Crux Mathematicorum

Published by the Canadian Mathematical Society.

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 20 #2

February
février
1994

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Canadian Mathematical Society

Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

SUBSCRIPTION INFORMATION

Crux Mathematicorum is published monthly (except July and August). The subscription rates for ten issues are detailed on the inside back cover. Information on Crux Mathematicorum back issues is also provided on the inside back cover.

RENSEIGNEMENTS SUR L'ABONNEMENT

Crux Mathematicorum est publié mensuellement (sauf juillet et août). Les tarifs d'abonnement pour dix numéros figurent sur la couverture arrière. On peut également y retrouver de plus amples renseignements sur les volumes antérieurs de Crux Mathematicorum

ACKNOWLEDGEMENTS

The support of the Department of Mathematics & Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

REMERCIEMENTS

Nous rendons hommage à l'appui du département de mathématique et statistique de l'Université de Calgary et du département de mathématiques de l'Université d'Ottawa.

Printed at / imprimé à: Synergy Print and Copy, Ottawa.

ISSN 0705-0348

Publication Mail Registration No. 5432

A COROLLARY TO GAUSS'S LEMMA

Andy Liu

Let p be a prime and a be an integer not divisible by p. We say that a is a *quadratic residue* of p if the congruence $x^2 \equiv a \pmod{p}$ has solutions, and a *quadratic non-residue* if no solutions exist.

Gauss's Lemma provides a necessary and sufficient condition which distinguishes the quadratic residues from the non-residues of an odd prime. *Let the integers ak, $1 \le k \le (p-1)/2$, be reduced modulo p so that $|ak| \le (p-1)/2$. Then a is a quadratic residue of p if and only if the number of these integers which are negative is even.*

EXAMPLE 1: Let $a = 3$ and $p = 17$. Then $(p-1)/2 = 8$. The integers $3, 6, 9, 12, 15, 18, 21$ and 24 reduce modulo 17 to $3, 6, -8, -5, -2, 1, 4$ and 7 respectively. Since an odd number of them are negative, 3 is a quadratic non-residue of 17.

When p and a are much larger, Gauss's Lemma is not very convenient to use. Thus its principal application lies in proving the most practical Quadratic Reciprocity Law. Nevertheless, it may pay dividends to work through a more involved case using Gauss's Lemma.

EXAMPLE 2: Let $a = 19$ and $p = 461$. We have to reduce $(p-1)/2 = 230$ integers. However, it is not necessary to handle them one at a time. We have $19 \times 1 \equiv 19 \pmod{461}$ and $19 \times 12 \equiv 228 \pmod{461}$. Hence all of $19k$, $1 \le k \le 12$, are positive. Now $19 \times 13 \equiv -214 \pmod{461}$ and $19 \times 24 \equiv -5 \pmod{461}$. Hence all $19k$, $13 \le k \le 24$, turn negative. Thereafter, we have alternate blocks of 12 positive and 12 negative integers, until the pattern is broken at $19 \times 97 \equiv -1 \pmod{461}$. This pattern is disrupted one more time, at $19 \times 194 \equiv -2 \pmod{461}$. Of the 228 integers in blocks of 12, an even number of them are negative. The two extra integers -1 and -2 are both negative, so that the overall total remains even. By Gauss's Lemma, 19 is a quadratic residue of 461.

We generalize the above example to the following result.

COROLLARY TO GAUSS'S LEMMA: Let p be an odd prime and a be an integer not divisible by p. Let $(p-1)/2 = aq + r$, $0 \le r \le a-1$. Let $s = 0$ if $r = 0$. If $r > 0$, let b be the inverse modulo p of a. Let the integers bi, $1 \le i \le r$, be reduced modulo p so that $|bi| \le (p-1)/2$. Let s be the number of these integers which are negative. Then a is a quadratic residue of p if and only if $s \equiv q[a/2] \pmod{2}$, where $[x]$ denotes the greatest integer $\le x$.

Before we prove this result, let us use it to work out Examples 1 and 2 again.

EXAMPLE 1: We have $8 = 3 \times 2 + 2$. Hence $q[a/2] = 2 \times 1$ is even. Now the inverse modulo 17 of 3 is 6. From $6 \times 1 \equiv 6 \pmod{17}$ and $6 \times 2 \equiv -5 \pmod{17}$, $s = 1$. Hence 3 is a quadratic non-residue of 17.

EXAMPLE 2: We have $230 = 19 \times 12 + 2$. Hence $q[a/2] = 12 \times 9$ is even. Now the inverse modulo 461 of 19 is 364. From $364 \times 1 \equiv -97 \pmod{461}$ and $364 \times 2 \equiv -194 \pmod{461}$, $s = 2$. Hence 19 is a quadratic residue of 461.

PROOF OF COROLLARY: Consider the integers ak, $1 \le k \le (p-1)/2$, reduced modulo p so that $|ak| \le (p-1)/2$. Since $(p-1)/2 = aq+r$, these integers can be divided into a blocks of q or $q+1$ consecutive integers, such that all integers in the same block have the same sign. Moreover, there are exactly r blocks of size $q+1$, and the blocks alternate in signs, starting with a block of positive integers.

If $r = 0$, the total number of negative integers is exactly $q[a/2]$. By Gauss's Lemma, a is a quadratic residue of p if and only if $q[a/2]$ is even, or $q[a/2] \equiv 0 = s \pmod 2$.

Suppose $r > 0$. Consider the r blocks of integers of size $q+1$ and the integer with the smallest absolute value in each block. These r absolute values must be $1, 2, \cdots, r$. The number of negative integers not including any from these r is $q[a/2]$ as before. It remains to count the number of negative integers among these r.

For $1 \le i \le r$, we have $ak_i \equiv \pm i \pmod p$ for some integers k_i, where $1 \le k_i \le (p-1)/2$. Multiplying by b, we have $\pm k_i \equiv bi \pmod p$. Note that $|\pm k_i| \le (p-1)/2$, and $\pm k_i$ has the same sign as $\pm i$. It follows that the number of negative integers among the r in question is equal to the number s of negative integers among bi, $1 \le i \le r$, reduced modulo p so that $|bi| \le (p-1)/2$.

It now follows from Gauss's Lemma that a is a quadratic residue of p if and only if $s \equiv q[a/2] \pmod 2$. This completes the proof of the corollary. □

This result is particularly useful when a is fixed and p varies. We conclude with a further application.

EXAMPLE 3: We wish to determine the odd primes p for which $a = 3$ is a quadratic residue. Note that $[3/2] = 1$. We consider four cases.

Let $p = 12k+1$. Then $(p-1)/2 = 6k$, $q = 2k$, $r = 0$ and $s = 0$. Since $2k \equiv 0 \pmod 2$, 3 is a quadratic residue of p.

Let $p = 12k+5$. Then $(p-1)/2 = 6k+2$, $q = 2k$ and $r = 2$. The inverse modulo p of 3 is $4k+2$. From $(4k+2)\cdot 1 \equiv 4k+2 \pmod p$ and $(4k+2)\cdot 2 \equiv -(4k+1) \pmod p$, $s = 1$. Since $2k \not\equiv 1 \pmod 2$, 3 is a quadratic non-residue of p.

Let $p = 12k+7$. Then $(p-1)/2 = 6k+3$, $q = 2k+1$ and $r = 0$ and $s = 0$. Since $2k+1 \not\equiv 0 \pmod 2$, 3 is a quadratic non-residue of p.

Let $p = 12k+11$. Then $(p-1)/2 = 6k+5$, $q = 2k+1$ and $r = 2$. The inverse modulo p of 3 is $4k+4$. From $(4k+4)\cdot 1 \equiv 4k+4 \pmod p$ and $(4k+4)\cdot 2 \equiv -(4k+3) \pmod p$, $s = 1$. Since $2k+1 \equiv 1 \pmod 2$, 3 is a quadratic residue of p.

In summary, $a = 3$ is a quadratic residue of an odd prime p if and only if $p = 12k \pm 1$.

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CONGRUENCES IDENTIFYING THE PRIMES

Richard J. McIntosh

Gauss (*Disquisitiones Arithmeticae*, 1801, art. 329) wrote:

The problem of distinguishing prime numbers from composite numbers ... is known to be one of the most important and useful in arithmetic. ... The dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated.

Wilson's Theorem, which was actually discovered by Leibniz and first proved by Lagrange, states that if p is prime then $(p-1)! \equiv -1 \pmod{p}$. It is easy to see that the converse of Wilson's Theorem also holds. Thus, Wilson's Theorem can be used to identify the primes: an integer $n \geq 2$ is prime if and only if $(n-1)! \equiv -1 \pmod{n}$. Another congruence identifying the primes, communicated to me in 1980 by Professor James P. Jones of the University of Calgary, is the following.

THEOREM 1. *An integer $n \geq 2$ is prime if and only if*

$$(n+1)(2n+1)(3n+1)\cdots((n-1)n+1) \equiv 0 \pmod{(n-1)!}.$$

Before proving this theorem let us examine the divisibility of products of integers in arithmetic progressions by factorials. It is easy to see that the product $n(n+1)$ $(n+2)\cdots(n+k-1)$ is divisible by k because one of the numbers in this product is divisible by k. It is much more difficult to show that this product is divisible by $k!$, unless one makes the clever observation that the binomial coefficient

$$\binom{n+k-1}{k} = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$$

is an integer. We claim that this product of consecutive integers can be replaced by a product of integers in arithmetic progression. In particular, we have

THEOREM 2. *If n is relatively prime to $k!$ and a is any integer, then the product $a(n+a)(2n+a)\cdots((k-1)n+a)$ is divisible by $k!$.*

We will need the following lemma.

