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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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THE OLYMPIAD CORNER

No. 122

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

It has been some time since we published a problem set from France, and in this issue we give the problems of the 1988 sitting of the French national competition. Thanks to Bruce Shawyer, Memorial University of Newfoundland, for sending them in.

COMPOSITION DE MATHÉMATIQUES

Session de 1988

Time: 5 hours

Problem I. Let N, p, n be non-zero whole numbers. Consider a rectangular matrix T having n lines numbered 1 through n, and p columns numbered 1 through p. For $1 \leq i \leq n$ and $1 \leq k \leq p$, the entry in row i and column k is an integer a_{ik} satisfying $1 \leq a_{ik} \leq N$. Let E_i be the set of integers appearing in row i.

Answer Question 1 or 2.

Question 1.

In this question two further conditions are imposed on T:

i. for $1 \leq i \leq n$, E_i has exactly p elements;

ii. for different values of i and j, the sets E_i and E_j are different.

Let m be the smallest value of N for which, given values for n and p, one can form a matrix T having the preceding properties.

(a) Calculate m for $n = p + 1$.

(b) Calculate m for $n = 10^{30}$ and $p = 1988$.

(c) Determine the limit of m^p/n where p is fixed and n tends to infinity.

Question 2.

In this question we replace the two extra conditions of Question 1 by the two following conditions:

i. $p = n$;

ii. for every ordered pair of positive integers (i, k) with $i + k \leq n$, the integer a_{ik} does not belong to the set E_{i+k}.

(a) Show that for distinct i and j the sets E_i and E_j are different.

(b) Show that if n is at least 2^q, where q is a positive integer, then $N \geq q + 1$.

(c) Suppose that $n = 2^r - 1$, where r is a fixed positive integer. Show that $N \geq r$.

Exercise II. Determine, for n a positive integer, the sign of $n^6 + 5n^5 \sin n + 1$. For which positive integers n is it true that

$$\frac{n^2 + 5n\cos n + 1}{n^6 + 5n^5 \sin n + 1} \geq 10^{-4}?$$

Exercise III. Consider two spheres Σ_1 and Σ_2 and a straight line Δ which does not meet them. For $i = 1$ and $i = 2$, let C_i be the centre of Σ_i, H_i the orthogonal projection of C_i on Δ, r_i the radius of Σ_i, and let d_i be the distance of C_i to Δ. Let M be a point on Δ, and for $i = 1$ and $i = 2$, let T_i be the point of contact with Σ_i of a plane tangent to Σ_i and passing through M; set $\delta_i(M) = MT_i$. Situate M on Δ so that $\delta_1(M) + \delta_2(M)$ is minimized.

Exercise IV. Consider five points M_1, M_2, M_3, M_4, M situated on a circle C in the plane. Show that the product of the distances of M from the lines M_1M_2 and M_3M_4 equals the product of the distances from M to the lines M_1M_3 and M_2M_4. What can one deduce about $2n+1$ distinct points $M_1, \ldots, M_{2n}, M$ situated on C?

* * *

Now we turn to solutions sent in response to the appeal to help "tidy up" the archives.

K797. [1983: 270] *Problems from Kvant.*

It is well known that the last digit of the square of an integer is one of the following: 0,1,4,5,6,9. Is it true that any finite sequence of digits may appear before the last one, that is, for any sequence of n digits $\{a_1, a_2, \ldots, a_n\}$ there exists an integer whose square ends with the digits $a_1a_2\ldots a_nb$, where b is one of the digits listed above?

Solutions by Andy Liu, University of Alberta and by Richard K. Guy, University of Calgary.

The answer is no. The sequence of digits 101 cannot precede the units digit in any square. Suppose on the contrary that there exist integers k and y, and a digit x in $\{0, 1, \ldots, 9\}$, such that $k^2 = 10^4y + 1010 + x$. We know that $k^2 \equiv 0$ or $1 \bmod 4$. Now $10^4y + 1010 + x \equiv 2 + x \bmod 4$. Hence $x = 2$, 3, 6 or 7. However, k^2 cannot end in 2, 3, or 7. Hence $k^2 = 10^4y + 1016$. We must have $k = 2h$ for some integer h, and then $h^2 = 2500y + 254 \equiv 2 \bmod 4$, a contradiction.

Alternatively, consider any sequence of the form $40k + 39$. Squares cannot end as $\ldots 90, \ldots 91, \ldots 94, \ldots 95$ or $\ldots 99$, and $400k + 396 = (10x \pm 4)^2$ gives $20k + 19 = 5x^2 \pm 4x$, which implies that x is odd and $19 \equiv 5 \bmod 4$, a contradiction.

*

Turning now to the 1985 problems from the Corner, I would like to thank Murray Klamkin who sent in a large number of solutions as well as references to some solutions which have appeared in his book *International Mathematical Olympiads 1979–1985*, MAA, Washington, D.C., 1986. Here is a listing of those problems from the 1985 numbers for which solutions are discussed in his book.

p.38/#16	Listed as G.I./5	on pp. 13, 88-91.
p.38/#21	Listed as S.G./2	on pp. 12, 82.
p.72/#40	Listed as G.I./1	on pp. 12, 83–84.
p.72/#41	Listed as P.G./4	on pp. 12, 80–81.

p.72/#42	Listed as N.T./5	on pp. 11, 76–77.
p.73/#44	Listed as A/1	on pp. 9, 61–63.
p.73/#45	Listed as A/5	on pp. 10, 67–68.
p.103/#58	Listed as A/9	on pp. 10, 72–73.
p.104/#62	Listed as C/1	on pp. 14, 99–100.
p.104/#65	Listed as A/10	on pp. 10, 73–75.
p.105/#69	Listed as A/8	on pp. 10, 71–72.
p.105/#70	Listed as A/6	on pp. 10, 68–70.
p.305/#7	Listed as P.G./3	on pp. 12, 79–80.
p.305/#12	Listed as G.I./6	on pp. 13, 91.
p.306/#16	Listed as I/2	on pp. 13, 96–97.

*

The next block of solutions are for problems from the unused IMO proposals given in the 1985 numbers of *Crux.*

17. [1985: 38] *Proposed by Poland.*

Given nonnegative real numbers $x_1, x_2, \ldots, x_k$ and positive integers k, m, n such that $km \le n$, prove that

$$n\left(\prod_{i=1}^{k} x_i^m - 1\right) \le m\sum_{i=1}^{k}(x_i^n - 1).$$

Solution by Murray S. Klamkin, University of Alberta.

Let $P = x_1x_2\cdots x_k$. Since $\sum_{i=1}^{k} x_i^n \ge kP^{n/k}$, it suffices to show that

$$nP^m - n \le mkP^{n/k} - mk$$

or that

$$\frac{P^m - 1}{m} \le \frac{P^r - 1}{r} \tag{1}$$

where $r = n/k \ge m$. (1) follows from the known result that $(P^x - 1)/x$ is increasing in x for $x \ge 0$. A proof follows immediately from the integral representation

$$\frac{P^x - 1}{x\ln P} = \int_0^1 e^{xt\ln P}dt.$$

Finally it is to be noted that m and n need not be positive integers, just being positive reals with $n \ge km$ suffices.

23. [1985: 39] *Proposed by the USSR.*

A tetrahedron is inscribed in a unit sphere. The tetrahedron is such that the center of the sphere lies in its interior. Show that the sum of the edge lengths of the tetrahedron exceeds 6.

Comment by Murray S. Klamkin, University of Alberta.

It is shown more generally in [1] that "The total edge length of a simplex inscribed in a unit sphere in E^n with the center in the interior is greater than $2n$". Also, still more general results are given.

Reference:

[1] G. D. Chakerian and M. S. Klamkin, Inequalities for sums of distances, *Amer. Math. Monthly* **80** (1973) 1009–1017.

25. [1985: 39] *Proposed by the USSR.*

A triangle T_1 is constructed with the medians of a right triangle T. If R_1 and R are the circumradii of T_1 and T, respectively, prove that $R_1 > 5R/6$.

Solution by Murray S. Klamkin, University of Alberta.

The inequality should be $R_1 \geq 5R/6$ since equality occurs for an isosceles right triangle. If a,b,c are the sides of T with $c^2 = a^2 + b^2$, then

$$4{m_a}^2 = 4b^2 + a^2, \quad 4{m_b}^2 = 4a^2 + b^2, \quad 4{m_c}^2 = a^2 + b^2,$$

where m_a, m_b, m_c are the medians corresponding to a, b, c respectively. Also,

$$R_1 = \frac{m_a m_b m_c}{4F_1} = \frac{m_a m_b m_c}{3F}$$

where $F = ab/2$ is the area of T, F_1 the area of T_1. Also $R = c/2$. The given inequality now becomes $8m_a m_b m_c \geq 5abc$ or, squaring,

$$(a^2 + b^2)(a^2 + 4b^2)(4a^2 + b^2) = 64(m_a m_b m_c)^2 \geq 25a^2b^2c^2 = 25a^2b^2(a^2 + b^2).$$

Expanding out and factoring, we obtain the obvious inequality

$$(a^2 + b^2)(a^2 - b^2)^2 \geq 0.$$

There is equality if and only if $a = b$.

48. [1985: 73] *Proposed by the USSR.*

Let O be the center of the axis of a right circular cylinder; let A and B be diametrically opposite points in the boundary of its upper base; and let C be a boundary point of its lower base which does not lie in the plane OAB. Show that

$$\angle BOC + \angle COA + \angle AOB = 2\pi.$$

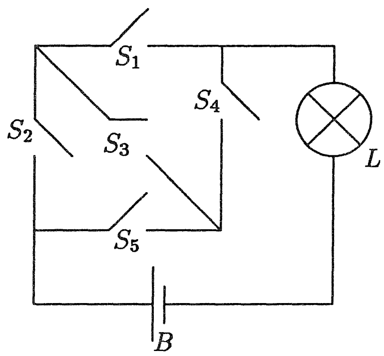
Correction and solution by Murray S. Klamkin, University of Alberta.

The inequality should read $\angle BOC + \angle COA + \angle AOB \leq 2\pi$, and is immediate by the known result that the sum of the face angles of a convex polyhedral angle is less than 2π. See, e.g., p. 61 of my book *USA Mathematical Olympiads 1972–1986*, MAA, 1988.

*

The last seven solutions are to problems from other Olympiads which appeared in the 1985 issues.

2. [1985: 168] *1984 Dutch Olympiad.*



In the given diagram, B is a battery, L is a lamp, and S_1, S_2, S_3, S_4, S_5 are switches. The probability that switch S_3 is on is 2/3, and it is 1/2 for the other four switches. These probabilities are independent. Compute the probability that the lamp is on.

Solution by Murray S. Klamkin, University of Alberta.

