Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
- University of the Fraser Valley

IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Welcome to issue 4. As you notice, it is coming out almost on time. The entire Editorial Board has been hard at work to catch up to a small backlog created by delay in production (due to uncertain future) last year and tighter deadlines between published problems and solution (in response to new online format) this year. To catch up to deadlines, over the next couple of issues we will publish more solutions and I am extremely grateful to my editors for taking on a larger workload.

In this issue, we are introducing a new column to *MathemAttic* called *Teaching Problems*. This feature will be of interest to high school teachers as well as their students trying to explore "meaty" problems or problems one can approach from many different angles. You can read more about the column and the column itself on pages 178-180. We welcome contributions to this column, so please send us correspondence to MathemAttic@cms.math.ca.

Kseniya Garaschuk

"Of course you have problems! You're a math teacher."

© 2012 Jonny Hawkins

MathemAttic

No. 4

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2019**.

MA16. Prove that every year must have a month in which the 13th of some month is a Friday. What is the maximum number of Friday the 13ths in one year?

MA17. Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

MA18. Twenty calculus students are comparing grades on their first two quizzes of the year. The class discovers that every pair of students received the same grade on at least one of the two quizzes. Prove that the entire class received the same grade on at least one of the two quizzes.

MA19. Choose N elements of $\{1, 2, 3, \dots, 2N\}$ and arrange them in increasing order. Arrange the remaining N elements in decreasing order. Let D_i be the absolute value of the difference of the ith elements in each arrangement. Prove that

$$D_1 + D_2 + \cdots + D_N = N^2.$$

MA20.

a) A line segment of length 11 is randomly cut into three pieces, each of integer length. What is the probability that the three pieces can be formed into a (non-degenerate) triangle?

b) Can you find the probability above for a line segment of any odd integer length $n = 2k + 1$?

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

MA16. Montrer que toutes les années doivent contenir au moins un treizième du mois qui est un vendredi. Quel est le nombre maximal possible de vendredis 13 dans une année ?

MA17. Soit $f(n) = 25^n - 72n - 1$. Trouver, avec preuve, le plus grand entier M tel que $f(n)$ est divisible par M pour tout entier positif n.

MA18. Vingt étudiants d'une classe de calcul comparent leurs notes de leurs deux premiers tests de l'année. Les étudiants constatent que chaque paire d'étudiants a obtenu la même note sur au moins un des deux tests. Montrer que la classe entire a obtenu la même note pour au moins un des deux tests.

MA19. On choisit N éléments de $\{1, 2, 3, \ldots, 2N\}$, on les ordonne en ordre croissant et on ordonne les N éléments restants en ordre décroissant. Soit D_i la valeur absolue de la différence des $i$èmes éléments de chaque arrangement. Montrer que

$$D_1 + D_2 + \cdots + D_N = N^2.$$

MA20.

a) Un segment de droite de longueur 11 est coupé aléatoirement en trois segments, chacun d'une longueur entière. Quelle est la probabilité que les trois segments puissent former un triangle (non-dégénéré) ?

b) Est-il possible de trouver la probabilité ci-dessus pour n'importe quelle longueur entière impaire $n = 2k + 1$?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(7), p. 279; and 44(8), p. 318–319.

CC331. Consider triangle ABC with $\angle B = \angle C = 70^\circ$. On the sides AB and AC, we take the points F and E, respectively, so that $\angle ABE = 15^\circ$ and $\angle ACF = 30^\circ$. Find $\angle AEF$.

Originally 2016 BC Exam, Texas A&M High School Math Contest, #20.

We received 12 correct solutions. We present the solution by Ivko Dimitrić.

Since

$$\angle FCB = \angle ACB - \angle ACF = 70^\circ - 30^\circ = 40^\circ,$$

in $\triangle BCF$ we have

$$\angle BFC = 180^\circ - (40^\circ + 70^\circ) = 70^\circ.$$

Therefore $\triangle BCF$ is isosceles with $CB = CF$. Then,

$$\angle EBC = \angle ABC - \angle ABE = 70^\circ - 15^\circ = 55^\circ$$

and hence

$$\angle BEC = 180^\circ - (55^\circ + 70^\circ) = 55^\circ.$$

Thus, $\triangle BCE$ is also isosceles with $CE = CB$. It follows hence that $CE = CF$, i.e. $\triangle CEF$ is itself isosceles, so

$$\angle CEF = \frac{1}{2}(180^\circ - \angle ECF) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ.$$

Finally, $\angle AEF = 180^\circ - \angle CEF = 105^\circ$.

CC332. Find the largest integer k such that 135^k divides $2016!$. Note that $n! = 1 \cdot 2 \cdot 3 \cdots n$.

Originally 2016 AB Exam, Texas A&M High School Math Contest, #20.

We received five correct solutions and three incorrect solutions. We present the solution of the Missouri State University Problem Solving Group.

For any prime p, the number of factors of p in $n!$ can be calculated by

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor .$$

Now, $135 = 3^3 \cdot 5$. The highest power of 3 that divides 2016! is

$$\begin{aligned}\sum_{i=1}^{\infty}\left\lfloor\frac{2016}{3^i}\right\rfloor &= \left\lfloor\frac{2016}{3}\right\rfloor+\left\lfloor\frac{2016}{9}\right\rfloor+\left\lfloor\frac{2016}{27}\right\rfloor+\left\lfloor\frac{2016}{81}\right\rfloor+\left\lfloor\frac{2016}{243}\right\rfloor+\left\lfloor\frac{2016}{729}\right\rfloor\\ &= 672+244+74+24+8+2\\ &= 1004.\end{aligned}$$

So the largest k such that 3^{3k} divides 2016! is 334.

The highest power of 5 that divides 2016! is

$$\begin{aligned}\sum_{i=1}^{\infty}\left\lfloor\frac{2016}{5^i}\right\rfloor &= \left\lfloor\frac{2016}{5}\right\rfloor+\left\lfloor\frac{2016}{25}\right\rfloor+\left\lfloor\frac{2016}{125}\right\rfloor+\left\lfloor\frac{2016}{625}\right\rfloor\\ &= 403+80+16+3\\ &= 502.\end{aligned}$$

Since this is larger than our previous k, we conclude 135^{334} is the largest power dividing 2016!.

CC333. Let $\theta = \arctan 2 + \arctan 3$. Find $\dfrac{1}{\sin^2\theta}$ and simplify fully.

Originally 2016 DE Exam, Texas A&M High School Math Contest, #8.

We received thirteen correct solutions and one incorrect solution. We present the solution of Charles Justin Shi.

Using the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, we have

$$\frac{1}{\sin^2\theta} = 1 + \frac{1}{\tan^2\theta}.$$

Using the formula $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$, we find

$$\tan\theta = \tan(\arctan 2 + \arctan 3) = \frac{2+3}{1-2\cdot 3} = -1.$$

Combining our two results we get $\dfrac{1}{\sin^2\theta} = 1 + \dfrac{1}{(-1)^2} = 2.$

CC334. Find the sum of all positive integers x for which $x+56$ and $x+113$ are perfect squares.

Originally 2016 BC Exam, Texas A&M High School Math Contest, #11.

We received eleven correct solutions and two incorrect solutions. We present the solution of Maria Pilehrood.

We start with a system of equations,

$$\begin{aligned} x+56 &= a^2 \\ x+113 &= b^2 \end{aligned}$$

Then, $b^2 - a^2 = (x+113)-(x+56) = 57 \Rightarrow (b-a)(b+a) = 57$. We get two possibilities from here:

$$\begin{aligned} b-a &= 1 \\ b+a &= 57 \end{aligned}$$

or

$$\begin{aligned} b-a &= 3 \\ b+a &= 19 \end{aligned}$$

This first case gives $2b = 58$, so $b = 29$ and thus $a = 28$. In the second case, $2b = 22$, so $b = 11$ and thus $a = 8$. Solving for x in each of these cases we find

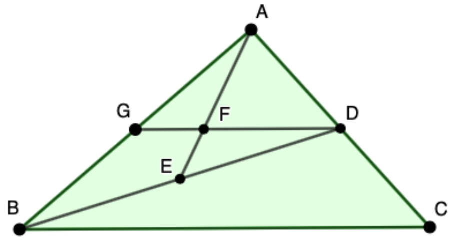
$$x = a^2 - 56 = 28^2 - 56 = 728$$

or $8^2 - 56 = 8$, so the sum of all such positive integers x is $8 + 728 = 736$.

CC335. In the triangle ABC, BD is the median to the side AC, DG is parallel to the base BC (G is the point of intersection of the parallel with AB). In the triangle ABD, AE is the median to the side BD and F is the intersection point of DG and AE. Find $\dfrac{BC}{FG}$.

Originally 2016 BC Exam, Texas A&M High School Math Contest, #16.

We received thirteen submissions, twelve correct and one incorrect. We present the solution by Daniel Văcaru.



Point F is the intersection point for medians AE and DG. It follows that

$$\frac{GF}{GD} = \frac{1}{3} \implies GD = 3 \cdot FG.$$

On the other hand, $BC = 2 \cdot GD = 6 \cdot FG$ gives

$$\frac{BC}{FG} = \frac{6 \cdot FG}{FG} = 6.$$

CC336. Define the $n \times n$ *Pascal matrix* as follows: $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Show that every Pascal matrix is invertible.

Originally from the 2017 Science Atlantic Math Competition.

This problem also appeared as CC308 with solution published in ***Crux*** *45(1). We received one additional solution.*

CC337. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

Originally from the 2017 Science Atlantic Math Competition.

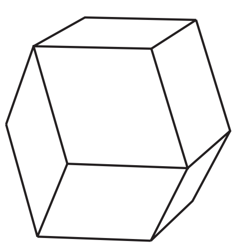
This problem also appeared as CC309 with solution published in ***Crux*** *45(1). We received one additional solution.*

CC338. Find (with proof) all integer solutions (x, y) to $x^2 - xy + 2017y = 0$.

Originally from the 2017 Science Atlantic Math Competition.

This problem also appeared as CC307 with solution published in ***Crux*** *45(1). We received two additional solutions.*

CC339. A *rhombic dodecahedron* has twelve congruent rhombic faces; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\dfrac{p+\sqrt{q}}{r}$, where p, q, and r are natural numbers and r has no factor in common with p or q.



Originally from the 2018 Science Atlantic Math Competition.

We received no solutions to this problem.

CC340. Let S be the set of natural numbers dividing 2018^{2018}. In how many ways can one select three numbers $\{x, y, z\}$ (not necessarily distinct, but order being irrelevant) from S so that $y = \sqrt{xz}$?

Originally from the 2018 Science Atlantic Math Competition.

We received one solution by CR Pranesachar, which is presented below lightly edited.

Let $M = 2018^{2018}$ and $p = 1009$ (note that p is a prime). Then

$$M = (2p)^{2p} = 2^{2p}p^{2p}.$$

If $y = \sqrt{xz}$, then x, y, z are in a geometric progression (possibly $x = y = z$). Let

$$x = 2^{a_1}p^{b_1},\ y = 2^{a_2}p^{b_2},\ z = 2^{a_3}p^{b_3},$$

where $0 \le a_j, b_j \le 2p$ for $1 \le j \le 3$. Since x, y, z are in geometric progression, a_1, a_2, a_3 and b_1, b_2, b_3 are in arithmetic progression.

Suppose, $c_1 < c_2 < c_3$ is an arithmetic progression with $c_i \in \{0, 1, \ldots, 2p\}$ for $1 \le i \le 3$. Then c_1 and c_3 can be any two numbers of the same parity. As there are $p+1$ even numbers and p odd numbers in $\{0, 1, \ldots, 2p\}$, the number of arithmetic progressions is $\binom{p+1}{2} + \binom{p}{2} = p^2$.

To avoid duplicate triples we may assume $a_1 \le a_2 \le a_3$ and, if $a_1 = a_2 = a_3$, then $b_1 \le b_2 \le b_3$. This leaves us with the following cases.

(a) $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$: There are $2p+1$ choices for the a_i's and $2p+1$ choices for the b_j's, giving us $(2p+1)^2$ for the number of pairs of such triples;

(b) $a_1 = a_2 = a_3$, $b_1 < b_2 < b_3$: There are $2p+1$ choices for the a_i and p^2 choices for the b_j, thus $(2p+1)p^2$ for the number of pairs of triples;

(c) $a_1 < a_2 < a_3$, $b_1 > b_2 > b_3$: There are $p^2p^2 = p^4$ pairs of triples;

(d) $a_1 < a_2 < a_3$, $b_1 = b_2 = b_3$: There are $p^2(2p+1)$ pairs of triples;

(e) $a_1 < a_2 < a_3$, $b_1 < b_2 < b_3$: There are p^4 pairs of triples.

Thus the number of triples $\{x, y, z\}$ is

$$(2p+1)^2 + 2(2p+1)p^2 + 2p^4 = (p+1)^4 + p^4 = 1010^4 + 1009^4.$$

PROBLEM SOLVING VIGNETTES

No. 4

Shawn Godin

Utilizing Congruences

Don Rideout's article *Arithmetic of Remainders* in [2019: 45(3), p. 118 - 121] introduced a powerful problem solving tool – congruences. In this issue we revisit some open problems from earlier columns and introduce a few more to get a feel for congruences in action. All of these problems come from the course C&O 380 that I took from Ross Honsberger eons ago when I was a student at the University of Waterloo.

The first problem appeared in [2018: 44(4), p. 157 - 159]: *Find the smallest natural number composed only of 1's and 0's which is divisible by 225. (#1)*

As $225 = 9 \times 25$, our number is divisible by both 25 and 9. Examining multiples of 25 modulo 100 we see that they are congruent to either 0, 25, 50, or 75. Thus, for our problem, the number must end with 00 to be a multiple of 25. Don Rideout pointed out in problem 2 from his article that the sum of the digits of any multiple of 9 is itself a multiple of 9. Thus, the smallest possible number that satisfies both conditions is 11 111 111 100.

