Crux

Mathematicorum

Published by the Canadian Mathematical Society.

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 17 #3

March / mars

1991

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Canadian Mathematical Society

Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1991 subscription rate for ten issues is \$ 17.50 for members of the Canadian Mathematical Society and \$35.00, for non-members. Back issues: \$3.50 each. Bound Volumes with index: volumes 1 & 2 (combined) and each of 3, 7, 8 & 9: \$10.00 (Volumes 4, 5, 6 & 10 are out-of-print). All prices are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

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Canadian Mathematical Society
577 King Edward
Ottawa, Ontario, Canada K1N 6N5

ACKNOWLEDGEMENTS

The support of the Department of Mathematics and Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

Published by the Canadian Mathematical Society
Printed at Ottawa Laser Copy

ISSN 0705-0348 Second Class Mail Registration Number 5432

THE OLYMPIAD CORNER

No. 123

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first contest for this issue is the *12th Austrian-Polish Mathematics Competition.* Many thanks to Walther Janous, Ursulinengymnasium, Innsbruck, Austria, for sending it to me.

12TH AUSTRIAN-POLISH MATHEMATICS COMPETITION

Individual Competition

1st Day—June 28, 1989 ($4\frac{1}{2}$ hours)

1. Let $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n$ be positive real numbers. Show that

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^3 \le \left(\sum_{k=1}^{n} a_k^3\right)\left(\sum_{k=1}^{n} b_k^3\right)\left(\sum_{k=1}^{n} c_k^3\right).$$

2. Each point of the plane ($\mathbf{R}^2$) is coloured by one of the two colours A and B. Show that there exists an equilateral triangle with monochromatic vertices.

3. Determine all natural numbers N (in decimal representation) satisfying the following properties:
(1) $N = (aabb)_{10}$, where $(aab)_{10}$ and $(abb)_{10}$ are primes.
(2) $N = P_1 \cdot P_2 \cdot P_3$, where P_k $(1 \le k \le 3)$ is a prime consisting of k (decimal) digits.

2nd Day—June 29, 1989 ($4\frac{1}{2}$ hours)

4. Let P be a convex polygon in the plane having $A_1, A_2, \ldots, A_n$ $(n \ge 3)$ as its vertices. Show that there exists a circle containing the entire polygon P and having at least three adjacent vertices of P on its boundary.

5. Let A be a vertex of a cube ω circumscribed about a sphere κ of radius 1. We consider lines g through A containing at least one point of κ. Let P be the point of $g \cap \kappa$ having minimal distance from A. Furthermore, $g \cap \omega$ is AQ. Determine the maximum value of $\overline{AP} \cdot \overline{AQ}$ and characterize the lines g yielding the maximum.

6. We consider sequences $\{a_n : n \ge 1\}$ of squares of natural numbers (> 0) such that for each n the difference $a_{n+1} - a_n$ is a prime or the square of a prime. Show that all such sequences are finite and determine the longest sequence $\{a_n : n \ge 1\}$.

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Team Competition

June 30, 1989 (4 hours)

7. Functions f_0, f_1, f_2, ... are recursively defined by
(1) $f_0(x) = x$, for $x \in \mathbf{R}$;
(2) $f_{2k+1}(x) = 3^{f_{2k}(x)}$, where $x \in \mathbf{R}$, $k = 0, 1, 2, \ldots$;
(3) $f_{2k}(x) = 2^{f_{2k-1}(x)}$, where $x \in \mathbf{R}$, $k = 1, 2, 3, \ldots$.
Determine (with proof) the greater one of the numbers $f_{10}(1)$ and $f_9(2)$.

8. We are given an acute triangle ABC. For each point P of the interior or boundary of ABC let P_a, P_b, P_c be the orthogonal projections of P to the sides a, b and c, respectively. For such points we define the function

$$f(P) = \frac{\overline{AP_c} + \overline{BP_a} + \overline{CP_b}}{\overline{PP_a} + \overline{PP_b} + \overline{PP_c}}.$$

Show that $f(P)$ is constant if and only if ABC is an equilateral triangle.

9. Determine the smallest odd natural number N such that N^2 is the sum of an odd number (> 1) of squares of adjacent natural numbers (> 0).

*

The second group of problems are from the other side of the globe, and we thank S. C. Chan of Singapore for forwarding them to us.

SINGAPORE MATHEMATICAL SOCIETY INTERSCHOOL MATHEMATICAL COMPETITION 1989

Part B, Saturday, 17 June 1989

1. Let $n \geq 5$ be an integer. Show that n is a prime if and only if $n_i n_j \neq n_p n_q$ for every partition of n into 4 positive integers, $n = n_1 + n_2 + n_3 + n_4$, and for each permutation (i, j, p, q) of $(1, 2, 3, 4)$.

2. Given arbitrary positive numbers a, b and c, prove that at least one of the following inequalities is false:

$$a(1-b) > \frac{1}{4}\,, \quad b(1-c) > \frac{1}{4}\,, \quad c(1-a) > \frac{1}{4}.$$

3.(a) Show that

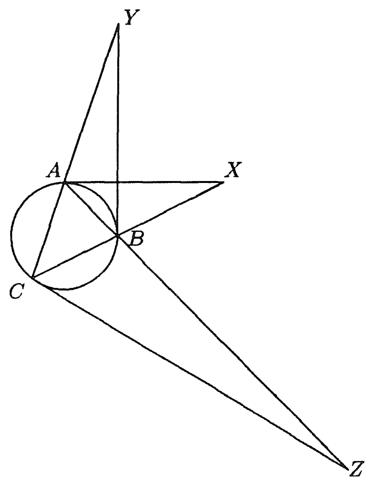
$$\tan(\frac{\pi}{12}) = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}.$$

(b) Given any thirteen distinct real numbers, show that there exist at least two, say x and y, which satisfy the inequality

$$0 < \frac{x-y}{1+xy} < \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}.$$

4. There are n participants in a conference. Suppose (i) every 2 participants who know each other have no common acquaintances; and (ii) every 2 participants who do not know each other have exactly 2 common acquaintances. Show that every participant is acquainted with the same number of people in the conference.

5. In the following diagram, ABC is a triangle, and X, Y and Z are respectively the points on the sides CB, CA and BA extended such that XA, YB and ZC are tangents to the circumcircle of ΔABC. Show that X, Y and Z are collinear.



* * *

Before giving solutions, a remark. Seung-Jin Bang, Seoul, Republic of Korea, notes that problem 2 [1989: 4] from the 1985–86 Flanders Mathematics Olympiad, for which we discussed a solution on [1990: 233], had appeared in Loren C. Larson's book *Problem-Solving Through Problems*, Springer-Verlag, 1983, p. 253, problem 7.2.9(b). Thanks for pointing this out.

Having given over last month's number of the Corner to problems from the 'archives', we concentrate this month on solutions to problems given in the 1989 numbers of *Crux*. First an acknowledgement. 'Also solved' status should have been given solutions sent in by Michael Selby, University of Windsor, for problems 1, 2 and 7 from the 24th Spanish Olympiad [1989: 67–68], for which we gave solutions in January [1991: 9-10]. His solutions arrived just as the issue was going to press.

*

When we gave the solutions, in the December 1990 number, to the 1986 Swedish Mathematical Competition, one problem remained unanswered. An alert reader spotted the gap and sent in a solution, which follows.

5. [1989: 34] *1986 Swedish Mathematical Competition.*
In the rectangular array

$$\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array}$$

of $m \times n$ real numbers, the difference between the maximum and the minimum element in each row is at most d, where $d > 0$. Each column is then rearranged in decreasing order so that the maximum element of the column occurs in the first row, and the minimum element occurs in the last row. Show that in the rearranged array the difference between the maximum and the minimum elements in each row is still at most d.

Solution by Andy Liu, University of Alberta.

Suppose to the contrary that in the reorganized array (whose elements we denote b_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$) the difference between the largest element and the smallest element in row i exceeds d. Let these elements be b_{ij} and b_{ik} respectively. Then

$$b_{1j} \geq b_{2j} \geq \dots \geq b_{ij} > b_{ik} \geq b_{i+1\ k} \geq \dots \geq b_{mk}.$$

By the Pigeon Hole Principle, two of these $n+1$ elements must be in the same row in the original array. One of them must be b_{pj} for some $p \leq i$ and the other b_{qk} for some $q \geq i$. However $b_{pj} - b_{qk} \geq b_{ij} - b_{ik} > d$. Hence the difference between the largest element and the smallest in that row also exceeds d. This is a contradiction.

*

In most of the remainder of this column we give the solutions we've received to problems of the *1987 Hungarian National Olympiad*, given in the April 1989 number of the Corner [1989: 100-101].

1. The surface area and the volume of a cylinder are equal to each other. Determine the radius and the altitude of the cylinder if both values are even integers.

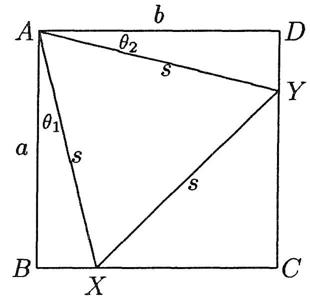
Solution by Bob Prielipp, University of Wisconsin–Oshkosh, and by Michael Selby, University of Windsor.

Let r be the radius and h be the altitude of the given cylinder. Then $2\pi rh + 2\pi r^2$ is the surface area of the cylinder and $\pi r^2 h$ is its volume. Since the surface area and the volume of the cylinder are equal, $2\pi rh + 2\pi r^2 = \pi r^2 h$ so $(r-2)(h-2) = 4$. Because r and h are both even positive integers, $r - 2 = 2$ and $h - 2 = 2$. Thus $r = 4$ and $h = 4$.

2. Cut the regular (equilateral) triangle AXY from rectangle $ABCD$ in such a way that the vertex X is on side BC and the vertex Y is on side CD. Prove that among the three remaining right triangles there are two, the sum of whose areas equals the area of the third.

Solution by Michael Selby, University of Windsor, and by D.J. Smeenk, Zaltbommel, The Netherlands.

