

# Crux

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# Mathematicorum

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Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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**REMERCIEMENTS**

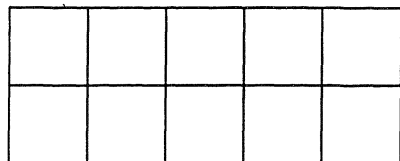
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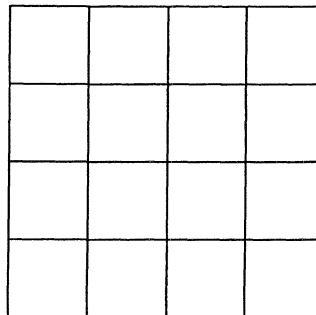
## ENUMERATING 3-, 4-, 6-GONS WITH VERTICES AT LATTICE POINTS: PART II

Y. Arzoumanian and W. O. J. Moser

[*Editor's note.* This is the concluding part of the article. Part I appeared in the previous issue of *Cruz.*]



$L_{2,5}$



$L_{4,4}$

The figures above indicate without further elaboration what is meant by the lattices (points and lines)  $L_{m,n}$ ,  $m \leq n$ . Recall from last time the notation  $\underbrace{x}$ , which indicates that  $x$  is an integer and  $x \geq 0$ . In this part we will solve problems concerning numbers of triangles and squares with vertices at lattice points of  $L_{m,n}$ , once again using the

**Lemma.**  $\underbrace{a_1} + \underbrace{a_2} + \cdots + \underbrace{a_k} \leq n$  has  $\binom{n+k}{k}$  solutions  $(a_1, a_2, \dots, a_k)$ .

**Problem 6.** (a) How many squares have vertices at lattice points of  $L_{m,m}$  and sides along lattice lines?

(b) How many squares have vertices at lattice points of  $L_{m,n}$ ,  $m \leq n$ , and sides along lattice lines?

**Solution.** (a) To each square there corresponds a triple  $(a, b, w)$  satisfying

$$\underbrace{a} + w \leq m, \quad \underbrace{b} + w \leq m, \quad w \geq 1. \quad (3)$$

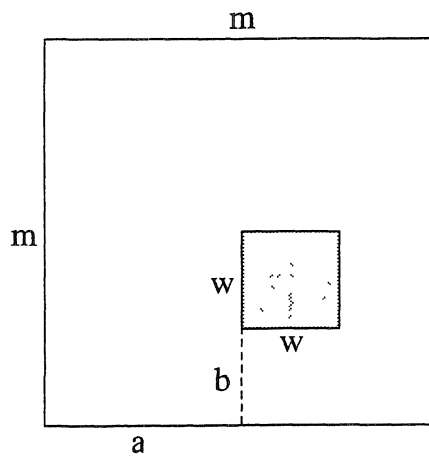
We count these triples in 2 subsets, according to  $a \leq b$  or  $b < a$ . In the first subset  $(a, b, w)$  satisfies (3) and  $a \leq b$ , and these are equivalent to

$$\underbrace{a} + \underbrace{b-a} + \underbrace{w-1} \leq m-1,$$

so by the Lemma this subset contains  $\binom{m-1+3}{3}$

triples. In the second subset  $(a, b, w)$  satisfies (3) and  $b < a$ , which are equivalent to

$$\underbrace{a-b-1} + \underbrace{b} + \underbrace{w-1} \leq m-2,$$



so by the Lemma this subset contains  $\binom{m-2+3}{3}$  triples. Altogether,

$$\binom{m+2}{3} + \binom{m+1}{3} = \frac{1}{6}m(m+1)(2m+1) \tag{4}$$

squares.

(b) To each square there corresponds a triple  $(a, b, w)$  satisfying

$$\underbrace{a} + w \leq n, \quad \underbrace{b} + w \leq m, \quad w \geq 1. \tag{5}$$

We count these triples in two subsets, according to  $a < n - m$  or  $a \geq n - m$ . In the first subset we have  $(a, b, w)$  satisfying (5) and  $a < n - m$ , and these are equivalent to

$$\underbrace{a} \leq n - m - 1, \quad \underbrace{b} + \underbrace{w - 1} \leq m - 1,$$

so by the Lemma this subset contains  $\binom{n-m-1+1}{1} \binom{m-1+2}{2}$  triples. In the second subset we have  $(a, b, w)$  satisfying (5) and  $a \geq n - m$ , and these are equivalent to

$$\underbrace{a - n + m} + w \leq m, \quad \underbrace{b} + w \leq m, \quad w \geq 1,$$

so by (3) and (4) this subset contains  $\frac{1}{6}m(m+1)(2m+1)$  triples. Altogether we have

$$(n-m) \binom{m+1}{2} + \frac{1}{6}m(m+1)(2m+1) = \frac{1}{6}m(m+1)(3n-m+1) \tag{6}$$

squares.

**Problem 7.** (a) How many squares have vertices at lattice points of  $L_{m,m}$ ? (The sides need not be along lattice lines.)

(b) How many squares have vertices at lattice points of  $L_{m,n}$ ,  $m \leq n$ ? (The sides need not be along lattice lines.)

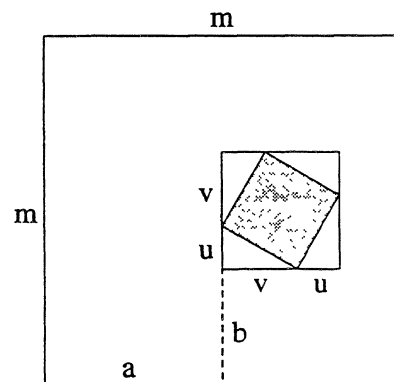
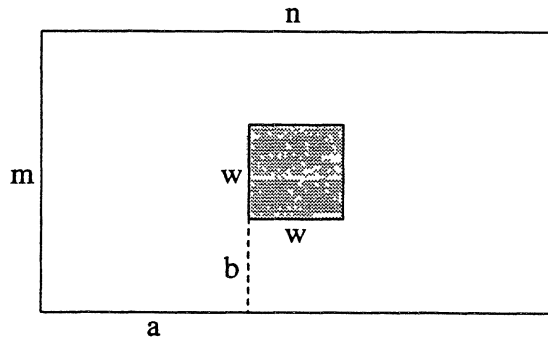
**Solution.** (a) To each square there corresponds a 4-tuple  $(a, b, u, v)$  satisfying

$$\underbrace{a} + \underbrace{u} + v \leq m, \quad \underbrace{b} + \underbrace{u} + v \leq m, \quad v \geq 1. \tag{7}$$

We count these 4-tuples in 2 subsets, according to  $a \leq b$  or  $b < a$ . In the first subset we have  $(a, b, u, v)$  satisfying (7) and  $a \leq b$ , and these are equivalent to

$$\underbrace{a} + \underbrace{b - a} + \underbrace{u} + \underbrace{v - 1} \leq m - 1,$$

so by the Lemma this subset contains  $\binom{m-1+4}{4}$  4-tuples. In the second subset we have



$(a, b, u, v)$  satisfying (7) and  $b < a$ , and these are equivalent to

$$\underbrace{a - b - 1} + \underbrace{b} + \underbrace{u} + \underbrace{v - 1} \leq m - 2,$$

so by the Lemma this subset contains  $\binom{m-2+4}{4}$  4-tuples. Altogether (see Dawes (1972), Fox (1984))

$$\binom{m+3}{4} + \binom{m+2}{4} = \frac{1}{12}m(m+1)^2(m+2) \tag{8}$$

squares.

(b) To each square there corresponds a 4-tuple  $(a, b, u, v)$  satisfying

$$\begin{aligned} \underbrace{a} + \underbrace{u} + \underbrace{v - 1} &\leq n - 1 \\ \underbrace{b} + \underbrace{u} + \underbrace{v - 1} &\leq m - 1. \end{aligned} \tag{9}$$

If  $a < n - m$  then  $(a, b, u, v)$  satisfies (9) and  $a < n - m$ , and these are equivalent to

$$\underbrace{a} \leq n - m - 1, \quad \underbrace{b} + \underbrace{u} + \underbrace{v - 1} \leq m - 1,$$

for a count of  $\binom{n-m-1+1}{1} \binom{m-1+3}{3}$ . If  $a \geq n - m$ , the 4-tuple satisfies (9) and  $a \geq n - m$ , and these are equivalent to

$$\underbrace{a - n + m} + \underbrace{u} + v \leq m, \quad \underbrace{b} + \underbrace{u} + v \leq m, \quad v \geq 1,$$

for a count, by (7) and (8), of  $\binom{m+3}{4} + \binom{m+2}{4}$ . The number of squares is

$$(n - m) \binom{m+2}{3} + \binom{m+3}{4} + \binom{m+2}{4} = \frac{1}{12}m(m+1)(m+2)(2n - m + 1).$$

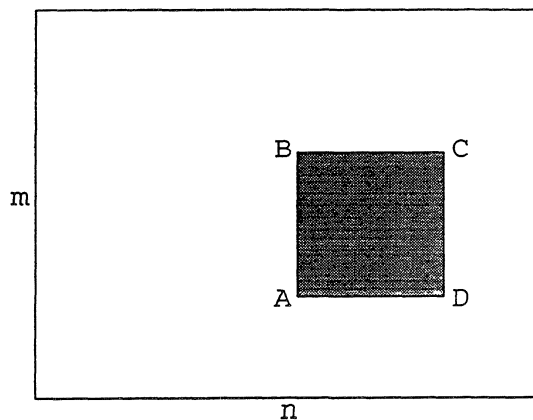
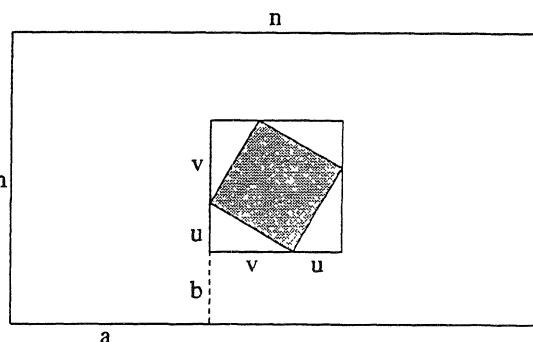
**Problem 8.** How many isosceles right triangles have vertices at lattice points of  $L_{m,n}$ ,  $1 \leq m \leq n$ , and legs along lattice lines?