Lemma. *If n is relatively prime to m and t is any integer, then the set $\{t, n+t, 2n+t, \ldots, (m-1)n+t\}$ forms a complete set of residues modulo m.*

Proof of Lemma. It suffices to show that for each $b \in \{0,1,2,\ldots,m-1\}$ the congruence $xn+t \equiv b \pmod{m}$ has a solution. Since n is relatively prime to m, n has an inverse modulo m, and thus $x \equiv (b-t)n^{-1} \pmod{m}$. $\square$

Proof of Theorem 2. By the fundamental theorem of arithmetic it suffices to show that for each prime p the multiplicity to which p divides the product $a(n+a)(2n+a)\cdots((k-1)n+a)$ is greater than or equal to the multiplicity to which p divides $k!$.

Let $p^i \leq k$, where p is prime and $i \geq 1$. We construct $[k/p^i]$ sets of p^i elements, where $[x]$ denotes the greatest integer $\leq x$. The first set consists of the first p^i terms of our product, the second set consists of the next p^i terms, and so on. We ignore any numbers left over at the end. Since n is relatively prime to $k!$, and thus relatively prime to p^i, by our lemma each set forms a complete set of residues modulo p^i, and hence exactly one element in each set will be divisible by p^i. Therefore the number of terms in the product $a(n+a)(2n+a)\cdots((k-1)n+a)$ that are divisible by p^i is $\geq [k/p^i]$. Recall that some of the ignored terms at the end may be divisible by p^i. However, the number of terms in the set $\{1, 2, 3, \ldots, k\}$ that are divisible by p^i is exactly $[k/p^i]$. This completes the proof of Theorem 2. $\square$

Returning to Theorem 1, it follows that if n is prime, then n is relatively prime to $(n-1)!$, and thus by Theorem 2, with a replaced by $n+1$ and k replaced by $n-1$, we have

$$(n+1)(2n+1)(3n+1)\cdots((n-1)n+1) \equiv 0 \pmod{(n-1)!}.$$

Conversely, if n is not prime, then n has a prime factor p and p divides $(n-1)!$. Since $n+1, 2n+1, 3n+1, \ldots, (n-1)n+1$ are congruent to 1 modulo p, it follows that p, and hence $(n-1)!$, does not divide the product $(n+1)(2n+1)(3n+1)\cdots((n-1)n+1)$. This completes the proof of Theorem 1. $\square$

We end with a conjectured congruence for primes, this time involving binomial coefficients. It is not difficult to show that $\binom{2n-1}{n-1} \equiv 1 \pmod{n}$ for all primes n. This congruence is also satisfied by squares of odd primes and cubes of primes ≥ 5. The only other solutions found so far are $n = 29\cdot 937, 787\cdot 2543, 69239\cdot 231433, 16843^4$ and 2124679^4. In 1819 Babbage observed that the stronger congruence $\binom{2n-1}{n-1} \equiv 1 \pmod{n^2}$ holds for all primes $n \geq 3$, and Wolstenholme, in 1862, proved that $\binom{2n-1}{n-1} \equiv 1 \pmod{n^3}$ for all primes $n \geq 5$. (See Dickson, *History of the Theory of Numbers*, vol. 1, p. 271.) I conjecture that the only composite solutions of

$$\binom{2n-1}{n-1} \equiv 1 \pmod{n^2}$$

are of the type $n = p^2$, where p is a prime satisfying

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4};$$

the only primes $p < 10^8$ satisfying this last congruence are 16843 and 2124679. Jones has conjectured that

$$n \geq 5 \quad \text{is prime if and only if} \quad \binom{2n-1}{n-1} \equiv 1 \pmod{n^3}.$$

(See Guy, *Unsolved Problems in Number Theory*, problem B31, p. 47, and Ribenboim, *The Book of Prime Number Records*, 2ed., p. 21.) I have verified this for all $n < 3 \times 10^7$. I've also shown that if $n = p^2$ satisfies the congruence modulo n^3, then $n = p^3$ satisfies the congruence modulo n^2. Thus, the first conjecture implies the second. The details of these computations and conjectures should soon be published elsewhere in a research paper titled "On primality and Wolstenholme's Theorem".

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THE OLYMPIAD CORNER

No. 152

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this number with a "pre-Olympiad" set, the 2nd U.K. Schools Mathematical Challenge which was written February 2nd, 1989. This is a multiple choice format for which the students must not have been over 15 years of age by August 31, 1989. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for collecting the set.

2nd U.K. SCHOOLS MATHEMATICAL CHALLENGE

2nd February 1989—Time allowed: 1 hour

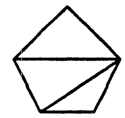
1. If the following fractions are written in order of size, which will be in the middle?

A. $\frac{1}{3}$ **B.** $\frac{3}{10}$ **C.** 31% **D.** 0.03 **E.** 0.303

2. I start counting at 19 and go on to 89, taking one second to say each number. How long do I take altogether?

A. 1 min 10 sec **B.** 1 min 29 sec **C.** 1 min 11 sec **D.** 1 min 19 sec **E.** 1 min exactly

3. The diagram shows a regular pentagon with two of its diagonals. If *all* the diagonals are drawn in, into how many areas will the pentagon be divided?

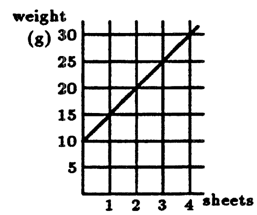


A. 4 **B.** 8 **C.** 11 **D.** 10 **E.** 5

4. The names of the whole numbers from one to twelve are written down in the order they occur in a dictionary. What is the fourth number on the list?

A. four **B.** five **C.** six **D.** seven **E.** nine

5. The graph shows how the weight of a letter (including the envelope) varies with the number of sheets of paper used. What is the weight of a single sheet of paper?



A. 5g **B.** 10g **C.** 15g **D.** 20g **E.** 25g

6. In a magic square, the three numbers in each row, in each column, and in each diagonal add up to the same number. When the magic square shown here is completed which of the following numbers is not used?

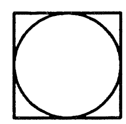
13		
	10	
9		7

A. 6 **B.** 8 **C.** 12 **D.** 14 **E.** 15

7. Which of these numbers is the average (mean) of the other four?

A. 28 **B.** 30 **C.** 26 **D.** 37 **E.** 29

8. The square has area 36cm^2. What is the area (in cm^2) of its inscribed circle?



A. 6π **B.** 9π **C.** 12π **D.** 36π **E.** 81π

9. An ant is crawling in a straight line from one corner of the table to the opposite corner when he bumps into a one centimetre cube of sugar. Instead of crawling round it, or eating his way through it, he climbs straight up and over it before continuing on his intended route. How much does this detour add to the expected length of his journey?

A. 1cm **B.** 2cm **C.** 3cm **D.** 4cm **E.** Can't be sure

10. When the 8th of December 1988 is written in abbreviated form 8/11/88, we see that $8 \times 11 = 88$. How many such dates will there be in 1990?

A. 5 **B.** 4 **C.** 3 **D.** 2 **E.** 1

11. I want to read the coded message shown here. I know that each number stands for a letter, but I have lost the 'key'. All I can remember is that 26 stands for 'A' and that 5 stands for 'V'. What does 23 stand for?

26 5 12 18 23

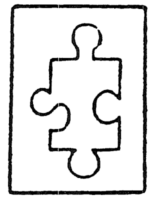
26 25 24 22

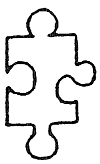
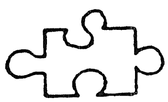
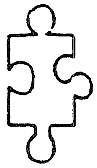
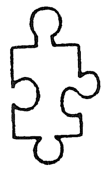
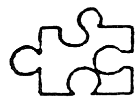
A. **B.** **C.** **D.** **E.**

12. One million is approximately the number of

A. grains of sand on a beach **B.** seconds in a day **C.** people in England and Wales **D.** grains of sugar in a cupful **E.** blades of grass on a football pitch

13. The Kryptor Faction required competitors to choose one of these five jigsaw pieces and to fit it into the hole on the right without turning the piece over. Which is the correct piece?

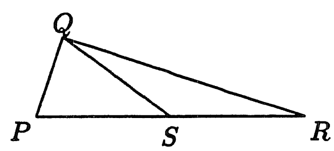


A.  **B.**  **C.**  **D.**  **E.** 

14. Baby's nearly 1 now. We've worked out how to weigh her, but nurse and I still have trouble measuring her height. She just *will* not stand up straight against our measuring chart. In fact she can't stand up at all yet! So we measure her upside down. Last year nurse held Baby's feet, keeping them level with the 140cm mark, while I read off the mark level with the top of Baby's head: 97cm. This year it was my turn to hold the feet. Being taller than nurse I held them against the 150cm mark while nurse crawled on the floor to read the mark level with the top of Baby's head: 84cm. How many centimetres has Baby grown in her first year?

A. 13 **B.** 237 **C.** 53 **D.** 23 **E.** 66

15. In the diagram the lengths SP, SQ and SR are equal and the angle SRQ is $x°$. What is the size (in degrees) of angle PQR?



A. 90 **B.** $2x$ **C.** $3x$ **D.** $180-x$ **E.** $180-2x$

16. This grid can be filled up using only the numbers 1, 2, 3, 4, 5 so that each number appears just once in a row, once in each column, and once in each diagonal. Which number goes in the centre square?

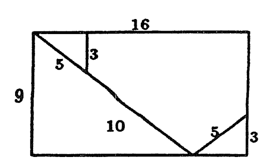
3	4			5
2				
				4

A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5

17. A car with five tyres (four road tyres and a spare) travelled 30,000 km. All five tyres were used equally. How many kilometres wear did each tyre receive?