Next, we will look at the following problem, from the same issue: *By factoring, show that* $2222^{5555} + 5555^{2222}$ *is divisible by 7. (#3)*

Rather than solve this by factoring, we will use congruences. First note that $2222 \equiv 3 \pmod 7$, $5555 \equiv 4 \equiv -3 \pmod 7$ and $3^3 = 27 \equiv -1 \pmod 7$. Thus

$$\begin{aligned}
2222^{5555} + 5555^{2222} &\equiv 3^{5555} + (-3)^{2222} \pmod 7\\
&\equiv 3^{5555} + 3^{2222} \pmod 7\\
&\equiv 3^{2222}\left(3^{3333} + 1\right) \pmod 7\\
&\equiv 3^{2222}\left(\left(3^3\right)^{1111} + 1\right) \pmod 7\\
&\equiv 3^{2222}\left((-1)^{1111} + 1\right) \pmod 7\\
&\equiv 3^{2222}(-1 + 1) \pmod 7\\
&\equiv 0 \pmod 7
\end{aligned}$$

and hence $2222^{5555} + 5555^{2222}$ is divisible by 7. Note, the original problem used the fact that $(x + y) \mid (x^n + y^n)$ for odd positive integers n and $(x - y) \mid (x^n - y^n)$ for all positive integers n. You may want to see if you can prove these facts.

The next problem comes from [2018: 44(4), p. 157 - 159] and states: *Prove that* $n^2 + 3n + 5$ *is never divisible by* 121 *for any natural number* n. *(#6)*

If we let $f(n) = n^2 + 3n + 5$, then since $121 = 11^2$ we know that if $f(n)$ is going to be divisible by 121, it must also be divisible by 11. Computing $f(n) \pmod{11}$ we get

n	0	1	2	3	4	5	6	7	8	9	10
$f(n) \pmod{11}$	5	9	4	1	0	1	4	9	5	3	3

Hence, $f(n)$ is divisible by 11 only when $n \equiv 4 \pmod{11}$. Let $n = 11k + 4$, that is we are making $n \equiv 4 \pmod{11}$. Then

$$\begin{aligned} f(11k+4) &= (11k+4)^2 + 3(11k+4) + 5 \\ &= 121k^2 + 121k + 33 \\ &\equiv 33 \pmod{121} \end{aligned}$$

So when $f(n)$ is divisible by 11 it is not divisible by 121. Therefore $f(n)$ is never divisible by 121. You can also solve this problem by contradiction, by setting $f(n) = 121k$ for some $k \in \mathbb{Z}$ and using the quadratic formula. I will leave this to you as an exercise.

Our final problem comes from the list below:

#11. Prove that no matter how the points of a closed unit square are coloured red or blue, either some two red points or some two blue points are at least $\frac{\sqrt{5}}{2}$ units apart.

#12. There are 120 different ways of selecting a set of 5 numbers from the given table such that a number is taken from each row and from each column. In each of these sets a smallest number occurs. Find the largest number among these 120 minimal values.

11	17	25	19	16
24	10	13	15	3
12	5	14	2	18
23	4	1	8	22
6	20	7	21	9

#13. Six points in three-dimensional space are taken so that no three are collinear and no four are coplanar. The fifteen segments obtained by joining these points in pairs are coloured either red or blue at random. Prove that some triangle has all its sides the same colour.

#14. What are the last two digits of 2^{999}?

#15. Place n points around a circle and draw chords which join them in pairs. Suppose that no three of the chords are concurrent inside the circle. Into how many regions is the interior of the circle divided?

We will examine problem #14.

Solution #1: Since we are interested in the last two digits, we are interested in the behavior of 2^n modulo 100. After a bit of computation you should find that

$2^{22} \equiv 2^2 \pmod{100}$. This implies that $2^n \equiv 2^{n+20}$ for all integers $n \geq 2$. Hence

$$2^{999} \equiv 2^{979} \equiv \cdots \equiv 2^{19} \equiv 88 \pmod{100}$$

which means the last two digits are 88.

Solution #2: Note that $2^{10} = 1024 \equiv -1 \pmod{25}$. Thus

$$\begin{aligned} 2^{997} &= 2^{990} \cdot 2^7 \\ &= \left(2^{10}\right)^{99} \cdot 128 \\ &\equiv (-1)^{99} \cdot 3 \pmod{25} \\ &\equiv -3 \pmod{25} \\ &\equiv 22 \pmod{25} \end{aligned}$$

so we can write $2^{997} = 25k + 22$ and therefore

$$2^{999} = 2^2 \cdot 2^{997} = 4(25k + 22) = 100k + 88 \equiv 88 \pmod{100}$$

which means the last two digits are 88.

Through these examples hopefully you have seen how solving problems involving divisibility or remainders can be greatly aided by the use of congruences. In many cases we have to be clever in the ways we use the congruences to solve problems. Keep your eye out for where this tool can be used, and enjoy the remaining problems from Professor Honsberger's course.

TEACHING PROBLEMS

John McLoughlin

Welcome to *Teaching Problems*. This regular feature of *MathemAttic* is being introduced here with the expectation that it will appear in most issues.

Teaching problems in my vocabulary refer to problems that are pedagogically effective. The value may be rooted in the elegance of a solution, a surprising "unsolvability" characteristic, hidden structural similarities to familiar problems, or counterintuitive results or .... The opening example in this issue brings attention to a problem that offers a variety of approaches to solution.

Good teaching problems are not usually identified until our experience awakens us to an insight exposing richness that was not immediately evident. Intellectual curiosity is sparked with a desire to revisit a problem, share insights with others, delve into related patterns, or possibly transfer the conceptual aspect to another mathematical area. The common element is that something about a problem is perceived to be extraordinary.

My teaching problems are not likely to be yours. Hence, the value of this feature will be enriched with contributions from people like yourselves, the readers of this introductory note. Please feel free to send along your ideas and/or contribute a piece to be considered for publication in a future issue. Correspondence can be directed to `MathemAttic@cms.math.ca`.

TEACHING PROBLEMS

No. 1

John McLoughlin

The Piggy Bank Problem

> A piggy bank contains 100 coins worth a total of \$9.50. The coins consist of only nickels, dimes, and quarters. What is the maximum number of quarters that the piggy bank could contain?

The piggy bank problem is a good teaching problem, in my opinion, as it is easy to understand while being widely accessible. This latter feature invites a range of entry points leading to a variety of approaches. Advanced mathematical ideas can be introduced through subsequent discussion. Terminology such as lattice points, the greatest integer function, and upper bound figure into different methods. Further, illustration of the use of a system of two equations to resolve a problem with three unknowns along with the interplay of graphical and algebraic representations add to the merits of this particular example. In addition, the problem draws attention to the under-utilized problem solving strategy of *Considering an Extreme*.

Readers are challenged to consider the above problem and attempt to solve it in at least two different ways before proceeding to read the solutions offered here. Additional comments on the problem are welcomed.

Solution A

Letting N, D, and Q represent the numbers of nickels, dimes, and quarters respectively, we set up two equations - a value equation and a quantity equation:

$$5N + 10D + 25Q = 950 \text{ (value)}$$

dividing by 5, we get

$$N + 2D + 5Q = 190, \tag{1}$$

$$N + D + Q = 100 \text{ (quantity)}. \tag{2}$$

$(1) - (2)$ gives $D + 4Q = 90$. Since D, Q are positive integers and we wish to maximize Q, we let $Q = 22$. (It follows that $D = 2$ and $N = 76$).

Solution B

\$9.50 = the value of 38 quarters.

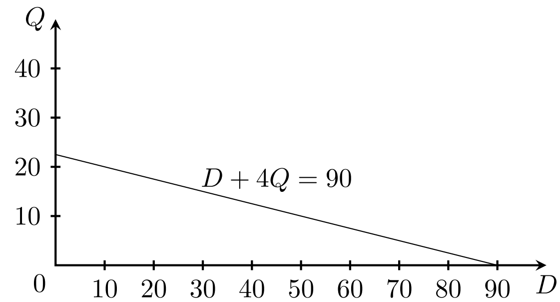
Some of these quarters can be exchanged for 5 nickels to increase the total number of coins. Each exchange increases the coin count by 4: $38, 42, 46, \ldots$ until being forced to stop at 98. By then 15 quarters have been exchanged for 75 nickels, leaving one with 23 quarters and 75 nickels. In order to get 2 more coins, another quarter is exchanged for 2 dimes and 1 nickel. This leaves 22 quarters (as many as possible).

Solution C

The 100 coins are worth at least \$5 because that's the value of 100 nickels. Hence, the remaining \$4.50 or 450¢ is made up by adding 5¢ for each dime and 20¢ for each quarter. The problem can be reframed as: Maximize Q such that $5D + 20Q = 450$ where D, Q are the respective numbers of dimes and quarters. (This is equivalent to saying $D + 4Q = 90$ as in Solution A.) Here, the maximum value of Q is the whole number (or greatest integer) portion of $\frac{450}{20}$, namely 22.

Solution D

The essence of the solution is the same as in either Solution A or C. That is, an equation with D and Q is formed. Then the line $D + 4Q = 90$ or $5D + 20Q = 450$ is graphed in the first quadrant, as shown, with D and Q as the respective axes. The lattice point with the largest value of Q gives us the result. (A lattice point is a point with integer coordinates.)



Solution E

The most common approach is a form of trial and error that does not connect the underlying algebraic or graphical concepts of the preceding solutions. It is important to acknowledge that the solver has usually gained insight into the intricacies of the problem situation without formalizing them in mathematical notation. Exposure to various other forms of solution will validate the result while offering an invitation to conceptually grasp the problem on another level.

Closing Comments

It is hoped that the variety of solutions shared here have enhanced your appreciation of the merits of such a problem. Additional comments on the problem are welcomed via email `johngm@unb.ca`. Finally, anyone who wishes to revisit one or more of the methods here may apply the same thinking to the original problem statement with a notable change. What would be the maximum number of nickels or dimes (rather than quarters) to meet the required condition of having such a collection of 100 coins with a total value of \$9.50?

OLYMPIAD CORNER

No. 372

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **July 15, 2019**.

OC426. Find all pairs of natural numbers a and k such that for every positive integer n relatively prime to a, the number $a^{k^n+1} - 1$ is divisible by n.

OC427. In a scalene triangle ABC, $\angle C = 60^\circ$ and Ω is its circumcircle. On the angle bisectors of $\angle A$ and $\angle B$ take two points A' and B', respectively such that $AB' \parallel BC$ and $BA' \parallel AC$. The line $A'B'$ intersects Ω at points D and E. Prove that triangle CDE is isosceles.

OC428.

(a) Prove that there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f$, $f \circ f = g \circ g$ and $\forall x \in \mathbb{R}\ f(x) \neq g(x)$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f \circ g = g \circ f$ and $\forall x \in \mathbb{R}\ f(x) \neq g(x)$, then $\forall x \in \mathbb{R}\ (f \circ f)(x) \neq (g \circ g)(x)$.

OC429. Let $A \in \mathcal{M}_n(\mathbb{C})$ $(n \geq 2)$ with $\det A = 0$ and let A^* be its adjoint. Prove that $(A^*)^2 = \mathrm{tr}(A^*)A^*$, where $\mathrm{tr}(A^*)$ is the trace of the matrix A^*.

OC430.

(a) Prove that for any choice of n rational numbers a_i/b_i from the interval (0,1) with distinct pairs (a_i, b_i) of positive integers, the sum of the denominators is at least $\dfrac{2\sqrt{2}}{3} \cdot n^{\frac{3}{2}}$.

(b) Prove that if we add the restriction that the rational numbers are distinct, then the sum of the denominators is at least $2 \cdot \left(\dfrac{2}{3}n\right)^{\frac{3}{2}}$.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC426. Déterminer tous les couples de nombres naturels a et k tels que, pour tout entier positif n relativement premier avec a, le nombre $a^{k^n+1}-1$ soit divisible par n.

OC427. Soit ABC un triangle scalène tel que $\angle C = 60^\circ$ et soit Ω son cercle circonscrit. Les points A' et B' se trouvent sur les bissectrices de $\angle A$ et $\angle B$, respectivement, de façon à ce que $AB' \parallel BC$ et $BA' \parallel AC$. La ligne $A'B'$ intersecte Ω en D et E. Démontrer que le triangle CDE est isocèle.

OC428.

(a) Démontrer qu'il existe des fonctions $f : \mathbb{R} \to \mathbb{R}$ et $g : \mathbb{R} \to \mathbb{R}$ telles que $f \circ g = g \circ f$, $f \circ f = g \circ g$ et, pour tout $x \in \mathbb{R}$, $f(x) \neq g(x)$.

(b) Démontrer que si $f : \mathbb{R} \to \mathbb{R}$ et $g : \mathbb{R} \to \mathbb{R}$ sont des fonctions continues telles que $f \circ g = g \circ f$ et $f(x) \neq g(x) \forall x \in \mathbb{R}$, alors, $(f \circ f)(x) \neq (g \circ g)(x)\ \forall x \in \mathbb{R}$.

OC429. Soit $A \in \mathcal{M}_n(\mathbb{C})$ $(n \geq 2)$ telle que dét $A = 0$ et soit A^* son adjointe. Démontrer que $(A^*)^2 = \mathrm{tr}(A^*)A^*$, où $\mathrm{tr}(A^*)$ est la trace de la matrice A^*.

OC430.

(a) Démontrer que pour tout choix de n nombres rationnels a_i/b_i dans l'intervalle $(0, 1)$, avec des paires distinctes d'entiers positifs (a_i, b_i), la somme des dénominateurs est au moins $\dfrac{2\sqrt{2}}{3} \cdot n^{\frac{3}{2}}$.

(b) Démontrer que si on oblige les nombres à être distincts, alors la somme des dénominateurs est au moins $2 \cdot \left(\dfrac{2}{3}n\right)^{\frac{3}{2}}$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(4), p. 145–146; 44(5), p.195–196.

OC376. Let m be a positive integer and a and b be distinct positive integers strictly greater than m^2 and strictly less than m^2+m. Find all integers d such that $m^2 < d < m^2+m$ and d divides ab.

Originally Problem 4 of Day 2 of the 2016 Spain Mathematical Olympiad.

We received 4 correct submissions. We present the solution by Oliver Geupel.

