Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 19 #8

October
octobre
1993

CONTENTS / TABLE DES MATIÈRES

Canadian Mathematical Society

Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

SUBSCRIPTION INFORMATION

Crux Mathematicorum is published monthly (except July and August). The subscription rates for ten issues are detailed on the inside back cover. Information on Crux Mathematicorum back issues is also provided on the inside back cover.

ACKNOWLEDGEMENTS

The support of the Department of Mathematics & Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

RENSEIGNEMENTS SUR L'ABONNEMENT

Crux Mathematicorum est publié mensuellement (sauf juillet et août). Les tarifs d'abonnement pour dix numéros figurent sur la couverture arrière. On peut également y retrouver de plus amples renseignements sur les volumes antérieurs de Crux Mathematicorum

REMERCIEMENTS

Nous rendons hommage à l'appui du département de mathématique et statistique de l'Université de Calgary et du département de mathématiques de l'Université d'Ottawa.

Printed at / imprimé à: Ottawa Laser Copy

ISSN 0705-0348

Publication Mail Registration No. 5432

A TRIANGULAR TRIANGLE PROBLEM

K.R.S. Sastry

A triangle ABC right angled at C is called a *Pythagorean* triangle if its legs a, b and the hypotenuse c are all natural numbers. Then the famous Pythagorean relation $a^2 + b^2 = c^2$ may be looked upon as an equality relation among the square numbers a^2, b^2, c^2, i e. the a-th square number + the b-th square number = the c-th square number. For this reason we say more accurately that the natural number triple (a, b, c) such that $a^2 + b^2 = c^2$ forms a Pythagorean triangle of *square* numbers.

There is a large variety of interesting problems involving Pythagorean triangles of square numbers. One of them is:

> Suppose the prime factorization of a natural number N is $N = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where the p_i are distinct odd primes and the α_i are non-negative integers. If L is the number of Pythagorean triangles of square numbers in which N can be a leg, what is the value of L?

A.H. Beiler [1, pp. 116] gives the answer as

$$L = \frac{1}{2}[(2\alpha_0 - 1)(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_k + 1) - 1]. \tag{1}$$

The r-th *triangular* number is $\frac{1}{2}r(r+1)$. Analogous to the well known Pythagorean triangle of square numbers we now define a *Pythagorean triangle of triangular numbers* (PTTN) — also called a *triangular triangle* by R.K. Guy.

A natural number triple (a, b, c) such that $a \le b < c$ is said to form a PTTN or a triangular triangle if the a-th triangular number + the b-th triangular number = the c-th triangular number. That is,

$$\frac{1}{2}a(a+1) + \frac{1}{2}b(b+1) = \frac{1}{2}c(c+1),$$

or

$$a^2 + a + b^2 + b = c^2 + c. \tag{2}$$

See [3] for details. For example, the triple $(3, 5, 6)$ forms a triangular triangle because when we add the 3rd triangular number, $\frac{1}{2} \times 3 \times 4 = 6$, to the 5th triangular number, $\frac{1}{2} \times 5 \times 6 = 15$, we get the 6th triangular number, $\frac{1}{2} \times 6 \times 7 = 21$. The aim of this article is to solve the following problem analogous to (1).

PROBLEM: *Suppose a natural number N is given. Determine the number $T(N)$ of non-degenerate triangular triangles having N as a leg.*

Solution: Suppose (a, b, c) is a triangular triangle having N as a leg. Without loss of generality let $a = N$. Then from (2) we have $N^2 + N + b^2 + b = c^2 + c$, that is,

$$(c - b)(c + b + 1) = N(N + 1).$$

Now if $c - b$ is odd then exactly one of b and c is odd and the other even. This makes $c + b$ also odd and hence $c + b + 1$ will be even. Likewise if $c - b$ is even then $c + b + 1$ will be odd. Therefore to determine all possible values of b and c we refactor $N(N + 1)$ into exactly two factors of opposite parity in all possible ways. We then equate the larger one to $c + b + 1$ and the smaller one to $c - b$. So we need to consider the prime factorization *not of N but of $N(N + 1)$*.

Suppose then that $N(N + 1) = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. As explained in the preceding paragraph we need to re-factor $N(N + 1)$ into exactly two factors (of which one is odd) in all possible ways. Since the p_i are distinct odd primes, the entire contribution of the odd factors comes from the factors of $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ only. It is well known that this number of factors is $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \cdots (\alpha_k+1)$. See [1, p. 7]. However, the particular triangular triangle obtained by equating $c + b + 1 = N + 1$, $c - b = N$, that is $a = N, b = 0, c = N$, degenerates. Hence

$$T(N) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) - 1. \qquad \square$$

We provide more focus by considering a numerical example.

EXAMPLE 1: How many triangular triangles have 14 as a leg? Determine all these triangles.

Solution: Here $N = 14$ and $N(N + 1) = 210 = 2^1 \times 3^1 \times 5^1 \times 7^1$. Hence

$$T(N) = (1 + 1)(1 + 1)(1 + 1) - 1 = \mathbf{7}.$$

To determine all these seven non-degenerate triangular triangles we factorize 210 into exactly two factors of which one is odd. One such factorization is 210×1. Then we solve the equations $c + b + 1 = 210$ and $c - b = 1$ to obtain $b = 104$, $c = 105$. Of course $a = 14$. This yields the triangular triangle triple $(14, 104, 105)$. Continuing this way with other factorizations of 210, viz. 105×2, 70×3, 42×5, 35×6, 30×7, 21×10 we obtain the other triples $(14, 51, 53)$; $(14, 33, 36)$; $(14, 18, 23)$; $(14, 14, 20)$; $(14, 11, 18)$; $(14, 5, 15)$. Observe that the factorization $210 = 15 \times 14$ leads to the degenerate triangle. $\square$

On the other hand, 14 may not be the *only* natural number that can be a leg of precisely seven triangular triangles. In fact the converse problem of finding N from a given $I(N)$ is more interesting as well as more challenging due to the additional constraint on N that $N(N + 1)$ be of special form. We see this in

EXAMPLE 2: Find the least natural number N that can be a leg of precisely seven non-degenerate triangular triangles.

Solution: Here $T(N) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) - 1 = 7$. Since

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) = 8 = \mathbf{2} \times \mathbf{2} \times \mathbf{2} \quad \text{or} \quad \mathbf{4} \times \mathbf{2} \quad \text{or} \quad \mathbf{8} \times \mathbf{1},$$

we have three cases to consider.

Case (i). $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) = 2\times2\times2 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 1$ and $N(N+1) = 2^{\alpha_0} p_1 p_2 p_3$. This equation can have several solutions. For example, $\alpha_0 = 1, p_1 = 3$, $p_2 = 5, p_3 = 7$ gives $N = 14$; $\alpha_0 = 2, p_1 = 3, p_2 = 5, p_3 = 7$ gives $N = 20$; $\alpha_0 = 1$, $p_1 = 5, p_2 = 7, p_3 = 31$ gives

$N = 70$; and so on. Obviously to seek the minimum N we have to choose $\alpha_0 = 1, p_1 = 3, p_2 = 5, p_3 = 7$.

Case (ii). $(\alpha_1 + 1)(\alpha_2 + 1) = 4 \times 2 \Rightarrow \alpha_1 = 3, \alpha_2 = 1$ and $N(N+1) = 2^{\alpha_0} p_1^3 p_2$. If $\alpha_0 = 1, p_1 = 3$ then $p_2 = 13$ for the smallest N in this class which implies $N = 26 > 14$.

Case (iii). $(\alpha_1 + 1)(\alpha_2 + 1) = 8 \times 1 \Rightarrow \alpha_1 = 7, \alpha_2 = 0, N(N+1) = 2^{\alpha_0} p_1^7$. Since $\alpha_0 \geq 1$ and $p_1 \geq 3$ it again follows that $N > 14$.

We have completed the analysis of all the three cases and so we conclude that $N = \mathbf{14}$ is the required minimum value of N. □

Further questions. The r-th pentagonal number is $\frac{1}{2}r(3r-1)$. See [2] and also *Crux* 1756 [1993: 172–174]. A natural number triple (a, b, c) is said to form a *pentagonal* Pythagorean triangle with a, b as legs and c as hypotenuse if the a-th pentagonal number + the b-th pentagonal number = the c-th pentagonal number; that is, if $3a^2 - a + 3b^2 - b = 3c^2 - c$. Note that $(4, 7, 8)$ is a pentagonal Pythagorean triangle. Show that $N = 4$ can be a leg of just one non-degenerate pentagonal Pythagorean triangle. Can $N = 6$ be a leg of such a triangle? What form must a natural number N have in order to be a leg of at least one non-degenerate pentagonal Pythagorean triangle? Suppose a natural number N is given. Let $T(N)$ be the number of non-degenerate pentagonal Pythagorean triangles in which N is a leg. What is the value of $T(N)$?

More generally, the r-th n-gonal number of side $n \geq 3$ (or the r-th polygonal number of side $n \geq 3$) is

$$P(n, r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}\ .$$

A natural number triple (a, b, c) is said to form an n-gonal Pythagorean triangle if $P(n, a) + P(n, b) = P(n, c)$ [3]. The search to find answers to the kinds of questions we asked about the pentagonal Pythagorean triangles can be extended to the case of n-gonal Pythagorean triangles, $n > 5$.

Acknowledgement: The author thanks the referee for constructive suggestions to improve the presentation.

References:

[1] A.H. Beiler, *Recreations in the Theory of Numbers*, Dover (1964), pp. 7, 116, 185–199.

[2] K.R.S. Sastry, Cubes of natural numbers in arithmetic progression, *Crux Mathematicorum* **18** (1992), pp. 161–164.

[3] K.R.S. Sastry, Pythagorean triangles of the polygonal numbers, *Mathematics and Computer Education Journal* **27** (Spring 1993) No. 2, pp. 135–142.

K.R.S. Sastry
Box 21862
Addis Ababa, Ethiopia

*   *   *   *   *

THE OLYMPIAD CORNER

No. 148

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Last issue we gave the "official solutions" to the *Canadian Mathematical Olympiad*, but I had not received a copy of the official results, which I now have. The contest was written on Wednesday, April 7, 1993 with a total of 213 competitors. To give some idea of the difficulty, over sixty percent of those writing scored 10 or less out of a possible 50 points. Congratulations to all the competitors.