**Solution.** For each square counted in Problem 6(b), for example  $ABCD$ , there are 4 triangles, namely  $ABC$ ,  $BCD$ ,  $CDA$ ,  $DAB$ . Hence the number of triangles is (see (6))

$$\frac{2}{3}m(m+1)(3n - m + 1).$$

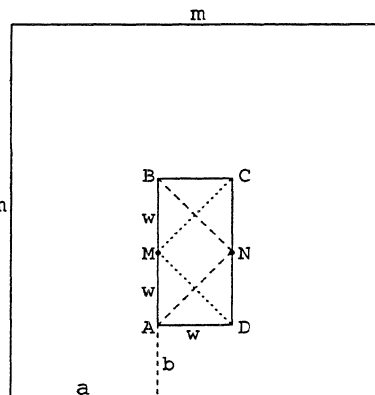
In particular, the number of isosceles right triangles with vertices at lattice points of  $L_{m,m}$  and legs along lattice lines is

$$\frac{2}{3}m(m+1)(2m+1). \tag{10}$$



**Problem 9.** How many isosceles right triangles have vertices at lattice points of  $L_{m,m}$  and hypotenuse along a lattice line?

**Solution.** We count the triangles with vertical hypotenuse. (There are an equal number with horizontal hypotenuse.) Every  $2w \times w$  rectangle (vertices at lattice points, vertical side-length  $2w$ , horizontal side-length  $w$ ) covers two of these triangles. For example rectangle  $ABCD$  covers the two triangles  $ANB$  and  $DMC$  with vertical hypotenuse. Hence there are 2 triangles for every triple  $(a, b, w)$  satisfying



$$\underbrace{a} + w \leq m, \quad \underbrace{b} + 2w \leq m, \quad w \geq 1. \quad (11)$$

We count these triples in subsets: (i)  $a \leq b$ , (ii)  $b < a \leq b + w$ , (iii)  $b + w < a$ . In the first subset we have (11) and  $a \leq b$ , and these are equivalent to

$$\underbrace{a} + \underbrace{b - a} + 2(\underbrace{w - 1}) \leq m - 2,$$

so (compare with (1) and (2) in Problem 1 of Part I) this subset contains

$$t_{II}(m) = \begin{cases} \frac{1}{24}m(m+2)(2m-1) & \text{if } m \text{ is even,} \\ \frac{1}{24}(m-1)(m+1)(2m+3) & \text{if } m \text{ is odd,} \end{cases}$$

triples.

In the second subset the triples satisfy (11) and  $b < a \leq b + w$ , and these are equivalent to

$$\underbrace{b} + 2(\underbrace{a - b - 1}) + 2(\underbrace{b + w - a}) \leq m - 2.$$

When  $b \equiv 0 \pmod 2$  (resp.  $b \equiv 1 \pmod 2$ ) we pick up  $\binom{\lfloor \frac{m-2}{2} \rfloor + 3}{3}$  (resp.  $\binom{\lfloor \frac{m-3}{2} \rfloor + 3}{3}$ ) triples. Thus the second subset contains

$$\binom{\lfloor \frac{m}{2} \rfloor + 2}{3} + \binom{\lfloor \frac{m-1}{2} \rfloor + 2}{3}$$

triples.

In the third subset, the triples satisfy (11) and  $b + w < a$ , and these are equivalent to

$$\underbrace{b} + \underbrace{a - b - w - 1} + 2(\underbrace{w - 1}) \leq m - 3,$$

so (compare with (1) and (2) Problem 1 Part I) this subset contains

$$t_{II}(m-1) = \begin{cases} \frac{1}{24}(m-2)(m)(2m+1) & \text{if } m \text{ is even,} \\ \frac{1}{24}(m-1)(m+1)(2m-3) & \text{if } m \text{ is odd,} \end{cases}$$

triples.

The answer to the problem is

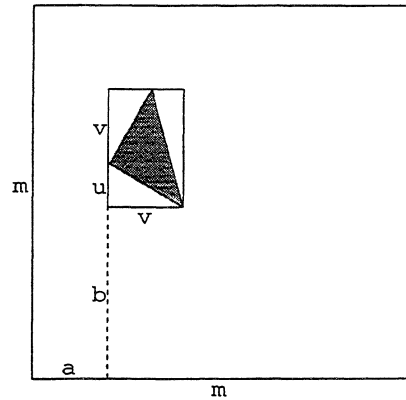
$$4 \left\{ t_{II}(m) + \binom{\lfloor \frac{m}{2} \rfloor + 2}{3} + \binom{\lfloor \frac{m-1}{2} \rfloor + 2}{3} + t_{II}(m-1) \right\} = \begin{cases} \frac{1}{8}m(m+1)(5m-2) & \text{if } m \text{ is even,} \\ \frac{1}{8}(m+1)(m-1)(5m+3) & \text{if } m \text{ is odd.} \end{cases} \tag{12}$$

**Problem 10.** How many isosceles right triangles have vertices at lattice points of  $L_{m,m}$  and no sides on lattice lines?

**Solution.** For each 4-tuple  $(a, b, u, v)$  satisfying

$$\underbrace{a} + v \leq m, \quad \underbrace{b} + u + v \leq m, \quad 1 \leq u < v, \tag{13}$$

there are 8 triangles, namely those obtained by applying to the triangle in the figure the symmetries of the  $m \times m$  square. There are 8 symmetries: the identity, 4 reflections (2 in the diagonals and 2 in the right-bisectors of the sides) and 3 rotations (the one-quarter, one-half and three-quarter turns about the center of the square).



We count these 4-tuples in subsets: (i)  $a \leq b$ , (ii)  $b < a \leq b + u$ , (iii)  $b + u < a$ .

In the first subset the 4-tuples satisfy (13) and  $a \leq b$ , and these are equivalent to

$$\underbrace{a} + \underbrace{b-a} + \underbrace{v-u-1} + 2(\underbrace{u-1}) \leq m-3,$$

We count these in 8 subsets according to  $a \equiv i \pmod 2, b-a \equiv j \pmod 2, v-u-1 \equiv k \pmod 2, i, j, k \in \{0, 1\}$  and are led to

$$\binom{\lfloor \frac{m-3}{2} \rfloor + 4}{4} + 3 \binom{\lfloor \frac{m-4}{2} \rfloor + 4}{4} + 3 \binom{\lfloor \frac{m-5}{2} \rfloor + 4}{4} + \binom{\lfloor \frac{m-6}{2} \rfloor + 4}{4} \tag{14}$$

4-tuples.

In the second subset the 4-tuples satisfy (13) and  $b < a \leq b + u$ , and these are equivalent to

$$\underbrace{b} + 2(\underbrace{a-b-1}) + 2(\underbrace{b+u-a}) + \underbrace{v-u-1} \leq m-3.$$

We count these in 4 subsets according to  $b \equiv i \pmod 2, v-u-1 \equiv j \pmod 2, i, j \in \{0, 1\}$ , and are led to

$$\binom{\lfloor \frac{m-3}{2} \rfloor + 4}{4} + 2 \binom{\lfloor \frac{m-4}{2} \rfloor + 4}{4} + \binom{\lfloor \frac{m-5}{2} \rfloor + 4}{4} \tag{15}$$

4-tuples.

In the third subset, the 4-tuples satisfy (13) and  $b + u < a$ , and these are equivalent to

$$\underbrace{a - b - u - 1} + \underbrace{b} + \underbrace{v - u - 1} + 2(\underbrace{u - 1}) \leq m - 4.$$

We count these in 8 subsets according to  $a - b - u - 1 \equiv i \pmod 2$ ,  $b \equiv j \pmod 2$ ,  $v - u - 1 \equiv k \pmod 2$ ,  $i, j, k \in \{0, 1\}$  and are led to

$$\binom{\lfloor \frac{m-4}{2} \rfloor + 4}{4} + 3 \binom{\lfloor \frac{m-5}{2} \rfloor + 4}{4} + 3 \binom{\lfloor \frac{m-6}{2} \rfloor + 4}{4} + \binom{\lfloor \frac{m-7}{2} \rfloor + 4}{4} \tag{16}$$

4-tuples.

Adding (14), (15) and (16), we find that the number of 4-tuples satisfying (13) is

$$\begin{aligned} & 2 \binom{\lfloor \frac{m-1}{2} \rfloor + 3}{4} + 6 \binom{\lfloor \frac{m}{2} \rfloor + 2}{4} + 7 \binom{\lfloor \frac{m-1}{2} \rfloor + 2}{4} + 4 \binom{\lfloor \frac{m}{2} \rfloor + 1}{4} + \binom{\lfloor \frac{m-1}{2} \rfloor + 1}{4} \\ & = \begin{cases} \frac{1}{96} m^2 (m - 2) (5m + 4) & \text{if } m \text{ is even,} \\ \frac{1}{96} (m - 1) (m + 1) (5m^2 - 6m - 3) & \text{if } m \text{ is odd,} \end{cases} \end{aligned}$$

and the answer to the problem is

$$\begin{cases} \frac{1}{12} m^2 (m - 2) (5m + 4) & \text{if } m \text{ is even,} \\ \frac{1}{12} (m - 1) (m + 1) (5m^2 - 6m - 3) & \text{if } m \text{ is odd.} \end{cases} \tag{17}$$

**Problem 11.** How many isosceles right triangles have vertices at lattice points of  $L_{m,m}$ ? (None of the sides need lie on lattice lines.)