The numbers $d=a$ and $d=b$ obviously satisfy the required conditions. We show that there are no further solutions. Let d be any integer such that

$$m^2 < d < m^2+m, \quad d \neq a \quad \text{and} \quad d \neq b.$$

Then $0 < |a-d| < m$, and $0 < |b-d| < m$, that is,

$$\gcd(a,d) < m \quad \text{and} \quad \gcd(b,d) < m.$$

Hence, $\gcd(ab,d) < m^2 < d$, which implies that d does not divide ab.

OC377. Prove that $x-\frac{1}{x}+y-\frac{1}{y}=4$ has no solutions over the rationals.

Originally Problem 3 of Day 1 of the 2016 Final Round Korea.

We received 4 correct submissions. We present a solution based on the submissions of Mohammed Aassila and C.R. Pranesachar.

We prove the statement by contradiction. Assume, that x, y are rationals. Without loss of generality, assume that $x>0$ and $y>0$. Let $x=a/b$, and $y=c/d$ with $a,b,c,d \in \mathbb{N}$, and $\gcd(a,b)=1$, $\gcd(c,d)=1$. The equation is equivalent to

$$cd(a^2-b^2)+ab(c^2-d^2)=4abcd. \tag{1}$$

We obtain $ab \mid cd(a^2-b^2)$, and because $\gcd(a,b)=1$, we have $ab \mid cd$. Similarly, $cd \mid ab$. Thus $ab=cd$. Since $\gcd(a,b)=1$ and $\gcd(c,d)=1$, there exists pairwise relatively prime numbers m,n,p,q such that $a=mp$, $c=mq$, $b=nq$, and $d=np$. The equation (1) becomes

$$(p^2+q^2)(m^2-n^2)=4mnpq. \tag{2}$$

Next, we show that there are no non-zero natural numbers m,n,p,q satisfying (2).

As before, because $\gcd(m,n)=1$ we have $\gcd(m^2-n^2,mn)=1$. Indeed, let t be a prime common divisor of m^2-n^2 and mn. Then either $t \mid m$, or $t \mid n$. If $t \mid m$

then $t \mid -(m^2 - n^2) + m^2 = n^2$, and $t \mid n$. If $t \mid n$, then $t \mid m^2$, and $t \mid m$. It follows that $t = 1$.

In order for equation (2) to hold, we must have an integer number, k, such that

$$p^2 + q^2 = mnk \quad \text{and} \quad 4pq = (m^2 - n^2)k. \tag{3}$$

This implies

$$\begin{aligned} 8(p^2+q^2)^2 &= 4(p^2+q^2)^2 - 16p^2q^2 + 4(p^2+q^2)^2 + 16p^2q^2 \\ &= 4(p^2-q^2)^2 + 4m^2n^2k^2 + (m^2-n^2)^2k^2 \\ &= 4(p^2-q^2)^2 + (m^2+n^2)^2k^2. \end{aligned} \tag{4}$$

We claim that there are no integers m, n, p, q and k that satisfy (4). We use the fact that the maximum power of 2 in a perfect square is always an even number. The maximum power of 2 in $8(p^2 + q^2)^2$ is odd, due to factor 8, where as the maximum power of 2 in the sum of perfect squares $4(p^2 - q^2)^2 + (m^2 + n^2)^2k^2$ is even. This is a contradiction and completes the proof.

Editor's Comment. C.R. Pranesachar pointed out that relations (3) cannot hold because u^2+v^2 and u^2-v^2 are discordant forms, in other words, for any non-zero integers u and v, $u^2 + v^2$ and $u^2 - v^2$ cannot be both squares. In our case we can select $u = 2(p^2 + q^2)$ and $v = 4pq$. One reference on concordant-discordant forms is *History of the Theory of Numbers*, L.E. Dickson, Carnegie Institution of Washington, Washington, Vol II, chapter XVI, page 473, 1920.

OC378. Define a sequence $\{a_n\}$ by

$$S_1 = 1,\ S_{n+1} = \frac{(2+S_n)^2}{4+S_n}(n = 1,\ 2,\ 3,\ \ldots),$$

where S_n the sum of the first n terms of sequence $\{a_n\}$. For any positive integer n, prove that

$$a_n \geq \frac{4}{\sqrt{9n+7}}.$$

Originally Problem 5 of Day 2 of the 2016 China Girls Mathematical Olympiad.

We received 6 correct submissions. We present two solutions.

Solution 1, by Mohammed Aassila

First $a_1 = S_1 = 1$ and $a_{n+1} = S_{n+1} - S_n = \dfrac{4}{4+S_n}$ for all integers $n \geq 1$. Thus

$$\frac{4}{a_{n+1}} - \frac{4}{a_n} = 4 + S_n - 4 - S_{n-1} = S_n - S_{n-1} = a_n \quad \text{and}$$

$$\frac{4}{a_{n+1}} = a_n + \frac{4}{a_n}. \tag{1}$$

Apply AM-GM inequality to (1) to find that $4/a_{n+1} \geq 4$, and $1 \geq a_{n+1}$ for all integers $n \geq 1$. Square both sides of (1) to get that

$$\frac{16}{a_{n+1}^2} = \frac{16}{a_n^2} + a_n^2 + 8 \quad \text{for all} \quad n \geq 1. \tag{2}$$

Apply successively (2) to find

$$\frac{16}{a_{n+1}^2} = \frac{16}{a_1^2} + a_1^2 + \cdots + a_n^2 + 8n \quad \text{for all} \quad n \geq 1. \tag{3}$$

Use $1 \geq a_n$ and $a_1 = 1$ in (3) to find

$$\frac{16}{a_{n+1}^2} \leq 16 + n + 8n = 9(n+1) + 7.$$

Hence $a_n \geq 4/\sqrt{9n+7}$ for all $n \geq 2$. Since $a_1 = 1$, this is true for $n = 1$.

Solution 2, by IISER Mohali Problem Solving Group.

Clearly, $a_1 = 1 \geq \frac{4}{\sqrt{16}} = 1$. We define a new sequence $\{T_n\}$ by $T_n = 4 + S_n$. Make note that $T_{n+1} - T_n = 4/T_n = a_{n+1}$. Thus, proving $a_n \geq 4/\sqrt{9n+7}$ for all $n \geq 1$ reduces to showing

$$\frac{4}{\sqrt{9(n+1)+7}} \leq \frac{4}{T_n}, \quad \text{or} \quad T_n^2 \leq 9n + 16$$

for all $n \geq 1$.

We prove this by induction. The base case holds trivially: $9 \cdot 1 + 16 = 25 \geq 25 = T_1^2$. Assume that for some integer $k > 1$, the following holds:

$$T_k^2 \leq 9k + 16. \tag{1}$$

We know that $T_{k+1} - T_k = 4/T_k$, and hence,

$$T_{k+1}^2 = T_k^2 + 8 + 16/T_k^2.$$

We add $8 + 16/T_k^2$ to both sides of inequality (1), and we get

$$T_k^2 + 8 + \frac{16}{T_k^2} \leq 9k + 8 + \frac{16}{T_k^2} + 16.$$

Moreover, $T_k = 4 + S_k \geq 4$ hence, we have $16/T_k^2 \leq 1$ and

$$T_{k+1}^2 = T_k^2 + 8 + \frac{16}{T_k^2} \leq 9k + 8 + \frac{16}{T_k^2} + 16 \leq 9(k+1) + 16.$$

This solves the problem.

OC379. Let $n \geq 3$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$, such that

$$\frac{1}{1+a_1^4} + \frac{1}{1+a_2^4} + \cdots + \frac{1}{1+a_n^4} = 1.$$

Prove that:

$$a_1 a_2 \cdots a_n \geq (n-1)^{\frac{n}{4}}.$$

Originally Problem 5 of the 2016 Macedonia National Olympiad.

We received 4 correct submissions. We present a solution that was submitted independently by Paolo Perfetti and Sundara Narasimhan.

Let $x_k = 1/(1+a_k^4)$. Then $x_1 + \cdots + x_n = 1$ and

$$a_k = \left(\frac{1-x_k}{x_k}\right)^{\frac{1}{4}}.$$

The inequality becomes

$$\frac{1-x_1}{x_1} \times \frac{1-x_2}{x_2} \times \cdots \times \frac{1-x_n}{x_n} \geq (n-1)^n,$$

which is equivalent to

$$\frac{x_2 + \cdots + x_n}{x_1} \times \frac{x_3 + \cdots + x_n + x_1}{x_2} \times \cdots \times \frac{x_1 + \cdots + x_{n-1}}{x_n} \geq (n-1)^n. \qquad (1)$$

Apply AM-GM inequality to obtain

$$x_2 + \cdots + x_n \geq (n-1) \times (x_2 \cdots x_n)^{\frac{1}{n-1}}.$$

Consequently, the left hand side of (1) is greater than or equal to

$$\frac{(n-1)(x_2 \cdots x_n)^{\frac{1}{n-1}}}{x_1} \times \frac{(n-1)(x_3 \cdots x_n x_1)^{\frac{1}{n-1}}}{x_2} \times \cdots \times \frac{(n-1)(x_1 \cdots x_{n-1})^{\frac{1}{n-1}}}{x_n},$$

which equals $(n-1)^n$ establishing the inequality.

OC380. Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

Originally Problem 5 of the 2016 Canadian Mathematical Olympiad.

We received 2 submissions. We present the solution by Mohammed Aassila.

Let f be the inversion with center M and radius $r = MA$. Let $\odot ABC$ denote the circumcircle of triangle ABC.

Let $\{U, V\} = MH \cap (\odot ABC)$ with U in the arc $\widehat{ACB}$ and denote $X' = f(X)$. Then,

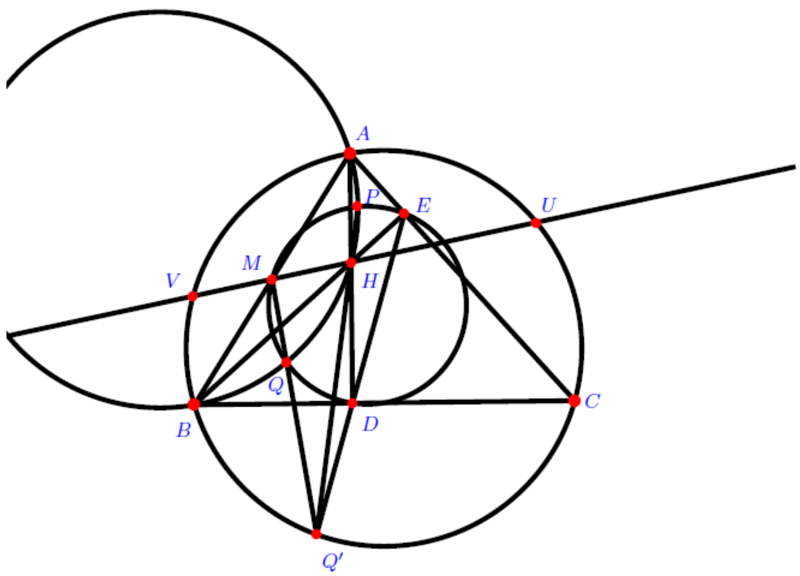
$$MU \cdot MV = MA \cdot MB = MH \cdot MU \implies f(H) = U,$$

thus we get $f(\odot AHB) = \odot ABC$. Furthermore P', E, D, Q' are collinear with $P', Q' \in \odot ABC$. Hence, $\{Q'\} = MQ \cap DE$.

Now, consider g as the composition of an inversion with center H and radius $\sqrt{HA \cdot HD}$, and reflection with respect to H, then $g(\odot ABC) = \odot MED$ and $g(DE) = \odot ABH$. Therefore,

$$\{g(Q')\} = \odot MED \cap \odot ABH = \{P\}$$

(the latter follows since $g(Q')$ is on the same side of A with respect to MH), hence we proved that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.



OC381. The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with one copy of their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

a) Prove that there is a sequence of replacements that will make the final number equal to 2.

b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

Originally Problem 1 of the 2016 Canadian Mathematical Olympiad.

We received 3 *submissions. We present the solution by Ramanujan Srihari.*

For brevity, we use an arrow to show a replacement as described in the problem. As an example,

$$(a, b, c, d) \to (a, (b + c)/2, d) \to ((a + d)/2, (b + c)/2).$$

Further, we write $\xrightarrow{S}$ to denote a sequence S of such replacements.

For $n \geq 3$, we claim that there is a sequence L_n of such replacements which replaces the numbers $1, 2, \ldots, n$ with $n - 1$, or $(1, 2, \ldots, n) \xrightarrow{L_n} (n - 1)$. To prove this statement, we will use induction on n. The following replacements are obvious: $(1, 2, 3) \to (2, 2) \to (2)$ so that $(1, 2, 3) \xrightarrow{L_3} (2)$. Now for $k > 2$, let us assume (induction hypothesis) that there exists L_k such that $(1, 2, \ldots, k) \xrightarrow{L_k} (k - 1)$. Then we have

$$(1, 2, \ldots, k, k + 1) \xrightarrow{L_k} (k - 1, k + 1) \to (k)$$

where we replaced $k - 1$ and $k + 1$ with their average, k.

Similarly, for $n \geq 3$, we claim that there is a sequence R_n of such replacements so that $(2016-n+1, 2016-n+2, \ldots, 2016) \xrightarrow{R_n} (2016-n+2)$. One can easily prove this claim using induction, and the argument is very similar to the one given for L_n. Then we have:

a) $(1, 2, \ldots, 2016) \xrightarrow{R_{2016}} (2)$

b) $(1, 2, \ldots, 2016) \xrightarrow{L_{999}} (998, 1000, 1001, \ldots, 2016) \xrightarrow{R_{1016}} (998, 1000, 1002)$ $\to (1000, 1000) \to (1000)$

and we are done.

Generalization. It is not hard to see that if $1, 2, \ldots, m$ are written on a board, then the following numbers can be left on the board after $m - 1$ replacements:

a) 2; $(1, 2, \ldots, m) \xrightarrow{R_m} (2)$

b) $m - 1$; $(1, 2, \ldots, m) \xrightarrow{L_m} (m - 1)$

c) n, where $3 < n < m - 2$; $(1, 2, \ldots, m) \xrightarrow{L_{n-1}} (n - 2, n, n + 1, \ldots, m) \xrightarrow{R_{m-n}}$ $(n - 2, n, n + 2) \to (n, n) \to (n)$

where L_k replaces $1, 2, \ldots, k$ by $k - 1$ and R_k replaces $m - k + 1, m - k + 2, \ldots, m$ by $m - k + 2$. In other words,

$$(1, 2, \ldots, k) \xrightarrow{L_k} (k - 1) \quad \text{and} \quad (m - k + 1, m - k + 2, \ldots, m) \xrightarrow{R_k} (m - k + 2).$$

On the other hand, the number m cannot be "obtained" since the average of any two numbers among $1, 2, \ldots, m$ is always *less* than m. Similarly 1 cannot be obtained through such replacements.