A. 6,000 **B.** 7,500 **C.** 24,000 **D.** 30,000 **E.** 150,000

18. When a 16 by 9 rectangle is cut as shown here the pieces can be rearranged to make a square of perimeter



A. 32 **B.** 36 **C.** 40 **D.** 48 **E.** 51

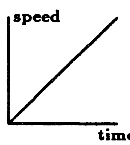
19. Our school dinners offer the same basic choice each day: **Starters**: soup or fruit juice; **Main course**: meat, fish, salad, chicken or curry; **Afters**: crumble, cake or sponge. I change my choice of course each day, trying each option in turn, going back to soup after fruit juice, to meat after curry, and to crumble after sponge. Today, to celebrate the 2nd SMC, I shall sit down to Soup, Meat and Crumble. How many school dinners will I have eaten before I next sit down to the same combination?

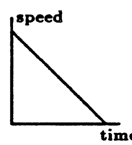
A. 3 **B.** 5 **C.** 10 **D.** 15 **E.** 30

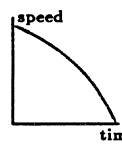
20. A knitted scarf uses three balls of wool. I start the day with b balls of wool and knit s scarves. How many balls of wool do I have at the end of the day?

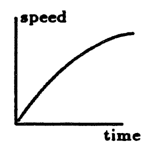
A. $3(b-s)$ **B.** $b-3-s$ **C.** $b-3s$ **D.** $3bs$ **E.** $s+3b$

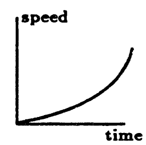
21. A stone is dropped from the top of a high tower. Which of the following graphs shows how its speed changes with time up to the moment when it hits the ground?

A. 

B. 

C. 

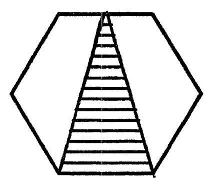
D. 

E. 

22. When Ann stands on Ben's shoulders she can just see over a wall. When Ben stands on Con's shoulders he can see nothing but bricks. When Con stands on Den's shoulders she can see over easily. Who are the tallest and the shortest?

A. Ann & Ben **B.** Den & Con **C.** Den & Ben **D.** Ann & Con **E.** Can't be sure

23. What fraction of the area of the regular hexagon is the shaded triangle?



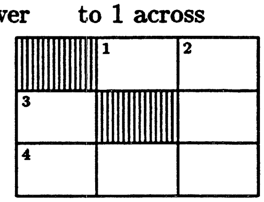
A. $\frac{1}{4}$ **B.** $\frac{1}{3}$ **C.** $\frac{3}{8}$ **D.** $\frac{5}{12}$ **E.** $\frac{1}{2}$

24. A crossnumber is like a crossword except that the answers are numbers, with one digit in each square. What is the sum of all the digits in the solution to this crossnumber?

A. 15 **B.** 17 **C.** 23 **D.** 26 **E.** Can't be sure

Across
1. See 3 down
3. Cube
4. 5 times answer to 3 down

Down
2. Square
3. 4 times answer to 1 across



25. How many patches (pentagons and hexagons) were stitched together to make this football?

A. 15 **B.** 19 **C.** 30 **D.** 32 **E.** 34

* * *

As an Olympiad level contest this issue we give the Final Round of the Czechoslovak Mathematical Olympiad for 1992. As there is no longer a country of that name, this also marks the last. Thanks again go to Georg Gunther who collected this contest (and others) while Canadian I.M.O. Team leader in Moscow.

CZECHOSLOVAK MATHEMATICAL OLYMPIAD

Final Round 1992

1. Let $p = (a_1, a_2, \ldots, a_{17})$ be any permutation of numbers $1, 2, \ldots, 17$. Let k_p denote the greatest index k for which the inequality

$$a_1 + a_2 + \cdots + a_k < a_{k+1} + a_{k+2} + \cdots + a_{17}$$

holds. Find the greatest and the smallest possible value of k_p and find the sum of all numbers k_p corresponding to all different permutations p.

2. Let a, b, c, d, e, f be the lengths of edges of a given tetrahedron and S be its surface area. Prove that

$$S \le \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2).$$

3. Find all natural numbers n which satisfy equalities

$$S(n) = S(2n) = S(3n) = \cdots = S(n^2)$$

if $S(x)$ denotes the sum of digits of the number x (in decimal).

4. Find all solutions of the equation

$$\cos 12x = 5 \sin 3x + 9 \tan^2 x + \cot^2 x.$$

5. A function f is defined on the interval $(0,1)$ by

$$f(x) = \begin{cases} x, & \text{for } x \text{ irrational} \\ \frac{p+1}{q}, & \text{for } x = \frac{p}{q} \text{ with } p, q \text{ coprime, positive and } p < q. \end{cases}$$

Find the maximum value of f on the interval $(7/8, 8/9)$.

6. In a plane the acute triangle ABC is given. Its altitude through vertex B intersects the circle with diameter AC in points P, Q and the altitude through point C intersects the circle with diameter AB in points M, N. Prove that all the points M, N, P, Q lie on the same circle.

* * *

First an apology. I did not list D.J. Smeenk, Zaltbommel, The Netherlands, when I gave the solution to problem 4 of the 1990 Dutch Mathematical Olympiad, Second Round [1994: 13]. His solution was inadvertently stuck with solutions he sent in to problems from the 1991 numbers of *Crux* which I am saving to use later on.

* * *

Now we turn to the remaining solutions sent in to problems in the December 1992 number of *Crux*. These are problems of the *Vietnamese National Olympiad in Mathematics For Secondary Schools*, 1991 [1992: 296–297].

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

holds for arbitrary $x, y, z \in \mathbb{R}$.

Solutions by Seung-Jin Bang, Albany, California; by Beatriz Margolis, Paris, France; by Pavlos Maragoudakis, Pireas, Greece; by Gottfried Perz, Pestalozzigymnasium, Graz, Austria; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Clearly the constant function $1/2$ satisfies the given equality. We show this is the only possibility.

Setting $x = y = z = 0$ one gets

$$\frac{1}{2}f(0) + \frac{1}{2}f(0) - f^2(0) \geq \frac{1}{4}$$

so $[f(0) - 1/2]^2 \leq 0$ and $f(0) = 1/2$. Similarly $f(1) = 1/2$.

Now setting $y = z = 0$ one obtains

$$\frac{1}{2}f(0) + \frac{1}{2}f(0) - f(x)f(0) \geq \frac{1}{4}\,,$$

giving $f(x) \leq 1/2$.

From setting $y = z = 1$,

$$\frac{1}{2}f(x) + \frac{1}{2}f(x) - f(x) \cdot f(1) \geq \frac{1}{4}$$

and $f(x) \geq 1/2$. Thus $f(x) = 1/2$ for all $x \in \mathbb{R}$.

2. Let k be an odd integer ($k > 1$). For every positive integer n, denote by $f(n)$ the greatest non-negative integer such that $(k^n - 1) \mid 2^{f(n)}$. Find a formula for $f(n)$ in terms of k and n.

Comment and solution by Seung-Jin Bang, Albany, California.

It is clear that $(k^n - 1)|2^{f(n)}$ should be replaced by $2^{f(n)}|(k^n - 1)$.

Let $n = 2^s m$ (where m is an odd integer). Since

$$k^{2^s m} - 1 = (k^{2^s} - 1)((k^{2^s})^{m-1} + \cdots + k^{2^s} + 1) = (k^{2^s} - 1) \cdot (\text{odd integer})$$

we have $f(n) = f(2^s)$.

Let $k = 2^p l + 1 = 2^q m - 1$, where l, m are odd integers. Since

$$(2r+1)^{2^t} + 1 = \sum_{i=1}^{2t} \binom{2^t}{i} (2r)^i + 2 = 2 \cdot (\text{odd integer})$$

for $t = 1, 2, \dots$ we have

$$\begin{aligned} k^{2^s} - 1 &= (k-1)(k+1)(k^2+1)(k^{2^2}+1)\dots(k^{2^{s-1}}+1) \\ &= (k-1)(k+1)2^{s-1} \cdot (\text{odd integer}) \end{aligned}$$

and $f(2^s) = p + q + s - 1$.

Answer: $f(n) = \alpha(k-1) + \alpha(k+1) + \alpha(n) - 1$, where $\alpha(a)$ is the greatest integer b such that $2^b | a$.