First Prize:	Naoki Sato
Second Prize:	Edward Leung
Third Prize:	Alex Lee
Fourth Prize:	Howard Jonathan Feldman
	Ka-Ping Yee

* * *

Next we begin the list of problems proposed to the jury but not selected for the 33rd International Mathematical Olympiad held in Moscow, Russia in July, 1992. Many thanks to Georg Gunther, the Canadian Team leader, who collected the problems and sent them on to us.

1. *Proposed by Australia.*

Let m be a positive integer and x_0, y_0 be integers such that

(i) x_0 and y_0 are relatively prime; (ii) y_0 divides x_0^2+m; and (iii) x_0 divides y_0^2+m. Prove that there exist positive integers x and y such that

(I) x and y are relatively prime; (II) y divides x^2+m; (III) x divides y^2+m; and (IV) $x+y \leq m+1$.

Alternative Problem. Prove that for any positive integer m there exist an infinite number of pairs of integers (x, y) such that

(1) x and y are relatively prime; (2) y divides x^2+m; and (3) x divides y^2+m.

2. *Proposed by China.*

Let $\mathbb{R}^+$ be the set of all non-negative real numbers. Two positive real numbers a and b are given. Suppose that a mapping $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the functional equation

$$f(f(x)) + af(x) = b(a+b)x.$$

Prove that there exists a unique solution of this equation.

3. *Proposed by China.*

Outside a convex quadrilateral $ABCD$ with perpendicular diagonals, four squares, $AEFB$, $BGHC$, $CIJD$, $DKLA$, are constructed (vertices are given in a counterclockwise order). Prove that the quadrilaterals Q_1 and Q_2 formed by the lines AG, BI, CK, DE and AJ, BL, CF, DH, respectively, are congruent.

4. *Proposed by Colombia.*

Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, O_4 be the centroids of the triangles constructed on AB, BC, CD, DA respectively. Show that O_1O_3 is perpendicular to O_2O_4.

5. *Proposed by India.*

Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(a) its sides are $1, 2, 3, \ldots, 1992$ in some order;

(b) the polygon circumscribes a circle.

6. *Proposed by Iran.*

Let $f(x)$ be a polynomial with rational coefficients and α be a real number such that $\alpha^3 - \alpha = (f(\alpha))^3 - f(\alpha) = 33^{1992}$. Prove that for each $n \geq 1$

$$(f^{(n)}(\alpha))^3 - f^{(n)}(\alpha) = 33^{1992},$$

where $f^{(n)}(x) = f(f(\ldots f(x)))$, and n is a positive integer.

7. *Proposed by North Korea.*

Does there exist a set M with the following properties:

(1) the set M consists of 1992 natural numbers;

(2) every element in M and the sum of any number of elements have the form m^k $(m, k \in \mathbb{N}, k \geq 2)$?

*   *   *

Next, a generalization of problem 3 of the Canadian Mathematical Olympiad for which the official solution appeared last issue.

3. [1993: 195] *Canadian Mathematical Olympiad.*

In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq 2/3$.

Generalization by D.J. Smeenk, Zaltbommel, The Netherlands.

Let G be the centroid of ΔABC, and set $\angle BGC = \varphi$. Show that

$$\cot\beta + \cot\gamma \geq \frac{2}{3}\tan\frac{1}{2}\varphi.$$

[*Editor's Note.* The first published solution can be generalized to give a solution.]

* * *

To complete this month's Corner we turn to solutions sent in by the readers to problems proposed to the jury, but not used, at the 32nd I.M.O. in Sigtuna, Sweden [1992: 195–196].

1. *Proposed by the Philippines.*

Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the two sides AC and BC. Draw AP and BP and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. Prove that the lines Q_1P_2, Q_2P_1 and AB are concurrent.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by George Evagelopoulos, Athens, Greece.

The information given concerning right angles imply that P_1, P_2, Q_1, Q_2, P, C lie on a circle with PC as diameter. Consider the hexagon $PQ_2P_1CP_2Q_1$ inscribed in this circle. The intersection of PQ_2 and CP_2 is B; the intersection of Q_2P_1 and P_2Q_1 is R; and the intersection of P_1C and Q_1P is A. So, by Pascal's theorem A, R, B are collinear.

Comment: The question should add ($Q_2 \neq P_1$, $Q_1 \neq P_2$).

2. *Proposed by North Korea.*

Let S be any point on the circumcircle of ΔPQR. Then the feet of the perpendiculars from S to the three sides of the triangle lie on the same straight line. Denote this line by $\ell(S, PQR)$. Suppose that the hexagon $ABCDEF$ is inscribed in a circle. Show that the four lines $\ell(A, BDF)$, $\ell(B, ACE)$, $\ell(D, ABF)$ and $\ell(E, ABC)$ intersect at one point if and only if $CDEF$ is a rectangle.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by George Evagelopoulos, Athens, Greece. We give Bradley's solution.

Take the centre of the circumcircle to be O and write $\overrightarrow{OA} = \mathbf{a}$, etc.

Theorem 1. If X, Y, Z lie on the circumcircle then the vector position of the orthocentre H of ΔXYZ is $\mathbf{x}+\mathbf{y}+\mathbf{z}$.

[This is easy since $(\mathbf{x}+\mathbf{y}+\mathbf{z}-\mathbf{x})\cdot(\mathbf{y}-\mathbf{z}) = R^2 - R^2 = 0$ shows $\overrightarrow{XH} \perp \overrightarrow{YZ}$, etc.]

Theorem 2. If P lies on the circle through XYZ then $\ell(P, XYZ)$ bisects the line from P to the orthocentre of ΔXYZ. [This is Thm 327, p. 207, of *Advanced Euclidean Geometry* by Roger A. Johnson, Dover Publications.]

Hence $\ell(P, XYZ)$ contains the point with vector position $(1/2)[\mathbf{p}+(\mathbf{x}+\mathbf{y}+\mathbf{z})] = (1/2)[\mathbf{p}+\mathbf{x}+\mathbf{y}+\mathbf{z}]$.

Theorem 3. $\ell(A, BDF)$ and $\ell(D, ABF)$ are concurrent at $(1/2)(\mathbf{a}+\mathbf{b}+\mathbf{d}+\mathbf{f})$, and $\ell(B, ACE)$ and $\ell(E, ABC)$ are concurrent at $(1/2)(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{e})$.

Main Theorem.

All of $\ell(A, BDF)$, $\ell(D, ABF)$, $\ell(B, ACE)$ and $\ell(E, ABC)$ are concurrent

iff $\mathbf{d}+\mathbf{f} = \mathbf{c}+\mathbf{e}$

iff $CDEF$ is a parallelogram

iff $CDEF$ is a rectangle, since the only parallelograms inscribed in a circle are rectangles.

3. *Proposed by China.*

Let O be the centre of the circumsphere of a tetrahedron $ABCD$. Let L, M, N be the midpoints of BC, CA, AB respectively, and assume that $AB+BC=AD+CD$, $BC+CA=BD+AD$ and $CA+AB=CD+BD$. Prove that $\angle LOM=\angle MON=\angle NOL$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and George Evagelopoulos, Athens, Greece. We give the latter solution.

Solving the system of equations gives that $BC=AD$, $AB=CD$ and $CA=BD$. Let L_1, M_1, N_1 be the midpoints of AD, BD, and CD respectively.

The above equalities give

$$L_1M_1=\frac{1}{2}AB=LM, \qquad L_1M_1\|AB\|LM$$

and

$$L_1M=\frac{1}{2}CD=LM_1, \qquad L_1M\|CD\|LM_1.$$

So L_1, M_1, L, M are coplanar and L_1M_1LM is a rhombus, and so are M_1N_1MN, L_1N_1LN.

Let Q be the midpoint of L_1L. Then Q is also the midpoint of M_1M and N_1N and $QN\perp QM$, $QN\perp QL$, so $NN_1\perp CD$, $NN_1\perp AB$ and Q is just the centre O. Thus $\angle LOM=\angle MON=\angle NOL=90°$.

4. *Proposed by The Netherlands.*

S is a set of n points in the plane. No three points of S are collinear. Prove that there exists a set P containing $2n-5$ points satisfying the condition: the interior of each triangle with three vertices from S contains an element of P.

Solution by George Evagelopoulos, Athens, Greece.

Let $P_1(x_1,y_1),P_2(x_2,y_2),\ldots,P_n(x_n,y_n)$ be the n points of S in the plane. We may assume $x_1<x_2<\cdots<x_n$ (choosing adequate axes and possibly renumbering the points).

Define d to be $1/2$ the minimum distance of P_i to the line through P_j and P_k, where (i,j,k) runs through all triples of distinct numbers from 1 to n.

Let T be the following set of $2n-4$ points:

$$\{(x_2,y_2-d),(x_2,y_2+d),(x_3,y_3-d),(x_3,y_3+d),\ldots,(x_{n-1},y_{n-1}-d),(x_{n-1},y_{n-1}+d)\}.$$

Consider for a fixed l, $1<l<n$, all the triangles $P_kP_lP_m$ with $1\le k<l<m\le n$. The interior of each of these triangles must contain one of (x_l,y_l-d), (x_l,y_l+d). Thus T is a set of $2n-4$ points with the required property.

At least one of the points of T can be left out. The convex hull of S is a polygon, so it has at least three points of S as vertices. Let P_j be a vector of that hull with $1<j<n$. It is clear that one of the points (x_j,y_j-d), (x_j,y_j+d) lies outside the convex hull. That point can be left out. Therefore we have a set with the required property containing $2n-5$ points.

5. *Proposed by France.*

In the plane we are given a set E of 1991 points; certain pairs of these points are joined by a path. We suppose that for every point of E, there exists at least 1593 other points of E to which it is joined by a path. Show that there exist six points of E every pair of which are joined by a path.

Solutions by George Evagelopoulos, Athens, Greece; and by John Morvay, Springfield, Missouri.

