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- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
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E U R E K A

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Send all communications to the editor: Léo Sauvé, Math-Architecture, Algonquin College, Col. By Campus, 281 Echo Drive, Ottawa, Ont., K1S 5G2.

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NOTES FROM THE EDITOR

1. One hundred seventy-one copies of EUREKA No. 4 were mailed a few days ago, most of them going to the Ottawa region, a few dozen elsewhere in Ontario, a few in Quebec, and one to the United States. New subscribers are welcome. Get your mathematically-minded friends to send the editor their name, address, and place of employment. He will add them to the mailing list and send them back issues.

2. Even more than additional names on the mailing list, the editor would like to see greater participation by existing readers, in the form of solutions to problem proposals. To make his job easier, he would like to remind you that solutions, if not typewritten, should be neatly handwritten on signed, separate sheets, in a form suitable for publication.

The expression "in a form suitable for publication" means that a handful of mathematical symbols sprinkled on a page does not constitute a proof. The words and mathematical symbols appearing in your solutions should collectively form recognizable, grammatical, properly punctuated English (or French) sentences, accompanied by neatly drawn diagrams where appropriate. Careful phrasing is conducive to clear thinking. It is not easy to write a proof in which you are convinced that every last comma is exactly where it should be; but what is worth doing is worth doing well.

Regardless of anything said in the last paragraph, the editor would like to encourage readers who are young or inexperienced in mathematical writing to send in their solutions in spite of a fancied lack of elegance in their presentation. After all, in launching EUREKA, one of the principal aims of the founding members was precisely to help these people in their struggle to achieve mathematical maturity. So do the best you can and send your solutions in; what is worth doing is worth doing badly.

3. The mathematical elite in our community, the university professors and graduate students, can help a great deal by submitting interesting articles and model solutions to the more difficult problems. They will themselves benefit in the end, for increasingly competent high school teachers will eventually result in increasingly competent high school graduates and university students.

4. EUREKA will normally be published ten times a year, every month except July and August, when most of its readers (and the editor) are on holidays. Since the first issue of EUREKA was in March 1975, it will this year, exceptionally, be published in July and August, so that Volume 1 (1975) will have a full complement of ten issues.

In the future, at the request of several readers, solutions will be published three months, instead of two, after the problems are proposed. In order to make this possible, the August issue will contain new problem proposals but no solutions.

5. The editor hopes that readers enjoyed his report on Martin Gardner's hoaxes in the last issue of EUREKA. Here, as promised, is further information on the matter:

i) Gardner has promised to give, in the July issue of *Scientific American*, a report on the public reaction to his hoaxes.

ii)Dr. John D. Brillhart, the presumed *alter ego* of John Brillo, is a widely known number theorist from the University of Arizona. Among his discoveries is the factorization of the Mersenne number $2^{103}-1$. It is

$$2^{103} - 1 = 2550183799 \times 3976656429941438590393.$$

He has also proved that $2^{131} - 1$ is a product of two primes, one of which is 263.

And *that* is no April Fool's joke.

* * *

P R O B L E M S - - P R O B L È M E S

Problem proposals, preferably accompagnied by a solution, should be sent to the editor, whose address appears on page 37.

For the problems given below, solutions, if available, will appear in EUREKA No. 8, to be published around October 15, 1975. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than October 1, 1975.

41. *Proposé par Léo Sauvé, Collège Algonquin.*

Ayant donné $\log_8 3 = p$ et $\log_3 5 = q$, exprimer $\log_{10} 5$ et $\log_{10} 6$ en fonction de p et q.

42. *Proposed by Viktors Linis, University of Ottawa.*

Find the area of a quadrilateral as a function of its four sides, given that the sums of opposite angles are equal.

43. *Proposed by André Bourbeau, École Secondaire Garneau.*

In a 3×3 matrix, the entries a_{ij} are randomly selected integers such that $0 \le a_{ij} \le 9$. Find the probability that

(a) the three-digit numbers formed by each row will be divisible by 11;

(b) the three-digit numbers formed by each row and each column will be divisible by 11.

44. *Proposed by Viktors Linis, University of Ottawa.*

Construct a square *ABCD* given its centre and any two points *M* and *N* on its two sides *BC* and *CD* respectively.

45. *Proposed by H.G. Dworschak, Algonquin College.*

Find constants A, B, C, D, p, q such that

$$A(x-p)^2 + B(x-q)^2 = 5x^2 + 8x + 14,$$
$$C(x-p)^2 + D(x-q)^2 = x^2 + 10x + 17.$$

46. *Proposed by F.G.B. Maskell, Algonquin College.*

If p_1, p_2, p_3 are the altitudes of a triangle and r is the radius of its inscribed circle, show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}.$$

47. *Proposé par Jacques Sauvé, étudiant, Université d'Ottawa.*

Si $a > 1$, calculer la somme de la série $\sum_{k=1}^{\infty} \frac{k^2}{a^k}$.