More generally, let p_i and q_i denote switch S_i being on or off, respectively, and also their probabilities so $0 \le p_i, q_i \le 1$, $p_i + q_i = 1$. The probability that the lamp is on is given by

$$p_1p_2p_3p_4p_5 + \sum q_1p_2p_3p_4p_5 + \sum q_1q_2p_3p_4p_5 - q_2q_5p_1p_3p_4 - q_1q_4p_2p_3p_5$$

$$+p_1p_2q_3q_4q_5 + p_4p_5q_1q_2q_3.$$

The sums are symmetric over the p's and q's. Note that if at most one switch is off, the lamp is on; if any two switches except q_2q_5 or q_1q_4 are off, the lamp is on, and finally if only two switches are on they must be p_1p_2 or p_4p_5. [*Editor's note:* this gives a probability of 25/48 for the above problem.]

*

4. [1985: 237] *1981 Leningrad High School Olympiad.*

A plane is partitioned into an infinite set of unit squares by parallel lines. A triangle ABC is constructed with vertices at line intersections. Show that if $|AB| > |AC|$, then $|AB| - |AC| > 1/p$, where p is the perimeter of the triangle. (Grades 8, 9, 10)

Solution by Murray S. Klamkin, University of Alberta.

Without loss of generality we can take the rectangular coordinates of A,B,C to be $(0,0)$, (x,y), (u,v), respectively, where x,y,u,v are integers. Then letting $c^2 = |AB|^2 = x^2+y^2$, $b^2 = |AC|^2 = u^2+v^2$, $a^2 = |BC|^2 = (x-u)^2+(y-v)^2$, we have to show that

$$(c-b)(c+b+a) = c^2 - b^2 + (c-b)a > 1.$$

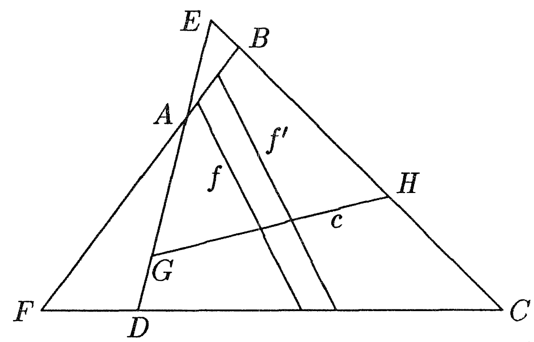
Finally, since $c^2 - b^2$ is a positive integer it is ≥ 1; also $(c-b)a > 0$.

6. [1985: 237] *1981 Leningrad High School Olympiad.*

In a convex quadrilateral the sum of the distances from any point within the quadrilateral to the four straight lines along which the sides lie is constant. Show that the quadrilateral is a parallelogram. (Grade 9)

Solution by Murray S. Klamkin, University of Alberta.

Assume for the quadrilateral $ABCD$ that lines AD and BC are not parallel; so they must meet in a point E as in the figure. Now draw a chord $c = GH$ of $ABCD$ perpendicular to the angle bisector of angle E. It follows by considering the area of the isosceles triangle EGH that the sum of the distances from any point on c to lines AD and BC is constant. Consequently, by hypothesis, the sum of the distances from any point of c to lines AB and CD is constant. If AB and CD are not parallel, let their point of intersection be F and draw 2 chords f and f' of $ABCD$ perpendicular to the angle bisector of angle F, with f nearer to F. It follows easily that the sum of the perpendiculars to lines AB and CD from a point on f is less than from a point on f'. Consequently, AB must be parallel to CD. Proceeding in a similar way using another chord c' parallel to c, it follows also that AD must be parallel to BC, whence the figure must be a parallelogram.



It is to be noted that a non-convex quadrilateral cannot have the given property. Also as a rider, show that if the sum of the six distances from any point within a hexahedron is constant, the hexahedron must be a parallelepiped.

*

5. [1985: 239] *1984 Bulgarian Mathematical Olympiad.*

Let $0 \le x_i \le 1$ and $x_i + y_i = 1$, for $i = 1, 2, \ldots, n$. Prove that

$$(1 - x_1 x_2 \cdots x_n)^m + (1 - y_1^m)(1 - y_2^m) \cdots (1 - y_n^m) \ge 1$$

for all positive integers m and n.

Comment by Murray S. Klamkin, University of Alberta.

In my editorial note to Problem 68-1 (*SIAM Review* 11(1969) 402–406), I gave the more general inequality

$$\prod_{i=1}^{n}\left(1 - \prod_{j=1}^{m} p_{ij}\right) + \prod_{j=1}^{m}\left(1 - \prod_{i=1}^{n} q_{ij}\right) > 1 \qquad (1)$$

where $p_{ij} + q_{ij} = 1$, $0 < p_{ij} < 1$, and m, n are positive integers greater than 1. (The cases $m, n = 1$ are trivial.) Just let $p_{ij} = x_j$ for all i and interchange m and n to get the given inequality. Other particularly nice special cases of the above inequality are

$$\left(1 - \frac{1}{2^n}\right)^m + \left(1 - \frac{1}{2^m}\right)^n > 1$$

and

$$\frac{1}{2^m} + \frac{1}{2^{1/m}} < 1.$$

Further extensions of (1) have been given by Joel Brenner.

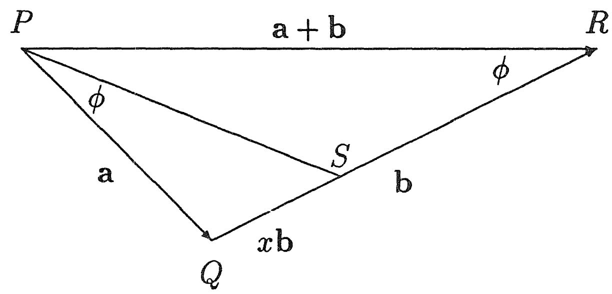
*

2. [1985: 239] *1983 Annual Greek High School Competition.*
If $\mathbf{a}$ and $\mathbf{b}$ are given nonparallel vectors, solve for x in the equation

$$\frac{\mathbf{a}^2 + x\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{a}+x\mathbf{b}|} = \frac{\mathbf{b}^2 + \mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}||\mathbf{a}+\mathbf{b}|}.$$

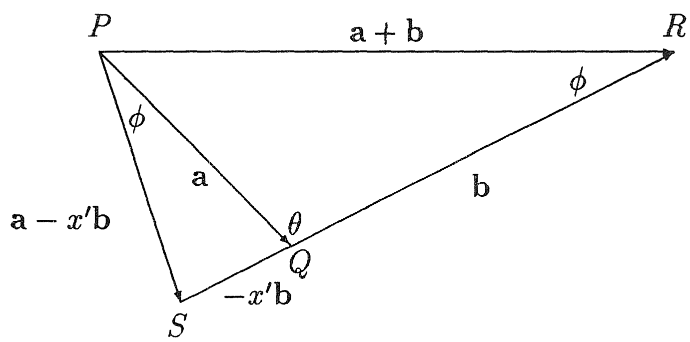
Solution by Murray S. Klamkin, University of Alberta.

One can solve by squaring out and solving the rather messy quadratic in x. However, it is easier to proceed geometrically. Referring to the figure, the given equation requires that $\angle SPQ = \angle PRQ$ $(= \phi$, say). Thus $\Delta PQS \sim \Delta RQP$.



Thus, putting $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$, $xb/a = a/b$ or $x = a^2/b^2$.

Also, $x = -x'$ can be negative as in the following figure.



By the law of sines, $a/\sin(\theta-\phi) = x'b/\sin\phi$, and $a/\sin\phi = b/\sin(\theta+\phi)$. Hence,

$$\sin(\theta+\phi) - \sin(\theta-\phi) = \frac{b\sin\phi}{a} - \frac{a\sin\phi}{x'b}$$

or $2\cos\theta = b/a - a/x'b$. Finally,

$$x = -x' = \frac{-a^2}{b(b-2a\cos\theta)} = \frac{-a^2}{b^2 + 2\mathbf{a}\cdot\mathbf{b}}.$$

*

3. [1985: 304] *Proposed by Canada.*
Determine the maximum value of

$$\sin^2\theta_1+\sin^2\theta_2+\cdots+\sin^2\theta_n,$$

where $\theta_1+\theta_2+\cdots+\theta_n=\pi$ and all $\theta_i\geq 0$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $f(n)$ denote the maximum value. Then obviously $f(1)=0$ and $f(2)=2$, the latter attained if and only if $\theta_1=\theta_2=\pi/2$. When $n=3$, it is well known that

$$\sin^2\theta_1+\sin^2\theta_2+\sin^2\theta_3\leq\frac{9}{4}$$

with equality just when $\theta_1=\theta_2=\theta_3=\pi/3$. (See O. Bottema et al, *Geometric Inequalities*, item 2.3 on p. 18 or the equivalent inequality in item 2.23 on p. 25.) Thus $f(3)=9/4$. We show that $f(n)=9/4$ *for all* $n\geq 3$ by proving the following theorem.

Theorem: Let $\theta_i\geq 0$ for all $i=1,2,\ldots,n$ be such that $\sum_{i=1}^n\theta_i=\pi$ with $n\geq 3$. Then $\sin^2\theta_1+\sin^2\theta_2+\cdots+\sin^2\theta_n\leq 9/4$ with equality holding if and only if $\theta_1=\theta_2=\theta_3=\pi/3$ and $\theta_i=0$ for all $i=4,5,\ldots,n$, where we have, without loss of generality, renumbered the indices so that $\theta_1\geq\theta_2\geq\theta_3\geq\ldots\geq\theta_n$.

We first give a simple lemma.

Lemma: If $A\geq 0$ and $B\geq 0$ are such that $A+B\leq\pi/2$, then $\sin^2A+\sin^2B\leq\sin^2(A+B)$ with equality if and only if $A=0$ or $B=0$ or $A+B=\pi/2$.