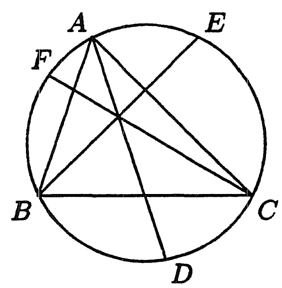
5. Let a triangle ABC with centre G be inscribed in a circle of radius R. Medians from vertices A, B, C meet the circle at D, E, F, respectively. Prove the inequalities

$$\frac{3}{R} \le \frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \le \sqrt{3}\left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right).$$

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let O be the centre and R the radius of the circumscribed circle. Then $AG \cdot GD = R^2 - OG^2$ so

$$\frac{1}{GD} = \frac{AG}{R^2 - DB^2}.$$



Therefore

$$\frac{1}{GD} = \frac{2}{3}\frac{r_a}{R^2 - OG^2}.$$

Similarly

$$\frac{1}{GE} = \frac{2}{3}\frac{r_b}{R^2 - OG^2} \quad \text{and} \quad \frac{1}{GF} = \frac{2}{3}\frac{r_c}{R^2 - OG^2}.$$

So

$$\frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} = \frac{2}{3} \cdot \frac{r_a + r_b + r_c}{R^2 - GO^2}. \tag{1}$$

By Leibnitz's Theorem $GO^2 = R^2 - (1/9)(a^2 + b^2 + c^2)$. Now (1) becomes

$$\frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} = 6\frac{r_a + r_b + r_c}{a^2 + b^2 + c^2}.$$

Finally it is enough to prove

$$\frac{3}{R} \le 6\frac{r_1 + r_b + r_c}{a^2 + b^2 + c^2} \le \sqrt{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \tag{2}$$

(i) We have $a^2+b^2=2r_c^2+c^2/2$ so

$$a^2+b^2+c^2=2r_c^2+\frac{3c^2}{2}\geq 2\sqrt{2r_c^2\frac{3c^2}{2}}.$$

So $(a^2+b^2+c^2)/c\geq 2\sqrt{3}\,r_c$. Similarly $(a^2+b^2+c^2)/a\geq 2\sqrt{3}\,r_a$ and $(a^2+b^2+c^2)/b\geq 2\sqrt{3}\,r_b$. By addition $(a^2+b^2+c^2)(1/a+1/b+1/c)\geq 2\sqrt{3}(r_a+r_b+r_c)$. This gives

$$6\frac{r_a+r_b+r_c}{a^2+b^2+c^2}\leq\sqrt{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(ii) We shall use the inequality

$$4Rr_a\geq b^2+c^2. \tag{3}$$

Now we have the following:

$$(3)\iff 4R^2\frac{2b^2+2c^2-a^2}{4}\geq(b^2+c^2)^2\iff\left(\frac{abc}{E}\right)^2\frac{2b^2+2c^2-a^2}{4}\geq(b^2+c^2)^2.$$

This is equivalent to

$$4a^2b^2c^2(2b^2+2c^2-a^2)\geq(a+b+c)(a+c-b)(a+b-c)(b+c-a)(b^2+c^2)^2$$

or

$$\begin{aligned}8a^2b^4c^2+8a^2b^2c^4-4a^4b^2c^2&\geq[(b+c)^2-a^2][a^2-(b-c)^2](b^2+c^2)^2\\&=[2a^2(b^2+c^2)-a^4-(b^2-c^2)^2](b^2+c^2)^2\\&=2a^2(b^2+c^2)^3-a^4(b^2+c^2)^2-(b^2+c^2)^2(b^2-c^2)^2.\end{aligned}$$

This becomes

$$2a^2b^4c^2+2a^2b^2c^4-2a^2b^6-2a^2c^6+(b^2+c^2)^2(b^2-c^2)^2+a^4(b^2-c^2)^2\geq 0$$

or

$$2a^2[b^4(c^2-b^2)+c^4(b^2-c^2)]+[(b^2+c^2)^2+a^4](b^2-c^2)^2\geq 0$$

which becomes

$$-2a^2(b^2-c^2)^2(b^2+c^2)+[(b^2+c^2)^2+a^4](b^2-c^2)^2\geq 0$$

whence

$$(b^2-c^2)^2[(b^2+c^2)^2+a^4-2a^2(b^2+c^2)]\geq 0$$

which is $(b^2-c^2)^2((b^2+c^2)-a^2)^2\geq 0$ and that is true!

Similarly $4Rr_b\geq a^2+c^2$ and $4Rr_c\geq a^2+b^2$. On addition $4R(r_a+r_b+r_c)\geq 2(a^2+b^2+c^2)$ so

$$\frac{3}{R}\leq 6\frac{r_a+r_b+r_c}{a^2+b^2+c^2}$$

as required to complete the solution.

6. Let x, y, z be positive real numbers with $x \geq y \geq z$. Prove

$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\geq x^2+y^2+z^2.$$

Solution by Pavlos Maragoudakis, Pireas, Greece.

$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\geq x^2+y^2+z^2$$

just in case

$$\frac{x^2(y-z)}{z}+\frac{z^2(x-y)}{y}\geq y^2\left(\frac{x-z}{z}\right)=\frac{y^2}{x}(x-y)+\frac{y^2}{x}(y-z).$$

The problem is equivalent to showing

$$(y-z)\left(\frac{x^2}{z}-\frac{y^2}{x}\right)\geq (x-y)\left(\frac{y^2}{x}-\frac{z^2}{y}\right)$$

or

$$(y-z)\left(\frac{x^2}{z}-\frac{y^2}{z}+\frac{y^2}{z}-\frac{y^2}{x}\right)\geq (x-y)\left(\frac{y^2}{x}-\frac{z^2}{x}+\frac{z^2}{x}-\frac{z^2}{y}\right).$$

This becomes

$$\frac{(y-z)(x-y)(x+y)}{z}+\frac{(y-z)y^2(x-z)}{xz}\geq\frac{(x-y)(y-z)(y+z)}{x}+\frac{(x-y)z^2(y-x)}{xy}.$$

Now this is

$$(x-y)(y-z)\left(\frac{x+y}{z}-\frac{y+z}{x}\right)+\frac{y^2(y-z)(x-z)}{xz}+\frac{z^2(x-y)^2}{xy}\geq 0.$$

Finally we obtain

$$\frac{(x-y)(y-z)(x-z)(x+y+z)}{xz}+\frac{y^2(y-z)(x-z)}{xz}+\frac{z^2(x-y)^2}{xy}\geq 0$$

which is true since $x \geq y \geq z$. Note that equality holds just if $x = y = z$.

Alternate solution (using calculus) by Seung-Jin Bang, Albany, California; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The given inequality is equivalent to

$$x^2y+\frac{y^2}{x}+\frac{x}{y}\geq x^2+y^2+1 \quad \text{for } x\geq y\geq 1.$$

Let $f(x)=x^3y+y^2+x^2/y-x^3-(y^2+1)x$ (for y fixed).

Then $f'(x) = 3x^2y + (2x)/y - 3x^3 - (y^2+1)$ and $f''(x) = 6xy + 2/y - 6x = 6x(y-1) + 2/y > 0$. From $f'(y) = (3y^2 - y - 1)(y-1) = (y-1)((2y+1)(y-1) + y^2) \geq 0$ we have $f'(x) \geq 0$ (for $x \geq y$).

Since $f(y) = y^2(y-1)^2 \geq 0$ we conclude that $f(x) \geq 0$ for $x \geq y$, which completes the proof. Note that the equality holds only if $x = y = z$.

9. Given is a sequence of positive real numbers $x_1, x_2, \ldots, x_n, \ldots$ defined by the formula: $x_1 = 1$, $x_2 = 9$, $x_3 = 9$, $x_4 = 1$,

$$x_{n+4} = \sqrt[n]{x_n x_{n+1} x_{n+2} x_{n+3}} \qquad \text{if } n \geq 1.$$

Prove that this sequence is convergent and find its limit.

Solutions by Seung-Jin Bang, Albany, California; and by Pavlos Maragoudakis, Pireas, Greece. We give the solution of Maragoudakis.

From the formula we have $x_1 = 1$, $x_2 = 9$, $x_3 = 9$, $x_4 = 1$, $x_5 = 3^4$, $x_6 = 3^4$, $x_7 = 3^{10/3}$.

First we shall prove by induction that $1 \leq x_n \leq 3^4$ for all $n \geq 1$. It is enough to prove that $1 \leq x_k \leq 3^4$, $\forall k < n$, for all $n \geq 1$.

For $n \leq 7$, the statement is true.

Now suppose that

$$1 \leq x_k \leq 3^4, \quad \forall k < n \qquad \text{where } n \geq 1. \tag{1}$$

We argue that $1 \leq x_k \leq 3^4$, $\forall k < n+1$. It is enough to prove that $1 \leq x_n \leq 3^4$. From the first remark we can assume $n \geq 8$. Then

$$x_n = \sqrt[n-4]{x_{n-4} \cdot x_{n-3} \cdot x_{n-2} \cdot x_{n-1}}\ .$$

By (1), $1 \leq x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1} \leq 3^4$. So

$$1 \leq x_n \leq \sqrt[n-4]{3^{16}} = 3^{16/(n-4)}.$$

Also $n \geq 8 \Rightarrow 16/(n-4) \leq 4$ so $1 \leq x_n \leq 3^4$. This completes the induction.

Now for $n \geq 5$

$$1 \leq x_n = \sqrt[n-4]{x_{n-4} \cdot x_{n-3} \cdot x_{n-2} \cdot x_{n-1}} \leq 3^{16/(n-4)}.$$

But $\lim_{n\to\infty} 3^{16/(n-4)} = 1$ so $\lim_{n\to\infty} x_n = 1$.

* * *

Next an update on one of the errors that I missed, and for which a corrected solution by Stan Wagon was published earlier. Another faithful reader spotted the earlier flaw, gave us a solution and a reference.

8. [1992: 196; 1993: 228, 260] *Proposed by the U.S.S.R.*

Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence become periodic after a finite number of terms?

Comment by Waldemar Pompe, student, University of Warsaw, Poland.

The problem has appeared earlier. I found it in a Russian book "Foreign Mathematical Olympiads," published in 1987!

*   *   *

Next we give a nice geometric solution to a problem of the Canadian Mathematical Olympiad. The official solution we published involved more trigonometry.

3. [1993: 159, 196] 1993 *Canadian Mathematical Olympiad.*

In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

Let AH be the altitude, G be the centroid and M the midpoint of BC of a given triangle. Since M is a circumcentre of ΔBCG, we have $MB = MC = MG = x$ and $AG = 2x$. Therefore

$$\cot B + \cot C = \frac{BH}{AH} + \frac{CH}{AH} = \frac{BC}{AH} \geq \frac{BC}{AM} = \frac{2x}{3x} = \frac{2}{3}\,.$$

Equality holds only if $AB = AC$.