48. *Proposé par Léo Sauvé, Collège Algonquin.*

La fonction $f: R \to R$ est définie par les relations

$$f(x) = 2 + \sin x \cos \frac{1}{x}, \quad \text{si } x \neq 0,$$
$$f(0) = 2.$$

Pour tout entier $n \geq 1$, on considère l'intégrale

$$I_n = \int_{-\frac{2}{n}}^{\frac{2}{n}} \{n + (\tfrac{1}{n} - n)\chi_n(x)\}\, f(x)dx,$$

où χ_n désigne la fonction caractéristique de l'intervalle $\left[-\frac{1}{n}, \frac{1}{n}\right]$. Calculer I_n en fonction de n et en déduire la valeur de $\lim_{n\to\infty} I_n$.

49. *Proposed by H.G. Dworschak, Algonquin College.*

The series

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \ldots + \frac{1}{6n-5} - \frac{1}{6n-1} + \ldots$$

clearly converges. Find its sum.

50. *Proposed by John Thomas, Digital Methods Ltd.*

I found the following fascinating two-part problem in Martin Eisen's *Introduction to Mathematical Probability Theory* (Prentice-Hall 1969). Information as to its origin and history would be appreciated.

(a) *Show that 2^n can begin with any sequence of digits.*

(b) *Let N be an r-digit number. What is the probability that the first r digits of 2^n represent N?*

S O L U T I O N S

21. *Proposed by H.G. Dworschak, Algonquin College.*

What single standard mathematical symbol can be used with the digits 2 and 3 to make a number greater than 2 but less than 3?

Solutions by Keith Bateman, Algonquin College; Don Hull, Hillcrest High School; André Ladouceur, École Secondaire De La Salle; Viktors Linis, University of Ottawa; F.G.B. Maskell, Algonquin College; Léo Sauvé, Collège Algonquin; John Thomas, Digital Methods Ltd; and the proposer.

Each of the above solvers gave one or more of the following solutions:

2.3, $\sqrt{2^3} \doteq 2.83$, $3\sin 2 \doteq 2.73$, $2\arctan 3 \doteq 2.50$, $3\,\ell n\, 2 \doteq 2.08$, $2\,\ell n\, 3 \doteq 2.20$, $\csc 3^2 \doteq 2.43$.

Comment by G.D. Kaye, Department of National Defence.

Readers of Kurt Vonnegut's science fiction will recall a similar message conveyed across the universe by a robot. In robot language, it signified "Greetings".

Thanks, Mr. Kaye, and 2.3 to you too. (Ed.)

22. *Proposed by H.G. Dworschak, Algonquin College.*

Numbers are written on little paper squares as shown in the figure.

$$\boxed{1}\quad\boxed{2}\quad\boxed{7}\quad\boxed{9}$$
$$\boxed{3}\quad\boxed{4}\quad\boxed{5}\quad\boxed{8}$$

Show how to make the sums of the two rows equal by moving just two of the pieces.

I. *Solution de Nicole Trudel-Marion.*

On renverse le 9 pour en faire un 6 et on l'échange ensuite avec le 8. La somme de chaque rangée devient alors 18.

C'est un problème parfait pour les femmes de mathématiciens!

Editor's comment. Nicole Trudel-Marion is the recent bride of Jacques Marion, a graduate student in mathematics at the University of Ottawa.

II. *Solution by F.G.B. Maskell, Algonquin College.*

Move the 1 to the bottom row and raise the 3 slightly. The result is

$$2 + 7 + 9 = 18,$$

$$1^3 + 4 + 5 + 8 = 18.$$

Also solved by John Hayes; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; and the proposer.

23. *Proposé par Léo Sauvé, Collège Algonquin.*

Déterminer s'il existe une suite $\{u_n\}$ d'entiers naturels telle que, pour $n = 1, 2, 3, \ldots$, on ait

$$2^{u_n} < 2n + 1 < 2^{1+u_n}$$

I. *Solution de John Thomas, Digital Methods Ltd.*

La fonction logarithme de base 2 étant strictement croissante, la relation proposée équivaut à la suivante:

$$u_n < \log_2(2n + 1) < 1 + u_n.$$

S'il existe une telle suite, l'entier u_n est nécessairement le plus grand entier inférieur à $\log_2(2n + 1)$, que l'on note, comme d'habitude,

$$u_n = \left[\log_2(2n + 1)\right].$$

L'inégalité est stricte, car 2^{u_n} est un nombre pair.

