ and we can write the given equation as

$$6(3^{x-1} - 1) = 7(5^y - 1). \qquad (1)$$

If $y > 0$, then $7|(3^{x-1} - 1)$ and $6|(x - 1)$.

The last sentence can be expressed by saying that 3 is a *primitive root* of 7. If you were born, as one of us was, before Abel and Galois, and don't know any group theory, then this is a good way to begin to learn some. The primitive root 3 generates the (cyclic) multiplicative group of nonzero residues, modulo 7:

| exponent $e$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $3^e$ mod 7 | 1 | 3 | 2 | 6 | 4 | 5 | 1 | ... |

The *order* of 3 and 5 (the other primitive root mod 7) is 6 (= $\phi(7)$), the order of 2 and 4 is 3, that of 6 is 2, and that of 1 is 1.

Lemma 2.31 of [4] states that if $a$ has order $h$ mod $m$, then the positive integers $k$ such that $a^k \equiv 1 \bmod m$ are precisely those for which $h|k$. Theorem 2.40 is that if $p$ is an odd prime and $g$ is a primitive root modulo $p^2$, then $g$ is a primitive root modulo $p^\alpha$ for $\alpha = 3, 4, 5, \ldots$.

Euler's generalization of Fermat's "little" theorem states that

$$\gcd(a, m) = 1 \quad \text{implies} \quad a^{\phi(m)} \equiv 1 \bmod m.$$

[Note that $\phi(p^\alpha) = p^{\alpha-1}(p - 1)$.]

Before we interrupted ourselves, we had just seen that $6|(x-1)$. In fact

$$x = 7, \quad y = 4$$

is a solution, and we will show that there are no others. Write equation (1) as

$$2 \cdot 3^7(3^a - 1) = 7 \cdot 5^4(5^b - 1) \qquad (2)$$

where $a = x - 7$ and $b = y - 4$. We will obtain a contradiction by assuming that $a$ and $b$ are positive.

Conveniently, 3 and 5 are primitive roots of each other and of each other's squares, and hence, by the theorem quoted above, of all higher powers.

| exponent $e$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $5^e$ mod $3^2$ | 1 | 5 | 7 | 8 | 4 | 2 | 1 | ... |

| exponent $e$ | 0 | 1 | 2 | 3 | 4 | 5 | 10 |
|---|---|---|---|---|---|---|---|
| $3^e$ mod $5^2$ | 1 | 3 | 9 | 2 | 6 | 18 | $24 \equiv -1$ |

(a) From (2), $5^4 \| (3^a - 1)$, so $4 \cdot 5^3 | a$ and $5^3 \| a$.
[We use the symbol $\|$ for "exactly divides" in the sense that $5^4$ divides $3^a - 1$, but $5^5$ does not divide $3^a - 1$, so $5^3$ divides $a$, but $5^4$ does not divide $a$.]
(b) $3^7 \| (5^b - 1)$, so $2 \cdot 3^6 | b$ and $3^6 \| b$.
(c) From (a) $2^4 | (3^4 - 1) | (3^a - 1)$, so $2^5 | (5^b - 1)$ and $8 | b$.
(d) From (b) & (c) $24 | b$, so $390001 | (5^{12}+1) | (5^b - 1)$ and $390001 | (3^a - 1)$ implies that $625 | a$. But this contradicts (a).

End of proof. You may have a couple of objections: that the calculations in (d) are beyond the capabilities of your hand calculator, and where did 390001 come from? We admit that it's very useful to have the tables [1]. But the calculations are easy: the prime

$$390001 = 1 + 5^4(5^4 - 1) = 1 - 5^4 + 5^8 = \frac{5^{12} + 1}{5^4 + 1}$$

divides $5^b - 1 = 5^{24k} - 1$ which is a multiple of $5^{24} - 1 = (5^{12} - 1)(5^{12} + 1)$. We also want to check that the order of 3 modulo 390001 is divisible by 625. Work in the scale of $625 = 5^4$ (note that $5^8 \equiv 5^4 - 1$ and $5^{12} \equiv -1 \bmod 390001$): $3^6 = 5^4 + 104$ and we already know that $2 \cdot 3^7 = 7 \cdot 5^4 - 1$, leading to $3^{13} \equiv 55 \cdot 5^4 - 56$ and squaring and multiplying (which can be done by hand calculation or a hand calculator) lead eventually to $3^{4875} \equiv 91 \cdot 5^4 - 185$ which is not $\equiv 1$, but whose fifth power is.

*References:*

[1] John Brillhart, D. H. Lehmer, J. L. Selfridge, Bryant Tuckerman & S. S. Wagstaff, *Factorizations of $b^n \pm 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ up to high powers*, Contemporary Math. 22, Amer. Math. Soc., Second edition 1988 & updating by S. S. Wagstaff.

[2] R. K. Guy, C. B. Lacampagne & J. L. Selfridge, Primes at a glance, *Math. Comput.*, 48(1987) 183-202.

[3] D. H. Lehmer, On a problem of Størmer, *Illinois J. Math.*, 8(1964) 59–79.

[4] Ivan Niven, Herbert S. Zuckerman & Hugh L. Montgomery, *An Introduction to the Theory of Numbers*, Wiley, Fifth edition 1991.

*Also solved by KEE-WAI LAU, Hong Kong; and the proposer. The solution $x = 7, y = 4$ was found by Richard I. Hess, Rancho Palos Verdes, California; J. A. McCallum, Medicine Hat, Alberta; and Panos E. Tsaoussoglou, Athens, Greece, but without a proof that there are no other solutions. There was also one incorrect solution sent in.*

*Lau's solution is essentially the same as the above, which is actually one of* seven*(!) submitted by Guy and Selfridge.*

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