Choose two points in E, P_1 and P_2 such that P_1 and P_2 are connected. Each of P_1 and P_2 are connected to at least 1592 points of $E\backslash\{P_1, P_2\}$. Since $1592+1592-1989 = 1195$, we conclude that there is a subset $A \subset E\backslash\{P_1, P_2\}$ with $|A| = 1195$ such that each element of A is connected to both P_1 and P_2. Now with $A' = E\backslash A$ we have $|A'| = 796$. Each point of A is connected to at most 796 elements of A' and hence to at least 797 elements of A. Thus there are two elements, P_3, P_4, say, of A which are connected. Notice that $\{P_1, P_2, P_3, P_4\}$ are pairwise connected. Now each of P_3 and P_4 is connected to at least 796 members of $A\backslash\{P_3, P_4\}$. Again, since $796 + 796 - 1193 = 399$ there is a subset $B \subset A\backslash\{P_3, P_4\}$ with $|B| = 399$ such that each point of B is connected to both P_3 and P_4. Let $B' = E\backslash B$. Then $|B'| = 1592$. Choose P_5 in B. It is connected to at most 1592 elements of B' and thus there is P_6 in B to which P_5 is connected. Thus $\{P_1, P_2, \ldots, P_6\}$ are all pairwise connected.

6. *Proposed by Australia.*
Prove that

$$\frac{1}{1991}\binom{1991}{0} - \frac{1}{1990}\binom{1990}{1} + \frac{1}{1989}\binom{1989}{2} - \cdots + \frac{(-1)^m}{1991-m}\binom{1991-m}{m} + \\ + \cdots - \frac{1}{996}\binom{996}{995} = \frac{1}{1991}\,.$$

Solutions by Seung-Jin Bang, Albany, California; by Curtis Cooper, Central Missouri State University, Warrensburg; by George Evagelopoulos, Athens, Greece; and by William Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We given Chen and Wang's solution.

We establish the following general result:

Theorem. For all natural numbers n,

$$\sum_{k=0}^{[n/2]}(-1)^k\frac{n}{n-k}\binom{n-k}{k} = \begin{cases} 1 & \text{if } n \equiv 1,5 \mod 6 \\ -1 & \text{if } n \equiv 2,4 \mod 6 \\ -2 & \text{if } n \equiv 3 \mod 6 \\ 2 & \text{if } n \equiv 0 \mod 6. \end{cases}$$

The given identity follows from the special case when $n = 1991$. Since $1991 \equiv 5 \bmod 6$ we get

$$\sum_{k=0}^{995}(-1)^k\frac{1991}{1991-k}\binom{1991-k}{k} = 1.$$

Proof. Let $f(n) = \sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}$.

Then for $n \geq 3$

$$f(n) - f(n-1) = \sum_{k\geq 0}(-1)^k \left(\binom{n-k}{k} - \binom{n-1-k}{k}\right) = \sum_{k\geq 1}(-1)^k \binom{n-1-k}{k-1}$$
$$= \sum_{k\geq 0}(-1)^{k+1}\binom{n-2-k}{k} = -f(n-2).$$

Thus $f(n) = f(n-1) - f(n-2)$.

Using the initial values $f(1) = 1$, and $f(2) = 0$ we find easily by induction that

$$f(n) = \begin{cases} 1 & \text{if } n \equiv 0,1 \mod 6 \\ 0 & \text{if } n \equiv 2,5 \mod 6 \\ -1 & \text{if } n \equiv 3,4 \mod 6 \end{cases} \qquad (*)$$

Let $S(n) = \sum_{k=0}^{[n/2]}(-1)^k \frac{n}{n-k}\binom{n-k}{k}$. Then clearly $S(1) = 1$, $S(2) = -1$, and for all $n \geq 3$, we have

$$S(n) = \sum_{k\geq 0}(-1)^k \frac{n-k+k}{n-k}\binom{n-k}{k} = f(n) + \sum_{k\geq 1}(-1)^k \frac{k}{n-k}\binom{n-k}{k}$$
$$= f(n) + \sum_{k\geq 1}(-1)^k \binom{n-1-k}{k-1} = f(n) + \sum_{k\geq 0}(-1)^{k+1}\binom{n-2-k}{k}$$
$$= f(n) - f(n-2).$$

Our claim now follows easily from this relation and $(*)$.

7. *Proposed by Poland.*

Let n be any integer, $n \geq 2$. Assume that the integers $a_1, a_2, \ldots, a_n$ are not divisible by n and, moreover, that n does not divide $a_1 + a_2 + \cdots + a_n$. Prove that there exist at least n different sequences $(e_1, e_2, \ldots, e_n)$ consisting of zeros or ones such that $e_1a_1 + e_2a_2 + \cdots + e_na_n$ is divisible by n.

Solution by George Evagelopoulos, Athens, Greece.

Lemma. Let $A = (a_{ij})$ be a $k \times n$ matrix such that (i) a_{ij} is 0 or 1, (ii) $k \leq n-2$, (iii) there is at least one 0 in every row, (iv) there are at least two ones in each row.

Then it is possible to permute the columns of A in such a way that the 1's do not form a single block in any row (i.e. they are separated by at least one zero).

Proof. By induction on k. The case $k = 1$, ($n \geq 3$ arbitrary) is trivial. Suppose that $k \geq 2$ and for $k-1$ and every $n \geq (k-1)+2 = k+1$ the lemma is true. Let $n \geq k+2 \geq 4$. Consider a matrix A which satisfies the conditions (i)–(iv). An element a_{ij} is called *special* if either it is the only zero in the ith row or in this row there are exactly two 1's and a_{ij} is one of them. Since $n \geq 4$, in each row there are at most two special elements. The total number of special elements in A does not exceed $2k$ and since $k < n$, there exists a column with at most one special element. We may assume that the first column has this property and that elements a_{i1} $(i \geq 2)$ are not special. Let $B = (a_{ij})$ $2 \leq i \leq k$, $i \leq j \leq n$. It is obvious that B satisfies (i)–(iv). Therefore, by the induction assumption, we can permute the columns of B and get the required form. The permutation may be considered as a permutation of the columns of A. If $a_{11} = 1$ and the 1's in the first row of A form a block, then it is sufficient to put the first column in the last place. If $a_{11} = 0$, we put the first column between any two columns having a 1 in the first row. □

Now to finish the problem let $(e_1^i, e_2^i, \ldots, e_n^i)$ $(1 \leq i \leq k)$ be a nonempty system such that

$$\sum e_j^i a_n \equiv 0 \bmod n \qquad \text{for } 1 \leq i \leq n,$$

with $e_j^i \in \{0, 1\}$. If $k \leq n-2$, then the conditions of the problem ensure that the matrix $A = (e_j^i)$ satisfies (i)–(iv). Let σ be a permutation of its columns as guaranteed by the lemma. Consider the $n+1$ numbers

$$0, a_{\sigma(1)}, a_{\sigma(1)} + a_{\sigma(2)}, \ldots, a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(n)}.$$

There are only n possible remainders mod n, so the difference of some pair, which is of the form $a_{\sigma(i)} + a_{\sigma(i+1)} + \cdots + a_{\sigma(j)}$ is divisible by n. By defining $e_\mu^{k+1} = \begin{cases} 1 & i \leq \mu \leq j \\ 0 & \text{otherwise} \end{cases}$ we properly extend the system (because in permutation σ the 1's form a block in the $(k+1)$st row, but in no other).

Thus there are $n-1$ distinct nontrivial solutions. Of course setting $e_i = 0$, $1 \leq i \leq n$ gives the remaining solution to finish the problem.

8. *Proposed by the U.S.S.R.*

Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

Solution by Seung-Jin Bang, Albany, California.

From the definition of a_n, we have $a_{10m+1} \equiv (10m+1)a_{10m} \equiv a_{10m} \pmod{10}$ and $a_{10m+2} \equiv (10m+2)a_{10m+1} \equiv 2a_{10m} \bmod 10$. Similarly $a_{10m+3} \equiv 6a_{10m}$, $a_{10m+4} \equiv 4a_{10m}$, $a_{10m+5} \equiv 2a_{10m}$, $a_{10m+6} \equiv 2a_{10m}$, $a_{10m+7} \equiv 4a_{10m}$, $a_{10m+8} \equiv 2a_{10m}$, $a_{10m+9} \equiv 8a_{10m}$, $a_{10m+10} \equiv 8a_{10m} \bmod 10$. This shows that $a_{10m+1}, a_{10m+2}, \ldots, a_{10m+9}$ are completely determined by a_{10m}. Next $a_{10m+10} \equiv 8a_{10m} \bmod 10$ implies $a_{10m} \equiv 3^m \bmod 10$. Since $\{8^m \bmod 10\}$ is a pure modulo period sequence with period 5, we have $a_{10(m+5)} \equiv 8^{m+5} \equiv 8^m \equiv a_{10m} \pmod{10}$ and so $a_{10(m+5)} = a_{10m}$. It follows that the sequence $a_1, a_2, \ldots, a_n, \ldots$ is a periodic sequence, with period 50.

9. *Proposed by Ireland.*

Let a be a rational number with $0 < a < 1$ and suppose that

$$\cos(3\pi a) + 2\cos(2\pi a) = 0.$$

(Angle measurements are in radians.) Prove that $a = 2/3$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; and by Stephen D. Hnidei, Windsor, Ontario. We use the solution sent in by Evagelopoulos.

Set $x = \cos(\pi a)$. The given equation is equivalent to $4x^3 + 4x^2 - 3x - 2 = 0$. Factoring yields $(2x+1)(2x^2+x-2) = 0$.

The solution $2x+1 = 0$ yields $\cos(\pi a) = -1/2$, and since $0 < a < 1$, we get $a = 2/3$. It remains to show that if x satisfies $2x^2 + x - 2 = 0$ then a is not rational.

The solutions of $2x^2 + x - 2 = 0$ are $x = (-1 \pm \sqrt{17})/4$ and since $x = \cos \pi a$ we obtain

$$\cos \pi a = \frac{\sqrt{14} - 1}{4} .$$

Claim. For every integer $n \geq 0$, $\cos(2^n \pi a) = (a_n + b_n\sqrt{17})/4$ where a_n, b_n are odd integers.