First, such a triangle is not possible in every rectangle. In fact a necessary condition is that the sides $a \le b$ of the rectangle must satisfy $a \ge b\sqrt{3}/2$.



Let us assume such a triangle is possible. Let s be the length of the side of the equilateral triangle. We claim that $[XYC] = [ABX] + [ADY]$, where $[T]$ denotes the area of triangle T. Indeed

$$[ABX] = \frac{1}{2}s^2 \cos\theta_1 \sin\theta_1 = \frac{1}{4}s^2 \sin 2\theta_1$$

and

$$\begin{aligned}[ADY] &= \frac{1}{2}s^2 \cos\theta_2 \sin\theta_2 = \frac{1}{4}s^2 \sin 2\theta_2 = \frac{1}{4}s^2 \sin\left(\frac{\pi}{3} - 2\theta_1\right) \\ &= \frac{1}{4}s^2\frac{\sqrt{3}}{2}\cos 2\theta_1 - \frac{1}{8}s^2 \sin 2\theta_1,\end{aligned}$$

thus

$$[ABX] + [ADY] = \frac{1}{8}s^2 \sin 2\theta_1 + \frac{\sqrt{3}}{8}s^2 \cos 2\theta_1. \tag{1}$$

Now for ΔXYC, $\angle YXC = \pi/6 + \theta_1$. Therefore,

$$\begin{aligned}[XYC] &= \frac{s^2}{4}\sin\left(\frac{\pi}{3} + 2\theta_1\right) \\ &= \frac{s^2}{4}\frac{\sqrt{3}}{2}\cos 2\theta_1 + \frac{s^2}{4}\frac{1}{2}\sin 2\theta_1.\end{aligned}$$

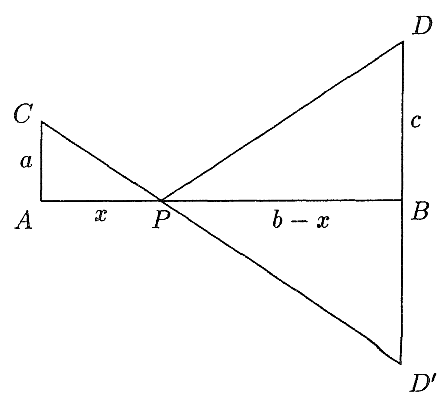
This is exactly (1), the sum of the areas.

3. Determine the minimum of the function

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b - x)^2 + c^2}$$

where a, b, c are positive numbers.

Solution by Mangho Ahuja, Southeast Missouri State University, and by D.J. Smeenk, Zaltbommel, The Netherlands.



Let $AC = a$, $AB = b$ and $BD = c$. Let P be a point on AB and let $x = AP$, so that $BP = b - x$. Then $f(x) = CP + PD$. To minimize $CP + PD$, we follow the method of reflection (see Z. A. Melzak, *Companion to Mathematics*, John Wiley & Sons, 1973, pp. 26–27). Let D' be the reflection of D in the line AB. Since triangles PBD and PBD' are congruent, $PD = PD'$ and $f(x) = CP + PD'$. As x varies, P changes its position. But the distance $CP + PD'$ will be a minimum when P lies on the line CD'. The minimum value of $f(x)$ is then $CP + PD' = CD'$. Let CL be the perpendicular from C to the line DD'. From the right triangle CLD',

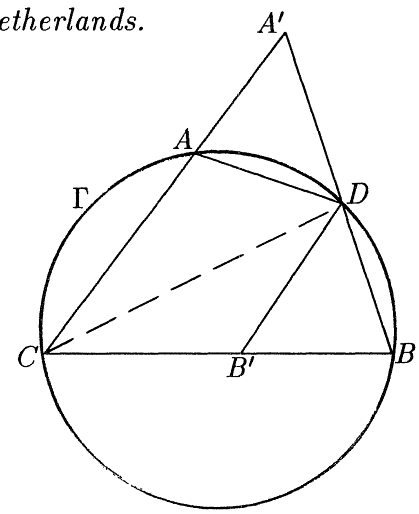
$$CD' = \sqrt{(LD)^2 + (CL)^2} = \sqrt{(a+c)^2 + b^2}.$$

[*Editor's note.* A solution via the calculus was submitted by Michael Selby, University of Windsor.]

4. Consider points A and B on given rays (semilines) starting from C, such that the sum $CA + CB$ is a given constant. Show that there is a point $D \neq C$ such that for each position of A and B the circumcircle of triangle ABC passes through D.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Let A and B be given and let Γ be the circumcircle of ΔABC. The interior bisector of $\angle ACB$ intersects Γ for the second time in the required point D. To see this, consult the figure. If $CA' + CB' = CA + CB$ then $AA' = BB'$ (1). Also $DA = DB$ (2) since $\angle ABD = \angle ACD = \angle DCB = \angle DAB$. Assume without loss that $CB \geq CA$. Quadrilateral $DBCA$ is inscribed in a circle, hence $\angle DAA' = \angle DBB'$ (3). From (1), (2) and (3) $\Delta DAA' \cong \Delta DBB'$. Thus $\angle ADA' = \angle BDB'$ and $\angle A'DB' = \angle ADB$. Thus D is a point on the circumcircle of $\Delta A'B'C$.



6. N is a 4-digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find N.

Solutions by Stewart Metchette, Culver City, California; John Morvay, Springfield, Missouri; Bob Prielipp, University of Wisconsin-Oshkosh; D.J. Smeenk, Zaltbommel, The Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $N = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$, where a, b, c and d are integers such that $1 \leq a \leq 6$ and $0 \leq b, c, d \leq 6$. By hypothesis $N = n^2$ for some positive integer n. Hence $n^2 = N \leq 6666$ so $n \leq 81$. Also by hypothesis

$$(a+3) \cdot 10^3 + (b+3) \cdot 10^2 + (c+3) \cdot 10 + (d+3) = m^2$$

for some positive integer m. Thus $m^2 - n^2 = 3333$, making $(m+n)(m-n) = 3 \cdot 11 \cdot 101$. Because $m + n > m - n$ and $n \leq 81$, it follows that $m + n = 101$ and $m - n = 33$, so $n = 34$. Therefore $N = 1156$.

7. Let a, b, c be the sides and α, β, γ be the opposite angles of a triangle. Show that if

$$ab^2 \cos\alpha = bc^2 \cos\beta = ca^2 \cos\gamma$$

then the triangle is equilateral.

Solutions by George Evagelopoulos, Athens, Greece; Michael Selby, University of Windsor; and D.J. Smeenk, Zaltbommel, The Netherlands.

From the equality $ab^2\cos\alpha = bc^2\cos\beta$ and the law of cosines, we get

$$ab\left(\frac{b^2+c^2-a^2}{2bc}\right) = c^2\left(\frac{a^2+c^2-b^2}{2ac}\right),$$

equivalently

$$a^2(b^2+c^2-a^2) = c^2(a^2+c^2-b^2)$$

and

$$a^2b^2+b^2c^2 = a^4+c^4. \tag{1}$$

Similarly we obtain

$$b^2c^2+a^2c^2 = a^4+b^4 \tag{2}$$

and

$$a^2b^2+a^2c^2 = b^4+c^4. \tag{3}$$

Adding equations (1), (2) and (3) we have

$$2a^2b^2+2a^2c^2+2b^2c^2 = 2a^4+2b^4+2c^4.$$

Equivalently

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 = 0,$$

so $a^2=b^2=c^2$ and $a=b=c$.

8. Let u and v be two real numbers such that u, v and uv are roots of a cubic polynomial with rational coefficients. Prove or disprove that uv is rational.

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh; Michael Selby, University of Windsor; D.J. Smeenk, Zaltbommel, The Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We disprove.

Clearly we may assume, without any loss of generality, that u, v and uv are roots of a monic cubic polynomial

$$P(x) = x^3+ax^2+bx+c$$

where a, b and c are rationals. Since $P(x)=(x-u)(x-v)(x-uv)$ we have

$$\begin{aligned} u+v+uv &= -a,\\ uv+uv(u+v) &= b,\\ u^2v^2 &= -c. \end{aligned}$$

From this we have

$$auv = -uv(u+v)-u^2v^2 = uv-b+c$$

or $(a-1)uv = c-b$. Thus uv is rational if $a \neq 1$. However, if $a = 1$, then uv **need not** be rational since in this case $b = c$ and

$$P(x) = x^3 + x^2 + cx + c = (x+1)(x^2+c)$$

and thus $\{u, v, uv\} = \{-1, \sqrt{-c}, -\sqrt{-c}\}$ which are real for $c \leq 0$. Thus if we take for example $c = -2$, then $u = -1$, $v = \sqrt{2}$ and $uv = -\sqrt{2}$ are roots of the cubic polynomial $x^3 + x^2 - 2x - 2$.

9. The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

Editor's note. An equivalent problem appeared as problem 1 of the 1985 Canadian Mathematics Olympiad (the answer, "one", was given in the problem). A solution is given on [1985: 272]. A solution to the present problem was sent in by Michael Selby, University of Windsor.

11. The domain of function f is $[0,1]$, and for any $x_1 \neq x_2$

$$|f(x_1) - f(x_2)| < |x_1 - x_2|.$$

Moreover, $f(0) = f(1) = 0$. Prove that for any x_1, x_2 in $[0,1]$,

$$|f(x_1) - f(x_2)| < \frac{1}{2}.$$

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh, and by Michael Selby, University of Windsor.

