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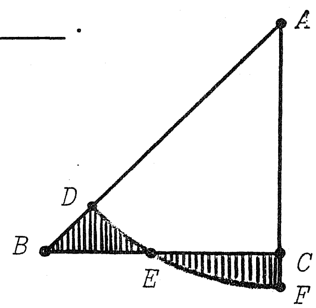
THE OLYMPIAD CORNER
No. 95
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this column with a set of problems which comes via Professor V.N. Murty and Dr. Frank Swetz of Penn State University, Harrisburg, PA. It constitutes the 1986 Nationwide Junior High School Mathematics Competition of The People's Republic of China. This two hour exam was written April 6, 1986.

Part I. Place your answer in the blank space following each question: 8 points for each question.

1. If one root of the equation $x^2 + px + q = 0$ is twice as large as the other root, the relationship between p and q is ______________________ .



2. In the figure, ABC is an isosceles right triangle with $AC = BC$. DEF is the arc of a circle with centre A. If the two shaded parts in the figure have equal area, then AD/CB is ______________ .

3. The natural number N, in decimal form, can be concatenated on the right of any natural number M, in decimal form, to produce a third natural number $M\hat{}N$ in decimal form. For example, if 2 is concatenated on the right side of 35 the number 352 is produced. We call N a "magic number" if N is always a divisor of $M\hat{}N$. Among the natural numbers less than 130, how many "magic numbers" are there? __________

4. If a, b, c are whole numbers and $m = a^2 + b^2$ and $n = c^2 + d^2$, then mn can also be expressed in the form of the sum of two squared whole numbers as $mn =$ ______________________ .

5. The parabola $y = -x^2 + 2x + 8$ meets the x–axis at points B and C, and the point D divides the segment BC into two equal parts. Point A is a moving point on the parabola (above the x–axis) and $\angle BAC$ is an acute angle. Over what interval does the length of segment AD vary? _________

Part II. Multiple choice questions. Place your answer in the box following the question. If your answer is correct, you will receive 8 points.

If your answer is wrong, you will receive no points. If you do not answer the question, you will get 2 points.

1. If $\log_x a = a$, where $a > 1$ and $a \in \mathbb{N}^+$, then x is

A. $10^{a \log_{10} a}$ B. $10^{(\log_{10} a)/a^2}$

C. $10^{(\log_{10} a)/a}$ D. $10^{a \log_{10}(1/a)}$.

2. If $a < b$, then $\sqrt{-(x+a)^3(x+b)}$ is

A. $(x+a)(\sqrt{-(x+a)(a+b)}$

B. $(x+a)\sqrt{(x+a)(x+b)}$

C. $-(x+a)\sqrt{-(x+a)(x+b)}$

D. $-(x+a)\sqrt{(x+a)(x+b)}$.

3. If the equation $||x-2|-1| = a$ has three whole number solutions (a is a constant), then a is

A. 0 B. 1

C. 2 D. 3.

4. Let $[x]$ be the largest whole number that does not exceed x. Let n be a natural number and

$$I = (n+1)^2 + n - [\sqrt{(n+1)^2 + n + 1}]^2.$$

Then

A. $I > 0$ B. $I < 0$

C. $I = 0$ D. When n takes different values, all the three cases A, B, C occur.

5. The lengths of the four sides AB, BC, CD, DA of the quadrilateral $ABCD$ are 1, 9, 8, 6 respectively. Which of the following is true?

(i) Quadrilateral $ABCD$ can be circumscribed about a circle.
(ii) Quadrilateral $ABCD$ cannot be inscribed in a circle.
(iii) The diagonal lines AC and BD are not perpendicular.
(iv) $\angle ADC \geq 90^\circ$.
(v) ΔBCD is an isosceles triangle.

A. (i) is true, (ii) is false, (iv) is true

B. (iii) is true, (iv) is false, (v) is true

C. (iii) is true, (iv) is false, (v) is false

D. (ii) is false, (iii) is false, (iv) is true.

* * *

Murray S. Klamkin of the University of Alberta has kindly provided us with a list of five "Quickies". These are problems which have a short solution, and knowing they are Quickies often helps to find the answer. Look to the end of this column for Murray's answers.

1. If two conic sections of the same type (e.g., two parabolas) have their axes perpendicular to one another and intersect in four points, prove that the points of intersection lie on a circle.

2. Prove that $\sqrt{\left[\frac{1987}{2\sin x}\right]^2 + \left[\frac{1989}{2\cos x}\right]^2} \geq 1988.$

3. Let $ABCD$ denote a tetrahedron with circumcenter O and centroid G. The point Q is taken on OG produced so that $\vec{OG} = 3\vec{GQ}$. Prove that Q is the center of a sphere passing through the centroids of the faces of $ABCD$.

4. If $P(x,y)$ is a symmetric polynomial in x, y and is divisible by $(x-y)^{2n-1}$ show that it is also divisible by $(x-y)^{2n}$.

5. Is it possible to have two congruent triangles inscribed in an ellipse (not a circle) which cannot be obtained from one another by reflection across the axes or the center? (This nice problem is due to George Szekeres.)

* * *

This column's Olympiad problems come to us from Professor Francisco Bellot of Valladolid, Spain. They are the *final round* problems of the 23rd Spanish Mathematical Olympiad, which was written in February 1987.

1. Let a, b, c be the lengths of the sides of a (non-isosceles) triangle. Let O_a, O_b and O_c be three concentric circles with radii a, b and c respectively.

(a) How many equilateral triangles with different areas can be constructed such that the lines containing the sides are tangent to the circles?

(b) Find the possible areas of such triangles.

2. Show that for each natural number $n > 1$

$$1\cdot\sqrt{\binom{n}{1}} + 2\cdot\sqrt{\binom{n}{2}} + \cdots + n\cdot\sqrt{\binom{n}{n}} < \sqrt{2^{n-1}n^3}\,.$$

(Here $\binom{n}{k}$ is, of course, the binomial coefficient.)

3. A given triangle is tiled by n triangles in such a way that:

(i) no two tiling triangles have interior points in common;

(ii) the union of all the tiling triangles is the given triangle;

(iii) any line segment which is a side of a tiling triangle is either a side of another tiling triangle or a side of the given triangle.

Let s be the total number of sides of tiling triangles (counted without multiplicity so that each side is counted only once even if it is common to two triangles). Let v be the total number of vertices (again counted without multiplicity).

(a) Show that, if n is odd, then there exist several such tilings, but that all have the same number v of vertices and the same number s of sides. Express v and s as functions of n.

(b) Show that, for n even, no such tiling is possible.

4. If a and b are distinct real numbers, solve the system

$$x + y = 1$$

$$(ax + by)^2 \leq a^2x + b^2y.$$

Also, solve the system

$$x + y = 1$$

$$(ax + by)^4 \leq a^4x + b^4y.$$

5. In a triangle ABC, D lies on AB, E lies on AC and $\angle ABE = 30^\circ$, $\angle EBC = 50^\circ$, $\angle ACD = 20^\circ$, $\angle DCB = 60^\circ$. Find $\angle EDC$.

6. For all natural numbers n, define the polynomial

$$P_n(x) = x^{n+2} - 2x + 1.$$

(a) Show that the equation $P_n(x) = 0$ has one and only one root c_n in the open interval (0,1).

(b) Find $\lim_{n\to\infty} c_n$.

* * *

As promised in the last Corner we next give the numerical answers for the 1988 AIME. These problems and their official solutions are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, CAMC Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

1. 770	2. 169	3. 027	4. 020	5. 634
6. 142	7. 110	8. 364	9. 192	10. 840
11. 163	12. 441	13. 987	14. 084	15. 704

* * *

We now turn to solutions received for problems from the 1986 Olympiad Corners. But first an alternate solution to a problem considered last year.

71. [1985: 105; 1987: 79] *Proposed by Spain.*

Construct a nonisosceles triangle ABC such that

$$a(\tan B - \tan C) = b(\tan A - \tan C)$$

where a and b are the side lengths opposite angles A and B, respectively.

Alternate solution by G.R. Veldkamp, De Bilt, The Netherlands.

The given equation is equivalent to

$$\sin(B + C)\sin(B - C)\cos A = \sin(A + C)\sin(A - C)\cos B. \qquad (1)$$

[To see this rewrite the given equation as

$$a(\sin B \cos C - \sin C \cos B)\cos A = b(\sin A \cos C - \sin C \cos A)\cos B.$$

Then apply the law of sines and the difference formula to write

$$\sin A \sin(B - C)\cos A = \sin B \sin(A - C)\cos B.$$

Finally notice that $\sin A = \sin(B + C)$ and $\sin B = \sin(A + C)$.]

Now (1) is equivalent to

$$(\sin^2 B \cos^2 C - \cos^2 B \sin^2 C)\cos A = (\sin^2 A \cos^2 C - \cos^2 A \sin^2 C)\cos B$$

from which we get the equivalent equation

$$(\cos^2 C - \cos^2 B)\cos A = (\cos^2 C - \cos^2 A)\cos B$$

or

$$(\cos^2 C + \cos A \cos B)(\cos A - \cos B) = 0.$$

Since we are looking for a nonisosceles triangle, we infer

$$\cos^2 C + \cos A \cos B = 0. \qquad (2)$$

This implies that either A or B is obtuse. We assume $A > 90^\circ$ and rewrite (2) as follows:

$$2 \cos^2 C - \cos C + \cos(A - B) = 0. \qquad (3)$$

(Note that $-\cos C = \cos(A + B) = \cos A \cos B - \sin A \sin B$.) We only need to give *one* constructible triangle to answer the question. By inspection (3) is satisfied by taking $2 \cos C - 1 = 0$ at the same time as $\cos(A - B) = 0$. This is the case if $C = 60^\circ$, $A - B = 90^\circ$. This leads to the easily constructible triangle with $A = 105^\circ$, $B = 15^\circ$ and $C = 60^\circ$.

*

2. [1986: 97] *1985 Spanish Mathematical Olympiad — 1st Round.*

Let n be a natural number. Prove that the expression

$$(n + 1)(n + 2) \ldots (2n - 1)(2n)$$

is divisible by 2^n.

Solution by John Morvay, Dallas, Texas.