OC382. There are $n > 1$ cities in a country and some pairs of cities are connected by two-way non-stop flights. Moreover, every two cities are connected by a unique route (possibly with stopovers). A mayor of every city X counted the number of labelings of the cities from 1 to n so that every route beginning with X has the rest of the cities on that route occurring in ascending order. Every mayor, except one, noticed that the resulting number of their labelings is a multiple of 2016. Prove that the last mayor's number of labelings is also a multiple of 2016.

Originally Problem 6 (Grade 11) of Day 2 of the 2016 AllRussian Olympiad.

We received 1 *incomplete submission.*

OC383. Let ABC be a triangle. Let r and s be the angle bisectors of $\angle ABC$ and $\angle BCA$, respectively. The points E in r and D in s are such that $AD \| BE$ and $AE \| CD$. The lines BD and CE cut each other at F. The point I is the incenter of ABC. Show that if A, F and I are collinear, then $AB = AC$.

Originally Problem 1 of Day 1 of the 2016 Brazil National Olympiad.

We received 6 *submissions. We present* 2 *solutions.*

Solution 1, by IISER Mohali Problem solving group.

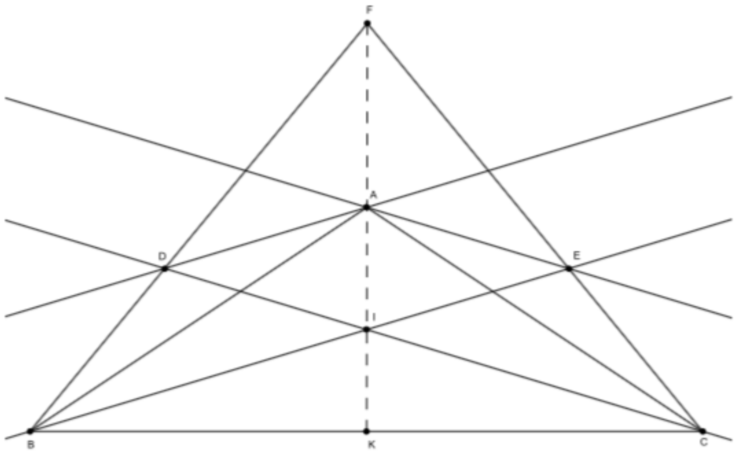
Let AI meet BC at K. Apply Ceva's Theorem to $\triangle FBC$, where the cevians FK, BE, CD meet at I. Then, we get

$$\frac{BK}{KC} \cdot \frac{CE}{EF} \cdot \frac{FD}{DB} = 1.$$

Since $\dfrac{FD}{DB} = \dfrac{FA}{AI}$ and $\dfrac{CE}{EF} = \dfrac{AI}{FA}$, then we see immediately that

$$BK = KC,$$

which implies that the angle bisector of $\triangle ABC$ is its median, which shows that $AB = AC$.



Solution 2, by Ivko Dimitrić.

We use barycentric coordinates in reference to triangle ABC. Let a, b and c be the side lengths of this triangle and $q = AI$, $r = BI$ and $s = CI$ be the internal angle bisectors at vertices $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$, respectively. Each of them is a cevian that passes through one of the vertices and the incenter $I(a:b:c)$. Since an equation of a general line is $ux + vy + wz = 0$, the equations of these bisectors are respectively given by

$$q: \ cy - bz = 0, \qquad r: \ cx - az = 0, \qquad s: \ bx - ay = 0.$$

Since the line at infinity has an equation $x + y + z = 0$, the points at infinity on bisectors r and s are $r_\infty = (-a : a + c : -c)$ and $s_\infty = (a : b : -a - b)$, respectively. Hence, the line through A parallel to r has an equation

$$\begin{vmatrix} 1 & 0 & 0 \\ -a & a+c & -c \\ x & y & z \end{vmatrix} = 0 \quad \Longleftrightarrow \quad cy + (a+c)z = 0.$$

This line intersects bisector s at point $D\Big(a : b : -\frac{bc}{a+c}\Big)$. Likewise, the line through A parallel to s has an equation

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & -a-b \\ x & y & z \end{vmatrix} = 0 \quad \Longleftrightarrow \quad (a+b)y + bz = 0$$

and intersects bisector r at point $E\Big(a : -\frac{bc}{a+b} : c\Big)$. Consequently, the line DB has an equation

$$\begin{vmatrix} a & b & -\frac{bc}{a+c} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = 0 \quad \Longleftrightarrow \quad \frac{bc}{a+c}x + az = 0$$

and the line EC an equation

$$\begin{vmatrix} a & -\frac{bc}{a+b} & c \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = 0 \quad \Longleftrightarrow \quad \frac{bc}{a+b}x + ay = 0.$$

These two lines intersect at point $F\Big(-a : \frac{bc}{a+b} : \frac{bc}{a+c}\Big)$. The points A, F and I are then collinear if and only if F belongs to bisector q iff

$$\frac{bc^2}{a+b} = \frac{b^2c}{a+c}.$$

This in turn reduces to

$$c(a+c) = b(a+b) \quad \Longleftrightarrow \quad a(b-c) + b^2 - c^2 = (a+b+c)(b-c) = 0.$$

Since $a + b + c \neq 0$, it follows that $c = AB$ is equal to $b = AC$.

OC384. Solve the equation $xyz + yzt + xzt + xyt = xyzt + 3$ over the set of natural numbers.

Originally Problem 3 of the 2016 Macedonia National Olympiad. We received 8 submissions, of which 6 were correct and complete.

We present the solution by Ivko Dimitrić, slightly modified.

The only solutions are the following quadruples for (x, y, z, t):

$$(1, 1, 1, 1),\ (2, 3, 9, 17),\ (2, 3, 7, 39),\ (3, 3, 4, 11),\ (3, 3, 5, 7)$$

and other quadruples obtained from these by arbitrary permutations of x, y, z and t, given that the equation is completely symmetric in the four variables. It is easily checked that these quadruples satisfy the equation. Conversely, we prove that any solution must be one of the listed or obtained from it by a permutation. Let $s = xyzt,\ \ u = yzt = s/x$ and assume without loss of generality that $x \le y \le z \le t$. If $x \ge 4$, then $y, z, t \ge 4$ and we would have

$$s + 3 = \frac{s}{x} + \frac{s}{y} + \frac{s}{z} + \frac{s}{t} \le 4 \cdot \frac{s}{4} = s,$$

which is not possible. Hence, $x \le 3$.

Case 1. If $x = 1$, from the given condition we would have $yz + zt + yt = 3$, and since the numbers involved are positive integers, it follows that $yz = zt = yt = 1$, implying $y = z = t = 1$, so we get one solution $(1, 1, 1, 1)$.

Case 2. If $x = 2$, the equation reduces to

$$2(yz + zt + yt) = yzt + 3 \tag{1}$$

or

$$u + 3 = 2\left(\frac{u}{y} + \frac{u}{z} + \frac{u}{t}\right). \tag{2}$$

Clearly, $u = yzt$ must be odd, so all three of y, z, t are odd. If $7 \le y \le z \le t$ from (2) we would have

$$u + 3 \le 2 \cdot 3 \cdot \frac{u}{7} = \frac{6}{7}u,$$

which is not possible. If $y = 5$ and $7 \le z \le t$, from (2) we get

$$u + 3 \le 2\left(\frac{u}{5} + \frac{2u}{7}\right) = \frac{34}{35}u,$$

which is not possible. Moreover, if $y = 5$ and $z = 5$, for any choice of $t \ge 5$ the left hand side of (1) is divisible by 5, whereas the right hand side is not, contradiction! Thus, $y = 3$ and the equation reduces to

$$6(z + t) = zt + 3 \iff (z - 6)(t - 6) = 33,$$

Since $3 \le z \le t$, then $-3 \le z - 6 \le t - 6$ and we get $(z - 6, t - 6) \in \{(1, 33), (3, 11)\}$, i.e. $(z, t) \in \{(7, 39), (9, 17)\}$. So, we get the solutions $(2, 3, 7, 39)$ and $(2, 3, 9, 17)$.

Case 2. If $x = 3$, the equation becomes

$$3(yz + zt + yt) = 2yzt + 3, \quad \text{or} \quad 2u + 3 = 3\left(\frac{u}{y} + \frac{u}{z} + \frac{u}{t}\right).$$

Thus, $3 \mid yzt$ and one of the numbers y, z or t must be divisible by 3. If $y \geq 4$, then one of z or t is ≥ 6 and the other one is ≥ 4. Then,

$$2u + 3 \leq 3\left(\frac{u}{4} + \frac{u}{4} + \frac{u}{6}\right) = 2u,$$

which is not possible. Hence, $y = 3$ and the equation is further reduced to

$$3(z + t) = zt + 1 \iff (z - 3)(t - 3) = 8.$$

Since $3 \leq z \leq t$, then $0 \leq z - 3 \leq t - 3$ and we get $(z - 3, t - 3) \in \{(1, 8), (2, 4)\}$, i.e. $(z, t) \in \{(4, 11), (5, 7)\}$. So, we get the solutions $(3, 3, 4, 11)$ and $(3, 3, 5, 7)$.

Therefore, all the solutions are those listed and their permutations.

OC385. A subset $S \subset \{0, 1, 2, \cdots, 2000\}$ satisfies $|S| = 401$. Prove that there exists a positive integer n such that there are at least 70 positive integers x such that $x, x + n \in S$.

Originally Problem 8 of Day 2 of the 2016 Korea National Olympiad.

We received no submissions.

APPLICATION OF CORRESPONDENCE: Counterparts

Olga Zaitseva

*Part 1 of this article entitled "Problem Solving Toolkit: Correspondence" appeared in **Crux** 45(2), p. 82–85.*

The idea of a *counterpart* is closely related to symmetry and therefore connected to one-to-one correspondence. Suppose we want to determine some parameter of the given set (to calculate the number of elements, to find the sum of all the elements, to determine the probability of an event, etc). If the direct approach is not applicable, we can try to get the answer indirectly.

The idea is to partition the given set into sets for which the required parameter can be determined. For instance, one can find "a natural counterpart" of the given set and to consider a set which combines the given set, the counterpart and possibly some additional "leftover" set.

Problem 1 (Tournament of Towns, 1985 Fall, Seniors)
Two boys toss a coin, one tosses 10 times, while the other tosses 11 times. Which of the boys, the first or the second, has a greater chance of tossing more heads than tails?

Solution. Toss a coin n times and consider all possible outcomes of the experiment. We are interested in chances of the event "more heads than tails" or, in other words, in the number of elementary events that belong to the event. Let us introduce a reciprocal event "more tails than heads". By symmetry, these two events are equally likely.

If n is odd, the number of tails and heads cannot be the same and therefore these two events combined give us all possible outcomes. Therefore, in this case the probability of each event is $1/2$; in particular, "more heads than tails" has the probability of $1/2$. If n is even, the probability of each event is less than $1/2$, since we need to account for the event "the number of tails and heads is the same". Hence, the boy who tosses the coin 11 times has greater chances than the boy who tosses the coin 10 times. □

Problem 2 (Tournament of Towns)
In the first quadrant of Cartesian coordinate system (including x and y axes) draw 90 rays corresponding to angles of degrees $0, 1, \ldots, 90$. Determine the total sum of x-coordinates of all intersection points of these rays with the line $x + y = 100$.

Solution. Let $(x_i, y_i), i = 0, 1, 2, ..., 90$ be coordinates of the points of intersections of rays with the line $y = 100 - x$. Observe that $x_i = y_{90-i}$ for $i = 0, 1, ..., 90$. Then

$$(x_0 + \cdots + x_{90}) + (y_0 + \cdots + y_{90}) = (x_1 + y_1) + \cdots + (x_{90} + y_{90})$$
$$= 91 \times 100.$$

Hence, $x_0 + \cdots + x_{90} = 4550$. □

Problem 3 (Follow-up to one of the Tournament of Towns problems)
Twelve students did not attend the award ceremony for the winners of Tournament of Towns. They were given a second chance to pick up their book prizes from the Math Department. The books, with name tags attached, were arranged at the front desk for pick up. First came Joshua, the youngest participant. Not paying attention to the tags, he took a book at random and left with it. Every student after him took their own book if it was there; otherwise, he or she picked up a book at random. Jenny was the last one to come. What are chances that she picked up the book assigned to her?

Solution. Let us partition the set of the books into three sets: the book assigned to Joshua, the book assigned for Jenny, and the set consisting of all the remaining books.

If Joshua takes his book, then every other student (including Jenny) would take their own book.

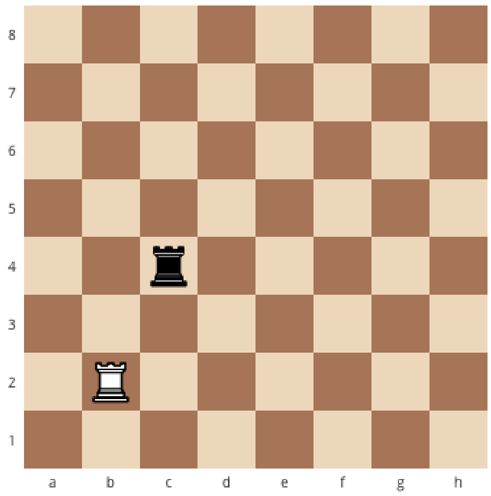
If Joshua takes Jenny's book, then every other student would take their own book while Jenny would take Joshua's book. Since Joshua has equal chances to choose either his or Jenny's book (given that he chooses a book from the first or the second set), Jenny also has equal chances to get her own book or Joshua 's book.