First $|f(x) - f(0)| \leq |x - 0|$, i.e. $|f(x)| \leq x$ and the inequality is strict for $x \neq 0$. Also $|f(x) - f(1)| \leq |x - 1|$, i.e. $|f(x)| \leq 1 - x$ with strict inequality for $x \neq 1$. Therefore

$$|f(x)| \leq \min(x, 1-x),$$

and the inequality is strict unless $x = 0$ or $x = 1$. Let $x_1, x_2 \in [0,1]$. If $|x_1 - x_2| \leq 1/2$, then

$$|f(x_1) - f(x_2)| \overset{*}{\leq} |x_1 - x_2| \leq 1/2.$$

This gives $|f(x_1) - f(x_2)| < 1/2$ since * is strict unless $x_1 = x_2$ and this case is trivial. So suppose $|x_1 - x_2| > 1/2$. Without loss of generality suppose that $x_1 \in (1/2, 1]$ and $x_2 \in [0, 1/2)$. Then

$$|f(x_1) - f(x_2)| \leq |f(x_1)| + |f(x_2)| \leq 1 - x_1 + x_2 = 1 - (x_1 - x_2) < \frac{1}{2}.$$

Therefore, in all cases $|f(x_1) - f(x_2)| < 1/2$ for all $x_1 \neq x_2$.

*

The last two solutions we present this number are to problems from the May 1989 column.

1. [1989: 129] *19th Austrian Mathematical Olympiad, 2nd Round.*

Determine the sum of all the divisors d of $N = 19^{88} - 1$ which are of the form $d = 2^a \cdot 3^b$ with $a, b > 0$.

Solution adapted by the editors from one by John Morvay, Springfield, Missouri.

Note that

$$19^{88} - 1 = (19^{11} - 1)(19^{11} + 1)(19^{22} + 1)(19^{44} + 1).$$

We look for integers m, n such that $2^m 3^n \mid 19^{88} - 1$, where m, n are the greatest such integers. It is clear from the above factorization that $n = 2$, as $3^2 \mid 19^{11} - 1$ while 3^3 does not divide into $19^{11} - 1$, and the other factors are $\equiv -1 \bmod 3$. Also $m = 5$, since $19^{11} + 1$ is divisible by 4 but not 8, and the other three factors are divisible by only 2. From this, the required sum of the divisors is $(3 + 9)(2 + 4 + 8 + 16 + 32) = 744$.

*

1. [1989: 130] *19th Austrian Mathematical Olympiad, Final Round.*

Let $a_1, \ldots, a_{1988}$ be positive real numbers whose arithmetic mean equals 1988. Show

$$\sqrt[1988]{\prod_{i=1}^{1988} \prod_{j=1}^{1988} (1 + \frac{a_i}{a_j})} \geq 2^{1988}$$

and determine when equality holds.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We shall prove the following general result which shows that the assumption about the arithmetic mean of the a_k's is redundant.

Theorem: If $a_i > 0$ for all $i = 1, 2, \ldots, n$ then

$$\left(\prod_{i=1}^{n} \prod_{j=1}^{n} (1 + \frac{a_i}{a_j})\right)^{1/n} \geq 2^n$$

with equality if and only if the a_k's are equal.

Proof. It is well known (cf [1], [2]) and easy to show that if $a_1 a_2 \cdots a_n = b^n$, then

$$\prod_{i=1}^{n} (1 + a_i) \geq (1 + b)^n$$

with equality if and only if the a_k's are equal. Using this, we obtain for each fixed i,

$$\prod_{j=1}^{n} (1 + \frac{a_i}{a_j}) \geq (1 + \frac{a_i}{b})^n$$

and hence

$$\left(\prod_{i=1}^{n} \prod_{j=1}^{n} (1 + \frac{a_i}{a_j})\right)^{1/n} \geq \prod_{i=1}^{n} (1 + \frac{a_i}{b}) \geq (1 + \frac{b}{b})^n = 2^n$$

with equality if and only if the a_k's are equal.

References:
[1] G. Chrystal, *Algebra*: Part II, Ex. V., # 30, p. 51.
[2] D. S. Mitrinović, *Analytic Inequalities*; §3.2.34, p. 208.

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The Olympiad season is upon us. Send me your national and regional contests, as well as your nice solutions to problems from the Corner.

* * * * *

MINI-REVIEWS

by ANDY LIU

INDIVIDUAL TITLES

How To Solve It, by G. Pólya, Princeton University Press, 1973. (paperback, 253 pp.)

This is the second edition of the first of five books by an outstanding scientist and educator on his theory and methods of problem-solving. Here, numerous examples illustrate the Pólya method which divides the task into four phases: understanding the problem, devising a plan, carrying out the plan and looking back.

Mathematics and Plausible Reasoning I, by G. Pólya, Princeton University Press, 1954. (hardcover, 210 pp.)

The subtitle of this volume is "Induction and Analogy in Mathematics". It is a continuation and elaboration of the ideas propounded in "How To Solve It". It discusses inductive reasoning and making conjectures, with examples mainly from number theory and geometry. The transition to deductive reasoning is via mathematical induction.

Mathematics and Plausible Reasoning II, by G. Pólya, Princeton University Press, 1954. (hardcover, 280 pp.)

The subtitle of this volume is "Patterns of Plausible Inference". It is primarily concerned with the role of plausible reasoning in the discovery of mathematical facts. Two chapters in probability provide most of the illustrative examples.

Mathematical Discovery I, by G. Pólya, Wiley, 1965. (hardcover, 208 pp.)

This volume contains part one of the work, titled "Patterns", and the first two chapters of part two, titled "Toward a General Method". The first part gives practices for pattern recognition in geometric loci, analytic geometry, recurrence relations and interpolation. The two chapters in the second part discuss general philosophy in problem-solving.

Mathematical Discovery II, by G. Pólya, Wiley, 1965. (hardcover, 213 pp.)

This volume contains the remaining nine chapters of part two of the work. More specific advice on the art and science of problem-solving is offered. There is a chapter on learning, teaching and learning teaching.

Discovering Mathematics, by A. Gardiner, Oxford University Press, 1987. (paperback, 206 pp.)

This is a do-it-yourself package through which the reader can learn and develop methods of problem-solving. The first part of the book contains four short investigations, and the second part two extended ones. Each investigation is conducted via a structured sequence of questions.

Mathematical Puzzling, by A. Gardiner, Oxford University Press, 1987. (paperback, 157 pp.)

This book contains thirty-one chapters. The first twenty-nine are groups of related puzzles. The chapters are independent, except for three on counting and three on primes. In each chapter, commentaries follow the statements of the puzzles. Answers are given at the end of the book. Chapter 30 reexamines four of the earlier puzzles while Chapter 31 discusses the role of puzzles in mathematics.

Selected Problems and Theorems in Elementary Mathematics, by D.O. Shklyarsky, N.N. Chentsov and I.M. Yaglom, Mir Publications, 1979. (hardcover, 427 pp.)

This excellent book contains three hundred and fifty problems in arithmetic and algebra, many from papers of the U.S.S.R. Olympiads. It is the first of three volumes, but unfortunately, the volumes on plane geometry and solid geometry have not been translated into English.

All the Best from the Australian Mathematics Competition, edited by J.D. Edwards, D.J. King and P.J. O'Halloran, Australian Mathematics Competition, 1986. (paperback, 220 pp.)

This book contains four hundred and sixty-three multiple-choice questions taken from one of the world's most successful mathematics competitions. They are grouped by subject area to facilitate the study of specific topics.

The First Ten Canadian Mathematics Olympiads 1969-1978, edited by E. Barbeau and W. Moser, Canadian Mathematical Society, 1978. (paperback, 90 pp.)

This booklet contains the questions and solutions of the first ten Canadian Mathematics Olympiads. Each of the first five papers consists of ten questions. The numbers of questions in the remaining five vary between six and eight.

The Canadian Mathematics Olympiads 1979-1985, edited by C.M. Reis and S.Z. Ditor, Canadian Mathematical Society, 1987. (paperback, 47 pp.)

This booklet contains the questions and solutions of the Canadian Mathematics Olympiads from 1979 to 1985. Each paper consists of five questions.

1001 Problems in High School Mathematics, edited by E. Barbeau, M.S. Klamkin and W. Moser, Canadian Mathematical Society, 1985. (paperback, 430 pp.)

To date, half of this work has appeared in a preliminary version in the form of five booklets. In addition to problems and solutions, a mathematical "tool chest" is appended to each booklet.

Winning Ways I, by E.R. Berlekamp, J.H. Conway and R.K. Guy, Academic Press, 1982. (hardcover & paperback, 426 pp.)

This is the definitive treatise on mathematical games. As the subtitle "Games in General" suggests, the general theory of mathematical games is presented in this first volume, but there are also plenty of specific games to be analysed, played and enjoyed. The book is written with a great sense of humor, and is profusely illustrated, often in bright colours.

Winning Ways II, by E.R. Berlekamp, J.H. Conway and R.K. Guy, Academic Press, 1982. (hardcover & paperback, 424 pp.)

The subtitle of this volume is "Games in Particular". Here, all kinds of mathematical games, classical as well as brand new, are presented attractively. Most of them are two-player games. There are two chapters devoted to one-player games or solitaire puzzles, and the book concludes with a chapter on a zero-player game!

Puzzles Old and New, by J. Slocum and J. Botermans, distributed by University of Washington Press, 1986. (hardcover, 160 pp.)

Jerry Slocum has probably the largest puzzle collection in the world. This book features a small subset of his mechanical puzzles, that is, puzzles made of solid pieces that must be manipulated by hand to obtain a solution. They are classified into ten broad categories, with enough information to make most of them and to solve some of them. The book is full of striking full-colour plates.

The Mathematical Gardner, edited by D.A. Klarner, Wadsworth International, 1981. (hardcover, 382 pp.)

This book contains thirty articles dedicated to Martin Gardner for his sixty-fifth birthday. They reflect part of his mathematical interest, and are classified under the headings Games, Geometry, Two-Dimensional Tiling, Three-Dimensional Tiling, Fun and Problems, and Numbers and Coding Theory.

Mathematical Snapshots, by H. Steinhaus, Oxford University Press, 1983. (paperback, 311 pp.)

This is an outstanding book on significant mathematics presented in puzzle form. Topics include dissection theory, the golden ratio, numeration systems, tessellations, geodesics, projective geometry, polyhedra, Platonic solids, mathematical cartography, spirals, ruled surfaces, graph theory and statistics.

Mathematics Can Be Fun, by Y.I. Perelman, Mir Publishers, 1979. (hardcover, 400 pp.)