Proof of Claim. By induction on n. With $n = 0$, there is nothing to prove. Assume $\cos(2^n\pi a) = (a_n + b_n\sqrt{17})/4$, with a_n, b_n odd integers. Then

$$\cos(2^{n+1}\pi a) = 2\cos^2(2^n\pi a) - 1 = \frac{2(a_n^2 + 17b_n^2 + 2a_nb_n\sqrt{17})}{16} - 1$$
$$= \frac{2[(a_n^2 + 17b_n^2) - 8] + 4a_nb_n\sqrt{17}}{16}.$$

Because a_n, b_n are both odd integers

$$a_n^2 + 17b_n^2 - 8 = 2 + 4t, \qquad \text{for some integer } t.$$

So $a_{n+1} = 1 + 2t$, $b_{n+1} = a_nb_n$ are both odd, proving the claim.

Note also that

$$a_{n+1} = \frac{1}{2}(a_n^2 + 17b_n^2 - 8) \geq \frac{1}{2}(a_n^2 + 9) > a_n$$

since a_n, b_n are integers. So the sequence $\{a_n\}$ is strictly increasing.

Since $\sqrt{17}$ is not rational, it follows that

$$\{\cos(2^n\pi a) : n = 0, 1, 2, \ldots\}$$

has infinitely many distinct elements. However, if a were rational, then $\{\cos(m\pi a) : m$ is an integer$\}$ would be a *finite* set. Hence a is irrational. So the only rational a satisfying the hypothesis is $a = 2/3$.

10. *Proposed by Hong Kong.*

Let $f(x)$ be a monic polynomial of degree 1991 with integer coefficients. Define $g(x) = f^2(x) - 9$. Show that the number of distinct integer solutions of $g(x)$ cannot exceed 1995.

Solutions by Seung-Jin Bang, Albany, California; and by George Evagelopoulos, Athens, Greece. We give Bang's answer.

Let $a_1, a_2, \ldots, a_m$ be all distinct integer solutions of $f(x-3)$. Then for some monic polynomial $h(x)$ with integer coefficients, $f(x) = (x-a_1)\ldots(x-a_m)h(x)+3$. Suppose now that $f(x)+3$ has an integer solution b. Then $(b-a_1)\ldots(b-a_m)h(b) = f(b)-3 = f(b)+3-6 = -6$. Since the $(b-a_i)$ and $h(b)$ are all integers, amongst $(b-a_1), \ldots, (b-a_m)$ at least $m-2$ take the value 1 or -1. It follows that $m \leq 4$, because $(b-a_1), \ldots, (b-a_m)$ are all distinct.

Since the number of roots of $f(x)+3$ cannot exceed 1991, we conclude that the number of solutions of $g(x) = f^2(x) - 9$ cannot outdo $1991 + 4 = 1995$.

11. *Proposed by India.*

Let f and g be two integer-valued functions defined on the set of all integers such that

(a) $f(m + f(f(n))) = -f(f(m+1)) - n$ for all integers m and n;

(b) g is a polynomial function with integer coefficients and $g(n) = g(f(n))$ for all integers n.

Determine $f(1991)$ and the most general form of g.

Solution by George Evagelopoulos, Athens, Greece.

Let $f^2(n)$ denote $f(f(n))$. From (2) replacing m by $f^2(m)$, we get

$$f(f^2(m) + f^2(n)) = -f^2(f^2(m)+1) - n.$$

Interchanging m and n we get

$$f(f^2(n) + f^2(m)) = -f^2(f^2(n)+1) - m.$$

From these two relations we get

$$f^2(f^2(m)+1) - f^2(f^2(n)+1) = m - n.$$

Again from (a)

$$f^2(f^2(m)+1) = f(f(f^2(m)+1)) = f(-m - f^2(2))$$

and similarly

$$f^2(f^2(n)+1) = f(-n - f^2(2)).$$

Putting $f^2(2) = k$, we get

$$f(-m-k) - f(-n-k) = m - n$$

for all integers m and n. Replacing m by $-m-k$ and n by $-k$ we obtain

$$f(m) - f(0) = -m - k + k = -m.$$

Therefore $f(m) = -m + f(0)$ for all $m \in Z$. So $f(f(m)) = -f(m) + f(0) = m - f(0) + f(0) = m$. Hence $f^2(m) = m$ for all integers m. Using this in (a), we get

$$f(m+n) = -m - 1 - n$$

for all integers m and n.

Setting $m = 0$ we finally get $f(n) = -n - 1$ for all integers n. Thus $f(1991) = -1992$.

From (b) we obtain $g(n) = g(-n-1)$, for all integers. Since g is a polynomial on the set of integers, extending the function to the reals gives a polynomial satisfying $g(x) = g(-x-1)$ for all real x. A polynomial in x can also be expressed as a polynomial in $x + \lambda$ for any real λ.

Taking, in particular, $\lambda = 1/2$ we get

$$g(x) = P\left(x + \frac{1}{2}\right) .$$

Replacing x by $-x-1$ we get $g(-x-1) = P(-x-1/2)$. Thus $P(x+1/2) = P(-x-1/2)$. Hence g is a polynomial in $(x+1/2)^2 = x^2 + x + 1/4$, and so a polynomial in $x^2 + x$.

The most general form of g is thus

$$g(n) = a_0 + a_1 n(n+1) + a_2 n^2(n+1)^2 + \cdots + a_P n^P (n+1)^P$$

where the a_i's are integers.

12. *Proposed by the U.S.A.*

Suppose that $n \geq 2$ and $x_1, x_2, \ldots, x_n$ are real numbers between 0 and 1 (inclusive). Prove that for some index i between 1 and $n-1$ the inequality

$$x_i(1 - x_{i+1}) \geq \frac{1}{4} x_1(1 - x_n)$$

holds.

Solution by Seung-Jin Bang, Albany, California.

It is clear that if $x_1 = 0$ or $x_n = 1$ then the fact holds. So let $x_1 \neq 0$ and $x_n \neq 1$. Suppose that $x_i(1 - x_{i+1}) < x_1(1 - x_n)/4$ for $i = 1, 2, \ldots, n-1$. From $x_1(1 - x_2) < x_1(1 - x_n)/4 < x_1/4$ we have $x_2 > 3/4 > 1/2$, and from $(1 - x_3)/2 < x_2(1 - x_3) < x_1(1 - x_n)/4 < 1/4$ we have $x_3 > 1/2$, etc. It follows that $x_n > 1/2$.

Let $b_1 = 1/2$ and let $x_n > b_m$.

From $x_1(1-x_2) < x_1(1-x_n)/4 < x_2(1-b_m)/4$ we have $x_2 > (3+b_n)/4 > (1+\sqrt{b_m})/2$ and from $(1 + \sqrt{b_m})(1 - x_3)/2 < x_2(1 - x_3) < (1 - b_m)/4$ we have $x_3 > (1 + \sqrt{b_m})/2$, etc. It follows that $x_n > (1 + \sqrt{b_m})/2$, for all $m = 1, 2, 3, \ldots$.

Since $\lim_{m\to\infty} b_m = 1$, we conclude that $x_m = 1$ which gives a contradiction. This completes the proof.

* * *

That's all the space we have this issue. Send me your problem sets as well as your nice solutions.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Index to Mathematical Problems 1980–1984. Edited by Stanley Rabinowitz. Foreword by Murray S. Klamkin. Published by MathPro Press, P.O. Box 713, Westford, MA 01886, USA. Hardcover, xii+532 pages, US$49.95 plus $5 shipping in the US and Canada, $10 shipping elsewhere. Phone orders (by credit card): (800)247-6553 US, +1 419-281-1802 elsewhere. *Reviewed by Marcin E. Kuczma, University of Warsaw, Poland.*

Prove that among any 13 *real numbers, there are two, x and y, such that*

$$|x-y| \leq (2-\sqrt{3})|1+xy|.$$

Nice inequality problem, isn't it? I came across it when browsing through a recent national mathematics contest of a certain country, feeling quite sure that I had seen it before. But when? And where? To dig through the heaps of paper collected on my desk and bookshelves seemed a hopeless task. Then the solution arrived, in the form of a huge mailing that contained a 8.5×11 in., 532-page book, now widely known under the pet-name "Big Red Book" (abbr. $\mathcal{BRB}$).

The cover is red. The book is bigger than big. It reproduces more than 5000 problems that were published during the years 1980–1984 in the problem columns of 28 journals and in 16 contests, international as well as national. The problems are classified by subject: Algebra, Analysis, Combinatorics, Geometry, Number Theory, Recreational Mathematics and Other Topics. Each chapter is divided into several dozen headings and sub-headings.

The problem I was searching for should come under "Inequalities". But where should one look for inequalities? Under Algebra, Analysis or elsewhere? It turns out to be everywhere. Each chapter has an *Inequalities* section. The problem in question looks like an algebraic inequality. So I tried chapter *Algebra* under the heading *Inequalities* and the subheading *fractions*. On page 51 I found what I was looking for: Canadian Mathematics Olympiad 1984, problem 5. Although it is not *exactly* the same (with 7 numbers rather than 13, and the constant $1/\sqrt{3}$ instead of $2-\sqrt{3}$), it is *essentially* the same.

I got curious about the other four problems in the Canadian MO '84. I simply consulted the Problem Locator, pp. 15–31. It lists Canada 1984/1 (p. 242), Canada 1984/2 (p. 109), etc. It is very simple to use.

The Problem Locator is the most important index. The $\mathcal{BRB}$ is equipped with several indexes all with cross-references to the Problem Locator. Naturally, there is an Author Index. You will find your own name there if you are a problemist, which may make you feel very proud. You may also search there if your uncle is a problemist, or a friend of yours is. It may happen that a certain problemist is of particular interest to you at the moment and you would like to look at her/his problem proposals. In the Author Index, the problems and solutions by that person are listed with cross-references to the Problem Locator. Suppose you wish to see the problems devised by K. S. Murray. In the

Author Index you will find MI 83-10, MI 84-1 and MI 84-12. In the Problem Locator, you will learn how to decipher the coding and locate the texts for those problems by page numbers. At the end of the Author Index, you will find a list of pseudonyms, from which you can learn the real names of, say, Lewis Carroll, or K. M. Seymour.

Some journals assign titles to the problems they publish. The alphabetic Title Index is the next section of the $\mathcal{BRB}$. A title consisting of several significant words is cited several times, in cyclic permutation. Suppose you vaguely remember having seen a problem entitled "*... something about...* Privacy". In the Title Index, you will find "Privacy • Invasion of", which is of course an equivalent form of "Invasion of Privacy".