This solution also works when one of the angles B or C is obtuse.

*   *   *

We finish this month's Corner with three comments/solutions by Murray Klamkin regarding problems from the 1991 numbers of the Corner.

2. [1986: 202; 1991: 200] 1986 *USAMO Training Session.*

Determine the maximum value of

$$x^3 + y^3 + z^3 - x^2y - y^2z - z^2x$$

for $0 \leq x, y, z \leq 1$.

Comment by Murray S. Klamkin, University of Alberta.

Here we give a generalization with a simpler solution.

We find the maximum value over x, y, z of

$$S = a(x^p + y^p + z^p) - b(x^q y^r + y^q z^r + z^q x^r) - c(x^q y^r + y^q z^r + z^q x^r)$$

where $a, b, c \geq 0$, $p \geq q \geq r \geq 0$, and $0 \leq x, y, z \leq 1$.

Clearly,

$$S \leq a(x^q + y^q + z^q) - (b + c)(x^q y^q + y^q z^q + z^q x^q).$$

Since the latter right hand expression is linear in x^q, y^q, z^q, it takes on its maximum value at the end points $x^q, y^q, z^q = 0$ or 1. Hence,

$$S_{\max} = \max\{3(a - b - c), 2a - b - c, a\}.$$

We can extend the result further by having different suitable exponents, e.g., $x^{p_1} + y^{p_2} + z^{p_3}$ instead of $x^p + y^p + z^p$ where $p_1 \geq p_2 \geq p_3 \geq q$, etc. Also, one can increase the number of variables.

3. [1989: 65; 1991: 8] 1987 *Annual Greek High School Competition.*

Let A be an $n \times n$ matrix such that $A^2 - 3A + 2I = 0$ where I is the identity matrix. Prove that for all natural numbers k,

$$A^{2k} - (2^k + 1)A^k + 2^k I = 0.$$

Comment by Murray S. Klamkin, University of Alberta.

More generally,

$$\text{if} \quad \prod (A - x_i I) = 0, \quad \text{then} \quad \prod (A^{m_i} - x_i^{m_i} I) = 0$$

where the products are over $i = 1$ to n, the m_i's are positive integers and the x_i's are scalars. This follows immediately from the factorization

$$(A^{m_i} - x_i^{m_i} I) = \prod (A - \omega^j x_i I)$$

where the product is over 1 to m_i, ω is a primitive m_ith root of 1 and that the factors on the right hand side commute, i.e., $(A - xI)(A - yI) = (A - yI)(A - xI)$.

3. [1991: 2; 1992: 102] *Celebration of Chinese New Year Contest.*

Let $f(x) = x^{99} + x^{98} + x^{97} + \cdots + x^2 + x + 1$. Determine the remainder when $f(x^{100})$ is divided by $f(x)$.

Comment by Murray S. Klamkin, University of Alberta.

A generalization with a simpler proof than the previous one is as follows:

Let $f(x) = 1 + x + \cdots + x^{n-1}$. Then the remainder when $f(x^{mn})$ is divided by $f(x)$ is n where m is any positive integer. Here, $f(x) = (x^n - 1)/(x - 1)$ and we rewrite $f(x^{mn})$ in the form

$$f(x^{mn}) = (x^{mn(n-1)} - 1) + (x^{mn(n-2)} - 1) + \cdots + (x^{mn} - 1) + n.$$

Then since $x^{kn} - 1$ is divisible by $x^n - 1$, the remainder is immediately n.

The case when $n = 5$ with similar proof appears in a problem by Norman Anning, *School Science and Math.*, **54** (1954), 576.

* * *

That completes this number of the Corner. The Olympiad season approaches. Send me your national/regional contests, and your nice solutions. Also please send any suitable pre-Olympiad material.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **September 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1890. [1993: 265] **(corrected)** *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let n be a positive integer and let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad g(x) = \frac{k}{a_n}x^n + \frac{k}{a_{n-1}}x^{n-1} + \cdots + \frac{k}{a_1}x + \frac{k}{a_0},$$

where k and the a_i's are positive real numbers. Prove that

$$f(g(1))g(f(1)) \geq 4k.$$

When does equality hold?

[*Editor's note.* The error in the previous statement of this problem was due to the editor's overzealous "simplification" of the proposer's original version. Sorry about that.]

1911. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let B and E be two opposite vertices of the regular icosahedron. Consider the random walk over the edge-skeleton of the icosahedron, beginning at B. Each time he arrives at a vertex, the walker continues the walk along any one of the five edges emanating from that vertex, with equal probability of each choice. The walk ends when the walker reaches vertex E. Find the expected length of the walk (in number of edges).

1912. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB \neq AC$. Similar triangles ABD and ACE are drawn outwardly on the sides AB and AC of ΔABC, so that $\angle ABD = \angle ACE$ and $\angle BAD = \angle CAE$. CD and BE meet AB and AC at P and Q respectively. Prove that $AP = AQ$ if and only if

$$[ABD] \cdot [ACE] = [ABC]^2,$$

where $[XYZ]$ denotes the area of triangle XYZ. (This problem is an extension of *Crux* 1537 [1991: 182].)

1913. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

I was at a restaurant for lunch the other day. The bill came, and I wanted to give the waiter a whole number of dollars, with the difference between what I give him and the bill being the tip. I always like to tip between 10 and 15 percent of the bill. But if I gave him a certain number of dollars, the tip would have been **less** than 10% of the bill, and if instead I gave him one dollar more, the tip would have been **more** than 15% of the bill. What was the largest possible amount of the bill? [*Editor's note to non-North American readers*: your answer should be in dollars and cents, where there are (reasonably enough) 100 cents in a dollar.]

1914. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Let $A_1A_2\ldots A_n$ be a regular n-gon, with $M_1, M_2, \ldots, M_n$ the midpoints of the sides. Let P be a point in the plane of the n-gon. Prove that

$$\sum_{i=1}^{n} PM_i \geq \cos(180°/n)\sum_{i=1}^{n} PA_i.$$

1915. *Proposed by Richard K. Guy, University of Calgary.*

Is $\binom{n}{r}$ ever relatively prime to $\binom{n}{s}$ for $0 < r < s < n$? (This is not a new problem. Its history will be revealed when a solution is published.)

1916. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2\ldots A_{n+1}$ be a regular simplex inscribed in a unit sphere in n-dimensional space, and let P be a point on the sphere. Prove that

$$\sum_{i=1}^{n+1}(A_iP)^4 = \frac{4(n+1)^2}{n}\ .$$

1917. *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

For any positive integer n, evaluate a_n/b_n where

$$a_n = \sum_{k=1}^{n}\tan^2\frac{k\pi}{2n+1}\ , \qquad b_n = \prod_{k=1}^{n}\tan^2\frac{k\pi}{2n+1}\ .$$

1918. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with circumcentre O and incentre I, and K, L, M are the midpoints of BC, CA, AB respectively. Let E and F be the feet of the altitudes from B and C respectively.

(a) If $\overline{OK}^2 = \overline{OL}^2 + \overline{OM}^2$, show that E, F and O are collinear, and determine all possible values of $\angle BAC$.

(b) If instead $\overline{OK} = \overline{OL} + \overline{OM}$, show that E, F and I are collinear, and determine all possible values of $\angle BAC$.

1919. *Proposed by H.N. Gupta, University of Regina.*

In Lotto 6/49, six balls are randomly drawn (without replacement) from a bin holding balls numbered from 1 to 49. Find the expected value of the kth lowest number drawn, for each $k = 1, 2, \ldots, 6$.

1920. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle.

(a) Prove that, for any $0 < \lambda \le 2$,

$$\frac{1}{(1+\lambda)^2} < \frac{(a+b)(b+c)(c+a)}{(\lambda a+b+c)(a+\lambda b+c)(a+b+\lambda c)} \le \left(\frac{2}{2+\lambda}\right)^3,$$

and that both bounds are best possible.

(b)* What are the bounds for $\lambda > 2$?

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1695. [1991: 301; 1992: 287] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ with $a_0 > 0$ and

$$a_0 + \frac{a_0+a_2}{3} + \frac{a_2+a_4}{5} + \frac{a_4}{7} < 0.$$

Prove that there exists at least one zero of $p(x)$ in the interval $(-1, 1)$.

III. *Comment by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Here is an answer to the editor's question appearing on [1992: 288]: the interval $(-3/\sqrt{14}, 3/\sqrt{14})$, shown on [1992: 287] to contain a zero of $p(x)$, can not be further shortened.

We take $a_1 = a_3 = a_5 = 0$, $a_0 = 1$ and

$$a_2 = \frac{14}{\varepsilon}, \quad a_4 = -\frac{196 + 35\varepsilon + \varepsilon^2}{9\varepsilon}$$

where $\varepsilon > 0$ is a constant. It is easy to verify that

$$a_0 + \frac{a_0+a_2}{3} + \frac{a_2+a_4}{5} + \frac{a_4}{7} < 0,$$

because this is equivalent to $35a_0 + 14a_2 + 9a_4 < 0$, which is true. Also $p(-x) = p(x)$, and we now prove that $p(x) > 0$ for all $0 \le x \le 3/\sqrt{14+\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, this establishes the result.