This is a translation of two books in Russian, "Figures for Fun" and "Algebra Can Be Fun". The former is an excellent collection of simple puzzles. The latter is a general discourse of algebra with quite a few digressions into number theory.

Fun with Maths and Physics, by Y.I. Perelman, Mir Publishers, 1984. (hardcover, 374 pp.)

The first half of this beautiful book describes a large number of interesting experiments in physics. The second half consists of a large collection of mathematical puzzles.

The Moscow Puzzles, by B.A. Kordemsky, Charles Scribner's Sons, 1972. (paperback, 309 pp.)

This is the translation of the outstanding single-volume puzzle collection in the history of Soviet mathematics. Many of the three hundred and fifty-nine problems are presented in amusing and charming story form, often with illustrations.

The Tokyo Puzzles, by K. Fujimura, Charles Scribner's Sons, 1978. (paperback, 184 pp.)

This is the translation of one of many books from the leading puzzlist of modern-day Japan. It contains ninety-eight problems, most of them previously unfamiliar to the western world.

536 Puzzles and Curious Problems, by H.E. Dudeney, Charles Scribner's Sons, 1967. (paperback, 428 pp.)

This book is a combination of two out-of-print works of the author, "Modern Puzzles" and "Puzzles and Curious Problems". Together with Dover's "Amusements in Mathematics", they constitute a substantial portion of Dudeney's mathematical problems. Those in this book are classified under three broad headings, arithmetic and algebra, geometry, and combinatorics and topology.

Science Fiction Puzzle Tales, by Martin Gardner, Clarkson N. Potter, 1981. (paperback, 148 pp.)

This is the first of three collections of Martin Gardner's contribution to Isaac Asimov's *Science Fiction Magazine*. The book contains thirty-six mathematical puzzles in science fiction settings. When solutions are presented, related questions are often raised.

Puzzles From Other Worlds, by Martin Gardner, Vintage Books, 1984. (paperback, 189 pp.)

This is the sequel to "Science Fiction Tales" and the predecessor of New Mathematical Library's "Riddles of the Sphinx". It contains thirty-seven mathematical puzzles plus further questions raised in the answer sections.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **October 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1621*. *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to Jack Garfunkel.)

Let P be a point within or on an equilateral triangle and let $c_1 \le c_2 \le c_3$ be the lengths of the three concurrent cevians through P. Determine the minimum value of c_2/c_3 over all P.

1622. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let n be a positive integer.

(a) Prove the inequality

$$\frac{a^{2n}+b^{2n}}{2} \le \left(\left(\frac{a+b}{2}\right)^2 + (2n-1)\left(\frac{a-b}{2}\right)^2\right)^n$$

for real a, b, and find conditions for equality.

(b) Show that the constant $2n-1$ in the right-hand expression is the best possible, in the sense that on replacing it by a smaller one we get an inequality which fails to hold for some a, b.

1623. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.* (Dedicated to Jack Garfunkel.)

Let ℓ be any line through vertex A of triangle ABC that is external to the triangle. Two circles with radii r_1 and r_2 are each external to the triangle and each tangent to ℓ and to line BC, and are respectively tangent to AB and AC.

(a) If $AB = AC$, prove that as ℓ varies, $r_1 + r_2$ remains constant and equal to the height of A above BC.

(b) If ΔABC is arbitrary, find constants k_1 and k_2, depending only on the triangle, so that $k_1r_1 + k_2r_2$ remains constant as ℓ varies.

1624*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $[n] = \{1, 2, \ldots, n\}$. Choose independently and at random two subsets A, B of $[n]$. Find the expected size of $A \cap B$. What if A and B must be *different* subsets?

1625. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Isosceles triangles $A_3A_1O_2$ and $A_1A_2O_3$ (with $O_2A_1 = O_2A_3$ and $O_3A_1 = O_3A_2$) are described on the sides A_3A_1 and A_1A_2 outside the triangle $A_1A_2A_3$. Point O_1 outside $\Delta A_1A_2A_3$ is such that $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$ and $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$. Prove that $A_1O_1 \perp O_2O_3$, and that

$$\overline{A_1O_1} : \overline{O_2O_3} = 2\overline{O_1T} : \overline{A_2A_3},$$

where T is the foot of the perpendicular from O_1 to A_2A_3.

1626. *Proposed by P. Penning, Delft, The Netherlands.*

Determine the average number of throws of a standard die required to obtain each face of the die at least once.

1627. *Proposed by George Tsintsifas, Thessaloniki, Greece.* (Dedicated to Jack Garfunkel.)

Two perpendicular chords MN and ET partition the circle (O, R) into four parts Q_1, Q_2, Q_3, Q_4. We denote by (O_i, r_i) the incircle of Q_i, $1 \le i \le 4$. Prove that

$$r_1 + r_2 + r_3 + r_4 \le 4(\sqrt{2} - 1)R.$$

1628*. *Proposed by Remy van de Ven, student, University of Sydney, Sydney, Australia.*

Prove that

$$(1-r)^k \sum_{i=1}^{\infty} \binom{k+i-1}{i} r^i \left(\frac{1}{k^2} + \frac{1}{(k+1)^2} + \cdots + \frac{1}{(k+i-1)^2} \right) = \sum_{i=1}^{\infty} \frac{r^i}{i^2 \binom{k+i-1}{i}},$$

where k is a positive integer.

1629. *Proposed by Rossen Ivanov, student, St. Kliment Ohridsky University, Sofia, Bulgaria.*

In a tetrahedron x and v, y and u, z and t are pairs of opposite edges, and the distances between the midpoints of each pair are respectively l, m, n. The tetrahedron has surface area S, circumradius R, and inradius r. Prove that, for any real number a with $0 \le a \le 1$,

$$x^{2a} v^{2a} l^2 + y^{2a} u^{2a} m^2 + z^{2a} t^{2a} n^2 \ge \left(\frac{\sqrt{3}}{4} \right)^{1-a} (2S)^{1+a} (Rr)^a.$$

1630. *Proposed by Isao Ashiba, Tokyo, Japan.*

Maximize

$$a_1 a_2 + a_3 a_4 + \cdots + a_{2n-1} a_{2n}$$

over all permutations $a_1, a_2, \ldots, a_{2n}$ of the set $\{1, 2, \ldots, 2n\}$.

* * * * *

SOLUTIONS

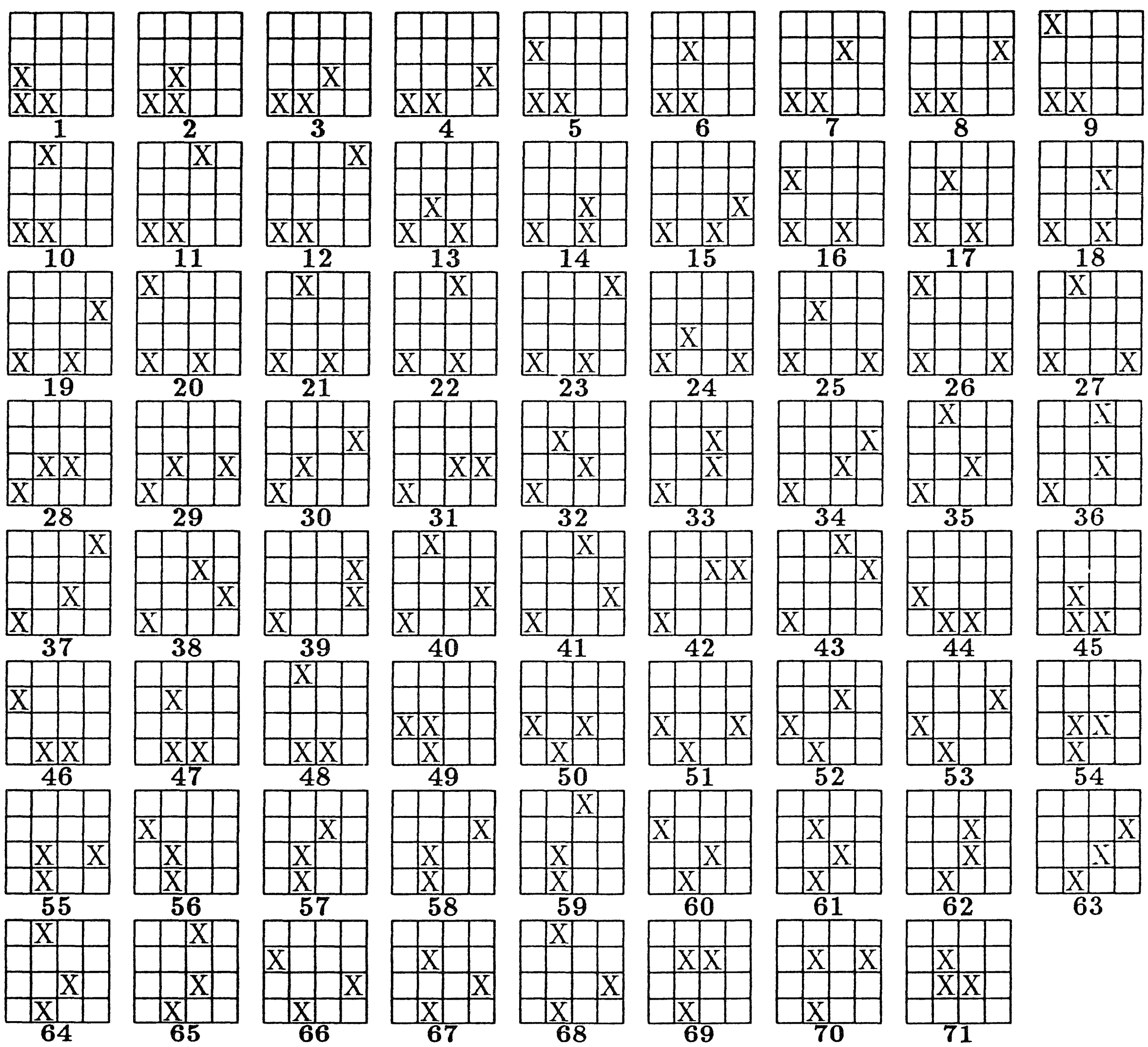
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

693*. [1981: 301; 1982: 314] *Proposed by Ferrell Wheeler, student, Texas A & M University.*

On a 4×4 tick-tack-toe board, a winning path consists of four squares in a row, column, or diagonal. In how many ways can three X's be placed on the board, not all on the same winning path, so that if a game is played on this partly-filled board, X going first, then X can absolutely force a win?