Suppose Joshua takes a book assigned for student X. Then every student who comes before X will take their own book, so Joshua's and Jenny's books will remain. Suppose student X chooses the book assigned for student Y. Observe that from Jenny's point of view, it does not matter who (Joshua or X) takes the book assigned for Y. Thus, we can assume that student X and Joshua exchange their roles (X becomes Joshua while Joshua becomes X) so that X takes their book while Joshua takes the book assigned for Y. This means that sooner or later Joshua has to choose either his book or Jenny's book and it is not important what happened before this point. At this point, Joshua's chances to pick up his own book are 50-50, so Jenny's chances to pick up her own book are also 50-50. □

Practice Problems (for Correspondence and Counterparts)

Problem 4 (Tournament of Towns, 2009 Fall, Juniors)
There are forty weights: $1, 2, \ldots, 40$ grams. Ten even weights are placed on the left side of a balance scale while ten odd weights are placed on the right side. It so happens that the left and the right sides are in equilibrium. Prove that one of the sides contains two weights that differ by exactly 20 grams.

Problem 5 (Tournament of Towns, 2012, Spring, O-level, Seniors)
Two players take turns moving rooks on an 8×8 chessboard. The first player moves the white rook while the second player moves the black rook. Initially, the white and the black rooks are positioned on b2 and c4, respectively (see the diagram below). Rooks move vertically or horizontally any number of squares. At any time, the rook must not land on a square which is under attack by the second rook and it must not land on a square already visited by either rook. The player who can not make a move loses the game. Which of the players has a winning strategy and what is the winning strategy?



Problem 6 (Tournament of Towns, 2012 Spring, O-level, Seniors)
Consider points of intersection of the graphs of $y = \cos x$ and $x = 100\cos(100y)$ for which both coordinates are positive. Let a and b be the sums of their x-coordinates and y-coordinates, respectively. Determine the value of a/b.

Problem 7 (Tournament of Towns 2019, Spring, Juniors A-level)
2019 point-size grasshoppers sit on a line. At each move one of the grasshoppers jumps over another one and lands at the point the same distance away from it. Jumping only to the right, the grasshoppers are able to position themselves so that two of them are exactly 1 mm apart. Prove that the grasshoppers can achieve the same jumping only to the left and starting from the same position.

Problem 8 (Tournament of Towns 2018, Fall Round, Seniors O-level)
Pete is placing 500 kings on a 100×50 chessboard so that none of them attack one another. Basil is placing 500 kings on white squares of a 100×100 chessboard so that none of them attack one another. Which of the boys has more ways to place the kings? (A king attacks all adjacent squares).

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2019**.

4431. *Proposed by Şefket Arslanagić.*

Let $x, y \geq 0$ and $x + y = 2$. Prove that

$$\sqrt{x^2+8} + \sqrt{y^2+8} + \sqrt{xy+8} \geq 9.$$

4432. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let $x_1, x_2, \ldots, x_k \geq 0$, $n \in \mathbb{N}$ be such that

$$(x_1+x_2)(x_2+x_3)\ldots(x_k+x_1) \geq (x_1+1)(x_2+1)\ldots(x_k+1)$$

Prove that

$$\frac{x_1^{n+1}}{x_2+1} + \frac{x_2^{n+1}}{x_3+1} + \cdots + \frac{x_k^{n+1}}{x_1+1} \geq \frac{x_1^n}{x_2+1} + \frac{x_2^n}{x_3+1} + \cdots + \frac{x_k^n}{x_1+1}.$$

4433. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a tetrahedron and let M be an interior point of $ABCD$. If $AB \geq AC \geq AD \geq BC \geq BD \geq CD$, then prove that

$$MA + MB + MC + MD < AB + AC + AD.$$

4434. *Proposed by Michel Bataille.*

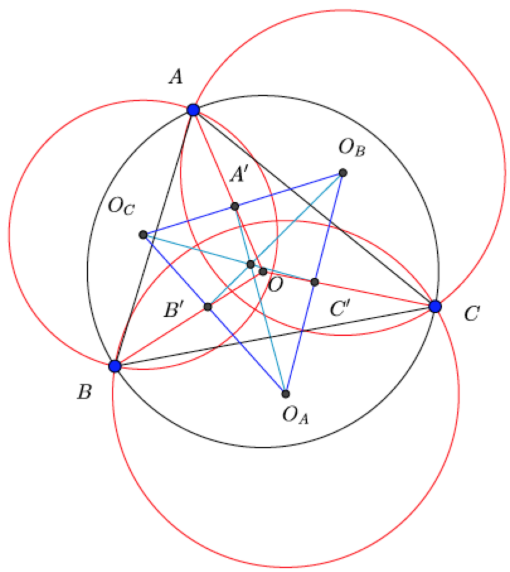
Let ABC be a scalene triangle with $BC = a, CA = b, AB = c$. The internal bisectors of $\angle A, \angle B, \angle C$ meet the opposite sides at D, E, F and the perpendicular bisectors of AD, BE, CF intersect the lines BC, CA, AB at U, V, W, respectively. Prove that the sum

$$(b^2-c^2)PU^2 + (c^2-a^2)PV^2 + (a^2-b^2)PW^2$$

is independent of the point P chosen in the plane of ABC and find its value in terms of a, b, c.

4435. *Proposed by Dao Thanh Oai.*

Let ABC be a triangle with circumcenter O. Let C_{BC}, C_{CA}, C_{AB} be three arbitrary circles through B and C, C and A, A and B with centers O_A, O_B and O_C, respectively. Suppose OA, OB and OC meet O_BO_C, O_CO_A and O_AO_B at A', B' and C', respectively. Show that O_AA', O_BB' and O_CC' are concurrent.



4436. *Proposed by Leonard Giugiuc.*

Find all positive solutions to the following equation:

$$(x+y+z)^2(10x+4y+z) = \frac{243xyz}{2}.$$

4437. *Proposed by Leonard Giugiuc and Shafiqur Rahman.*

Let a be a real number with $a > 1$. Find $\lim_{n\to\infty}(n\sqrt[n]{an} - n - \ln n)$.

4438. *Proposed by Michel Bataille.*

Let a, b, c be nonzero distinct real numbers and let A be the matrix

$$\begin{pmatrix} 2(a^2-bc) & b^2-a^2 & (c+b)(c-a) \\ (a+c)(a-b) & 2(b^2-ca) & c^2-b^2 \\ a^2-c^2 & (b+a)(b-c) & 2(c^2-ab) \end{pmatrix}.$$

Prove that A is invertible.

4439★. *Proposed by Cristi Savescu.*

Let M be a set of at least six points in the plane such that any of its six point subsets represents, in a certain order, the vertices of two triangles with the same centroid. Prove that M has exactly six points.

4440. *Proposed by Mihaela Berindeanu.*

Let H be the orthocenter of triangle ABC and let AA_1, BB_1, CC_1 be the altitudes; define the points X to be the intersection of AA_1 with B_1C_1 and Y to be where the perpendicular from X to AC intersects AB. Prove that the line YA_1 passes through the midpoint of BH.

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4431. *Proposé par Şefket Arslanagić.*

Soient $x, y \geq 0$ tels que $x + y = 2$. Démontrer que

$$\sqrt{x^2+8} + \sqrt{y^2+8} + \sqrt{xy+8} \geq 9.$$

4432. *Proposé par Marian Cucoaneş et Marius Drăgan.*

Soient $x_1, x_2, \ldots, x_k \geq 0$, $n \in \mathbb{N}$ tels que

$$(x_1 + x_2)(x_2 + x_3) \ldots (x_k + x_1) \geq (x_1 + 1)(x_2 + 1) \ldots (x_k + 1)$$

Démontrer que

$$\frac{x_1^{n+1}}{x_2+1} + \frac{x_2^{n+1}}{x_3+1} + \cdots + \frac{x_k^{n+1}}{x_1+1} \geq \frac{x_1^n}{x_2+1} + \frac{x_2^n}{x_3+1} + \cdots + \frac{x_k^n}{x_1+1}.$$

4433. *Proposé par Leonard Giugiuc.*

Soit $ABCD$ un tétraèdre et soit M un point dans son intérieur. Démontrer que si $AB \geq AC \geq AD \geq BC \geq BD \geq CD$ alors

$$MA + MB + MC + MD < AB + AC + AD.$$

4434. *Proposé par Michel Bataille.*

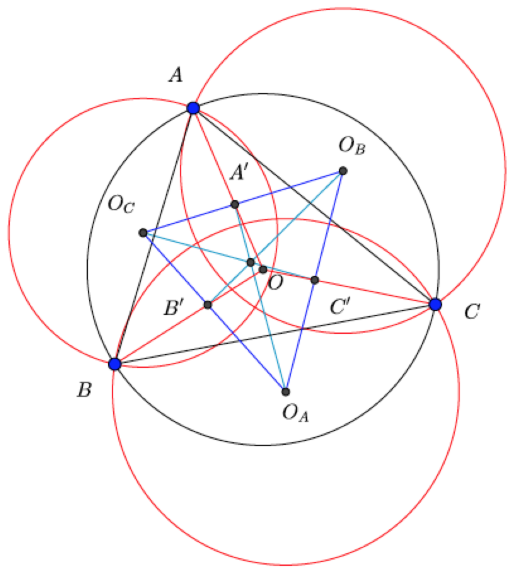
Soit ABC un triangle scalène où $BC = a, CA = b$ et $AB = c$. Les bissectrices de $\angle A, \angle B$ et $\angle C$ rencontrent les côtés opposés en D, E et F, respectivement ; les bissectrices perpendiculaires de AD, BE et CF intersectent les lignes BC, CA et AB en U, V et W, respectivement. Démontrer que la somme

$$(b^2 - c^2)PU^2 + (c^2 - a^2)PV^2 + (a^2 - b^2)PW^2$$

est indépendante du choix de point P dans le plan; donner la valeur de cette somme en termes de a, b et c.

4435. *Proposé par Dao Thanh Oai.*

Soit ABC un triangle dont le centre du cercle circonscrit est O. Soient C_{BC}, C_{CA}, C_{AB} trois cercles arbitraires passant par B et C, C et A, puis A et B, les centres de ces cercles étant O_A, O_B et O_C, respectivement. Supposons que OA, OB et OC rencontrent O_BO_C, O_CO_A et O_AO_B en A', B' et C' respectivement. Démontrer que O_AA', O_BB' et O_CC' sont concurrents.



4436. *Proposé par Leonard Giugiuc.*

Déterminer les solutions positives à l'équation suivante:

$$(x+y+z)^2(10x+4y+z) = \frac{243xyz}{2}.$$

4437. *Proposé par Leonard Giugiuc et Shafiqur Rahman.*

Soit a un nombre réel tel que $a > 1$. Déterminer $\lim_{n\to\infty}(n\sqrt[n]{an} - n - \ln n)$.

4438. *Proposé par Michel Bataille.*

Soient a, b et c des nombres réels distincts et non nuls et soit A la matrice

$$\begin{pmatrix} 2(a^2-bc) & b^2-a^2 & (c+b)(c-a) \\ (a+c)(a-b) & 2(b^2-ca) & c^2-b^2 \\ a^2-c^2 & (b+a)(b-c) & 2(c^2-ab) \end{pmatrix}.$$

Démontrer que A est inversible.

4439★. *Proposé par Cristi Savescu.*

Soit M un ensemble d'au moins six points dans le plan, tel que pour tout sous-ensemble consistant d'exactement six points, ceux-ci, dans un certain ordre, forment les sommets de deux triangles de même centroïde. Démontrer que M consiste d'exactement six points.

4440. *Proposé par Mihaela Berindeanu.*

Soit H l'orthocentre du triangle ABC et soient AA_1, BB_1 et CC_1 les altitudes de ce même triangle. Dénotons X l'intersection de AA_1 et B_1C_1, puis Y l'endroit où la perpendiculaire de X vers AC intersecte AB. Démontrer que la ligne YA_1 passe par le mi point de BH.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(4), p. 160–163, and 44(5), p. 214–218.

4336. *Proposed by Michel Bataille.*

For non-negative integers m and n, evaluate in closed form

$$\sum_{k=0}^{n}\sum_{j=0}^{m}(j+k+1)\binom{j+k}{j}.$$

We received 7 submissions, all correct. We present the solution by Paul Bracken, modified and enhanced by the editor.

Denote the given sum by S. Since both summations are finite, they can be interchanged to yield:

$$S=\sum_{j=0}^{m}j\sum_{k=0}^{n}\binom{j+k}{j}+\sum_{k=0}^{n}k\sum_{j=0}^{m}\binom{j+k}{k}+\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{j+k}{j} \qquad (1)$$

The two formulae below are known for $m,n,r,k\in\mathbb{N}\cup\{0\}$.

$$(a)\quad \sum_{k=0}^{n}\binom{r+k}{r}=\binom{r+k+1}{r+1}, \qquad (2)$$

$$(b)\quad \sum_{k=0}^{m}\binom{n+k+1}{k+1}=\frac{m(n+1)-1}{n+2}\binom{n+m+2}{n+1}+1, \qquad (3)$$

From $(1)-(3)$ we then have

$$\begin{aligned}S&=\sum_{j=0}^{m}j\binom{j+n+1}{j+1}+\sum_{k=1}^{n}k\binom{k+m+1}{k+1}+\sum_{j=0}^{m}\binom{j+n+1}{j+1}\\&=\frac{m(n+1)-1}{n+2}\binom{m+n+2}{n+1}+1+\frac{n(m+1)-1}{m+2}\binom{m+n+2}{m+1}+1\\&\quad+\binom{m+n+2}{n+1}-1\\&=\left(\frac{m(n+1)-1}{n+2}+\frac{n(m+1)-1}{m+2}+1\right)\binom{m+n+2}{n+1}+1 \qquad (4)\end{aligned}$$

Now by straightforward but tedious computations, we have

$$\begin{aligned}&(m+2)(n+1)m-(m+2)+(m+1)(n+2)n-(n+2)+(m+2)(n+2)\\&=mn^2+m^2n+m^2+n^2+5mn+3m+3n\\&=mn(m+n+3)+m(m+n+3)+n(m+n+3)\\&=(m+n+3)(mn+m+n).\end{aligned} \tag{5}$$

Substitute (5) into (4) we finally obtain

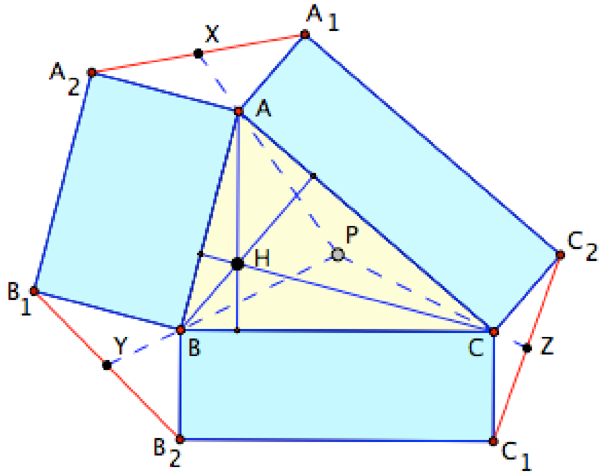
$$\begin{aligned}S&=\frac{(m+n+3)(mn+m+n)}{(m+2)(n+2)}\cdot\binom{m+n+2}{n+1}+1\\&=\frac{mn+m+n}{m+2}\cdot\binom{m+n+3}{n+2}+1.\end{aligned}$$

4337. *Proposed by Mihaela Berindeanu.*

Let h_a, h_b, h_c be the altitudes from vertices A, B, C, respectively, of triangle ABC. Erect externally on its sides three rectangles ABB_1A_2, BCC_1B_2, CAA_1C_2, whose widths are k times as long as the parallel altitudes; that is,

$$\frac{CC_1}{h_a}=\frac{AA_1}{h_b}=\frac{BB_1}{h_c}=k.$$

If X, Y, Z are the respective midpoints of the segments A_1A_2, B_1B_2, C_1C_2, prove that the lines AX, BY, CZ are concurrent.