Proof: $\sin^2(A+B)-(\sin^2A+\sin^2B)$

$$\begin{aligned}&=\sin^2A\cos^2B+2\sin A\sin B\cos A\cos B+\cos^2A\sin^2B-\sin^2A-\sin^2B\\&=2\sin A\sin B\cos A\cos B-\sin^2A(1-\cos^2B)-\sin^2B(1-\cos^2A)\\&=2\sin A\sin B(\cos A\cos B-\sin A\sin B)\\&=2\sin A\sin B\cos(A+B)\geq 0.\quad\Box\end{aligned}$$

To prove the theorem, we use induction on n. The case $n=3$ is the classical result mentioned above. Suppose the theorem holds for some $n\geq 3$ and suppose $\theta_i\geq 0$ with $\sum_{i=1}^{n+1}\theta_i=\pi$ and $\theta_1\geq\theta_2\geq\ldots\geq\theta_{n+1}$. Let $\phi_i=\theta_i$ for $i=1,2,\ldots,n-1$ and $\phi_n=\theta_n+\theta_{n+1}$. Then $\sum_{i=1}^n\phi_i=\pi$ and $\phi_n\leq\pi/2$, for otherwise $\theta_n+\theta_{n+1}>\pi/2$ together with $n\geq 3$ would imply that $\sum_{i=1}^{n+1}\theta_i>\pi$, a contradiction. Hence by the lemma and the induction hypothesis we obtain

$$\sum_{i=1}^{n+1}\sin^2\theta_i=\sum_{i=1}^{n-1}\sin^2\phi_i+\sin^2\theta_n+\sin^2\theta_{n+1}\leq\sum_{i=1}^{n}\sin^2\phi_i\leq\frac{9}{4}.$$

If equality holds it must hold in the lemma. This implies that either $\theta_{n+1}=0$ or $\theta_n+\theta_{n+1}=\pi/2$. However, if $\theta_n+\theta_{n+1}=\pi/2$, then $\theta_1+\theta_2\geq\pi/2$ and since $\sum_{i=1}^{n+1}\theta_i=\pi$, we obtain $\theta_1+\theta_2=\pi/2$ and $n=4$. Since $\theta_1\geq\theta_2\geq\theta_3\geq\theta_4$, it follows immediately that $\theta_i=\pi/4$, $i=1,2,3,4$, from which we obtain $\sum_{i=1}^4\sin^2\theta_i=2$, a contradiction. Thus

$\theta_{n+1} = 0$ which implies $\phi_i = \theta_i$ for all $i = 1, 2, \ldots, n$. The induction hypothesis then yields $\phi_1 = \phi_2 = \phi_3 = \pi/3$ and $\phi_i = 0$ for $i = 4, \ldots, n$. On the other hand, the sufficiency of the equality condition is obvious. This completes the proof.

4. [1985:304] *Proposed by Canada.*

Prove that

$$\frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_4} + \cdots + \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_1} + \frac{x_n^2}{x_n^2 + x_1x_2} \le n - 1,$$

where all $x_i > 0$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let S_n denote the cyclic sum. Clearly

$$S_2 = \frac{x_1^2}{x_1^2 + x_2x_1} + \frac{x_2^2}{x_2^2 + x_1x_2} = 1.$$

We prove by induction that $S_n < n - 1$ for all $n \ge 3$. Consider the case $n = 3$. Since the sum is cyclic and since the expression is unchanged when x_2 and x_3 are interchanged, we may assume that $x_1 \le x_2 \le x_3$. Then

$$S_3 = \frac{x_1^2}{x_1^2 + x_2x_3} + \frac{x_2^2}{x_2^2 + x_3x_1} + \frac{x_3^2}{x_3^2 + x_1x_2} < \frac{x_1^2}{x_1^2 + x_2x_1} + \frac{x_2^2}{x_2^2 + x_2x_1} + 1 = 2.$$

Now suppose $S_n < n - 1$ for some $n \ge 3$ and consider S_{n+1} for $n + 1$ positive numbers $x_1, x_2, \ldots, x_{n+1}$. Without loss of generality, we may assume that $x_{n+1} = \max\{x_i : i = 1, 2, \ldots, n + 1\}$. Note that

$$S_{n+1} = S_n + \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_{n+1}} + \frac{x_n^2}{x_n^2 + x_{n+1}x_1} + \frac{x_{n+1}^2}{x_{n+1}^2 + x_1x_2} - \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_1} - \frac{x_n^2}{x_n^2 + x_1x_2}.$$

Since

$$\frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_{n+1}} \le \frac{x_{n-1}^2}{x_{n-1}^2 + x_nx_1}, \quad \frac{x_n^2}{x_n^2 + x_{n+1}x_1} \le \frac{x_n^2}{x_n^2 + x_1x_2}, \quad \text{and} \quad \frac{x_{n+1}^2}{x_{n+1}^2 + x_1x_2} < 1,$$

we conclude that $S_{n+1} < S_n + 1 < n$, completing the induction.

Remark. If we set $x_i = t^i$ for $i = 1, 2, \ldots, n - 1$, and $x_n = t^{2n}$, then

$$S_n = \frac{t^2}{t^2 + t^5} + \frac{t^4}{t^4 + t^7} + \cdots + \frac{t^{2n-4}}{t^{2n-4} + t^{3n-1}} + \frac{t^{2n-2}}{t^{2n-2} + t^{2n+1}} + \frac{t^{4n}}{t^{4n} + t^3}$$

$$\longrightarrow n - 1 \quad \text{as } t \longrightarrow 0^+.$$

That is, though the bound $n - 1$ can not be attained for $n > 2$, it is nonetheless *sharp*.

* * *

This completes the "archive" material for 1985 and the space available this number. Contest season is upon us. Send me your contests and solutions!

BOOK REVIEW

30th International Mathematical Olympiad, Braunschweig, 1989 - Problems and Results, edited by Hanns-Heinrich Langmann. K.H. Bock, 5340 Bad Honnef, Germany, 1990. ISBN 3-87066-213-1. 79 pp., softcover. *Reviewed by Richard Nowakowski, Dalhousie University.*

This report on the 1989 IMO first lists the rules and regulations governing the competition. The program of events comes next followed by the competition problems and solutions. All 109 proposed problems are listed next with solutions given to the 32 that were shortlisted by the jury. (These 32 problems have appeared in *Crux* in [1989: 196–197, 225–226, 260–262].) The contestants' scores and awards are given last. The book is an excellent source of problems, but it would have been nice to see the solutions given to more of the difficult problems which were not shortlisted. For example, I have yet to see a good proof (one reasonably explained and under four pages long) of the following proposed question:

HEL 2. In a triangle ABC for which $6(a+b+c)r^2 = abc$ holds and where r denotes the inradius of ABC, we consider a point M of its inscribed circle and projections D, E, F of M on the sides BC, AC, AB, respectively. Let S and S_1 denote the areas of the triangles ABC and DEF, respectively. Find the maximum and minimum values of the quotient S/S_1.

* * * * *

CALL FOR PAPERS – ICME

There will be two 90–minute sessions at the ICME in Quebec City (August 1992) on **mathematical competitions**. Papers are solicited on this topic which are of general interest to the mathematics education community.

Enquiries and proposals for papers should be sent to

Dr. E.J. Barbeau
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 1A1
Canada

email: barbeau@math.toronto.edu
fax: (416)978-4107
phone: (416)978-7200

Please send notice of your desire to present a paper no later than **May 31, 1991.**

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **September 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1611. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with angles A, B, C (measured in radians), sides a, b, c, and semiperimeter s. Prove that

$$(i)\sum\frac{b+c-a}{A}\geq\frac{6s}{\pi};\qquad(ii)\sum\frac{b+c-a}{aA}\geq\frac{9}{\pi}.$$

1612*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\sum\frac{y^2-x^2}{z+x}\geq 0,$$

where the sum is cyclic over x, y, z, and determine when equality holds.

1613. *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that

$$\left(\frac{\sin x}{x}\right)^{2p}+\left(\frac{\tan x}{x}\right)^{p}\geq 2$$

for $p \geq 0$ and $0 < x < \pi/2$. (The case $p = 1$ is problem E3306, *American Math. Monthly,* solution in March 1991, pp. 264–267.)

1614. *Proposed by Toshio Seimiya,Kawasaki, Japan.*

Let D and E be points on side BC of a triangle ABC. Draw lines through D, E parallel to AC, AB respectively, meeting AB and AC at F and G. Let P and Q be the intersections of line FG with the circumcircle of ΔABC. Prove that D, E, P and Q are concyclic.

1615. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Consider the following array:

$\boxed{1}$	2	3	4	5	6	7	8	9	10	$\cdots$
2	$\boxed{3}$	4	5	6	7	8	9	10	11	$\cdots$
4	2	$\boxed{5}$	6	7	8	9	10	11	12	$\cdots$
6	2	7	$\boxed{4}$	8	9	10	11	12	13	$\cdots$
8	7	9	2	$\boxed{10}$	6	11	12	13	14	$\cdots$
6	2	11	9	12	$\boxed{7}$	13	8	14	15	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

For example, to produce row 5 from row 4, write down, in order: the 1st number to the right of 4, the 1st number to the left of 4, the 2nd to the right of 4, the 2nd to the left of 4, the 3rd to the right of 4, the 3rd to the left of 4, and then the 4th, 5th, ... numbers to the right of 4.

Notice that a number will be expelled from a row if and only if it is the diagonal element in the previous row (the boxed numbers in the array), and once missing it of course never reappears.

(a) Is 2 eventually expelled?

(b)* Is every positive integer eventually expelled?

1616. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In triangle ABC, angles α and γ are acute, D lies on AC so that $BD \perp AC$, and E and F lie on BC so that AE and AF are the interior and exterior bisectors of $\angle BAC$. Suppose that BC is the exterior bisector of $\angle ABD$. Show that $AE = AF$.

1617. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

If p is a prime and a and k are positive integers such that $p^k|(a-1)$, then prove that

$$p^{n+k}|(a^{p^n}-1)$$

for all positive integers n.

1618. *Proposed by Jordan Tabov, Sofia, Bulgaria.* (Dedicated to Dan Pedoe on the occasion of his 80th birthday.)

The sides of a given angle α intersect a given circle $O(r)$ in four points. Four circles $O_k(r_k)$, $k = 1,2,3,4$, are inscribed in α so that $O_1(r_1)$ and $O_4(r_4)$ touch $O(r)$ externally, and $O_2(r_2)$ and $O_3(r_3)$ touch $O(r)$ internally. Prove that $r_1r_4 = r_2r_3$. (This is more general than problem 1.8.9, page 20, of *Japanese Temple Geometry Problems* by H. Fukagawa and D. Pedoe.)

1619. *Proposed by Hui-Hua Wan and Ji Chen, Ningbo University, Zhejiang, China.*

Let P be an interior point of a triangle ABC and let R_1, R_2, R_3 be the distances from P to the vertices A, B, C, respectively. Prove that, for $0 < k < 1$,

$$R_1^k + R_2^k + R_3^k < (1 + 2^{\frac{1}{k-1}})^{1-k}(a^k + b^k + c^k).$$

1620. *Proposed by Ilia Blaskov, Technical University, Gabrovo, Bulgaria.*

Find a finite set S of (at least two) points in the plane such that the perpendicular bisector of the segment joining any two points in S passes through exactly two points of S.