Let

$$P_n = (n + 1)(n + 2)\ldots(2n - 1)(2n)$$

and let $k(N)$ be the greatest whole number k such that 2^k divides N. We will show that $k(P_n) = n$. The statement is trivially true for $n = 1$. Assume that $k(P_n) = n$ for some $n \geq 1$. Observe that

$$P_{n+1} = (n+2)...(2n)(2n+1)(2(n+1)) = 2(2n+1)P_n.$$

Hence $k(P_{n+1}) = 1 + 0 + k(P_n) = n + 1$ and the theorem is proved by induction.

3. [1986: 97] *1985 Spanish Mathematical Olympiad — 1st Round.*

Let a, b, and c be positive real numbers. Prove that

$$(b+c)(c+a)(a+b) \geq 8abc.$$

Solution by John Morvay, Dallas, Texas.

We use the easily proved fact that for $x,y > 0$, $x + y \geq 2\sqrt{xy}$. From this it immediately follows that

$$(b+c)(c+a)(a+b) \geq (2\sqrt{bc})(2\sqrt{ca})(2\sqrt{ab}) = 8abc.$$

*

3. [1986: 98] *1985 Spanish Mathematical Olympiad — 2nd Round.*

Solve the equation

$$\tan^2 2x + 2\tan 2x \tan 3x = 1.$$

Solution by J.T. Groenman, Arnhem, The Netherlands.

Note that

$$\tan^2 2x + 2\tan 2x \tan 3x = 1$$

just in case

$$\tan^2 2x + 2\tan 2x \tan 3x + \tan^2 3x = 1 + \tan^2 3x = \sec^2 3x,$$

or equivalently

$$(\tan 2x + \tan 3x)^2 - \sec^2 3x = 0.$$

This becomes

$$\tan 2x + \tan 3x = \pm 1/\cos 3x$$

and we get

$$\frac{\sin 2x \cos 3x + \sin 3x \cos 2x}{\cos 2x \cos 3x} = \pm\frac{1}{\cos 3x}.$$

This yields

$$\sin 5x = \pm\cos 2x$$

or

$$\cos(\pi/2 - 5x) = \pm\cos 2x.$$

This gives the cases

(i) $2x = \pi/2 - 5x + k\cdot 2\pi$, i.e. $x = \pi/14 + k\cdot 2\pi/7$,

(ii) $2x = -\pi/2 + 5x + k\cdot 2\pi$, i.e. $x = \pi/6 + k\cdot 2\pi/3$,

(iii) $2x = \pi/2 + 5x + k\cdot 2\pi$, i.e. $x = -\pi/6 + k\cdot 2\pi/3$,

(iv) $2x = -\pi/2 - 5x + k\cdot 2\pi$, i.e. $x = -\pi/14 + k\cdot 2\pi/7$,
where k is an arbitrary integer. However we must rule out (ii) and (iii) for otherwise tan $3x$ is undefined. This leaves the solutions

$$x = \pm\pi/14 + k\cdot 2\pi/7.$$

4. [1986: 98] *1985 Spanish Mathematical Olympiad – 2nd Round.*

Prove that for each positive integer k there exists a triple (a,b,c) of positive integers such that $abc = k(a + b + c)$. In all such cases prove that $a^3 + b^3 + c^3$ is not a prime.

I. *Solutions by J.T. Groenman, Arnhem, The Netherlands and by Bob Prielipp, The University of Wisconsin–Oshkosh, U.S.A.*

Given the positive integer k, let $a = 2$, $b = k$ and $c = k + 2$. Then $abc = 2k(k + 2)$ and $a + b + c = 2(k + 2)$ so that $abc = k(a + b + c)$.

Now, suppose that $abc = k(a + b + c)$. Then

$$\begin{aligned} a^3 + b^3 + c^3 &= (a^2 + b^2 + c^2)(a + b + c) - (ab + ac + bc)(a + b + c) + 3abc \\ &= (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc + 3k). \end{aligned}$$

Thus $a + b + c$ divides $a^3 + b^3 + c^3$ where $a + b + c$ and $a^3 + b^3 + c^3$ are both positive integers. Hence $3 \le a + b + c \le a^3 + b^3 + c^3$. Suppose for a contradiction that $a + b + c = a^3 + b^3 + c^3$. Then by the Arithmetic Mean-Geometric Mean inequality

$$a + b + c = a^3 + b^3 + c^3 \ge 3abc = 3k(a + b + c).$$

Thus $3k \le 1$ contradicting the fact that k is a positive integer. It follows that $a^3 + b^3 + c^3$ is not a prime.

II. *Remark by J.T. Groenman, Arnhem, The Netherlands.*

With a, b, c, k as in the problem, one can prove that (i) $a + b + c$ does not divide $a^4 + b^4 + c^4$ while (ii) $a + b + c$ divides $a^n + b^n + c^n$ for all odd n.

To see (ii) by induction notice that for $n \ge 1$

$$\begin{aligned} a^{n+2} + b^{n+2} + c^{n+2} &= (a^{n+1}+b^{n+1}+c^{n+1})(a+b+c) - (a^n+b^n+c^n)(ab+ac+bc) \\ &\quad + (a^{n-1}+b^{n-1}+c^{n-1})(abc) \\ &= (a^{n+1}+b^{n+1}+c^{n+1})(a+b+c) - (a^n+b^n+c^n)(ab+ac+bc) \\ &\quad + (a^{n-1}+b^{n-1}+c^{n-1})k(a+b+c). \end{aligned}$$

On the other hand, to see that $a + b + c$ does not divide $a^4 + b^4 + c^4$, set $c = -(a + b)$ in $a^4 + b^4 + c^4$ to calculate the remainder on division by $a + b + c$ and get

$$a^4 + b^4 + (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) = 2(a^2 + ab + b^2)^2 > 0$$

for $a,b > 0$.

5. [1986: 98] *1985 Spanish Mathematical Olympiad – 2nd Round.*

Find the equation of the circle determined by the roots (in the Argand diagram) of the equation

$$z^3 + (-1 + i)z^2 + (1 - i)z + i = 0.$$

Solution by J.T. Groenman, Arnhem, The Netherlands.

By inspection we see that $z_1 = -i$ is a root since

$$(-i)^3 + (-1 + i)(-i)^2 + (1 - i)(-i) + i = -i^3 + (-1 + i)(-1) - i + i^2 + i$$
$$= i + 1 - i - i - 1 + i = 0.$$

Now on long division by $z + i$ we find that

$$z^3 + (-1 + i)z^2 + (1 - i)z + i = (z + i)(z^2 - z + 1).$$

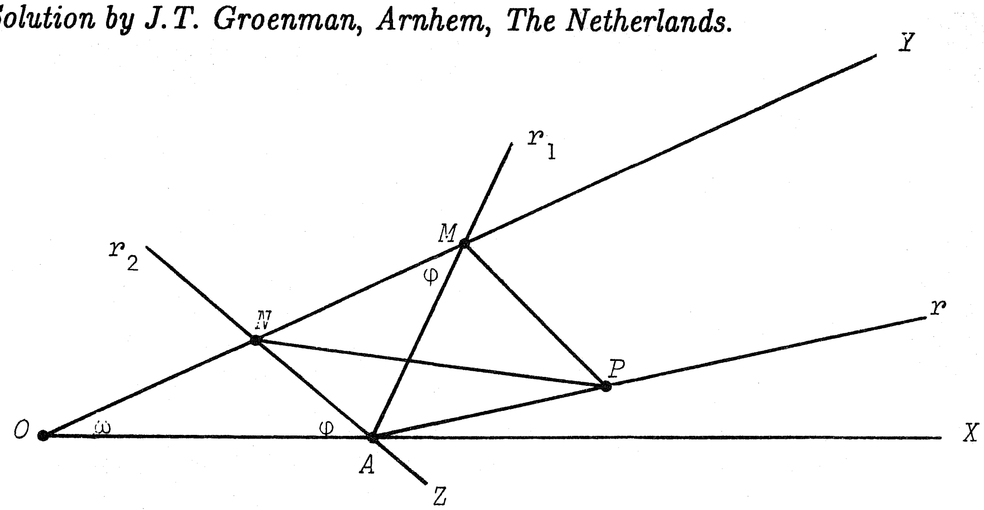
Hence the other two roots are $z_2 = \dfrac{1 + \sqrt{-3}}{2}$, $z_3 = \dfrac{1 - \sqrt{-3}}{2}$. The roots z_1, z_2, z_3 all have absolute value 1. Thus the circle is the circle $|z| = 1$.

6. [1986: 98] *1985 Spanish Mathematical Olympiad – 2nd Round.*

Let OX and OY be non-collinear rays. Through a point A on OX, draw two lines r_1 and r_2 that are antiparallel with respect to $\angle XOY$. Let r_1 cut OY at M and r_2 cut OY at N. (Thus, $\angle OAM = \angle ONA$.) The bisectors of $\angle AMY$ and $\angle ANY$ meet at P. Determine the location of P.

[*Editor's comment*: The last sentence of the problem should have been phrased "Determine the *locus* of P."]

Solution by J.T. Groenman, Arnhem, The Netherlands.



Denote $\angle XOY$ by ω and let $\angle OMA = \angle OAN = \varphi$. Notice that φ varies as M varies on OY. Notice too that if r_1 and r_2 are interchanged the unordered pair $\{M,N\}$ remains unchanged. The points M and N coincide when $OM = ON = OA$. With this observation we can think of P as a function of the point M (or N) when $OM > OA$.

As MP is a bisector of $\angle AMY$ and NP is a bisector of $\angle ANY$, P is the center of one of the three escribed circles of ΔMAN. This means that AP is the bisector of the exterior angle MAZ. Also it means that the mapping taking M to P is 1–1 since there is a unique circle κ with center P and tangent to OY. Through A there are at most two tangents to κ. These determine the lines r_1 and r_2 (and the degenerate case where $r_1 = r_2$ and $OM = OA$). Now observe that

$$\angle MNA = \varphi + \omega,$$
$$\angle NAM = \pi - \varphi - (\varphi + \omega) = \pi - 2\varphi - \omega,$$
$$\angle MAZ = 2\varphi + \omega,$$
$$\angle MAP = \varphi + \omega/2,$$
$$\angle XAZ = \varphi,$$

and therefore

$$\angle PAX = \angle PAZ - \varphi = \omega/2,$$

a constant. Thus P lies on the ray r from A which is parallel to the bisector of $\angle XOY$. Moreover it is clear that as $\overline{OM}$ grows very large $\overline{AP}$ must go to infinity as well, and monotonically since the mapping is 1–1. Let P^* be the point on r which corresponds to $\overline{OM} = \overline{ON} = \overline{OA}$. The locus is then those points P on r satisfying $\overline{AP} \geq \overline{AP^*}$.