How can you search for a problem that had *not* been given a title, recalling just some keywords? Consult the Keyword Index! It contains words like "divisor" or "permutation", but also "Cheshire cat", "Fifibonacci" or "Lulucas" (also "Phibonacci"), "sanctum", "Yellow Pigs", and many others.

If a problem has been further developed, generalized or substantially employed in an article, the suitable reference can be found in the Citation Index, which also includes Contest References and Biographical Notes. In addition, there is a Bibliography containing nearly 500 items.

Other useful indexes are: Problem Chronology, Journal Issue Checklist, Journals with Problem Columns (this seems to be a complete list!), Notation, Glossary (of terms), and, very important, the list of Unsolved Problems, with a separate Author Index.

The main section (the "body") of the book is the Subject Index (pp. 33–298), in which the problems, sorted by topic, are quoted in full. What a wealth of mathematical thought : from very elementary to the frontiers of research, from the almost trivial to open problems, from rather ponderous statements to very witty offerings! It is a true pleasure to choose some specific topic and discover what has been invented there. Look, for instance, under the heading "Collatz problem" (iteration of $f(x) = x/2$ resp. $3x + 1$ for x even resp. odd). You will find several interesting variations of it. Each page of the book is a new source of immense pleasure.

The Author of $\mathcal{BRB}$, an excellent program designer and software engineer, seems to have squeezed out all that was possible (if not more) from a perfect database system, loaded with a great deal of information. Yet the book is manageable, readable and enjoyable! It is hard to believe that most of that monumental work has been done by just one man.

My opinion on this magnificent work is highly enthusiastic. My only regret is that the material of various nice contests and problem journals from outside the English-speaking regions has not been included. For instance, *Elemente der Mathematik*, and the national olympiads of "my" region, that is, East-European countries, are missing. However, *some* selection *had* to be made. While the larger the scale is, the better the tourist map is, yet a 1:1 map would be completely useless. Moreover, the Author himself claims modestly that "the index is a small start at solving the complete problem of indexing the world's mathematics problems". If this is a small start, then the "Whiteoaks" are a short story!

How needed such a publication was and how useful it is going to be, once the work is done, requires no comment. I am pleased to learn that the Author is preparing further volumes, each covering a time period of five years, working forward *and backward* (!): 1985–1989, 1975–1979, and so on. He promises to cover even more journals.

There is no better way to conclude this review than to quote M. S. Klamkin's words from his *Foreword*: "...a must book for problemists as well as problem editors". And I would add: great fun for all those who like mathematics in its most easily comprehensible form and who enjoy the challenge of mathematics problems.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **May 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1871. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be a variable point on the arc BC (not containing A) of the circumcircle of ΔABC, and let I_1, I_2 be the incenters of triangles PAB and PAC, respectively. Prove that:

(a) the circumcircle of ΔPI_1I_2 passes through a fixed point;
(b) the circle with diameter I_1I_2 passes through a fixed point;
(c) the midpoint of I_1I_2 lies on a fixed circle.

1872*. *Proposed by Murray S. Klamkin, University of Alberta.*

Are there any integer sided non-equilateral triangles whose angles are in geometric progression?

1873. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two circles externally tangent at point P. Let $A_1 \neq P$ be a variable point on $\mathcal{C}_1$ and let $A_2 \neq P$ be on $\mathcal{C}_2$ so that A_1, P, A_2 are collinear. Point A_3 is in the plane of $\mathcal{C}_1$ and $\mathcal{C}_2$ so that $A_1A_2A_3$ is directly similar to a given fixed triangle. Determine the locus of A_3.

1874. *Proposed by Pedro Melendez, Belo Horizonte, Brazil.*

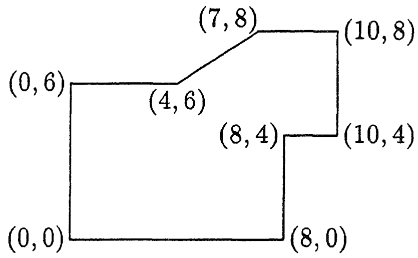
Find the smallest positive integer n such that $n!$ is divisible by 1993^{1994}.

1875. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

The iterates $f^2, f^3, \ldots$ of a mapping f of a set A into itself are defined by: $f^2(x) = f(f(x))$, $f^3(x) = f(f^2(x))$, and in general $f^{i+1}(x) = f(f^i(x))$. Assuming that A has n elements ($n \geq 3$), find the number of mappings $f : A \to A$ such that f^{n-2} is a constant while f^{n-3} is not a constant.

1876. *Proposed by Bernardo Recamán, United World College of Southern Africa, Mbabane, Swaziland.*

Find four points on the boundary of the following octagon which are the corners of a square.



1877. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Let $B_1, B_2, \ldots, B_b$ be k-element subsets of $\{1, 2, \ldots, n\}$ such that $|B_i \cap B_j| \leq 1$ for all $i \neq j$. Show that

$$b \leq \left[\frac{n}{k}\left[\frac{n-1}{k-1}\right]\right],$$

where $[x]$ denotes the greatest integer $\leq x$.

1878*. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Given two triangles ABC and $A'B'C'$, prove or disprove that

$$\frac{\sin A'}{\sin A} + \frac{\sin B'}{\sin B} + \frac{\sin C'}{\sin C} \leq 1 + \frac{R}{r},$$

where r, R are the inradius and circumradius of triangle ABC.

1879. *Proposed by Jisho Kotani, Akita, Japan.*

Show that for any integer $n \geq 3$ there is a polynomial $x^n + ax^2 + bx + c$, with $a, b, c \neq 0$, which has three of its roots equal.

1880. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

AD and BE are angle bisectors of triangle ABC, with D on BC and E on AC. Suppose that $AD = AB$ and $BE = BC$. Determine the angles of ΔABC.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1678. [1991: 238; 1992: 245] *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Show that

$$\sqrt{s}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \le \sqrt{2}(r_a+r_b+r_c),$$

where a,b,c are the sides of a triangle, s the semiperimeter, and r_a,r_b,r_c the exradii.

III. *Comment by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*
Murray Klamkin extended the proposed inequality to

$$2^n(r_a^{2n}+r_b^{2n}+r_c^{2n}) \ge s^n(a^n+b^n+c^n) \tag{1}$$

for $n \ge 1/2$ or $n \le 0$. We give a unified proof of (1) for $n \ge 1/2$.

Assume $a \ge b \ge c$. Then $r_a \ge r_b \ge r_c$ (because $r_a = s\tan(A/2)$ etc.). We first note that

$$\sqrt{2}\,r_a \ge \sqrt{sa}, \tag{2}$$

$$\sqrt{2}\,r_a+\sqrt{2}\,r_b \ge \sqrt{sa}+\sqrt{sb}, \tag{3}$$

$$\sqrt{2}\,r_a+\sqrt{2}\,r_b+\sqrt{2}\,r_c \ge \sqrt{sa}+\sqrt{sb}+\sqrt{sc}, \tag{4}$$

where (4) is the original proposed inequality. Since

$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}\,,$$

(2) is equivalent to

$$2(s-b)(s-c) \ge a(s-a). \tag{5}$$

Letting $s-a=x$, $s-b=y$, $s-c=z$, we get $x \le y \le z$, $s=x+y+z$, and $a=y+z$ etc.; so (5) becomes $2yz \ge (y+z)x$ or

$$y(z-x)+z(y-x) \ge 0,$$

which is true, and (2) follows. In similar fashion we may prove that $\sqrt{sc} \ge \sqrt{2}\,r_c$ (which is equivalent to $x(z-y)+y(z-x) \ge 0$). So (3) follows from (4).

Therefore

$$(\sqrt{2}\,r_a,\sqrt{2}\,r_b,\sqrt{2}\,r_c) \succ_w (\sqrt{sa},\sqrt{sb},\sqrt{sc}).$$

[Here $\succ_w$ means that the first triple *weakly majorizes* the second.—*Ed.*] It is easy to see that the function $f(x)=x^{2n}$ $(n \ge 1/2)$ is convex and increasing. By applying the majorization inequality [e.g., see 3.C.1.b, page 64 of Marshall and Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, 1979], we have

$$f(\sqrt{2}\,r_a)+f(\sqrt{2}\,r_b)+f(\sqrt{2}\,r_c) \ge f(\sqrt{sa})+f(\sqrt{sb})+f(\sqrt{sc}),$$

which is (1).

For $n < 0$ we give a simple proof too. Change n to $-n$; then (1) is equivalent to

$$\frac{1}{2^n}\left(\frac{x^n}{y^n z^n}+\frac{y^n}{z^n x^n}+\frac{z^n}{x^n y^n}\right) \ge \frac{1}{(x+y)^n}+\frac{1}{(y+z)^n}+\frac{1}{(z+x)^n},$$

where again $x = s - a$ etc., and now $n > 0$. Note that for any real x_1, x_2, x_3

$$x_1^2 + x_2^2 + x_3^2 \ge x_1x_2 + x_2x_3 + x_3x_1$$

[e.g., multiply both sides by 2 and rearrange everything into squares on the left side.—*Ed.*]. So

$$\begin{aligned}\frac{1}{2^n}\left(\frac{x^n}{y^n z^n}+\frac{y^n}{z^n x^n}+\frac{z^n}{x^n y^n}\right) &\ge \frac{1}{2^n}\left(\frac{1}{x^n}+\frac{1}{y^n}+\frac{1}{z^n}\right)\\ &\ge \frac{1}{2^n}\left(\frac{1}{(\sqrt{xy})^n}+\frac{1}{(\sqrt{yz})^n}+\frac{1}{(\sqrt{zx})^n}\right)\\ &\ge \frac{1}{(x+y)^n}+\frac{1}{(y+z)^n}+\frac{1}{(z+x)^n}.\end{aligned}$$

[*Editor's note*: (1) is still open for $1/5 \le n < 1/2$.]

* * * * *

1781. [1992: 274] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $a > 0$ and $x_1, x_2, \ldots, x_n \in [0, a]$ $(n \ge 2)$ such that

$$x_1x_2\ldots x_n = (a-x_1)^2(a-x_2)^2\ldots(a-x_n)^2.$$

Determine the maximum possible value of the product $x_1x_2\ldots x_n$.

Solution by P.E. Tsaoussoglou, Athens, Greece.