On voit facilement que cette suite convient; elle est par conséquent la seule.

II. *Solution by G.D. Kaye, Department of National Defence.*

For $n = 1, 2, 3, \ldots$, let u_n be the number of digits in the binary representation of n. If

$$n = 1xx\,x \ldots x \qquad (u_n \text{ digits}),$$

where x is 0 or 1, then

$$2n + 1 = 1xx\,x \ldots x1 \qquad (u_n + 1 \text{ digits}).$$

This is obviously greater than $1\,000 \ldots 0$ ($u_n + 1$ digits), which is 2^{u_n}, and less than $1\,000 \ldots 00$ ($u_n + 2$ digits), which is 2^{u_n+1}, so that

$$2^{u_n} < 2n + 1 < 2^{1+u_n}.$$

Editor's comment.

The uniqueness of the result, together with the different representations of it given in solutions I and II, gives

$$\left[\log_2(2n + 1)\right] = \text{number of digits in the binary representation of } n,$$

which is otherwise not immediately obvious.

Also solved by André Ladouceur, École Secondaire De La Salle; Viktors Linis, University of Ottawa; F.G.B. Maskell, Algonquin College; and the proposer.

24. *Proposed by Viktors Linis, University of Ottawa.*

A paper triangle has base 6 cm and height 2 cm. Show that by three or fewer cuts the pieces can cover a cube of edge 1 cm.

Solution by G.D. Kaye, Department of National Defence.

The triangle can be dissected, by two or three cuts, and the pieces rearranged into two 3 cm × 1 cm strips which will cover a cube of edge 1 cm. This is shown in the figures below, where the cuts are numbered and the rearranged pieces are represented by dotted lines.

In Figure 1, the longest side of the triangle has length 6 cm. In Figure 2, one of the shorter sides of the triangle has length 6 cm. In this case, it will be observed, the third cut goes through two different pieces of the triangle. This may be considered a violation of the conditions of the problem, but I don't know how to get around it. In Figure 3, the triangle is right-angled and one of the legs has length 6 cm. Two cuts suffice in this case.

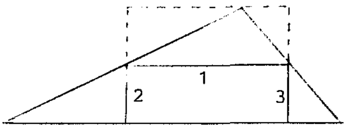


Figure 1

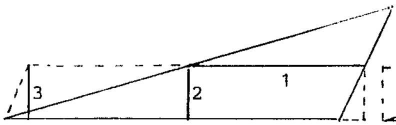


Figure 2

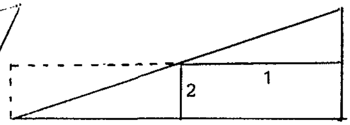


Figure 3

Also solved by F.G.B. Maskell, Algonquin College; and Léo Sauvé, Collège Algonquin.

25. *Proposed by Viktors Linis, University of Ottawa.*

Find the smallest positive value of $36^k - 5^\ell$ where k and ℓ are positive integers.

I. *Solution by F.G.B. Maskell, Algonquin College.*

We have

$$6^{2k} - 1 - 5^\ell = (6^k - 1)(6^k + 1) - 5^\ell \neq 0,$$

since $6^k + 1$ has at least one factor which is not 5, and 5^ℓ does not; hence

$$36^k - 5^\ell = 6^{2k} - 5^\ell \neq 1.$$

But the final digit of $36^k - 5^\ell$ must be 1, since the final digits of 36^k and 5^ℓ are 6 and 5 respectively. Hence the minimum value is 11, which occurs when $k = 1$ and $\ell = 2$.

II. *Solution d'André Ladouceur, École Secondaire De La Salle.*

Les puissances de 36 se terminent par 16, 36, 56, 76, ou 96, i.e. elles ont la forme $(2n + 1)10 + 6$. En effet, par induction, on voit que

$$36\left[(2n + 1)10 + 6\right] = (72n + 57)10 + 6,$$

et $72n + 57$ est impair. On voit de même que les puissances de 5 supérieures à 5 finissent par 25, car

$$5(100n + 25) = (5n + 1)100 + 25.$$

La quantité $36^k - 5^\ell$ doit donc finir par 91, 11, 31, 51, où 71; elle ne peut donc être inférieure à 11. Or elle prend la valeur 11 quand $k = 1$ et $\ell = 2$.

III. *Solution by the proposer.*

Since $36^k - 5^\ell \equiv 1 \pmod{10}$ and $36^k - 5^\ell \equiv 3 \pmod 4$, we must have $36^k - 5^\ell \geq 11$; but for $k = 1$ and $\ell = 2$, $36^k - 5^\ell = 11$, which gives then the required minimum.

