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CONTENTS



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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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# THE OLYMPIAD CORNER

## No. 100

R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

We begin with problems from two European Olympiads. Thanks go to Bruce Shawyer for collecting and relaying these questions to me.

### 1ST NORDIC MATHEMATICAL OLYMPIAD

March 30, 1987

Time: 4 hours

1. Nine foreign journalists meet at a press conference. Each of them speaks at most three different languages, and any two of them can speak a common language. Show that at least five of them speak the same language.

2. Let $ABCD$ be a parallelogram in the plane. Draw two circles with common radius $R$, one through the points $A$ and $B$, and the other through the points $B$ and $C$. Let $E$ be the second point of intersection of the two circles. Assume that $E$ does not coincide with any vertex of the parallelogram. Show that the circle through the points $A$, $D$ and $E$ also has radius $R$.

3. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function with $f(2) = a > 2$ and
$$f(mn) = f(m)f(n) \text{ for all } m, n \in \mathbb{N}.$$
Determine the least possible value for $a$.

4. Let $a$, $b$, $c$ be positive real numbers. Prove that
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \le \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

*

We go from Northern Europe to Southern Europe with the next set of problems.

### 4TH BALKAN OLYMPIAD

May 5, 1987

Time: 4 1/2 hours

1. Let $a$ be a real number, and let $f$ be a real-valued function defined on the set of all real numbers such that

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x)$$

for all reals $x$, $y$. Also assume that $f(0) = 1/2$. Show that $f$ is a constant function. (Yugoslavia)

2. Let $x \geq 1$ and $y \geq 1$ be such that the numbers

$$a = \sqrt{x-1} + \sqrt{y-1}$$

and

$$b = \sqrt{x+1} + \sqrt{y+1}$$

are non-consecutive integers. Show that $b = a + 2$ and $x = y = 5/4$. (Romania)

3. In a triangle $ABC$ such that

$$\sin^{23}(\alpha/2)\cos^{48}(\beta/2) = \sin^{23}(\beta/2)\cos^{48}(\alpha/2),$$

where $\alpha$ and $\beta$ are the angles with vertices at $A$ and $B$ respectively, compute the ratio $AC/BC$. (Cyprus)

4. Two circles $\kappa_1$, $\kappa_2$ with centres $O_1$, $O_2$ and radii 1, $\sqrt{2}$, respectively, intersect in two points $A$ and $B$. Also $O_1O_2$ has length 2. Let $AC$ be a chord in $\kappa_2$. Find the length of $AC$ if the midpoint of $AC$ lies on $\kappa_1$. (Bulgaria)

* * *

Now we return to solutions received for problems posed in the May 1987 number of the Corner.

7. [1987: 139] *Bulgarian Spring Competition-Kazanlik, 1985.*

Let $S_n = \sum_{k=0}^{n} \binom{3n}{3k}$. Prove that $\lim_{n\to\infty}(S_n)^{1/3n} = 2$. (Grade 11)

*First solution by George Evagelopoulos, Law student, Athens, Greece.*

For $n > k \geq 1$ we have that

$$\begin{aligned}\binom{3n}{3k} &= \binom{3n-1}{3k-1} + \binom{3n-1}{3k} \\ &= \binom{3n-2}{3k-2} + 2\binom{3n-2}{3k-1} + \binom{3n-2}{3k} \\ &> \binom{3n-2}{3k-2} + \binom{3n-2}{3k-1} + \binom{3n-2}{3k}.\end{aligned}$$

Thus for $n > 1$ we have that

$$S_n = \sum_{k=0}^{n} \binom{3n}{3k} = 1 + \sum_{k=1}^{n-1} \binom{3n}{3k} + 1$$

$$> 2 + \sum_{k=1}^{n-1} \left[ \binom{3n-2}{3k-2} + \binom{3n-2}{3k-1} + \binom{3n-2}{3k} \right]$$

$$= 2 + \sum_{k=1}^{3n-3} \binom{3n-2}{k} = \sum_{k=0}^{3n-2} \binom{3n-2}{k} = 2^{3n-2} = 2^{3n}/4.$$

Clearly

$$S_n = \sum_{k=0}^{n} \binom{3n}{3k} < \sum_{k=0}^{3n} \binom{3n}{k} = 2^{3n}.$$

Thus

$$2^{3n}/4 \leq S_n \leq 2^{3n}$$

and

$$2/4^{1/3n} \leq (S_n)^{1/3n} \leq 2.$$

But $\lim_{n\to\infty} 4^{1/3n} = 1$. Thus $\lim_{n\to\infty} (S_n)^{1/3n} = 2$.

*Second solution by M. Selby, Department of Mathematics, The University of Windsor, Ontario, and also by Robert E. Shafer of Berkeley, California.*

By the Binomial Theorem, we have

$$(1+x)^{3n} = \sum_{j=0}^{3n} \binom{3n}{j} x^j.$$

With $x = 1$ we get

$$2^{3n} = \sum_{j=0}^{3n} \binom{3n}{j}.$$

Setting $x = \omega = e^{\pi i/3}$,

$$(1+\omega)^{3n} = \sum_{j=0}^{3n} \binom{3n}{j} \omega^j$$

and also

$$(1+\omega^2)^{3n} = \sum_{j=0}^{3n} \binom{3n}{j} \omega^{2j}.$$

Adding these three,

$$2^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n} = \sum_{j=0}^{3n} \binom{3n}{j} (1 + \omega^j + \omega^{2j}).$$

Now since $\omega^3 = 1$,

$$1+\omega^j+\omega^{2j}=\begin{cases}3 & \text{if } j=3s\\ 1+\omega+\omega^2 & \text{if } j=3s+1\\ 1+\omega^2+\omega & \text{if } j=3s+2\end{cases}$$

$$=\begin{cases}3 & \text{if } 3\mid j\\ 0 & \text{otherwise.}\end{cases}$$

Using this and the fact that $1+\omega+\omega^2=0$ again we obtain

$$3\sum_{k=0}^{n}\binom{3n}{3k}=\sum_{j=0}^{3n}\binom{3n}{j}(1+\omega^j+\omega^{2j})$$

$$=2^{3n}+(1+\omega)^{3n}+(1+\omega^2)^{3n}$$
$$=2^{3n}+(-1)^{3n}\omega^{6n}+(-1)^{3n}\omega^{3n}$$
$$=2^{3n}+(-1)^{3n}\cdot 2.$$

Therefore

$$S_n=\frac{1}{3}[2^{3n}+(-1)^{3n}\cdot 2].$$

Thus

$$\frac{1}{3}(2^{3n}-2)\le S_n\le\frac{1}{3}(2^{3n}+2).$$

Since

$$\lim_{n\to\infty}\left(\frac{1}{3}(2^{3n}-2)\right)^{1/3n}=2=\lim_{n\to\infty}\left(\frac{1}{3}(2^{3n}+2)\right)^{1/3n}$$

the result follows.

* * *

The next solutions are for problems from the *Bulgarian Spring Mathematics Competition-Gambol*, March 1986.

**3.** [1987: 139] *Bulgarian Spring Competition, 1986.*

For any integer $m$, let $\tau(m)$ denote the number of positive integers which divide $m$. Prove that there exist infinitely many positive integers $n$ such that $\tau(2^n-1)>n$. (Grades 10, 11)

*Solution by M. Selby, Department of Mathematics, The University of Windsor, Ontario, and also by George Evagelopoulos, Law student, Athens, Greece.*

First observe that if $n$ is an integer such that $\tau(2^n-1)>n$, then $k=2n$ satisfies $\tau(2^k-1)>k$. To see this notice that

$$2^k-1=(2^n)^2-1=(2^n+1)(2^n-1).$$

Also the greatest common divisor of $2^n-1$ and $2^n+1$ is 1 since they are both odd and differ by 2. Therefore

$$\tau(2^k - 1) = \tau(2^n + 1)\tau(2^n - 1) \geq 2\tau(2^n - 1) > 2n = k$$

since for any integer $l$ greater than 1, $\tau(l) \geq 2$.

We can therefore construct an infinite sequence of solutions once we have found one. Now $\tau(2^n - 1) \leq n$ for $n \leq 11$, but $\tau(2^{12} - 1) = 24$, so $\tau(2^n - 1) > n$ for every $n = 2^k \cdot 3$ with $k \geq 2$.

4. [1987: 139] *Bulgarian Spring Competition, 1986.*

Prove that for every real number $x$ the inequality

$$x^2 - x + 0.96 > \sin x$$

holds. (Grade 11)

*First solution by Robert E. Shafer, Berkeley, California.*

In this solution we avoid using calculus, except the intermediate value theorem, as follows. For a contradiction suppose $x^2 - x + 0.96 \leq \sin x$ for some $x$. Then noticing that $0^2 - 0 + 0.96 > 0$ we conclude that there is $\xi$ with

$$\xi^2 - \xi + 0.96 = \sin \xi.$$

Now observe that

$$x^2 - x + 0.96 > 1$$

for $x < 0.5 - \sqrt{0.29}$ or $x > 0.5 + \sqrt{0.29}$, since

$$x^2 - x - 0.04 = [(x - 0.5) + \sqrt{0.29}][(x - 0.5) - \sqrt{0.29}].$$

Note

$$-0.04 < 0.5 - \sqrt{0.29} \text{ and } 0.5 + \sqrt{0.29} < 1.04,$$

since $(0.54)^2 = 0.2916$. Thus we have $-0.04 < \xi < 1.04$. Also $0.04 < \pi$ so $\sin x$ is negative on $[-0.04,0)$. This is impossible since $x^2 - x + 0.96 > 0$. Thus $0 < \xi < 1.04$.