**Solution.** This is the sum of the numbers (10), (12), (17) obtained in Problems 8, 9, 10 respectively. This sum reduces to

$$\begin{cases} \frac{1}{12} m (m + 2) (5m^2 + 10m + 2) & \text{if } m \text{ is even,} \\ \frac{1}{12} (m + 1)^2 (5m^2 + 10m - 3) & \text{if } m \text{ is odd.} \end{cases}$$

### References

J. Dawes (1972), Squares in a square lattice, *Mathematical Gazette* **56**, 129.  
 M. Fox (1984), Solution to problem 68.C, *Mathematical Gazette* **68**, 294.

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\* \* \* \* \*



## THE OLYMPIAD CORNER

No. 150

R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

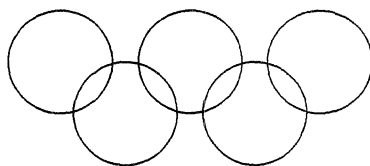
We begin this issue with a set of pre-Olympiad problems sent in by Andy Liu. Even though these problems are for the preliminary round of a mathematics olympiad for elementary school, the problems would challenge many high school students as well.

### 1993 JAPANESE MATHEMATICAL OLYMPIAD

for Elementary School Students

Time: 2 Hours

**1.** There are 9 regions inside the 5 rings of the Olympics. Put a different whole number from 1 to 9 in each so that the sum of the numbers in each ring is the same. What are the largest and the smallest values of this common sum?



**2.** A country has only \$6 and \$7 bills. How many different amounts, in whole numbers of dollars, cannot be paid for exactly, and what is the highest such amount?

**3.** Peter bought some pigs at \$590 each and some horses at \$670 each. He bought more horses than pigs. He paid with a \$10,000 bill and got back some \$100 bills and some \$10 bills. Had the numbers of pigs and horses bought been interchanged, so would the numbers of \$100 bills and \$10 bills obtained as change. How many pigs and how many horses did Peter buy?

**4.** On each of three cards was written a whole number from 1 to 10. These cards were shuffled and dealt to three people who recorded the numbers on their respective cards. The cards were collected, and the process was repeated again. After a few times, the three people computed the totals of their numbers. They turned out to be 13, 15 and 23. What were the numbers on the cards?

**5.** Each of  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$  was told in secrecy a different whole number from 1 to 5. The teacher asked  $A$ : “whose number is the largest?”  $A$  said: “I don’t know.” The

teacher then asked  $B$ : “Is  $C$ ’s number larger than yours?”  $B$  said: “I don’t know.” The teacher asked  $C$ : “Is  $D$ ’s number larger than yours?”  $C$  said: “I don’t know.” The teacher then asked  $D$ : “Is  $B$ ’s number larger than yours?”  $D$ ’s answer was not recorded. Finally, the teacher asked  $B$  again: “Is  $C$ ’s number larger than yours?”  $B$  said: “No.” From the above information, it is possible to deduce which number was told to whom, and was  $D$ ’s answer “Yes”, “No” or “I don’t know”?

\*                    \*                    \*

The Olympiad problems we give this number are from an intercontinental contest, the Iberoamerican Olympiad. Thanks go to Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain, who forwarded them to us.

### PROBLEMS PROPOSED AT THE 7TH IBEROAMERICAN MATHEMATICAL OLYMPIAD

Caracas, Venezuela, September 22–23, 1992

First Day: Time 4.5 Hours

1. For each positive integer  $n$ , let  $a_n$  be the last digit of the number

$$1 + 2 + 3 + \cdots + n.$$

Calculate  $a_1 + a_2 + \cdots + a_{1991}$ .

2. Given the set of  $n$  real numbers such that

$$0 < a_1 < a_2 < \cdots < a_n$$

and given the function

$$f(x) = \frac{a_1}{x + a_1} + \frac{a_2}{x + a_2} + \cdots + \frac{a_n}{x + a_n}$$

determine the sum of the lengths of all the pairwise disjoint intervals formed by all the  $x$  such that  $f(x) \geq 1$ .

3. In an equilateral triangle  $ABC$  (of side length 2) consider the incircle  $\Gamma$ .

(a) Show that for all points  $P$  of  $\Gamma$ ,

$$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 5.$$

(b) Show that for all points  $P$  of  $\Gamma$ , it is possible to construct a triangle of sides  $PA$ ,  $PB$ ,  $PC$ , with area  $\sqrt{3}/4$ .

## Second Day: Time 4.5 Hours

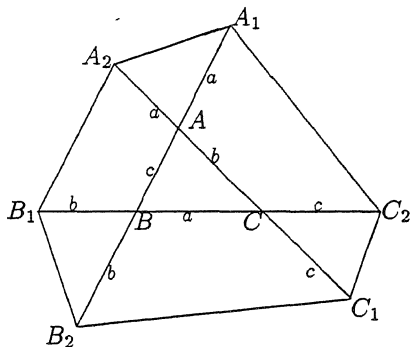
4. Let  $(a_n)$  and  $(b_n)$  be two sequences of integer numbers which satisfy the following conditions:

- (i)  $a_0 = 0, b_0 = 8$ ;
- (ii)  $a_{n+2} = 2a_{n+1} - a_n + 2; b_{n+2} = 2b_{n+1} - b_n$ ;
- (iii)  $a_n^2 + b_n^2$  is a perfect square, for all  $n$ .

Determine at least two possible values for the pair  $(a_{1992}, b_{1992})$ .

5. Given the circumference  $\Gamma$  and positive numbers  $h$  and  $m$  such that there is a trapezoid  $ABCD$  inscribed in  $\Gamma$  with altitude  $h$  and for which the sum of the basis  $AB$  and  $CD$  is  $m$ , construct the trapezoid  $ABCD$ .

6. From the triangle  $T$  with vertices  $A, B$  and  $C$ , the hexagon  $H$  with vertices  $A_1, A_2, B_1, B_2, C_1, C_2$  is constructed, as shown in the figure. Show that the area of  $H$  is at least thirteen times the area of  $T$ .



[Editor's note. This is similar to *Cruze* 1887 [1993: 265].]

\* \* \*

Next we have a "letter to the editor" from a faithful reader with some views about remarks that we made about reusing problems.

#### Letter to the Editor

Many thanks for your continued interest in the Austrian–Polish Mathematics Competition; for publishing its problems and their solutions.

In the solution column in *Cruze* [1993: 69–70], problems 5 and 6 of the 13-th A.–P.M.C. appear with the comment that they had already appeared before; the first, in the Training Test for the 1991 U.S.S.R. I.M.O. Team, and the second at a previous U.S.A.M.O.(!).

It is no secret that many of the problems used at the A.–P.M.C. have been proposed by me; in particular, I am "responsible" for the two just mentioned. However, in devising

these problems I was working quite independently, and they occurred to me in a very natural way. The quantity  $M_n$  in problem 5 expresses  $n$  times the “mean displacement” of an element in a random permutation of  $\{1, \dots, n\}$ , while the statement in problem 6 is just a simple property of integer-valued closed iteration orbits of integer polynomials (one arrives at this small “discovery” almost as soon as one tries to invent an exercise on iteration of polynomials). Incidentally, I have also found the latter problem a year or two ago in the materials of yet another mathematical competition in some country.

The intention of this letter is not to “swear having played fair” (*qui s’excuse, s’accuse!*). I just wished to call attention to the fact that situations like that are not at all rare. For instance, one of the problems at the 1989 I.M.O. was a special case of *Cruz* 1382 [1988: 268]. Many other examples can be given; just look at S. Rabinowitz’s *Index*. Probably, in some cases problems are being borrowed, from one competition to another. But I am sure that in most cases very similar ideas occur simultaneously to different people, working in distant parts of the world. This can seem surprising at first glance. After a moment’s thought, this is no surprise at all.

I wish to assure you that I appreciate the *Olympiad Corner* highly and like it very much.

Sincerely yours,

*Marcin E. Kuczma*

\* \* \*

Next we have a correction to a solution which we printed in the March number. There was an error in case 3 of the solution to problem 2 of the 13th Austrian–Polish Mathematics Competition. This led to a loss of several solutions. The corrected solution comes from a preprint of a book on the competition which Marcin E. Kuczma is preparing.

**2.** [1992: 3; 1993: 67–68] *13th Austrian–Polish Mathematics Competition*.

Determine all triples  $(x, y, z)$  of positive integers such that

$$x^{(y^z)} \cdot y^{(z^x)} \cdot z^{(x^y)} = 1990^{1990} xyz.$$

*Corrected solution by Marcin E. Kuczma, Warszawa, Poland.*

Let as usual  $x^{y^z} = x^{(y^z)}$  etc. Rewrite the equation as

$$x^{y^z-1} \cdot y^{z^x-1} \cdot z^{x^y-1} = 1990^{1990}. \quad (1)$$

Suppose the integers  $x \geq 1$ ,  $y \geq 1$ ,  $z \geq 1$  satisfy (1). Consider two cases.

*Case 1.*  $x, y, z$  are greater than 1. Since  $1990 = 2 \cdot 5 \cdot 199$  is the prime factorization of 1990, one of the three numbers  $x, y, z$  has to be divisible by 199 — hence not smaller than 199. Let e.g.  $x \geq 199$ . Then

$$z^x - 1 \geq 2^{199} - 1 > 11 \cdot 1990$$

and so

$$y^{z^x-1} \geq 2^{z^x-1} > 2^{11 \cdot 1990} > 1990^{1990},$$

in contradiction to (1).

Case 2. One of the numbers  $x, y, z$  equals 1; say,  $z = 1$ . The equation becomes

$$x^{y-1} = 1990^{1990}. \quad (2)$$

Thus  $x = 2^k 5^l 199^m$ . Equation (2) now yields

$$k(y-1) = l(y-1) = m(y-1) = 1990,$$

whence  $k = l = m$  and  $x = 1990^k$ . The following table displays all solutions of  $k(y-1) = 1990$  in positive integers:

$k =$	1990	995	398	199	10	5	2	1
$y - 1 =$	1	2	5	10	199	398	995	1990

This yields the following solutions  $(x, y, z)$  of equation (1):

$$\begin{aligned} &(1990^{1990}, 2, 1), \quad (1990^{995}, 3, 1), \quad (1990^{398}, 6, 1), \quad (1990^{199}, 11, 1), \\ &(1990^{10}, 200, 1), \quad (1990^5, 399, 1), \quad (1990^2, 996, 1), \quad (1990^1, 1991, 1). \end{aligned}$$

Assuming that, instead of  $z$ , either  $x$  or  $y$  equals 1, we obtain 16 further solutions  $(x, y, z)$  of (1); they arise from those listed above by means of a cyclic shift, the same within each triple. As there were no solutions in case 1, these  $8 + 8 + 8 = 24$  triples  $(x, y, z)$  constitute the complete solution of the given equation.