Solution by Sam Maltby, student, University of Calgary.

There are $\binom{16}{3} = 560$ possible arrangements for the first three X's, but 40 of these have them on a winning path (four possibilities for each of the four rows, four columns and two diagonals). The remaining 520 cases may be reduced to the following 71, by eliminating equivalent boards obtained by rotations and reflections.



These may be further reduced to 25 by applying the following two operations:

(i) interchanging the second and third columns and then the second and third rows, i.e.,

$$\begin{array}{|cccc|}\hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ \hline \end{array} \longrightarrow \begin{array}{|cccc|}\hline 1 & 3 & 2 & 4 \\ 9 & 11 & 10 & 12 \\ 5 & 7 & 6 & 8 \\ 13 & 15 & 14 & 16 \\ \hline \end{array} ;$$

(ii) the "inversion"

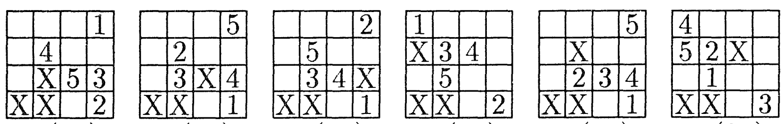
$$\begin{array}{|cccc|}\hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ \hline \end{array} \longrightarrow \begin{array}{|cccc|}\hline 6 & 5 & 8 & 7 \\ 2 & 1 & 4 & 3 \\ 14 & 13 & 16 & 15 \\ 10 & 9 & 12 & 11 \\ \hline \end{array} .$$

It is easy to verify that each of these operations leaves the set of ten winning rows, columns and diagonals fixed. Therefore if there is a winning strategy for X on a certain board, there will also be a winning strategy on any board resulting from these operations. Thus we need only examine one board from each equivalence class determined by these operations. There are 25 such equivalence classes, namely:

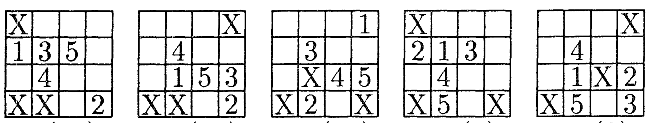
{**1**, 16, 49, 70}	{**2**, 18}	{**3**, 17, 31, 36}	{**4**, 19, 56, 67}	{**5**, 55}
{**6**, 14}	{**7**, 13, 29, 42}	{8, 15, 50, **58**}	{**9**, 20, 54, 69}	{**10**, 22, 45, 47}
{**11**, 21, 59, 65}	{**12**, 23, 57, 61}	{**24**, 25, 28, 33}	{**26**, 71}	{**27**, 62}
{**30**, 38}	{32, **37**}	{**34**, 35}	{**39**, 64}	{**40**, 43, 52, 63}
{**41**, 60}	{**44**, 46, 51, 68}	{**48**}	{**53**}	{**66**}

The representatives which we use are given in bold print.

For eleven of these boards, X can force a win by determining all of O's moves (i.e., making three X's in a line so that O must take the fourth), and ending up with two lines with three X's each so that O cannot block them both. In the following figure, as well as the original three X's on each of these eleven boards we also show X's successive moves by numbers 1.2,3 and so on. Also shown (in parentheses) is the number of the original 520 boards equivalent to each board.



2 (16) **3** (32) **4** (32) **5** (16) **6** (16) **7** (32)

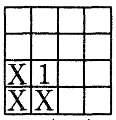
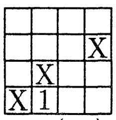
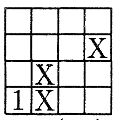
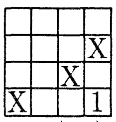
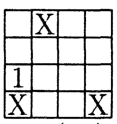
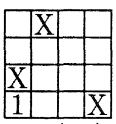


9 (32) **12** (32) **24** (32) **26** (8) **37** (8)

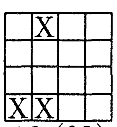
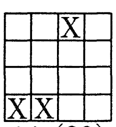
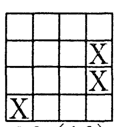
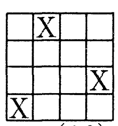
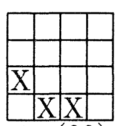
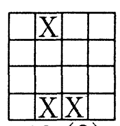
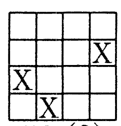
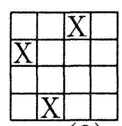
For six other cases, X has a winning strategy, but he cannot dictate O's moves. The positions in the following table are given by coordinates as follows:

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

Also given is X's first move.

	If O takes	**X takes**
 **1** (16)	33,34,43 or 44	11,12 and 14
	11,12,21 or 22	14,34 and 44
	13,14,23 or 24	11,22 and 33
 or  **30** (16) **58** (32)	11,13,31,33,34,43 or 44	23 and 22
	12,21 or 22	14 and 44
	14	44,22,21 and 11
	23	44,34,33 and 22
 **34** (16)	12,13,31,42 or 43	14,23 and 22
	14,32 or 34	22,23 and 43
	11,21 or 22	14,23 and 43
	23	34,31 and 11
 or  **27** (16) **41** (16)	13,14,23,24 or 34	42,22 and 11
	32	43,11,33 and 13
	33	11,32 and then either 14 or 42
	11 or 21	42,32,34 and 14
	42 or 43 { then 11,13 or 21	14 and { 34 and 32
	else	11
	22 { then 21	14 and { 23,13 and 43
	then 11 or 13	32 and 34
	else	11

This leaves eight classes:

 **10** (32)  **11** (32)  **39** (16)  **40** (16)  **44** (32)  **48** (8)  **53** (8)  **66** (8)

Having been unable to find a winning strategy for any of them and having convinced myself that there were none, I put the problem to my Atari 1040ST computer. After about 5 hours, it agreed that O can always force a tie in these situations. Therefore X has a winning strategy in 368 of the 520 starting positions.

*   *   *   *   *

1423*. [1989: 73; 1990: 145] *Proposed by Murray S. Klamkin, University of Alberta.*

Given positive integers k, m, n, find a polynomial $p(x)$ with real coefficients such that

$$(x-1)^n | (p(x))^m - x^k.$$

What is the least possible degree of p (in terms of k, m, n)?

II. *Comment by Rex Westbrook, University of Calgary.*

This is in response to the editor's request for a proof that the published answer

$$1+\sum_{j=1}^{n-1}\frac{\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-j+1)}{j!}(x-1)^j$$

can be written

$$\frac{\frac{k}{m}(\frac{k}{m}-1)\dots(\frac{k}{m}-n+1)}{(n-1)!}\sum_{i=0}^{n-1}\frac{(-1)^{n-1-i}}{\frac{k}{m}-i}\binom{n-1}{i}x^i.$$

Set the former expression equal to $f_n(x)$ and the latter equal to $g_n(x)$. The asked equality obviously holds if $k/m = t \le n-1$, t an integer, because then both expressions reduce to x^t. Otherwise, consider

$$\begin{aligned}
\frac{d}{dx}[x^{-k/m}f_n(x)] &= -\frac{k}{m}x^{-(\frac{k}{m}+1)} \\
&\quad+\sum_{j=1}^{n-1}\frac{\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-j+1)}{j!}\left[j(x-1)^{j-1}x^{-k/m}-\frac{k}{m}(x-1)^jx^{-\frac{k}{m}-1}\right] \\
&= x^{-(\frac{k}{m}+1)}\left[-\frac{k}{m}+\sum_{j=1}^{n-1}\frac{\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-j+1)}{j!}(x-1)^{j-1}\left(jx-\frac{k}{m}(x-1)\right)\right] \\
&= x^{-(\frac{k}{m}+1)}\left[-\frac{k}{m}+\sum_{j=1}^{n-1}\frac{\frac{k}{m}\cdots(\frac{k}{m}-j+1)}{j!}\left[-\left(\frac{k}{m}-j\right)(x-1)^j+j(x-1)^{j-1}\right]\right] \\
&= x^{-(\frac{k}{m}+1)}\left[-\frac{k}{m}+\sum_{j=1}^{n-1}\frac{\frac{k}{m}\cdots(\frac{k}{m}-j+1)}{(j-1)!}(x-1)^{j-1}\right. \\
&\qquad\left.-\sum_{j=1}^{n-1}\frac{\frac{k}{m}\cdots(\frac{k}{m}-j+1)(\frac{k}{m}-j)}{j!}(x-1)^j\right] \\
&= x^{-(\frac{k}{m}+1)}\left[\frac{-\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-(n-1))}{(n-1)!}(x-1)^{n-1}\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}[x^{-k/m}g_n(x)] &= \frac{\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-(n-1))}{(n-1)!}\frac{d}{dx}\left[\sum_{i=0}^{n-1}\frac{(-1)^{n-1-i}\binom{n-1}{i}}{\frac{k}{m}-i}x^{i-k/m}\right] \\
&= \frac{\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-(n-1))}{(n-1)!}(-1)x^{-(\frac{k}{m}+1)}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}x^i
\end{aligned}$$

$$= x^{-(\frac{k}{m}+1)}\left[\frac{-\frac{k}{m}(\frac{k}{m}-1)\cdots(\frac{k}{m}-(n-1))}{(n-1)!}(x-1)^{n-1}\right].$$

Therefore

$$\frac{d}{dx}\left[x^{-k/m}(g_n(x)-f_n(x))\right]=0,$$

so

$$x^{-k/m}(g_n(x)-f_n(x)) = \text{ constant.}$$

If either k/m is not an integer, or is an integer $\geq n$, then since $g_n - f_n$ is a polynomial (of degree at most $n-1$) the above constant must be 0, i.e. $g_n \equiv f_n$.