By the A.M.–G.M. inequality,

$$\begin{aligned}(x_1x_2\ldots x_n)^{1/2n} &= [(a-x_1)(a-x_2)\ldots(a-x_n)]^{1/n}\\ &\le \frac{(a-x_1)+(a-x_2)+\cdots+(a-x_n)}{n} = a - \frac{x_1+x_2+\cdots+x_n}{n}\\ &\le a - (x_1x_2\ldots x_n)^{1/n}. \qquad (1)\end{aligned}$$

Let $(x_1x_2\ldots x_n)^{1/2n} = k > 0$; then (1) says that $k \le a - k^2$ or $k^2 + k - a \le 0$, so

$$0 \le k \le \frac{-1+\sqrt{4a+1}}{2}.$$

Thus

$$\max(x_1x_2\ldots x_n) \le \left(\frac{-1+\sqrt{4a+1}}{2}\right)^{2n}.$$

Equality holds if

$$x_1 = x_2 = \cdots = x_n = \left(\frac{-1+\sqrt{4a+1}}{2}\right)^2.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; KEE-WAI LAU, Hong Kong; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Geretschläger, Huang and the proposer had solutions similar to the above.

* * * * *

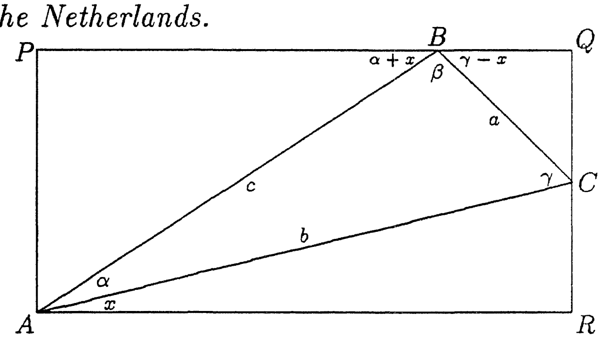
1782. [1992: 274] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangle ABC, with angles α, β, γ, is inscribed in rectangle $APQR$ so that B lies on PQ and C lies on QR. Prove that

$$\cot\alpha \cdot [BCQ] = \cot\beta \cdot [ACR] + \cot\gamma \cdot [ABP],$$

where $[XYZ]$ denotes the area of triangle XYZ. (This problem is an extension of problem 2 of the 1987 Hungarian National Olympiad [1991: 68].)

Solution by P. Penning, Delft, The Netherlands.



Introduce $x = \angle CAR$. Then at vertex B, the angles with PQ are $\alpha + x$ and $\gamma - x$ as shown. Introduce the circumradius R of ΔABC. Then, since

$$a = 2R\sin\alpha, \quad \text{etc.},$$

we have

$$\begin{aligned}\cot\alpha\cdot[BCQ] &= \frac{QC\cdot BQ\cdot\cot\alpha}{2} = \frac{a^2\sin(\gamma-x)\cos(\gamma-x)\cos\alpha}{2\sin\alpha}\\ &= \frac{a^2\sin(2\gamma-2x)\sin 2\alpha}{8\sin^2\alpha} = \frac{R^2\sin(2\gamma-2x)\sin 2\alpha}{2}\\ &= \frac{R^2}{4}[\cos(2\gamma-2\alpha-2x)-\cos(2\alpha+2\gamma-2x)].\end{aligned}$$

Similarly,

$$\begin{aligned}\cot\beta\cdot[ACR] &= \frac{b^2\sin 2x\sin 2\beta}{8\sin^2\beta} = \frac{R^2\sin 2x\sin 2\beta}{2} = \frac{R^2}{4}[\cos(2\beta-2x)-\cos(2\beta+2x)]\\ &= \frac{R^2}{4}[\cos(2\alpha+2\gamma+2x)-\cos(2\alpha+2\gamma-2x)]\end{aligned}$$

and

$$\cot\gamma \cdot [ABP] = \frac{c^2 \sin(2\alpha + 2x)\sin 2\gamma}{8\sin^2\gamma} = \frac{R^2 \sin(2\alpha+2x)\sin 2\gamma}{2}$$
$$= \frac{R^2}{4}[\cos(2\gamma - 2\alpha - 2x) - \cos(2\alpha + 2\gamma + 2x)].$$

Substitution in the relation to be proved leads to an identity.

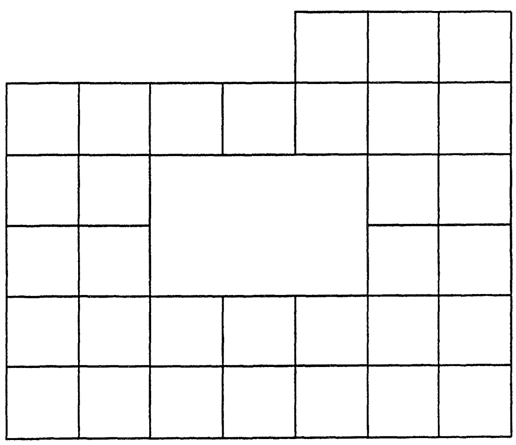
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER BRADLEY, Clifton College, Bristol, U.K.; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEE LIAN KIM, Messiah College, Grantham, Pennsylvania; KEE-WAI LAU, Hong Kong; JOSEPH LING, The University of Calgary; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. One other reader handled only a special case.

Readers will be glad to note that regular contributor Šefket Arslanagić, a "refugee from Bosnia and Hercegovina", has found a safer place.

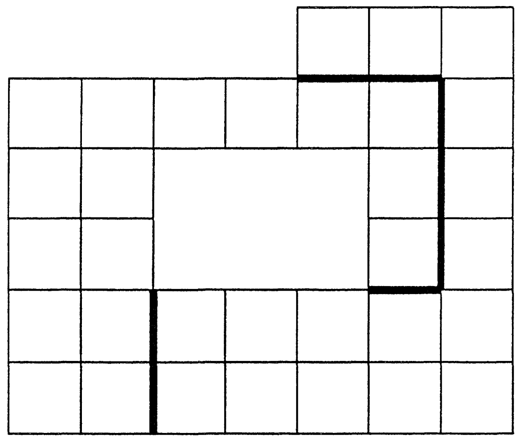
* * * * *

1783. [1992: 274] *Proposed by Andy Liu, University of Alberta, and Daniel van Vliet, student, Salisbury Composite H.S., Sherwood Park, Alberta.*

Dissect the figure into two congruent pieces.



Solution.



Found by SAM BAETHGE, Science Academy, Austin, Texas; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEE LIAN KIM, Messiah College, Grantham, Pennsylvania; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; KENNETH M. WILKE, Topeka, Kansas; and the proposers.

Perz even included a proof that this dissection is the only one!

For some similar dissection problems, see Chapter 12 *of Martin Gardner's* Penrose Tiles to Trapdoor Ciphers, *W.H. Freeman,* 1989.

*　　*　　*　　*　　*

1784. [1992: 275] *Proposed by Murray S. Klamkin, University of Alberta, and Dale Varberg, Hamline University, St. Paul, Minnesota.*

A point in 3–space is at distances 9, 10, 11 and 12 from the vertices of a tetrahedron. Find the maximum and minimum possible values of the sum of the squares of the edges of the tetrahedron.

Solution by the proposers.

Let the tetrahedron be $A_0A_1A_2A_3$, the point P, and set $PA_i = a_i$, so that in the given case $a_0 = 9$, $a_1 = 10$, $a_2 = 11$, $a_3 = 12$. Let S denote the sum of the squares of the edges and let $\mathbf{A}_i = \overrightarrow{PA_i}$, so that

$$S = \frac{1}{2}\sum_{i,j} |\mathbf{A}_i - \mathbf{A}_j|^2 = 3\sum_{i=0}^{3} a_i^2 - 2\sum_{i<j} \mathbf{A}_i \cdot \mathbf{A}_j. \tag{1}$$

We now let $\mathbf{G} = \mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3$, which on squaring gives

$$\mathbf{G}^2 = \sum_{i=0}^{3} a_i^2 + 2\sum_{i<j} \mathbf{A}_i \cdot \mathbf{A}_j. \tag{2}$$

Hence from (1) and (2),

$$S = 4\sum_{i=0}^{3} a_i^2 - \mathbf{G}^2. \tag{3}$$

Clearly S will be a maximum if $\mathbf{G}$ can be $\mathbf{0}$. In this case P will be the centroid of the tetrahedron. Since $a_i = |\mathbf{A}_i|$, a necessary and sufficient condition that $\mathbf{G}$ can be $\mathbf{0}$ is that a_0, a_1, a_2, a_3 can be the sides of a quadrilateral, i.e., $a_0 + a_1 + a_2 + a_3 \geq 2a_i$ for all i. Since this is true for $(a_0, a_1, a_2, a_3) = (9, 10, 11, 12)$, from (3) the maximum S is

$$S_{\max} = 4\sum_{i=0}^{3} a_i^2 = 4(9^2 + 10^2 + 11^2 + 12^2) = \mathbf{1784}.$$

To minimize S, we need to maximize $\mathbf{G}^2$. Hence

$$S_{\min} = 4\sum_{i=0}^{3} a_i^2 - \left(\sum_{i=0}^{3} a_i\right)^2 = 1784 - 42^2 = \mathbf{20}.$$

[This value is achieved when P, A_0, A_1, A_2, A_3 are collinear, say at the points $0, 9, 10, 11, 12$ respectively on the x-axis.]

[*Editor's note.* The proposers' original problem and solution concerned arbitrary a_i's, and was actually for an n-dimensional simplex rather than a tetrahedron, but their proof in this general case isn't much different from that given above.]

Also solved by P. PENNING, Delft, The Netherlands; and (maximum value of S only) CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.

* * * * *

1785. [1992: 275] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

(a) Show that the set of numbers of the form $2^r/5^n$, where r is a nonnegative rational number and n is a nonnegative integer, is dense in the set of nonnegative real numbers [that is, any nonnegative real number can be approximated arbitrarily closely by numbers of the form $2^r/5^n$].

(b)* What if r and n must both be nonnegative integers?