\*                    \*                    \*

As promised last issue we continue with solutions to the problems proposed but not used at the 32nd I.M.O. at Sigtuna, Sweden [1992: 225–226]. The astute reader will note that we haven't yet given a solution to problem 15.

**16.** *Proposed by Hong Kong.*

Find all positive integer solutions  $x, y, z$  of the equation  $3^x + 4^y = 5^z$ .

*Solutions by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.*

Since  $5^z \equiv 1 \pmod{4}$  while  $3^x \equiv (-1)^x \pmod{4}$ , we obtain that  $x$  is even. Since  $4^y \equiv 1 \pmod{3}$  while  $5^z \equiv -1^z \pmod{3}$ ,  $z$  is also even. Let  $x = 2u$  and  $z = 2v$ . Then the equation becomes  $3^{2u} + 2^{2y} = 5^{2v}$  which implies that  $2^{2y} = (5^v + 3^u)(5^v - 3^u)$ . Clearly this implies that  $5^v + 3^u = 2^s$  and  $5^v - 3^u = 2^{2y-s}$  for some integer  $s$  with  $y < s < 2y$ . Now adding the two equations we get

$$2 \cdot 5^v = 2^s + 2^{2y-s} \quad \text{or} \quad 5^v = 2^{2y-s-1}(2^{2s-2y} + 1)$$

so that

$$2y - s = 1 \quad \text{and} \quad 5^v - 3^u = 2. \quad (1)$$

In a similar way, beginning with  $3^{2u} = (5^v + 2^y)(5^v - 2^y)$  we get that  $5^v + 2^y = 3^t$  and  $5^v - 2^y = 3^{2u-t}$  for some integer  $t$  with  $u < t \leq 2u$ . These equations give  $2 \cdot 5^v = 3^t + 3^{2u-t} = 3^{2u-t}(3^{2t-2u} + 1)$  so

$$2u - t = 0 \quad \text{and} \quad 5^v - 2^y = 1. \quad (2)$$

From (1) and (2) we get  $3^u = 2^y - 1$ . From  $5^v + 3^u = 2^s$  and  $5^v - 3^u = 2$  we get  $2 \cdot 3^u = 2^{2y-1} - 2$ . Comparing these we conclude that  $y = 2y - 2$ , so  $y = 2$ . It then follows easily that  $u = 1$ ,  $x = 2$  and  $z = 2$ . Thus  $x = y = z = 2$  is the only solution.

*Remark.* The problem is well known. It appeared, among other places, as problem P341 in the *Canadian Mathematical Bulletin* [26(2), 1983, p. 252; solution in 28(2), 1985, p. 255]. In fact, it asked for all solutions in nonnegative integers and it is fairly easy to show that in this case, there is one more solution given by  $x = 0$ ,  $y = z = 1$ . The published solution in CMB however, contains a few inaccuracies, e.g., statements like "since  $3^p \equiv 1 \pmod{4}$ , and  $5^n \equiv 1 \pmod{3} \dots$ ".

*Comment by Seung-Jin Bang, Albany, California.*

This problem appeared in the April 1991 number of *Mathematics Magazine* as part (2) of problem 1369. A solution was published in the April 1992 number.

**17. Proposed by Bulgaria.**

Find the highest degree  $k$  of 1991 for which  $1991^k$  divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

*Solution by George Evagelopoulos, Athens, Greece.*

First we shall prove that for every odd number  $a \geq 3$  and for every positive integer  $n$ ,

$$(1 + a)^{a^n} = 1 + s_n a^{n+1}$$

where  $s_n$  is an integer and  $a$  does not divide  $s_n$ . If  $n = 1$

$$\begin{aligned} (1 + a)^a &= 1 + \binom{a}{1}a + \binom{a}{2}a^2 + \dots + \binom{a}{a}a^a \\ &= 1 + a^2 \left[ 1 + \binom{a}{2} + \binom{a}{3}a + \dots \right] \\ &= 1 + s_1 a^2. \end{aligned}$$

Since  $a$  is odd,  $a \mid \binom{a}{2}$  and therefore  $a$  does not divide  $s_1 = 1 + \binom{a}{2} + \binom{a}{3}a + \dots$ . If the proposition is true for  $n$ , then for  $n + 1$  we have

$$\begin{aligned} (1 + a)^{a^{n+1}} &= (1 + s_n a^{n+1})^a \\ &= 1 + \binom{a}{1} s_n a^{n+1} + \binom{a}{2} s_n^2 a^{n+2} + \dots \\ &= 1 + a^{n+1} \left( s_n + \binom{a}{2} s_n^2 a^n + \dots \right) \\ &= 1 + s_{n+1} a^{n+2} \end{aligned}$$

and since  $a \nmid s_n$ , then  $a \nmid s_{n+1}$ .

Analogously we can prove that for every odd number  $b \geq 3$  and every positive integer  $n$

$$(b-1)^{b^n} = -1 + t_n b^{n+1},$$

where  $t_n$  is an integer and  $b \nmid t_n$ .

From these two propositions we obtain that

$$1991^{1991} \mid 1992^{1991^{1990}} - 1$$

and

$$1991^{1992} \nmid 1992^{1991^{1990}}$$

( $a = 1991$ ,  $n = 1990$ ) and

$$1991^{1993} \mid 1990^{1991^{1992}} + 1$$

( $b = 1991$ ,  $n = 1992$ ).

Therefore

$$1991^{1991} \mid 1992^{1991^{1990}} + 1990^{1991^{1992}}$$

and

$$1991^{1992} \nmid 1992^{1991^{1990}} + 1990^{1991^{1992}},$$

i.e. the number we are seeking is  $k = 1991$ .

### 18. Proposed by Ireland.

Let  $\alpha$  be the positive root of the equation  $x^2 = 1991x + 1$ . For natural numbers  $m$  and  $n$  define

$$m * n = mn + [\alpha m][\alpha n]$$

where  $[x]$  is the greatest integer not exceeding  $x$ . Prove that for all natural numbers  $p$ ,  $q$  and  $r$ ,

$$(p * q) * r = p * (q * r).$$

*Solution by George Evagelopoulos, Athens, Greece.*

We prove the result holds with 1991 replaced by any positive integer  $k$ . For natural numbers  $p$ ,  $q$ , let

$$\varepsilon = (\alpha p - [\alpha p])(\alpha q - [\alpha p]).$$

Then  $0 < \varepsilon < 1$  and

$$\varepsilon = \alpha^2 pq - \alpha(p[\alpha q] + q[\alpha p]) + [\alpha p][\alpha q].$$

So, using  $\alpha^2 = k\alpha + 1$ , i.e.  $\alpha(\alpha - k) = 1$ ,

$$(\alpha - k)\varepsilon = \alpha pq + \alpha[\alpha p][\alpha q] - (p[\alpha q] + q[\alpha p] + k[\alpha p][\alpha q]).$$

Since  $0 < (\alpha - k)\varepsilon < 1$ , we have

$$0 < \alpha pq + \alpha[\alpha p][\alpha q] - (p[\alpha q] + q[\alpha p] + k[\alpha p][\alpha q]) < 1. \quad (1)$$

But now

$$\begin{aligned}(p * q) * r &= (p * q)r + [\alpha(p * q)][\alpha r] \\ &= pqr + [\alpha p][\alpha q]r + (p[\alpha q] + q[\alpha p] + k[\alpha p][\alpha q])[\alpha r] \\ &= pqr + [\alpha p][\alpha q]r + p[\alpha q][\alpha r] + q[\alpha p][\alpha r] + k[\alpha p][\alpha q][\alpha r].\end{aligned}$$

Replacing  $p, q$  by  $q, r$  in (1) we have

$$[\alpha qr + \alpha[\alpha q][\alpha r]] = q[\alpha r] + r[\alpha q] + k[\alpha q][\alpha r].$$

Thus

$$p * (q * r) = pqr + p[\alpha q][\alpha r] + q[\alpha p][\alpha r] + r[\alpha p][\alpha q] + k[\alpha p][\alpha q][\alpha r] = (p * q) * r.$$

**19. Proposed by the U.S.A.**

Real constants  $a, b, c$  are such that there is exactly one square all of whose vertices lie on the cubic curve  $y = x^3 + ax^2 + bx + c$ . Prove that the square has side  $\sqrt[4]{72}$ .

*Solution by George Evagelopoulos, Athens, Greece.*

By performing the horizontal translation  $x' = x + a/3$ , we may assume that  $a = 0$ . Then we may perform a vertical translation ( $y' = y - c$ ) to assume  $c = 0$ . The square must be centered at the origin or else its rotation by  $180^\circ$  about the origin would be a second square lying on the cubic.

Let  $(r, s)$  be one vertex of the square. The other vertices are then  $(-s, r)$ ,  $(-r, -s)$ , and  $(s, -r)$  and all four vertices lie on the cubic  $-x = y^3 + by$  which is obtained by a rotation through  $90^\circ$  of  $y = x^3 + bx$ . Conversely, if  $P$  is any point other than the origin which lies on both the cubic  $y = x^3 + bx$  and its rotation by  $90^\circ$ ,  $-x = y^3 + by$ , then the rotations of  $P$  by  $90^\circ$ ,  $180^\circ$ , and  $270^\circ$  also lie on the cubic, forming a square. Since there is only one square the cubic must meet its rotation exactly once in the interior of each quadrant. Clearly there are no points of intersection on the axes except for  $(0, 0)$ . In particular the original cubic must pass through each quadrant, so  $b < 0$ .

