# Crux

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Léo Sauvé, Editor


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## Notice

Since Léo Sauvé was on vacation during the month of July, the present edition of EUREKA was prepared by John Thomas, Jacques Marion and G. D. Kaye.

# PROBLEMS -- PROBLÈMES

*Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose address appears on page 47.*

*Solutions to the problems appearing in this issue, if available, will appear in* EUREKA NO. 9 *to be published around* November *15, 1975. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed separate sheets, should be mailed to the editor no later than 1 October 1975.*

51. *Proposed by H.G. Dworschak, Algonquin College.*

Solve the following equation for the positive integers $x$ and $y$:

$$(360 + 3x)^2 = 492{,}y04 \quad .$$

52. *Proposed by Viktors Linis, University of Ottawa.*

The sum of one hundred positive integers, each less than 100, is 200. Show that one can select a partial sum equal to 100.

53. *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que la somme de tous les entiers positifs inférieurs à $10n$ qui ne sont pas des multiples de 2 ou 5 est $20n^2$.

54. *Proposé par Léo Sauvé, Collège Algonquin.*

Si $a$, $b$, $c>o$ et $a < b + c$, montrer que

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c} \quad .$$

55. *Proposed by Viktors Linis, University of Ottawa.*

What is the last digit of $1+2+ \ldots +n$ if the last digit of $1^3+ 2^3+ \ldots +n^3$ is 1?

56. *Proposed by F.G.B. Maskell, Algonquin College.*

The area of a triangle in terms of its sides is $\sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$ .

What is the area in terms of its medians $m_1$, $m_2$, $m_3$?

57. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $G$ un groupe d'ordre $p^n$ ou $p$ est premier et $p \geq n$. Si $H$ est un sous-groupe d'ordre $p$ alors $H$ est normal dans $G$.

58. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $f: \{z : Re z = o\} \to R$ une fonction continue et bornée. Si l'on définit $\mu : \{ z : Re z > o \} \to R$ par

$$\mu(z) = \mu(x+iy) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x f(it)}{x^2+(y-t)^2}\, dt,$$

montrer que $f(ic) = \lim_{z \to ic} \mu(z)$ .

59. *Proposed by John Thomas.*

Find the shortest proof to the following proposition: every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.

60. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $f$ une fonction analytique sur le disque fermé $\overline{B}(o,R)$ telle que $|f(z)|<M$, et $|f(o)| = a>o$. Montrer que le nombre de zéros de $f$ dans $B(o,\frac{R}{3})$ est inférieur ou égal à $\frac{1}{\log 2} \log \frac{M}{a}$ .

* * *

Une Note Concernant Le Problème 10.

par J. Marion

On démontre de façon plus générale que la fonction $e^{\lambda z} - p(z)$, où $p(z)$ est un polynôme, possède une infinité de zéros.

Démonstration: Posons $f(z) = e^{\lambda z} - p(z)$. Cette fonction est entière. D'Après le théorème de factorization d'Hadamard on peut écrire $f(z) = z^m \exp(g(z))\, P(z)$, où $g(z)$ est un polynôme de dégré inférieur ou égal à l'ordre exponentiel de $f(z)$ et $P(z) = \prod_{n=1}^{\infty} E_p(z/a_n)$ est le produit canonique de Weierstrass correspondant aux zéros $a_1, a_2, \ldots$ de $f(z)$.

(Voir J.B. Conway, "Functions of One Complex Variable", Springer Verlag, 1973, p. 164-165, p. 285-291). En particulier si $f(z)$ ne possède qu'un nombre fini de zéros, $P(z)$ devient alors un polynôme. De plus l'ordre exponential de $f(z)$ est egal à 1 (puisque l'ordre d'un polynôme est 0). Donc on peut écrire $f(z) = (\exp(cx))\, P_o(z)$, ou $P_o(z)$ est le polynôme $z^m\, P(z)$ et $c$ une constante. Par dérivée logarithmique on obtient

$$\frac{\lambda e^{\lambda z} - p'(z)}{e^{\lambda z} - p(z)} = \frac{P_o'(z)}{P_o(z)} + c,$$

ce qui implique que $e^{\lambda z}$ est une fonction rationnelle. Cette contradiction prouve l'existence d'une infinité de zéros de $f(z)$.

* * *

## CLASSROOM NOTES

### Convergence of "P" Series with Missing Terms

by J. THOMAS

Let $k$ and $q$ be integers with $q \geqslant 2$ and $o < k < q-1$. For any subset $D_k$ consisting of $k$ elements of the set of $q$-adic digits $\{o, 1, \ldots, q-1\}$ let $E_k$ denote the set of numbers whose $q$-adic representations do not involve any of the digits of $D_k$.

THEOREM. The Exponent of Convergence, $\alpha$, of the Series $\sum_{n \in E_k} n^{-p}$ Equals $\dfrac{\ln(q-k)}{\ln q}$.

The proof of this theorem will involve properties of growth of the counting functions $E(x) = \sum_{\substack{n<x \\ n\in E}} 1$.