Next recall that for $x > 0$, $\sin x < x$. Thus since

$$\xi^2 - \xi + 0.96 - \sin \xi = 0$$

we have

$$\xi^2 - 2\xi + 1 < 0.04,$$
$$(\xi - 1)^2 < (0.2)^2,$$

so

$$0.8 < \xi < 1.2.$$

With the above this gives $0.8 < \xi < 1.04$. Now, since $0.8 < \xi$

$$\xi^2 - \xi + 0.96 > (0.8)^2 - 0.8 + 0.96 = 0.8.$$

Let $x$ with $0 < x < \pi/2$ satisfy $\sin x = 0.8$. Then $x < \xi$. Now $\sin \pi/4 = \sqrt{2}/2 < 0.8$ so let $\epsilon > 0$ be such that $x = \pi/4 + \epsilon$. Then

$$0.8 = \sin x = \sin\frac{\pi}{4}\cos\epsilon + \cos\frac{\pi}{4}\sin\epsilon = \frac{1}{\sqrt{2}}(\cos\epsilon + \sin\epsilon).$$

Now $1 > \cos \epsilon$ and $\epsilon > \sin \epsilon$ so

$$0.8 < \frac{1 + \epsilon}{\sqrt{2}}.$$

This gives $\epsilon > (0.8)\sqrt{2} - 1 \geq 0.13124$. From this $\xi > \pi/4 + \epsilon > 0.916$. Next we have $1.04 < \pi/3$, however $\sin \pi/3 = \sqrt{3}/2 > 0.866$. But

$$\xi^2 - \xi + 0.96 < \sqrt{3}/2 \Rightarrow \xi^2 - \xi + 0.094 < 0$$
$$\Rightarrow (\xi - 1/2)^2 < 0.156$$

so

$$|\xi - 1/2| < 0.4$$

and $\xi < 0.9$, contradicting the previous estimate.

*Second solution by George Evagelopoulos, Law student, Athens, Greece.*

This solution uses calculus.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 - x + 0.96 - \sin x.$$

Then

$$f'(x) = 2x - 1 - \cos x$$

and

$$f''(x) = 2 + \sin x > 0.$$

Since $f''(x) > 0$, $f'(x)$ is strictly increasing. Since $f'(0) = -2$ and $f'(1) = 1 - \cos 1 > 0$, there is a unique $p$ with $f'(p) = 0$, and $0 < p < 1$.

Furthermore $f'(x) < f'(p) = 0$ for $x < p$ and $f'(x) > f'(p) = 0$ for $x > p$. Hence $f$ has a minimum at $p$. We shall prove that $f(p) > 0$.

First

$$f'(p) = 0 \Leftrightarrow 2p - 1 - \cos p = 0$$
$$\Leftrightarrow p = (1 + \cos p)/2.$$

Using this

$$\begin{aligned} f(p) &= \left(\frac{1 + \cos p}{2}\right)^2 - \left(\frac{1 + \cos p}{2}\right) + \frac{96}{100} - \sin p \\ &= \cos^4\left(\frac{p}{2}\right) - \cos^2\left(\frac{p}{2}\right) + \frac{24}{25} - \sin p \\ &= -\cos^2\left(\frac{p}{2}\right)\sin^2\left(\frac{p}{2}\right) + \frac{24}{25} - \sin p \\ &= -\frac{\sin^2 p}{4} - \sin p + \frac{24}{25} \\ &= -\frac{1}{4}\left(\sin p - \frac{4}{5}\right)\left(\sin p + \frac{24}{5}\right). \end{aligned}$$

To prove that $f(p) > 0$ it suffices to show $\sin p < 4/5$. But since $\cos p = 2p - 1$ and $0 < p < 1$ we have either

(i) $0 < p \leq 4/5$, and thus $\sin p < 4/5$ (as $\sin p < p$),

or

(ii) $4/5 < p$, whence $\cos p = 2p - 1 > 3/5$, and so

$$\sin p = \sqrt{1 - \cos^2 p} < 4/5.$$

In either case $\sin p < 4/5$ so that $f(p) > 0$. Thus for $x \in \mathbb{R}$

$$f(x) \geq f(p) > 0$$

and

$$x^2 - x + 0.96 > \sin x.$$

*Editor's note*: Comparing these two solutions, it appears the setter had the second one using the Calculus in mind. A similar solution was also submitted by M. Selby of the University of Windsor, Ontario.

* * *

1. [1987: 210] *28th I.M.O. Havana, 1987.*

Let $P_n(k)$ be the number of permutations of the set $\{1,\ldots,n\}$, $n \geq 1$, which have exactly $k$ fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot P_n(k) = n!.$$

(*Remark*: A permutation $f$ of a set $S$ is a one-to-one mapping of $S$ onto itself. An element $i$ in $S$ is called a fixed point of the permutation $f$ if $f(i) = i$.)

*Editor's comment.*

Professor V.N. Murty, Penn State, points out that this is a very well known classical problem, and so not suited to the Olympiad. He supplies the following standard solution.

One can view $P_n(k)/n!$ as the probability that a randomly chosen permutation has exactly $k$ fixed points. Thus

$$\sum_{k=0}^{n} \frac{k \cdot P_n(k)}{n!}$$

is just the expected number of fixed points. To see that this equals 1 let $X_i$ denote a random variable which takes the value 1 if $i$ is a fixed point of the permutation and 0 otherwise. Then

$$E(X_1 + \cdots + X_n) = \sum_{i=1}^{n} E(X_i).$$

But $E(X_i)$ is just the probability that $X_i = 1$, i.e.

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

Thus

$$E(X_1 + \cdots + X_n) = n \cdot \frac{1}{n} = 1.$$

Since $X_1 + X_2 + \cdots + X_n$ gives the total number of fixed points of a permutation,

$$\sum_{k=0}^{n} \frac{k\cdot P_n(k)}{n!} = 1$$

and

$$\sum_{k=0}^{n} k\cdot P_n(k) = n!$$

* * *

R.K. Guy, The University of Calgary, comments on the conjecture [1987: 212] that there are only two solutions to $a^2 + b^2 = n!$ for positive integers $a$, $b$ and $n$, with $a \le b$. He points out that the proof given for $n < 14$ generalizes if one has a strengthened version of "Bertrand's Postulate" (that there is always a prime between $x$ and $2x$). This would read:

"for all $x \geq 2$ there is a prime $p = 4k + 3$ such that $x \leq p < 2x$."

Is there an elementary proof of this?

* * *

The problems proposed but not used for the 28th I.M.O., Havana, 1987, elicited a good response. I shall begin this month to discuss the solutions submitted but will continue them into the next number.

**Australia 1**. [1987: 245]

Let $x_1,x_2,\ldots,x_n$ be $n$ integers and let $p$ be a positive integer less than $n$. Put

$$\begin{aligned} S_1 &= x_1 + x_2 + \cdots + x_p\,, & T_1 &= x_{p+1} + x_{p+2} + \cdots + x_n, \\ S_2 &= x_2 + x_3 + \cdots + x_{p+1}\,, & T_2 &= x_{p+2} + \cdots + x_n + x_1\,, \\ & \vdots & & \\ S_n &= x_n + x_1 + \cdots + x_{p-1}\,, & T_n &= x_p + x_{p+1} + \cdots + x_{n-1} \end{aligned}$$

(so the $x_i$ "wrap around", that is, after $x_n$ there comes $x_1$ again). Next let $m(a,b)$ be the number of numbers $i$ for which $S_i$ leaves the remainder $a$ and $T_i$ leaves the remainder $b$ on division by 3, where each of $a$ and $b$ is 0, 1, or 2.

Show that $m(1,2)$ and $m(2,1)$ leave the same remainder when divided by 3.

*Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Note first that

$$S_i + T_i = x_1 + x_2 + \cdots + x_n \text{ for } i = 1,2,\ldots,n. \tag{1}$$

Also

$$S_1 + S_2 + \cdots + S_n = p(x_1 + x_2 + \cdots + x_n). \tag{2}$$

*Case* 1. $x_1 + x_2 + \cdots + x_n \not\equiv 0 \bmod 3$. Then (1) implies $m(1,2) = m(2,1) = 0$.

*Case* 2. $x_1 + x_2 + \cdots + x_n \equiv 0 \bmod 3$. Then (1) and (2) imply

$$S_i + T_i \equiv 0 \bmod 3, \quad i = 1,2,\ldots,n \tag{3}$$

and

$$S_1 + \cdots + S_n \equiv 0 \bmod 3. \tag{4}$$

From (3) we have $S_i \equiv 1 \bmod 3$ iff $T_i \equiv 2 \bmod 3$ and $S_i \equiv 2 \bmod 3$ iff $T_i \equiv 1 \bmod 3$. Therefore, from (4) we have

$$2m(2,1) + m(1,2) \equiv 0 \bmod 3.$$

Thus

$$2m(2,1) \equiv -m(1,2) \equiv 2m(1,2) \bmod 3$$

from which $m(2,1) \equiv m(1,2) \bmod 3$ follows.