Letting $t = x^2$, we have $p(x) = f(t) = a_4t^2 + a_2t + 1$ where $0 \le t \le 9/(14+\varepsilon)$. Since $f(0) = 1 > 0$ and

$$\begin{aligned}
f\left(\frac{9}{14+\varepsilon}\right) &= \frac{9}{(14+\varepsilon)^2}\left(\frac{(14+\varepsilon)^2}{9} + \frac{14}{\varepsilon}(14+\varepsilon) - \frac{196+35\varepsilon+\varepsilon^2}{\varepsilon}\right) \\
&= \frac{9}{(14+\varepsilon)^2}\left(\frac{(14+\varepsilon)^2}{9} - 21 - \varepsilon\right) = \frac{9}{(14+\varepsilon)^2}\left(\frac{7}{9} + \frac{19\varepsilon}{9} + \frac{\varepsilon^2}{9}\right) > 0,
\end{aligned}$$

and since $a_4 < 0$ [so f is a parabola opening downward], hence

$$f(t) \ge \min\left\{f(0), f\left(\frac{9}{14+\varepsilon}\right)\right\} > 0$$

for $0 \le t \le 9/(14+\varepsilon)$. So $p(x) > 0$ for all $0 \le x \le 3/\sqrt{14+\varepsilon}$.

* * * * *

1816. [1993: 49] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

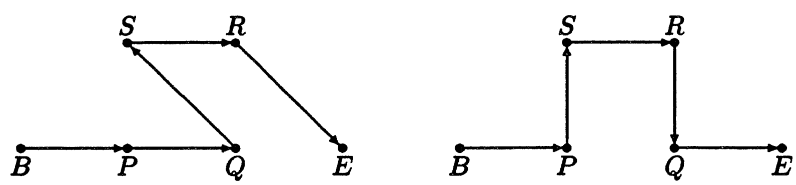
Given a finite set $\mathcal{S}$ of $n+1$ points in the plane, with two distinguished points B and E in $\mathcal{S}$, consider all polygonal paths $\mathcal{P} = P_0P_1\ldots P_n$ whose vertices are all points of $\mathcal{S}$, in any order except that $P_0 = B$ and $P_n = E$. For such a path $\mathcal{P}$ define $l(\mathcal{P})$ to be the length of $\mathcal{P}$ and

$$a(\mathcal{P}) = \sum_{i=1}^{n-1} \theta(\overrightarrow{P_{i-1}P_i}, \overrightarrow{P_iP_{i+1}}),$$

where $\theta(\mathbf{v},\mathbf{w})$ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$, $0 \le \theta(\mathbf{v},\mathbf{w}) \le \pi$. Prove or disprove that the minimum values of $l(\mathcal{P})$ and of $a(\mathcal{P})$ are attained for the same path $\mathcal{P}$.

Solution by the proposer.

The statement is false; here is a counterexample. Let $n = 5$, and let $\mathcal{S} = \{B,E,P,Q,R,S\}$ where P and Q trisect BE and $PQRS$ is a square. Then of the two paths $BPQSRE$ and $BPSRQE$, one has $a(\mathcal{P})$ greater and the other has $l(\mathcal{P})$ greater.



[*Editor's note.* In fact it is clear that $BPSRQE$ has the minimum $l(\mathcal{P})$ over all paths $\mathcal{P}$, while

$$a(BPQSRE) = \frac{3\pi}{4} + \frac{3\pi}{4} + \frac{\pi}{4} = \frac{7\pi}{4} < 2\pi = a(BPSRQE).$$

So as the proposer says, the shortest path need not be the "least curved" one.]

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1817. [1993: 50] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

For a natural number n, $d(n)$ denotes the number of positive integer divisors of n (including n itself) and $\phi(n)$ denotes the number of positive integers less than n and relatively prime to n. Find all positive integers n so that $d(n) + \phi(n) = n$.

Composite solution by Nick Lord, Tonbridge School, Kent, England, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We show that

$$d(n) + \phi(n) = \begin{cases} n & \text{if } n = 6, 8, 9, \\ n+1 & \text{if } n = 1, 4 \text{ or a prime,} \end{cases}$$

and $d(n) + \phi(n) < n$ otherwise.

Clearly $d(1)+\phi(1) = 1+1 = 2$ and if n is a prime, then $d(n)+\phi(n) = 2+n-1 = n+1$. We assume then $n \geq 4$ and is a composite. Since 1 is the only positive integer counted by both $d(n)$ and $\phi(n)$, we have $d(n) + \phi(n) < n$ if there exist distinct $a, b \in \{2, 3, \cdots, n-1\}$ which are neither divisors of n nor relatively prime to n. If $n \geq 10$ is even, then we can take $a = n-4$, $b = n-2$ since $b > a > n/2$ implies neither of them are divisors of n. If $n > 16$ is odd, then it has a prime factor p with $p \leq \sqrt{n} < n/4$ and hence we can take $a = 2p$, $b = 4p$. Directly checking the remaining cases when $n = 4, 6, 8, 9$ and 15 reveals that $d(4) + \phi(4) = 3 + 2 = 5$, $d(6) + \phi(6) = 4 + 2 = 6$, $d(8) + \phi(8) = 4 + 4 = 8$, $d(9) + \phi(9) = 3 + 6 = 9$, and $d(15) + \phi(15) = 4 + 8 = 12$.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Albany, California; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, Pireas, Greece; HENRY J. RICARDO, Tappan, New York; SHAILESH SHIRALI, Rishi Valley School, India; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Three other readers submitted the correct solution without proof, and one reader sent in an incorrect solution.

Actually, strictly according to the definition given in the problem, $\phi(1) = 0$ (there are no positive integers less than 1!), so $n = 1$ should also be a solution. However, readers apparently realized that $\phi(1) = 1$ is the correct definition.

* * * * *

1821. [1993: 77] *Proposed by Gerd Baron, Technische Universität, Vienna, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine all pairs (a, b) of nonnegative real numbers such that the functional equation

$$f(f(x)) + f(x) = ax + b$$

has a unique continuous solution $f : \mathbb{R} \to \mathbb{R}$.

Solution by the 1993 U.K. I.M.O. team.

The required pairs are precisely: all $(2, b)$ where $b > 0$.

There exists a *linear* solution $f(x) = cx + d$ precisely when $c(cx+d)+d+(cx+d) = ax + b$ for all x; i.e., when

$$c(c+1) = a \tag{1}$$

and

$$(c+2)d = b. \tag{2}$$

For any value of $a \geq 0$, equation (1) yields two possible values of c. If $c \neq -2$, then each value of c corresponds, via equation (2), to a unique value of d, namely $d = b/(c+2)$. Hence for any $a \geq 0$, if $c \neq -2$, there are at least two linear (hence continuous) functions satisfying the given functional equation. Thus this equation can have a unique continuous solution only for values of a which give rise to $c = -2$ — that is, when $a = 2$. We therefore assume from now on that, for some $b \geq 0$, f satisfies

$$f(f(x)) + f(x) = 2x + b \quad \text{for all } x. \tag{3}$$

If $b = 0$, then we get two possible linear functions, namely $f(x) = -2x$ and $f(x) = x$. Thus we need only consider the case where $b \neq 0$: $c = -2$ is then impossible, so $c = 1$, and there is a unique linear function, $f(x) = x + b/3$. We prove that, for each $b > 0$, this is the only continuous function satisfying (3).

Let f be any such function. Since $b \neq 0$, (3) shows that for every x, $f(x) \neq x$, so the graph of f lies wholly above or wholly below $y = x$. Since $b > 0$, we cannot have $f(x) < x$ for all x; hence $f(x) > x$ for all x. By (3), f must be one-one. Since f is continuous, f is either increasing or decreasing, and must in fact be increasing (otherwise the graphs $y = f(x)$ and $y = x$ would cross).

Lemma. *For all x, we have*

(i) $x + b/4 < f(x) < x + b/2$;

(ii) *more generally, for all integers $n \geq 1$,*

$$x + b\left(\frac{1 - 4^{-n}}{3}\right) < f(x) < x + b\left(\frac{1 + 2 \cdot 4^{-n}}{3}\right).$$

Proof. We prove part (i); the induction step for part (ii) is almost identical. We know that $f(x) > x$ for all x. Suppose $f(x_0) \geq x_0 + b/2$ for some x_0. Then

$$f(f(x_0)) + f(x_0) > f(x_0) + x_0 + b/2 > 2x_0 + b,$$

contradicting (3). Hence $x < f(x) < x + b/2$. If $f(x_0) \leq x_0 + b/4$ for some x_0, then

$$f(f(x_0)) + f(x_0) < f(x_0) + b/2 + x_0 + b/4 \leq 2x_0 + b,$$

which also contradicts (3). $\square$

Taking limits in part (ii) shows that the only possible f is $f(x) = x + b/3$.

Also solved by MARCIN E. KUCZMA, Warszawa, Poland; WALDEMAR POMPE, student, University of Warsaw, Poland; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; and the proposer. Three other readers gave the correct set of pairs (a, b) *without a complete proof.*

Sinefakopoulos's solution was the same as the above one.

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1822. [1993: 77] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

AB and AC are tangent to a circle Γ at B and C respectively. Let D be a point on AB produced beyond B, and let P be the second intersection of Γ with the circumcircle of ΔACD. Let Q be the foot of the perpendicular from B to CD. Prove that $\angle DPQ = 2\angle ADC$.

Combination of solutions of Waldemar Pompe, student, University of Warsaw, Poland, and the proposer.