7. [1986: 98] *1985 Spanish Mathematical Olympiad — 2nd Round.*

Determine the value of p such that the equation $x^5 - px - 1 = 0$ has two roots r and s which are the roots of an equation $x^2 - ax + b = 0$ where a and b are integers.

Solution by J.T. Groenman, Arnhem, The Netherlands.

If p is chosen so that two of the roots of $x^5 - px - 1 = 0$ are roots of $x^2 - ax + b = 0$, then $x^5 - px - 1$ is divisible by $x^2 - ax + b$. On long division we find that

$$x^5 - px - 1 = (x^2 - ax + b)(x^3 + ax^2 + (a^2 - b)x + (a^3 - 2ab)) + [x(-p + a^4 - 3a^2b + b^2) + (-1 - a^3b + 2ab^2)].$$

On setting the remainder to zero this gives

$$p = a^4 - 3a^2b + b^2$$
$$a^3b - 2ab^2 = -1.$$

Now

$$ab(a^2 - 2b) = -1$$

for *integer* a, b gives

$$ab = 1, \quad a^2 - 2b = -1$$

or

$$ab = -1, \quad a^2 - 2b = 1.$$

We then consider the four possibilities from $a,b = \pm 1$.

$a = 1$, $b = 1$, giving $p = 1 - 3 + 1 = -1$;

$a = 1$, $b = -1$, giving $a^2 - 2b = 3 \neq -1$, impossible;

$a = -1$, $b = 1$, giving $a^2 - 2b = -1 \neq 1$, impossible;

and

$a = -1$, $b = -1$, giving $a^2 - 2b = 3 \neq -1$, impossible.

We conclude that $p = -1$.

* * *

We close this article with the solutions to the Klamkin Quickies posed earlier.

1. *Case 1, two parabolas.* By choosing an appropriate set of rectangular coordinates, we may take the equations of the two parabolas as

$$y = ax^2 \quad \text{and} \quad x - h = b(y-k)^2.$$

The intersection points also lie on the curve

$$b(y - ax^2) + a(x - h - b(y-k)^2) = 0,$$

which is a circle since the coefficients of x^2 and y^2 are the same.

Case 2, two ellipses. We can take the equations of the ellipses as

$$b^2x^2 + a^2y^2 = b^2a^2 \quad \text{and} \quad c^2(x-h)^2 + d^2(y-k)^2 = c^2d^2$$

with $a \neq b$, $c \neq d$, and $a/b \neq c/d$. (The last condition is necessary since two homothetic ellipses can intersect in at most two points.) The four points of intersection will also lie on the curve

$$u[b^2x^2 + a^2y^2 - b^2a^2] + v[c^2(x-h)^2 + d^2(y-k)^2 - c^2d^2] = 0$$

where u and v are arbitrary constants. We now choose u and v to satisfy

$$ub^2 + vc^2 = ua^2 + vd^2$$

or

$$u(b^2 - a^2) = v(d^2 - c^2).$$

Case 3, two hyperbolas. We can take the equation of the hyperbolas as

$$b^2x^2 - a^2y^2 = b^2a^2 \quad \text{and} \quad c^2(x-h)^2 - d^2(y-k)^2 = c^2d^2$$

and proceed as in the previous case.

2. Applying Cauchy's inequality $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$, we have

$$\sqrt{\left[\frac{1987}{2\sin x}\right]^2 + \left[\frac{1989}{2\cos x}\right]^2} \cdot \sqrt{\sin^2 x + \cos^2 x} \geq 1987/2 + 1989/2$$
$$= 1988.$$

3. The solutions of many problems become "messy" if an inappropriate representation is used. It turns out here that a vector representation is particularly appropriate. (For further comments and examples on using appropriate representations, see "Vector proofs in solid geometry", *Amer. Math. Monthly* 77 (1970) 1051–1065.)

Vectors from O to the vertices A, B, C, D are denoted by **A**, **B**, **C**, **D**, respectively, etc. Then the centroid of $ABCD$ is given by

$$\mathbf{G} = (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})/4,$$

and

$$\mathbf{Q} = (4/3)\mathbf{G} = (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})/3.$$

The centroid G_A of the face opposite A is then given by $(\mathbf{B}+\mathbf{C}+\mathbf{D})/3$. Thus,

$$QG_A = |(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D})/3-(\mathbf{B}+\mathbf{C}+\mathbf{D})/3| = |\mathbf{A}/3| = R/3$$

where R is the circumradius. Similarly, $QG_B = QG_C = QG_D = R/3$.

4. Since $(x-y)^{2n-1}$ divides $P(x,y)$, $P(x,y) = (x-y)^{2n-1}Q(x,y)$ where Q is some other polynomial. Interchanging x and y, we get

$$P(y,x) = (y-x)^{2n-1}Q(y,x).$$

Since $P(x,y)$ is symmetric it equals $P(y,x)$. Hence,

$$(x-y)^{2n-1}Q(x,y) = (y-x)^{2n-1}Q(y,x)$$

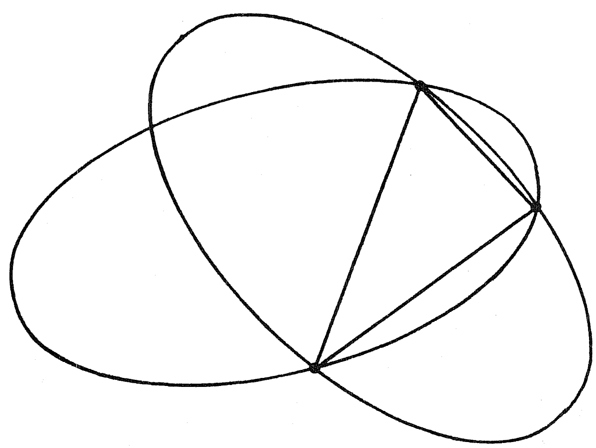
or

$$(x-y)^{2n-1}[Q(x,y)+Q(y,x)] = 0$$

for all x, y. Thus $Q(x,y)+Q(y,x) = 0$, that is, $Q(x,y)$ is skew symmetric in x, y. Now let $x = y$ to give $Q(x,x) = 0$. By the Factor Theorem, $Q(x,y)$ is divisible by $x-y$ which gives the required result.

In a similar way, if $P(x,y)$ is a skew symmetric polynomial and is divisible by $(x-y)^{2n}$, then it is also divisible by $(x-y)^{2n+1}$; if $P(x,y,z)$ is a symmetric polynomial in x, y, z and is divisible by $(x+y-z)^n$, then it is also divisible by $(y+z-x)^n(z+x-y)^n$; etc.

5. The answer is yes and follows immediately by considering the intersection of two congruent ellipses as in the following figure.



It follows that the affirmative result also holds for congruent inscribed quadrilaterals.

*   *   *

As always we solicit your problem sets and solutions. Please send me your local Olympiads and National competitions.

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PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **December 1, 1988,** *although solutions received after that date will also be considered until the time when a solution is published.*

1341. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

An ellipse has center O and the ratio of the lengths of the axes is $2 + \sqrt{3}$. If P is a point on the ellipse, prove that the (acute) angle between the tangent to the ellipse at P and the radius vector PO is at least 30°.

1342. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle and let D and E be the midpoints of BC and AC respectively. Suppose that DE is tangent to the incircle of ΔABC. Prove that $r_c = 2r$, where r is the inradius of ΔABC and r_c is the exradius to AB.

1343. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is an acute triangle and D, E are the feet of the altitudes to BC, AC respectively. Suppose DE is tangent to the incircle. Show that $r_c = 2R$, where R is the circumradius and r_c is the exradius to AB.

1344. *Proposed by Florentin Smarandache, Craiova, Romania.*

There are given $mn + 1$ points such that among any $m + 1$ of them there are two within distance 1 from each other. Prove that there exists a sphere of radius 1 containing at least $n + 1$ of the points.

1345. *Proposed by P. Erdos, Hungarian Academy of Sciences, and Esther Szekeres, University of New South Wales, Kensington, Australia.*

Given a convex n–gon $X_1X_2...X_n$ of perimeter p, denote by $f(X_i)$ the sum of the distances of X_i to the other n–1 vertices.

(a) Show that if $n \geq 6$, there is a vertex X_i such that $f(X_i) > p$.

(b) Is it true that for n large enough, the *average value* of $f(X_i)$, $1 \leq i \leq n$, is greater than p?

1346. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = 12^\circ$. Let D on AC and E on AB be such that $\angle CBD = 42^\circ$ and $\angle BCE = 18^\circ$. Prove that $\angle EDB = 12^\circ$. (This problem came to me via a student; I don't know the source.)

1347. *Proposed by Lanny Semenko, Erewhon, Alberta.*

The positive integer 275 has the property that

$$275^\circ C = 527^\circ F,$$

where 527 is obtained by moving the rightmost digit of 275 to the left end. Find another positive integer with this property.

1348*. *Proposed by Murray S. Klamkin, University of Alberta.*

Two congruent convex centrosymmetric planar figures are inclined to each other (in the same plane) at a given angle. Prove or disprove that their intersection has maximum area when the two centers coincide.

1349. *Proposed by Josep Rifa i Coma, Institut "Jaume Callis", Barcelona, Spain.*

(a) Show that, if n is an even positive integer,

$$x^n(y-z) + y^n(z-x) + z^n(x-y) = 0 \qquad (1)$$

has no solution in distinct nonzero real numbers.

(b) Show that (1) does have a solution in distinct nonzero real numbers if $n = 3$.

1350. *Proposed by Peter Watson-Hurthig, Columbia College, Burnaby, British Columbia.*

(a) Dissect an equilateral triangle into three polygons that are similar to each other but all of different sizes.

(b) Do the same for a square.