**Australia 2**. [1987: 245]

$a_1$, $a_2$, $a_3$, $b_1$, $b_2$, $b_3$ are positive real numbers. Prove that

$$(a_1b_2 + b_1a_2 + a_2b_3 + b_2a_3 + a_3b_1 + b_3a_1)^2$$
$$\geq 4(a_1a_2 + a_2a_3 + a_3a_1)(b_1b_2 + b_2b_3 + b_3b_1)$$

and show that the two sides of the inequality are equal if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

*Solution by George Evagelopoulos, Law student, Athens, Greece.*

We may assume that $a_3 = b_3 = 1$. For otherwise divide the inequality by $a_3^2b_3^2$ and replace $a_1$, $a_2$, $b_1$, $b_2$ by

$$\frac{a_1}{a_3}, \frac{a_2}{a_3}, \frac{b_1}{b_3}, \frac{b_2}{b_3},$$

respectively.

Let $b_1 = a_1 + d_1$, $b_2 = a_2 + d_2$. Then the inequality to be proved, after some regrouping of the terms, becomes

$$[2(a_1a_2 + a_1 + a_2) + d_1(a_2 + 1) + d_2(a_1 + 1)]^2$$
$$\geq 4(a_1a_2 + a_1 + a_2)[(a_1a_2 + a_1 + a_2) + d_1d_2 + d_1(a_2 + 1) + d_2(a_1 + 1)].$$

Cancelling identical terms on the two sides, we are left with

$$[d_1(a_2 + 1) + d_2(a_1 + 1)]^2 \geq 4(a_1a_2 + a_1 + a_2)d_1d_2.$$

Now rewriting the right-hand side we get the equivalent form

$$[d_1(a_2 + 1) + d_2(a_1 + 1)]^2 \geq 4(a_1 + 1)(a_2 + 1)d_1d_2 - 4d_1d_2.$$

This becomes

$$[d_1(a_2 + 1) - d_2(a_1 + 1)]^2 \geq -4d_1d_2.$$

This inequality clearly holds if $d_1d_2 \geq 0$, since the left-hand side is non-negative and the right-hand side is non-positive.

So, suppose that $d_1d_2 < 0$ and without loss of generality that $d_1 > 0$, $d_2 < 0$. Set $d = -d_2 > 0$. Then we have to prove

$$[d_1(a_2 + 1) + d(a_1 + 1)]^2 > 4dd_1.$$

This is clearly the case since $a_1$, $a_2 > 0$ and the left-hand side is greater than $(d_1 + d)^2$ which, in turn, is greater than or equal to $4dd_1$.

Equality holds only in the first case and when $d_1 = 0 = d_2$. This gives $b_1 = a_1$ and $b_2 = a_2$, which in terms of the original numbers is equivalent to the stated conditions.

*Editor's note*: Zun Shan and Edward T.H. Wang of Wilfrid Laurier University, Waterloo, Ontario, sent in two solutions, the first based on expansion and regrouping and a second which neatly gives the inequality, but which does not so readily give the conditions for equality. The essence is to look at the quadratic $p(x) = Ax^2 - Bx + C$ where

$$A = b_1b_2 + b_2b_3 + b_3b_1,$$
$$B = a_1b_2 + a_2b_1 + a_2b_3 + a_3b_2 + a_3b_1 + a_1b_3,$$
$$C = a_1a_2 + a_2a_3 + a_3a_1.$$

Then the given inequality is equivalent to $B^2 - 4AC \geq 0$. It is straightforward to factor

$$p(x) = (b_1x - a_1)(b_2x - a_2) + (b_2x - a_2)(b_3x - a_3) + (b_3x - a_3)(b_1x - a_1).$$

Since interchanging $a_i$ and $a_j$ and simultaneously $b_i$ and $b_j$ leaves the inequality unaltered one may assume that $a_1/b_1 \leq a_2/b_2 \leq a_3/b_3$. Since $A > 0$ while

$$P(a_2/b_2) = (a_2b_3 - a_3b_2)(a_2b_1 - a_1b_2)/b_2^2 \leq 0,$$

the polynomial has real roots and its discriminant is non-negative.

Yet another solution was sent in by Murray Klamkin.

*Alternate solution by Murray S. Klamkin, The University of Alberta, Edmonton.*

We will show that the given inequality reduces to a special case of the known [1] triangle inequality

$$(ax + by + cz)^2 \geq 16(yz/bc + zx/ca + xy/ab)F^2 \qquad (1)$$

where $a$, $b$, $c$ are sides of a triangle $ABC$ of area $F$, and $x$, $y$, $z$ are arbitrary real numbers. Equality holds in (1) if and only if

$$\frac{a(b^2 + c^2 - a^2)}{x} = \frac{b(c^2 + a^2 - b^2)}{y} = \frac{c(a^2 + b^2 - c^2)}{z}$$

or equivalently

$$\frac{x}{\cos A} = \frac{y}{\cos B} = \frac{z}{\cos C}.$$

Letting $b_2 + b_3 = 2x_1^2$, $b_3 + b_1 = 2x_2^2$, $b_1 + b_2 = 2x_3^2$, the given inequality reduces (after some algebra) to

$$(a_1x_1^2 + a_2x_2^2 + a_3x_3^2)^2 \geq 16(a_2a_3 + a_3a_1 + a_1a_2)\ F^2 \qquad (2)$$

where here

$$16\ F^2 = 2(x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2^2) - (x_1^4 + x_2^4 + x_3^4).$$

Since

$$16\ F^2 = (x_1 + x_2 + x_3)(x_2 + x_3 - x_1)(x_3 + x_1 - x_2)(x_1 + x_2 - x_3),$$

inequality (2) is obviously satisfied if $x_1$, $x_2$, $x_3$ are not sides of a triangle (in which case $16\ F^2 \leq 0$). So we can assume that $x_1$, $x_2$, $x_3$ are sides of a triangle and then $F$ is its area.

The inequality (2) is obtained from (1) simply by letting

$$(a,b,c) = (x_1,x_2,x_3) \text{ and } (x,y,z) = (a_1x_1,a_2x_2,a_3x_3).$$

Also, the equality in (1) reduces to

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

*Reference:*

[1] M.S. Klamkin, *Asymmetric triangle inequalities*, Publ. Electrotehn. Ser. Mat. Fiz. Univ. Beograd, No. 357–380 (1971) p.35.

**Belgium 1**. [1987: 245]

If $f:(0,\infty) \to \mathbb{R}$ is a function having the property that $f(x) = f(1/x)$ for all $x > 0$, prove that there exists a function $u$: $[1,\infty) \to \mathbb{R}$ such that

$$u\left(\frac{x + 1/x}{2}\right) = f(x) \text{ for all } x > 0.$$

*Solution by George Evagelopoulos, Law student, Athens, Greece and also by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

If $y = \frac{x + 1/x}{2} \geq 1$ then $x^2 - 2xy + 1 = 0$ and we have $x = y \pm \sqrt{y^2 - 1}$. Notice that $h(x) = x + \sqrt{x^2 - 1}$ maps $[1,\infty)$ to $[1,\infty)$.

Now let $u(x) = f(x + \sqrt{x^2 - 1})$. Then $u$: $[1,\infty) \to \mathbb{R}$ and moreover for $x > 0$

$$\begin{aligned} u\left(\frac{x + 1/x}{2}\right) &= f\left(\frac{x + 1/x}{2} + \sqrt{\frac{(x + 1/x)^2}{4} - 1}\right) \\ &= f\left(\frac{x + 1/x}{2} + \frac{\sqrt{(x - 1/x)^2}}{2}\right) \\ &= \begin{cases} f(x) & x \geq 1 \\ f(1/x) & x < 1 . \end{cases} \end{aligned}$$

However, since $f(x) = f(1/x)$ we have $u\left(\frac{x + 1/x}{2}\right) = f(x)$, for $x > 0$.

**Finland 2**. [1987: 246]

Does there exist a second degree polynomial $p(x,y)$ in two variables such that every non-negative integer $n$ equals $p(k,m)$ for one and only one ordered pair $(k,m)$ of non-negative integers?

*Solution by George Evagelopoulos, Law student, Athens, Greece.*

Yes! Enumerate such points $(k,m)$ by setting

$p(k,m) = k +$ "the number of points $(r,s)$ with non-negative integers $r$, $s$ that lie *below* the line $x + y = k + m$".

Then

$$p(k,m) = k + (0 + 1 + \cdots + (k + m)) = k + \frac{(k + m)(k + m + 1)}{2}.$$

So

$$p(x,y) = \frac{1}{2}[(x + y)^2 + 3x + y]$$

is a polynomial of the desired type.