* * * * *

SOLUTIONS

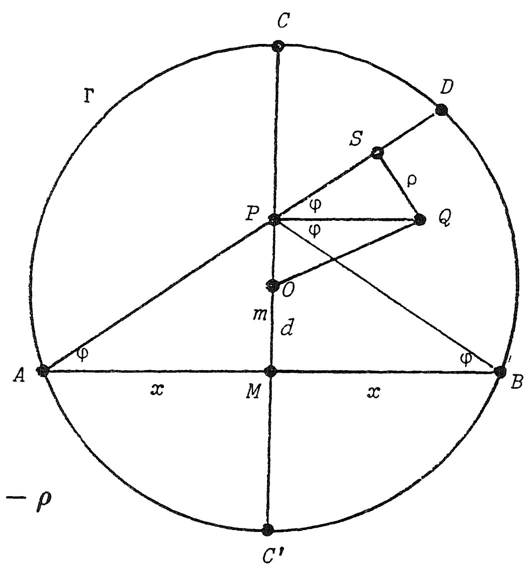
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1390. [1988: 269; 1990: 27] *Proposed by H. Fukagawa, Aichi, Japan.*

A, B, C are points on a circle Γ such that CM is the perpendicular bisector of AB [where M lies on AB]. P is a point on CM and AP meets Γ again at D. As P varies over segment CM, find the largest radius of the inscribed circle tangent to segments PD, PB, and arc DB of Γ, in terms of the length of CM.

II. *Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let $(O, R) = \Gamma$ be the circumcircle of ΔABC, and (Q, ρ) the given inscribed circle. We denote



$$\varphi = \angle PAB = \angle PBA = \angle BPQ = \angle SPQ,$$

S being the projection of Q to PD. Let $MA = MB = x$, $OM = d$, $PM = m$. C' is the second intersection point of CP and Γ.

In right triangle OPQ we have

$$PQ = \frac{\rho}{\sin\varphi}\ , \ OP = m - d\ , \ OQ = R - \rho$$

and so

$$\frac{\rho^2}{\sin^2\varphi} + (m-d)^2 = (R-\rho)^2,$$

or

$$\rho^2 \cot^2\varphi + 2R\rho - R^2 + (m-d)^2 = 0. \tag{1}$$

Now

$$\cot^2\varphi = \frac{x^2}{m^2} = \frac{R^2 - d^2}{m^2},$$

and (1) becomes

$$\rho^2(R^2 - d^2) + 2Rm^2\rho - m^2[R^2 - (m-d)^2] = 0. \tag{2}$$

It is clear that $R > m - d$, so $R^2 - (m-d)^2 > 0$, and we conclude that (2) has two different roots $\rho_1 > 0$ and $\rho_2 < 0$. D being the discriminant of (2) we have

$$\begin{aligned}\frac{D}{4} &= R^2m^4 + m^2(R^2-d^2)[R^2-(m-d)^2] \\ &= m^2(R^4 - 2R^2d^2 + d^4 + 2R^2dm - 2d^3m + d^2m) \\ &= m^2(R^2 - d^2 + dm)^2.\end{aligned}$$

So

$$\begin{aligned}\rho_1 &= \frac{-Rm^2 + m(R^2-d^2+dm)}{R^2-d^2} \\ &= \frac{m(R^2-d^2) - m^2(R-d)}{R^2-d^2} \\ &= \frac{m(R+d)-m^2}{R+d} = \frac{m(R+d-m)}{R+d},\end{aligned}$$

and thus

$$\frac{1}{\rho_1} = \frac{R+d}{m(R+d-m)} = \frac{1}{m} + \frac{1}{R+d-m} = \frac{1}{PM} + \frac{1}{PC}, \tag{3}$$

the "harmonic mean" property given on [1990: 27], with a maximum for ρ_1 when $PM = PC$. Also

$$\begin{aligned}\rho_2 &= \frac{-Rm^2 - m(R^2-d^2+dm)}{R^2-d^2} \\ &= \frac{-m(R^2-d^2) - m^2(R+d)}{R^2-d^2} \\ &= \frac{-m(R-d)-m^2}{R-d} = \frac{-m(R-d+m)}{R-d},\end{aligned}$$

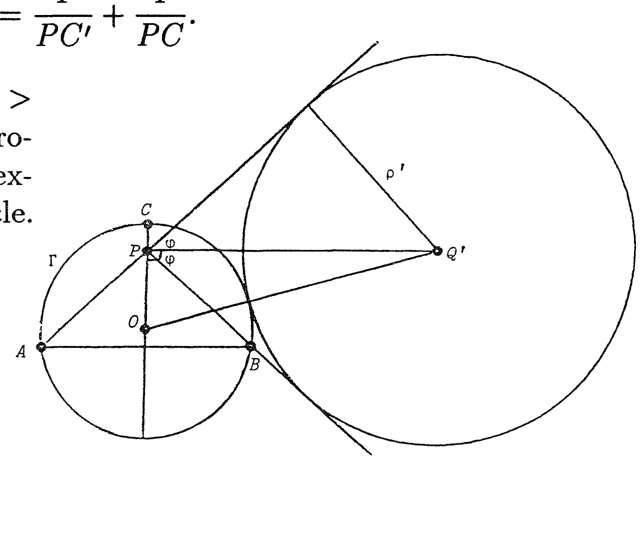
so

$$\frac{1}{\rho_2} = \frac{1}{R-d+m} - \frac{1}{m} = \frac{1}{PC'} - \frac{1}{PM}. \tag{4}$$

From (3) and (4) follows

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{PC'} + \frac{1}{PC}.$$

ρ_2 has a geometrical meaning: $-\rho_2 > 0$ is the radius of the circle touching the productions of AP and PB, and touching Γ externally. In the figure, (Q', ρ') is this circle. We have



$$\frac{\rho'^2}{\sin^2\varphi} + (m-d)^2 = (R+\rho')^2$$

or (as above)

$$\rho'^2(R^2-d^2)-2Rm^2\rho'-m^2[R^2-(m-d)^2]=0. \tag{5}$$

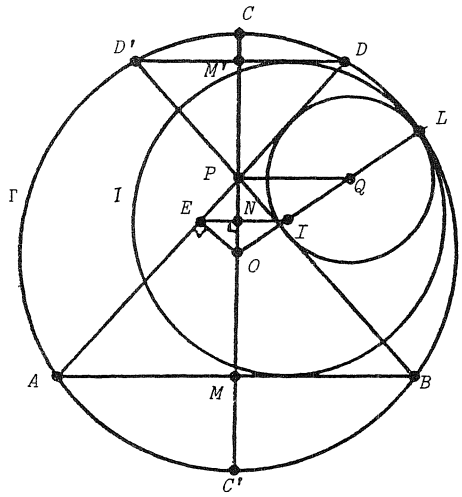
Comparing (2) and (5) we conclude: if (2) has the roots ρ_1 and ρ_2, then (5) has the roots $-\rho_1$ and $-\rho_2$.

III. *Solution by Albert W. Walker, Toronto, Ontario.*

We use the following result, due to Poncelet (e.g., [2]):

THEOREM. *Let points A, B, C, D lie on one circle O of a coaxial family of circles, and suppose that a second circle I of the family touches lines AB and CD. Then there are circles J and K of the family, J touching AC and BD, and K touching AD and BC. Moreover the six contact points of I, J, K with these lines are collinear.* □

[*Editor's note*: expert colleague Chris Fisher, University of Regina, also suggests the reference [1].]



In the present problem we draw the parallel to AB through D, letting it meet CM at M' and Γ again at D', and let $\mathcal{I}$ (with centre I) be the circle tangent to these parallels and internally tangent to Γ at a point L in the minor arc BD. The coaxial family containing Γ and $\mathcal{I}$ is thus the family of all circles mutually tangent at L. By the above theorem of Poncelet, the incircle (Q,ρ) tangent to AD, BD', and minor arc BD of Γ will be tangent to Γ at L. A fourth circle, exterior to Γ, will be tangent to Γ at L and also to AD' and BD.

Let E be the midpoint of AD, so that $OE \perp AD$ and $EI \perp MM'$, and EI meets MM' at its midpoint N. Let CC' be a diameter of Γ. Put $AP = a, CP = c, MP = m$. Then $C'M + m = C'P = R + OP$, or

$$C'M - OP = R - m. \tag{1}$$

We have

$$(AM)^2 = CM \cdot C'M \tag{2}$$

so that

$$\frac{ON}{OP} = \frac{ON}{OE}\cdot\frac{OE}{OP} = \left(\frac{AM}{AP}\right)^2 = \frac{CM\cdot C'M}{a^2}, \tag{3}$$

or

$$a^2 ON = OP \cdot CM \cdot C'M. \tag{4}$$

Also

$$\frac{ON}{OP} = \frac{OI}{OQ} = \frac{R-IL}{R-QL} = \frac{R-MN}{R-\rho},$$

and thus from (3),

$$\frac{R-MN}{R-\rho}=\frac{CM\cdot C'M}{a^2}. \tag{5}$$

Next $R+ON=C'N=C'M+MN$, and so

$$R-MN=C'M-ON. \tag{6}$$

Now we have from (5),(6),(4),(2) and (1),

$$\begin{aligned}CM\cdot C'M\cdot(R-\rho)&=a^2(R-MN)=a^2(C'M-ON)\\&=C'M(a^2-OP\cdot CM)=C'M(m^2+(AM)^2-OP\cdot CM)\\&=C'M(m^2+CM(C'M-OP))=C'M(R\cdot CM-(CM-m)m)\\&=C'M(R\cdot CM-cm).\end{aligned}$$

Thus $CM(R-\rho)=R\cdot CM-cm$, or

$$\rho=\frac{cm}{CM}=\frac{cm}{c+m}.$$

Therefore the diameter 2ρ of the incircle is the harmonic mean of CP and MP (and, incidentally, also of $C'P$ and $M'P$). The result then follows as before.

References:

[1] Marcel Berger, *Geometry II*(English translation), Springer, Berlin, 1987, Prop. 16.6.4 and Fig. 16.6.4(2), pp. 203 and 204.

[2] H.G. Forder, *Higher Course Geometry*, 1931/49, p. 224.

Editor's note. Walker also observes that, letting CQ and AMB meet at S, and CL and PQ meet at T, one gets $MS=PT=PA$, and therefore OL,CS,PT concur (at Q). He uses this approach to give a second solution, not using the Poncelet theorem.

A further solution has been received from *TOSHIO SEIMIYA*, Kawasaki, Japan. Seimiya first establishes the following lemma. *ABC is a triangle with incenter I and circumcircle Γ_1. Let D be a point on side BC. Let Γ_2 be an inscribed circle tangent to the segments AD and DC, at E and F respectively, and to the arc AC of Γ_1. Then E,F and I are collinear.* Seimiya uses this lemma to prove the present problem, as well as to give another proof of *Crux* 1260 [1988: 237; 1989: 51]. He also points out that the harmonic mean relationship given above occurs as problem 1.2.7, p. 5 of H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems* (reviewed on [1990: 203]).

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1430. [1989: 74; 1990: 158] *Proposed by Mihály Bencze, Brasov, Romania.*

AD,BE,CF are (not necessarily concurrent) cevians in triangle ABC, intersecting the circumcircle of ΔABC in the points P,Q,R. Prove that

$$\frac{AD}{DP}+\frac{BE}{EQ}+\frac{CF}{FR}\geq 9.$$

When does equality hold?