We received 8 solutions, all of which were correct. We present the solution by Ivko Dimitrić, who notes that this problem is very similar to the ***Crux*** *problem 4258 by the same proposer, and the published solutions of that problem – in particular* *solution 2 – are readily adapted to provide the solution of the present problem. The solution given here gives an alternative approach.*

Let O be the circumcenter of triangle ABC. Then

$$\angle OAC = \frac{1}{2}(180^\circ - \angle AOC) = 90^\circ - \angle B.$$

Let E and F be the feet of the altitudes from B and C, respectively, and let EK be the straight line segment equal in length and parallel to CF that intersects AB perpendicularly. Then the quadrilateral $KFCE$ is a parallelogram and $KF \parallel AC$ so that

$$\angle FKE = \angle KEA = 90^\circ - \angle A.$$

Moreover,

$$\angle KEB = \angle EHC = \angle A$$

since $AFHE$ is cyclic. Since

$$\tan A = \frac{h_b}{AE} = \frac{h_c}{AF},$$

we have

$$\frac{EB}{EK} = \frac{h_b}{h_c} = \frac{AE}{AF},$$

so $\triangle BEK$ is similar to $\triangle EAF$ (and to $\triangle BAC$) and consequently

$$\angle BKE = \angle AFE = \angle C.$$

Since $BCEF$ is cyclic we have

$$\angle BKF = \angle BKE - \angle FKE = \angle C - (90^\circ - \angle A) = 90^\circ - \angle B = \angle OAC,$$

so $BK \parallel OA$. Further,

$$\overrightarrow{AX} = \frac{1}{2}(\overrightarrow{AA_1} + \overrightarrow{AA_2}) = \frac{k}{2}(\overrightarrow{h_b} + \overrightarrow{h_c}) = \frac{k}{2}\overrightarrow{BK}$$

and hence $AX \parallel BK \parallel OA$, meaning that O, A, X are collinear. In the same manner, each triple O, B, Y and O, C, Z consists of collinear points, which implies that the lines AX, BY and CZ are concurrent at the circumcenter O, the isogonal conjugate of the orthocenter H.

4338. *Proposed by Daniel Sitaru.*

Prove that for any triangle ABC, we have

$$2\sum\left|\cos\frac{A-B}{2}\right| \leq 3 + \sqrt{3 + 2\sum\cos(A-B)}.$$

We received five submissions, all of which are correct. We present two different solutions.

Solution 1, by Shuborno Das, modified and enhanced by the editor.

Note first that the absolute value sign in the statement is redundant since

$$-\frac{\pi}{2} < \frac{A-B}{2}, \frac{B-C}{2}, \frac{C-A}{2} < \frac{\pi}{2}.$$

Since

$$\cos(A-B) = 2\cos^2\frac{A-B}{2} - 1,$$

we have

$$2\cos^2\frac{A-B}{2} = 1 + \cos(A-B),$$

so

$$2\cos\frac{A-B}{2} = \sqrt{2+2\cos(A-B)}.$$

Similarly,

$$2\cos\frac{B-C}{2} = \sqrt{2+2\cos(B-C)} \tag{1}$$

and

$$2\cos\frac{C-A}{2} = \sqrt{2+2\cos(C-A)}.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\sum_{cyc}\sqrt{2+2\cos(A-B)} &\le \sqrt{\sum_{cyc}(1+1+1)(2+2\cos(A-B))} \\ &= \sqrt{6\sum_{cyc}(1+\cos(A-B))}. \qquad (2)\end{aligned}$$

It thus suffices to show that

$$\sqrt{6\sum_{cyc}(1+\cos(A-B))} \le 3 + \sqrt{3+2\sum_{cyc}\cos(A-B)}$$

or

$$6\sum_{cyc}(1+\cos(A-B)) \le 12 + 2\sum_{cyc}\cos(A-B) + 6\sqrt{3+2\sum_{cyc}\cos(A-B)}$$

or

$$3 + 2\sum_{cyc}\cos(A-B) \le 3\sqrt{3+2\sum_{cyc}\cos(A-B)}. \tag{3}$$

Let $S = \sum_{cyc}\cos(A-B)$. Then (3) becomes

$$(2s+3)^2 \le 9(2s+3) \iff 2s+3 \le 9 \iff s \le 3,$$

which is obviously true. From (1) – (3), the conclusion follows.

Solution 2, by Leonard Giugiuc.

Clearly, the absolute value sign in the question is redundant.

Let $u = \cos A + i\sin A$, $v = \cos B + i\sin B$, and $w = \cos C + i\sin C$.

Then $|u| = |v| = |w|$ and

$$\begin{aligned}|u+v| &= |(\cos A+\cos B)+i(\sin A+\sin B)|\\ &= \left|2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+2i\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right|\\ &= 2\left|\left(\cos\frac{A-B}{2}\right)\left(\sin\frac{C}{2}+i\cos\frac{C}{2}\right)\right|\\ &= 2\cos\frac{A-B}{2}.\end{aligned}$$

Similarly, $|v+w| = 2\cos\dfrac{B-C}{2}$ and $|w+u| = 2\cos\dfrac{C-A}{2}$.

Also,

$$\begin{aligned}|u+v+w| &= \sqrt{(\cos A+\cos B+\cos C)^2+(\sin A+\sin B+\sin C)^2}\\ &= \sqrt{3+2\sum_{cyc}\cos(A-B)}.\end{aligned}$$

Since

$$|u+v|+|v+w|+|w+u| \le |u|+|v|+w|+|u+v+w|$$

by Hlawka's Inequality, we have

$$2\sum_{cyc}\cos\frac{A-B}{2} \le 3+\sqrt{3+2\sum_{cyc}\cos(A-B)},$$

completing the proof.

4339. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Suppose ABC is an acute-angled triangle, DEF is an orthic triangle of ABC, S is the symmedian point of ABC, G is the barycenter of DEF. If D is the foot of the altitude from A and K is the point of intersection of AS and FE, prove that D, G and K are collinear.

We received four submissions, all of which were correct, and feature the solution by Ivko Dimitrić.

We use barycentric coordinates based on $\triangle ABC$ where the coordinates of the vertices are $A(1:0:0)$, $B(0:1:0)$ and $C(0:0:1)$ and a, b, c denote the corresponding side lengths. The vertices of the orthic triangle are the feet of the altitudes of $\triangle ABC$. Hence, we get

$$E\,(S_C : 0 : S_A) \qquad \text{and} \qquad F\,(S_B : S_A : 0\,),$$

where

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2},$$

according to Conway's triangle notation. Since $S_C + S_A = b^2$ and $S_B + S_A = c^2$, we get the respective normalized barycentric coordinates as

$$E\left(\frac{S_C}{b^2},\, 0,\, \frac{S_A}{b^2}\right) \qquad \text{and} \qquad F\left(\frac{S_B}{c^2},\, \frac{S_A}{c^2},\, 0\right).$$

Then the midpoint M of EF is

$$M = \frac{1}{2}(E+F) = \left(\frac{b^2S_B + c^2S_C}{2b^2c^2},\, \frac{S_A}{2c^2},\, \frac{S_A}{2b^2}\right).$$

Since the symmedian point has homogeneous coordinates $S = (a^2 : b^2 : c^2)$, the line through A and S has an equation $c^2y - b^2z = 0$. The coordinates of M clearly satisfy this equation, so M belongs to AS and, therefore, $M \equiv K$. Since the centroid G of $\triangle DEF$ belongs to the median from D, namely $DM = DK$, the points D, G, and K are collinear.

Editor's Comments. In other words Dimitrić has shown that each symmedian of a triangle passes through the midpoint of a side of the orthic triangle. This result is an immediate consequence of the "bisection property" discussed by Michel Bataille at the start of his ***Crux*** article, *"Characterizing a Symmedian"* [Vol. 43(4), April 2017, 145-150], namely that the A-symmedian is the locus of the midpoints of the antiparallels to BC bounded by the lines AB and AC. A very simple coordinate-free proof is given there.

4340. *Proposed by Digby Smith.*

Let a, b, c and d be positive real numbers such that

$$a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Show that

$$a+b+c+d \geq max\left\{4\sqrt{abcd}, \frac{4}{\sqrt{abcd}}\right\}.$$

We received 11 submissions of which all but two were correct and complete. We present a solution followed by a generalization.

Solution by Roy Barbara.

First, it suffices to prove that for any $a > 0$, $b > 0$, $c > 0$, and $d > 0$ that satisfy the hypothesis, $a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$, the following inequality holds

$$a+b+c+d \geq 4\sqrt{abcd}. \tag{1}$$

Indeed, positive numbers x, y, z, and t defined by

$$x = \frac{1}{a},\ y = \frac{1}{b},\ z = \frac{1}{c} \quad \text{and} \quad t = \frac{1}{d}$$

satisfy the hypothesis and consequently the inequality (1). However, inequality (1) for x, y, z, and t is

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq \frac{4}{\sqrt{abcd}}, \tag{2}$$

which proves the statement inequality.

Second, we establish inequality (1). Let

$$S = a + b + c + d \quad \text{and} \quad T = abc + bcd + cda + dab.$$

The following identity relates S and T:

$$S^3 - 16T = S(a + b - c - d) + 4(a + b)(c - d)^2 + 4(c + d)(a - b)^2.$$

Therefore $S^3 - 16T \geq 0$. (Ed.:This inequality is known as Maclaurin's inequality, as several solvers indicated.) The hypothesis, is equivalent to $abcd = T/S$. Hence

$$16abcd = \frac{16T}{S} \leq \frac{S^3}{S} = S^2$$

and (1) follows.

Generalization by Leonard Giugiuc, Ardak Mirzakhmedov, and Roy Barbara. Let a_1, a_2, $\ldots$, and a_n be n strictly positive numbers that satisfy

$$a_1 + \cdots + a_n = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

Then

$$a_1 + \cdots + a_n \geq \max\left\{ n \sqrt[n-2]{a_1 \ldots a_n}, \frac{n}{\sqrt[n-2]{a_1 \ldots a_n}} \right\}. \tag{3}$$

Solution for generalized inequality.

Maclaurin's inequality is a refinement of the AM-GM inequality, which you can find at `https://en.wikipedia.org/wiki/Maclaurin%27s_inequality`

Due to Maclaurin's inequality for n positive numbers,

$$(a_1 + \cdots + a_n)^{n-1} \geq n^{n-2}(a_1 \ldots a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right).$$

However, $a_1 + \cdots + a_n = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$, leading to

$$(a_1 + \cdots + a_n)^{n-1} \geq n^{n-2}(a_1 \ldots a_n)(a_1 + \cdots + a_n),$$

$$(a_1 + \cdots + a_n)^{n-2} \geq n^{n-2}(a_1 \ldots a_n),$$

$$a_1 + \cdots + a_n \geq n \sqrt[n-2]{a_1 \ldots a_n}. \tag{4}$$

Denote by $x_i = 1/a_i$ for $i = 1, \dots, n$. Then

$$x_1 + \cdots + x_n = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

and according to (4)

$$\begin{aligned} x_1 + \cdots + x_n &\geq n \sqrt[n-2]{x_1 \dots x_n}, \quad \text{or} \\ a_1 + \cdots + a_n &\geq n \frac{1}{\sqrt[n-2]{a_1 \dots a_n}}. \end{aligned} \tag{5}$$

Based on (4) and (5), generalized inequality (3) follows.

4341. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let ABC be an arbitrary triangle. Show that

$$\sum_{\text{cyc}} \sin A(|\cos A| - |\cos B \cos C|) = \sin A \sin B \sin C.$$

Eleven correct solutions were received. Most of the solvers used the approach in the solution given here.

From the expansion of

$$0 = \tan \pi = \tan(A + B + C),$$

we obtain

$$\tan A \tan B \tan C = \tan A + \tan B + \tan C,$$

whence

$$\sin A \sin B \sin C = \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B.$$

Also,

$$\begin{aligned} &2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C \\ &= \sin 2B + \sin 2C + 2 \sin A \cos A \\ &= 2 \sin(B + C) \cos(B - C) + 2 \sin A \cos A \\ &= 2 \sin A(\cos(B - C) - \cos(B + C)) \\ &= 4 \sin A \sin B \sin C \\ &= 2(\sin A \sin B \sin C + \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B). \end{aligned}$$

This yields the result for acute and right triangles. If, say, $\angle A > \pi/2$, then cosine of A is negative and all the other trigonometric ratios are positive. Note that

$$\cos A + \cos B \cos C = -\cos(B + C) + \cos B \cos C = \sin B \sin C,$$

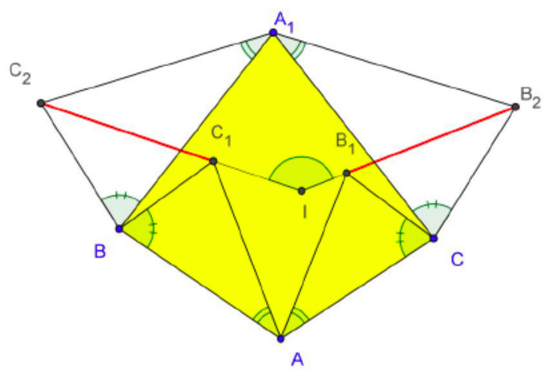
with analogous identities for other arrangements of A, B, C.