**France 1**. [1987: 246]

Let $t_1, t_2, \ldots, t_n$ be $n$ real numbers satisfying $0 < t_1 \le t_2 \le \cdots \le t_n < 1$. Prove that

$$(1 - t_n)^2 \left[\frac{t_1}{(1 - t_1^2)^2} + \frac{t_2^2}{(1 - t_2^3)^2} + \cdots + \frac{t_n^n}{(1 - t_n^{n+1})^2}\right] < 1.$$

*Solutions by George Evagelopoulos, Law student, Athens, Greece; M.A. Selby, University of Windsor, Ontario; and also by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For any integer $k \ge 1$ and for $0 < t \le s < 1$ we have $0 < t^k \le s^k < 1$ and $(1 - s^{k+1})^2 \le (1 - t^{k+1})^2$. Thus

$$\frac{t^k}{(1 - t^{k+1})^2} \le \frac{s^k}{(1 - s^{k+1})^2}.$$

Thus

$$(1 - t_n)^2 \sum_{k=1}^{n} \frac{t_k^k}{(1 - t_k^{k+1})^2} \le \sum_{k=1}^{n} t_n^k \left(\frac{1 - t_n}{1 - t_n^{k+1}}\right)^2 = \sum_{k=1}^{n} \frac{t_n^k}{(1 + t_n + \cdots + t_n^k)^2}$$

$$\le \sum_{k=1}^{n} \frac{t_n^k}{(1 + t_n + \cdots + t_n^{k-1})(1 + t_n + \cdots + t_n^k)}$$

$$= \sum_{k=1}^{n} \left(\frac{1}{1 + \cdots + t_n^{k-1}} - \frac{1}{1 + \cdots + t_n^k}\right)$$

$$= 1 - \frac{1}{1 + \cdots + t_n^n} < 1.$$

*

Next month we will continue with the solutions to these unused I.M.O. problems. In the meantime send me your contest problems and solutions!

* * *

## THANK YOU, KEN WILLIAMS

This issue marks the end of Kenneth Williams' tenure as managing editor and, more recently, technical editor of *Crux Mathematicorum*. Fred Maskell's sudden illness in December 1984 meant that *Crux* had to find a new managing editor on short notice. Ken generously agreed to take on this duty, despite many other demands on his time (for instance he was, and still is, an active researcher in number theory). Ken's presence on the job three years ago was for this novice editor a source of encouragement, and the service he has performed for *Crux* over four years essential, if not very visible. The contribution readers may most vividly remember him for is his long, affectionate obituary of Léo Sauvé [1987: 240]. Ken has also contributed a number of problems to *Crux* (#1176 [1988: 19] is a personal favourite of the editor) as well as solutions; here's hoping he continues to do so.

Ken's duties as technical editor will in future be divided between the managing editor and the staff of the Canadian Mathematical Society (in particular Claudine Le Quellec), and the position of technical editor will disappear.

# PROBLEMS

*Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk* (*) *after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **July 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

**1391**. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $ABC$ be a triangle and $D$ the point on $BC$ so that the incircle of $\Delta ABD$ and the excircle (to side $DC$) of $\Delta ADC$ have the same radius $\rho_1$. Define $\rho_2$, $\rho_3$ analogously. Prove that

$$\rho_1 + \rho_2 + \rho_3 \ge \frac{9}{4}r,$$

where $r$ is the inradius of $\Delta ABC$.

**1392**. *Proposed by Angel Dorito, Geld, Ontario.*

An immense spherical balloon is being inflated so that it constantly touches the ground at a fixed point $A$. A boy standing at a point at unit distance from $A$ fires an arrow at the balloon. The arrow strikes the balloon at its nearest point (to the boy) but does not penetrate it, the balloon absorbing the shock and the arrow falling vertically to the ground. What is the longest distance through which the arrow can fall, and how far from $A$ will it land in this case?

1393. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle with incenter $I$, excenters $I_1$, $I_2$, $I_3$, and median point $G$. Let $H_1$ be the orthocenter of $\Delta I_1A_2A_3$, and define $H_2$, $H_3$ analogously. Prove that $A_1H_1$, $A_2H_2$, $A_3H_3$ are concurrent at a point collinear with $G$ and $I$.

1394. *Proposed by Murray S. Klamkin, University of Alberta.*

If $x$, $y$, $z > 0$, prove that

$$\sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} + \sqrt{x^2 + xy + y^2} \geq 3\sqrt{yz + zx + xy}.$$

1395. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given an equilateral triangle $ABC$, find all points $P$ in the same plane such that $(PA)^2$, $(PB)^2$, $(PC)^2$ form a triangle.

1396. *Proposed by Colin Springer, student, Waterloo, Ontario.*

Evaluate

$$\prod_{k=1}^{n-1}\left(1 - \cos\frac{2k\pi}{n}\right)$$

where $n$ is a positive integer, $n \geq 2$.

1397. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Find the equation of the circle passing through the points other than the origin which are common to the two conics

$$x^2 + 6xy + 10x - 2y = 0,$$
$$x^2 + 3y^2 - 7x + 5y = 0.$$

1398. *Proposed by Ravi Vakil, student, University of Toronto.*

Let $p$ be an odd prime. Show that there are at most $(p + 1)/4$ consecutive quadratic residues mod $p$. For which $p$ is this bound attained?

1399. *Proposed by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Prove that

$$\sigma(n!) \leq \frac{(n + 1)!}{2}$$

for all natural numbers $n$ and determine all cases when equality holds. (Here $\sigma(k)$ denotes the sum of all positive divisors of $k$.)

1400. *Proposed by Robert E. Shafer, Berkeley, California.*

In a recent issue of the *American Mathematical Monthly* (June–July 1988, page 551), G. Klambauer showed that if $x^s e^{-x} = y^s e^{-y}$ $(x, y, s > 0)$ then $x + y > 2s$. Show that if $x^s e^{-x} = y^s e^{-y}$ and $x, y, s > 0$ then $xy(x + y) < 2s^3$.

*　　　　*　　　　*

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1122**. [1986: 50; 1987: 197; 1988: 204] *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*

Find a dissection of a $6 \times 6 \times 6$ cube into a small number of connected pieces which can be reassembled to form cubes of sides 3, 4, and 5, thus demonstrating that $3^3 + 4^3 + 5^3 = 6^3$. One could ask this in at least four forms:

(a) the pieces must be bricks, with integer dimensions;

(b) the pieces must be unions of $1 \times 1 \times 1$ cells of the cube;

(c) the pieces must be polyhedral;

(d) no restriction.

*Comment by Hans Havermann, Weston, Ontario.*

Please refer to *Knotted Doughnuts* by Martin Gardner (W.H. Freeman, 1986), pp. 198–201. Shown there are three solutions to part (b) wherein the 3–cube, 4–cube and 5–cube, respectively, are left intact (all are 8–piece dissections). It is also stated that Thomas H. O'Beirne showed that an eight–piece dissection into rectangular blocks is not possible, but that he did find a nine–block dissection (given on p.201), this back in 1971. It is not known whether it is unique.

* * *

**1281***. [1987: 289] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find the area of the largest triangle whose vertices lie in or on a unit $n$–dimensional cube.

*Solution by Murray S. Klamkin, University of Alberta.*

Since the distance from a point $P$ to a given line is a convex function of $P$, it follows that the vertices of the maximum area triangle (which exists by continuity) must be vertices of the cube.

Let the vertices of the unit cube be $(\pm 1/2, \pm 1/2, \ldots, \pm 1/2)$. Without loss of generality, we can let the point $A : (-1/2, -1/2, \ldots, -1/2)$ be one of the vertices of the maximum area triangle. If $B$ and $C$ denote the other two vertices, then let

$$\mathbf{x} = \overrightarrow{AB} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = \overrightarrow{AC} = (y_1, y_2, \ldots, y_n).$$

It follows that all the components of $\mathbf{x}$ and $\mathbf{y}$ are 0 or 1. The area $F_n$ of $\Delta ABC$ is $|\mathbf{x} \times \mathbf{y}|/2$ so that

$$\begin{aligned} 4F_n^2 &= \mathbf{x}^2\mathbf{y}^2 - (\mathbf{x}\cdot\mathbf{y})^2 \\ &= (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) - (x_1y_1 + \cdots + x_ny_n)^2. \end{aligned}$$

Let $r$ and $s$ denote the number of zero components in $\mathbf{x}$ and $\mathbf{y}$ respectively, where $r \geq s$ without loss of generality. For given $r$ and $s$, $\mathbf{x}\cdot\mathbf{y}$ will be a minimum when the zero components of $\mathbf{x}$ are paired with the 1 components of $\mathbf{y}$ and vice versa. Thus

$$\min(\mathbf{x}\cdot\mathbf{y}) = n - r - s$$

(it will turn out that $r + s < n$) and

$$\max(4F_n^2) = \max\{(n-r)(n-s) - (n-r-s)^2\}.$$

We now maximize over $r$, $s$ by completing the square, i.e.

$$\max(4F_n^2) = \max\left\{\frac{n^2}{3} - \frac{3}{4}\left(s - \frac{n}{3}\right)^2 - \frac{(2r + s - n)^2}{4}\right\}.$$

For $n = 3m$, the maximum occurs for $r = s = m$. For $n = 3m + 1$, it occurs for $s = m$, $r = m$ or $m + 1$. For $n = 3m + 2$, it occurs for $r = m + 1$, $s = m$ or $m + 1$. In all cases we can choose $r = s = [(n + 1)/3]$, [ ] denoting the greatest integer function, so that

$$\max F_n = \frac{1}{2}\sqrt{\left[\frac{n+1}{3}\right]\left[2n - 3\left[\frac{n+1}{3}\right]\right]}. \tag{1}$$