II. *Solution by Ji Chen, Ningbo University, China.*
The result may be sharpened to

$$\frac{AD}{DP}+\frac{BE}{EQ}+\frac{CF}{FR}\geq\left(4-\frac{2r}{R}\right)^2.$$

From the inequality (given in the published solution of Seimiya [1990: 159])

$$\frac{AD}{DP}+\frac{BE}{EQ}+\frac{CF}{FR}\geq\sum\left(\frac{(b+c)^2}{a^2}-1\right)=\sum a\cdot\sum\frac{b+c-a}{a^2},$$

where the sums are cyclic over a,b,c, and the fact that

$$4-\frac{2r}{R}=4-\prod\frac{b+c-a}{a}=\frac{\sum a(b+c-a)^2}{\prod a},$$

it is enough to show that

$$\sum a\cdot\sum b^2c^2(b+c-a)\geq\left(\sum a(b+c-a)^2\right)^2.$$

Putting

$$x=b+c-a,\ y=c+a-b,\ z=a+b-c,$$

this becomes

$$\sum x\cdot\sum x\left(\frac{x+y}{2}\right)^2\left(\frac{x+z}{2}\right)^2\geq\left(\sum x^2\left(\frac{y+z}{2}\right)\right)^2,$$

or

$$\sum x\cdot\sum x(x+y)^2(x+z)^2-4(\sum x^2(y+z))^2\geq 0, \tag{1}$$

where the sums are cyclic over x,y,z. The left side of (1) becomes

$$\begin{aligned}&\sum x^6+3\sum(x^5y+x^5z)-\sum(x^4y^2+x^4z^2)-6\sum x^3y^3\\&\qquad+2\sum(x^3y^2z+x^3z^2y)-9x^2y^2z^2\\&=\left[\sum x^6-\sum(x^4y^2+x^4z^2)+3x^2y^2z^2\right]+3\left[\sum(x^5y+x^5z)-2\sum y^3z^3\right]\\&\qquad+2\left[\sum(x^3y^2z+x^3z^2y)-6x^2y^2z^2\right]\\&=\frac{\sum(y^2+z^2-x^2)^2(y^2-z^2)^2+2\sum y^2z^2(y^2-z^2)^2}{2\sum x^2}\\&\qquad+3\sum yz(y^2-z^2)^2+2\sum x^2yz(y-z)^2\\&\geq 0,\end{aligned}$$

so (1) follows.

Editor's note. Murray Klamkin has obligingly supplied the editor with the following simpler proof of inequality (1). By Cauchy's inequality,

$$\sum x\cdot\sum x(x+y)^2(x+z)^2\geq\left(\sum x(x+y)(x+z)\right)^2,$$

so it suffices to prove the stronger inequality

$$\sum x(x+y)(x+z) \geq 2\sum x^2(y+z),$$

which simplifies to

$$\sum x(x-y)(x-z) \geq 0. \qquad (2)$$

But assuming without loss of generality that $x \geq y \geq z$, (2) follows from

$$x(x-y)(x-z) \geq y(x-y)(y-z) \qquad \text{and} \qquad z(z-x)(z-y) \geq 0.$$

(2) is the special case $n = 1$ of Schur's inequality

$$\sum x^n(x-y)(x-z) \geq 0$$

which can be established the same way (see pp. 49–50 of M.S. Klamkin, *International Mathematical Olympiads 1979–1985*, M.A.A., 1986).

* * * * *

1486. [1989: 269] *Proposed by Jordi Dou, Barcelona, Spain.*

Given three triangles T_1, T_2, T_3 and three points P_1, P_2, P_3, construct points X_1, X_2, X_3 such that the triangles $X_2X_3P_1, X_3X_1P_2, X_1X_2P_3$ are directly similar to T_1, T_2, T_3, respectively.

Solution by the proposer.

Denote by S_i the dilative rotation with centre P_i, angle $\gamma_i = \angle A_iC_iB_i$ (in the sense C_iA_i to C_iB_i), and ratio $r_i = C_iB_i/C_iA_i$, for $i = 1,2,3$. For each point X, the triangle $XS_i(X)P_i$ will be similar to $A_iB_iC_i$. Let $S = S_1S_2S_3$, i.e. $S(X) = S_3(S_2(S_1(X)))$. S is a dilative rotation of angle $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, ratio $r = r_1r_2r_3$ and with fixed point O, say. If we put $X_2 = O, S_1(X_2) = X_3$ and $S_2(X_3) = X_1$, then $S_3(X_1) = S(X_2) = X_2$. Thus the problem reduces to constructing the fixed point of S and this point will be X_2. X_3 and X_1 will then be $S_1(X_2)$ and $S_3^{-1}(X_2)$ respectively. If $\gamma \neq 0°$ or $360°$, or $r \neq 1$, the problem has a unique solution. [*Editor's note.* Dou ended with a construction of the required points. A construction of the unique fixed point given three noncollinear points and their images under a similarity can be found in references such as Coxeter's *Introduction to Geometry*, page 73.]

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1492. [1989: 297] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC, B' \in CA, C' \in AB$. Suppose also that $BA' = CB' = AC'$.

(a) If either the centroids G, G' or the circumcenters O, O' of the triangles coincide, prove that ΔABC is equilateral.

(b)* If either the incenters I, I' or the orthocenters H, H' of the triangles coincide, characterize ΔABC.

I. *Solution to part* (a) *by Murray S. Klamkin, University of Alberta.*

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ denote vectors from a common origin to the respective vertices A, B, C. It then follows that

$$\mathbf{A}' = \mathbf{B} + \frac{\alpha}{a}(\mathbf{C}-\mathbf{B}) = \frac{\alpha}{a}\mathbf{C} + \left(1-\frac{\alpha}{a}\right)\mathbf{B}$$

and similarly

$$\mathbf{B}' = \frac{\alpha}{b}\mathbf{A} + \left(1-\frac{\alpha}{b}\right)\mathbf{C}, \quad \mathbf{C}' = \frac{\alpha}{c}\mathbf{B} + \left(1-\frac{\alpha}{c}\right)\mathbf{A},$$

where a,b,c are the respective sides of ABC and α is a nonzero constant.

For the case $G = G'$, we choose the origin to be the centroid of ΔABC so that $\mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0}$ and also

$$\mathbf{0} = \mathbf{A}'+\mathbf{B}'+\mathbf{C}' = \left(\frac{\alpha}{b}-\frac{\alpha}{c}+1\right)\mathbf{A} + \left(\frac{\alpha}{c}-\frac{\alpha}{a}+1\right)\mathbf{B} + \left(\frac{\alpha}{a}-\frac{\alpha}{b}+1\right)\mathbf{C}.$$

Thus

$$\mathbf{0} = \left(\frac{1}{b}-\frac{1}{c}\right)\mathbf{A} + \left(\frac{1}{c}-\frac{1}{a}\right)\mathbf{B} + \left(\frac{1}{a}-\frac{1}{b}\right)\mathbf{C} = \left(\frac{2}{b}-\frac{1}{c}-\frac{1}{a}\right)\mathbf{A} + \left(\frac{1}{b}+\frac{1}{c}-\frac{2}{a}\right)\mathbf{B},$$

and so

$$\frac{2}{b}-\frac{1}{c}-\frac{1}{a} = \frac{1}{b}+\frac{1}{c}-\frac{2}{a} = 0,$$

which implies $a = b = c$.

For the case $O = O'$, we choose the origin to be the circumcenter of ΔABC so that $|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = R$, the circumradius. Then also $|\mathbf{A}'|^2 = |\mathbf{B}'|^2 = |\mathbf{C}'|^2 = R'^2$, or

$$\begin{aligned}
R'^2 &= \frac{\alpha^2}{a^2}|\mathbf{C}|^2 + \left(1-\frac{\alpha}{a}\right)^2|\mathbf{B}|^2 + 2\cdot\frac{\alpha}{a}\left(1-\frac{\alpha}{a}\right)\mathbf{B}\cdot\mathbf{C} \\
&= \frac{\alpha^2}{a^2}R^2 + \left(1-\frac{\alpha}{a}\right)^2 R^2 + \frac{2\alpha(a-\alpha)}{a^2}R^2\cos 2A \\
&= \left(\frac{2\alpha^2}{a^2}-\frac{2\alpha}{a}+1\right)R^2 + \frac{2\alpha(a-\alpha)}{a^2}R^2 - \frac{4\alpha(a-\alpha)R^2\sin^2 A}{a^2} \\
&= R^2 - \alpha(a-\alpha), \quad \text{etc.,}
\end{aligned}$$

or

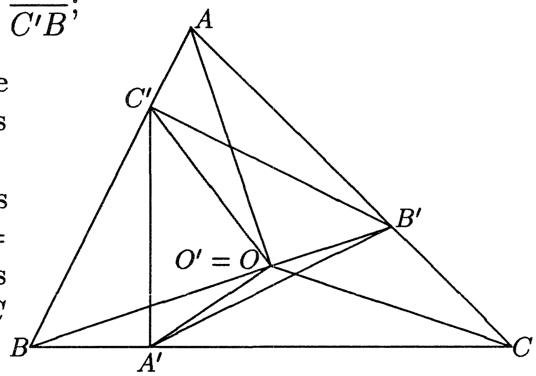
$$\alpha(a-\alpha) = R^2 - R'^2 = \alpha(b-\alpha) = \alpha(c-\alpha).$$

Hence $a = b = c$.

II. *Solution to part* (a) *by the proposer.*

It is known (*Crux* 1464 [1990: 282]) that $G = G'$ if and only if

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B};$$



therefore according to our assumption we have $A'C = B'A = C'B$, and hence that ΔABC is equilateral.

If $O = O'$, we easily find that the triangles $OA'B$, $OB'C$, $OC'A$ are congruent, so $\angle OBA' = \angle OCB'$. But also $\angle OBA' = \angle OCA'$, so CO bisects $\angle C$. Similarly we find that $O = I$, hence ΔABC is equilateral.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and by D.J. SMEENK, Zaltbommel, The Netherlands.*

Part (b) *remains unsolved, as did part* (c) *of the proposer's earlier problem* Crux 1464 [1990: 282]*; in fact Janous comments that in trying* (b) *he "got stuck in quite awkward and disgusting expressions", the same predicament that befell him in* Crux 1464 *! Looks like these problems need a new idea from someone.*

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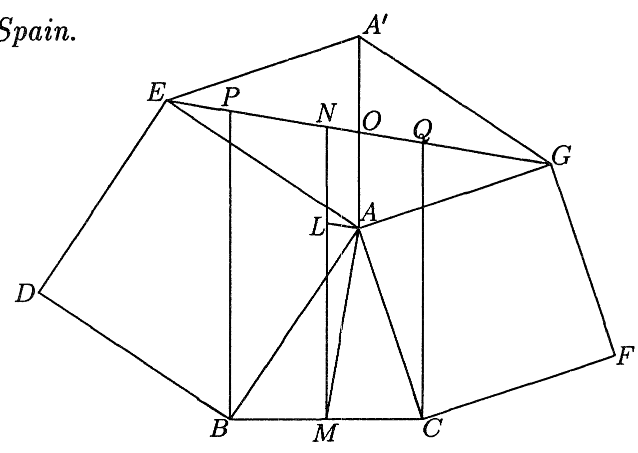
1493. [1989: 297] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Two squares $ABDE$ and $ACFG$ are described on AB and AC outside the triangle ABC. P and Q are on line EG such that BP and CQ are perpendicular to BC. Prove that

$$BP + CQ \geq BC + EG.$$

When does equality hold?