IV. *Solution by Léo Sauvé, Algonquin College.*

Since every power of 36 ends in 6 and every power of 5 ends in 5, the difference $36^k - 5^\ell$ must end in 1. For $k = 1$ and $\ell = 2$, the difference is 11. This will be the required value, unless we can find values of k and ℓ for which the difference is 1. Suppose such values of k and ℓ exist; then

$$36^k - 1 = 5^\ell.$$

The left side is divisible by $36 - 1 = 5\cdot 7$; hence 5^ℓ is divisible by 7, which is impossible. The answer is 11.

Comment by the proposer.

Are there other values of k and ℓ, besides $k = 1$ and $\ell = 2$, which solve the equation $36^k - 5^\ell = 11$? The answer is no. For if $36^k - 5^\ell = 11$, then $(-1)^\ell \equiv 1 \pmod 6$ and ℓ must be even, say $2m$, and

$$36^k - 5^{2m} = 6^{2k} - 5^{2m} = (6^k - 5^m)(6^k + 5^m) = 11.$$

Since 11 is a prime, $6^k - 5^m = 1$ and $6^k + 5^m = 11$, so that $6^k = 6$, $5^m = 5$, and $k = 1$, $m = 1$ ($\ell = 2$) is the only solution.

26. *Proposed by Viktors Linis, University of Ottawa.*

Given n integers. Show that one can select a subset of these numbers and insert plus or minus signs so that the number obtained is divisible by n.

I. *Solution d'André Ladouceur, École Secondaire De La Salle.*

Il y a n classes d'équivalence modulo n. Si un des n nombres est équivalent à $0 \pmod n$, il est divisible par n. Dans le cas contraire, il doit y avoir deux des nombres, disons α et β, qui sont dans la même classe d'équivalence, et leur différence $\alpha - \beta$ est alors divisible par n.

II. *Solution by Léo Sauvé, Algonquin College.*

Let the given integers be a_i, $i = 1, 2, \ldots, n$. With appropriately inserted + and — signs, we form the sum

$$S_n = b_1 + b_2 + \ldots. + b_n \qquad (1)$$

where $b_i = |a_i|$. For $k = 1, 2, \ldots, n$, let r_k be the remainder obtained when S_k is divided by n. If $r_k = 0$ for some k, then the corresponding S_k is divisible by n. Otherwise, since there are only n possible remainders, there must exist positive integers i and j, $i < j$, such that $r_i = r_j$, and then

$$s_j - s_i = b_{i+1} + b_{i+2} + \ldots + b_j$$

is divisible by n. We have thus shown the interesting result that in any case there exists a subset of *consecutive* terms of (1) whose sum is divisible by n.

Also solved by the proposer.

27. *Proposé par Léo Sauvé, Collège Algonquin.*

Soient A, B, et C les angles d'un triangle. Il est facile de vérifier que si $A = B = 45°$, alors

$$\cos A \cos B + \sin A \sin B \sin C = 1$$

La proposition réciproque est-elle vraie?

Solution by Viktors Linis University of Ottawa.

For angles A, B, C in a triangle, we have, using the given condition,

$$1 = \cos A \cos B + \sin A \sin B \sin C \le \cos A \cos B + \sin A \sin B = \cos(A - B) \le 1.$$

Therefore the equality signs must hold throughout, and this implies $\sin C = 1$ and $\cos(A - B) = 1$. Hence $C = 90°$ and $A = B = 45°$.

Also solved by G.D. Kaye Department of National Defence; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; and the proposer.

28. *Proposed by Léo Sauvé, Algonquin College.*

If 7% of the population escapes getting a cold during any given year, how many days must the average inhabitant expect to wait from one cold to the next?

I. *Solution by the proposer.*

We assume that the required time is n years from the start of one cold to the start of the next, and that all times of year are equally likely for a cold to begin.

Suppose a year is divided into k equal periods, where k is large, so that the required time contains nk periods. The probability of escape in any one period is approximately $1 - \frac{1}{nk}$, and the probability P of escaping for a whole year or k periods is approximately $(1 - \frac{1}{nk})^k$. Hence

$$P = \lim_{k \to \infty} (1 - \frac{1}{nk})^k = e^{-\frac{1}{n}}.$$

If we equate this to 0.07, we obtain

$$n = (\ln \frac{100}{7})^{-1} \text{ years} \doteq 137 \text{ days.}$$

II. *Solution by G.D. Kaye, Department of National Defence.*

If it is assumed that the incidence of colds is random and independent, the distribution of colds is Poisson, which is a one-parameter distribution and hence convenient for this problem. The assumption is adequate only if the duration of each cold is relatively short so that each individual may be exposed to a new cold several times a year. Under the Poisson distribution, if the average number of colds per head per year is x, the proportion of the population with k colds per year is given by the term in x^k of

$$e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots\right).$$

In this case we have $e^{-x} \cdot 1 = 0.07$, giving $x \doteq 2.659$. Thus the average man can expect to wait $\frac{365}{2.659} \doteq 137$ days from the onset of one cold to the onset of the next.