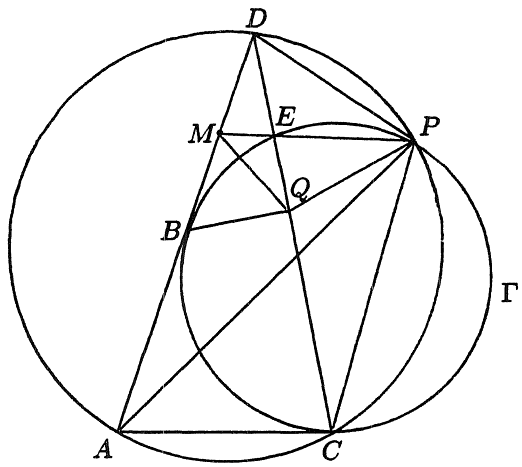
Let E be the second intersection of Γ with the line CD. Then

$$\angle APD = \angle ACD = \angle CPE$$

since AC is tangent to Γ at C, so

$$\angle EPD = \angle CPA = \angle CDA.$$

This means that the circumcircle of ΔDEP is tangent to AD at D. Let PE and AD meet at M. Then



$$(DM)^2 = ME \cdot MP = (MB)^2$$

since MB is tangent to Γ at B, which gives us $DM = MB$. Therefore M is the midpoint of BD and is also the circumcentre of ΔDBQ. This means that

$$\angle DPE = \angle EDM = \angle DQM.$$

Thus D, M, Q, P are concyclic. Hence

$$\angle DPQ = \angle DPE + \angle MPQ = \angle ADC + \angle ADC = 2\angle ADC.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; P. PENNING, Delft, The Netherlands; and A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands.

The solutions of Pompe and the proposer were almost identical.

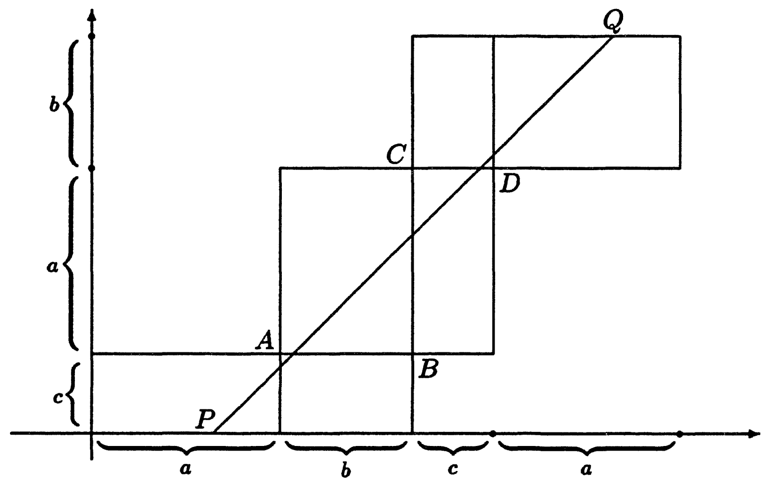
* * * * *

1823. [1993: 77] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A rectangular box is to be decorated with a ribbon that goes across the faces and makes various angles with the edges. If possible, the points where the ribbon crosses the edges are chosen so that the length of the closed path is a local minimum. This will ensure that the ribbon can be tightened and tied without slipping off. Is there always a minimal path that crosses all six faces just once?

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let the box have size $a \times b \times c$, where $a = \max\{a, b, c\}$. Unfold the surface and place it on the coordinate plane as shown:



Segment PQ illustrates the ribbon; rectilinearity is imposed by the minimality of its length. The labelling (a, b, c) may always be chosen so that the ribbon crosses the faces in the order as in the diagram.

Let $P = (p, 0)$, $0 < p < a$. Then $Q = (a + b + c - p, a + b + c)$ and hence the line PQ is given by the equation $y = x - p$. The existence of the minimal path crossing all faces just once is equivalent to the requirement that (by a suitable choice of p) the points A and C should lie above line PQ and points B and D should lie below it.

As $A = (a, c)$, $B = (a + b, c)$, $C = (a + b, a + c)$, $D = (a + b + c, a + c)$, this requirement yields the system of inequalities

$$a - p < c < a + b - p < a + c < a + b + c - p;$$

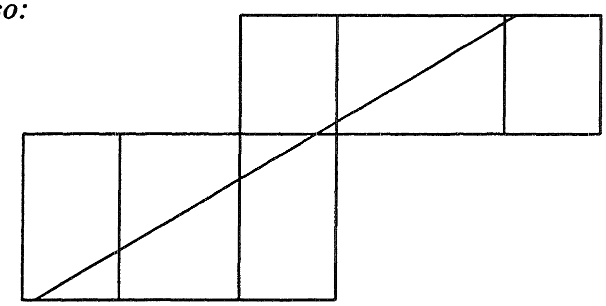
equivalently,

$$p > \max\{a - c, b - c\} = a - c, \qquad p < \min\{a + b - c, b\} = b.$$

Thus p exists if and only if $a - c < b$; that is, when a, b, c are the lengths of sides of a triangle!

Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; P. PENNING, Delft, The Netherlands; and the proposer.

Dou's solution, a seven–page tour de force *of case analysis and three-dimensional diagrams, includes examples of minimal paths which do* **not** *cross the faces of a box in the above manner! Rather the ribbon traverses a pair of opposite faces from one side to the opposite side, like so:*



However, it seems that no box can be decorated this way that cannot also be decorated the other way.

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1824. [1993: 77] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

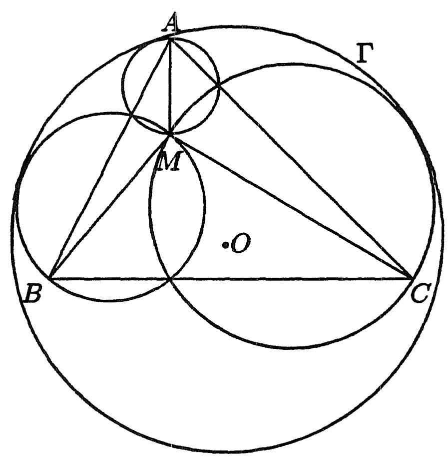
Let ABC be a triangle and M a point in its plane. We consider the circles with diameters AM, BM, CM and the circle containing and internally tangent to these three circles. Show that the radius P of this large circle satisfies $P \geq 3r$, where r is the inradius of ΔABC.

Correction by Toshio Seimiya, Kawasaki, Japan.

The conclusion $P \geq 3r$ is not right, the correct inequality must be $P \geq 2r$.

We denote the large circle of radius P by Γ. Let O be the center of Γ. As the circle with diameter AM is contained in Γ, A lies on or within Γ. Therefore $P \geq AO$. Similarly we get $P \geq BO$ and $P \geq CO$. Hence we have

$$3P \geq AO + BO + CO. \qquad (1)$$



Using item 12.14 of Bottema et al, *Geometric Inequalities*, we get

$$AO + BO + CO \geq 6r. \qquad (2)$$

From (1) and (2) we have $3P \geq 6r$, thus $P \geq 2r$. Equality holds when ΔABC is equilateral and M is its circumcenter.

Counterexample also found by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; and WALDEMAR POMPE, student, University of Warsaw, Poland.

The proposer's solution contained a minor arithmetical error which, if corrected, leads to the result $P \geq 2r$.

1825. [1993: 77] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose that the real polynomial $x^4 + ax^3 + bx^2 + cx + d$ has four positive roots. Prove that $abc \geq a^2d + 5c^2$.

I. *Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Given that the real polynomial has four positive roots it must factorize over the reals into the form

$$(x^2 - Ax + B)(x^2 - Cx + D)$$

where

$$A^2 \geq 4B > 0 \quad \text{and} \quad C^2 \geq 4D > 0, \tag{1}$$

and we may take $A, C > 0$. Comparing coefficients gives

$$a = -(A + C), \quad b = B + D + AC, \quad c = -(AD + BC), \quad d = BD.$$

From (1), evidently

$$B^2 + D^2 + A^2D + C^2B \geq B^2 + D^2 + 8BD \geq 10BD.$$

Hence

$$ACB^2 + ACD^2 + A^3CD + AC^3B \geq 10ABCD.$$

Also from (1),

$$A^2BC^2 \geq 4B^2C^2 \quad \text{and} \quad A^2C^2D \geq 4A^2D^2.$$

Putting these results together,

$$\begin{aligned} ACB^2 + ACD^2 + A^3CD + AC^3B + A^2D^2 + B^2C^2 + A^2BC^2 + A^2C^2D \\ \geq 5A^2D^2 + 10ABCD + 5B^2C^2. \end{aligned} \tag{2}$$

After some elementary algebra this gives $abc - a^2d \geq 5c^2$. [*Editor's note.* For instance, the left side of (2) is

$$\begin{aligned} &(A + C)(B^2C + AD^2 + A^2CD + ABC^2) \\ &\quad = (A + C)[AD(D + AC) + BC(B + AC)] \\ &\quad = (A + C)[(B + D + AC)(AD + BC) - BAD - DBC] \\ &\quad = (A + C)(B + D + AC)(AD + BC) - (A + C)^2BD \\ &\quad = abc - a^2d, \end{aligned}$$

while the right side is clearly $5c^2$.]

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In familiar notation, $-a$, b, $-c$, d are respectively the elementary symmetric functions S_1, S_2, S_3, S_4 of the four roots. Using the also familiar notation $S_k = \binom{n}{k} p_k$ with (for us) $n = 4$, we get $S_1 = 4p_1$, $S_2 = 6p_2$, $S_3 = 4p_3$, $S_4 = p_4$, and thus the claimed inequality is equivalent to

$$6p_1p_2p_3 \geq p_1^2p_4 + 5p_3^2. \tag{1}$$

Now the Newton–Maclaurin inequalities state that

$$r < s \quad \Rightarrow \quad p_{r-1}p_s \leq p_r p_{s-1} \tag{2}$$

(see e.g. pp. 95–97 of D.S. Mitrinović, *Analytic Inequalities*, Berlin, 1968). Putting $r = 2$ and $s = 4$ in (2) we get $p_1p_4 \leq p_2p_3$, and (1) would follow from the sharper inequality $6p_1p_2p_3 \geq p_1p_2p_3 + 5p_3^2$, i.e.

$$p_1p_2 \geq p_3. \tag{3}$$

[*Editor's comments.* Janous then gives a proof of (3) using Lagrange multipliers. However, this is just the special case $r = 1$, $s = 3$ of (2), where we define $p_0 = 1$.