The left side of the equation is then

$$\begin{aligned}
&-\sin A\cos A-\sin A\cos B\cos C+\sin B\cos B\\
&\qquad+\sin B\cos C\cos A+\sin C\cos C+\sin C\cos A\cos B\\
&=-\sin A(\cos A+\cos B\cos C)+\sin B(\cos B+\cos C\cos A)\\
&\qquad+\sin C(\sin C+\cos A\cos B)\\
&=-\sin A\sin B\sin C+2\sin A\sin B\sin C\\
&=\sin A\sin B\sin C.
\end{aligned}$$

4342. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

In a convex quadrilateral ABA_1C, construct four similar triangles ABC_1, A_1BC_2, ACB_1 and A_1CB_2 as shown in the figure.



Show that $C_1C_2 = B_1B_2$ and that the directed angles satisfy

$$\angle(C_1C_2, B_1B_2) = 2\angle C_1BA.$$

We received 6 submissions, all of which were correct, including fixing the typographical error in the problem's statement (which had one unwanted subscript). We present a composite of the similar solutions by AN-anduud Problem Solving Group and Michel Bataille.

Let us denote the pairs of equal oriented angles at B and at C by ψ; more precisely,

$$\psi = \angle C_1BA = \angle C_2BA_1 = \angle ACB_1 = \angle A_1CB_2.$$

Denote by k the common ratio of the corresponding adjacent sides of the given similar triangles:

$$k = \frac{BA}{BC_1} = \frac{BA_1}{BC_2} = \frac{CA}{CB_1} = \frac{CA_1}{CB_2}.$$

The spiral similarity about center B with angle ψ and ratio k takes C_1 to A and C_2 to A_1, while the spiral similarity about center C with angle $-\psi$ and ratio k takes B_1 to A and B_2 to A_1.

Consequently, $C_1C_2 = B_1B_2$ (because they equal $\frac{1}{k}AA_1$). Moreover,

$$\psi = \angle(\overrightarrow{C_1C_2}, \overrightarrow{AA_1}) = \angle(\overrightarrow{AA_1}, \overrightarrow{B_1B_2}),$$

Therefore

$$\begin{aligned}\angle(\overrightarrow{C_1C_2}, \overrightarrow{B_1B_2}) &= \angle(\overrightarrow{C_1C_2}, \overrightarrow{AA_1}) + \angle(\overrightarrow{AA_1}, \overrightarrow{B_1B_2})\\ &= 2\psi\\ &= 2\angle C_1BA.\end{aligned}$$

4343. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and E be the center of the excircle tangent at F and G to the extended sides AB and AC, respectively. If $GF \cap BE = \{B_1\}$, $FG \cap CE = \{C_1\}$ and B' and C' are feet of the altitudes from B, respectively C, show that $B_1C_1B'C'$ is a cyclic quadrilateral.

All of the 11 submissions we received were correct. Seven were essentially the same as our featured solution by Shuborno Das, and most solvers observed that the restriction to acute triangles is not required.

Note that

$$\begin{aligned}\angle CEB_1 &= \angle CEB\\ &= 180^\circ - \left(90^\circ - \frac{B}{2} + 90^\circ - \frac{C}{2}\right)\\ &= 90^\circ - \frac{A}{2}.\end{aligned}$$

Moreover, because $AF = AG$,

$$\angle CGB_1 = \frac{180^\circ - A}{2} = 90^\circ - \frac{A}{2}.$$

Thus $CGEB_1$ is cyclic.

Since $EG \perp CG$ it follows that $\angle EB_1C = 90^\circ$. Similarly we get $\angle EC_1B = 90^\circ$.

Hence, the points B_1 and C_1 lie on the circle whose diameter is BC. But B' and C' are the feet of the altitudes from B and C (that is, $BB' \perp B'C$ and $CC' \perp C'B$), so they also must lie on the same circle. In other words, for any given triangle ABC, acute or not, the quadrilateral $B_1C_1B'C'$ is inscribed in the circle whose diameter is BC, and we are done.

4344. *Proposed by Michel Bataille.*

Let n be a positive integer. Find all polynomials $p(x)$ with complex coefficients and degree less than n such that $x^{2n} + x^n + p(x)$ has no simple root.

We received 4 correct solutions. We present 3 of them.

Solution 1, by the proposer.

The only polynomial is the constant $p(x) = \frac{1}{4}$.

Let $q(x) = x^{2n} + x^n + p(x)$ have s distinct roots $r_1, r_2, \ldots, r_s$ with respective multiplicities $k_1, k_2, \ldots, k_s$, all exceeding 1. Then

$$2s \leq k_1 + k_2 + \cdots + k_s = 2n,$$

whence $s \leq n$. Suppose that

$$p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

and that $f(x) = 2nq(x) - xq'(x)$. Then $f(x)$ is a polynomial of degree n equal to

$$nx^n + (n+1)a_{n-1} + (n+2)a_{n-2} + \cdots + (2n-1)a_1 x + 2na_0.$$

Since each of the roots r_i has multiplicity exceeding 1, it is a root of $q'(x)$ with multiplicity $k_i - 1$. Thus, each of the roots r_i is a root of $f(x)$ with multiplicity at least $k_i - 1$, so that

$$2n - s = (k_1 + k_2 + \cdots + k_s) - s = (k_1 - 1) + (k_2 - 1) + \cdots + (k_s - 1) \leq n,$$

whence $s \geq n$. Therefore $n = s$, $k_1 = k_2 = \cdots = k_n = 2$, and

$$q(x) = \prod_{i=1}^{n}(x - r_i)^2 = (x^n + u_{n-1}x^{n-1} + \cdots + u_1 x + u_0)^2,$$

for some complex coefficients u_i. Plugging this into the equation defining q and noting that all the coefficients of powers of x between x^n and x^{2n} vanish, we find that

$$\begin{aligned}
2u_{n-1} &= 0;\\
2u_{n-2} + u_{n-1}^2 &= 0;\\
2u_{n-3} + 2u_{n-1}u_{n-2} &= 0;\\
&\vdots\\
2u_1 + 2u_2 u_{n-1} + \cdots &= 0;\\
&\vdots\\
2u_0 + 2u_1 u_{n-1} + \cdots &= 1.
\end{aligned}$$

Thus $q(x) = (x^n + \frac{1}{2})^2$ and $p(x) = \frac{1}{4}$ is the unique solution to the problem.

Solution 2, AN-anduud Problems Solving Group.

Using the notation and argument of the first solution, we have that

$$q(x) = x^{2n} + x^n + p(x) = (u(x))^2$$

for some monic polynomial $u(x)$ of degree n. Then

$$p(x) - \frac{1}{4} = u(x)^2 - \left(x^n + \frac{1}{2}\right)^2 = \left(u(x) - x^n - \frac{1}{2}\right)\left(u(x) + x^n + \frac{1}{2}\right).$$

Either $p(x) = 1/4$ and $u(x) = x^n + 1/2$ or the polynomials on both sides of the equation are nontrivial. But the latter possibility is precluded by the fact that the degree of $p(x) - 1/4$ is less than n while that of $u(x) + x^n + 1/2$ is equal to n.

Solution 3, by Leonard Giugiuc and Ramanujan Srihari (done independently).

With the notation of the previous solutions and from the fact that $s \leq n$, we note that the coefficients of the powers x^{2n-s+1} up to x^{2n-1} in $q(x)$ vanish, along with the corresponding symmetric functions of the roots. From Newton's formulae for the powers of the roots, we obtain the equation

$$k_1 + k_2 + \cdots + k_s = 2n$$

as well as the equations in the system

$$S = \{k_1 r_1^t + k_2 r_2^t + \cdots + k_s r_s^t = 0 : 1 \leq t \leq s - 1 \leq n - 1 < 2n - n\},$$

for integers k_1, k_2, …, k_s in terms of distinct complex numbers r_1, r_2, …, r_s.

If, say, $r_s = 0$, then S becomes a system of $s - 1$ equations in $s - 1$ unknowns with a nonzero Vandermonde determinant of its coefficients, so it has the unique solution $k_1 = k_2 = \cdots = k_{s-1} = 0$. But this forces $k_s = 2n$ and $q(x) = x^{2n}$, a palpable falsehood.

Thus, all the r_i are nonzero. If $s < n$, we can append to S the equation

$$k_1 r_1^s + k_2 r_2^s + \cdots + k_s r_s^s = 0$$

and obtain the solution $k_1 = k_2 = \cdots = k_s = 0$, which is again false. Therefore $s = n$, $k_1 = k_2 = \cdots = k_n = 2$, and $x^{2n} + x^n + p(x)$ is the square of a polynomial of degree n. Since the roots of this polynomial satisfy

$$r_1^t + r_2^t + \cdots + r_n^t = 0$$

for $1 \leq t \leq n - 1$, we must have that

$$x^{2n} + x^n + p(x) = (x^n + c)^2$$

for some constant c. A check of the coefficient of x^n reveals that $c = \frac{1}{2}$.

4345. *Proposed by Mihai Miculiţa and Titu Zvonaru.*

Let ABC be a triangle with $AB < BC$ and incenter I. Let F be the midpoint of AC. Suppose that the C-excircle is tangent to AB at E. Prove that the points E, I and F are collinear if and only if $\angle BAC = 90^\circ$.

We received 16 submissions, all of which were correct, and will feature two of them.

Solution 1, by Ivko Dimitrić (typical of the five submissions that used barycentric coordinates).

We will see that that the assumption $AB < BC$ is superfluous. We use barycentric coordinates based on $\triangle ABC$ where $A(1:0:0)$, $B(0:1:0)$ and $C(0:0:1)$ are the coordinates of the vertices and a, b, c denote the corresponding side lengths. As usual, $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter. Then $F(1:0:1)$ and $I(a:b:c)$. Since $BE = s-a$ and $AE = s-b$, it follows that $E(s-a : s-b : 0)$. Hence, F, I and E are collinear if and only if

$$\begin{vmatrix} 1 & 0 & 1 \\ a & b & c \\ s-a & s-b & 0 \end{vmatrix} = 0.$$

Expanding this determinant gives

$$-c(s-b) + a(s-b) - b(s-a) = 0$$

or

$$bc - (-a+b+c)\,s = 0.$$

This, in turn, is equivalent to

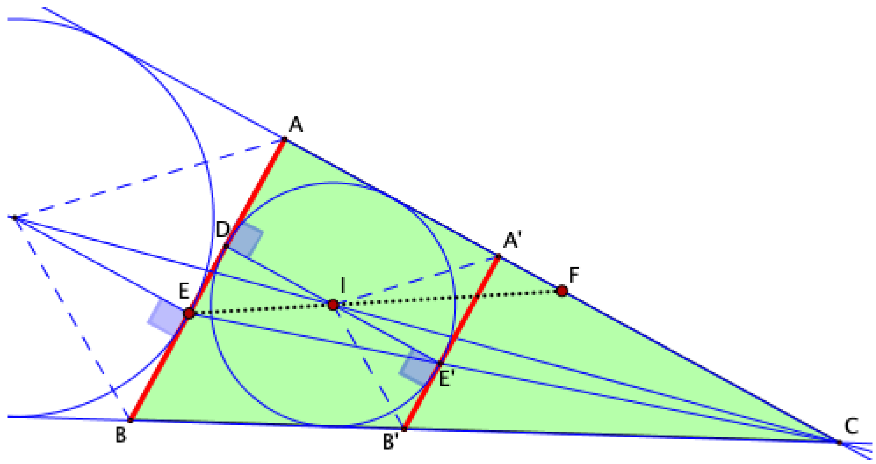
$$\begin{aligned} 0 &= (-a+b+c)(a+b+c) - 2bc \\ &= (b+c)^2 - a^2 - 2bc \\ &= b^2 + c^2 - a^2, \end{aligned}$$

which reduces to the Pythagorean condition $b^2 + c^2 = a^2$. In other words, starting with an arbitrary triangle ABC, we have that F, I and E are collinear if and only if $\triangle ABC$ is a right-angled triangle with $\angle BAC = 90^\circ$.

Solution 2, by AN-anduud Problem Solving Group, with some details supplied by the editor.

Let σ be the dilatation with center C that takes the C-excircle to the incircle. Side AB of ΔABC, which is tangent to the excircle at E, is taken by σ to a line segment $A'B'$ (with A' on AC and B' on BC) that is tangent to the incircle at a point E'. If D is the point where the incircle is tangent to AB, then DE' is a diameter of the incircle.

Assume now that $\angle BAC = 90^\circ$. Then $\Delta EDE'$ and ΔEAC are homothetic right triangles that share the angle at E; because I and F are the midpoints of the parallel sides opposite E, namely DE' and AC, we deduce that E, I, and F are collinear.



Conversely, we assume that E, I, and F are collinear. Consider the dilatation with center E that takes I to F. It takes E' to a point E'' on the line EE' and D to a point D' on AB such that $E''D' \perp BD'$ and F is the midpoint of $D'E''$. This forces D' to coincide with A and E'' with C as follows: as a moving point X slides along the line AB, the locus of points Y for which F is the midpoint of XY is the line through C that is parallel to AB, and that line intersects EE' in the unique point C. We conclude that $\angle BAC = \angle BD'E'' = 90°$, as desired.

4346. *Proposed by Daniel Sitaru.*

Find all $x, y, z \in (0, \infty)$ such that

$$\begin{cases} 64(x+y+z)^2 = 27(x^2+1)(y^2+1)(z^2+1), \\ x+y+z = xyz. \end{cases}$$

We received 17 correct solutions, along with 1 incorrect solution. Eight solutions proceeded as in Solution 1; they all followed the same strategy, some depending on the Hermite-Hadamard inequality. We present 3 solutions in total.