The number of times that one cubic actually crosses the other in a given quadrant is even, so the only way for there to be one intersection point is for the two cubics to be tangent. Hence the  $x$ -coordinate of each vertex is a multiple root of

$$-x = (x^3 + bx)^3 + b(x^3 + bx).$$

These four  $x$  coordinates are distinct and nonzero since each determines the  $y$ -coordinate for a point on the cubic, so counting zeros shows that each nonzero root must be only a double zero of the polynomial  $p(x) = (x^3 + bx)^3 + b(x^3 + bx) + x$ . Hence

$$\frac{p(x)}{x} = x^8 + 3bx^6 + 3b^2x^4 + b(b^2 + 1)x + (b^2 + 1) = [(x - r)(x + r)(x - s)(x + s)]^2.$$

This gives that

$$3b = -2(r^2 + s^2), \quad 3b^2 = (r^2 + s^2)^2 + 2r^2s^2,$$



$$b(b^2 + 1) = -2r^2s^2(r^2 + s^2), \quad \text{and} \quad b^2 + 1 = r^4s^4.$$

Thus  $3b(b^2 + 1) = -2(r^2 + s^2)r^4s^4$  and  $3b(b^2 + 1) = -6r^2s^2(r^2 + s^2)$  so  $r^2s^2 = 3$  and  $b^2 = 8$ . Finally  $b = -2\sqrt{2}$  and  $r^2 + s^2 = 3\sqrt{2}$ . The line segment from  $(0, 0)$  to  $(r, s)$  is half a diagonal of the square, so a side of the square has length  $\sqrt{2(r^2 + s^2)} = \sqrt[4]{72}$ .

**20.** *Proposed by India.*

An odd integer  $n \geq 3$  is said to be “nice” if and only if there is at least one permutation  $a_1, a_2, \dots, a_n$  of  $1, 2, \dots, n$  such that the  $n$  sums

$$\begin{aligned} a_1 - a_2 + a_3 - \dots - a_{n-1} + a_n, \\ a_2 - a_3 + a_4 - \dots - a_n + a_1, \\ a_3 - a_4 + a_5 - \dots - a_1 + a_2, \\ \vdots \\ a_n - a_1 + a_2 - \dots - a_{n-2} + a_{n-1} \end{aligned}$$

are all positive. Determine the set of all “nice” integers.

*Solution by George Evagelopoulos, Athens, Greece.*

Let  $y_1 = a_1 - a_2 + a_3 - \dots + a_n$ ,  $y_2 = a_2 - a_3 + a_4 - \dots + a_1$ ,  $\dots$ ,  $y_n = a_n - a_1 + a_2 - \dots + a_{n-1}$ . We have  $y_1 + y_2 = 2a_1$ ,  $y_2 + y_3 = 2a_2$ ,  $\dots$ ,  $y_n + y_1 = 2a_n$ .

*Case A.* Let  $n$  be a natural number of the form  $4k + 1$ ,  $k \geq 1$ . Setting  $y_1 = 1$ ,  $y_2 = 3$ ,  $y_3 = 5, \dots, y_{2k+1} = 4k + 1$ ,  $y_{2k+2} = 4k + 1$ ,  $y_{2k+3} = 4k - 3, \dots, y_{4k-1} = 5$ ,  $y_{4k} = 5$ ,  $y_{4k+1} = 1$  we get  $a_1 = 2$ ,  $a_2 = 4, \dots, a_{2k} = 4k$ ,  $a_{2k+1} = 4k + 1$ ,  $a_{2k+2} = 4k - 1$ ,  $a_{2k+3} = 4k - 3, \dots, a_{4k-1} = 5$ ,  $a_{4k} = 3$ ,  $a_{4k+1} = 1$ . Thus we have exhibited the desired permutation, namely  $2, 4, \dots, 4k, 4k + 1, 4k - 1, \dots, 3, 1$  of  $1, 2, 3, \dots, 4k + 1$  satisfying the given property, since  $y_1, \dots, y_{4k+1}$  are all positive. Hence the numbers of the form  $4k + 1$ ,  $k \geq 1$  are all nice.

*Case B.* Let  $n$  be a natural number of the form  $4k - 1$ . Then for each  $i$ ,  $1 \leq i \leq 4k - 1$ , we have

$$\begin{aligned} y_i &= (a_i + a_{i+1} + \dots + a_{i-1}) - 2(a_{i+1} + a_{i+3} + \dots + a_{i-2}) \\ &= (1 + 2 + \dots + 4k - 1) - (\text{an even number}) = \text{an even number.} \end{aligned}$$

Suppose now that there is a permutation witnessing that  $n$  is nice. Then each  $y_i$  is not only positive but also even. But for some  $t$ ,  $1 \leq t \leq 4k - 1$ , we must have  $a_t = 1$ , and so  $y_t + y_{t+1} - 2a_t = 2$ , which is impossible as both  $y_t$  and  $y_{t+2}$  are positive and even, so numbers of the form  $4k - 1$  are not nice.

\*                    \*                    \*

That completes this month’s Corner. We will continue the solutions to the problems proposed at Sigtuna next number. Send me your nice solutions, as well as Olympiad and pre-Olympiad contests.

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## PROBLEMS

*Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (\*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **July 1, 1994**, although solutions received after that date will also be considered until the time when a solution is published.*

**1891.** *Proposed by Toshio Seimiya, Kawasaki, Japan.*

*ABC is a non-right-angled triangle with orthocenter H. A line  $\ell$  through H meets AB and AC at  $D \neq B$  and  $E \neq C$  respectively. Let P be a point such that  $AP \perp \ell$ . Prove that  $[PBD] : [PCE] = DH : HE$ , where  $[XYZ]$  denotes the area of triangle XYZ.*

**1892.** *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

*Let  $n \geq 4$  be an integer. Find the exact upper and lower bounds for the cyclic sum*

$$\sum_{i=1}^n \frac{x_i}{x_{i-1} + x_i + x_{i+1}}$$

*(where of course  $x_0 = x_n$ ,  $x_{n+1} = x_1$ ), over all  $n$ -tuples of nonnegative numbers  $(x_1, \dots, x_n)$  without three zeros in cyclic succession. Characterize all cases in which either one of these bounds is attained.*

**1893.** *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

*For a non-obtuse triangle ABC, D is the foot of the perpendicular from A to BC, E is the point of intersection of BC and the internal bisector of  $\angle A$ , and M is the midpoint of the segment BC. Suppose that the lengths of the segments BD, DE, EM, MC (in that order) are in arithmetic progression.*

(a) *Describe all such triangles ABC.*

(b) *Let F be the point of intersection of AC and the line passing through E that is perpendicular to AE. Show that triangle EMF is isosceles.*

**1894.** *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India. (Dedicated in memoriam to K.V.R. Sastry.)*

*Evaluate*

$$\sum_{k=0}^n \frac{\binom{n+1}{k} \binom{n}{k}}{\binom{2n}{2k}}$$

*where  $n$  is a nonnegative integer.*

**1895.** *Proposed by Ji Chen and Gang Yu, Ningbo University, China.*

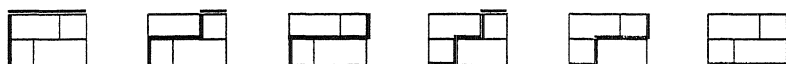
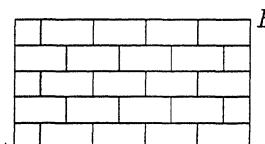
Let  $P$  be an interior point of a triangle  $A_1A_2A_3$ ;  $R_1, R_2, R_3$  the distances from  $P$  to  $A_1, A_2, A_3$ ; and  $R$  the circumradius of  $\Delta A_1A_2A_3$ . Prove that

$$R_1R_2R_3 \leq \frac{32}{27}R^3,$$

with equality when  $A_2 = A_3$  and  $PA_2 = 2PA_1$ .

**1896.** *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Consider an  $m \times n$  "brick wall" grid of  $m$  rows and  $n$  columns, made up of  $1 \times 2$  bricks with  $1 \times 1$  bricks at the ends of rows where needed, and so that we always have a  $1 \times 1$  brick in the lower left corner. The diagram shows the case  $m = 5, n = 9$ . Let  $f(m, n)$  denote the number of walks of minimum length (using the grid lines) from  $A$  to  $B$ , so for example  $f(2, 3) = 6$ :



Prove that

$$f(m, n) = f(m, n - 2) + f(m - 1, n - 1)$$

for all  $m > 1$  and  $n > 2$ .

**1897.** *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

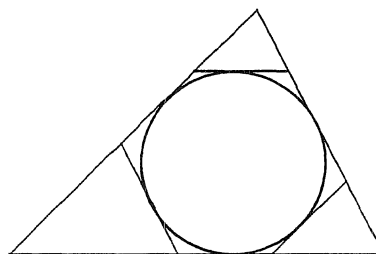
In triangle  $ABC$  the angle bisectors of angles  $B$  and  $C$  meet the median  $AD$  at points  $E$  and  $F$  respectively. If  $BE = CF$  then prove that  $ABC$  is isosceles.

**1898.** *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

The faces of a tetrahedron  $ABCD$  are cut by a line in four distinct points  $A', B', C', D'$  (with  $A'$  opposite  $A$ , etc.). Prove that the midpoints of the segments  $AA', BB', CC', DD'$  are coplanar.

**1899.** *Proposed by Neven Jurić, Zagreb, Croatia.*

In a triangle with circumradius  $R$ , three lines are drawn tangent to the inscribed circle and parallel to the sides, cutting three small triangles off the corners of the given triangle as shown. Let the circumradii of these triangles be  $R_a, R_b, R_c$ . Show that  $R_a + R_b + R_c = R$ .