I. *Solution to* (a) *by Margherita Barile, student, Universität Essen, Germany.*

We prove that every interval (x, y) with $0 \leq x$ contains a number of the given form. If $x = 0$ this is true, since $\lim_{n\to\infty} 2/5^n = 0$. Now suppose $x > 0$. Since $\lim_{n\to\infty} 2^{1/n} = 1$, there exist nonnegative integers m, v such that

$$\frac{2}{5^m} < x \quad \text{and} \quad 2^{1/v} - 1 < \frac{y-x}{x}\ .$$

Let u be a nonnegative integer maximal with respect to the property:

$$\frac{2^{u/v}}{5^m} \leq x.$$

Then
$$x < \frac{2^{\frac{u+1}{v}}}{5^m} = \frac{2^{u/v}}{5^m}(2^{1/v}-1) + \frac{2^{u/v}}{5^m} < x\left(\frac{y-x}{x}\right) + x = y.$$
This proves the claim.

II. *Solution by H.L. Abbott, University of Alberta.*

It suffices to solve part (b). A theorem of Dirichlet states the following: If α is an irrational real number, β any real number and ϵ any positive number then there are infinitely many pairs of integers n and m, $n > 0$, such that $|n\alpha - m - \beta| < \epsilon$. It follows from Dirichlet's Theorem that the sequence $\{\langle n\alpha\rangle\}_{n=1}^{\infty}$ is dense in $[0,1]$, where $\langle x\rangle = x - \lfloor x\rfloor$, the fractional part of x [e.g., see Theorem 439, page 376 of Hardy and Wright's *An Introduction to the Theory of Numbers* (4th Edition)]. Since $\log_5 2$ is irrational, $\{\langle n\log_5 2\rangle\}_{n=1}^{\infty}$ is dense in $[0,1]$. This implies that $\{5^{\langle n\log_5 2\rangle}\}_{n=1}^{\infty}$ is dense in $[1,5]$. It now follows that for each integer k, $\{5^{\langle n\log_5 2\rangle - k}\}_{n=1}^{\infty}$ is dense in $[5^{-k}, 5^{-k+1}]$. Thus $\cup_{k=-\infty}^{\infty} \cup_{n=1}^{\infty} \{5^{\langle n\log_5 2\rangle - k}\}$ is dense in $[0,\infty)$. That $\{2^n/5^m\}_{n,m=1}^{\infty}$ is dense in $[0,\infty)$ may now be seen by noting that

$$5^{\langle n\log_5 2\rangle} \cdot 5^{\lfloor n\log_5 2\rfloor} = 5^{\langle n\log_5 2\rangle + \lfloor n\log_5 2\rfloor} = 5^{n\log_5 2} = 2^n$$

and thus
$$5^{\langle n\log_5 2\rangle - k} = \frac{2^n}{5^{\lfloor n\log_5 2\rfloor + k}}\ .$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; and CHRIS WILDHAGEN, Rotterdam, The Netherlands. Part (a) *only solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; and the proposer.*

Meyers commented that the problem is related to a well known theorem about the distribution of multiples of irrational numbers modulo 1. *He showed that the result holds in general if* 2 *and* 5 *are replaced by arbitrary positive real numbers x and y, respectively, provided that $\log_y x$ is irrational and positive. Note that the proof by Abbott given above could also be modified to yield the same conclusion.*

* * * * *

1786. [1992: 275] *Proposed by R.P. Sealy, Mount Allison University, Sackville, New Brunswick.*

For which values of n can one construct a sequence of n consecutive positive integers so that the mean and variance are both integers?

Solution by Leroy F. Meyers, The Ohio State University.

Let the n consecutive integers be $a + j$ for $j = 0, 1, 2, \ldots, n-1$. Then the mean of these integers is
$$\frac{1}{n}\sum_{j=0}^{n-1}(a+j) = a + \frac{(n-1)n}{2n} = a + \frac{n-1}{2}\ .$$
Hence the mean is an integer if and only if $n-1$ is even, i.e., n is odd.

Hence we have $n = 2k+1$ for some nonnegative integer k, and the integers may be taken to be $c+j$ for $j = -k, -k+1, \ldots, k-1, k$. The mean of these is the integer c. The variance of the set of integers is

$$\frac{1}{n}\sum_{j=-k}^{k}((c+j)-c)^2 = \frac{2}{2k+1}\cdot\sum_{j=1}^{k} j^2 = \frac{2}{2k+1}\cdot\frac{k(k+1)(2k+1)}{6} = \frac{k(k+1)}{3}\,.$$

Hence the variance is an integer if and only if k or $k+1$ is divisible by 3, which occurs just when $2k+1$ is *not* divisible by 3. Hence both mean and variance of a set of n consecutive integers is an integer if and only if n is divisible by neither 2 nor 3, i.e., just when n is congruent to 1 or 5 modulo 6.

Sometimes the variance is defined with denominator $n-1$ instead of n. In that case, the variance is $(k+1)(2k+1)/6 = (n+1)n/12$, and the condition that this be an integer for odd n is that $n+1$ be divisible by 4 and that either n or $n+1$ be divisible by 3, in other words, that n be congruent to either 3 or 11 modulo 12.

Also solved by H.L. ABBOTT, University of Alberta; MARGHERITA BARILE, student, Universität Essen, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Solved, assuming the denominator in the definition of variance is $n-1$, by CHARLES ASHBACHER, Cedar Rapids, Iowa; and KENNETH M. WILKE, Topeka, Kansas.

Another reader misread the problem as asking for the mean and standard deviation to be integers, and noted that that problem appeared as E3302 *in the* American Math. Monthly*; the solution, which gets into Pell equations, appears on p. 432 of the May* 1990 *issue of the* Monthly. *In fact that problem inspired the current proposal! After solving our problem, Bradley also solved the "standard deviation" problem, suggesting it may be more interesting from a teacher's point of view.*

* * * * *

1788. [1992: 275] *Proposed by C.J. Bradley, Clifton College, Bristol, England.*

A pack of cards consists of m red cards and n black cards. The pack is thoroughly shuffled and the cards are then laid down in a row. The number of colour changes one observes in moving from left to right along the row is k. (For example, for $m = 5$ and $n = 4$ the row RRBRBBRBR exhibits $k = 6$.) Prove that k is more likely to be even than odd if and only if

$$|m-n| > \sqrt{m+n}.$$

Solution by P. Penning, Delft, The Netherlands.

Note that k is even if and only if the first and last card have the same colour and that k is odd if and only if the first and last card have different colours. So we must find the condition that the probability for the same colour, S, is greater than the probability D of different colours for the first and last cards.

It is not necessary to consider the first and last card as a special pair. Since the colour is not correlated, we may consider any pair. The number of possible choices for a pair are:

$$\begin{array}{rcr} \text{Red, Red} & : & m(m-1) \\ \text{Red, Black} & : & mn \\ \text{Black, Red} & : & nm \\ \text{Black, Black} & : & n(n-1) \end{array}$$

So $S > D$ if and only if $m(m-1) + n(n-1) > 2mn$, that is, if $m^2 - 2mn + n^2 > m + n$, and taking the square root gives the required result.

Also solved by MARGHERITA BARILE, student, Universität Essen, Germany; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, Ohio State University; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

*　　*　　*　　*　　*

1789*. [1992: 275] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Let a_1, a_2, a_3 be the sides of a triangle, w_1, w_2, w_3 the angle bisectors, F the area, and s the semiperimeter. Prove or disprove that

$$w_1^{a_1} w_2^{a_2} w_3^{a_3} \leq (F\sqrt{3})^s.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The claimed inequality is valid. Changing notation to call the sides a, b, c and the bisectors w_a, w_b, w_c, and starting from

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \leq \sqrt{s(s-a)}$$

[e.g., see page 222 of D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*], we get

$$\prod w_a^a \leq \prod [s(s-a)]^{a/2},$$

and we're thus done if we show the stronger inequality

$$\prod [s(s-a)]^a \leq (F\sqrt{3})^{2s}. \tag{1}$$

Now (1) is equivalent to

$$s^{2s} \prod (s-a)^a \leq 3^s s^s \prod (s-a)^s,$$

i.e.

$$\left(\frac{s}{3}\right)^s \leq \prod (s-a)^{s-a}. \tag{2}$$

We now consider the function $f(t) = t \ln t$, $t > 0$, which is strictly convex. Therefore Jensen's inequality yields

$$\frac{s}{3}\ln\left(\frac{s}{3}\right) = f\left(\frac{s}{3}\right) = f\left(\frac{1}{3}\sum(s-a)\right) \leq \frac{1}{3}\sum f(s-a) = \frac{1}{3}\sum(s-a)\ln(s-a),$$

i.e., (2). Furthermore, there occurs equality if and only if the triangle is equilateral.

* * * * *

1790. [1992: 275] *Proposed by Neven Jurić, Zagreb, Croatia.*

$\mathcal{S}$ is the set of all finite sequences of 0's and 1's. For each $x \in \mathcal{S}$ let $\varphi(x)$ be the sequence obtained if each 1 in x is transformed into 01 and each 0 in x into 10. For example, $\varphi(01) = 1001$. Let $\varphi^2(x) = \varphi(\varphi(x))$ and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$, $n \geq 3$. How many pairs 00 are there in $\varphi^n(1)$?

Comment by the Editor.

As this problem is the same as #2 of the Second Test of the 1988 Chinese Olympiad Training Camp, and a solution of it has already appeared in *Crux* [1993: 105–106], the solution will not be repeated here. The editor apologizes for the duplication.

Solved by H.L. ABBOTT, University of Alberta; MARGHERITA BARILE, student, Universität Essen, Germany; KATHIE CAMERON and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; LEROY F. MEYERS, Ohio State University; P. PENNING, Delft, The Netherlands; MARIA JESUS VILLAR RUBIO, I.B. Leonardo Torres Quevedo, Santander, Spain; LAMARR WIDMER, Messiah College, Grantham, Pennsylvania; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

* * * * *

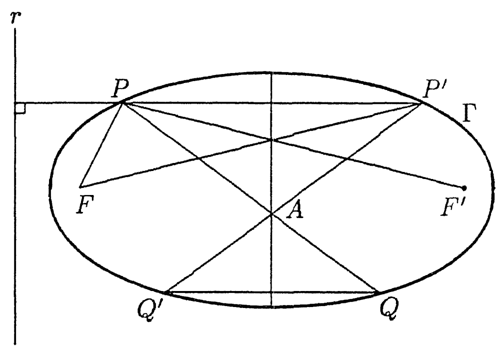
1791. [1992: 304] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Γ is an ellipse with foci F and F'. Let P, Q be points on Γ, and let A be the intersection of PQ with the minor axis of Γ. Prove that

$$\left|\frac{PF - PF'}{QF - QF'}\right| = \frac{AP}{AQ}\,.$$

Solution by Emilio Fernández Moral, I.B. Sagasta, Logroño, Spain.