We now consider the related problem of determining the maximum perimeter of a triangle inscribed in a unit $n$–cube. Since the sum of the distances from a point $P$ to the endpoints of a given line segment is a convex function of $P$, it follows as before that the vertices of the maximum perimeter triangle (which exists by continuity) must be vertices of the cube. Using the same representation as before, the perimeter $P_n$ of $\Delta ABC$ is

$$|\mathbf{x}| + |\mathbf{y}| + |\mathbf{x} - \mathbf{y}|.$$

Thus

$$\begin{aligned} \max P_n &= \max\{\sqrt{\mathbf{x}^2} + \sqrt{\mathbf{y}^2} + \sqrt{\mathbf{x}^2 + \mathbf{y}^2 - 2\mathbf{x}\cdot\mathbf{y}}\} \\ &= \max\{\sqrt{n - r} + \sqrt{n - s} + \sqrt{r + s}\}. \end{aligned}$$

Since $\sqrt{x}$ is concave, $\max P_n \leq 3\sqrt{2n/3}$, with equality if $r = s = n/3$. Thus for $n = 3m$, the maximum occurs at $r = s = m$; for $n = 3m + 1$, at $s = m$, $r = m$ or $m + 1$; and for $n = 3m + 2$, at $r = m + 1$, $s = m$ or $m + 1$. In all cases we can choose $r = s = [(n + 1)/3]$ so that

$$\max P_n = 2\sqrt{n - \left[\frac{n+1}{3}\right]} + \sqrt{2\left[\frac{n+1}{3}\right]}.$$

As expected, $\max P_2 = 1 + 1 + \sqrt{2}$ and $\max P_3 = \sqrt{2} + \sqrt{2} + \sqrt{2}$.

Open problems would be determination of the maximum (i) volume, (ii) surface area, and (iii) total edge length of a tetrahedron inscribed in a unit $n$–cube, and more generally the same problems for an inscribed $r$–dimensional simplex, where $r \leq n$.

*Also solved by P. PENNING, Delft, The Netherlands; and G. TSINTSIFAS, Thessaloniki, Greece.*

*The problem of finding the largest (in volume) r–simplex inscribed in a unit n–sphere was also proposed by Tsintsifas. He gave an argument that an inscribed tetrahedron has volume at most*

$$\frac{1}{3}\sqrt{\frac{n(n-1)(n-2)}{6}}\,,$$

*but claimed equality only for* $n = 3$.

*Penning's solution expressed the maximum area of a triangle inscribed in a unit n–cube as*

$$\begin{cases} \dfrac{n}{2\sqrt{3}}\,, & n \equiv 0 \mod 3, \\[2ex] \dfrac{\sqrt{n^2-1}}{2\sqrt{3}}\,, & \text{otherwise}, \end{cases}$$

*which, to the editor's eyes, is more pleasing than* (1). *Penning also gave a not-quite-rigorous argument that the maximum area of a (plane) quadrangle inscribed in a unit n–cube is*

$$\begin{cases} n/2\,, & n \text{ even} \\ \sqrt{n^2-1}/2\,, & n \text{ odd}. \end{cases}$$

* * *

**1282**. [1987: 289] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $ABC$ be a triangle, $I$ the incenter, and $A'$, $B'$, $C'$ the intersections of $AI$, $BI$, $CI$ with the circumcircle. Show that

$$IA' + IB' + IC' - (IA + IB + IC) \leq 2(R - 2r)$$

where $R$ and $r$ are the circumradius and inradius of $\Delta ABC$.

*Solution par C. Festraets–Hamoir, Brussels, Belgium.*

$I_a$, $I_b$, $I_c$ sont les centres des cercles ex–inscrits au $\Delta ABC$. Par la relation d'Erdős-Mordell, on a, dans $\Delta I_a I_b I_c$,

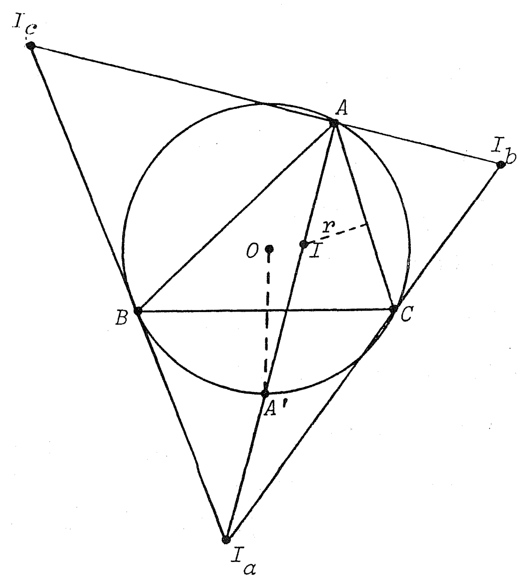
$$\sum II_a \geq 2 \sum IA.$$

Aussi $II_a = 2IA'$, d'où

$$\sum (IA' - IA) \geq 0. \qquad (1)$$

On sait que

$AI \sin \frac{A}{2} = r$, $AI \cos \frac{A}{2} = s - a$,



donc

$$\sum AI\sin\frac{A}{2} = 3r\,,\quad \sum AI\cos\frac{A}{2} = 3s - a - b - c = s. \tag{2}$$

On a aussi

$$A'I = 2R\sin\frac{A}{2}\,,$$

donc

$$\sum A'I\sin\frac{A}{2} = \sum R(1-\cos A) = R(3-(1+\frac{r}{R})) = 2R - r \tag{3}$$

et

$$\sum A'I\cos\frac{A}{2} = \sum R\sin A = \sum \frac{a}{2} = s. \tag{4}$$

Maintenant, on sait

$$\cos\frac{A}{2} + \sin\frac{A}{2} \ge 1$$

et on peut toujours supposer que

$$\cos\frac{A}{2} + \sin\frac{A}{2} \le \cos\frac{B}{2} + \sin\frac{B}{2} \le \cos\frac{C}{2} + \sin\frac{C}{2}\,.$$

On a donc, par (1) et (2)–(4),

$$\begin{aligned}\sum(A'I - AI) &\le \left[\sum(A'I - AI)\right]\cdot(\cos\frac{A}{2} + \sin\frac{A}{2})\\ &\le \sum\left[(A'I - AI)(\cos\frac{A}{2} + \sin\frac{A}{2})\right]\\ &= s + 2R - r - s - 3r = 2(R - 2r).\end{aligned}$$

*Also solved by SVETOSLAV J. BILCHEV and EMILIA A. VELIKOVA, Technical University, Russe, Bulgaria; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and the proposer.*

* * *

**1283**. [1987: 289] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Show that the polynomial

$$P(x,y,z) = (x^2 + y^2 + z^2)^3 - (x^3 + y^3 + z^3)^2 - (x^2y + y^2z + z^2x)^2 - (xy^2 + yz^2 + zx^2)^2$$

is nonnegative for all real $x$, $y$, $z$.

*Solution by Jorg Harterich, student, Winnenden, Federal Republic of Germany.*

This can be seen by writing the expression in another way:

$$\begin{aligned}&(x^2 + y^2 + z^2)^3 - (x^3 + y^3 + z^3)^2 - (x^2y + y^2z + z^2x)^2 - (xy^2 + yz^2 + zx^2)^2\\ &\quad = 2x^4y^2 + 2x^4z^2 + 2y^4x^2 + 2y^4z^2 + 2z^4x^2 + 2z^4y^2 + 6x^2y^2z^2 - 2x^3y^3\\ &\qquad - 2y^3z^3 - 2x^3z^3 - 2x^3z^2y - 2x^3y^2z - 2y^3x^2z - 2y^3z^2x - 2z^3x^2y - 2z^3y^2x\end{aligned}$$

$$= (x^2y^2 + y^2z^2 + x^2z^2)(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz)$$
$$= (x^2y^2 + y^2z^2 + x^2z^2)((x-y)^2 + (y-z)^2 + (x-z)^2).$$

It is obvious that this is nonnegative for all real $x$, $y$, $z$.

*Also solved by BENO ARBEL, Tel–Aviv University; FRANCISCO BELLOT ROSADO, E. Ferrari High School, Valladolid, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; GEORGE EVAGELOPOULOS, Athens, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; Z.F. LI, University of Regina; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; P. PENNING, Delft, The Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.*

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**1284**. [1987: 290] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

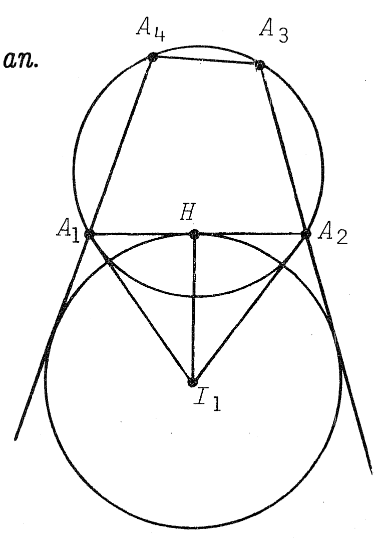
Let $A_1A_2A_3A_4$ be a cyclic quadrilateral with $\overline{A_1A_2} = a_1$, $\overline{A_2A_3} = a_2$, $\overline{A_3A_4} = a_3$, $\overline{A_4A_1} = a_4$. Let $\rho_1$ be the radius of the circle outside the quadrilateral, tangent to the segment $A_1A_2$ and the extended lines $A_2A_3$ and $A_4A_1$. Define $\rho_2$, $\rho_3$, $\rho_4$ analogously. Prove that

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_4} \ge \frac{8}{\sqrt[4]{a_1a_2a_3a_4}}.$$

When does equality hold?