Before proceeding we will simplify our notation as follows: if E is an infinite set of positive integers and $x$ and $p$ real numbers with $x \geqslant 1$ we write

$$E(x) = \sum_{n<x} X_E(n) \text{ and } L_E(p,x) = \sum_{n<x} \frac{X_E(n)}{n^p},$$

where $X_E$ is the characteristic function of $E$, that is, the arithmetical function defined by $X_E(n) = 1$ if $n \in E$ and $X_E(n) = o$ otherwise.

If $p<o$, lim $L_E(p,x) = +\infty$ so we shall restrict our attention to positive values of $p$ and write $\lim_{x\to+\infty} L_E(p,x) = L(p,E)$ .

If $f$, $g$ and $h$ are positive real-valued function then by writing

$$f(x,\varepsilon) \ll_\varepsilon g(x) \ll_\varepsilon h(x,\varepsilon)$$

we shall mean that given $\varepsilon > o$, there exists positive constants $\eta_\varepsilon$ and $\mu_\varepsilon$ such that

$$\eta_\varepsilon f(x,\varepsilon) \leq g(x) \leq \mu_\varepsilon h(x,\varepsilon)$$

for all $x \geq 1$.

<u>LEMMA</u> If E is an infinite set of positive integers and $\alpha$ a positive real constant such that

(1) $$x^{\alpha-\varepsilon} \ll_\varepsilon E(x) \ll_\varepsilon x^{\alpha+\varepsilon}$$

then $\alpha$ is the exponent of convergence of the series

$$L(p,E) = \sum_{m\in E} \frac{1}{m^p}$$

<u>PROOF</u> From the definition of the Riemann-Stieltjes integral we can write

$$L_E(p,x) = \int_1^x t^{-p} \, dE(t) .$$

Upon integrating by parts we therefore obtain

$$(2) \qquad L_E(p,x) = \frac{E(x)}{x^p} + p\int_1^x t^{-p-1}\, E(t)\, dt$$

Now let $\varepsilon > \varepsilon_1 > o$ and set $p = \alpha+\varepsilon$.

From Relation (1) we have

$$\frac{E(x)}{x^p} < \mu_\varepsilon \quad \text{for all } x > 1,$$

and

$$\int_1^x t^{-p-1}\, E(t)dt = \int_1^x t^{-(\alpha+\varepsilon_1+\varepsilon-\varepsilon_1)-1} \cdot E\ (t)dt$$

$$\leqslant \mu_{\varepsilon_1} \int_1^x t^{-(\varepsilon-\varepsilon_1)-1}\, dt$$

$$< \mu_{\varepsilon_1} \int_1^\infty t^{-(\varepsilon-\varepsilon_1)-1}\, dt$$

$$= \mu_{\varepsilon_1} \frac{1}{\varepsilon-\varepsilon_1} < +\infty.$$

Thus both terms on the R.H.S. of Relation (2) are bounded. Therefore $L_E(p,x)$ is bounded and since it increases with $x$ it follows that

$$(3) \qquad \lim_{x\to+\infty} L_E(p,x) < +\infty$$

Now set $p = \alpha-\varepsilon$ . Again from Relation (1)

$$\int_1^x t^{-(\alpha-\varepsilon)-1} E(t)dt > \eta_\varepsilon \ln x \ , \ x\geq 1$$

and therefore (in view of Relation (2)),

(4) $\lim_{x\to\infty} L_E(p,x) = +\infty$ .

From Relations (3) and (4) we deduce the lemma.

PROOF OF THE THEOREM We shall establish inequalities of the form (1) for the function $E_k(x)$.

Observe that for each $n$ the number of $q$-adic representations

$$a_n q^n + a_{n-1}q^{n-1} + \ldots + a_1 q + a_0, \ a_j \in \{0,1,\ldots,q-1\}\setminus D_k,$$

is $(q-k)^{n+1}$. Therefore with $x=q^u$ where $u\in\mathbb{R}$,

$$E_k(x) = E_k(q^u) \leq E_k(q^{[u]+1}) \leq (q-k)^{u+1},$$

and

$$E_k(x) = E_k(q^u) \geq E_k(q^{[u]}) \geq (q-k)^{u-1}$$

setting $\alpha = \dfrac{\ln(q-k)}{\ln q}$ in the identity

$$(q-k)^u = (q^u)^{\frac{\ln(q-k)}{\ln q}},$$

we can write the preceding inequalities in the form

(5) $(q-k)^{-1} x^\alpha \leq E_k(x) \leq (q-k)x^\alpha, \ x \geq 1.$

Applying the lemma to the set of inequalities (5) we conclude that $\alpha$ is the exponent of convergence of the series $L(p,E_k)$. Q.E.D.

REMARK If we apply the set of inequalities (5) to Relation (2) we obtain another set of inequalities, namely,

$$(q-k)^{-1}(1+\alpha\ell n x) \leqslant L_{E_k}(\alpha,x) \leqslant (q-k)(1+\alpha\ell n x), \quad x\geqslant 1,$$

from which we can conclude that

$$\sum_{n\varepsilon E_k} \frac{1}{n^{\alpha}} = +\infty.$$

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