There is insufficient information to determine the period from the *end* of one cold to the start of the next.

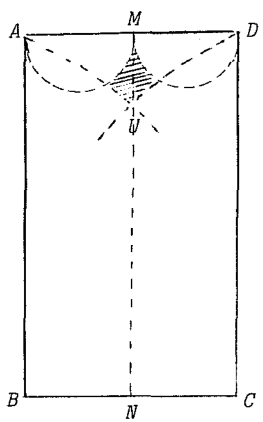
Also solved by H.G. Dworschak, Algonquin College.

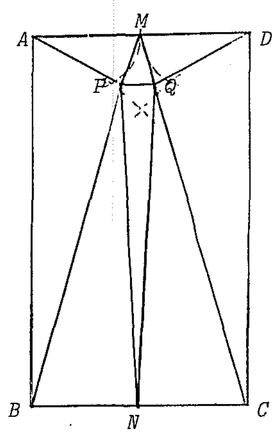
29. *Proposed by Viktors Linis, University of Ottawa.*

Cut a square into a minimal number of triangles with all angles acute.

Solution by the proposer.

The answer is 8 for the square, and the same holds for any rectangle. (For an obtuse triangle the answer is 7.) The following construction explains the procedure for any rectangle $ABCD$. Draw two semicircles on AB and DC (longest sides) and two semicircles on DM and MA, where M is the midpoint of AD (shortest side). The shaded region gives the location of the "permissible" vertices P and Q such that the eight triangles as shown in the drawing are all acute. (N is the midpoint of CB.) The shaded region is never an empty set: the point U where the two larger semicircles intersect each other lies *within* the rectangle and *outside* the two smaller semicircles, since it lies on MN.

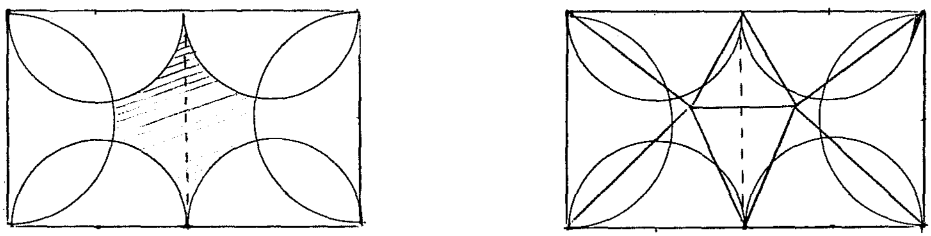




Editor's comments.

(i) The minimality of number 8 for the rectangle is a nearly immediate consequence of the fact, mentioned by the proposer, that for a non-acute triangle the minimal number is 7. For a proof of this fact, see *American Mathematical Monthly*, Vol. 67 (1960), page 923.

(ii) A "permissible" area for locating vertices P and Q can also be found by drawing semicircles on the shortest sides, and then drawing semicircles on the two halves of both longest sides, as shown in the figures below where the "permissible" area is shaded.



Also solved by F.G.B. Maskell, Algonquin College, and Léo Sauvé, Collège Algonquin. One incorrect solution was received.

30. *Proposed by Léo Sauvé, Algonquin College.*

Let a, b, and c denote three distinct integers and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$. (*Third USA Mathematical Olympiad - May 7, 1974*)

Solution by Viktors Linis, University of Ottawa.

A nice, short solution is given in *The Mathematics Teacher*, Vol. 68, No. 1 (Jan. 1975). It can hardly be improved.

Editor's comment.

The solution referred to above appeared in a report on the Third USA Mathematical Olympiad written by Samuel L. Greitzer, Chairman of the Olympiad Committee.

There is insufficient space to reproduce this solution here in its entirety. But *The Mathematics Teacher* is widely available, and interested readers should be able to look it up there.

Also solved by G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; and the proposer.

* * *

Six is a number perfect in itself, and not because God created all things in six days; rather the inverse is true, that God created all things in six days because this number is perfect, and would remain perfect, even if the work of the six days did not exist.

ST. AUGUSTINE