*Solution by Tosio Seimiya, Kanagawa, Kawasaki, Japan.*

We label as shown in the figure.




Then

$$\frac{a_1}{\rho_1} = \frac{A_1H + HA_2}{I_1H}$$
$$= \cot(\angle HA_1I_1) + \cot(\angle HA_2I_1)$$
$$= \cot(A_3/2) + \cot(A_4/2)$$
$$\ge 2\sqrt{\cot(A_3/2)\ \cot(A_4/2)}$$

by the A.M.–G.M. inequality, and therefore

$$\frac{1}{\rho_1} \ge \frac{2}{a_1}\sqrt{\cot(A_3/2)\ \cot(A_4/2)}.$$

For the same reason,

$$\frac{1}{\rho_2} \ge \frac{2}{a_2}\sqrt{\cot(A_4/2)\ \cot(A_1/2)},\quad \frac{1}{\rho_3} \ge \frac{2}{a_3}\sqrt{\cot(A_1/2)\cot(A_2/2)},$$
$$\frac{1}{\rho_4} \ge \frac{2}{a_4}\sqrt{\cot(A_2/2)\cot(A_3/2)}.$$

Since $A_1 + A_3 = A_2 + A_4 = \pi$,

$$\cot(A_1/2) = \cot(\pi/2 - A_3/2) = \tan(A_3/2)$$

and similarly

$$\cot(A_2/2) = \tan(A_4/2).$$

Thus

$$\begin{aligned}\frac{1}{\rho_1} + \frac{1}{\rho_3} &\geq \frac{2}{a_1}\sqrt{\cot(A_3/2)\ \cot(A_4/2)} + \frac{2}{a_3}\sqrt{\tan(A_3/2)\ \tan(A_4/2)} \\ &\geq 2\sqrt{\frac{2}{a_1}\cdot\frac{2}{a_3}\sqrt{\cot(A_3/2)\ \cot(A_4/2)\ \tan(A_3/2)\ \tan(A_4/2)}} \\ &= \frac{4}{\sqrt{a_1 a_3}},\end{aligned}$$

and similarly

$$\frac{1}{\rho_2} + \frac{1}{\rho_4} \geq \frac{4}{\sqrt{a_2 a_4}}.$$

Hence

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_4} \geq 4\left[\frac{1}{\sqrt{a_1 a_3}} + \frac{1}{\sqrt{a_2 a_4}}\right] \geq 8\sqrt{\frac{1}{\sqrt{a_1 a_3}}\cdot\frac{1}{\sqrt{a_2 a_4}}} = \frac{8}{\sqrt[4]{a_1 a_2 a_3 a_4}}.$$

Equality holds when $a_1 = a_2 = a_3 = a_4$ and

$$\cot(A_1/2) = \cot(A_2/2) = \cot(A_3/2) = \cot(A_4/2),$$

which means that equality holds when $A_1A_2A_3A_4$ is a square.

*Also solved by FRANCISCO BELLOT ROSADO, E. Ferrari High School and MARIA ASCENSION LOPEZ CHAMORRO, L. Cano High School, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, The Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.*

*The proposer (and also Smeenk) actually showed that*

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_4} \geq \frac{4}{\sqrt[4]{a_1 a_2 a_3 a_4}}\sqrt{2(\csc A_1 + \csc A_2)}.$$

*Janous observed that*

$$a_1 a_2 a_3 a_4 \geq 16\rho_1\rho_2\rho_3\rho_4,$$

*which can be derived from the above proof.*

* * *

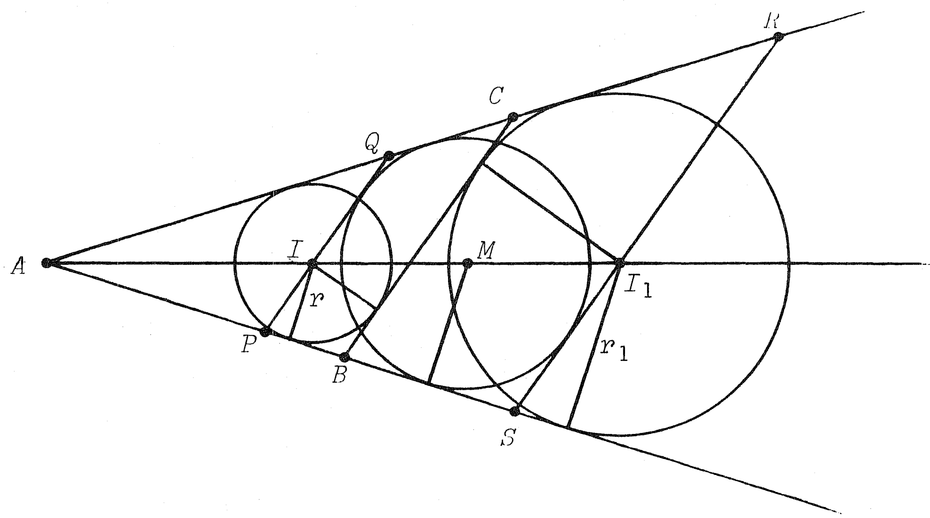
<u>1285</u>. [1987: 290] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

$I$ is the incenter of a triangle $ABC$ and $I_1$ is the excenter opposite $A$. Lines through $I$ and $I_1$ parallel to $BC$ meet $AB$ at $P$, $S$ and $AC$ at $Q$, $R$ respectively.

(a) Show that the trapezium $PQRS$ has an inscribed circle.

(b) Find the length of $BC$ in terms of the lengths of $PQ$ and $RS$.

I. *Solution by P. Penning, Delft, The Netherlands.*




(a) Let $r$ and $r_1$ be the inradius and exradius (to $BC$), respectively, of $\Delta ABC$. Consider the circle with centre $M$, midway between $I$ and $I_1$, that touches $AS$ and $AR$. Its radius must be the average of $r$ and $r_1$, i.e. $(r + r_1)/2$. Also note that the parallel lines $PQ$ and $RS$ lie a distance $r + r_1$ from one another. Thus, since $M$ lies midway between $PQ$ and $RS$, the circle also touches $PQ$ and $RS$, and so is inscribed in $PQRS$.

(b) The triangles $APQ$, $ABC$, and $ASR$ are similar. Hence, from the inscribed circles of $\Delta ABC$ and $\Delta ASR$,

$$\frac{|BC|}{r} = \frac{2|RS|}{r + r_1}, \tag{1}$$

and from the excircles of $\Delta APQ$ and $\Delta ABC$,

$$\frac{2|PQ|}{r + r_1} = \frac{|BC|}{r_1}. \tag{2}$$

From (1) and (2),

$$|PQ| = \frac{r}{r_1}\cdot|RS| = \left[\frac{2|PQ|}{|BC|} - 1\right]\cdot|RS|$$

and thus

$$\frac{2}{|BC|} = \frac{1}{|PQ|} + \frac{1}{|RS|},$$

i.e. $|BC|$ is the harmonic mean of $|PQ|$ and $|RS|$.

II. *Solution par C. Festraets–Hamoir, Brussels, Belgium.*

(a) $BI$ est bissectrice de $\angle ABC$, et $PQ \| BC$, donc

$$\angle IBP = \angle IBC = \angle PIB,$$

et $\Delta BPI$ est isocèle. De même, pour les triangles $IQC$, $CRI_1$ et $I_1SB$. D'où

$$\begin{aligned} |PQ| + |RS| &= |PI| + |IQ| + |RI_1| + |I_1S| \\ &= |PB| + |QC| + |CR| + |SB| \\ &= |PS| + |QR|, \end{aligned}$$

condition nécessaire et suffisante pour que le trapèze $PQRS$ soit circonscriptible.

(b) De

$$\frac{|PQ|}{|RS|} = \frac{|AI|}{|AI_1|} = \frac{r}{r_1}$$

on a

$$|BC| = \frac{|PQ|\cdot r_1 + |RS|\cdot r}{r + r_1} = \frac{|PQ| + |RS|\cdot r/r_1}{r/r_1 + 1}$$
$$= \frac{|PQ| + |RS|\cdot|PQ|/|RS|}{|PQ|/|RS| + 1} = \frac{2|PQ|\cdot|RS|}{|PQ| + |RS|}.$$

Donc, $|BC|$ est la moyenne harmonique de $|PQ|$ et $|RS|$.

*Also solved by JORDI DOU, Barcelona, Spain; JORG HARTERICH, Winnenden, Federal Republic of Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer. Part (a) solved by FRANCISCO BELLOT, Emilio Ferrari High School and MARIA ASCENSION LOPEZ CHAMORRO, Leopoldo Cano High School, Valladolid, Spain; and J.T. GROENMAN, Arnhem, The Netherlands.*

*Dou notes that, from (b), letting $A'$ be the intersection of the bisector of $A$ with $BC$, the distance of $A'$ from sides $AB$ and $AC$ is the harmonic mean of $r$ and $r_1$.*

* * *

**1286**. [1987: 290] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x$, $y$, $z$ be positive real numbers. Show that

$$\prod\left[\frac{x(x+y+z)}{(x+y)(x+z)}\right]^x \le \left[\frac{\left[\sum yz\right]^2}{4xyz(x+y+z)}\right]^{x+y+z},$$

where $\prod$ and $\sum$ are to be understood cyclically.