* * * * *

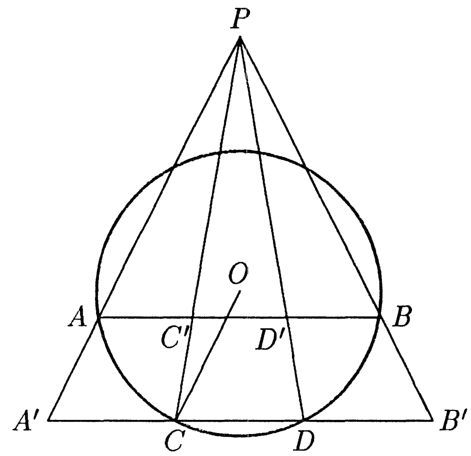
1502. [1990: 19] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

AB is a chord, not a diameter, of a circle with centre O. The smaller arc AB is divided into three equal arcs AC, CD, DB. Chord AB is also divided into three equal segments $AC', C'D', D'B$. Let CC' and DD' intersect in P. Show that $\angle APB = \frac{1}{3}\angle AOB$.

Solution by Mark Kisin, student, Monash University, Clayton, Australia.

Let

$$A' = PA \cap CD, \quad B' = PB \cap CD.$$



Then

$$\Delta PAC' \sim \Delta PA'C$$

and

$$\Delta PC'D' \sim \Delta PCD,$$

so that

$$\frac{AC'}{A'C} = \frac{PC'}{PC} = \frac{C'D'}{CD}.$$

But $AC' = C'D'$, so $A'C = CD = AC$. Therefore

$$\angle BAC = \angle ACA' = \pi - \angle CAA' - \angle CA'A = \pi - 2\angle CA'A$$
$$= \pi - \angle CA'A - \angle BB'D = \angle APB.$$

But C and D trisect the arc AB, so

$$\angle APB = \angle BAC = \frac{1}{2}\angle BOC = \frac{1}{2}\cdot\frac{2}{3}\angle AOB = \frac{1}{3}\angle AOB.$$

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; L. J. HUT, *Groningen, The Netherlands; SHIKO IWATA, Gifu, Japan; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; T. LEINSTER, Lansing College, England; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; HUME SMITH, Chester, Nova Scotia; and the proposer.*

The nice solution of Kisin was also given by Festraets-Hamoir.

The proposer found the problem in Mathématiques Élémentaires, *Paris, 1915, no. 8076.*

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1504. [1990: 19] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $A_1A_2\cdots A_n$ be a circumscribable n-gon with incircle of radius 1, and let $F_1, F_2, \ldots, F_n$ be the areas of the n corner regions inside the n-gon and outside the incircle. Show that

$$\frac{1}{F_1}+\cdots+\frac{1}{F_n} \geq \frac{n^2}{n\tan(\pi/n)-\pi}.$$

Equality holds for the regular n-gon.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Write $\angle A_{i-1}OA_i = 2x_i$ ($A_0 = A_n$), where O is the incenter. Then $\sum x_i = \pi$, $0 < x_i < \pi/2$ and $F_i = \tan x_i - x_i$. Applying Jensen's inequality

$$\sum f(x_i) \geq nf(\sum x_i/n)$$

to the (strictly) convex function

$$f(x) = \frac{1}{\tan x - x},$$

we get the claim. The convexity of f is justified, for example, by

$$f'(x) = -(1 - x\cot x)^{-2}.$$

As x grows from 0 to $\pi/2$, $x\cot x$ falls, hence $f'(x)$ grows.

Also solved (the same way) by DUANE M. BROLINE, Eastern Illinois University, Charleston; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; and the proposer.

The proposer has since discovered that the result occurs (without solution) as item 2.38, *p.* 689 *in the Appendix of Mitrinović, Pečarić, and Volenec,* Recent Advances in Geometric Inequalities, *where it is mentioned that the case $n = 3$ was a problem in* Math. Gazette *(#* 70.G, *solution in volume* 71(1987) 148–149*).*

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1505. [1990: 19] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $x_1 = 1$ and

$$x_{n+1} = \frac{1}{x_n}\left(\sqrt{1+x_n^2}-1\right).$$

Show that the sequence $(2^n x_n)$ converges and find its limit.

Solution by Jeff Higham, student, University of Toronto.

We first show by induction on n that

$$x_n = \tan(\pi/2^{n+1}) \tag{1}$$

for all $n \in \mathbf{N}$. Since $x_1 = 1 = \tan(\pi/4)$, (1) is true for $n = 1$. Suppose (1) is true for $n = k$. Then

$$\begin{aligned} x_{k+1} &= \frac{1}{\tan(\pi/2^{k+1})}\left(\sqrt{1+\tan^2(\pi/2^{k+1})}-1\right) \\ &= \frac{\sec(\pi/2^{k+1})-1}{\tan(\pi/2^{k+1})} = \frac{1-\cos(\pi/2^{k+1})}{\sin(\pi/2^{k+1})} \\ &= \tan(\frac{1}{2}\cdot\pi/2^{k+1}) = \tan(\pi/2^{k+2}), \end{aligned}$$

so (1) is true for all n.

Let $y = 1/2^n$. Then as $n \to \infty$, $y \to 0^+$, so by L'Hôpital's rule

$$\lim_{n\to\infty} 2^n x_n = \lim_{y\to 0+}\frac{\tan(\pi y/2)}{y} = \lim_{y\to 0+}\frac{\pi}{2}\cdot\frac{\sec^2(\pi y/2)}{1} = \frac{\pi}{2}.$$

Also solved by H. L. ABBOTT, University of Alberta; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; DUANE M. BROLINE, Eastern Illinois University, Charleston; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARK KISIN, student, Monash University, Clayton, Australia; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University; and the proposer. Two incorrect solutions were sent in.

All solutions were somewhat like the above. Abbott and the proposer note that x_n is half the side of a regular polygon of 2^{n+1} sides which is circumscribed about a unit circle, so $2^n x_n$ will approach $1/4$ of the circumference of this circle. Battles points out the similar problem 1214 *of* Mathematics Magazine, *solution in Vol.* 59*(April 1986), pp.* 117–118.

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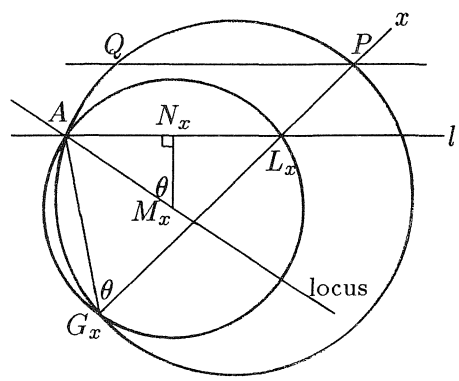
1506. [1990: 19] *Proposed by Jordi Dou, Barcelona, Spain.*

Let A and P be points on a circle Γ. Let l be a fixed line through A but not through P, and let x be a variable line through P which cuts l at L_x and Γ again at G_x. Find the locus of the circumcentre of ΔAL_xG_x.

Solution by P. Penning, Delft, The Netherlands.



Let $\theta = \angle AG_xP = \text{arc}(AP)/2$, which is independent of x. Let M_x be the circumcentre of ΔAG_xL_x and N_x the projection of M_x on l. Since $\text{arc}(AL_x) = 2\theta$, $\angle AM_xN_x$ is also equal to θ, and so

$$\angle M_xAN_x = \frac{\pi}{2} - \theta,$$

independent of x. So all circumcentres lie on one and the same straight line through A. [*Editor's note.* The diagram illustrates the case that θ is acute. All this still works, with minor changes, if G_x lies on the minor arc AP.]

To construct the locus: when x is parallel to l let $G_x = Q$, so that PQ is parallel to l; the circumcircle is then the straight line AQ, and the locus the straight line through A perpendicular to AQ.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; DAN PEDOE, Minneapolis, Minnesota; D.J. SMEENK, Zaltbommel, The Netherlands; HUME SMITH, Chester, Nova Scotia; and the proposer.

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1507. [1990: 20] *Proposed by Nicos D. Diamantis, student, University of Patras, Greece.*

Find a real root of

$$y^5 - 10y^3 + 20y - 12 = 0.$$

I. *Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

Making the substitution

$$y = 2\sqrt{2}\cosh\theta$$

in the equation of the problem statement and dividing each term by $8\sqrt{2}$ gives

$$16\cosh^5\theta - 20\cosh^3\theta + 5\cosh\theta = \frac{3\sqrt{2}}{4}.$$

Using the well-known multiple angle formula for hyperbolic functions, we obtain

$$\cosh 5\theta = \frac{3\sqrt{2}}{4}.$$

Now, since

$$\cosh^{-1} u = \ln(u \pm \sqrt{u^2 - 1}),$$

we find

$$5\theta = \ln\left(\frac{3\sqrt{2}}{4} \pm \frac{\sqrt{2}}{4}\right) = \pm\frac{1}{2}\ln 2,$$

so $\theta = \pm 0.1 \ln 2$. Thus

$$\begin{aligned} y &= 2\sqrt{2}\cdot\frac{1}{2}(e^{\theta} + e^{-\theta}) = \sqrt{2}(e^{0.1\ln 2} + e^{-0.1\ln 2}) \\ &= \sqrt{2}(2^{0\,1} + 2^{-0.1}) = 2^{0.6} + 2^{0.4}. \end{aligned}$$

II. *Solution by Murray S. Klamkin, University of Alberta.*

More generally ([1], [2]), it is known that the equation

$$y^5 - 5py^3 + 5p^2y - a = 0$$

is solvable in radicals. Letting $y = t + p/t$, the equation reduces to

$$t^5 + \frac{p^5}{t^5} = a$$

so that

$$2t^5 = a \pm \sqrt{a^2 - 4p^5}.$$

Here $p = 2$ and $a = 12$, so $t^5 = 8$ or 4 and the real root is

$$y = 2^{2/5} + 2^{3/5}.$$

Similarly, by using the same substitution, we can solve

$$y^7 - 7py^5 + 14p^2y^3 - 7p^3y - a = 0$$

in terms of the seventh roots of unity (due to L. E. Dickson).