I. *Solution by Jordi Dou, Barcelona, Spain.*

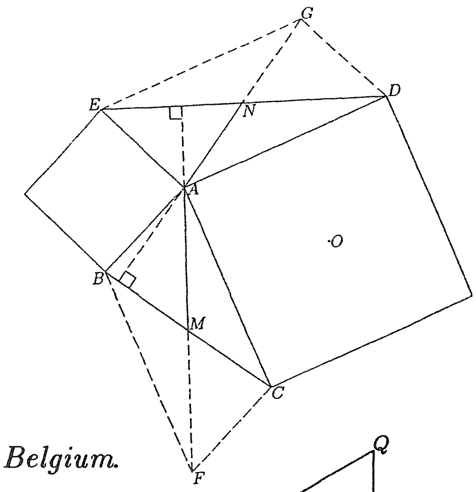


Let M, N, O be the midpoints of BC, PQ, EG, respectively. Let A' be symmetric to A with respect to O. We have that AE is equal and perpendicular to BA, and analogously EA' (= AG) to AC. Therefore $AA' = BC$, $AA' \perp BC$ and $OE = MA$, $OE \perp MA$. Let L on MN be such that $AL \| OE$. Then $AO = LN$, and $ML > MA$ because MA is perpendicular to AL. Hence $MN = LN + ML \geq AO + MA$, and

$$BP + CQ = 2MN \geq 2(AO + MA) = 2(BM + OE) = BC + EG.$$

Equality occurs if and only if $MN = MA + AO$, i.e., MN coincides with MO, i.e., $AB = AC$.

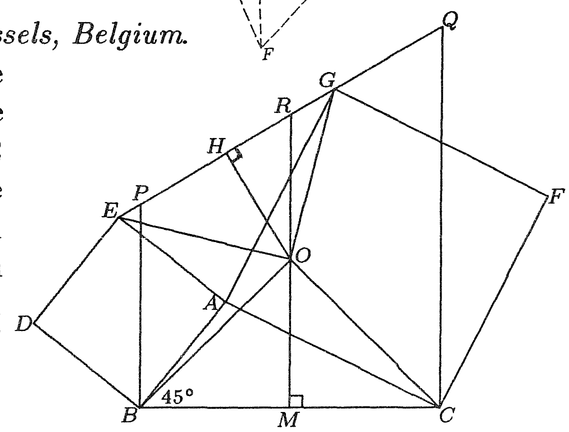
Note. Here is a nice property of triangles between two squares with a common vertex:



The median of one triangle is perpendicular to the base of the other.

The proof is: a rotation of 90° with centre O moves $ABFC$ onto $DGEA$, consequently $AM \perp DE$ and $AN \perp BC$. This property is a "leitmotif" in the above solution and also in my solution of *Crux* 1496.

II. *Solution par C. Festraets-Hamoir, Brussels, Belgium.*



Construisons, sur BC et du même côté que A, le triangle rectangle isocèle BOC. La similitude σ_1 de centre B, d'angle 45° et de rapport $\sqrt{2}/2$ applique E sur A et O sur M. La similitude σ_2 de centre C, d'angle 45° et de rapport $\sqrt{2}$ applique A sur G et M sur O. $\sigma_2 \circ \sigma_1$ est donc une rotation d'angle 90° (le rapport est $\sqrt{2} \cdot \sqrt{2}/2 = 1$) qui applique E sur G et dont le point fixe est O. EOG est ainsi un triangle rectangle isocèle.

M étant le milieu de BC,

$$BP + CQ = 2MR = 2MO + 2OR = BC + 2OR \geq BC + 2OH = BC + EG.$$

L'égalité a lieu si et seulement si $OR = OH$, c'est-à-dire $EG \parallel BC$, le triangle ABC est alors isocèle ($AB = AC$).

Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria: MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and the proposer.

The solutions of Kuczma and the proposer were very similar to solution I.

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1494. [1989: 298] *Proposed by Murray S. Klamkin, University of Alberta.*

Three numbers x, y, z are chosen independently at random and uniformly in [0,1]. What is the probability that x, y, z can be the lengths of the sides of a triangle whose altitudes are also the sides of some triangle?

Solution by P. Penning, Delft, The Netherlands.

The probability is equal to the volume in x, y, z–space where triangles meeting the constraint can be found. Since the size of the triangle cannot play a part, the boundaries of the volume must be straight lines passing through $(0,0,0)$.

First we impose another constraint,

$$x \leq y \leq z, \tag{1}$$

thereby reducing the relevant volume by a factor of $3! = 6$. These numbers form a triangle if

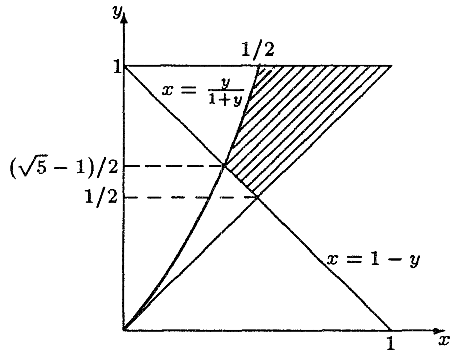
$$x \geq z - y. \tag{2}$$

The altitudes are proportional to $1/x \geq 1/y \geq 1/z$ (the area of the triangle is the proportionality factor), and they form a triangle if $1/x \leq 1/y + 1/z$, i.e.

$$x \geq \frac{yz}{y+z}. \tag{3}$$

The region where conditions (1)–(3) are met is a pyramid, with top at $(0,0,0)$ and base in the plane $z = 1$. (The base is shown in the figure.) With integral calculus the area is easily determined to be

$$\ln(\sqrt{5}-1) + \frac{2-\sqrt{5}}{4}.$$



The probability is then equal to

$$\begin{aligned} 6(\text{volume of pyramid}) &= 2(\text{area of base}) \\ &= 2\ln(\sqrt{5}-1) + 1 - \sqrt{5}/2 \\ &\approx 0.3058. \end{aligned}$$

If only the sides have to form a triangle, then the probability is 0.5 .

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer. A solution similar to Penning's, but with a trivial error at the end, was also sent in by C. FESTRAETS-HAMOIR, Brussels, Belgium.

* * * * *

1495. [1989: 298] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Show that there exist infinitely many positive integer solutions to

$$c^2 = \binom{a}{2} - \binom{b}{2}.$$

(b)* If $k > 2$ is an integer, are there infinitely many solutions in positive integers to

$$c^k = \binom{a}{k} - \binom{b}{k}?$$

I. *Solution to part* (a) *by Hayo Ahlburg, Benidorm, Spain.*
The equation becomes $a^2 - a - b^2 + b = 2c^2$ or

$$(a+b-1)(a-b) = 2c^2.$$

Introducing $D = a - b$, we find $D^2 + (2b-1)D = 2c^2$, so

$$b = \frac{c^2}{D} - \frac{D-1}{2}\ , \quad a = b + D = \frac{c^2}{D} + \frac{D+1}{2}.$$

To make a and b integers, we have two possibilities.

Case (i): $c^2 = kD$. Then

$$a = k + \frac{D+1}{2}\ , \quad b = k - \frac{D-1}{2},$$

where D is odd and k is any integer which makes kD a square. For example ($D = 1$):

$$a = c^2 + 1\ , \quad b = c^2\ , \quad c \text{ arbitrary.} \tag{1}$$

Case (ii): $c^2 = mD + D/2$. Here $D = 2^{2n+1}d$ with d odd, and m is any number which makes $(2m+1)d$ a square. In this case we have

$$c^2 = (2m+1)2^{2n}d\ , \ a = m + 1 + 2^{2n}d\ , \ b = m + 1 - 2^{2n}d.$$

For example ($D = 2$):

$$a = \frac{c^2+3}{2}\ , \ b = \frac{c^2-1}{2}\ , \quad c \text{ arbitrary odd.} \tag{2}$$

II. *Partial solution to part* (b) *by Kenneth M. Wilke, Topeka, Kansas.*
[Wilke first solved part (a)—*Ed.*]
Consider $k = 3$. Then $c^3 = \binom{a}{3} - \binom{b}{3}$ becomes

$$\begin{aligned} 6c^3 &= (a-1)^3 - (a-1) - [(b-1)^3 - (b-1)] \\ &= (a-b)(a^2 + ab + b^2 - 3(a+b) + 2). \end{aligned}$$

Taking $a = b + 2$, this becomes $6c^3 = 2 \cdot 3b^2$, whence we can take

$$c = t^2\ , \ b = t^3\ , \ a = t^3 + 2 \ \ (t \text{ arbitrary})$$

to produce an infinite number of solutions. These are not all the solutions when $k = 3$. Others include $(a, b, c) = (9, 6, 4)$, $(12, 4, 6)$ and $(25, 19, 11)$.

Also solved (both parts) by H. L. ABBOTT, University of Alberta; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; and RICHARD K. GUY, *University of Calgary. Part* (a) *only was solved by MATHEW ENGLANDER, Toronto, Ontario; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; CHRIS WILDHAGEN, Breda, The Netherlands; and the proposer.*

The remarkable solution II *was also found by Engelhaupt and by Guy. Abbott's solution to part* (b) *was a recursively defined infinite sequence of solutions, the first being the trivial*

$$1^3 = \binom{3}{3} - \binom{2}{3},$$

and the second being the already formidable

$$16199^3 = \binom{30527}{3} - \binom{14328}{3} \quad !$$

These all have the additional property that $a = b + c$. Guy has since found other solutions with this property, for instance

$$26^3 = \binom{50}{3} - \binom{24}{3}.$$

Example (1) *of case* (i) *in Ahlburg's solution was also found by Engelhaupt, Englander and Gibbs, while example* (2) *of case* (ii) *was again given by Engelhaupt. Another special case of Ahlburg's case* (i), *namely $k = D(odd)$, was found by Hess, Kierstead and Klamkin. This gives the nice solution*

$$a = 3b - 1 , \quad c = 2b - 1 , \quad b \text{ arbitrary.}$$

No solutions for $k > 3$ were sent in. However, Guy has come up with the single (and singular!) example

$$6^5 = \binom{18}{5} - \binom{12}{5}$$

for $k = 5$. Is it part of some infinite family?