(c)* Can you do the same for any other regular polygon? (Allow yourself more than three pieces if necessary.)

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1218*. [1987: 53] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*

Let F_1 be the area of the orthic triangle of an acute triangle of area F and circumradius R. Prove that

$$F_1 \le \frac{4F^3}{27R^4}.$$

Editor's comment.

This problem has already been dealt with in the solution of *Crux* 1199 [1988: 87]. Solvers include all solvers of *Crux* 1199 plus GEORGE TSINTSIFAS, Thessaloniki, Greece.

*   *   *

1222. [1987: 85] *Proposed by George Szekeres, University of New South Wales, Kensington, Australia.*

Evaluate the symmetric $n \times n$ determinant D_n in which $d_{i,i+2} = d_{i+2,i} = -1$ for $i = 1,\ldots,n-2$, $d_{ij} = 1$ otherwise. Also evaluate $\overline{D}_n$ in which $\overline{d}_{ij} = -d_{ij}$ for $i \neq j$, $\overline{d}_{ii} = d_{ii}$. [See the solution to *Crux* 1033 [1987: 89].]

Solution by the proposer.

Denote by E_n the determinant D_{n-1} bordered by a row of 1's on the top and a column of 1's on the left, i.e.,

$$E_n = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & -1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & -1 & 1 & \\ 1 & -1 & 1 & 1 & 1 & -1 & \cdots \\ 1 & 1 & -1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & -1 & 1 & 1 & \\ & & & \vdots & & & \ddots \end{vmatrix}.$$

Subtract the fifth row and column of D_n from the first row and column, then the sixth row and column from the second row and column. We obtain

$$D_n = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & \\ 0 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & \cdots \\ 2 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & \\ 0 & 2 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & \\ & & & & & \vdots & & & & & \ddots \end{vmatrix} = 16\, D^*_{n-4},$$

where

$$\overset{*}{D}_n = \begin{vmatrix} 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & \\ & & & \vdots & & & & & \ddots \end{vmatrix}$$

($\overset{*}{D}_n$ differs from D_n in only four places). Using similar trivial row and column operations on $\overset{*}{D}_{n-4}$, the first of which is subtracting the third row and column from the first row and column, we obtain

$$D_n = 16 \begin{vmatrix} 4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 4 & 0 & -2 & 0 & 0 & 0 & 0 & \\ -2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & \\ 0 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & \\ & & & \vdots & & & & & \ddots \end{vmatrix}$$

$$= 256 \begin{vmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & & & \\ 0 & 0 & 1 & & E_{n-7} & \\ 0 & 0 & 1 & & & \end{vmatrix} = 256 \begin{vmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & & & \\ 0 & 0 & 0 & & E_{n-7} & \\ 0 & 0 & 0 & & & \end{vmatrix}$$

$$= 256 \begin{vmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & & & \\ 0 & 0 & & E_{n-7} & \\ 0 & 0 & & & \end{vmatrix} - 256 \begin{vmatrix} 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & & & \\ 0 & 0 & & E_{n-7} & \\ 0 & 0 & & & \end{vmatrix}$$

$$= -256 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & & & \\ 0 & & E_{n-7} & \\ 0 & & & \end{vmatrix} - 256 \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & & & \\ 0 & & E_{n-7} & \\ 0 & & & \end{vmatrix}$$

$$= -256E_{n-7} - 256E_{n-7} - 256 \begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & & & \\ 0 & & D_{n-8} & \\ 0 & & & \end{vmatrix}$$

$$= -512E_{n-7} + 256D_{n-8}. \qquad (1)$$

Similarly, in E_n subtract the first row and column from the second and third; we easily find

$$E_n = \begin{vmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & \\ 1 & -2 & 0 & 1 & 1 & -1 & 1 & 1 & \\ 1 & 0 & -2 & 1 & 1 & 1 & -1 & 1 & \cdots \\ 1 & 0 & 0 & -1 & 1 & 1 & 1 & -1 & \\ 1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & \\ 1 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & \\ & & & \vdots & & & & & \ddots \end{vmatrix} = 16E_{n-4}\,.$$

Since $E_1 = 1$ and $E_2 = E_3 = E_4 = 0$, we get

$$E_n = \begin{cases} 2^{n-1} & \text{if} \quad n \equiv 1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Setting this into (1) we obtain

$$D_n = \begin{cases} 2^8 D_{n-8} - 2^{n+1} & \text{if} \quad n \equiv 0 \bmod 4, \\ 2^8 D_{n-8} & \text{otherwise.} \end{cases}$$

With the easily verifiable initial values

$$D_1 = 1, \quad D_2 = 0, \quad D_3 = -4, \quad D_4 = -16,$$
$$D_5 = D_6 = D_7 = 0, \quad D_8 = -256,$$

we finally get

$$D_{8m} = -(2m-1)2^{8m}, \quad D_{8m+1} = 2^{8m},$$
$$D_{8m+3} = -2^{8m+2}, \quad D_{8m+4} = -(2m+1)2^{8m+4},$$
$$D_n = 0 \quad \text{for} \quad n \equiv 2, 5, 6, 7 \bmod 8.$$

To determine $\overline{D}_n$ we indicate briefly the row and column operations: third from first, fourth from second, followed by fourth from third. This finally yields the recursion

$$\overline{D}_n = 64\overline{E}_{n-5} - 64\overline{D}_{n-6}\,.$$

In $\overline{E}_n$ the operations are: fourth from second, fifth from third, followed by second from first in the new determinant. We obtain

$$\overline{E}_n = 192\overline{E}_{n-6} - 256\overline{D}_{n-7}\,.$$

Substituting from the first into the second recursion we obtain

$$\overline{D}_n = 2^7\overline{D}_{n-6} - 2^{12}\overline{D}_{n-12}\,,$$

or, writing $F_n = 2^{-n+1}\overline{D}_n$,

$$F_n = 2F_{n-6} - F_{n-12}\,.$$

The initial values for F_n are easily calculated to be

$$F_1 = 1, \quad F_2 = F_3 = F_4 = F_5 = F_6 = 0,$$
$$F_7 = -1, \quad F_8 = -2, \quad F_9 = F_{10} = F_{11} = 0, \quad F_{12} = -2,$$

giving

$$F_{6m} = -2m + 2, \quad F_{6m+1} = -2m + 1, \quad F_{6m+2} = -2m,$$
$$F_{6m+3} = F_{6m+4} = F_{6m+5} = 0,$$

and finally

$$\mathcal{D}_{6m} = -(m-1)2^{6m}, \qquad \mathcal{D}_{6m+1} = -(2m-1)2^{6m},$$
$$\mathcal{D}_{6m+2} = -m\cdot 2^{6m+2},$$
$$\mathcal{D}_n = 0 \quad \text{for } n \equiv 3, 4, 5 \bmod 6.$$

* * *

1224. [1987: 86] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

$A_1A_2A_3$ is a triangle with circumcircle Ω. Let $x_1 < X_1$ be the radii of the two circles tangent to A_1A_2, A_1A_3, and arc A_2A_3 of Ω. Let x_2, X_2, x_3, X_3 be defined analogously. Prove that:

(a) $$\sum_{i=1}^{3} \frac{x_i}{X_i} = 1;$$

(b) $$\sum_{i=1}^{3} X_i \geq 3\sum_{i=1}^{3} x_i \geq 12r,$$

where r is the inradius of $\Delta A_1A_2A_3$.

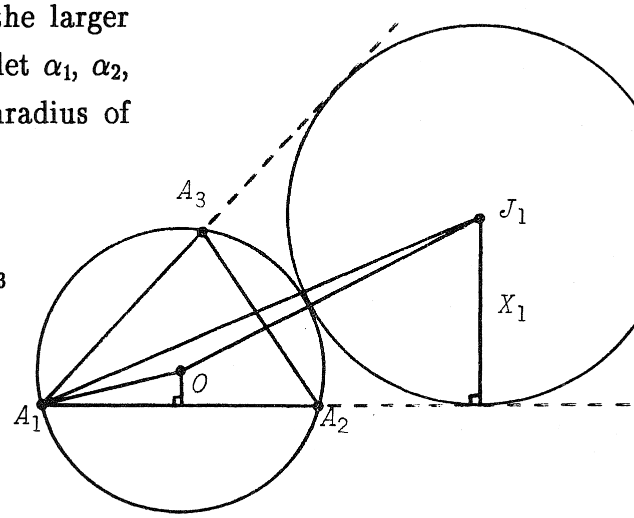
I. *Solution to part (a) by J.T. Groenman, Arnhem, The Netherlands.*

Let J_1 be the centre of the larger circle (of radius X_1) in the problem, and let α_1, α_2, α_3 and R be the angles and the circumradius of $\Delta A_1A_2A_3$. Then

$$\angle J_1A_1O = \alpha_1/2 - (90^\circ - \alpha_3)$$
$$= \frac{180^\circ - \alpha_2 - \alpha_3}{2} - 90^\circ + \alpha_3$$
$$= \frac{\alpha_3 - \alpha_2}{2},$$

and

$$A_1J_1 = \frac{X_1}{\sin(\alpha_1/2)}.$$



Using the cosine rule in ΔA_1J_1O,

$$(R + X_1)^2 = R^2 + \frac{X_1^2}{\sin^2(\alpha_1/2)} - \frac{2RX_1}{\sin(\alpha_1/2)}\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right]$$

from which we get

$$2R + X_1 = \frac{X_1}{\sin^2(\alpha_1/2)} - \frac{2R}{\sin(\alpha_1/2)}\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right],$$