**1900.** *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

Given a cube of edge 1, choose four of its vertices forming a regular tetrahedron of edge  $\sqrt{2}$  (the face diagonal of the cube). The other four vertices form another such tetrahedron. Find the volume of the union of these two tetrahedra.

\* \* \* \* \*

## SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

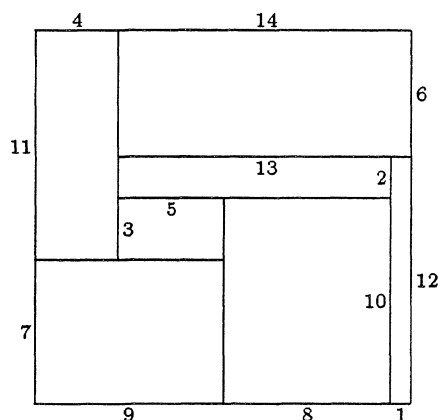
**1707.** [1992: 13; 1993: 18] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

What is the largest integer  $m$  for which an  $m \times m$  square can be cut up into 7 rectangles whose dimensions are  $1, 2, \dots, 14$  in some order?

II. *Comment by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*  
Here are the numbers of such squares I've found:

side $m$	17	18	19	20	21	22
number of solutions	none	3	12	2	5	2

[*Editor's note.* Engelhaupt then gave drawings of all these solutions. For example, here is one of the three  $18 \times 18$  squares he found:



As the smallest possible sum of areas of 7 rectangles of dimensions  $1, 2, \dots, 14$  in some order is

$$1 \cdot 14 + 2 \cdot 13 + 3 \cdot 12 + 4 \cdot 11 + 5 \cdot 10 + 6 \cdot 9 + 7 \cdot 8 = 280 > 16^2,$$

the minimum possible side of such a square is 17, and apparently there are no solutions of this size.]

\*                    \*                    \*                    \*                    \*

**1715.** [1992: 44; 1993: 28] *Proposed by Seung-Jin Bang, Albany, California.*  
Evaluate the sum

$$\sum_{k=0}^{n-2} \frac{1}{k!} \left( \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-k} \frac{1}{(n-k)!} \right)$$

for  $n \geq 2$ .

II. *Solution by Denis Hanson, University of Regina.*

It follows easily from the Principle of Inclusion and Exclusion that the number of derangements of  $n$  objects, permutations with no fixed points, is

$$D_n = n! \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$

Thus the number of ways to assign  $n$  given answers to  $n$  questions and get exactly  $k$  correct is

$$\binom{n}{k} D_{n-k} = \binom{n}{k} (n-k)! \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-k} \frac{1}{(n-k)!} \right),$$

since we may first pick the  $k$  correctly answered questions and then derange the remaining  $n-k$  answers. Let  $f(n, m)$  be the probability of getting at most  $m$  answers correct when answering in a random fashion. Then

$$\begin{aligned} f(n, n-2) &= \frac{1}{n!} \sum_{k=0}^{n-2} \binom{n}{k} D_{n-k} \\ &= \frac{1}{n!} \sum_{k=0}^{n-2} \binom{n}{k} (n-k)! \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-k} \frac{1}{(n-k)!} \right) \\ &= \sum_{k=0}^{n-2} \frac{1}{k!} \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-k} \frac{1}{(n-k)!} \right), \end{aligned}$$

so  $f(n, n-2)$  is the sum in question. Since  $f(n, n)$  is clearly 1 we have

$$f(n, n-2) = \frac{1}{n!} (n! - 0 - 1) = 1 - \frac{1}{n!}$$

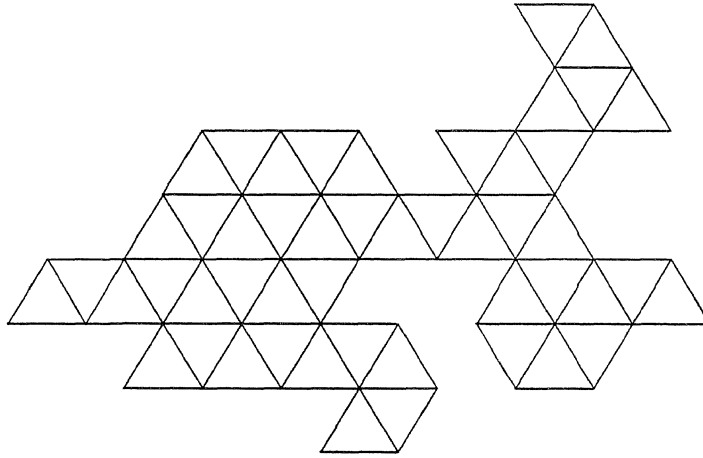
since we cannot get exactly  $n-1$  correct and there is only one way to get them all correct. One could find  $f(n, n-3)$  etc. in a similar manner.

For those students who like to guess on such “matching questions”, look at the sizes of  $f(n, 0)$  and  $f(n, 1)$ .  $f(n, 0) = D_n/n!$  which is about  $1/e$ , and  $f(n, 1)$  is about  $2/e$  which is almost  $3/4$ . In fact it is not too difficult to show that, if one guesses the answers to the questions randomly, the expected number correct is precisely 1 (independent of  $n$ ), so that a simple strategy, as good as any, would be to answer all questions with the same answer — this way one avoids the embarrassment of getting 0 right!

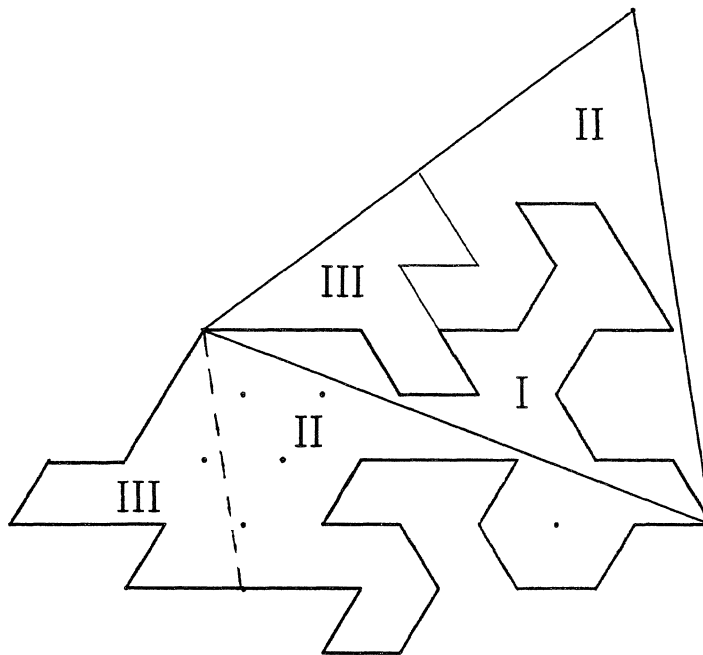
\* \* \* \* \*

**1806.** [1993: 15] *Proposed by Andy Liu, University of Alberta.*

Dissect the figure into three pieces which can be reassembled into an equilateral triangle. [*Warning: the dissecting lines need not lie along the grid lines!*]



*Solution by P. Penning, Delft, The Netherlands.*



*Also solved by the proposer, who uses the fact (perhaps visible from the above picture) that the given shape tiles the plane, a standard method in Harry Lindgren's Geometric Dissections, Dover, 1972.*

\* \* \* \* \*

**1807.** [1993: 15] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

The infinite series  $a_0 + a_1 + a_2 + \cdots$  is generated recursively by:  $a_0 = 5678$  and for  $n \geq 1$ ,

$$a_n = \sin(a_0 + \cdots + a_{n-1})$$

(where the quantity  $a_0 + \cdots + a_{n-1}$  is considered to be in radians). Show that the series converges and find its sum.

*Composite solution based on the ones submitted by P. Penning, Delft, The Netherlands, and Waldemar Pompe, student, University of Warsaw, Poland.*

For  $n \geq 0$  let  $S_n = a_0 + a_1 + \cdots + a_n$ . Then

$$S_{n+1} = S_n + \sin S_n. \quad (1)$$

If  $(2k+1)\pi < S_n < (2k+2)\pi$  where  $k$  is an integer, then clearly  $S_{n+1} < S_n$ . Furthermore, from the easily established fact that  $x < \sin x$  for  $x < 0$ , we get

$$\sin S_n = \sin((2k+1)\pi - S_n) > (2k+1)\pi - S_n$$

and so  $S_{n+1} > (2k+1)\pi$  from (1). Hence  $\{S_n\}_{n=1}^{\infty}$  is a monotonically decreasing sequence which is bounded below and so converges. Let  $l = \lim_{n \rightarrow \infty} S_n$ . Then from (1)  $l$  must satisfy  $l = l + \sin l$ , and we find easily that  $l = (2k+1)\pi$ . Since  $S_0 = a_0 = 5678 \approx 1807\pi + 1.142074$  we conclude that the series  $a_0 + a_1 + a_2 + \cdots$  converges to  $1807\pi$ .

In the case when  $2k\pi < S_n < (2k+1)\pi$ , a similar argument shows that the sequence  $\{S_n\}_{n=1}^{\infty}$  increases monotonically to the limit  $(2k+1)\pi$ .

Finally, if  $a_0 = k\pi$  for some integer  $k$  then it is readily verified that  $S_n = k\pi$  for all  $n$  and hence  $a_0 + a_1 + a_2 + \cdots = k\pi$ .

*Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MATTEO FERRANTI, student, University of Bologna, Italy; RICHARD I. HESS, Rancho Palos Verdes, California; NEVEN JURIC, Zagreb, Croatia; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; SHAILESH SHIRALI, Rishi Valley School, India; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

*The proposer's original question and solution was for an arbitrary  $a_0$ . The above solution does the same, and several other solvers gave such a generalization too.*

*One uninteresting fact emerging from this problem is that  $5678/1807$  is close to  $\pi$ . However, Shirali notes that the "nearby" fraction  $5680/1808$  is equal to  $355/113$ , which is a well known (and much better) approximation to  $\pi$ .*

\* \* \* \* \*

**1808.** [1993: 16] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Three congruent circles that pass through a common point meet again in points  $A, B, C$ .  $A'B'C'$  is the triangle containing the three circles and whose sides are each tangent to two of the circles. Prove that  $[A'B'C'] \geq 9[ABC]$ , where  $[XYZ]$  denotes the area of triangle  $XYZ$ .