Of course, (3) is true for any value of n, but Janous then suggests a different generalization of (3). Writing it as $S_1S_2 \geq 6S_3$ and then in terms of the four positive roots (denoted a, b, c, d by Janous with the apologetic remark "hopefully causing no confusion!"), he gets

$$\left(\sum a\right)\left(\sum ab\right) \geq 6\sum abc$$

(with the sums symmetric over a, b, c, d), which he then expands into

$$\sum a^2(b+c+d) + 3\sum abc \geq 6\sum abc$$

or

$$\frac{1}{3}\sum a^2(b+c+d) \geq \sum abc.$$

Thus he asks whether, for any $n \geq 2$ and $a_1, a_2, \ldots, a_n \geq 0$ with $\sum_{i=1}^n a_i = S$,

$$\frac{1}{n-1}\sum_{i=1}^n a_i^{n-2}(S - a_i) \geq \prod_{i=1}^n a_i \sum_{i=1}^n \frac{1}{a_i}\ . \tag{4}$$

Equality holds when $n = 2$, and $n = 4$ is the case handled above. Can any reader settle (4) in general?]

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; ED BARBEAU, University of Toronto; TIM CROSS, Wolverley High School, Kidderminster, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; and the proposer.

Barbeau's book Polynomials *(Springer–Verlag, 1989) contains a similar problem (#7 on page 225). With the same assumption as* Crux 1825, *it could be written: if* $c + d = 0$, *then* $a + b \geq 80$.

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1827. [1993: 78] *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D.M. Milošević, Pranjani, Yugoslavia.*

Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{bc}{A(s-a)} \geq \frac{12s}{\pi},$$

where the sums here and below are cyclic.

(ii)* It follows easily from the proof of *Crux* 1611 (see [1992: 62] and the correction on [1993: 79]) that also

$$\sum \frac{b+c}{A} \geq \frac{12s}{\pi}.$$

Do the two summations above compare in general?

I. *Solution to part* (i) *by Kee-Wai Lau, Hong Kong.*

Let R and r be the circumradius and inradius of the triangle. It is well known that

$$abc = 4Rrs, \quad \tan\frac{A}{2} = \frac{r}{s-a}, \quad a = 2R\sin A.$$

Hence

$$\sum \frac{bc}{A(s-a)} = 2s\sum \frac{\tan(A/2)}{A\sin A} = s\sum \frac{\sec^2(A/2)}{A}. \tag{1}$$

By Cauchy's inequality,

$$\sum \frac{\sec^2(A/2)}{A} \geq \frac{(\sum \sec(A/2))^2}{\sum A} = \frac{(\sum \sec(A/2))^2}{\pi}. \tag{2}$$

Since $\sec x$ is convex for $0 \leq x < \pi/2$,

$$\sum \sec(A/2) \geq 3\sec\left(\frac{A+B+C}{6}\right) = 2\sqrt{3}. \tag{3}$$

The inequality of the problem now follows from (1), (2) and (3).

II. *Solution to part* (ii) *by Richard I. Hess, Rancho Palos Verdes, California.*

Let

$$S_1 = \sum \frac{bc}{A(s-a)}, \quad S_2 = \sum \frac{b+c}{A}.$$

For $a = 5$, $b = 6$, $c = 10$ we get $A \approx .38976$, $B \approx .47345$, $C \approx 2.27838$, and so

$$S_1 - S_2 \approx 77.792 - 77.561 > 0.$$

For $a = 6$, $b = 7$, $c = 10$ we get $A \approx .6315$, $B \approx .75976$, $C \approx 1.75033$, and so

$$S_1 - S_2 \approx 53.700 - 55.406 < 0.$$

Therefore the two summations don't compare in general.

Part (i) *also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; TIM CROSS, Wolverley High School, Kidderminster, U.K.; and the proposers.*

* * * * *

1829. [1993: 78] *Proposed by C.J. Bradley, Clifton College, Bristol, U.K.*

The quadrilateral $ABCD$ is inscribed in a circle with centre O. AD and BC meet at P. L and M are the midpoints of AD and BC respectively. Q and R are the feet of perpendiculars from O and P respectively to LM. Prove that $LQ = RM$.

I. *Solution by Jordi Dou, Barcelona, Spain.*

A quick glimpse at Figure 1 shows the solution. Let ω be the circle through P, L, O, M with centre I, the midpoint of OP. [Note that $\angle OLP = \angle OMP = 90°$, so ω exists.—*Ed.*] The secants PR and OQ are equidistant from I [since they are parallel], and are perpendicular to the chord LM. Therefore the segments LQ and MR of the chord are equal.

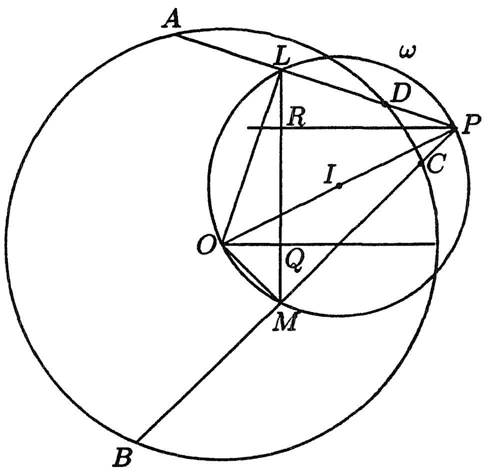

Figure 1

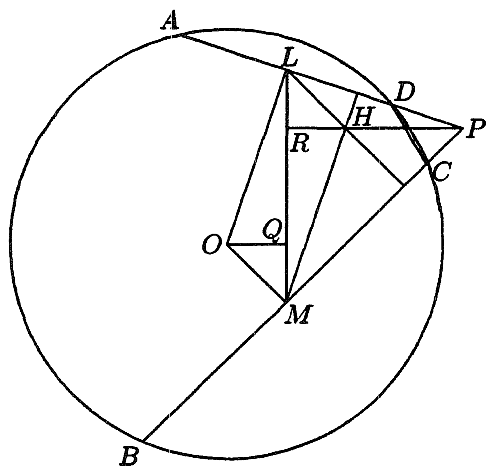

Figure 2

II. *Solution by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.*

See Figure 2. Let H be the orthocenter of ΔLMP. [Note H lies on PR.] Then $OM \| LH$ and $OL \| MH$ since $OM \perp BP$ and $OL \perp AP$. From this follows that in parallelogram $LOMH$ we have ΔHMR and ΔOLQ are congruent, and we conclude that $LQ = MR$.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Albany, California; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; and the proposer. One incorrect solution was received.

The solutions of Pompe, Seimiya and Smeenk were all similar to Solution I.

* * * * *

1830. [1993: 79] *Proposed by P. Tsaoussoglou, Athens, Greece.*

If $a > b > c > 0$ and $a^{-1} + b^{-1} + c^{-1} = 1$, prove that

$$\frac{4}{c^2} + \frac{1}{(a-b)b} + \frac{1}{(b-c)c} \geq \frac{4}{3}\,.$$

Solution by Šefket Arslanagić, Nyborg, Denmark.

From $a > b > 0$ and $b > c > 0$, by the Arithmetic-Mean Geometric-Mean inequality we have

$$a = (a-b) + b \geq 2\sqrt{(a-b)b} \quad \text{and} \quad b = (b-c) + c \geq 2\sqrt{(b-c)c}\,,$$

or

$$\frac{1}{(a-b)b} \geq \frac{4}{a^2} \quad \text{and} \quad \frac{1}{(b-c)c} \geq \frac{4}{b^2}\,.$$

Now, we have

$$\frac{4}{c^2} + \frac{1}{(a-b)b} + \frac{1}{(b-c)c} \geq 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right). \tag{1}$$

Equality in (1) holds if and only if $a = 2b = 4c$ (i.e., $a = 7$, $b = 7/2$, $c = 7/4$ because $a^{-1} + b^{-1} + c^{-1} = 1$). Since

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2 > 0$$

implies that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} > \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 = \frac{1}{3} \tag{2}$$

(strict inequality holds here because $a > b > c > 0$), it follows from (1) that

$$\frac{4}{c^2} + \frac{1}{(a-b)b} + \frac{1}{(b-c)c} > \frac{4}{3}\,,$$

i.e., the given inequality is correct, but only strict inequality holds.

[*Editor's note.* As many solvers pointed out, (2) also follows by Cauchy's inequality.]

Also solved (often the same way) by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, Wolverley High School, Kidderminster, U.K.; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; HENRY J. RICARDO, Medgar Evers College, Brooklyn, N.Y.; and the proposer. Two other readers sent in incomplete solutions.

Most solvers noted that the inequality is not sharp. Richard I. Hess, Rancho Palos Verdes, California, and Kuczma separately calculated that the minimum value of

$$\frac{4}{c^2} + \frac{1}{(a-b)b} + \frac{1}{(b-c)c}$$

under the given conditions is about 1.5434, *occurring at the uninteresting values* $a \approx 4.724$, $b \approx 3.227$, $c \approx 2.09$.

* * * * *

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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