References:

[1] W.S. Burnside and A.W. Panton, *The Theory of Equations I*, Dover, 1960, p. 104.
[2] Problem 86-3, *Mathematical Intelligencer* 8 (1986) 31,33.

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MATHEW ENGLANDER, Toronto, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; JEFF HIGHAM, student, University of Toronto; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. Four other readers sent in approximate solutions (one taken to over 70 decimal places!). There was also one incorrect solution received.

Klamkin also mentions the related Monthly *problem* 4568 *of A.W. Walker (solution in Vol.* 62 (1955) *pp.* 189–190*). Williams notes his short article in* Mathematical Gazette 46 (1962) 221–223, *which proves the same results given above by Klamkin. Higham observes that the polynomial*

$$y = x^5 - 10x^3 + 20x - 12$$

of the problem has the interesting property that its two local maxima have equal y-values and its two local minima also have equal y-values. According to expert colleague Len Bos, this is true because the above polynomial is actually

$$4\sqrt{8}\cdot T_5(x/\sqrt{8})-12,$$

where T_5 is the fifth Chebyshev polynomial, which is well known to have this property.

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1508. [1990: 20] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $a \le b < c$ be the lengths of the sides of a right triangle. Find the largest constant K such that

$$a^2(b+c)+b^2(c+a)+c^2(a+b)\ge Kabc$$

holds for all right triangles and determine when equality holds. It is known that the inequality holds when $K=6$ (problem 351 of the *College Math. Journal*; solution on p. 259 of Volume 20, 1989).

Solution by T. Leinster, Lansing College, England.

Let θ be the angle opposite the side with length a; then $0<\theta\le\pi/4$. Now

$$\begin{aligned}\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{abc} &= \frac{a}{c}+\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}\\ &= \sin\theta+\tan\theta+\cot\theta+\cos\theta+\sec\theta+\csc\theta\\ &= f(\theta),\text{ say.}\end{aligned}$$

Then

$$f'(\theta)=\cos\theta+\sec^2\theta-\csc^2\theta-\sin\theta+\sec\theta\tan\theta-\csc\theta\cot\theta,$$

so putting $s=\sin\theta$, $c=\cos\theta$, $f'(\theta)=0$ implies

$$\begin{aligned}0 &= s^2c^3+s^2-c^2-s^3c^2+s^3-c^3\\ &= (s-c)(s+c+s^2+sc+c^2-s^2c^2)\\ &= (s-c)(s+c+s^2+sc+c^4).\end{aligned}$$

Since each component of the second term is ≥ 0 for $0<\theta\le\pi/4$, and they cannot all be 0 at once, the only turning point of $f(\theta)$ in this range is where $s-c=0$, i.e., $\theta=\pi/4$. $f(\theta)$ is continuous over $(0,\pi/4]$, and $f(\theta)\to+\infty$ as $\theta\to 0$ from above, therefore $\theta=\pi/4$ gives the lowest possible value for $f(\theta)$ in this range. So

$$K_{\max}=f(\pi/4)=2+3\sqrt{2},$$

and equality holds for the isosceles right-angled triangle.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; C. FESTRAETS-HAMOIR, *Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; and the proposer.*

For a related problem, see Crux 1227 [1988: 148].

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1509. [1990: 20] *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

Professor Chalkdust teaches two sections of a mathematics course, with the same material taught in both sections. Section 1 runs on Mondays, Wednesdays, and Fridays for 1 hour each day, and Section 2 runs on Tuesdays and Thursdays for 1.5 hours each day. Normally Professor Chalkdust covers one unit of material per hour, but if she is teaching some material for the second time she teaches twice as fast. The course began on a Monday. In the long run (i.e. after N weeks as $N \to \infty$) will one section be taught more material than the other? If so, which one, and how much more?

Solution by Jeff Higham, student, University of Toronto.

Let x be the number of units of material covered by Section 1 (from the start of the course) minus the number of units of material covered by Section 2. We will show by induction that, in week n,

$$x = \begin{cases} (7 - 2^{5-4n})/5 & \text{after Monday,} \\ (-4 - 2^{4-4n})/5 & \text{after Tuesday,} \\ (3 - 2^{3-4n})/5 & \text{after Wednesday,} \\ (-6 - 2^{2-4n})/5 & \text{after Thursday,} \\ (2 - 2^{1-4n})/5 & \text{after Friday.} \end{cases} \tag{1}$$

(1) is easily verified when $n = 1$. Suppose (1) is true for $n = k$. Then on Monday of week $k+1$, Section 1 covers only "new material" (material not yet covered by the other section), since $x = (2 - 2^{1-4k})/5 > 0$ on the previous Friday by the induction hypothesis. Thus, after this Monday,

$$x = \frac{2 - 2^{1-4k}}{5} + 1 = \frac{7 - 2^{1-4k}}{5} = \frac{7 - 2^{5-4(k+1)}}{5}.$$

On Tuesday of week $k+1$, Section 2 covers the $(7 - 2^{5-4(k+1)})/5$ units of "old material" (material already covered by the other section) in $(7 - 2^{5-4(k+1)})/10 < 1.5$ hours. Thus, for the remaining

$$\frac{3}{2} - \frac{7 - 2^{5-4(k+1)}}{10} = \frac{8 + 2^{5-4(k+1)}}{10} = \frac{4 + 2^{4-4(k+1)}}{10}$$

hours, $(4 + 2^{4-4(k+1)})/5$ units of new material is covered, so $x = (-4 - 2^{4-4(k+1)})/5$ after this Tuesday. On Wednesday of week $k+1$, old material is covered in $(4+2^{4-4(k+1)})/10 < 1$ hours and new material in

$$1 - \frac{4 + 2^{4-4(k+1)}}{10} = \frac{6 - 2^{4-4(k+1)}}{10} = \frac{3 - 2^{3-4(k+1)}}{5}$$

hours, so $x = (3 - 2^{3-4(k+1)})/5$ after this Wednesday. On Thursday, old material is covered in $(3 - 2^{3-4(k+1)})/10 < 1.5$ hours, and new material in

$$\frac{3}{2} - \frac{3 - 2^{3-4(k+1)}}{10} = \frac{12 + 2^{3-4(k+1)}}{10} = \frac{6 + 2^{2-4(k+1)}}{5}$$

hours, so $x = (-6 - 2^{2-4(k+1)})/5$ after this Thursday. Finally, on Friday, old material is covered in $(6 + 2^{2-4(k+1)})/10 < 1$ hours, and new material in

$$1 - \frac{6 + 2^{2-4(k+1)}}{10} = \frac{4 - 2^{2-4(k+1)}}{10} = \frac{2 - 2^{1-4(k+1)}}{5}$$

hours, so $x = (2 - 2^{1-4(k+1)})/5$ after this Friday. This completes the induction argument.

Clearly as $n \to \infty$, $x \to 2/5$ after Friday; therefore in the long run Section 1 is ahead of Section 2 by 2/5 units after Friday.

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; CURTIS COOPER, Central Missouri State University, Warrensburg; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; HUME SMITH, student, University of British Columbia; and the proposer. One incorrect solution was received which was apparently due to the writer not understanding the problem.

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1510*. [1990: 20] *Proposed by Jack Garfunkel, Flushing, New York.*

P is any point inside a triangle ABC. Lines PA, PB, PC are drawn and angles PAC, PBA, PCB are denoted by α, β, γ respectively. Prove or disprove that

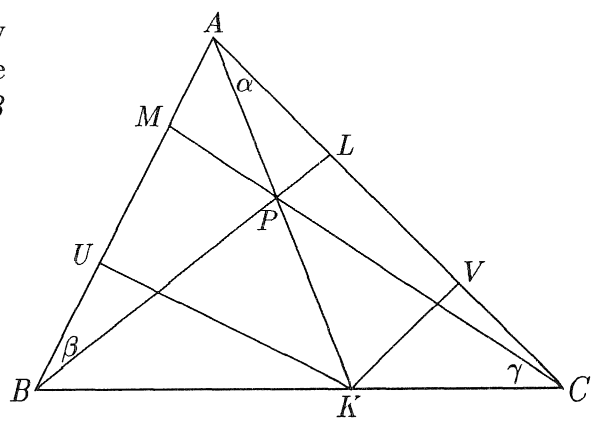
$$\cot\alpha + \cot\beta + \cot\gamma \geq \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2},$$

with equality when P is the incenter of ΔABC.

Solution by Marcin E. Kuczma, Warszawa, Poland.

I choose to disprove.

Lines AP, BP, CP cut the boundary of triangle ABC in K, L, M. Let U, V be the feet of perpendiculars from K to lines AB and CA, respectively. Then



$$\begin{aligned}\cot\alpha - \cot A &= \frac{\sin(A-\alpha)}{\sin A \sin\alpha} \\ &= \frac{1}{\sin A} \cdot \frac{KU}{KV} \\ &= \frac{1}{\sin A} \cdot \frac{BK \sin B}{KC \sin C}.\end{aligned}$$

Hence (in cyclic notation)

$$\sum(\cot\alpha - \cot A) = \sum \frac{BK}{KC}\cdot\frac{\sin B}{\sin C\sin A}$$
$$\geq 3\left(\prod \frac{BK}{KC}\cdot\frac{\sin B}{\sin C\sin A}\right)^{1/3}$$
$$= 3(\sin A\sin B\sin C)^{-1/3}, \qquad (1)$$

by the means inequality and Ceva's theorem. Equality holds if and only if each one of the summands in (1) equals $(\sin A\sin B\sin C)^{-1/3}$, i.e. (writing a, b, c for the side lengths), when

$$\frac{BK}{KC} = \left(\frac{ca}{b^2}\right)^{2/3}, \quad \frac{CL}{LA} = \left(\frac{ab}{c^2}\right)^{2/3}, \quad \frac{AM}{MB} = \left(\frac{bc}{a^2}\right)^{2/3}. \qquad (2)$$

In *every* triangle ABC there exists a unique point P for which (2) holds (pick K, L, M to partition BC, CA, AB in ratios as in (2); lines AK, BL, CM are concurrent by the inverse Ceva theorem). For this point the sum $\sum\cot\alpha$ attains minimum, equal by (1) to

$$\sum\cot A \;+\; 3(\sin A\sin B\sin C)^{-1/3}.$$

When P is the incenter, the ratios BK/KC, etc. are c/b, a/c, b/a, hence they differ from those of (2), unless the triangle is regular.