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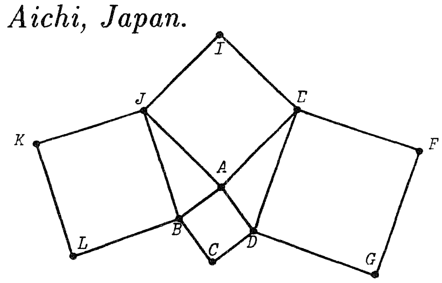
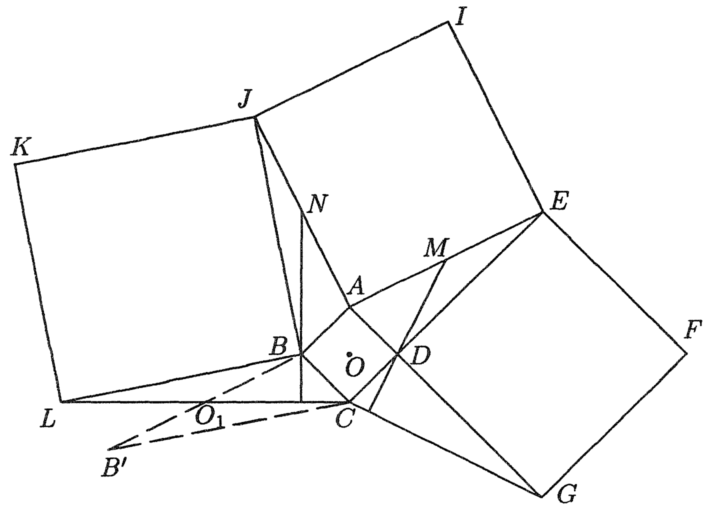
1496. [1989: 298] *Proposed by H. Fukagawa, Aichi, Japan.*

There are four squares $ABCD$, $DEFG$, $AEIJ$, $BJKL$ as shown in the figure. Show that L, C, G are collinear if and only if $2AD = AE$.



I. *Solution by Jordi Dou, Barcelona, Spain.*

Let M, N, O_1 be the midpoints of AE, AJ, CL, respectively. B' is symmetric to B with respect to O_1, and O is the centre of $ABCD$.



Triangle CBB' is obtained from triangle BAJ by a rotation of 90° about centre O. Therefore $BN \perp CL$. [*Editor's note:* see also Dou's comment at the end of his solution to *Crux* 1493, this issue!] Analogously, $DM \perp CG$. If $AE = 2AD$ then $AB = AN = AM = AD$ and $\angle NAM = \angle BAD$ $(= 90°)$, so BN is parallel to DM, and therefore L, C, G are collinear. It is clear that if $AE \neq 2AD$, then DM is not parallel to BN and L, C, G are not aligned.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Consider the vectors $\overrightarrow{AD} = \mathbf{u}$, $\overrightarrow{AE} = \mathbf{v}$. Let h denote rotation by 90° anticlockwise. Since h is a linear map,

$$\overrightarrow{CL} = \overrightarrow{CB} + \overrightarrow{BL} = -\mathbf{u} + h(\overrightarrow{BJ}) = -\mathbf{u} + h(\overrightarrow{AJ}) - h(\overrightarrow{AB}) = -\mathbf{u} - \mathbf{v} - \mathbf{u} = -2\mathbf{u} - \mathbf{v}$$

and

$$h(\overrightarrow{CG}) = h(\overrightarrow{CD}) + h(\overrightarrow{DG}) = \overrightarrow{DA} + \overrightarrow{DE} = -\mathbf{u} - \mathbf{u} + \mathbf{v} = \mathbf{v} - 2\mathbf{u}.$$

Therefore

$$\begin{aligned} L, C, G \text{ are in line} &\iff \overrightarrow{CG} \| \overrightarrow{CL} \iff \overrightarrow{CL} \perp h(\overrightarrow{CG}) \\ &\iff (2\mathbf{u} + \mathbf{v})(2\mathbf{u} - \mathbf{v}) = 0 \\ &\iff |\mathbf{v}| = 2|\mathbf{u}|. \end{aligned}$$

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland (a second solution); KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. One incorrect solution was sent in.

The above solutions are related, but the different approaches are interesting.

The problem was taken from a lost sangaku dated 1826, and appears as problem 4.21, page 47, of Fukagawa and Pedoe's Japanese Temple Geometry Problems. *Also given there is the relationship between the sides of the four squares.*

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1497. [1989: 298] *Proposed by Ray Killgrove and Robert Sternfeld, Indiana State University, Terre Haute.*

A *translate* g of a function f is a function $g(x) = f(x+a)$ for some constant a. Suppose that one translate of a function $f : \mathbf{R} \to \mathbf{R}$ is odd and another translate is even. Show that f is periodic. Is the converse true?

Solution by the St. Olaf Problem Solving Group, St. Olaf College, Northfield, Minnesota.

We know that there exist a and b such that for every x, $f(x+a) = f(-x+a)$ and $f(x+b) = -f(-x+b)$. This implies that $f(2a+x) = f(-x)$ and $f(2b+x) = -f(-x)$. Thus,

$$\begin{aligned} f(x+4(a-b)) &= f(2a+(x+2a-4b)) = f(4b-x-2a) \\ &= -f(2a+x-2b) = -f(2b-x) = f(x). \end{aligned}$$

Thus $f(x)$ has period $4(a-b)$. (If $a=b$, then $f(x+a) = f(a-x) = -f(a+x)$ which implies that $f(x)$ is the zero function.)

The converse is not true. The function $f(x) = x - [x]$ provides a counterexample.

Also solved (in about the same way) by SEUNG-JIN BANG, Seoul, Republic of Korea; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; DAVID E. MANES, SUNY at Oneonta, New York; CHRIS WILDHAGEN, Breda, The Netherlands; and the proposers.

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1498*. [1989: 298] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Show that

$$\prod_{i=1}^{3} h_i^{a_i} \le (3r)^{2s},$$

where a_1, a_2, a_3 are the sides of a triangle, h_1, h_2, h_3 its altitudes, r its inradius, and s its semiperimeter.

I. *Solution by Mark Kisin, student, Monash University, Clayton, Australia.*

By the generalized A.M.-G.M. inequality,

$$\left(\prod_{i=1}^{3} h_i^{a_i}\right)^{1/2s} = \left(\prod_{i=1}^{3} h_i^{a_i}\right)^{\frac{1}{a_1+a_2+a_3}} \le \frac{a_1h_1 + a_2h_2 + a_3h_3}{a_1+a_2+a_3}.$$

But

$$a_1h_1 = a_2h_2 = a_3h_3 = 2(\text{Area}) = 2rs,$$

so

$$\left(\prod_{i=1}^{3} h_i^{a_i}\right)^{1/2s} \le \frac{6rs}{2s} = 3r,$$

and the result follows.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Using the relations $h_i = 2F/a_i$ $(i = 1,2,3)$ and $r = Fs$, where F is the area of the triangle, the stated inequality becomes

$$2^{2s}F^{2s}\prod_{i=1}^{3} a_i^{-a_i} \le 3^{2s}F^{2s}s^{-2s},$$

i.e.

$$\prod_{i=1}^{3} a_i^{a_i} \ge \left(\frac{2s}{3}\right)^{2s}. \tag{1}$$

This inequality is the special case $n = 3$, $p_1 = p_2 = p_3 = 1$, $x_i = a_i$ $(i = 1,2,3)$ of the following.

THEOREM. Let $p_1, \ldots, p_n$ be positive real numbers and put $P_n = p_1 + p_2 + \cdots + p_n$. Then for all positive real numbers $x_1, \ldots, x_n$,

$$\prod_{i=1}^{n} x_i^{p_i x_i} \ge \left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right)^{\sum_{i=1}^{n} p_i x_i}. \tag{2}$$

Proof. The function $f(x) = x\log x$, $x > 0$, is convex. Thus we get

$$\sum_{i=1}^{n} p_i f(x_i) \ge P_n f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right),$$

i.e., (2). □

Since $f(x)$ is strictly convex, equality holds in (1) if and only if $a_1 = a_2 = a_3$.

[*Editor's note.* Janous also generalizes the problem to n-dimensional simplices. Using the method of solution I he obtains

$$\prod_{i=1}^{n+1} h_i^{F_i} \le [(n+1)r]^F,$$

where h_i are the altitudes, F_i the $(n-1)$-dimensional areas of the faces, $F = \sum F_i$, and r is the inradius.]

Also solved by L. J. HUT, Groningen, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; and MARCIN E. KUCZMA, Warszawa, Poland.

Klamkin's solution was very similar to solution II, obtaining a slightly weaker result.

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1499. [1989: 298] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

A second-order linear recursive sequence $\{A_n\}_1^\infty$ is defined by $A_{n+2} = A_{n+1} + A_n$ for all $n \ge 1$, with A_1 and A_2 any integers. Select a set S of any $2m$ consecutive elements from this sequence, where m is an odd integer. Prove that the sum of the numbers in S is always divisible by the $(m+2)$nd element of S, and the multiplying factor is L_m, the mth Lucas number.

Solution by Chris Wildhagen, Breda, The Netherlands.

Let

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then $A_n = r\alpha^n + s\beta^n$ for appropriate constants r and s. Thus

$$\begin{aligned}\sum_{k=n}^{n+2m-1} A_k &= \sum_{k=n}^{n+2m-1} (r\alpha^k + s\beta^k) \\ &= r\alpha^n \sum_{k=0}^{2m-1} \alpha^k + s\beta^n \sum_{k=0}^{2m-1} \beta^k \\ &= r\alpha^n \left(\frac{\alpha^{2m}-1}{\alpha-1}\right) + s\beta^n \left(\frac{\beta^{2m}-1}{\beta-1}\right) \\ &= r\alpha^{n+1}(\alpha^{2m}-1) + s\beta^{n+1}(\beta^{2m}-1). \qquad (1)\end{aligned}$$

To prove the required result, note that

$$\begin{aligned}A_{n+m+1}L_m &= (r\alpha^{n+m+1} + s\beta^{n+m+1})(\alpha^m + \beta^m) \\ &= r\alpha^{n+2m+1} + r\alpha^{n+1}(\alpha\beta)^m + s\beta^{n+2m+1} + s\beta^{n+1}(\alpha\beta)^m,\end{aligned}$$

and this equals (1), since $\alpha\beta = -1$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; BOB PRIELIPP, University of Wisconsin–Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

López Chamorro points out a related problem, B–31 *of the* Fibonacci Quarterly, *solved on p.* 233 *of the October* 1964 *issue.*

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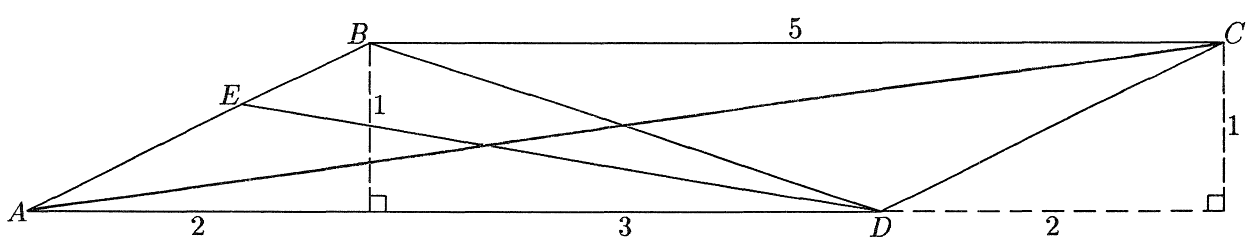
1500. [1989: 299] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

A parallelogram is called *self-diagonal* if its sides are proportional to its diagonals. Suppose that $ABCD$ is a self-diagonal parallelogram in which the bisector of angle ADB meets AB at E. Prove that $AE = AC - AB$.