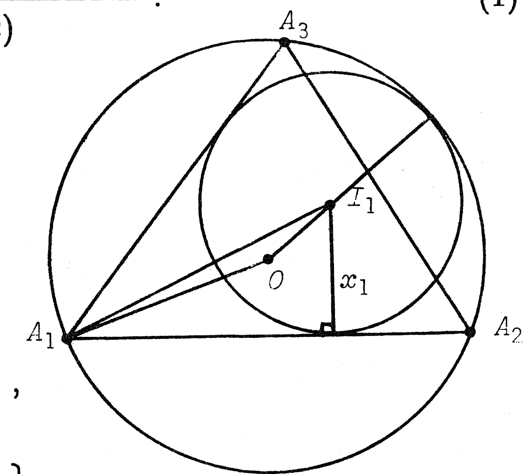
$$2R\sin^2(\alpha_1/2) + 2R\sin(\alpha_1/2)\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right] = X_1\cos^2(\alpha_1/2),$$

$$2R\sin(\alpha_1/2)\left[\cos\left[\frac{\alpha_3 + \alpha_2}{2}\right] + \cos\left[\frac{\alpha_3 - \alpha_2}{2}\right]\right] = X_1\cos^2(\alpha_1/2),$$

$$4R\sin(\alpha_1/2)\cos(\alpha_2/2)\cos(\alpha_3/2) = X_1\cos^2(\alpha_1/2),$$

and so

$$X_1 = \frac{4R\ \sin(\alpha_1/2)\cos(\alpha_2/2)\cos(\alpha_3/2)}{\cos^2(\alpha_1/2)}. \tag{1}$$



We use the same method on the smaller circle (centre I_1 and radius x_1). Applying the cosine rule in ΔA_1I_1O yields

$$(R - x_1)^2 = R^2 + \frac{x_1^2}{\sin^2(\alpha_1/2)} - \frac{2Rx_1}{\sin(\alpha_1/2)}\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right],$$

$$-2R + x_1 = \frac{x_1}{\sin^2(\alpha_1/2)} - \frac{2R}{\sin(\alpha_1/2)}\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right],$$

$$-2R\sin^2(\alpha_1/2) + 2R\sin(\alpha_1/2)\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right] = x_1\cos^2(\alpha_1/2),$$

$$2R\sin(\alpha_1/2)\left[\cos\left[\frac{\alpha_3 - \alpha_2}{2}\right] - \cos\left[\frac{\alpha_3 + \alpha_2}{2}\right]\right] = x_1\cos^2(\alpha_1/2),$$

$$4R\sin(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2) = x_1\cos^2(\alpha_1/2),$$

and finally

$$x_1 = \frac{4R\ \sin(\alpha_1/2)\ \sin(\alpha_2/2)\ \sin(\alpha_3/2)}{\cos^2(\alpha_1/2)}. \tag{2}$$

Thus from (1) and (2)

$$\frac{x_1}{X_1} = \frac{4R\ \sin(\alpha_1/2)\ \sin(\alpha_2/2)\ \sin(\alpha_3/2)}{4R\ \sin(\alpha_1/2)\ \cos(\alpha_2/2)\ \cos(\alpha_3/2)} = \frac{r}{r_1}, \tag{3}$$

where r_1 is the exradius to the side A_2A_3 of $\Delta A_1A_2A_3$. Similarly

$$\frac{x_2}{X_2} = \frac{r}{r_2}, \qquad \frac{x_3}{X_3} = \frac{r}{r_3}, \tag{4}$$

where r_2, r_3 are the other exradii. Using

$$r = \frac{\text{area } \Delta A_1A_2A_3}{s}, \qquad r_1 = \frac{\text{area } \Delta A_1A_2A_3}{s - a_1}, \qquad \text{etc.}, \tag{5}$$

where a_1, a_2, a_3 are the sides of $\Delta A_1A_2A_3$ and s is the semiperimeter, we have

$$\frac{x_1}{X_1} + \frac{x_2}{X_2} + \frac{x_3}{X_3} = \frac{r}{r_1} + \frac{r}{r_2} + \frac{r}{r_3}$$

$$= \frac{s - a_1}{s} + \frac{s - a_2}{s} + \frac{s - a_3}{s}$$

$$= 1.$$

II. *Solution to part (b) by the proposer.*

[The proposer first solved part (a) much as above. References to Groenman's solution have been added by the editor.]

Assume $a_1 \geq a_2 \geq a_3$. Then from (5) we see that $r_1 \geq r_2 \geq r_3$, and from (2) and

$$4R \sin(\alpha_1/2) \sin(\alpha_2/2) \sin(\alpha_3/2) = r$$

we see that

$$x_1 = r \sec^2(\alpha_1/2), \qquad \text{etc.} \tag{6}$$

and thus that $x_1 \geq x_2 \geq x_3$. By (3), (4) and Chebyshev's inequality,

$$\begin{aligned} X_1 + X_2 + X_3 &= \frac{r_1x_1 + r_2x_2 + r_3x_3}{r} \\ &\geq \frac{1}{3r}(r_1 + r_2 + r_3)(x_1 + x_2 + x_3) \\ &= \frac{4R + r}{3r}(x_1 + x_2 + x_3) \\ &\geq \frac{9r}{3r}(x_1 + x_2 + x_3) \\ &= 3(x_1 + x_2 + x_3). \end{aligned}$$

This establishes the first inequality of (b). For the second,

$$\begin{aligned} x_1 + x_2 + x_3 &= r[\sec^2(\alpha_1/2) + \sec^2(\alpha_2/2) + \sec^2(\alpha_3/2)] \\ &\geq 4r \end{aligned}$$

by (6) and item 2.48 of Bottema et al, *Geometric Inequalities.*

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. For part (b), Groenman did only the right–hand inequality.

* * *

1226. [1987: 86] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

Let $ABCD$ be a quadrilateral inscribed in a circle, and let O_1, O_2, O_3, O_4 be the inscribed circles of triangles BCD, CDA, DAB, ABC respectively.

(a) Show that the centers of these four circles are the vertices of a rectangle.

(b) Show that $r_1 + r_3 = r_2 + r_4$, where r_i is the radius of O_i.

Comments by the editor.

Both parts of this problem, it turns out, are already known. In fact, part (a) (as seems to have happened a lot lately) has already appeared in *Crux*, as part (b) of problem 483 (see [1980: 227] for a nice solution). It is also given on p.255 of [1], and in other places; Léo's comments on [1980: 229] list some of them. A generalization of part (b) is on p.193 of [1]. I thank those readers who sent in references for both parts.

At least we can now update one of Léo's remarks on [1980: 229]: the present problem came from a lost Japanese wooden tablet hung in 1800 and was recorded in the 1807 Japanese book *Zoku Sinheki Sanpo.* Thus it easily predates the 1874 Neuberg reference, given by Léo as the earliest he had found.

Reference:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960.

Solved by FRANCISCO BELLOT, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

* * *

1227. [1987: 86] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find all angles θ in $[0,2\pi)$ for which

$$\sin\theta + \cos\theta + \tan\theta + \cot\theta + \sec\theta + \csc\theta = 6.4.$$

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

Let

$$x = \cos\theta, \qquad y = \sin\theta.$$

Then the given equation is

$$x + y + \frac{x}{y} + \frac{y}{x} + \frac{1}{x} + \frac{1}{y} = 6.4, \tag{1}$$

subject to $x^2 + y^2 = 1$. (1) can be simplified to

$$x + y + \frac{x^2 + y^2}{xy} + \frac{x + y}{xy} = 6.4,$$

that is,

$$x + y + \frac{1 + x + y}{xy} = 6.4, \qquad x^2 + y^2 = 1. \tag{2}$$

This suggests the use of

$$u = x + y, \qquad v = xy, \qquad v \neq 0.$$

Then (2) becomes

$$u + \frac{1 + u}{v} = \frac{32}{5}, \qquad u^2 - 2v = 1,$$

or, simplifying,

$$5uv + 5u - 32v + 5 = 0, \qquad u^2 - 2v = 1.$$

Substituting

$$v = \frac{u^2 - 1}{2},$$

we obtain

$$5u^3 - 32u^2 + 5u + 42 = 0.$$

Since $u = -1$ is a root, we can factor to obtain

$$(u + 1)(5u - 7)(u - 6) = 0.$$

This gives three possibilities for u: $u = -1$, $u = 7/5$, or $u = 6$. Since $u = -1$ implies $v = 0$, we reject this. Also, since

$$u = x + y = \cos\theta + \sin\theta,$$

$u = 6$ is impossible. Hence $u = 7/5$ is the only possible solution, whence $v = 12/25$, and we must solve

$$x + y = \frac{7}{5}, \qquad xy = \frac{12}{25}.$$

This gives

$$x = \frac{3}{5}, \qquad y = \frac{4}{5}$$

or

$$x = \frac{4}{5}, \qquad y = \frac{3}{5},$$

and these satisfy the original equation. Hence the two angles are

$$\theta = \sin^{-1}\frac{3}{5} \quad \text{and} \quad \theta = \sin^{-1}\frac{4}{5},$$

the angles in the familiar 3–4–5 right triangle.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano and FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; KEE–WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. There was one partial solution.

Ahlburg, and Lopez and Bellot, solved the general equation

$$\sin\theta + \cos\theta + \tan\theta + \cot\theta + \sec\theta + \csc\theta = m$$

for m constant, determining that there is a maximum of four solutions, this occurring for

$$-3\sqrt{2} + 2 < m < -2\sqrt{2} + 1.$$

Lopez and Bellot located the above general equation in a problem on page 349 *of Rey Pastor and Gallega Diaz,* Norte de Problemas, *Dossat, Madrid, ca.* 1951.

* * *

<u>1228</u>. [1987: 87] *Proposed by J. Garfunkel, Flushing, New York and C. Gardner, Austin, Texas.*

If QRS is the equilateral triangle of minimum perimeter that can be inscribed in a triangle ABC, show that the perimeter of QRS is at most half the perimeter of ABC, with equality when ABC is equilateral.

Solution by Niels Bejlegaard, Stavanger, Norway.

Minimizing the perimeter of an equilateral triangle inscribed in ΔABC is equivalent to minimizing its area. It has been shown in the solution of *Crux* 624 (b) [1982: 109] that

$$\frac{\text{Area(least equilateral triangle inscribed in } \Delta ABC)}{\text{Area}(\Delta ABC)} \le \frac{1}{4}.$$

Let s be the side of the least equilateral triangle inscribed in ΔABC. Then its area is $s^2\sqrt{3}/4$. From 4.2 of Bottema et al, *Geometric Inequalities*, we get

$$\frac{s^2\sqrt{3}}{4} \le \frac{1}{4}\,\text{Area}(\Delta ABC) \le \frac{1}{4}\frac{1}{3\sqrt{3}}\left[\frac{a + b + c}{2}\right]^2,$$

that is,

$$3s \le \frac{a + b + c}{2},$$

as required. Equality holds only when ΔABC is equilateral.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; and the proposers. Klamkin's solution was the same as Bejlegaard's.

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<u>1229</u>. [1987: 87] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Characterize all positive integers a and b such that

$$(a,b)^{[a,b]} \le [a,b]^{(a,b)}$$

and determine when equality holds. (As usual, (a,b) and $[a,b]$ denote respectively the g.c.d. and l.c.m. of a and b.)

Solution by Robert E. Shafer, Berkeley, California.

It is a well–known theorem that $x^y \ge y^x$ for $e \le x \le y$, with equality if and only if $x = y$. Therefore we find:

if $(a,b) \ge 3$, $a \ne b$, the inequality is false;

if $(a,b) = 2$ and $[a,b] > 4$, the inequality is false;

if $a = 2$, $b = 4$ or $a = 4$, $b = 2$ we have equality (the only "interesting" case);

if $a = b \ge 1$, we have equality;

if $(a,b) = 1$, $ab > 1$, the inequality is evident.

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston, Illinois; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; MICHAEL SELBY, University of Windsor, Windsor, Ontario; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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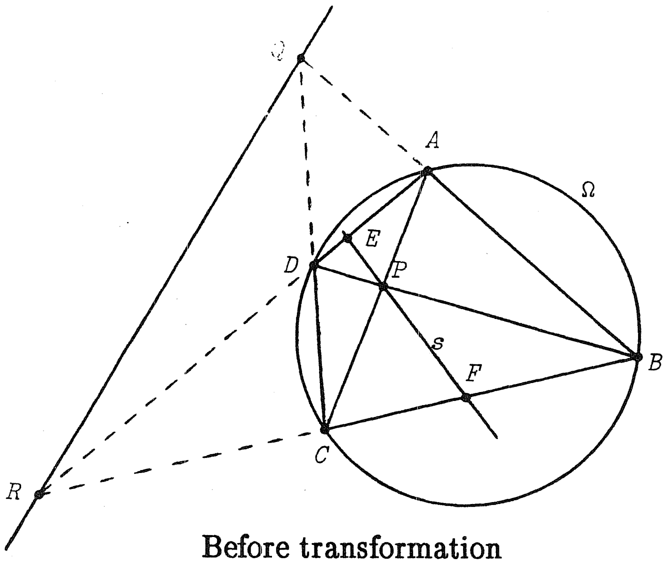
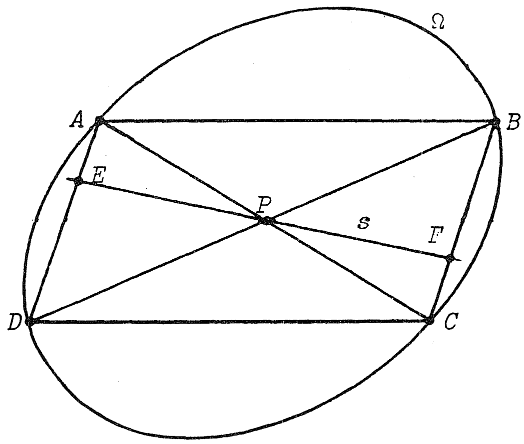
1230. [1987: 87] *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let $P = AC \cap BD$ and let s be a line through P cutting AD at E and BC at F. Prove that there exists a conic tangent to AD at E, to BC at F, and twice tangent to Ω.

Solution by Dan Pedoe, Minneapolis, Minnesota.



Before transformation



After transformation

This problem is one of affine geometry, and Ω being specified as a circle is irrelevant, unless the solution by the proposer depended on this fact.

Let $Q = AB \cap CD$ and $R = AD \cap CB$. An affine transformation which maps the line QR onto the "line at infinity" maps P onto the centre of the image of the circle Ω. We use the same notation for the mapped figure, noting that $ABCD$ is now a parallelogram, that P is the midpoint of EF, and of course Ω is not necessarily a circle.

There exists a conic Ω^* touching AD at E and BC at F, and since these are parallel tangents to Ω^*, the midpoint of EF (i.e. P) is the centre of Ω^*. If Ω^* touches Ω at the point X, say, then it also touches Ω at X', where $PX = PX'$. The problem therefore reduces to showing that there is a conic Ω^* in the pencil of conics touching AD at E and BC at F which touches the conic Ω. Since Ω has the same centre as conics of the pencil, this is intuitive and also correct, and easily proved.

Also solved by the proposer, who recognized (with Pedoe) that Ω need not be a circle, but could be any conic.

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1231. [1987: 118] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

On the planet of Lyre the inhabitants carefully recognize special years when their age is of the form p^2q where p and q are different prime numbers. On Lyre one is a student until he reaches a special year immediately following a special year; he then becomes a master until he reaches a year that is the third in a row of consecutive special years; he then becomes a sage until he dies in a special year that is the fourth in a row of consecutive special years.

(a) When does one become a master?

(b) When does one become a sage?

(c) How long do the inhabitants of Lyre live?

(d)* Do five special years ever occur consecutively?

Solution by the proposer.

(a) At age 45 he becomes a master: $44 = 2^2 \cdot 11$, $45 = 3^2 \cdot 5$.

(b) At age 605 he becomes a sage: $603 = 3^2 \cdot 67$, $604 = 2^2 \cdot 151$, $605 = 11^2 \cdot 5$.

(c) To have four special years in a row there must be two that are even with the intervening year a multiple of 3. Thus they must be of the form:

$$n = 2p_1^2\,, \qquad n+1 = 3p_2^2 \;\text{ or }\; 9p_2\,, \qquad n+2 = 4p_3\,, \tag{1}$$

with the fourth year either $n-1$ or $n+3$. [Note that

$$n = 4p_1\,, \qquad n+1 = 3p_2^2 \;\text{ or }\; 9p_2\,, \qquad n+2 = 2p_3^2$$

is impossible since this would imply either

$$3p_2^2 + 1 = 2p_3^2 \qquad \text{or} \qquad 9p_2 + 1 = 2p_3^2\,;$$

the first fails since $p_2^2 \equiv p_3^2 \equiv 1 \bmod 4$, the second since $p_3^2 \not\equiv 5 \bmod 9$.] From (1) we get either $2p_1^2 + 1 = 3p_2^2$, which can be solved as a Fermat-Pell equation, with a check for prime entries, up to very large numbers rapidly; or $2p_1^2 + 1 = 9p_2$ and $2p_1^2 + 2 = 4p_3$, which can also be checked for p_1, p_2, p_3 all prime. In either case $n-1$ and $n+3$ can then be factored to test if they are of the right form. The result is that the inhabitants die at age 17042641444:

$17042641441 = 49 \cdot 347809009$, $\quad 17042641442 = 2 \cdot 92311^2$,

$17042641443 = 9 \cdot 1893626827$, $\quad 17042641444 = 4 \cdot 4260660361$.

(d) A longer search produced

$10093613546512321 = 49 \cdot 205992113194129$,

$10093613546512322 = 2 \cdot 71040881^2$,

$10093613546512323 = 9 \cdot 1121512616279147$,

$10093613546512324 = 4 \cdot 2523403386628081$,

$10093613546512325 = 25 \cdot 403744541860493$.

Note: it is not possible to have six special years in a row because this would require three successive even numbers of the form p^2q when only two are possible (one with $p = 2$ and the other with $q = 2$).

Parts (a) and (b) were also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.A. MCCALLUM, Medicine Hat, Alberta; and P. PENNING, Delft, The Netherlands.

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1232. [1987: 118] *Proposed by Esther and George Szekeres, University of New South Wales, Kensington, Australia.*

Let n be a positive integer not equal to 1, 2, 3, 6, or 15. Show that there is a positive integer $x \le [n/2] - 1$ such that both x and $2x + 1$ are relatively prime to n.

I. *Solution by the proposers.*

Assume

$$n = 2^r 3^s 5^t (2k + 1),$$

where $2k + 1$ is relatively prime to 15. If $s = 0$ (and $n \ge 4$) then $x = 1$ is a solution, and if $r = t = 0$ and $n \ge 7$ then $x = 2$ is a solution. We may therefore assume that $s > 0$ and $r + t > 0$, and in particular $n \ge 6(2k + 1)$. We may also assume that $k \ne 0$ (and hence $k \ge 3$); for if $k = 0$ and $n \ge 24$ then $x = 11$ is a solution, and the only other cases to be considered when $k = 0$ are $n = 12$ and $n = 18$, for which $x = 5$ answers the question.

Consider now all possible values of $k \ge 3$ such that $(2k + 1,30) = 1$. These are

$$k \equiv 0, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 24, 26, 29 \pmod{30}.$$

For each of these residue classes we shall find $x = ak + b$, where a, b are integers and $a > 0$, which solves the problem. The following table gives these values of x :

k (modulo 30)	x	$2x + 1$
0, 24	$k - 1$	$2k - 1$
3	$k + 8$	$2k + 17$
5, 14, 20, 29	$2k + 13$	$4k + 27$
6, 18	$k + 5$	$2k + 11$
8, 26	$k + 3$	$2k + 7$
9	$k + 2$	$2k + 5$
11, 23	$k - 12$	$2k - 23$
15, 21	$2k - 1$	$4k - 1$

It is seen from the table that

$$x \equiv 29, \quad 2x + 1 \equiv 29 \quad \text{for} \quad k \equiv 0, 11, 15, 26,$$
$$x \equiv 11, \quad 2x + 1 \equiv 23 \quad \text{for} \quad k \equiv 3, 6, 8, 9, 14, 21, 23, 29,$$
$$x \equiv 23, \quad 2x + 1 \equiv 17 \quad \text{for} \quad k \equiv 5, 18, 20, 24.$$

Hence x and $2x + 1$ are both relatively prime to 30 in all cases. Also,

$$x \le 2k + 13 < 6k + 2 = 3(2k + 1) - 1 \le [n/2] - 1$$

in all cases, and $x \ge 1$ except for $k = 11$, in which case we can choose $x = 29$, $2x + 1 = 59$ (since $[n/2] - 1 \ge 3 \cdot 23 - 1 > 29$).

The only thing that remains to be verified is that with the chosen values of x both x

and $2x + 1$ are prime to $2k + 1$; this will imply that x and $2x + 1$ are prime to n. Now if $x = k - 1$ for instance, then

$$(x,2k+1) = (k-1,2k+1) = (2k+1,3) = 1 \text{ or } 3,$$
$$(2x+1,2k+1) = (2k-1,2k+1) = (2k+1,2) = 1 \text{ or } 2,$$

and here 2 and 3 are excluded since $(2k + 1,30) = 1$. Thus

$$(x,2k+1) = (2x+1,2k+1) = 1.$$

Similarly,

if	$x = k + 8$	then	$(k+8,2k+1) \mid 15,$	$(2k+17,2k+1) \mid 16,$
if	$x = 2k + 13$	then	$(2k+13,2k+1) \mid 12,$	$(4k+27,2k+1) \mid 25,$
if	$x = k + 5$	then	$(k+5,2k+1) \mid 9,$	$(2k+11,2k+1) \mid 10,$
if	$x = k + 3$	then	$(k+3,2k+1) \mid 5,$	$(2k+7,2k+1) \mid 6,$
if	$x = k + 2$	then	$(k+2,2k+1) \mid 3,$	$(2k+5,2k+1) \mid 4,$
if	$x = k - 12$	then	$(k-12,2k+1) \mid 25,$	$(2k-23,2k+1) \mid 24,$
if	$x = 2k - 1$	then	$(2k-1,2k+1) \mid 2,$	$(4k-1,2k+1) \mid 3.$

In all cases, since 2, 3, and 5 do not divide $2k + 1$, the only alternative left for the gcd's is 1.

II. *Editor's comment.*

There were three responses to this problem, and all three, after promising starts, failed to consider some residue class or otherwise had some serious omission. Perhaps the lengthy case analysis of the proposers is unavoidable!

Readers will observe that the above proof actually chooses x satisfying the stronger bound

$$x \le \frac{n}{6 - \epsilon}$$

for any $\epsilon > 0$, with only finitely many exceptions for each ϵ.

* * *

1233. [1987: 118] *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In the plane we have given the line l: $y = 43/25\ x + 25/43$. For $\epsilon > 0$ denote by S_ϵ the ϵ–neighbourhood of l, i.e. S_ϵ is the strip containing all points in the plane whose distance to l is not greater than ϵ. Find a value for ϵ such that S_ϵ contains no points with integer coordinates.

Solution by Kee–Wai Lau, Hong Kong.

We show that ϵ can be taken to be any positive number less than

$$\frac{20}{43\sqrt{2474}}.$$

The equation of the line l written in normal form is

$$\frac{1849x - 1075y + 625}{43\sqrt{2474}} = 0,$$

so that the distance of any point (a,b) to l is given by

$$\frac{|1849a - 1075b + 625|}{43\sqrt{2474}}.$$

It remains to show that the minimum of $|1849a - 1075b + 625|$ is 20 where a and b are integers. Now

$$|1849a - 1075b + 625| = |43z + 625|,$$

where $z = 43a - 25b$ is an integer. It is easy to see that the minimum of $|43z + 625|$ is 20 when $z = -15$. The minimum can be attained at any solution of $43a - 25b = -15$, for example when $(a,b) = (-105,-180)$.

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

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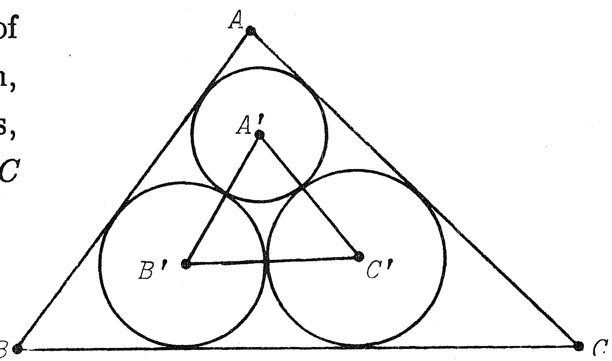
1234*. [1987: 119] *Proposed by Jack Garfunkel, Flushing, New York.*

Given the Malfatti configuration of three circles inscribed in triangle ABC as shown, let A', B', C' be the centers of the three circles, and let r and r' be the inradii of triangles ABC and $A'B'C'$ respectively. Prove that

$$r \le (1 + \sqrt{3})r'.$$

Equality is attained when ABC is equilateral.



Solution by G.P. Henderson, Campbellcroft, Ontario.

The inequality as stated is the wrong way around. We will show that

$$1 + \sqrt{3} \le \frac{r}{r'} < 3. \qquad (1)$$

Let the radii of the circles with centres A', B', C' be r_1, r_2, r_3 respectively. By equation (5) on [1982: 84] we have

$$r = \frac{(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 + r_2 + r_3})\sqrt{r_1r_2r_3}}{\sqrt{r_2r_3} + \sqrt{r_3r_1} + \sqrt{r_1r_2}}$$

$$= \frac{2\sqrt{r_1r_2r_3}}{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} - \sqrt{r_1 + r_2 + r_3}}.$$

Since the sides of $\Delta A'B'C'$ have lengths $r_1 + r_2$, $r_2 + r_3$, $r_3 + r_1$ and its semiperimeter is $r_1 + r_2 + r_3$,

$$r'(r_1 + r_2 + r_3) = \text{Area}(\Delta A'B'C') = \sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}$$

and thus

$$\frac{r}{r'} = \frac{2\sqrt{r_1 + r_2 + r_3}}{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} - \sqrt{r_1 + r_2 + r_3}}.$$

It is easily seen (by putting $r_1 + r_2 + r_3 = 1$) that the minimum value of this expression is $1 + \sqrt{3}$, obtained when $r_1 = r_2 = r_3$. This establishes the lower bound of (1).

For the upper bound, we can assume $1 = r_3 \le r_2 \le r_1$. Since the projection of $B'C'$ on BC has length $2\sqrt{r_2 r_3}$ (see (2) on [1982: 83]), we have $r_1 < 2\sqrt{r_2 r_3}$. Thus, putting $\sqrt{r_1} = x_1$, $\sqrt{r_2} = x_2$,

$$\frac{x_1^2}{2} < x_2 \le x_1$$

and so

$$1 \le x_1 < 2.$$

We are to prove that

$$\frac{2\sqrt{x_1^2 + x_2^2 + 1}}{x_1 + x_2 + 1 - \sqrt{x_1^2 + x_2^2 + 1}} < 3$$

or

$$5\sqrt{x_1^2 + x_2^2 + 1} < 3(x_1 + x_2 + 1)$$

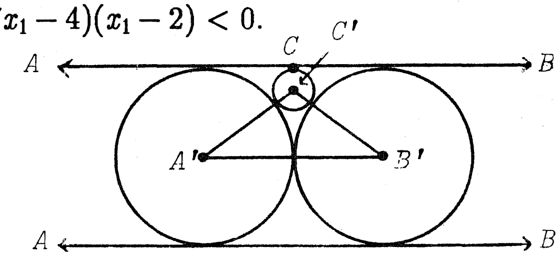
or

$$f(x_2) = 8x_2^2 - 9(x_1 + 1)x_2 + 8x_1^2 - 9x_1 + 8 < 0. \qquad (2)$$

Since, for each fixed x_1, the graph of $f(x_2)$ is convex up, we only need to verify (2) for the endpoints $x_2 = x_1^2/2$ and $x_2 = x_1$. Since $1 \le x_1 < 2$,

$$f(x_1^2/2) = \frac{1}{2}(4x_1^4 - 9x_1^3 + 7x_1^2 - 18x_1 + 16)$$
$$= \frac{1}{2}(x_1 - 1)(x_1 - 2)(4x_1^2 + 3x_1 + 8) \le 0,$$
$$f(x_1) = 7x_1^2 - 18x_1 + 8 = (7x_1 - 4)(x_1 - 2) < 0.$$

The upper bound of (1) is approached arbitrarily closely when $r_1{:}r_2{:}r_3 \to 4{:}4{:}1$. In the limit, ΔABC becomes a pair of parallel lines.



Also solved (and corrected) by ISAO NAOI, Seki, Gifu, Japan.

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1235. [1987: 119] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In triangle ABC, D and E are the feet of the altitudes of BC and AC respectively, K and L are the midpoints of BC and AC respectively, H is the orthocentre, O is the circumcentre. Prove that if $LD \| EK$ then $EK \| HO$. Does the converse hold?

Solution by Tosio Seimiya, Kawasaki, Japan.

If $LD \| EK$, then $CK{:}CD = CE{:}CL$ where $CD = |b \cos C|$, $CK = a/2$, $CL = b/2$,

$CE = |a\cos C|$. It follows that $\cos^2 C = 1/4$, i.e. $C = 60^\circ$ or 120°. If $C = 120^\circ$ (Figure 1), then $\angle DHE = 60^\circ$ and $\angle AOB = 120^\circ$. Thus the four points A, O, B, H are concyclic. Since $OA = OB$, $\angle OHB = 30^\circ$. On the other hand, $\angle BCE = 60^\circ$ and $BK = KC = KE$ shows that $\angle KEB = \angle KBE = 30^\circ = \angle OHB$. From this follows $EK \| HO$ as required. If $C = 60^\circ$ (Figure 2), then the same argument still works with the only change being that $\angle DHE = 120^\circ$.

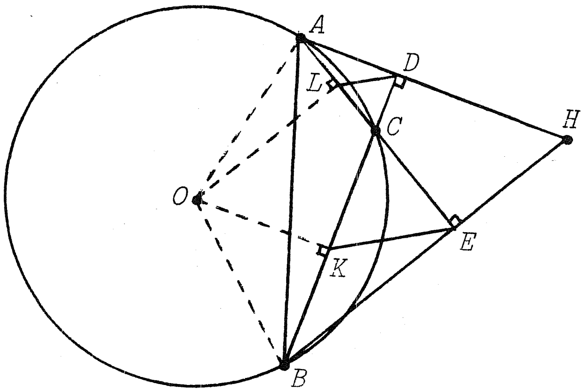


Figure 1

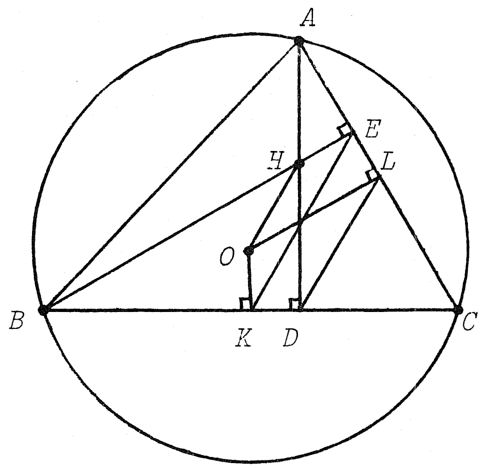
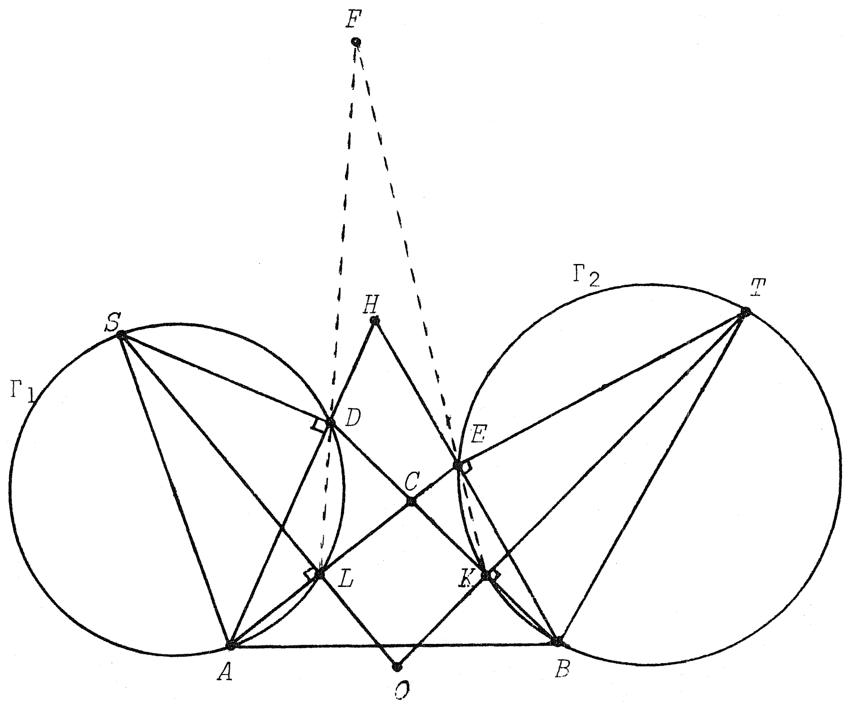


Figure 2

We give a second solution to the problem by showing that, in general, *the three lines LD, KE, and HO are concurrent.* Let Γ_1 be the circumcircle of ΔALD with diameter AS, and Γ_2 be the circumcircle of ΔBKE with diameter BT. Since the four points S, L, K, T are concyclic,

$$OL \cdot OS = OK \cdot OT.$$

Thus O lies on the radical axis of Γ_1 and Γ_2. Similarly, $ADEB$ is concyclic, and so H also lies on the radical axis of Γ_1 and Γ_2. But since L, D, E, K all lie on the nine-point circle of ΔABC, $F = LD \cap KE$ will also lie on the radical axis of Γ_1 and Γ_2, so LD, KE, and HO will concur at F.



Now if $LD \| EK$ then $EK \| HO$ is obvious unless LD and EK coincide. But in this case it is easily seen that $L = E$ and $K = D$, that is, ΔABC is equilateral. Hence $H = O$ and there is nothing to prove.

[*Editor's note*: Seimiya's nice solution (which was kindly translated and forwarded by H. Fukagawa) actually ended with the unfortunate claim that $LD \| EK$ if and only if $EK \| HO$, the author forgetting that two lines can both intersect and be parallel, namely if they coincide. This oversight has been corrected, free of charge, by the editor. To show that the converse, i.e. $EK \| HO \Rightarrow LD \| EK$, in fact *fails*, we give the example of Janous, an isosceles right triangle with $A = 90°$. Then $A = H = E$ and $O = K = D$, so that $EK = HO$ but $LD \nparallel EK$.]

Also solved (the first part) by FRANCISCO BELLOT, I.B. Emilio Ferrari, Valladolid, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. Only Janous showed correctly that the converse fails (see above). Bellot gave an example where $LD = HO$ and $EK \nparallel HO$. The others, including the proposer, claimed that the converse was true.

* * *

1236. [1987: 119] *Proposed by Gordon Fick, University of Calgary, Calgary, Alberta.*

Prove *without calculus* that if $0 \le \theta \le 1$, and $0 \le y \le n$ where y and n are integers, then

$$\theta^y(1-\theta)^{n-y} \le (y/n)^y(1-y/n)^{n-y}.$$

In statistics, this says that the sample proportion is the maximum likelihood estimator of the population proportion. To the best of my knowledge, all mathematical statistics texts prove this result with calculus.

I. *Solution by Leroy F. Meyers, The Ohio State University.*

By the arithmetic-geometric mean inequality, we have

$$[(n-y)\theta]^y[y(1-\theta)]^{n-y} \le \left[\frac{y(n-y)\theta + (n-y)y(1-\theta)}{n}\right]^n = \left[\frac{y(n-y)}{n}\right]^n,$$

from which the required inequality follows. Equality holds if and only if

$$(n-y)\theta = y(1-\theta),$$

that is,

$$\theta = y/n.$$

II. *Generalization by M.S. Klamkin, University of Alberta.*

We use Jensen's inequality for concave functions $F(x)$:

$$\frac{w_1F(x_1) + \cdots + w_mF(x_m)}{w_1 + \cdots + w_m} \le F\left[\frac{w_1x_1 + \cdots + w_mx_m}{w_1 + \cdots + w_m}\right],$$

where the x_i are arbitrary reals and the w_i arbitrary positive reals.

Let $F(x) = \ln x$ to give

$$x_1^{w_1} \dots x_m^{w_m} \le \left[\frac{w_1x_1 + \cdots + w_mx_m}{w_1 + \cdots + w_m}\right]^{w_1+\cdots+w_m},$$

and then let $x_i = \theta_i/y_i$, $w_i = y_i$ for θ_i, y_i arbitrary positive reals. This yields

$$\left[\frac{\theta_1}{y_1}\right]^{y_1} \cdots \left[\frac{\theta_m}{y_m}\right]^{y_m} \le \left[\frac{\theta_1 + \cdots + \theta_m}{y_1 + \cdots + y_m}\right]^{y_1+\cdots+y_m},$$

which, putting $\theta_1 + \cdots + \theta_m = s$ and $y_1 + \cdots + y_m = n$, can be written

$$\theta_1^{y_1} \cdots \theta_m^{y_m} \le \left[\frac{y_1}{n}\right]^{y_1} \cdots \left[\frac{y_m}{n}\right]^{y_m} s^n.$$

The given inequality corresponds to the special case $m = 2$, $s = 1$.

Also solved by RICHARD K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario. One other reader sent in a solution using calculus.

* * *

1237*. [1987: 119] *Proposed by Niels Bejlegaard, Stavanger, Norway.*

If m_a, m_b, m_c denote the medians to the sides a, b, c of a triangle ABC, and s is the semiperimeter of ABC, show that

$$\sum a \cos A \le \frac{2}{3} \sum m_a \sin A \le s,$$

where the sums are cyclic.

Combined solutions of S.J. Bilchev, Technical University, Russe, Bulgaria, and John Oman and Bob Prielipp, University of Wisconsin, Oshkosh.

We show in fact the slightly sharper inequality

$$\sum a \cos A \le \frac{2}{3} \sum m_a \sin A \le \frac{2s}{3}(1 + \frac{r}{R}), \tag{1}$$

where r is the inradius and R the circumradius. Let F be the area of ΔABC. From

$$\sum a \cos A = \frac{2F}{R}$$

and the fact that

$$2R \sin A = a, \quad \text{etc.},$$

(1) is equivalent to

$$6F \le \sum a m_a \le 2s(R + r).$$

Since m_a is greater than or equal to the altitude from A, the left-hand inequality is immediate (see [1987: 188]). Further, since

$$m_a \le R + d_a$$

where d_a is the distance from the circumcentre to BC,

$$\sum a m_a \le R \sum a + \sum a d_a = 2sR + 2F = 2s(R + r).$$

Also solved by AAGE BONDESEN, Royal Danish School of Educational Studies, Copenhagen; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; ZUN SHAN and EDWARD T.H.

WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

Bilchev proved the stronger inequality (1); *Oman and Prielipp conjectured and essentially proved it.*

* * *

ON SHORT ARTICLES IN CRUX MATHEMATICORUM

Now and then short articles appear in *Crux* alongside the regular fare of problems and solutions. This has been happening less frequently the last three years or so, partly due to the desire of the former and present editors to lessen the backlog of solutions, and partly because of a lack of suitable articles to publish. It seems the readers would like to see more such articles, and the editor (having made some progress on the backlog) agrees. Here are some guidelines.

(i) **Articles should be aimed at the intended audience of *Crux*,** which is the university undergraduate and high school community, and anyone else interested in mathematics at that level.

(ii) **Articles need not be "original research".** As one potential source of good *Crux* articles, teachers who assign written projects to their students might be on the lookout for any unusually interesting ones and encourage their authors to submit them to *Crux*.

(iii) **Articles should be short**, let's say three pages or less (when printed in *Crux*).

(iv) **Articles should have some connection with topics discussed in *Crux*.** In fact, the problems in *Crux* can furnish an unending supply of ideas for articles. Many problems can suggest extensions or related problems which might make good subjects for publication. (For example, just looking at the last issue, *Crux* 1211 [1988: 114] suggests: what happens if 6 is replaced by some other positive integer, or some base besides 10 is used? Or from the current issue: in *Crux* 1232 [1988: 153], what if $2x + 1$ is replaced, or joined, by say $3x + 1$? In the editor's comment following the solution, can the denominator $6 - \epsilon$ be further increased?) Although solutions to *Crux* problems should first appear in the solutions section, any later investigations extending, or otherwise inspired by, the original problem might be more fitting as a separate article.

Comments on the above, as on the content of *Crux* in general, are welcome.

* * *

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