*Solution by M.S. Klamkin, University of Alberta.*

Let

$$p = x + y + z\,,\quad q = yz + zx + xy\,,\quad r = xyz.$$

By the weighted A.M.–G.M. inequality,

$$\prod\left[\frac{xp}{(x+y)(x+z)}\right]^{x/p} \le \frac{1}{p}\sum x\left[\frac{xp}{(x+y)(x+z)}\right],$$

so it suffices to prove the stronger inequality

$$4rp\sum\frac{x^2}{(x+y)(x+z)} \le q^2.$$

Equivalently,

$$4rp\sum x^2(y+z) \le q^2\prod(y+z)$$

or

$$4rp(pq-3r) \le q^2(pq-r). \qquad (1)$$

By expanding out it can be shown that (1) is the same as

$$x^3(y-z)^2(xy+xz-yz) + y^3(z-x)^2(yz+yx-zx) + z^3(x-y)^2(zx+zy-xy) \ge 0.$$

We can assume that $x \ge y \ge z$ without loss of generality. Then if $zx + zy \ge xy$, we are done. If otherwise, then $xy > zx + zy$, and by direct comparison

$$y^3(z-x)^2(yz+yx-zx) > z^3(x-y)^2(xy-zx-zy)$$

and we are again done. There is equality if and only if $x = y = z$.

*Comment:* It is known that $q^2 \ge 3pr$. If we replace $q^2$ by $3pr$ in the given inequality we obtain the complementary (going the other way) inequality

$$\prod\left[\frac{x(x+y+z)}{(x+y)(x+z)}\right]^x \ge \left[\frac{3}{4}\right]^{x+y+z},$$

which can be written in the more appealing form

$$x^x y^y z^z\left[\frac{x+y+z}{3}\right]^{x+y+z} \ge \left[\frac{y+z}{2}\right]^{y+z}\left[\frac{z+x}{2}\right]^{z+x}\left[\frac{x+y}{2}\right]^{x+y}. \qquad (2)$$

To prove (2), write it in logarithmic form

$$3\left[\frac{x+y+z}{3}\log\left[\frac{x+y+z}{3}\right]\right] + x\log x + y\log y + z\log z$$
$$\ge 2\left[\frac{y+z}{2}\log\left[\frac{y+z}{2}\right] + \frac{z+x}{2}\log\left[\frac{z+x}{2}\right] + \frac{x+y}{2}\log\left[\frac{x+y}{2}\right]\right].$$

The latter inequality is the special case $F(t) = t\log t$ of Popoviciu's inequality [1] for convex functions $F$:

$$3F\left[\frac{x+y+z}{3}\right] + F(x) + F(y) + F(z) \ge 2\left[F\left[\frac{y+z}{2}\right] + F\left[\frac{z+x}{2}\right] + F\left[\frac{x+y}{2}\right]\right].$$

*Reference:*

[1] T. Popoviciu, Sur certaines inégalités qui caractèrisent les fonctions convexes, *Analele Stiintifice Univ., "Al.I. Cuza" Iasi*, Sect. I–a Mat. 11B (1965) 155–164.

*The proposer's solution was quite short but unfortunately contained an error which the editor was unable to fix.*

* * *

**1287**. [1987: 290] *Proposed by Leroy F. Meyers, The Ohio State University.*

Find all differentiable functions $f$ such that $f'(x) = f(3) + f(6)$ for all real $x$.

*Solution by Seung–Jin Bang, Seoul, Korea.*

Let $k = f(3) + f(6)$. Then we see that $f(x) = kx + c$ for some constant $c$. Since

$$k = f(3) + f(6) = 9k + 2c,$$

we have $c = -4k$. Thus the answer is

$$f(x) = k(x-4)$$

where $k$ is any real number.

*Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; M.A. SELBY, University of Windsor; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.*

*Gibbs and Janous found all solutions of $f'(x) = f(a) + f(b)$ where $a$ and $b$ are arbitrary real numbers.*

* * *

**1288**. [1987: 290] *Proposed by Len Bos, University of Calgary, Calgary, Alberta.*

Show that for $x_1, x_2, \ldots, x_n > 0$,

$$n(x_1^n + x_2^n + \cdots + x_n^n) \geq (x_1 + x_2 + \cdots + x_n)(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}).$$

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For the stated inequality we may and do assume $x_1 \geq x_2 \geq \ldots \geq x_n$. Then also

$$x_1^{n-1} \geq x_2^{n-1} \geq \ldots \geq x_n^{n-1},$$

and the inequality follows by Chebyshev's inequality.

We now show more generally that

$$\sum x_i^a \cdot \sum x_i^d \geq \sum x_i^b \cdot \sum x_i^c, \tag{1}$$

where $a + d = b + c$ and $a > b \geq c > d$ ($a$, $b$, $c$, $d$ real), and the sums are from 1 to $n$. Put

$$f(t) = \log \sum x_i^t .$$

Then

$$f'(t) = \frac{\sum x_i^t \log x_i}{\sum x_i^t}$$

and

$$f''(t) = \frac{\sum x_i^t \log^2 x_i \cdot \sum x_i^t - \left[\sum x_i^t \log x_i\right]^2}{\left[\sum x_i^t\right]^2}.$$

By the Cauchy–Schwarz inequality,

$$\left[\sum x_i^t \log x_i\right]^2 \leq \sum x_i^t \log^2 x_i \cdot \sum x_i^t ,$$

i.e. $f''(t) \geq 0$. Thus $f$ is convex and therefore

$$\frac{f(c) - f(d)}{c - d} \le \frac{f(a) - f(d)}{a - d} \le \frac{f(a) - f(b)}{a - b}.$$

Thus, since $c - d = a - b > 0$

$$f(c) - f(d) \le f(a) - f(b),$$

which is (1).

*Also solved by BENO ARBEL, Tel–Aviv University; SEUNG–JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; M.S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; Z.F. LI, University of Regina; M.M. PARMENTER, Memorial University of Newfoundland; JOSIP E. PECARIC, Zagreb, Yugoslavia; M.A. SELBY, University of Windsor; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.*

*Half the solvers noted that the problem follows from Chebyshev's inequality. This may indicate that Chebyshev's inequality is more widely known among* Crux *readers in general than among the set* {*proposer, editor*}!

* * *

<u>1289</u>. [1987: 290] *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

"To reward you for slaying the dragon", the Queen said to Sir George, "I grant you all the land you can walk around in a day."

She pointed to a pile of wooden stakes. "Take some of these stakes with you", she continued. "Pound them into the ground along the way, and be back at your starting point in 24 hours. All the land in the convex hull of your stakes will then be yours." (The Queen had read a little mathematics.)

Assume that it takes Sir George 1 minute to pound in a stake, and that he walks at constant speed between stakes. How many stakes should he use, to get as much land as possible?

*Solution by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia.*

Suppose Sir George drives $n$ stakes, and otherwise walks at unit speed for $24.60 - n = 1440 - n$ minutes. We may assume that connecting consecutively driven stakes with straight lines yields a regular, convex polygon of $n$ sides, since departing from this assumption cannot increase the area enclosed in the convex hull. The interior of the polygon is then the union of $n$ congruent isosceles triangles with bases of length $(1440 - n)/n$, apex angles $2\pi/n$, and hence altitudes

$$(1440 - n)\cot(\pi/n)/2n.$$

The area of the polygon is then

$$A(n) = \frac{(1440 - n)^2 \cot \pi/n}{4n}.$$

Then

$$4\frac{dA}{dn} = \frac{(n - 1440)^2}{n} \cdot \frac{\pi}{n^2} \csc^2\frac{\pi}{n} + \cot \frac{\pi}{n} \frac{2n(n - 1440) - (n - 1440)^2}{n^2}$$

$$= \frac{(n - 1440)\csc \pi/n}{n^3}[\pi(n - 1440)\csc \pi/n + n(n + 1440)\cos \pi/n],$$

which is zero only when the factor in square brackets is zero. An application of Newton's Method yields a root at approximately 16.82, and no others. Since the value of $dA/dn$ changes from positive to negative at the root, the root is a local maximum, and so an absolute maximum, by uniqueness. Since $n$ must be an integer, we compare $A(16) = 318572.45$ with $A(17) = 318600.39$, and conclude that Sir George should plant 17 stakes.

*Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; and the proposer. One other reader sent in an incorrect answer, likely due to a simple calculation error.*

*Klamkin proposes the three-dimensional analogue, in which we would want to maximize the volume of the convex hull of n points. Here we assume we can fix a point (a "stake") in space in one minute and "fly" to the next selected point at a constant rate. (Sounds like Sir George would have a considerably more difficult time with this task. But then, the stakes are higher.)*

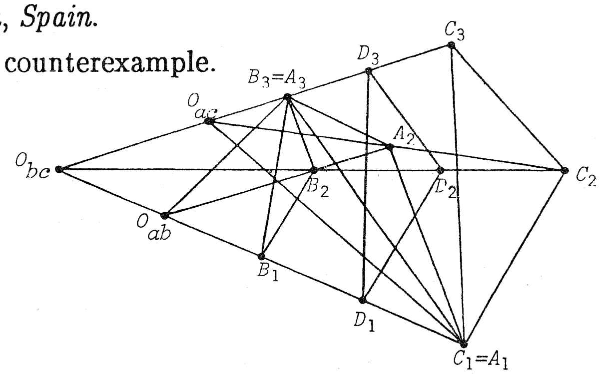
* * *

1290. [1987: 291] *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

The triangles $B_1B_2B_3$ and $C_1C_2C_3$ are homothetic and each of them is in perspective with the triangle $A_1A_2A_3$ (vertices with the same index correspond). $D_i$ $(i = 1,2,3)$ is the midpoint of the segment $B_iC_i$. Prove that the triangles $A_1A_2A_3$ and $D_1D_2D_3$ are in perspective.

I. *Comment by Jordi Dou, Barcelona, Spain.*

The proposition is false. Here is a counterexample.




Triangles $B_1B_2B_3$ and $C_1C_2C_3$ are perspective, with centre $O_{bc}$. $\Delta A_1A_2A_3$ is perspective to $\Delta B_1B_2B_3$ and $\Delta C_1C_2C_3$ with respective centres $O_{ab}$ and $O_{ac}$. But $\Delta A_1A_2A_3$ and $\Delta D_1D_2D_3$ are not perspective, since $A_1D_1$ and $A_3D_3$ concur at $O_{bc}$ and $A_2D_2$ cannot pass through $O_{bc}$.

The author has neglected one condition, which I believe is "$O_{ab}$, $O_{ac}$, and $O_{bc}$ are collinear", and which probably was used tacitly in his solution. With this additional condition the solution is immediate.

[*Editor's note.* A similar counterexample was discovered by C. FESTRAETS-HAMOIR, Brussels, Belgium. To make amends for his oversight in the planar case, the proposer sends a solution (see below) which works for non-coplanar triangles $B_1B_2B_3$ and $C_1C_2C_3$.]

II. *Partial solution by the proposer.*

We shall prove the stated result in the case when the triangles $B_1B_2B_3$ and $C_1C_2C_3$ do not lie in one and the same plane.

By $a_i$, $b_i$, $c_i$, $d_i$ ($i$ =1,2,3) we denote the lines through the sides opposite to $A_i$, $B_i$, $C_i$, $D_i$ in the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, $D_1D_2D_3$, respectively. Denote by $A_i'$ the intersection of $a_i$ and $b_i$, by $A_i''$ the intersection of $a_i$ and $c_i$, and by $A_i'''$ the intersection of $a_i$ and $d_i$, $i$ = 1,2,3. [Note that $A_i'$ exists, at least at infinity, for each $i$ since $a_i$ and $b_i$ are on the same plane, and similarly for $A_i''$ and $A_i'''$.]

Since $B_1B_2B_3$ and $C_1C_2C_3$ are homothetic, $b_i \| c_i$ for $i$ = 1,2,3; and since in addition $D_i$ is the midpoint of a segment joining the points $B_i$ and $C_i$, $d_i$ is parallel to, and equidistant from, $b_i$ and $c_i$. Consequently $A_i'''$ is the midpoint of the segment $A_i'A_i''$ for each $i$.

Let $a'$ and $a''$ be the lines of intersection of the plane $A_1A_2A_3$ with the planes $B_1B_2B_3$ and $C_1C_2C_3$ respectively. Clearly $A_i' \in a'$ and $A_i'' \in a''$, $i$ = 1,2,3. Since the triangles $B_1B_2B_3$ and $C_1C_2C_3$ are homothetic, their planes are parallel, and consequently the lines $a'$ and $a''$ are parallel. Then the midpoints $A_i'''$ of the segments $A_i'A_i''$ are collinear, and hence, using Desargues' theorem, we conclude that the triangles $A_1A_2A_3$ and $D_1D_2D_3$ are in perspective.

*Remark.* Let $O$ be the homothetic centre of $B_1B_2B_3$ and $C_1C_2C_3$. It may be proved in the same way that $A_1A_2A_3$ is in perspective with every triangle homothetic to $B_1B_2B_3$ with centre $O$.

* * *

__1291__. [1987: 320] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate

$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}}\, dx.$$

*Solution by Alex Grossman, Queen's University, Kingston, Ontario.*

Let

$$f(x) = \frac{(\cos\ x)^{\sin\ x}}{(\cos\ x)^{\sin\ x} + (\sin\ x)^{\cos\ x}}, \quad 0 \le x \le \frac{\pi}{2}.$$

Since

$$\sin(\pi/2 - x) = \cos\ x \quad \text{and} \quad \cos(\pi/2 - x) = \sin\ x,$$

we have

$$f(\pi/2 - x) = \frac{(\sin\ x)^{\cos\ x}}{(\sin\ x)^{\cos\ x} + (\cos\ x)^{\sin\ x}}.$$

Hence (and this is the key observation)

$$f(x) + f(\pi/2 - x) = 1$$

for all $x$. Viewing the given integral as the area under the curve $f(x)$, this means we can imagine placing the portion of this area from $\pi/4$ to $\pi/2$ "on top of" the portion from 0 to $\pi/4$, fitting the pieces together. Thus the required area is just a rectangle of base $\pi/4$ and height 1, i.e.

$$\int f(x)dx = \frac{\pi}{4}.$$

Note the similarity of this question to the first problem of the afternoon session of the 1987 Putnam Examination.

*Also solved by JOE ALLISON, Eastfield College, Mesquite, Texas; BENO ARBEL, Tel Aviv University; SEUNG–JIN BANG, Seoul, Korea; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; C. FESTRAETS–HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Yokosuka High School, Aichi, Japan; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; Z.F. LI, University of Regina; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; M.A. SELBY, University of Windsor; ROBERT E. SHAFER, Berkeley, California; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; D.J. SMEENK, Zaltbommel, The Netherlands; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.*

*Several solvers mentioned that this type of problem is by now well known and has appeared in many places, for instance the 1987 Putnam and earlier as problem 260 of the* (Two–Year) College Mathematics Journal (*solution in Vol. 16* (*1985*) *pp. 305–306*). *Several solvers gave generalizations of the problem.*

*                    *                    *

# PAST PROBLEMS AND SOLUTIONS

This being the last issue of 1988, it seems convenient to collect together at this point, some information on past *Crux* problems which the editor has received during the year.

**1137.** [1986: 79, 177; 1987: 228; 1988: 79].

Readers may have already noted the obvious generalization of inequality (1) on [1988: 79] to $n$ variables, namely: *find the best lower bound for*

$$S = \sum_{i=1}^{n} \frac{1}{\sqrt{x_i^2 + x_i x_{i+1} + x_{i+1}^2}},$$

*where* $x_1, x_2, \ldots, x_n$ *are positive reals* ($x_{n+1} = x_1$) *satisfying*

$$\sum_{i=1}^{n} x_i = 1.$$

T. ANDO of Hokkaido University has a simple proof that $S \geq n^2/2$. For $n$ even this is best possible, equality being attained by the choice

$$x_2 = x_4 = \cdots = 0,$$
$$x_1 = x_3 = \cdots \neq 0.$$

For $n$ odd, $n > 3$, the problem is still open. J. BRENNER, H. ALZER, and Ando conjecture that the same choice of $x_i$'s gives the best lower bound in this case as well. (Thanks to J. Brenner for this information.)

**1173.** [1986: 205; 1988: 47]

P. WINKLER, Emory University, points out that in the lemma on [1988: 47] it need only be assumed that $G$ has no vertices of degree one. This error can be blamed on the editor, who added the unnecessarily strong condition that $G$ be 2–connected.

**1179.** [1986: 206; 1988: 22, 85]

Further on van Aubel's theorem, FRANCISCO BELLOT (Valladolid, Spain) mentions the useful paper "Further remarks on concentric polygons" by D. Merriell, *American Math. Monthly* 72 (1965) 960–965, which contains many references. Bellot also believes that the prefix "van" is the correct one. J. Suck (Essen) confirms this, and also corrects the name of the author in question to "H. van Aubel", the initial "M." on [1988: 85] perhaps standing for "Monsieur".

1198. [1986: 283; 1988: 85, 179]

Readers should ignore the phrase "all lie on the line $x + sy - z = 0$, and so" which appears at the end of Solution II [1988: 179]. It was not, repeat NOT, part of solver G.R. Veldkamp's submission, but was added by the editor (he sheepishly admits), perhaps during some strange regression to his student days. Apologies are offered to Professor Veldkamp and the readers.

1276. [1987: 257; 1988: 279]

A (slightly) late solution was received from J. SUCK (Essen), who also becomes the second reader to have noticed the duplication of this problem in the German journal *Der Mathematische und Naturwissenschaftliche Unterricht.*

*   *   *   *   *   *   *   *   *   *

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HAPPY NEW YEAR

**and many inspired contributions to *Crux* in 1989.**

*   *   *   *   *   *   *   *   *   *

# INDEX TO VOLUME 14, 1988

## ARTICLES AND NOTES



## PROPOSALS AND SOLUTIONS



## PROPOSERS AND SOLVERS

The numbers refer to the pages in which the corresponding name appears with a problem proposal, a solution, or a comment.

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