I. *Editor's comment.*

Three readers, MURRAY S. KLAMKIN, University of Alberta, P. PENNING, Delft, The Netherlands, and KENNETH M. WILKE, Topeka, Kansas, point out that this result is **false** in some cases, namely for self-diagonal parallelograms $ABCD$ satisfying $AB/BC = BD/AC$. Here is an example.

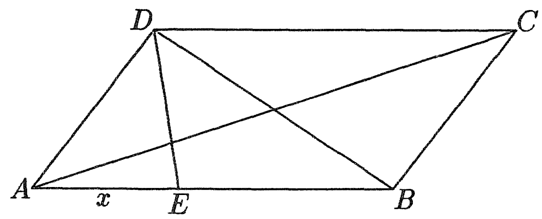


It is easy to see that $AB = \sqrt{5}$, $AD = 5$, $BD = \sqrt{10}$, and $AC = 5\sqrt{2}$, so $ABCD$ is self-diagonal. However, it is also pretty obvious that $AE < AC - AB$.

Penning and Wilke then showed that the problem is correct in the other case, that $AB/BC = AC/BD$, which was the case intended by the proposer and considered by all other solvers. Here are two of these solutions.

II. *Solution by Jack Garfunkel, Flushing, N.Y.*

Denote $AB = a$, $BC = b$, $AC = d_1$, $BD = d_2$, $AE = x$. We are given



$$\frac{a}{b} = \frac{d_1}{d_2}, \tag{1}$$

and since DE bisects $\angle ADB$,

$$\frac{x}{a-x} = \frac{b}{d_2},$$

or

$$x = \frac{ab}{b+d_2}.$$

We therefore have to show that

$$\frac{ab}{b+d_2} = d_1 - a,$$

which simplifies (by (1)) to showing that $2ab = d_1 d_2$. Now it is known that

$$d_1^2 + d_2^2 = 2(a^2+b^2)$$

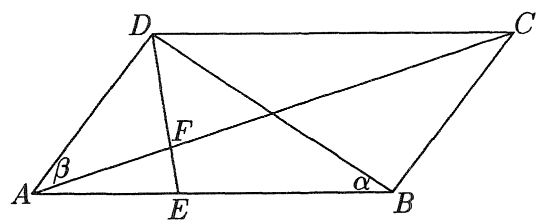
or, by (1) again,

$$2(a^2+b^2) = d_1^2 + \frac{b^2 d_1^2}{a^2} = \frac{d_1^2}{a^2}(a^2+b^2).$$

Hence $d_1 = a\sqrt{2}, d_2 = b\sqrt{2}$, and so $d_1 d_2 = 2ab$.

III. *Solution by Toshio Seimiya, Kawasaki, Japan.*

We assume that $AB : AD = AC : BD$, whence we get



$$AB \cdot BD = AD \cdot AC. \tag{2}$$

We put $\angle ABD = \alpha, \angle DAC = \beta$. Because $ABCD$ is a parallelogram,

$$\text{area}\,\Delta ABD = \text{area}\,\Delta ACD.$$

Therefore we get

$$\frac{1}{2}AB \cdot BD \sin\alpha = \frac{1}{2}AD \cdot AC \sin\beta.$$

Hence from (2) we have $\sin\alpha = \sin\beta$, and since $\alpha + \beta < \pi$ we get $\alpha = \beta$. Let F be the point of intersection of AC with DE. As

$$\angle AEF = \alpha + \angle EDB = \beta + \angle ADE = \angle AFE,$$

we get $AE = AF$. Because

$$\angle CDF = \angle AEF = \angle AFE = \angle CFD,$$

we have $CF = CD$. Hence

$$AE = AF = AC - CF = AC - CD = AC - AB.$$

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANTONIO LUIZ SANTOS, Rio de Janeiro, Brazil; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1501. [1990: 19] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Two circles K and K_1 touch each other externally. The equilateral triangle ABC is inscribed in K, and points A_1, B_1, C_1 lie on K_1 such that AA_1, BB_1, CC_1 are tangent to K_1. Prove that one of the lengths $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$ equals the sum of the other two. (The case when the circles are internally tangent was a problem of Florow in *Praxis der Mathematik* 13, Heft 12, page 327.)

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let O, O_1, r, r_1 be the centers and radii of K and K_1, and let T be the point of tangency. Assume without loss of generality that T belongs to the shorter arc AB. Then by Ptolemaeus, $\overline{AT} \cdot \overline{BC} + \overline{BT} \cdot \overline{CA} = \overline{CT} \cdot \overline{AB}$, whence

$$\overline{AT} + \overline{BT} = \overline{CT}. \tag{1}$$

[This is of course known. —*Ed.*] Produce AT to cut K_1 again in D. The isosceles triangles AOT and TO_1D are similar in ratio $r : r_1$, and so $\overline{AD}/\overline{AT} = (r + r_1)/r$. Thus

$$\overline{AA_1} = \sqrt{\overline{AT} \cdot \overline{AD}} = \sqrt{\frac{r + r_1}{r}} \cdot \overline{AT},$$

and similarly for $\overline{BB_1}$ and $\overline{CC_1}$. The claim results by (1).

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; JEFF HIGHAM, student, University of Toronto; L. J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria;

DAG JONSSON, Uppsala, Sweden; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; HUME SMITH, Chester, Nova Scotia; and the proposer. Several solvers gave the above solution.

Janous located the problem in two Bulgarian sources: problem 38, page 58 of Davidov, Petkov, Tonov, and Chukanov, Mathematical Contests, *Sofia, 1977; and problem 4.32, pages 55–56 of J. Tabov,* Homothety in Problems, *Sofia, 1989. He also found similar problems on pp. 172–175 of Honsberger's* Mathematical Morsels, *MAA, 1978.*

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1503. [1990: 19] *Proposed by M.S. Klamkin, University of Alberta.*

Prove that

$$1+2\cos(B+C)\cos(C+A)\cos(A+B)\geq\cos^2(B+C)+\cos^2(C+A)+\cos^2(A+B),$$

where A,B,C are nonnegative and $A+B+C\leq\pi$.

I. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

Put

$$B+C=\alpha,\ \ C+A=\beta,\ \ A+B=\gamma;$$

then the proposed inequality is $T\leq 0$ where

$$T=\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma-1.$$

Solving $T=0$ by considering T as a quadratic in $\cos\alpha$, we obtain

$$\begin{aligned}\cos\alpha&=\frac{2\cos\beta\cos\gamma\pm\sqrt{4\cos^2\beta\cos^2\gamma-4(\cos^2\beta+\cos^2\gamma-1)}}{2}\\&=\cos\beta\cos\gamma\pm\sqrt{(1-\cos^2\beta)(1-\cos^2\gamma)}\\&=\cos\beta\cos\gamma\pm\sin\beta\sin\gamma=\cos(\beta\mp\gamma).\end{aligned}$$

Thus

$$\begin{aligned}T&=(\cos\alpha-\cos(\beta+\gamma))(\cos\alpha-\cos(\beta-\gamma))\\&=4\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\alpha-\beta-\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\alpha-\beta+\gamma}{2}\\&=4\sin(A+B+C)\sin(-A)\sin C\sin B\\&=-4\sin(A+B+C)\sin A\sin B\sin C\\&\leq 0,\end{aligned}$$

since $A+B+C\leq\pi$ and A, B, C are non-negative.

II. *Solution by the proposer.*

Let

$$x = B + C, \quad y = C + A, \quad z = A + B.$$

Then $x+y+z \leq 2\pi$. Since $2A = y+z-x$, etc., x, y, z must satisfy the triangle inequality. Hence x, y, z are the angles of a (possibly degenerate) trihedral angle. If $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ denote unit vectors along the edges of this trihedral angle from the vertex, the volume V of the tetrahedron formed from $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ is given by $6V = \mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R})$ [p. 26 problem 37 of Spiegel, *Vector Analysis* (Schaum)]. Squaring, we obtain [p. 33 problem 89 of the same reference]

$$36V^2 = \begin{vmatrix} \mathbf{P}\cdot\mathbf{P} & \mathbf{P}\cdot\mathbf{Q} & \mathbf{P}\cdot\mathbf{R} \\ \mathbf{Q}\cdot\mathbf{P} & \mathbf{Q}\cdot\mathbf{Q} & \mathbf{Q}\cdot\mathbf{R} \\ \mathbf{R}\cdot\mathbf{P} & \mathbf{R}\cdot\mathbf{Q} & \mathbf{R}\cdot\mathbf{R} \end{vmatrix} = \begin{vmatrix} 1 & \cos y & \cos z \\ \cos y & 1 & \cos x \\ \cos z & \cos x & 1 \end{vmatrix}.$$

On expanding out the determinant, we see that the given inequality corresponds to $V^2 \geq 0$. There is equality if and only if $x+y+z = 2\pi$ or one of x, y, z equals the sum of the other two; correspondingly, if and only if $A+B+C = \pi$ or $ABC = 0$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; and KEE-WAI LAU, Hong Kong.

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JACK GARFUNKEL

One of *Crux*'s regular contributors since its earliest days, Jack Garfunkel, passed away on New Year's Eve, 1990. His many proposals in geometry, sometimes very difficult, but often beautiful, were well appreciated by *Crux* readers. He will be missed. The following information on his life was kindly furnished by his son, Sol Garfunkel.

Jack was born in Poland in 1910 and came to the United States at the age of nine. Although one of the top math majors at City College in New York, he left academic life after graduation to help his family weather the depression. For the next 25 years, he worked manufacturing candy. At the age of 45, he returned to his first love and became a high school mathematics teacher. Over the next 24 years, he taught at Forest Hills High School, supervising over two dozen Westinghouse finalist and semi-finalist winners of the talent search. When he "retired" from high school teaching, he immediately began work as an adjunct professor at Queens College and later at Queensborough Community College, where he taught until November of 1990.

Throughout his two careers as candyman and teacher, he continued to *do* mathematics, conjecturing and solving problems in synthetic geometry and geometric inequalities. While his formal education ended with his undergraduate work, he never lost his curiosity and love of his subject.

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