For instance, in the isosceles right triangle of vertices $A = (1,0)$, $B = (0,1)$, $C = (0,0)$, the optimal $P = (x,y)$ has $x = 2^{1/3} - 1 = 0.259\ldots$, $y = 2^{2/3} - 2^{1/3} = 0.327\ldots$ and gives the minimum value

$$\cot\alpha = 2 + 3\cdot 2^{1/3} = 5.779\ldots;$$

compare with the incenter $I = (u,v)$, $u = v = 1 - 2^{-1/2} = 0.292\ldots$, producing

$$\cot(A/2) = 3 + 2^{3/2} = 5.828\ldots.$$

Also solved by G. P. HENDERSON, Campbellcroft, Ontario.

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1511. [1990: 43] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*
Evaluate

$$\lim_{n\to\infty}\prod_{k=3}^{n}\left(1 - \tan^4\frac{\pi}{2^k}\right).$$

Solution by Beatriz Margolis, Paris, France.
Observe that

$$1 - \tan^4(\pi/2^k) = (1 - \tan^2(\pi/2^k))(1 + \tan^2(\pi/2^k))$$
$$= \frac{\cos^2(\pi/2^k) - \sin^2(\pi/2^k)}{\cos^2(\pi/2^k)}\cdot\frac{1}{\cos^2(\pi/2^k)}$$
$$= \frac{\cos(\pi/2^{k-1})}{\cos(\pi/2^k)}\cdot\left(\frac{2\sin(\pi/2^k)}{2\sin(\pi/2^k)\cos(\pi/2^k)}\right)^3$$
$$= \frac{\cos(\pi/2^{k-1})}{\cos(\pi/2^k)}\cdot\left(\frac{2\sin(\pi/2^k)}{\sin(\pi/2^{k-1})}\right)^3.$$

Hence

$$\begin{aligned}\prod_{k=3}^{n}(1-\tan^4(\pi/2^k)) &= \prod_{k=3}^{n}\frac{\cos(\pi/2^{k-1})}{\cos(\pi/2^k)}\cdot\left(\prod_{k=3}^{n}\frac{2\sin(\pi/2^k)}{\sin(\pi/2^{k-1})}\right)^3\\ &= \frac{\cos(\pi/2^2)}{\cos(\pi/2^n)}\cdot\left(2^{n-2}\frac{\sin(\pi/2^n)}{\sin(\pi/2^2)}\right)^3\\ &= A_n\cdot B_n^3.\end{aligned}$$

Now

$$\lim_{n\to\infty}A_n=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

and

$$\lim_{n\to\infty}B_n=\frac{1}{\sin(\pi/4)}\cdot\frac{\pi}{4}\cdot\lim_{n\to\infty}\frac{\sin(\pi/2^n)}{\pi/2^n}=\frac{\sqrt{2}\pi}{4},$$

so that

$$\lim_{n\to\infty}\prod_{k=3}^{n}\left(1-\tan^4\frac{\pi}{2^k}\right)=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}\pi}{4}\right)^3=2\left(\frac{\pi}{4}\right)^3.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; CURTIS COOPER, Central Missouri State University, Warrensburg; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; T. LEINSTER, Lansing College, England; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. One other reader sent in an approximation.

Janous, Klamkin, Penning, and the proposer in fact prove the more general result

$$\lim_{n\to\infty}\prod_{k=1}^{n}(1-\tan^4\frac{\pi}{2^k})=\frac{x^3\cos x}{\sin^3 x},$$

which with $x=\pi/4$ becomes the given problem, and which can be shown as above.

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1512*. [1990: 43] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given $r>0$, determine a constant $C=C(r)$ such that

$$(1+z)^r(1+z^r)\le C(1+z^2)^r$$

for all $z>0$.

I. *Solution by Richard Katz, California State University, Los Angeles.*
Let

$$f(x) = \frac{(1+x)^r(1+x^r)}{(1+x^2)^r}$$

and $S = S(r) = \sup\{f(x) \mid x \geq 0\}$. Clearly any constant C satisfying $C \geq S$ will solve the problem and S is the smallest possible value of C. Now

$$f(x) = \left(\frac{1+x}{1+x^2}\right)^r + \left(\frac{x+x^2}{1+x^2}\right)^r = (g(x))^r + (h(x))^r,$$

where

$$g(x) = \frac{1+x}{1+x^2} \quad , \quad h(x) = \frac{x+x^2}{1+x^2} = g\left(\frac{1}{x}\right).$$

Thus $f(x) = f(1/x)$, and so

$$S = \sup\{f(x) \mid 0 \leq x \leq 1\}.$$

Hence

$$\begin{aligned} S &= \sup_{0 \leq x \leq 1} \{(g(x))^r + (h(x))^r\} \\ &\leq \sup_{0 \leq x \leq 1} \{(g(x))^r\} + \sup_{0 \leq x \leq 1} \{(h(x))^r\} = [\sup_{0 \leq x \leq 1} g(x)]^r + [\sup_{0 \leq x \leq 1} h(x)]^r. \end{aligned}$$

Now

$$\sup\{g(x) \mid 0 \leq x \leq 1\} = g(\sqrt{2}-1) = \frac{1+\sqrt{2}}{2}$$

and

$$\sup\{h(x) \mid 0 \leq x \leq 1\} = h(1) = 1,$$

as are easily checked. Therefore

$$\left(\frac{1+\sqrt{2}}{2}\right)^r < S(r) < \left(\frac{1+\sqrt{2}}{2}\right)^r + 1$$

and so

$$C(r) = \left(\frac{1+\sqrt{2}}{2}\right)^r + 1$$

works and differs by less than 1 from the best possible value.

This result can be improved slightly. By writing the equation $f'(x) = 0$ in the form

$$-\frac{g'(x)}{h'(x)} = \left(\frac{h(x)}{g(x)}\right)^{r-1}$$

or

$$-\frac{x^2+2x-1}{x^2-2x-1} = x^{r-1},$$

the following can be deduced (details are straightforward but tedious):

(i) $S(r) = 2$ for $r \leq 3$;

(ii) $S(r) - (1+\sqrt{2})^r/2^r \sim 1/2^r$.

Hence for r sufficiently large (calculations suggest $r \geq 3$), one may take

$$C(r) = \left(\frac{1+\sqrt{2}}{2}\right)^r + \frac{1}{2^{r-1}},$$

and this differs from the best possible value by less than $1/2^r$.

II. *Partial solution by Murray S. Klamkin, University of Alberta.*

Letting $z = 1$, it follows that $C \geq 2$. We show that for $C = 2$, the inequality is at least valid for $r \leq 3$.

Letting $r = 3$, we have to show that

$$2(z^2+1)^3 - (z+1)^3(z^3+1) \geq 0.$$

Since the polynomial equals $(z-1)^3(z^3-1) = (z-1)^4(z^2+z+1)$, the inequality is valid. Equivalently,

$$\frac{1+z}{2} \cdot \left(\frac{1+z^3}{2}\right)^{1/3} \leq \frac{1+z^2}{2}.$$

We now establish the inequality for $0 < r < 3$, i.e.,

$$\frac{1+z}{2} \cdot \left(\frac{1+z^r}{2}\right)^{1/r} \leq \frac{1+z^2}{2}.$$

By the power mean inequality, for $0 < r < 3$,

$$\frac{1+z}{2} \cdot \left(\frac{1+z^r}{2}\right)^{1/r} \leq \frac{1+z}{2} \cdot \left(\frac{1+z^3}{2}\right)^{1/3} \leq \frac{1+z^2}{2}.$$

The given inequality for $C = 2$ and $r = 4$ is invalid. For by expanding out we should have

$$z^8 - 4z^7 + 2z^6 - 4z^5 + 10z^4 - 4z^3 + 2z^2 - 4z + 1 \geq 0.$$

This inequality is invalid since the polynomial factors into

$$(z-1)^2(z^6 - 2z^5 - 3z^4 - 8z^3 - 3z^2 - 2z + 1),$$

which takes on negative values for z close to 1.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; DAVID C. VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; and CHRIS WILDHAGEN, Breda, The Netherlands.

Kuczma and Vaughan show that

$$C(r) > \left(\frac{1+\sqrt{2}}{2}\right)^r + \frac{1}{2^r},$$

(Vaughan only for "large" r), and that ">" could be replaced by "$\sim$" as in (ii) *of Katz's proof. Wildhagen predicts the same approximation based on computer results. Kuczma also gives the explicit upper bound*

$$C(r) = \left(\frac{1+\sqrt{2}}{2}\right)^r \left(1+\frac{1}{2^r}\right)$$

for $r \geq 4$. *All solvers found* $C = 2$ *for* $0 < r \leq 3$.

* * * * *

LETTER TO THE EDITOR

Crux readers will be interested to learn that I am in the process of creating an index to the mathematical problems that appear in the problem columns of many mathematical journals. In particular, the problem column from *Crux* will be indexed in my book. The first volume of my index will cover the years 1980–1984 and will include all problems published in various problem columns in those years, sorted by topic, as well as an author and title index.

As part of the project, I have compiled a list of journals that include problem sections. The list currently contains almost 150 journals from around the world, with bibliographic data. The list includes both current journals and journals that have ceased publication. If any readers would like more information about my indexing project, and/or a free copy of my list of journals with problem columns, they should write to me at the address below.

Stanley Rabinowitz, *Crux* offer
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