Combination of solutions by Gottfried Perz, Pestalozzigymnasium, Graz, Austria, and Waldemar Pompe, student, University of Warsaw, Poland.

Let  $P$  be the common point of the three circles and  $D, E, F$  the centers of the circles (see Figure 1). Then  $P$  is the circumcenter of  $\triangle DEF$ . Since the midpoints  $A^*, B^*$  and  $C^*$  of  $PA, PB$  and  $PC$  respectively are as well midpoints of  $EF, FD$ , and  $DE$  ( $AFPE$  is a rhombus), triangles  $ABC$  and  $DEF$  are similar to triangle  $A^*B^*C^*$ . From  $AB : A^*B^* : DE = 2 : 1 : 2$  it follows that  $DEF$  and  $ABC$  are congruent.

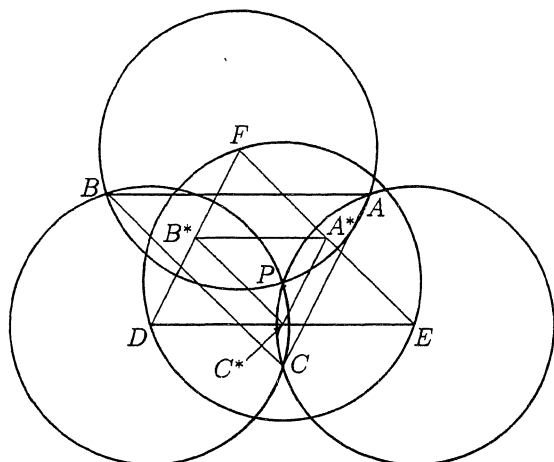


Figure 1

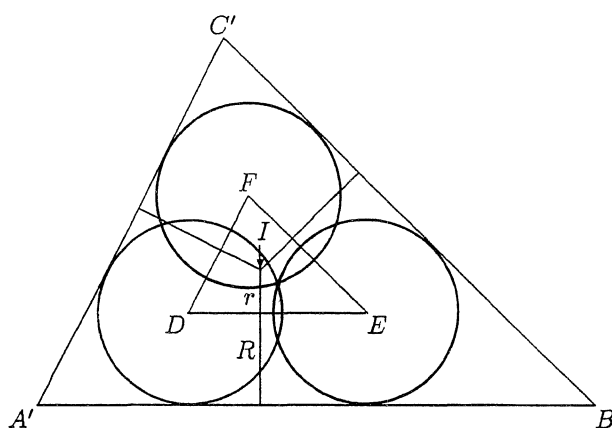


Figure 2

Let  $R$  and  $r$  be the circumradius and inradius, respectively, of  $\triangle DEF$  (or  $\triangle ABC$ ). It is clear that  $A'C' \parallel DF$ ,  $B'C' \parallel EF$ ,  $A'B' \parallel DE$  and that the distances between these pairs of parallel lines are all equal to  $R$  (see Figure 2). It's easy to see that  $\triangle DEF$  and  $\triangle A'B'C'$  have the same incenter  $I$ , and that  $r + R$  is the inradius of  $\triangle A'B'C'$ . Thus triangles  $A'B'C'$  and  $DEF \equiv ABC$  are similar with scale of similarity  $(R + r)/r$ . Therefore since  $R \geq 2r$ ,

$$\frac{[A'B'C']}{[ABC]} = \left(\frac{R+r}{r}\right)^2 \geq \left(\frac{2r+r}{r}\right)^2 = 9,$$

and this is our desired inequality.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MATTEO FERRANTI, student, University of Bologna, Italy; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; SHAIKESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Bellot and López note the related problem 5 of the 1981 I.M.O. [1981: 223], solution on pp. 35–36 of M.S. Klamkin, International Mathematical Olympiads 1979–1985, MAA, 1986; also Crux 694 [1982: 314].

Janous, and also Bellot and López, attribute the result that all four circles in this problem are congruent, to the Romanian mathematician Titeica.



\*                    \*                    \*                    \*                    \*

**1810.** [1993: 16] *Proposed by C.J. Bradley, Clifton College, Bristol, England.*

(a) Find all natural numbers  $n$  so that

$$S_n = 9 + 17 + 25 + \cdots + (8n + 1)$$

is a perfect square.

(b) Find all natural numbers  $n$  so that

$$T_n = 5 + 11 + 17 + \cdots + (6n - 1)$$

is a perfect square.

*Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

(a) Since

$$S_n = \sum_{k=1}^n (8k + 1) = 4n(n + 1) + n = 4n^2 + 5n,$$

we are to solve the Diophantine equation  $4n^2 + 5n = q^2$  where  $q$  is a natural number. Since

$$n = \frac{-5 + \sqrt{25 + 16q^2}}{8}$$

(by the quadratic formula), we have  $25 + 16q^2 = t^2$  for some natural number  $t$  such that  $t \equiv 5 \pmod{8}$ . Solving  $(t - 4q)(t + 4q) = 25$  we get, since  $t - 4q \neq t + 4q$ ,

$$t - 4q = 1 \quad \text{and} \quad t + 4q = 25$$

from which follows  $t = 13$  and  $n = 1$ . Thus  $S_n$  is a perfect square if and only if  $n = 1$ .

(b) Since

$$T_n = \sum_{k=1}^n (6k - 1) = 3n(n + 1) - n = 3n^2 + 2n,$$

we are to solve the Diophantine equation  $3n^2 + 2n = q^2$  where  $q$  is a natural number. Since

$$n = \frac{-2 + \sqrt{4 + 12q^2}}{6} = \frac{-1 + \sqrt{1 + 3q^2}}{3},$$

we have

$$1 + 3q^2 = t^2 \tag{1}$$

for some natural number  $t$  such that  $t \equiv 1 \pmod{3}$ . Now (1) is a Pell equation with  $t_1 = 2$  and  $q_1 = 1$  as the least solution in natural numbers and thus by well known results regarding solutions of Pell equations (cf. *Elementary Theory of Numbers* by W. Sierpiński, §2.17, Theorem 13) all solutions to (1) are given by  $(t_k, q_k)$  where  $t_k$  and  $q_k$  are defined recursively by

$$t_{k+1} = 2t_k + 3q_k, \quad q_{k+1} = t_k + 2q_k \tag{2}$$

for  $k = 1, 2, 3, \dots$ . (In fact,  $t^2 - 3q^2 = 1$  is the very equation discussed on p. 100 of Sierpiński's book.) From (2),  $t_{k+1} \equiv 2t_k \pmod{3}$ . Since  $t_1 = 2$ , induction shows that  $t_k \equiv 1 \pmod{3}$  if and only if  $k$  is *even* and so all admissible values of  $n$  are given by

$$n = \frac{-1 + t_{2k}}{3}, \quad k = 1, 2, 3, \dots$$

where the sequence  $\{t_k\}$  is determined from (2). Calculating the first few values of  $t_k$ , we find  $t_2 = 7$ ,  $t_4 = 97$  and  $t_6 = 1351$  yielding  $n = 2$ ,  $32$  and  $450$  respectively. The corresponding values of  $T_n$  are  $T_2 = 16 = 4^2$ ,  $T_{32} = 3136 = 56^2$  and  $T_{450} = 608400 = 780^2$ .

*Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Albany, California; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, Wolverley High School, Kidderminster, U.K.; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, N.Y.; YI-MING DING, student, Ningbo University, Ningbo, China; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, State University of New York, Oneonta; STEWART METCHETTE, Culver City, California; BOB PRIELIPP, University of Wisconsin-Oshkosh; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; SHAILESH SHIRALI, Rishi Valley School, India; LAWRENCE SOMER, Catholic University, Washington, D.C.; P. E. TSAOUSSOGLU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Part (a) only solved by MARGHERITA BARILE, student, Universität Essen, Germany; MATTEO FERRANTI, student, University of Bologna, Italy; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; and WALDEMAR POMPE, student, University of Warsaw, Poland.*

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**1811.** [1993: 48] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let  $d$  and  $k$  be natural numbers with  $d|k$ . Let  $X_k$  be the set of all  $k$ -tuples  $(x_1, \dots, x_k)$  of integers such that  $0 \leq x_1 \leq \dots \leq x_k \leq k$  and  $x_1 + \dots + x_k$  is divisible by  $d$ . Furthermore let  $Y_k$  be the set of all elements  $(x_1, \dots, x_k)$  of  $X_k$  such that  $x_k = k$ . What is the relationship between the sizes of  $X_k$  and  $Y_k$ ?

*Solution by Margherita Barile, student, Universität Essen, Germany.*

We show that

$$|Y_k| = \frac{1}{2}|X_k|.$$

Let  $0 \leq x_1 \leq x_2 \leq \dots \leq x_k \leq k$ . For all  $i, j \in \{1, \dots, k\}$  set

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$x_i = \sum_{j=1}^k a_{ij}$$

for all  $i$ . Moreover, set  $b_{ij} = 1 - a_{ij}$  and  $y_i = \sum_{j=1}^k b_{ji}$  for all  $i$ . Since  $a_{ij} \geq a_{i,j+1}$  for all  $i, j \in \{1, \dots, k\}$ , it follows that  $0 \leq y_1 \leq \dots \leq y_k \leq k$ . Also

$$\sum_{i=1}^k x_i + \sum_{i=1}^k y_i = \sum_{i=1}^k \sum_{j=1}^k (a_{ij} + 1 - a_{ij}) = k^2.$$

Thus  $(x_1, \dots, x_k) \in X_k$  if and only if  $(y_1, \dots, y_k) \in X_k$  [since  $d|k$ ]. Furthermore  $x_k = k$  if and only if  $a_{ik} = 1$  for some  $i$  [if and only if  $a_{kk} = 1$ ], which is the case if and only if

$$y_k = k - \sum_{i=1}^k a_{ik} < k.$$

Thus the injection  $\varphi : X_k \rightarrow X_k$  defined by the assignment

$$\varphi : (x_1, \dots, x_k) \rightarrow (y_1, \dots, y_k)$$

is such that  $\varphi(Y_k) \subseteq X_k \setminus Y_k$  and  $\varphi(X_k \setminus Y_k) \subseteq Y_k$ . Hence  $|Y_k| = |X_k \setminus Y_k|$ , so  $|X_k| = 2|Y_k|$ , as was to be proved.