Let r be the directrix of Γ corresponding to the focus F, that is, $P \in \Gamma$ if and only if $PF = e \cdot d(P, r)$ with $e < 1$ being the eccentricity of Γ. Letting P' and Q' be the points of Γ symmetric to P and Q with respect to the minor axis of Γ, we have that



$$PF' = P'F = e \cdot d(P', r),$$

which implies that

$$|PF - PF'| = e \cdot |d(P, r) - d(P', r)| = e \cdot PP'.$$

Similarly $|QF - QF'| = e \cdot QQ'$. These yield that

$$\left|\frac{PF - PF'}{QF - QF'}\right| = \frac{PP'}{QQ'} = \frac{AP}{AQ}$$

because of the similarity of the triangles APP' and AQQ'.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; SEUNG-JIN BANG, Albany, California; F. BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; JOSEPH LING, University of Calgary; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

* * * * *

1792. [1992: 304] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x, y \geq 0$ such that $x + y = 1$, and let $\lambda > 0$. Determine the best lower and upper bounds (in terms of λ) for

$$(\lambda + 1)(x^\lambda + y^\lambda) - \lambda(x^{\lambda+1} + y^{\lambda+1}).$$

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.

The best lower and upper bounds in this case are the absolute minimum and maximum of

$$f(x) = (\lambda + 1)[x^\lambda + (1 - x)^\lambda] - \lambda[x^{\lambda+1} + (1 - x)^{\lambda+1}]$$

on the interval $[0,1]$ for $\lambda > 0$. $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$, and $f(0) = f(1) = 1$, thus by Rolle's Theorem there exists at least one $c \in (0,1)$ such that $f'(c) = 0$. Now

$$f'(x) = \lambda(\lambda+1)[x^{\lambda-1} - (1-x)^{\lambda-1} - x^{\lambda} + (1-x)^{\lambda}].$$

The condition $f'(x) = 0$ yields

$$x^{\lambda-1}(1-x) = x^{\lambda-1} - x^{\lambda} = (1-x)^{\lambda-1} - (1-x)^{\lambda} = (1-x)^{\lambda-1}x,$$

or

$$\left(\frac{1-x}{x}\right)^{\lambda-2} = 1.$$

There is only one real solution to this equation, namely $x = 1/2$. Note $f(1/2) = (\lambda+2)/2^{\lambda}$, which equals 1 for $\lambda = 2$. Thus

for $0 < \lambda \le 2$, lower bound $= \mathbf{1}$ and upper bound $= (\boldsymbol{\lambda} + \mathbf{2})/\mathbf{2}^{\lambda}$;

for $2 \le \lambda$, lower bound $= (\boldsymbol{\lambda} + \mathbf{2})/\mathbf{2}^{\lambda}$ and upper bound $= \mathbf{1}$.

(Indeed, $f(x) \equiv 1$ for $\lambda = 2$.)

Also solved by SEUNG-JIN BANG, Albany, California; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; JOSEPH LING, University of Calgary; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; and the proposer. One incorrect solution was received.

The proposer wonders what the bounds are for the more general expression

$$\mu \sum_{i=1}^{n} x_i^{\lambda} - \lambda \sum_{i=1}^{n} x_i^{\mu},$$

in terms of λ, μ, n, *where* $\lambda, \mu > 0$ *and* $x_i \ge 0$, $x_1 + \cdots + x_n = 1$.

* * * * *

1793. [1992: 304] *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that in any n-dimensional simplex there is at least one vertex such that the n edges emanating from that vertex are possible sides of an n-gon.

Solution by Joseph Ling, University of Calgary.

We first note that n edges $e_1, e_2, \ldots, e_n$ $(n \ge 3)$ are possible sides of an n-gon if and only if the length of the longest edge is less than the sum of the lengths of the remaining $n-1$ edges. Now, let $X_0, X_1, X_2, \ldots, X_n$ be the $n+1$ vertices of an n-dimensional simplex. Without loss of generality, we may assume that X_0X_1 is the longest edge of the simplex. Consider the triangle $X_0X_1X_i$ $(2 \le i \le n)$. We have

$$X_0X_1 \le X_0X_i + X_1X_i \qquad (2 \le i \le n).$$

Hence

$$2X_0X_1 \le (n-1)X_0X_1 \le \sum_{i=2}^{n}(X_0X_i + X_1X_i) = \sum_{i=2}^{n} X_0X_i + \sum_{i=2}^{n} X_1X_i.$$

Therefore we must have

$$\sum_{i=2}^{n} X_0X_i \ge X_0X_1 \qquad \text{or} \qquad \sum_{i=2}^{n} X_1X_i \ge X_0X_1.$$

The first inequality shows that the edges emanating from X_0 are possible sides of an n-gon, and the second inequality shows that the edges emanating from X_1 are possible sides of an n-gon.

Also solved by the proposer, who notes that there are simplexes with only one vertex having the desired property, namely if one vertex is very far from all the others, which are bunched together.

* * * * *

1794. [1992: 304] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Pairs of numbers from the set $\{7, 8, \ldots, n\}$ are adjoined to each of the 20 different (unordered) triples of numbers from the set $\{1, 2, \ldots, 6\}$, to obtain twenty 5-element sets $A_1, A_2, \ldots, A_{20}$. Suppose that $|A_i \cap A_j| \le 2$ for all $i \ne j$. What is the smallest n possible?

Solution by Richard I. Hess, Rancho Palos Verdes, California.

We first claim that the 20 sets A_i cannot use any number $k \in \{7, 8, \ldots, n\}$ more than four times. For if a number k is appended to $m > 4$ triples then consider the $3m > 12$ integers from 1 to 6 contained in the triples. One of them, say 1, must be used at least three times, and the sets formed will contain something like $\{1,2,3,k\}$, $\{1,4,5,k\}$, $\{1,6,q,k\}$; but the integer $q \in \{1, 2, \ldots, 6\}$ cannot be chosen without causing two sets to have more than two elements in common.

Now 40 numbers must be added to the 20 triples to form the sets A_i. By the above we must have at least 10 different k's, so the minimum n is **16**. The table below shows a construction for $n = 16$.

$\{1,2,3,7,8\}$	$\{2,3,4,13,15\}$
$\{1,2,4,12,14\}$	$\{2,3,5,9,11\}$
$\{1,2,5,15,16\}$	$\{2,3,6,14,16\}$
$\{1,2,6,9,10\}$	$\{2,4,5,8,10\}$
$\{1,3,4,10,11\}$	$\{2,4,6,7,11\}$
$\{1,3,5,13,14\}$	$\{2,5,6,12,13\}$
$\{1,3,6,12,15\}$	$\{3,4,5,12,16\}$
$\{1,4,5,7,9\}$	$\{3,4,6,8,9\}$
$\{1,4,6,13,16\}$	$\{3,5,6,7,10\}$
$\{1,5,6,8,11\}$	$\{4,5,6,14,15\}$

Also solved by the proposer. Two other readers sent in non-minimal solutions with $n = 20$.

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

BACK ISSUES AND SUBSCRIPTION INFORMATION
INFORMATION SUR L'ABONNEMENT ET LES NUMÉROS PRÉCÉDENTS

1994 (Volume 20) SUBSCRIPTIONS / ABONNEMENTS 1994 (volume 20)
(10 issues per volume / 10 numéros par volume)

Regular / Régulier	$46.00
CMS Members / membres de la SMC	$23.00

These subscription rates include shipping and handling
Les frais de manutentions sont inclus dans ces tarifs
CANADIAN ADDRESSES ADD 7% GST / ADRESSES CANADIENNES, AJOUTER 7% DE TPS)

BACK ISSUES / NUMÉROS ANTÉRIEURS

Volume	$50.00
Issue / Numéro	$ 6.00

BOUND VOLUMES / VOLUMES RELIÉS
(available / disponible: 1&2, 3, 7, 8, 9)

Volume	$10.00 each/chacun

FOR BACK ISSUES AND BOUND VOLUMES ONLY
POUR NUMÉROS ANTÉRIEURS ET VOLUMES RELIÉS SEULEMENT
Canadian addresses, please add 7% GST (#R118 833 979).
Foreign addresses, please add 15% shipping and handling charges.

Les adresses canadiennes, veuillez ajouter 7% de TPS (#R118 833 979).
Les adresses étrangères, veuillez ajouter 15% pour les frais d'envois.

CANADIAN MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DU CANADA
577 King Edward
P.O. Box 450, Station A
Ottawa, Ontario K1N 6N5 CANADA
Tel: (613) 564-2223, Fax: (613) 565-1539

ORDER FORM / BON DE COMMANDE

CRUX MATHEMATICORUM

Subscription / Abonnement	______	X	$46.00	=	________	(1)
Back issues / Numéros antérieurs	______	X	$ 6.00	=	________	(2)
	______	X	$50.00	=	________	(3)
Bound volumes / Volumes reliés	______	X	$10.00	=	________	(4)
					________	(1+2+3+4)

Canadian addresses must add GST: 7% of total of lines 1 to 4 (# R118 833 979)
Les adresses canadiennes: veuillez ajouter 7% de TPS sur la somme des lignes 1 à 4 (# R118 833 979) ________

Foreign addresses must add Shipping and Handling: 15% of total of lines 2, 3 and 4
Les adresses étrangères: veuillez ajouter 15% pour les frais de manutention (15% de la somme des lignes 2, 3 et 4) ________

TOTAL $ ________

☐ Cheque enclosed / chèque inclus (micro-encoded cheques only chèques encodés seulement)

☐ Invoice / Facture

☐ Please charge / Portez à mon compte ☐ VISA ☐ MasterCard

______________________ Account no. / Numéro de compte

______________________ Expiry date / date d'expiration

______________________ Signature

MAILING ADDRESS / ADRESSE POSTALE

Name / Nom:
Address / Adresse:
City, Province / Ville, Province:
Postal Code, Country / Code postal, pays: