Crux Mathematicorum

Published by the Canadian Mathematical Society.

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Vol. 11, No. 8

October 1985

Publié par la Société Mathématique du Canada

The support of the Carleton University Department of Mathematics and Statistics and the University of Ottawa Department of Mathematics are gratefully acknowledged.

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CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is \$22.50 for members of the Canadian Mathematical Society and \$25 for nonmembers. Back issues: \$2.75 each. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-10: \$20. All prices quoted are in Canadian dollars. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

All communications about the content of the journal should be sent to the Editor. All changes of address and inquiries about subscriptions and back issues should be sent to the Managing Editor.

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ISSN 0705 - 0348.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

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CONTENTS

This issue is dedicated to Philip Killeen, President of Algonquin College.

A MESSAGE FROM THE MANAGING EDITOR

As announced in our September issue, *Crux Mathematicorum* became an official publication of the Canadian Mathematical Society on October 1, 1985.

In nearly eleven years, *Crux* has grown under the dedicated editorship of Léo Sauvé from a small local problems paper to an internationally recognized problem-solving journal. Many felt that the workload of the editor had become too heavy for any one individual to carry, and that *Crux* could benefit from having a nation-wide organization rather than a local organization as its sponsor. Therefore the Canadian Mathematical Society was approached to see if it would be willing to assume the responsibility of publishing *Crux*. The Society struck an ad hoc committee consisting of Marcel Déruaz (Chairman, University of Ottawa), Edgar Goodaire (Memorial University of Newfoundland), and Peter Taylor (Queen's University) to look into the matter and make recommendations to the C.M.S. The committee studied the matter carefully and recommended that the C.M.S. take over the responsibility for publishing *Crux*. It also recommended the creation of an editorial board to supervise the journal and assist its editor. No major changes in the contents of *Crux* are foreseen for the immediate future. Subscription rates for 1986 will also be held at this year's level.

It is appropriate at this point to express our thanks to Philip Killeen, President of Algonquin College, who has done so much to support *Crux* over the years. The editor agreed with great pleasure to dedicate this issue to him.

The office of the managing editor has now moved to the offices of the Canadian Mathematical Society. All communications intended for him should henceforth be addressed as follows:

Dr. Kenneth S. Williams, Managing Editor
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Canadian Mathematical Society
577 King Edward Avenue
Ottawa, Ontario
Canada K1N 6N5

Léo Sauvé has agreed to continue as editor until a new one is appointed by the C.M.S. His address remains unchanged (see front page).

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THE OLYMPIAD CORNER: 68

M.S. KLAMKIN

I present three new problem sets this month: the Third Round of the 1981 Leningrad High School Olympiad (transmitted by Alex Merkurjev and translated by Larry Glasser), the 1984 Bulgarian Mathematical Olympiad (transmitted by Jordan B. Tabov), and the 1983 Annual High School Competition of the Greek Mathematical Society (transmitted by Dimitris Vathis). As usual, I solicit from all readers, especially secondary school students, elegant solutions to all these problems with, if possible, extensions or generalizations. Readers submitting solutions should clearly identify the problems by giving their numbers as well as the year and page number of the issue where they appear. The solutions need not be typewritten, but they should be *easily legible*.

1981 LENINGRAD HIGH SCHOOL OLYMPIAD (Third Round)

1. Is there a sequence consisting of 1981 natural numbers whose sum is the cube of a natural number? (Grade 8)

2. A square is partitioned into several rectangles whose sides are parallel to those of the square. For each rectangle, the ratio of the length of the smaller side to that of the longer side is calculated. Show that the sum of these ratios is not less than 1. (Grades 8,9)

3. Show that if $a^2 + ab + ac < 0$, then $b^2 > 4ac$. (Grade 8)

4. A plane is partitioned into an infinite set of unit squares by parallel lines. A triangle ABC is constructed with vertices at line intersections. Show that if $|AB| > |AC|$, then $|AB| - |AC| > 1/p$, where p is the perimeter of the triangle. (Grades 8,9,10)

5. A $p{\times}p$ checkerboard has alternating black and white squares, the square in the top left corner being black (so that the number of black squares is one more than the number of white squares when p is odd). A piece moves among the black squares, crossing diagonally from one black square to a neighboring one. What is the smallest number of moves required to traverse all the black squares (a) if $p = 8$ (Grade 8), (b) if $p = 9$ (Grade 9), (c) if $p = 10$ (Grade 10)?

6. In a convex quadrilateral the sum of the distances from any point within the quadrilateral to the four straight lines along which the sides lie is constant. Show that the quadrilateral is a parallelogram. (Grade 9)

7. The sequence $\{a_i\}$ of natural numbers satisfies $1 < a_1 < a_2 < \ldots < a_n < \ldots$ and

$$a_{p+a_p} = 2a_p$$

for each natural number p. Show that there is a natural number c such that $a_n = n+c$ for any n. (Grade 9)

8. The integers a,b,c,d, and A are such that

$$a^2 + A = b^2 \quad \text{and} \quad c^2 + A = d^2.$$

Show that the number $2(a+b)(c+d)(ac+bd-A)$ is the square of a natural number. (Grades 9,10)

9. In each of two congruent regular 16-gons, seven vertices are designated. Show that it is possible to lay one polygon on top of the other in such a way that no fewer than four of the designated vertices of one polygon coincide with designated vertices of the other. (Grade 10)

10. On the edges AA_1, BB_1, CC_1 of the triangular prism $ABCA_1B_1C_1$ the points A_0, B_0, C_0 are chosen such that $|AA_0| = a$, $|BB_0| = b$, $|CC_0| = c$. Let M be the point of intersection of the planes A_0BC, B_0CA, and C_0AB. A line through M parallel to the edges of the prism meets the base ABC in P. If $|MP| = d$, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d}. \qquad \text{(Grade 10)}$$

11. Does there exist a positive integral power of 5 such that the first 100 digits of its decimal representation contain 30 consecutive zeros?

*

1984 BULGARIAN MATHEMATICAL OLYMPIAD

First day: May 19, 1984

1. Determine all nonnegative integer triples (x,y,z) such that

$$5^x \cdot 7^y + 4 = 3^z.$$

2. ABCD is a trapezoid with parallel sides BA and CD such that $BA/CD = k > 1$. The diagonals AC and BD intersect in O. The bisectors of the angles AOB, BOC, COD, and DOA meet the sides AB, BC, CD, and DA in the points K, L, M, N, respectively. Let $P = KL \cap MN$ and $Q = KN \cap LM$. Determine k if it is given that the area of trapezoid ABCD is equal to the area of triangle OPQ.

3. Let $P_1, P_2, \ldots, P_n, Q$ be $n+1$ points $(n \geq 4)$ no four of which are coplanar. Prove that if for every three points P_i, P_j, P_k there exists a point P_l such

that the tetrahedron $P_iP_jP_kP_l$ contains the point Q, then n is an even integer.

Rider. Prove that $n = 4$.

Second day: May 20, 1984

4. The numbers $a, b, a_2, a_3, \ldots, a_{n-2}$ are all real and $ab \neq 0$. It is known that all the roots of the equation

$$ax^n - ax^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots + a_{n-2}x^2 - n^2bx + b = 0$$

are real and positive. Prove that the roots are all equal.

5. Let $0 \le x_i \le 1$ and $x_i+y_i = 1$ for $i = 1,2,\ldots,n$. Prove that

$$(1 - x_1x_2\ldots x_n)^m + (1-y_1^m)(1-y_2^m)\ldots(1-y_n^m) \ge 1$$

for all positive integers m and n.

6. Let S-ABCD be a pyramid with a parallelogram ABCD as base. Let N be the midpoint of segment BC. A plane λ meets the lines SC,SA,AB in points P,Q,R, respectively, in such a way that

$$\frac{\overline{CP}}{\overline{CS}} = \frac{\overline{SQ}}{\overline{SA}} = \frac{\overline{AR}}{\overline{AB}}.$$

A point M on line SD is chosen such that line MN is parallel to λ. Prove that the locus of M, when λ describes all possibilities, is a line segment of length $\frac{1}{2}\sqrt{5}|SD|$.

*

1983 ANNUAL GREEK HIGH SCHOOL COMPETITION

1. The function $f\colon R \to R$ is defined by

$$f(x) = x^5 + x - 1.$$

(a) Prove that $f\colon R \to R$ is a bijection.

(b) Show that $f(1001^{999}) < f(1000^{1000})$.

(c) Determine the roots of the equation $f(x) = f^{-1}(x)$.

2. If $\vec{a}$ and $\vec{b}$ are given nonparallel vectors, solve for x the equation

$$\frac{\vec{a}^2 + x\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{a} + x\vec{b}|} = \frac{\vec{b}^2 + \vec{a}\cdot\vec{b}}{|\vec{b}|\,|\vec{a} + \vec{b}|}.$$

3. If the function $f\colon [0,+\infty) \to R$ satisfies the relation

$$f(x)e^{f(x)} = x$$

for all x in its domain, prove:

(a) the function f is monotonic over its entire domain;

(b) $\lim\limits_{x\to+\infty} f(x) = +\infty$;

(c) $\lim\limits_{x\to+\infty} \frac{f(x)}{\ln x} = 1$.

4. A *lattice point* in a given coordinate plane is a point both of whose coordinates are integers.

(a) Given five lattice points $A_1, A_2, \ldots, A_5$ in the plane, show that the midpoint of at least one of the segments A_iA_j $(i \neq j)$ is a lattice point.

(b) Given are two lattice points $B(k,\alpha)$ and $C(l,\alpha)$ in the plane, with $k < l$. If A is a point in the plane such that BC = CA = AB, prove that A is not a lattice point.

*

I now present solutions to some problems proposed in earlier columns.

1. [1984: 282] *From the 1984 Annual Greek High School Competition.*

Prove or disprove that there exists in space a pentagon all of whose sides are equal and all of whose angles are 90°.

Comment by Gali Salvatore, Perkins, Québec.

This problem appeared in the 1966 József Kürschák Mathematical Competition in Hungary. The following simple solution is given in the booklet *Problems of the József Kürschák Mathematical Competitions 1966-1981*, compiled (in English) in 1982 by László Csirmaz.

There is no such pentagon. For suppose, on the contrary, that such a pentagon ABCDE exists, all sides being of unit length, and let X be the fourth vertex of the unit square with adjacent sides AB and BC. Then AD = BD = $\sqrt{2}$ from isosceles right triangles AED and BCD, and $\angle BCD = 90^\circ$. Therefore the perpendicular projection D' of D upon the plane ABC bisects segment CX, and DD' = $\sqrt{3}/2$. Similarly, the perpendicular projection E' of E upon plane ABC bisects segment AX, and EE' = $\sqrt{3}/2$. Now DE = $1/\sqrt{2}$ or $\sqrt{3 + 1/2}$ according as D and E are on the same or on opposite sides of plane ABC, so we have a contradiction. □

More generally, it is proved in Problem 1110, *Mathematics Magazine*, 55 (1982) 47-48, that there is an n-gon in space with all sides equal and all angles 90° for all $n \geq 4$ except $n = 5$.

*

2. [1984: 283] *From the 1984 Annual Greek High School Competition.*

(a) Find the maximum and the minimum of $8x + 6y - 5z$, where x,y,z are real and $x^2 + y^2 + z^2 = 5$.

(b) Express the vector $\vec{x}$ in terms of the vectors $\vec{a},\vec{b}$ if

$$\vec{a}\cdot(\vec{x}+\vec{b}) = \vec{a}^2 \qquad \text{and} \qquad \vec{b}\cdot\vec{x} = 0.$$

It is assumed that $\vec{a} \neq \vec{0}$ and that the vectors $\vec{a},\vec{b},\vec{x}$ are coplanar.

I. *Solution to part (a) by René Schipperus, Western Canada High School, Calgary, Alberta.*

Let $\vec{a} = (x,y,z)$ and $\vec{b} = (8,6,-5)$, so that $|\vec{a}| = \sqrt{5}$ and $|\vec{b}| = 5\sqrt{5}$. By Cauchy's inequality,

$$|f(x,y,z)| \equiv |8x+6y-5z| = |\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}| = 25,$$

so -25 and 25 are lower and upper bounds for $f(x,y,z)$. These are in fact the minimum and maximum values for $f(x,y,z)$, for the lower and upper bounds are attained when

$$(x,\ y,\ z) = (-\tfrac{8}{5},\ -\tfrac{6}{5},\ 1) \quad \text{and} \quad (\tfrac{8}{5},\ \tfrac{6}{5},\ -1),$$

respectively.

II. *Solution to part (b) by K.S. Murray, Brooklyn, N.Y.*

Suppose $\vec{b} = \lambda\vec{a}$ for some scalar λ; then $\vec{x}$ is expressible in terms of $\vec{a}$ and $\vec{b}$ if and only if $\vec{x} = \mu\vec{a}$ for some scalar μ. The two equations then give

$$(\lambda+\mu)\vec{a}^2 = \vec{a}^2 \qquad \text{and} \qquad \lambda\mu\vec{a}^2 = 0.$$

Since $\vec{a} \neq \vec{0}$, we therefore have $\lambda+\mu = 1$ and $\lambda\mu = 0$, so $(\lambda,\mu) = (1,0)$ or $(0,1)$, and

$$\vec{x} = \vec{0} \qquad \text{or} \qquad \vec{x} = \vec{a}.$$

Suppose now that $\vec{b} \neq \lambda\vec{a}$ for any scalar λ; then, since the vectors $\vec{a},\vec{b},\vec{x}$ are coplanar, there is a unique representation $\vec{x} = m\vec{a}+n\vec{b}$. The two equations then give

$$m\vec{a}^2 + n\vec{a}\cdot\vec{b} = \vec{a}^2 - \vec{a}\cdot\vec{b} \qquad \text{and} \qquad m\vec{a}\cdot\vec{b} + n\vec{b}^2 = 0.$$

The unique solution (m,n) to this linear system is then easily found, and the representation $\vec{x} = m\vec{a}+n\vec{b}$ follows.

*

M857. [1984: 283] *Proposed by S. Stadnichenko in Kvant, April 1984.*

Among the first 1984 positive integers (from 1 to 1984) we underline those which may be represented as the sum of five nonnegative integer powers of 2 (i.e., of five not necessarily different numbers 1, 2, 4, 8, ...). Is the set of underlined numbers larger than that of the nonunderlined ones?

Solution by Curtis Cooper, Central Missouri State University.

The largest power of 2 that does not exceed 1984 is $2^{10} = 1024$, so the indices of the powers of 2 involved are all in the set

$$P = \{0, 1, 2, \ldots, 10\}.$$

The underlined numbers are all of the form

$$2^{n_1} + 2^{n_2} + 2^{n_3} + 2^{n_4} + 2^{n_5}, \tag{1}$$

where each $n_i \in P$ and $n_1 \le n_2 \le n_3 \le n_4 \le n_5$.

There are exactly $\binom{11}{5}$ distinct numbers of the form (1) in which all the n_i are distinct. There are exactly $\binom{11}{4}$ distinct numbers of the form

$$2^{n_1} + 2^{n_2} + 2^{n_3} + 2^{n_4} = 2^{n_1} + 2^{n_2} + 2^{n_3} + 2^{n_4-1} + 2^{n_4-1}$$

in which the four n_i are distinct. There are exactly $\binom{11}{3}$ distinct numbers of the form

$$2^{n_1} + 2^{n_2} + 2^{n_3} = 2^{n_1} + 2^{n_2-1} + 2^{n_2-1} + 2^{n_3-1} + 2^{n_3-1}$$

in which the three n_i are distinct. There are exactly $\binom{11}{2} - 1$ distinct numbers of the form

$$2^{n_1} + 2^{n_2} = 2^{n_1} + 2^{n_2-2} + 2^{n_2-2} + 2^{n_2-2} + 2^{n_2-2}$$

in which the two n_i are distinct and $n_2 \ge 2$. Hence the number of distinct underlined numbers is at least

$$\binom{11}{5} + \binom{11}{4} + \binom{11}{3} + \binom{11}{2} - 1 = 1011 > \frac{1984}{2},$$

so there are more underlined numbers than nonunderlined ones.

*

1. [1984: 310] *From the 1984 Balkanic Mathematical Olympiad.*

Let $x_1, x_2, \ldots, x_n$ $(n \ge 2)$ be positive numbers whose sum is 1. Prove that

$$\frac{x_1}{1+x_2+x_3+\ldots+x_n} + \frac{x_2}{1+x_1+x_3+\ldots+x_n} + \ldots + \frac{x_n}{1+x_1+x_2+\ldots+x_{n-1}} \ge \frac{n}{2n-1}.$$

Solution by M.S.K.

More generally, consider the sum

$$I = \sum_{i=1}^{n} \frac{x_i}{\lambda+s-x_i},$$

where $s = x_1+x_2+\ldots+x_n$. It is easy to see that

$$I = (\lambda+s)\sum_{i=1}^{n} \frac{1}{\lambda+s-x_i} - n.$$

By Cauchy's inequality,

$$\sum_{i=1}^{n}(\lambda+s-x_i)\cdot\sum_{i=1}^{n}\frac{1}{\lambda+s-x_i} \ge n^2,$$

with equality just when $x_1 = x_2 = \ldots = x_n$. Thus

$$\sum_{i=1}^{n}\frac{1}{\lambda+s-x_i} \ge \frac{n^2}{n(\lambda+s) - s},$$

and so

$$I \ge \frac{n^2(\lambda+s)}{n(\lambda+s) - s} - n = \frac{ns}{n(\lambda+s) - s}.$$

The proposed result corresponds to the case $\lambda = s = 1$.

In the same way, we can obtain a corresponding inequality for the cyclic sum

$$\sum\frac{x_i+x_{i+1}+\ldots+x_{i+r}}{\lambda+s-x_i-x_{i+1}-\ldots-x_{i+r}}.$$

*

2. [1984: 311] *From the 1984 Balkanic Mathematical Olympiad.*

Let $A_1A_2A_3A_4$ be a cyclic quadrilateral. If H_1, H_2, H_3, H_4 denote the orthocenters of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$, respectively, prove that quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.

Comment by Dimitris Vathis, Chalcis, Greece.

We give references where proofs of this theorem and related results can be found.

(i) What is in effect a proof of the theorem appears in Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 132, Art. 262.

(ii) The theorem is proved in solution I and in a subsequent comment to Problem E 1740, *American Mathematical Monthly*, 72 (1965) 1026-1027.

(iii) Our problem follows from the following theorem, which can be found in Flor Cartuyvels, "A special point in a quadrilateral, or how the ninepointcircle becomes a tenpointcircle", *American Mathematical Monthly*, 73 (1966) 616-619:

THEOREM I. The ortho-quadrilateral is congruent with its quadrilateral. A rotation over half a full turn around the orthopole makes the one coincide with the other.

(iv) There are other interesting results if the orthocenters are replaced by centroids or incenters. See Problems E 1739 and E 1740 in the reference given in (ii); Ross Honsberger, *Mathematical Morsels*, Mathematical Association of America, 1978, pp. 198-199; and Problem 483 in this journal [1980: 226-230].

*

3. [1984: 311] *From the 1984 Balkanic Mathematical Olympiad.*

Prove that for every positive integer m there exists an integer $n > m$ such that the decimal representation of 5^n can be obtained from the decimal representation of 5^m by including some digits on the left.

Solution by Curtis Cooper, Central Missouri State University.

It follows easily by induction that

$$5^{2^m} \equiv 1 \pmod{2^m}.$$

Thus

$$5^{m+2^m} \equiv 5^m \pmod{2^m},$$

and $n = m + 2^m$ gives the desired result.

*

4. [1984: 311] *From the 1984 Balkanic Mathematical Olympiad.*

Find all real solutions (x,y,z) of the system

$$\begin{cases} by + cz = (y-z)^2 \\ cz + ax = (z-x)^2 \\ ax + by = (x-y)^2 \end{cases}$$

where a,b,c are given positive numbers.

Solution by K.S. Murray, Brooklyn, N.Y.

From $(x-y)^2 - (y-z)^2 = ax - cz$ and $(z-x)^2 = ax + cz$, we obtain by addition the first of the following equations, and the other two are obtained in the same way:

$$(x-y)(x-z) = ax,$$
$$(y-z)(y-x) = by,$$
$$(z-x)(z-y) = cz.$$

Suppose x,y,z are all distinct. Then one of the products

$$(x-y)(x-z), \qquad (y-z)(y-x), \qquad (z-x)(z-y)$$

is negative, the other two are positive, and this is easily seen to lead to a contradiction. Therefore at least two of x,y,z are equal, and it follows that the only real solutions are

$$(x,y,z) = (0,0,0),\ (a,0,0),\ (0,b,0), \text{ and } (0,0,c).$$

*

3. [1984: 311] *From the 1984 Austrian-Polish Mathematical Competition.*

If $a,x_1,x_2,\ldots,x_n$ $(n \geq 2)$ are positive real numbers, prove that

$$\frac{a^{x_1-x_2}}{x_1+x_2}+\frac{a^{x_2-x_3}}{x_2+x_3}+\ldots+\frac{a^{x_n-x_1}}{x_n+x_1}\geq\frac{n^2}{2(x_1+x_2+\ldots+x_n)} \tag{1}$$

and determine when there is equality.

Solution by K.S. Murray, Brooklyn, N.Y.

Let I denote the left member of (1). By the A.M.-G.M. inequality,

$$\frac{I}{n}\geq\frac{1}{\{(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1)\}^{1/n}}\equiv\frac{1}{J}. \tag{2}$$

By the A.M.-G.M. inequality again,

$$\frac{2(x_1+x_2+\ldots+x_n)}{n}=\frac{(x_1+x_2)+(x_2+x_3)+\ldots+(x_n+x_1)}{n}\geq J, \tag{3}$$

and (1) follows from (2) and (3).

Equality holds in (3) just when

$$x_1+x_2=x_2+x_3=\ldots=x_n+x_1,$$

that is, just when

$$x_1=x_2=\ldots=x_n \text{ for any } n, \tag{4a}$$

or when $n=2m$ and

$$x_1\neq x_2,\quad x_1=x_3=\ldots=x_{2m-1} \quad\text{and}\quad x_2=x_4=\ldots=x_{2m}. \tag{4b}$$

Equality holds in (2) just when all the terms of I are equal. Therefore equality holds in (2) and (3), and hence in (1), just in the following two cases:

(i) when the x_i satisfy (4a) for any n (and any $a>0$); or

(ii) when $a=1$ and the x_i satisfy (4b) for even n.

*

1. [1985: 2] *From the 1984 Leningrad Olympiad.*

Diametrically opposite points A and B are chosen on a circle which touches the sides of an angle with vertex O (neither A nor B being a point of tangency). The tangent to the circle at B intersects the sides of the angle in C and D and the line OA in E. Prove that the segments BC and DE are equal in length.

Solution by Jordi Dou, Barcelona, Spain.

Let C'D' be the tangent to the given circle at A, with C' on OC and D' on OD. Consider the incircle and the excircle opposite vertex O of triangle OC'D'. One of these is the given circle. Let the other circle touch C'D' in B'. It is a known property of in- and ex-circles that B'C' = D'A, and then BC = DE follows from the fact that the two circles are homothetic with homothetic center O.

*

2. [1985: 2] *Proposed by S.V. Fomin at the 1984 Leningrad Olympiad.*

From a sheet of squared paper measuring 29×29, 99 squares, each of which consists of 4 unit squares, have been cut out. Prove that one more 2×2 square may be cut out.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

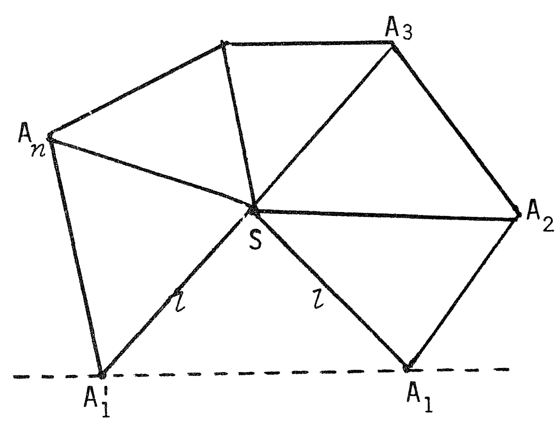
We can consider the lattice points of the squared paper as the points $(x,y) \in N\times N$ with $0 \le x,y \le 29$. The lattice points (x,y) such that $3|(x-1)$ and $3|(y-1)$, of which there are exactly 100, will be called *black points*. A closed 2×2 square cannot contain two black points. Consequently, after cutting out 99 2×2 squares, there is at least one black point left, and it is the center of a residual 2×2 square.

*

3. [1985: 2] *Proposed by Yu. I. Ionin and A.V. Smirnov at the 1984 Leningrad Olympiad.*

Prove that if the sum of the plane angles at the summit of a pyramid is more than 180°, then each lateral edge of the pyramid is shorter than half the perimeter of its base.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.



Let S-$A_1A_2\ldots A_n$ be the given pyramid and P the perimeter of its base. It suffices to show that $l < P/2$, where $l = SA_1$. The sum of the face angles at the summit S lies strictly between π and 2π. Consequently, if we cut along SA_1 and develop the pyramid (lay it flat, as shown in the figure), then $0 < \angle A_1'SA_1 < \pi$. It is well known and easy to show that if one convex figure strictly contains another convex figure, then the outer one has a larger perimeter. Therefore

$$2l + A_1'A_1 < P + A_1'A_1,$$

and $l < P/2$ follows.

*

4. [1985: 2] *Proposed by S.V. Fomin at the 1984 Leningrad Olympiad.*

The five integers a,b,c,d,e are chosen so that their sum and the sum of their squares are both divisible by the odd number p. Prove that

$$a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$$

is also divisible by p.

Solution by K.S. Murray, Brooklyn, N.Y.

The elementary symmetric functions T_r of a,b,c,d,e are defined by

$$(x-a)(x-b)(x-c)(x-d)(x-e) \equiv x^5 - T_1x^4 + T_2x^3 - T_3x^2 + T_4x - T_5. \qquad (1)$$

Let $s_k = a^k+b^k+c^k+d^k+e^k$. It follows from (1) that

$$T_1 = s_1, \qquad 2T_2 = s_1^2 - s_2, \qquad T_5 = abcde,$$

and we must show that if s_1 and s_2, and hence also T_1 and T_2, are divisible by the odd number p, then so is $s_5 - 5T_5$. But this follows immediately from the relation

$$s_5 - 5T_5 = T_1^5 - 5T_1^3T_2 + 5T_1^2T_3 + 5T_1T_2^2 - 5T_1T_4 - 5T_2T_3,$$

which is easily obtained from Newton's identities.

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5. [1985: 2] *Proposed by A.S. Merkuriev at the 1984 Leningrad Olympiad.*

In the sequence 1,0,1,0,1,0,..., each term beginning with the seventh equals the last digit of the sum of the previous six terms. Prove that the numbers 0,1,0,1,0,1 do not appear in that order in the sequence.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

Let the given sequence be $a_1,a_2,a_3,\ldots$ and, for $i \in N$, let

$$f(i) = a_i + 2a_{i+1} + 3a_{i+2} + 4a_{i+3} + 5a_{i+4} + 6a_{i+5},$$

where the addition, here and later, is modulo 10 (but not, of course, for the subscripts). Then

$$f(i+1) - f(i) = 6a_{i+6} - (a_i+a_{i+1}+a_{i+2}+a_{i+3}+a_{i+4}+a_{i+5}) = 5a_{i+6}.$$

It follows that $f(i) - f(j) = 0$ or 5 for all $i,j \in N$. In particular, $f(i) = f(1)$ or $f(1) + 5$ for all $i \in N$. Since $f(1) = 9$, we conclude that $f(i) = 9$ or 4 for all $i \in N$.

Now suppose that 0,1,0,1,0,1 appears somewhere in the sequence as $a_i,a_{i+1},\ldots,a_{i+5}$. We would then have $f(i) = 2$, a contradiction.

Editor' note. All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1,

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An Asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be sumitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly hand-written on signed, separate sheets, should preferably be mailed to the editor before April 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1027. [1985: 83] (Corrected) *Proposed by M.S. Klamkin, University of Alberta.*

Determine all quadruples (a,b,c,d) of nonzero integers satisfying the Diophantine equation

$$abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 = (a+b+c+d)^2$$

and such that $a^2+b^2+c^2+d^2$ is a prime.

1071. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

The cubic meter, or *stere*, is a measure of volume in the metric system:

```
    1
CUBIC
METER.
-----
STERE
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Solve this decimal addition without reusing the digit 1.

1072. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

For $n = 1,2,3,\ldots$, a sequence of triangles $A_nB_nC_n$ has sides (in the usual order)

$$a_n = n^2+n+1, \qquad b_n = 2n+1, \qquad c_n = n(n+2).$$

A point D_n is chosen on line A_nB_n such that $\angle A_nC_nD_n = 60^\circ$. Let

$$r_n = \frac{[D_nB_nC_n]}{[A_nD_nC_n]},$$

where the square brackets denote signed area. Find all pairs of positive integers m,n, if any, such that $r_mr_n = 1$.

1073. *Proposed by Jordi Dou, Barcelona, Spain.*

Let K be an interior point of triangle ABC. Through a point P in the plane of the triangle, parallels to the cevians AK,BK,CK are drawn to meet BC,CA, AB in L,M,N, respectively. If the points L,M,N are collinear,

(a) prove that the locus of P is an ellipse;

(b) construct the centre of this ellipse.

(This problem generalizes Crux 925 [1985: 154], which is the special case when K = G, the centroid of triangle ABC.)

1074. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with circumcenter O. Prove that

(a) there are two points P in the plane of the triangle such that

$$PA^2 : PB^2 : PC^2 = \sec^2 A : \sec^2 B : \sec^2 C;$$

(b) these two points and O are collinear;

(c) these two points are inverses with respect to the circumcircle of the triangle.

1075. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with circumcenter O and incenter I, and let DEF be the pedal triangle of an interior point M of triangle ABC (with D on BC, etc.). Prove that

$$OM \geq OI \iff r' \leq \frac{r}{2},$$

where r and r' are the inradii of triangles ABC and DEF, respectively.

1076. *Proposed by M.S. Klamkin, University of Alberta.*

Let x,y,z denote the distances from an interior point P of a given triangle ABC to the respective vertices A,B,C; and let K be the area of the pedal triangle of P with respect to ABC. Show that

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C + 8K$$

is a constant (independent of P).

1077*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

For i = 1,2,3, let C_i be the center and r_i the radius of the Malfatti circle nearest A_i in triangle $A_1A_2A_3$. Prove that

$$A_1C_1\cdot A_2C_2\cdot A_3C_3 \geq \frac{(r_1 + r_2 + r_3)^3 - 3r_1r_2r_3}{3}.$$

When does equality occur?

1078. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Prove that

$$\sum_{k=1}^{n} \binom{n}{k} \cdot \frac{1}{k} = \sum_{k=1}^{n} \frac{2^k - 1}{k}.$$

1079. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let

$$g(a,b,c) = \sum \frac{a}{a+2b} \cdot \frac{b-4c}{b+2c},$$

where the sum is cyclic over the sides a,b,c of a triangle.

(a) Prove that $-\frac{5}{3} < g(a,b,c) \le -1$.

(b)* Find the greatest lower bound of $g(a,b,c)$.

1080.* *Proposed by D.S. Mitrinović, University of Belgrade, Yugoslavia.*

Determine the maximum value of

$$f(x,y,z) = \left| \frac{y-z}{y+z} + \frac{z-x}{z+x} + \frac{x-y}{x+y} \right|,$$

where x,y,z are real numbers. Generalize.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

938. [1984: 115; 1985: 199] *Proposed by Charles W. Trigg, San Diego, California.*

Is there an infinity of pronic numbers of the form $a_n b_n$ in the decimal system? (A *pronic number* is the product ot two consecutive integers. The symbol x_n indicates x repeated n times. For example, $5_3 2_3 = 555222$.)

II. *Comment by Stewart Metchette, Culver City, California.*

The method used in solution I enables us to answer a question raised in the editor's comment [1985: 200]: for which positive integers d are there infinitely many d-pronic numbers of the form $a_n b_n$? We will show that $a_n b_n$ is d-pronic for all $n = 1,2,3,\ldots$ if and only if

$$(d,a,b) = (1,1,2),\ (1,4,2),\ (1,9,0),\ (2,1,5),\ (2,4,8),\ (2,9,9),\ \text{or}\ (3,1,8), \quad (1)$$

corresponding to the easily verified relations

$$3_n\cdot(3_n+1) = 1_n2_n, \qquad 3_n\cdot(3_n+2) = 1_n5_n, \qquad 3_n\cdot(3_n+3) = 1_n8_n.$$
$$6_n\cdot(6_n+1) = 4_n2_n, \qquad 6_n\cdot(6_n+2) = 4_n8_n,$$
$$9_n\cdot(9_n+1) = 9_n0_n, \qquad 9_n\cdot(9_n+2) = 9_n9_n,$$

(The three 1-pronic families are those already found in solution I.)

Let x be a positive integer such that

$$x(x + d) = a_nb_n \equiv \frac{1}{9}(a\cdot 10^n + b)(10^n - 1)$$

holds for some arbitrary but fixed positive integer n. By the quadratic formula,

$$x = \frac{1}{2}\left(-d + \frac{1}{3}\sqrt{4a\cdot 10^{2n} - 4(a-b)\cdot 10^n + 9d^2 - 4b}\right).$$

The expression under the radical is a quadratic in 10^n, and it is a perfect square for arbitrary n if and only if its discriminant vanishes, a condition equivalent to

$$(a + b)^2 = 9ad^2.$$

Therefore a is a perfect square, that is,

$$a = 1, \quad 4, \quad \text{or} \quad 9$$

and the corresponding values of b are

$$b = 3d-1, \quad 6d-4, \quad \text{or} \quad 9d-9,$$

from which it is clear that we must have $d \le 3$. For $d = 1$ we therefore have $(a,b) = (1,2)$, $(4,2)$, or $(9,0)$; for $d = 2$ we have $(a,b) = (1,5)$, $(4,8)$, or $(9,9)$; but for $d = 3$ we have only $(a,b) = (1,8)$. This confirms that the only values of the triple (d,a,b) are those given in (1).

* * *

939. [1984: 115; 1985: 224] (Corrected) *Proposed by George Tsintsifas, Thessaloniki, Greece.*

ABC is an acute triangle with AB < AC, altitude AD, and orthocenter H. M being an interior point of segment DH, lines BM and CM intersect sides CA and AB in B' and C', respectively. Prove that BB' < CC'.

II. *Comment by the proposer.*

Refer to Figure 2 in solution I [1985: 225]. We have seen that, for interior points M of triangle ABC, we have BB' < CC' if and only if M is an interior point of the shaded region between the branches of the rectangular hyperbola whose equation is

$$ax^2 + \frac{2(a^2+bc)}{b-c}xy - ay^2 + a(b-c)x + (a^2+bc)y - abc = 0.$$

It is easy to show that the equation of the tangent to the hyperbola at the point $A(0,a)$ is

$$y = \frac{a(2a^2+b^2+c^2)}{(b-c)(a^2-bc)}x + a.$$

This tangent meets side BC in the point K whose coordinates are

$$\left(\frac{(b-c)(bc-a^2)}{2a^2+b^2+c^2},\ 0\right).$$

Now we find that

$$\frac{BK}{KC} = \frac{a^2 + b^2}{a^2 + c^2} = \frac{AB^2}{AC^2},$$

and it follows that the tangent AK is the symmedian of triangle ABC from vertex A. The following facts, which might be difficult to establish directly, are immediate consequences: we have $BB' < CC'$ if M is any interior point of the symmedian AK, any interior point of the bisector of angle A, or any interior point of the median through A.

Editor's comment.

The interesting result in the above comment was inadvertently omitted from the proposer's solution I.

* * *

948. [1984: 156] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If a,b,c are the side lengths of a triangle of area K, prove that

$$27K^4 \le a^3b^3c^2,$$

and determine when equality occurs.

Solution by M.S. Klamkin, University of Alberta.

Using the well-known relations

$$a = 2R\sin A,\ \text{etc.} \qquad \text{and} \qquad K = 2R^2\sin A\sin B\sin C\,,$$

the proposed inequality is found to be equivalent to

$$\sin A\sin B\sin^2 C \le \frac{16}{27}. \tag{1}$$

If we set $\alpha = 1$, $\beta = 1$, and $\gamma = 2$ in Problem 908 [1985: 93], we immediately find that (1) holds for every triangle and that equality holds just when

$$\sin A = \sqrt{\frac{2}{3}},\quad \sin B = \sqrt{\frac{2}{3}},\quad \sin C = \frac{2\sqrt{2}}{3}. \tag{2}$$

It follows from (2) that A and B are both acute, and hence equal, and their common value (approximately 55°) shows that angle C is also acute. Therefore equality holds in the proposed inequality just when

$$a = b \quad \text{and} \quad C = \text{Arcsin}\,\frac{2\sqrt{2}}{3} = \text{Arccos}\,\frac{1}{3}.$$

Also solved by ROGER CUCULIÈRE, Paris, France; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

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949. [1984: 156] *Proposed by Charles W. Trigg, San Diego, California.*

In all bases, find all two-digit integers that are four times their reverses.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

Let $\overline{xy}$ be a number of the required type in base b. Then $bx+y = 4(by+x)$, or

$$(b - 4)x = (4b - 1)y. \tag{1}$$

A general solution of (1) is

$$x = \frac{k(4b - 1)}{d}, \qquad y = \frac{k(b - 4)}{d}, \tag{2}$$

where k is a natural number and $d = (4b-1, b-4)$. We assume that the proposer's intent is that initial (and hence also final) zeros be excluded. Hence $b > 4$, and the value of k must be restricted so that $0 < x,y < b$. Solutions (2) for which $k = 1$ will be designated *primitive solutions*.

Since $d|(4b-1)-4(b-4) = 15$, we must have $d = 1$, 3, 5, or 15. In fact, since $d = 1$ or 3 implies that $x > b$, the only possibilities are $d = 5$ or 15. In particular, there are no solutions in any base not congruent to 4 modulo 5, and hence no solutions in the commonly used binary, octal, decimal, duodecimal, and hexadecimal notations.

If $d = 5$, then $b \equiv 4 \pmod 5$, or, since $b > 4$, $b \equiv 9$ or $14 \pmod{15}$. The primitive solution

$$x = \frac{4b - 1}{5}, \qquad y = \frac{b - 4}{5}$$

is the only one in each case, since in each case $x > b$ when $k \ge 2$.

If $d = 15$, then $b \equiv 4 \pmod{15}$ and the primitive solution is

$$x = \frac{4b - 1}{15}, \qquad y = \frac{b - 4}{15}.$$

In this case there are also two nonprimitive solutions, corresponding to $k = 2$ and $k = 3$.

The following table gives all the solutions with $b < 50$.

b	9	14	19	19	19	24	29	34	34	34	39	44	49	49	49
x	7	11	5	10	15	19	23	9	18	27	31	35	13	26	39
y	1	2	1	2	3	4	5	2	4	6	7	8	3	6	9

Also solved by SAM BAETHGE, San Antonio, Texas; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; EDWIN M. KLEIN, University of Wisconsin-Whitewater; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; DANIEL ROPP, student, Stillman Valley High School, Illinois; MALCOLM A. SMITH, Georgia Southern College, Statesboro; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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950. [1984: 156] *Proposed by F.G.B. Maskell, Algonquin College, Ottawa.*

Let I be the incentre and Ω,Ω' the Brocard points of a nonequilateral triangle ABC, so that

$$\angle\Omega BC = \angle\Omega CA = \angle\Omega AB \qquad \text{and} \qquad \angle\Omega' CB = \angle\Omega' AC = \angle\Omega' BA.$$

Show that Ω, I, Ω' can never be collinear.

Solution by A.P. Guinand, Trent University, Peterborough, Ontario.

The theorem is false as stated, for Ω, I, Ω' *can* be collinear in a nonequilateral triangle.

Let a,b,c be the sides of any triangle ABC in the usual order. Homogeneous trilinear coordinates of the Brocard points are

$$\Omega\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right) \qquad \text{and} \qquad \Omega'\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right).$$

(This is easily proved, or see Casey, *Treatise on Conics* (1855), pp. 45, 107; or Sommerville, *Analytical Conics* (1955), p. 166.) For the incentre, we have I(1,1,1). Now I, Ω, Ω' are collinear if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{c}{b} & \frac{a}{c} & \frac{b}{a} \\ \frac{b}{c} & \frac{c}{a} & \frac{a}{b} \end{vmatrix} = 0,$$

or, equivalently, if and only if

$$(bc - a^2)(ca - b^2)(ab - c^2) = 0,$$

or, finally, if and only if the sides (in some order) are in geometric progression.

If $c = 16$, $a = 20$, and $b = 25$, for example, the Brocard angle ω is one-half angle A, and the points Ω, I, Ω' colline on the bisector of angle A.

A solution of the corrected problem was submitted by M.S. KLAMKIN, University of Alberta. Three other solvers "proved" the theorem as stated in the proposal.

* * *

951. [1984: 194] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In ancient times, a war galley was armed with a *rostrum*, the Latin word for the metal-shod beam jutting from the bow for use as a ram to sink enemy vessels. The Romans displayed the beaks of ships captured in battle around the platform in the Roman Forum where the populace assembled to hear orations, pleadings, etc. Because such war trophies adorned the area, the Romans called this platform the *Rostra*, which is the Latin plural of "rostrum". Later, because of its connection with the Forum, "rostra" evolved in Latin to denote any of several public speaking platforms in Rome. Hence an English word meaning "pulpit or platform for public speaking" was a Latin word meaning "naval ram".

Solve the following etymological multiplication in hexadecimal numbers

$$\begin{array}{r} \text{NAVAL} \\ \underline{\text{RAM}} \\ {*}{*}{*}{*}{*} \\ \underline{\text{ROME}{*}} \\ \text{ROSTRUM} \end{array}$$

given that this alphametic has digits such that

$$A < N < U < M < O < V < S < E < T < L < R.$$

Solution by Charles W. Trigg, San Diego, California.

We denote the digits ten, eleven, ..., fifteen by $\overline{10}$, $\overline{11}$, ..., $\overline{15}$. From the given inequalities,

$$9 \le L \le \overline{14}, \qquad 3 \le M \le 8, \qquad \text{and} \qquad L \ge M+6.$$

From the partial products, A = 0, N = 1, and $L \cdot M = UM$. The unique solution of the last equation is $\overline{13} \cdot 4 = 34$, so U = 3, M = 4, and L = $\overline{13}$. Now L < R, so R = $\overline{14}$ or $\overline{15}$. Also $6 \le V \le 9$ and $R \cdot V = OM$, from which uniquely O = 5, V = 6, and R = $\overline{14}$. It follows that the reconstructed multiplication is

$$\begin{array}{ccccccc} & & 1 & 0 & 6 & 0 & \overline{13} \\ & & & & \overline{14} & 0 & 4 \\ \hline & & 4 & 1 & 8 & 3 & 4 \\ \overline{14} & 5 & 4 & \overline{11} & 6 & 0 & \\ \hline \overline{14} & 5 & 8 & \overline{12} & \overline{14} & 3 & 4 \end{array},$$

and the last unidentified digits are S = 8, E = $\overline{11}$, and T = $\overline{12}$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

952. [1984: 194] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the following double inequality, where the sum and product are cyclic over the angles A,B,C of a triangle:

$$\Sigma\sin^2 A \le 2 + 16\Pi\sin^2\frac{A}{2} \le \frac{9}{4}.$$

The inequality between the first and third members is well known, and that between the second and third members is equivalent to the well-known $\Pi\sin(A/2) \le 1/8$. Prove the inequality between the first and second members.

I. *Solution by Kee-wai Lau, Hong Kong.*

If we use the familiar identity $\Sigma\sin^2 A = 2 + 2\Pi\cos A$, we find that the proposed inequality is equivalent to

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) \ge \cos A\cos B\cos C,$$

an inequality established in part (a) of Problem 836 [1984: 228].

II. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

If we multiply both sides of the proposed inequality by $4R^2$ and note that $a = 2R\sin A$, etc. and $\Pi\sin(A/2) = r/(4R)$, it is seen to be equivalent to

$$a^2 + b^2 + c^2 \le 8R^2 + 4r^2,$$

and this inequality is in Item 5.14 on page 53 of the Bottema bible.

III. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We have

$$\Sigma\sin^2 A \le 2 + 16\Pi\sin^2\frac{A}{2} \iff \frac{s^2 - 4Rr - r^2}{2R^2} \le 2 + \frac{r^2}{R^2}$$

$$\iff s^2 \le 4R^2 + 4Rr + 3r^2,$$

and the last inequality is established in Item 5.8 on page 50 of the Bottema bible.

IV. *Comment by M.S. Klamkin, University of Alberta.*

[The proposed inequality having been established, as in solutions I, II, III, or otherwise], we find, as a stronger result, the sharpest inequality of the form

$$\Sigma\sin^2 A \le \alpha + \beta\Pi\sin^2\frac{A}{2}, \qquad (1)$$

where α and β are constants, which holds for all triangles (even degenerate ones), with equality when the triangle is equilateral.

First of all, consideration of the equilateral triangle shows that we must have $\beta = 144 - 64\alpha$. Then $\alpha \geq 2$ follows from consideration of the degenerate triangle $(90^\circ, 90^\circ, 0^\circ)$. We show that (1) is sharpest when $\alpha = 2$ and $\beta = 16$, as in the proposed inequality. If $\alpha' > 2$, then

$$2 + 16\Pi\sin^2\frac{A}{2} \leq \alpha' + (144-64\alpha')\Pi\sin^2\frac{A}{2}$$

is equivalent to

$$(\alpha' - 2)(1 - 64\Pi\sin^2\frac{A}{2}) \geq 0 \quad \text{or to} \quad (\alpha' - 2)(1 - \frac{4r^2}{R^2}) \geq 0,$$

and the last is certainly true since $2r \leq R$.

If the right inequality in the proposal is to hold as well, then α must be restricted to $2 \leq \alpha \leq 9/4$.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

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953. [1984: 195] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

The *Lucas sequence* $\{L_n\}$ is defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_n + L_{n+1} = L_{n+2}, \quad n = 0,1,2,\ldots .$$

For any positive integer n, let

$$a = L_{n-1}L_{n+2} \quad \text{and} \quad b = 2L_nL_{n+1}$$

be the legs of a right triangle.

(a) Establish the following facts about this triangle.

(i) The hypotenuse, c, is integral (and so the triangle is Pythagorean).

(ii) The sum of the hypotenuse and one of the legs is the square of a Lucas number.

(iii) The semiperimeter s, the inradius r, and the exradius r_c are each the product of two consecutive Lucas numbers.

(iv) The exradii r_a and r_b are each the product of two Lucas numbers whose subindices differ by 2.

(b) Obtain the following limits:

(i) $\lim_{n\to\infty} r_c/r_b$, (ii) $\lim_{n\to\infty} r_c/r_a$, (iii) $\lim_{n\to\infty} r_b/r_a$.

Solution by Sam Baethge, San Antonio, Texas.

More generally, let $\{L_n\}$ be any sequence satisfying the given recurrence relation, L_0 and L_1 being arbitrary positive integers. To simplify the notation, we will when convenient write

$$l_1 = L_{n-1}, \qquad l_2 = L_n, \qquad l_3 = L_{n+1}, \qquad l_4 = L_{n+2}$$

for any $n \geq 1$, so that

$$a = l_1 l_4 = l_1^2 + 2l_1 l_2 \qquad \text{and} \qquad b = 2l_2 l_3 = 2l_1 l_2 + 2l_2^2.$$

(a) We establish the required facts about the right triangle with legs a,b and hypotenuse c.

We have

$$c^2 = a^2 + b^2 = (l_1^2+2l_1l_2)^2 + (2l_1l_2+2l_2^2)^2 = (l_1^2+2l_1l_2+2l_2^2)^2,$$

so c is an integer (and the triangle is Pythagorean). Now we get

$$b + c = (2l_1l_2+2l_2^2) + (l_1^2+2l_1l_2+2l_2^2) = (l_1+2l_2)^2 = l_4^2 = L_{n+2}^2.$$

The following results are also quickly established:

$$s = \tfrac{1}{2}(a+b+c) = l_1^2+3l_1l_2+2l_2^2 = (l_1+l_2)(l_1+2l_2) = l_3l_4 = L_{n+1}L_{n+2},$$

$$r = s - c = l_1l_2 = L_{n-1}L_n,$$

$$r_c = s = L_{n+1}L_{n+2},$$

$$r_a = s - b = l_1(l_1+l_2) = l_1l_3 = L_{n-1}L_{n+1},$$

$$r_b = s - a = l_2(l_1+2l_2) = l_2l_4 = L_nL_{n+2}.$$

(b) It is well known that $L_m = A\alpha^m + B\beta^m$ for $m = 1,2,3,\ldots$, where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ (the golden ratio)}, \qquad \beta = \frac{1-\sqrt{5}}{2},$$

and the constants A and B depend upon the values of L_0 and L_1; and that

$$\lim_{m\to\infty} \frac{L_{m+1}}{L_m} = \alpha. \tag{1}$$

Therefore

$$\lim_{n\to\infty} \frac{r_c}{r_b} = \lim_{n\to\infty} \frac{L_{n+1}L_{n+2}}{L_nL_{n+2}} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha,$$

$$\lim_{n\to\infty} \frac{r_b}{r_a} = \lim_{n\to\infty} \frac{L_n}{L_{n-1}}\cdot\frac{L_{n+2}}{L_{n+1}} = \alpha^2,$$

$$\lim_{n\to\infty} \frac{r_c}{r_a} = \lim_{n\to\infty} \frac{L_{n+1}L_{n+2}}{L_{n-1}L_{n+1}} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n}\cdot\frac{L_n}{L_{n-1}}\cdot\frac{L_{n+2}}{L_{n+1}} = \alpha^3.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; DONALD L. MUENCH, St. John Fisher College, Rochester, N.Y.; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

A proof of (1), if one is needed, follows from

$$\frac{L_{m+1}}{L_m} = \alpha + \frac{B\beta^m(\beta-\alpha)}{A\alpha^m+B\beta^m},$$

since $\alpha > 1$ and $|\beta| < 1$.

* * *

954. [1984: 195] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

The notation being the usual one, prove that each of the following is a necessary and sufficient condition for a triangle to be acute-angled:

(a) $\mathrm{IH} < r\sqrt{2}$,

(b) $\mathrm{OH} < R$,

(c) $\cos^2 A + \cos^2 B + \cos^2 C < 1$,

(d) $r^2 + r_a^2 + r_b^2 + r_c^2 < 8R^2$,

(e) $m_a^2 + m_b^2 + m_c^2 > 6R^2$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We start with two more necessary and sufficient conditions for a triangle to be acute-angled:

(f) $\cos A \cos B \cos C > 0$,

(g) $a^2 + b^2 + c^2 > 8R^2$.

The validity of (f) is obvious. As for (g), we have (all sums and products are cyclic over A,B,C)

(g) $\iff$ $4R^2\Sigma\sin^2 A > 8R^2$ $\iff$ $8R^2(1 + \Pi\cos A) > 8R^2$ $\iff$ (f).

Proofs of (a)-(e):

(a) $\iff$ $\mathrm{IH}^2 - 2r^2 < 0$ $\iff$ $(2r^2 - 4R^2\Pi\cos A) - 2r^2 < 0$ $\iff$ (f).

(b) $\iff$ $\mathrm{OH}^2 - R^2 < 0$ $\iff$ $(R^2 - 8R^2\Pi\cos A) - R^2 < 0$ $\iff$ (f).

(c) $\iff$ $1 - 2\Pi\cos A < 1$ $\iff$ (f).

(d) $\iff$ $16R^2 - (a^2+b^2+c^2) < 8R^2$ $\iff$ (g).

(e) $\iff$ $\frac{3}{4}(a^2+b^2+c^2) > 6R^2$ $\iff$ (g).

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

* * *

955. [1984: 195] *Proposed by Geng-zhe Chang, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.*

If the real numbers A,B,C,a,b,c satisfy

$$A+a \geq b+c, \quad B+b \geq c+a, \quad C+c \geq a+b,$$

show that

$$Q \equiv Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy \geq 0$$

holds for all real x,y,z such that $x+y+z = 0$.

Solution by M.S. Klamkin, University of Alberta.

We have

$$\begin{aligned} Q &\geq (b+c-a)x^2 + (c+a-b)y^2 + (a+b-c)z^2 + 2ayz + 2bzx + 2cxy \\ &= (x+y+z)\{(b+c-a)x + (c+a-b)y + (a+b-c)z\} \\ &= 0. \end{aligned}$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; JORDAN B. TABOV, Sofia, Bulgaria; and the proposer.

Editor's comment.

let $U = A+a-b-c$, $V = B+b-c-a$, and $W = C+c-a-b$. Our problem shows that

$$U,V,W \geq 0 \quad \Longrightarrow \quad Q \geq 0 \text{ on the plane } x+y+z = 0.$$

Tabov showed that the converse is not true. If $U = 3$, $V = 3$, $W = -1$, and $a = b = c = 0$, for example, then on the plane $z = -(x+y)$ we have

$$Q = 3x^2 + 3y^2 - z^2 = 2(x^2 - xy + y^2) \geq 0.$$

Tabov also showed that $Q \geq 0$ on the plane $x+y+z = 0$ if and only if

$$VW + WU + UV \geq 0$$

and $V+W \geq 0$ or $W+U \geq 0$ or $U+V \geq 0$. (The equivalences

$$VW+WU+UV \geq 0 \quad \Longleftrightarrow \quad (W+U)(U+V) \geq U^2 \quad \Longleftrightarrow \quad (U+V)(V+W) \geq V^2$$

show that if one of $V+W$, $W+U$, $U+V$ is nonnegative, then all three are nonnegative.) His proof was somewhat complicated, especially in comparison with Klamkin's solution given above. We give below a simple proof of Tabov's result. It is based partly on Tabov's own proof and partly on Klamkin's.

On the plane $z = -(x+y)$,

$$\begin{aligned} Q &= (U+b+c-a)x^2 + (V+c+a-b)y^2 + (W+a+b-c)z^2 + 2ayz + 2bzx + 2cxy \\ &= Ux^2 + Vy^2 + Wz^2 + (x+y+z)\{(b+c-a)x + (c+a-b)y + (a+b-c)z\} \\ &= Ux^2 + Vy^2 + Wz^2 \end{aligned}$$

$$= Ux^2 + Vy^2 + W(x+y)^2$$
$$= (W+U)x^2 + 2Wxy + (V+W)y^2.$$

This is nonnegative for all (x,y) if and only if $W+U \geq 0$ and

$$W^2 - (V+W)(W+U) \leq 0, \quad \text{or} \quad VW + WU + UV \geq 0.$$

* * *

956. [1984: 196] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let

$$S(n,p) = \sum_{k=0}^{n} \binom{n}{k} k^p,$$

where n is a positive integer and p a nonnegative integer. Prove that

(a) $S(n,0)\cdot S(n,3) - S(n,1)\cdot S(n,2)$ is the square of an integer for every n;

(b) $S(n,1)\cdot S(n,4) - S(n,2)\cdot S(n,3)$ cannot be the square of an integer for any n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

We first calculate the values of $S(n,p)$ for $p = 0,1,2,3,4$. Clearly

$$S(n,0) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

With the notation $(k)_i = k(k-1)(k-2)\ldots(k-i+1)$ for $i = 1,2,3,\ldots$, it is easy to see that

$$\sum_{k=0}^{n} \binom{n}{k} (k)_i = (n)_i \sum_{k=i}^{n} \binom{n-i}{k-i} = (n)_i \sum_{k=0}^{n-i} \binom{n-i}{k} = (n)_i \cdot 2^{n-i}. \tag{1}$$

Since

$$\begin{aligned} k &= (k)_1, \\ k^2 &= (k)_2 + (k)_1, \\ k^3 &= (k)_3 + 3(k)_2 + (k)_1, \\ k^4 &= (k)_4 + 6(k)_3 + 7(k)_2 + (k)_1, \end{aligned}$$

we therefore obtain from (1)

$$\begin{aligned} S(n,1) &= n\cdot 2^{n-1}, \\ S(n,2) &= n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} \\ &= n(n+1)\cdot 2^{n-2}, \\ S(n,3) &= n(n-1)(n-2)\cdot 2^{n-3} + 3n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} \\ &= n^2(n+3)\cdot 2^{n-3}, \end{aligned}$$

$$S(n,4) = n(n-1)(n-2)(n-3)\cdot 2^{n-4} + 6n(n-1)(n-2)\cdot 2^{n-3} + 7n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1}$$
$$= n(n+1)(n^2+5n-2)\cdot 2^{n-4}.$$

From these results we get

(a) $S(n,0)\cdot S(n,3) - S(n,1)\cdot S(n,2) = n^2\cdot 2^{2n-2} = (n\cdot 2^{n-1})^2$

and

(b) $S(n,1)\cdot S(n,4) - S(n,2)\cdot S(n,3) = n^2(n^2-1)\cdot 2^{2n-4} = (n\cdot 2^{n-2})^2(n^2-1)$.

It is clear that (a) is the square of an integer for every n and that (b) cannot be the square of an integer for any $n > 1$. (The restriction $n > 1$ for part (b) was omitted from the proposal.)

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The restriction $n > 1$ in part (b) was given by the proposer but inadvertently omitted by the editor.

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957. [1984: 196] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let a,b,c be the sides of a triangle with circumradius R and area K. Prove that

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \ge \frac{2K}{R},$$

with equality if and only if the triangle is equilateral.

Solution by M.S. Klamkin, University of Alberta.

We show, more generally, that

$$\frac{b^nc^n}{b^n+c^n} + \frac{c^na^n}{c^n+a^n} + \frac{a^nb^n}{a^n+b^n} \ge \frac{3}{2}\left(\frac{4K}{3R}\right)^n, \qquad 0 \le n \le \lambda, \tag{1}$$

where

$$\lambda = \frac{1}{2}\cdot\frac{\log(9/4)}{\log(4/3)} \approx 1.4094.$$

The proposed problem then corresponds to the case $n = 1$.

Starting from the Cauchy inequality, we get

$$\Sigma(b^{-n}+c^{-n})\cdot\Sigma(b^{-n}+c^{-n})^{-1} \ge 9,$$

where all sums, here and later, are cylic over a,b,c. From this we get

$$\sum\frac{b^nc^n}{b^n+c^n} \ge \frac{9(abc)^n}{2\Sigma b^nc^n}. \tag{2}$$

Now it is known (see O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, 1968, p. 55) that

$$a^{2n} + b^{2n} + c^{2n} \le 3^{n+1}\cdot R^{2n}, \qquad 0 \le n \le \lambda.$$

Then, since $\Sigma b^nc^n \le \Sigma a^{2n}$ and $abc = 4KR$, (1) follows from (2), with equality just when $n = 0$ or $a = b = c$.

With $a = 2R\sin A$, etc., (1) can be expressed trigonometrically as

$$\sum\frac{1}{\sin^n A(\sin^n B + \sin^n C)} \ge \frac{2^{2n-1}}{3^{n-1}}, \qquad 0 \le n \le \lambda.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

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958. [1984: 196] *Proposed by M.S. Klamkin, University of Alberta.*

If A_1, A_2, A_3 are the angles of a triangle, prove that

$$\tan A_1 + \tan A_2 + \tan A_3 \ge \text{ or } \le 2(\sin 2A_1 + \sin 2A_2 + \sin 2A_3)$$

according as the triangle is acute-angled or obtuse-angled, respectively. When is there equality?

Solution by Jack Garfunkel, Flushing, N.Y.

In this solution, all sums are cyclic over A_1, A_2, A_3, and the page numbers all refer to items in the Bottema bible, *Geometric Inequalities*.

For the obtuse triangle, the desired result is

$$\Sigma\tan A_i \le 2\Sigma\sin 2A_i.$$

Here the inequality is strict for all nondegenerate triangles. In fact, for all such triangles the left side is strictly negative (page 26) and the right side strictly positive (page 19).

For the acute triangle, we prove the stronger result

$$\Sigma\tan A_i \ge 2\Sigma\sin A_i \ge 2\Sigma\sin 2A_i.$$

The right inequality is established on pages 18-19, and the sharper left inequality follows from $\Sigma\tan A_i \ge 3\sqrt{3}$ (page 26) and $3\sqrt{3} \ge 2\Sigma\sin A_i$ (page 18). Here equality holds throughout just when the triangle is equilateral.

Also solved by CURTIS COOPER, Central Missouri State University, Warrensburg; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; JORDAN B. TABOV, Sofia, Bulgaria; and the proposer.

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959. [1984: 196] *Proposed by Sidney Kravitz, Dover, New Jersey.*

Two houses are located to the north of a straight east-west highway. House A is at a perpendicular distance a from the road, house B is at a perpendicular distance $b \geq a$ from the road, and the feet of the perpendiculars are one unit apart. Design a road system of minimum total length (as a function of a and b) to connect both houses to the highway.

Solution by Leroy F. Meyers, The Ohio State University.

A road system of minimum length must consist of either

(I) a straight road from A to some point R_1 on the east-west road l, and a straight road from B to some point R_2 on l; or

(II) straight roads connecting some point C with A, B, and some point R on l.

The shortest road system of Type (I) obviously has length

$$f_0(a,b) = a + b,$$

which occurs when R_1 and R_2 are the feet P and Q of the perpendiculars from A and B, respectively, to l (see Figure 1).

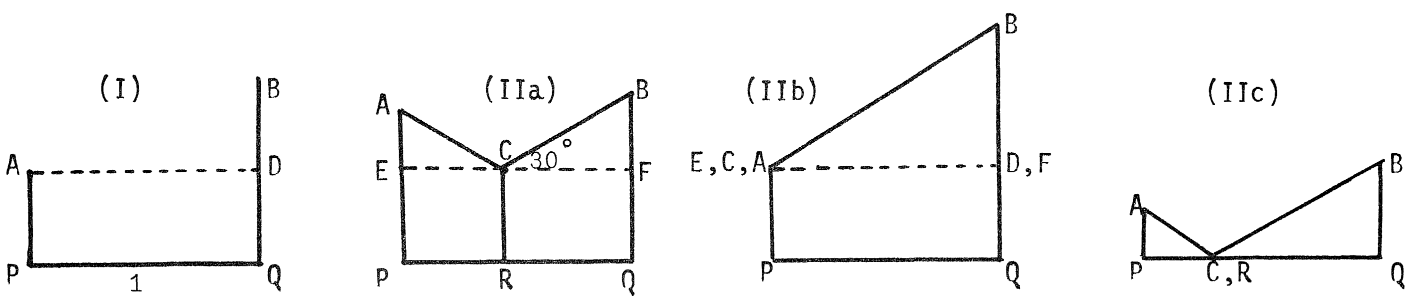

Figure 1

For the shortest road system of type (II), the point R must obviously be the foot of the perpendicular from C to l. Furthermore, C cannot lie outside the rectangle APQD. (Otherwise, if C is west of the line AP, east of the line BQ, or south of l, then C can be replaced by the foot of the perpendicular from C to AP, BQ, or l, respectively; if C is due north of some point G on the segment AB, then C can be replaced by G; and if C is in the triangle ABD but north of the segment AD, then C can be replaced by A.) Hence R lies on the segment PQ. Since C and R are constrained to lie in compact sets and the length of the road system depends continuously on C and R, there is a shortest road system of type (II). On the other hand, if R is fixed, then the point C determined by the condition that CA+CB+CR be minimized (Steiner's problem; see Courant and Robbins, *What is Mathematics?*, pp. 354-359) is such that

(IIa) if no angle of triangle ARB exceeds 120°, then $\angle ACB = \angle BCR = \angle RCA = 120^\circ$;

(IIb) if $\angle RAB \geq 120^\circ$, then C = A; and

(IIc) if $\angle ARB \geq 120^\circ$, then C = R.

(Note that $\angle ABR \leq \angle ABO \leq 90^\circ$, since $b \geq a$.)

Let E and F be the points of intersection of the segments AP and BQ, respectively, with the line parallel to l and passing through C; and let c be the length of the segment CR.

In case (IIa) we have $\angle ECA = \angle FCB = 30^\circ$, and so

$$1 = EC + CF = AE\sqrt{3} + BF\sqrt{3} = \{(a-c) + (b-c)\}\sqrt{3},$$

whence $c = \frac{1}{2}(a+b-1/\sqrt{3})$, provided that $0 \leq c \leq a$, i.e., $b-a \leq 1/\sqrt{3} \leq b+a$. Then the length of the road system, CA+CB+CR, is easily found to be

$$f_1(a,b) = \frac{1}{2}(a + b + \sqrt{3}), \qquad b-a \leq \frac{1}{\sqrt{3}} \leq b+a.$$

In case (IIb) we have $c = a$ and $\angle PAB \geq \angle RAB \geq 120^\circ$, so that $\angle BAD = \angle BCF \geq 30^\circ$ and $b-a \geq 1/\sqrt{3}$. Then the length of the road system is

$$f_2(a,b) = \sqrt{1 + (b-a)^2} + a, \qquad b-a \geq \frac{1}{\sqrt{3}}.$$

In case (IIc) we have $c = 0$ and $\theta = \angle PRA = \angle QRB \leq 30^\circ$ (law of reflection), so that

$$a + b = PR\tan\theta + RQ\tan\theta = \tan\theta \leq \frac{1}{\sqrt{3}}.$$

Then the length of the road system is

$$f_3(a,b) = \sqrt{1 + (b+a)^2}, \qquad b+a \leq \frac{1}{\sqrt{3}}.$$

The functions f_1, f_2, f_3 are defined on nonoverlapping sets whose union is $\{(a,b)\colon b \geq a \geq 0\}$, and their values agree where the sets intersect. Hence it is sufficient to compare each of f_1, f_2, f_3 with f_0. It is obvious that $f_0(a,b) < f_3(a,b)$ for all (a,b); hence case (IIc) never yields a shortest read system. Also

$$f_0(a,b) \gtreqless f_1(a,b) \quad \text{according as} \quad b + a \gtreqless \sqrt{3}.$$

Hence case (IIa) yields the shortest road system if and only if

$$b + a \geq \sqrt{3} \quad \text{and} \quad a \leq b \leq a + \frac{1}{\sqrt{3}}.$$

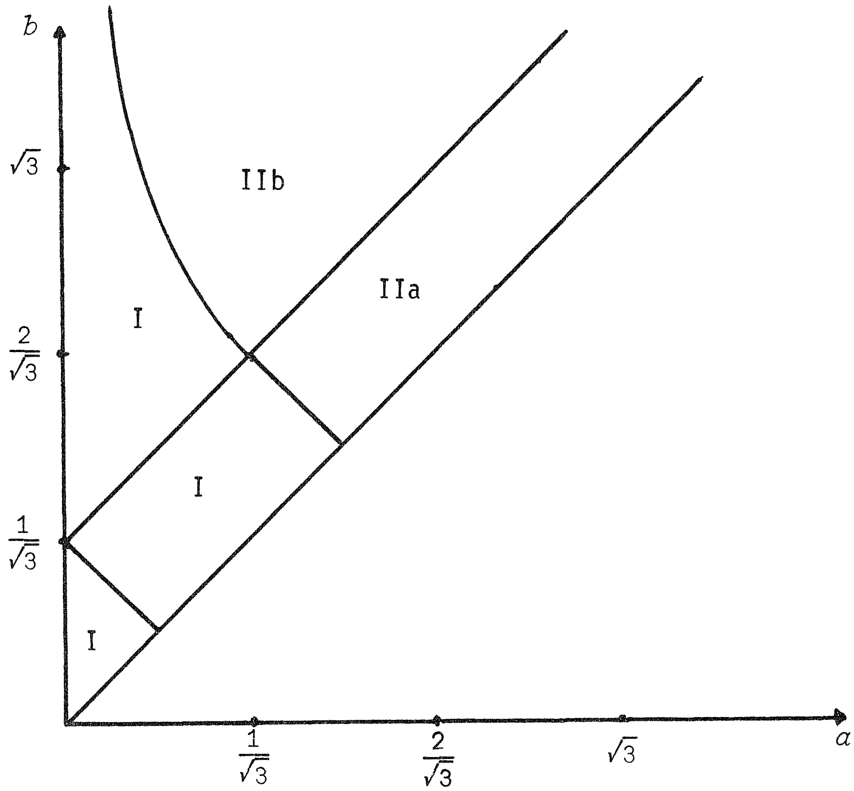

Figure 2

Furthermore,

$$f_0(a,b) \gtreqless f_2(a,b) \quad \text{according as} \quad b^2 \gtreqless 1 + (b-a)^2, \quad \text{or} \quad b \gtreqless \frac{a^2+1}{2a}.$$

Hence case (IIb) yields the shortest road system if and only if

$$b \ge \max\left(a + \frac{1}{\sqrt{3}}, \frac{a^2+1}{2a}\right).$$

In all other cases, type (I) yields the shortest road system, although there may be ties. Note that the graph of $b+a = \sqrt{3}$ is tangent to the graph of $b = (a^2+1)/2a$ at the point $(1/\sqrt{3}, 2/\sqrt{3})$. These results are all illustrated in Figure 2.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; DAN SOKOLOWSKY, College of William and Mary, Williamsburg; and the proposer. A comment was received from FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio.

Editor's comment.

The proposer wrote that this problem and several related ones are discussed in his article "Shortest Routes for System Connections", in *Design News*, 11 October 1968. But the article contained only the sketchiest of solutions to our problem, so we are pleased to be able to present now a full and complete solution to this interesting problem. For other related problems, see Cyril Isenberg, "Minimum Roadway Problems", *Parabola*, Vol. 20 (1984), No. 2, pp. 2-13 (published by the University of New South Wales, Australia).

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960. [1984: 196] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

If the altitude of a triangle is also a symmedian, prove that the triangle is either an isosceles triangle or a right triangle.

I. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

For any triangle ABC, let AH be the altitude and AM the median through A, and let O be the cirumcenter. The lines AH and AO are isogonal conjugates, as are AM and the symmedian through A. If AH coincides with the symmedian through A, then O must lie on AM. We now consider two cases.

If $O \neq M$, then BOM and COM are congruent right triangles, the altitude AH coincides with the median AM, and the triangle is isosceles.

If $O = M$, then $AM = BM = CM$ and ABC is a right triangle.

II. *Solution by the proposer.*

Let ABC be any triangle with sides a,b,c in the usual order, altitude AH and symmedian AS. It is well known that

$$\frac{BS}{SC} = \frac{c^2}{b^2} \qquad \text{and} \qquad \frac{BH}{HC} = \frac{c\cos B}{a - c\cos B}.$$

When S and H coincide, we therefore have

$$\frac{c^2}{b^2} = \frac{c\cos B}{a - c\cos B}. \tag{1}$$

With $\cos B = (c^2+a^2-b^2)/2ca$, we obtain from (1) an equation equivalent to

$$a^2(b^2 - c^2) = (b^2 + c^2)(b^2 - c^2).$$

Therefore, either $b = c$ and the triangle is isosceles, or else $a^2 = b^2+c^2$ and we have a right triangle.

Also solved by M.S. KLAMKIN, University of Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; DAN SOKOLOWSKY, College of William and Mary, Williamsburg; and KENNETH M. WILKE, Topeka, Kansas.

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961. [1984: 216] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

If the digit(s) of n can be reused in EGG and NEST, then in the decimal alphametic

$$n \cdot \text{EGG} = \text{NEST}$$

the maximum EGG is 988 and there are solutions when n is in geometric progression (n = 2, 4, 8) and when n is consecutive (n = 2, 3, 4):

$$2\cdot 988 = 1976,$$
$$3\cdot 988 = 2964,$$
$$4\cdot 988 = 3952,$$
$$8\cdot 988 = 7904.$$

Find the minimum EGG for which solutions exist when n is in arithmetic progression.

Solution by J.A. McCallum, Medicine Hat, Alberta.

We find *all* the values of EGG for which solutions exist when the n are in A.P. Since G ≠ 0, the smallest possible value is 122 and the largest is 988, and there are indeed solutions in both cases. In the case of 122, we find all its four-digit multiples that have a 1 in the hundreds' position and no duplicated digits. The only satisfactory values of n are 59, 67, and 75, and these *happen* to be in A.P. Proceeding likewise with 988, we find that the only satisfactory values of n are 2, 3, 4, and 8, and the first three are in A.P. Resorting now to a computer (as any one should who doesn't have a *lot* of time to waste), we find that the only satisfactory intermediate values of EGG are 166, 199, and 977. All these results are set out below.

59•122 = 7198	13•166 = 2158	43•166 = 7138
67•122 = 8174	19•166 = 3154	49•166 = 8134
75•122 = 9150	25•166 = 4150	55•166 = 9130
16•199 = 3184	2•977 = 1954	2•988 = 1976
26•199 = 5174	3•977 = 2931	3•988 = 2964
36•199 = 7164	4•977 = 3908	4•988 = 3952

Also solved by STEWART METCHETTE, Culver City, California; GLEN E. MILLS, Pensacola Junior College, Florida; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Metchette stated that the only solution with the n in G.P. is the one given in the proposal, with n = 2, 4, 8.

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KESIRAJU SATYANARAYANA (1897-1985)

We regret to inform our readers that Kesiraju Satyanarayana, a subscriber for many years and frequent contributor to *Crux Mathematicorum* and other journals, died on August 13, 1985, at the age of 88, exactly ten days after the death of his wife. He leaves to mourn him one son, one daughter, and three grandchildren. A biographical note about Professor Satyanarayana appeared in this journal a few years ago [1981: 294].

We extend our deepest sympathy to his family.