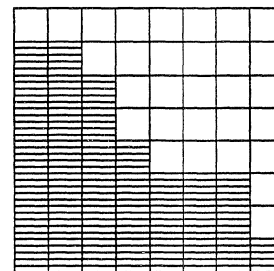
*Note.* The quantities  $y_i$  can be characterized as:  $y_i = \max\{j | x_j < i\}$ . Their relationship to the  $x_i$ 's can be pictured in the following way. For example for  $k = 8$ , if the shaded rows represent the quantities

$$x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 3, x_5 = 4, x_6 = 7, x_7 = 7, x_8 = 8,$$

then the quantities

$$y_1 = 1, y_2 = 1, y_3 = 2, y_4 = 4, y_5 = 5, y_6 = 5, y_7 = 5, y_8 = 7$$

appear in the unshaded parts of the corresponding columns.



[*Editor's note.* Thus in general, given a particular choice of the  $x_i$ 's, one draws a  $k \times k$  chessboard, and shades in the first  $x_i$  squares of the  $i$ th row for each  $i$ ; then the unshaded portions of the columns give the  $y_i$ 's. Since  $d|k$  (in fact it seems we need only assume  $d|k^2$ ), if the  $x_i$ 's add to a multiple of  $d$  the  $y_i$ 's must too. Moreover  $x_k = k$  if and only if  $y_k \neq k$ , so we get a one-to-one correspondence between those  $k$ -tuples  $(x_1, \dots, x_k)$  of  $X_k$  which satisfy  $x_k = k$  and those which don't. Hence each of these sets has exactly half the elements of  $X_k$ , and the first of these sets is just  $Y_k$ . Readers may well prefer this descriptive argument (which was given by the proposer as well as by Barile) to a more formal one!]

*Also solved by the proposer. One other reader misread the question.*

\* \* \* \* \*

## YEAR-END WRAPUP

Here it is December again, so time once again to record a few comments that readers have sent in during the last year on problems and other items in *CruX*.

**613** [1982: 67, 138]. Ji Chen, Ningbo University, China, points out that this inequality was established by A. Bager in [1], and is part of the graph on page 152 of [2]. Also, by using the triangle inequality

$$\frac{4}{9} \left( \sum \sin \frac{A}{2} \right)^2 \leq \frac{2\sqrt{3}}{9} \sum \cos \frac{A}{2}$$

which appears on [1991: 275], together with some more results of Bager ([1]; see again p. 152 of [2]), he obtains *CruX* **1083** [1987: 96].

[1] A. Bager, Another family of goniometric inequalities, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No. 412–460 (1973), 207–216.

[2] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*.

**972** [1985: 326]. Stanley Rabinowitz, Westford, Massachusetts, notes that the number of regular unit tetrahedra that has been packed without overlap into a unit cube was raised from two to three by George Evagelopoulos's solution of problem 8 of the 15th All Union Mathematical Olympiad, Tenth Grade [1992: 235]. In fact Rabinowitz built a model of the three-tetrahedra configuration to confirm it was really possible. He has no proof that four tetrahedra are impossible to pack in the unit cube.

**1370** [1989: 281; 1990: 317]. In a paper "On consecutive numbers of the same height in the Collatz problem", *Discrete Math.* **112** (1993) pp. 261–267, Guo-Gang Gao finds 35654 consecutive integers, starting at  $2^{500} + 1$  in fact, all with the same  $L$ -value (i.e., number of steps to reach 1 by the Collatz function).

**1565** [1991: 255]. Neven Jurić, Zagreb, Croatia, counts the number of *equilateral* triangles whose vertices are vertices of the  $n$ -dimensional cube, and gets

$$\frac{2^n}{6} \sum_{k=1}^{\lfloor n/3 \rfloor} \binom{n}{2k} \binom{2k}{k} \binom{n-2k}{k}.$$

**1627** [1992: 95, 317]. Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain, informs us that this problem has appeared in the French journal *Bulletin de l'APMEP*, solution on pages 76–81 of Number 387, with due credit given to *CruX* of course, and also with a couple of interesting generalizations.

**1663** [1992: 188]. Ji Chen points out that this problem is equivalent to the *Monthly* problem 3146 (solution on pages 767–769 in Volume 95, 1988), also proposed by Walther Janous. The equivalence is clear from Chen's proof [1992: 188].

**1699** [1992: 308]. Regarding the possible inequality (3) on [1992: 309], wondered about by the proposers Yang and Chen, Richard Guy, University of Calgary, calculates

that the triangle with sides 17, 17, 30 has  $r_a/h_a = r_b/h_b = 17/30$ ,  $r_c/h_c = 15/2$ , and  $R/2r = 289/120$ , so

$$\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} = 4.244158\dots = 3 \left(\frac{289}{120}\right)^\alpha \quad \text{with } \alpha \approx 0.3947177,$$

so no larger value of  $\alpha$  (in particular  $\alpha = 5/12$ ) will be valid. This may be close to the optimal value of  $\alpha$ , which likely does not have a “nice” value.

**1721** [1993: 55]. In answer to a request from the proposer Murray Klamkin, Francisco Bellot sends in *three* references for equation (1) on [1993: 56] (i.e., that the sums of the reciprocals of the squares of the altitudes and the bialtitudes of a tetrahedron are equal), the earliest being in an Editor’s note to the solution of Question 3683, pp. 458–463 of the Belgian journal *Mathesis*, Vol. LXV, 1956. By coincidence, a very recent article, “Geometric proofs of some recent results of Yang Lu”, by G. Cairns, M. McIntyre and J. Strantzen, in *Mathematics Magazine* **66** (1993) 263–265, also gives this result, but the authors were unable to find a reference for it earlier than 1991. Could *Mathesis* be the original source?

**1738** [1993: 91]. Francisco Bellot located this problem as part (2) of problem E1239 of the *American Math. Monthly*, solution on p. 368 of the May 1957 issue.

**1739** [1993: 93]. Tim Cross, Wolverley High School, Kidderminster, U.K., spotted an editor’s misprint in the last displayed equation of approach 1) on [1993: 93]:  $(1/2)^3$  should have been  $(1/3)^3$ .

**1740** [1993: 94]. Francisco Bellot found this problem, with solution but no information about its origin, as problem 22, pp. 26–27 of J.N. Kapur, *Mathematical Olympiad Problems, Book I*, Mathematical Sciences Trust Society, New Delhi, 1991. Miguel Amengual Covas, Cala Figuera, Mallorca, Spain, found it as problem 57, page 31 of V. Gusev, V. Litvinenko and A. Mordkovich, *Solving Problems in Geometry*, Mir Publishers, Moscow, 1988. Again no source, but a hint of the solution appears on page 203.

Regarding the article “Prime Pyramids” by Richard Guy in the April issue [1993: 97–99], Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, informs us that almost the same problem was proposed by him as Advanced Problem 6189 in the *Monthly* **85** (1978), p. 54. And according to Richard Guy, no solution of this problem ever appeared in the *Monthly*. Also, Charles Ashbacher notes that the problem more recently turned up as Problem 1664 in the *Journal of Recreational Mathematics*, with a solution published on pages 236–237 of Volume 21 (1989); however, the solution is just an algorithm which seems to work, and no doubt does, but there is no proof.

Going back to a much earlier article, Bob Prielipp, University of Wisconsin–Oshkosh, writes that he tried to use the algorithm for finding Easter dates contained in Viktors Linis’s “Gauss and Easter Dates” in [1977: 102–103], but that it didn’t seem to yield correct results; for example, he calculates using the article that in 1982 Easter was supposed to have fallen on April 9, while it actually occurred on April 11. Can any reader locate any misprints in this article which would account for such discrepancies?

Late solutions were received from Dag Jonsson, Uppsala, Sweden (1692); Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China (1695); Stanley Rabinowitz, Westford, Massachusetts (1716); Seung-Jin Bang, Albany, California (1733 and 1739); J.A. McCallum, Medicine Hat, Alberta (1759 and 1805); Walther Janous, Ursulinengymnasium, Innsbruck, Austria (1765, 1772 and 1778); Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia (1803); and Neven Jurić, Zagreb, Croatia (1804 and 1805).

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Of course the Editorial Board itself, whose members are not listed above, continues to make *Cruz* better and better. All the Editorial Board members have helped greatly in refereeing the pages of problems that the editor sends them. And that's only for starters! Thanks to Denis Hanson's efforts as Articles Editor, articles are practically a regular feature of *Cruz* issues now, and Book Reviews, under Andy Liu's dedicated guidance, are not far off that pace. Several solution writeups have been done by Chris Fisher, Richard Nowakowski and Edward Wang. Chris Fisher is still *Cruz*'s number one geometry expert, and as such has been invaluable in commenting on numerous solutions and proposals from *Cruz*'s high-powered geometric elite! Richard Guy's dependable memory (and library) has benefitted the editor constantly, in many different areas of mathematics.

Special thanks as always to *JOANNE LONGWORTH* for her lightning fingers and growing knowledge of  $\text{\LaTeX}$  (including better ways to draw pictures, which we hope soon to make use of).

A reminder to all readers that *Cruz* seems always to be in need of nice, interesting, elementary problems, especially in non-geometric areas. Keep us in mind! And finally, from *Cruz* to all its fans:

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 \*\*\*\*\*  
*HAPPY NEW YEAR!*  
 \*\*\*\*\*  
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\* \* \* \* \*

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