Solution 1, by Paul Bracken.

Let

$$(x, y, z) = (\tan A, \tan B, \tan C),$$

where $0 < A, B, C < \pi/2$. Then the two equations become

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

and
$$64(\tan A\tan B\tan C)^2 = 27(\sec^2 A)(\sec^2 B)(\sec^2 C).$$
These are equivalent to $A+B+C=\pi$ (expand $\tan(A+B+C)$) and
$$\sin^2 A\sin^2 B\sin^2 C = \frac{27}{64}.$$

Since $2\ln\sin t$ is a strictly concave function of t on $(0,\pi/2)$, by Jensen's inequality we get
$$\ln\sin^2 A + \ln\sin^2 B + \ln\sin^2 C \le 3\ln\sin^2\left(\frac{A+B+C}{3}\right) = 3\ln\sin^2\left(\frac{\pi}{3}\right).$$
Hence
$$\sin^2 A\sin^2 B\sin^2 C \le \left(\frac{3}{4}\right)^3 = \frac{27}{64},$$
with equality if and only if $A=B=C=\pi/3$.

Therefore the equations are satisfied if and only if $(x,y,z)=(\sqrt{3},\sqrt{3},\sqrt{3})$.

Solution 2, by Nghia Doan.

Let
$$p = x+y+z = xyz \quad \text{and} \quad q = xy+yz+zx.$$
Since
$$x^2+y^2+z^2 = p^2-2q$$
and
$$x^2y^2+y^2z^2+z^2x^2 = q^2-2p^2,$$
the first equation becomes
$$64p^2 = 27[p^2+(q^2-2p^2)+(p^2-2q)+1] = 27(q-1)^2.$$
Since
$$q = \left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)(xy+yz+zx) \ge 9,$$
then $8p = 3\sqrt{3}(q-1)$.

By the AM-GM inequality,
$$p^3 = (x+y+z)^3 \ge 27xyz = p,$$
so that $p \ge 3\sqrt{3}$, with equality if and only if $x=y=z=\sqrt{3}$.

On the other hand,
$$\begin{aligned}(xy+yz+zx)^2 &\ge 3((xy)(yz)+(yz)(zx)+(zx)(xy))\\ &= 3xyz(x+y+z)\\ &= 3p^2,\end{aligned}$$

so that $q \geq \sqrt{3}p$ with equality if and only if $xy = yz = zx$. Hence

$$8p \geq 3\sqrt{3}(\sqrt{3}p - 1) = 9p - 3\sqrt{3},$$

so that $p \leq 3\sqrt{3}$.

It follows that the only solution of the given system of equations is $x = y = z = \sqrt{3}$.

Solution 3, by Madhav Modak.

From the previous solution, we have that

$$p^2 = (27/64)(q-1)^2, \quad p^2 \geq 3q \quad \text{and} \quad q^2 \geq 3p^2.$$

Substituting for p^2 in these two inequalities yield respectively

$$0 \leq 9q^2 - 82q + 9 = (9q-1)(q-9)$$

and

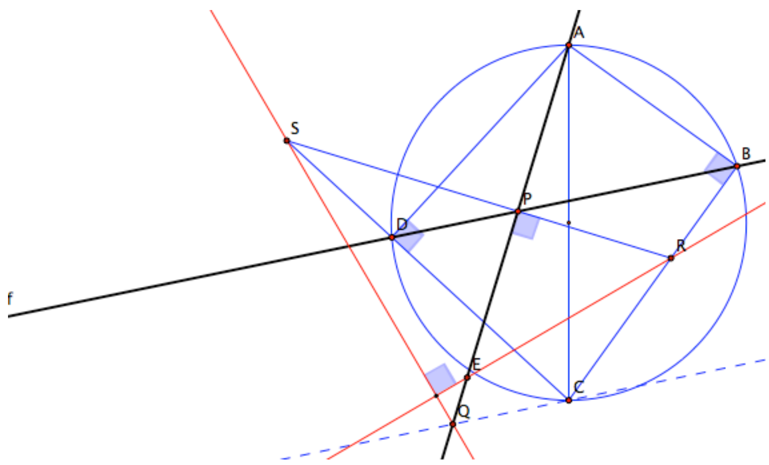
$$0 \leq -(17q^2 - 162q + 81) = (9-q)(17q-9).$$

The only value of the pair (p, q) that allows both inequalities to hold is $(\sqrt{3}, 9)$ and this in turn forces $x = y = z = 3\sqrt{3}$ as the unique solution of the system.

Editor's comment. Some solvers tackled the problem by examining when the polynomial $t^3 - pt^2 + qt - p$ has three real roots with $64p^2 = 27(q-1)^2$. This led to some straightforward technical gymnastics that are not sufficiently edifying to reproduce here.

4347. *Proposed by J. Chris Fisher.*

Given a cyclic quadrilateral $ABCD$ with diameter AC (and, therefore, right angles at B and D), let P be an arbitrary point on the line BD and Q a point on AP. Let the line perpendicular to AP at P intersect CB at R and CD at S. Finally, let E be the point where the line from R perpendicular to SQ meets AP. Prove that P is the midpoint of AE if and only if CQ is parallel to BD.



Comment by the proposer: This is a slightly generalized restatement of OC266 [2017:137-138], Problem 5 on the 2014 India National Olympiad. The solution featured recently in *Crux* was a long and uninformative algebraic verification; in particular, it failed to explain why the triangle had to be acute (it probably didn't), and it hid the true nature of the problem.

One of the five submissions we received was incomplete; the other four were correct and complete. We feature the solution by Michel Bataille.

Let γ be the circle with diameter AS. Since $AP \perp PS$ and $AD \perp DS$, the points P and D are on γ and it follows that

$$\angle(DP, DA) = \angle(SP, SA);$$

that is,

$$\angle(DB, DA) = \angle(SR, SA).$$

But

$$\angle(DB, DA) = \angle(CB, CA)$$

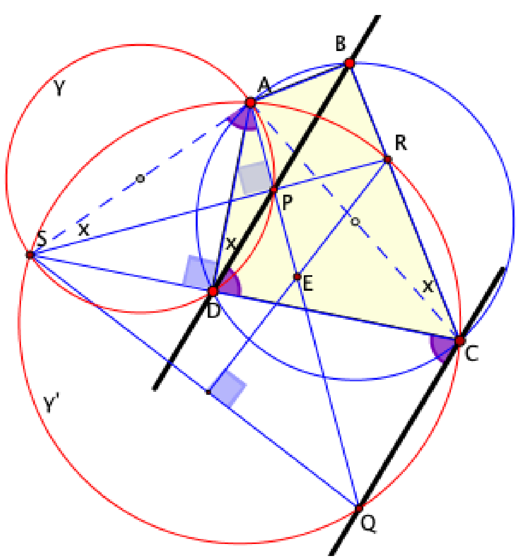
(since A, B, C, D are concyclic), hence

$$\angle(CB, CA) = \angle(SR, SA);$$

that is,

$$\angle(CR, CA) = \angle(SR, SA)$$

and so C, R, A, S are concyclic on a circle that we denote by γ'. (They cannot be collinear since otherwise B would be on CA.) Assuming that $Q \neq P$, we also observe that E is the orthocentre of ΔQRS (since $QP \perp RS$ and $RE \perp QS$).



Now, suppose that P is the midpoint of AE. Then A is the reflection of E in RS, which implies that A is on the circumcircle of ΔQRS, and therefore this circumcircle coincides with γ'. We deduce that

$$\angle(CD, CQ) = \angle(CS, CQ) = \angle(AS, AQ) = \angle(AS, AP).$$

Since

$$\angle(AS, AP) = \angle(DS, DP)$$

(A, D, S, P being concyclic) and

$$\angle(DS, DP) = \angle(CD, BD),$$

we finally obtain

$$\angle(CD, CQ) = \angle(CD, BD),$$

which proves that $CQ \parallel BD$.

Conversely, suppose that CQ is parallel to BD. Then, we have

$$\angle(CS, CQ) = \angle(DS, DP) = \angle(AS, AP) = \angle(AS, AQ),$$

so that Q is on the circumcircle γ' of ΔCSA. Since R is also on this circle, we see that γ' is the circumcircle of ΔQRS. Since A is on γ' and on the altitude QP, the point A is the reflection of the orthocentre E of ΔQRS and so P is the midpoint of AE.

Editor's comments. In terms of the notation of the foregoing problem, Problem OC266 (referred to by the proposer) states that

> For any point P on the side BD of an arbitrary triangle ABD, let O_1, O_2 denote the circumcentres of triangles ABP and APD, respectively. Prove that the line joining the circumcentre of triangle ABD to the orthocentre H' of triangle O_1O_2P is parallel to BC.

To see that this is a special case of the foregoing problem, define Q to be the intersection of AP with the line through C parallel to BD. Then, according to Problem 4347, the dilatation with center A and ratio 2 will take ΔO_1O_2P to ΔRSE, the orthocentre H' (of ΔO_1O_2P) to the orthocentre Q (of ΔRSE), and the circumcentre of ΔABD to C. Consequently the line joining the circumcentre of triangle ABD to the orthocentre H' is parallel to CQ, which is parallel to BC.

4348. *Proposed by Marius Drăgan.*

Let $p \in [0, 1]$. Then for each $n > 1$, prove that

$$(1-p)^n + p^n \geq (2p^2 - 2p + 1)^n + (2p - 2p^2)^n.$$

We received 11 correct solutions. We present the solution by Digby Smith.

Note that for $p = 0$ and $p = 1$, there is equality. Let $f(t) = t^n$ for $t \in [0, 1]$. Since $n > 1$, it follows that f is convex. Suppose that $0 < t < 1$. Applying convexity, it follows that

$$\begin{aligned} f(p) + f(1-p) &= [pf(p) + (1-p)f(1-p)] + [(1-p)f(p) + pf(1-p)] \\ &\geq f[p(p) + (1-p)(1-p)] + f[(1-p)p + p(1-p)] \\ &= f[p^2 + (1-p)^2] + f[p(1-p) + p(1-p)] \\ &= f(2p^2 - 2p + 1) + f(2p - 2p^2), \end{aligned}$$

with equality if and only if $p = 1 - p$. That is, there is equality if and only if $p = 1/2$.

It follows that

$$(1-p)^n + p^n \geq (2p^2 - 2p + 1)^n + (2p - 2p^2)^n,$$

with equality if and only if $p \in \{0, 1/2, 1\}$.

4349. *Proposed by Hoang Le Nhat Tung.*

Let x, y and z be positive real numbers such that $x+y+z = 3$. Find the minimum value of

$$\frac{x^3}{y\sqrt{x^3+8}} + \frac{y^3}{z\sqrt{y^3+8}} + \frac{z^3}{x\sqrt{z^3+8}}.$$

We received 7 correct solutions. We present the solution by Leonard Giugiuc.

Due to Cauchy's inequality

$$\left(\frac{x^3}{y \cdot \sqrt{x^3+8}} + \frac{y^3}{z \cdot \sqrt{y^3+8}} + \frac{z^3}{x \cdot \sqrt{z^3+8}}\right)(xy + yz + zx)$$
$$\geq \left(\frac{x^2}{\sqrt[4]{x^3+8}} + \frac{y^2}{\sqrt[4]{y^3+8}} + \frac{z^2}{\sqrt[4]{z^3+8}}\right)^2,$$

implying that

$$\frac{x^3}{y \cdot \sqrt{x^3+8}} + \frac{y^3}{z \cdot \sqrt{y^3+8}} + \frac{z^3}{x \cdot \sqrt{z^3+8}} \geq \frac{\left(\frac{x^2}{\sqrt[4]{x^3+8}} + \frac{y^2}{\sqrt[4]{y^3+8}} + \frac{z^2}{\sqrt[4]{z^3+8}}\right)^2}{3}.$$

But

$$3(xy + yz + ax) \leq (x + y + z)^2 = 9,$$

so

$$\frac{\left(\frac{x^2}{\sqrt[4]{x^3+8}} + \frac{y^2}{\sqrt[4]{y^3+8}} + \frac{z^2}{\sqrt[4]{z^3+8}}\right)^2}{xy + yz + zx} \geq \frac{\left(\frac{x^2}{\sqrt[4]{x^3+8}} + \frac{y^2}{\sqrt[4]{y^3+8}} + \frac{z^2}{\sqrt[4]{z^3+8}}\right)^2}{3}.$$

Thus

$$\frac{x^3}{y \cdot \sqrt{x^3+8}} + \frac{y^3}{z \cdot \sqrt{y^3+8}} + \frac{z^3}{x \cdot \sqrt{z^3+8}} \geq \frac{\left(\dfrac{x^2}{\sqrt[4]{x^3+8}} + \dfrac{y^2}{\sqrt[4]{y^3+8}} + \dfrac{z^2}{\sqrt[4]{z^3+8}}\right)^2}{3}.$$

We will show that

$$\frac{\left(\dfrac{x^2}{\sqrt[4]{x^3+8}} + \dfrac{y^2}{\sqrt[4]{y^3+8}} + \dfrac{z^2}{\sqrt[4]{z^3+8}}\right)^2}{3} \geq 1,$$

or equivalently

$$\frac{x^2}{\sqrt[4]{x^3+8}} + \frac{y^2}{\sqrt[4]{y^3+8}} + \frac{z^2}{\sqrt[4]{z^3+8}} \geq \sqrt{3}.$$

Consider the function $f : (0,3) \to \mathbf{R}$ defined by

$$f(t) = \frac{t^2}{\sqrt[4]{t^3+8}}.$$

For all $t \in (0,3)$, we have

$$f''(t) = k\left(5t^6 - 64t^3 + 2048\right)\left(t^3+8\right)^{-9/4},$$

where k is a positive constant, so that $f''(t) > 0$. Hence f is convex.

By Jensen's inequality,

$$f(x) + f(y) + f(z) \geq 3f(1) = \sqrt{3};$$

that is,

$$\frac{x^2}{\sqrt[4]{x^3+8}} + \frac{y^2}{\sqrt[4]{y^3+8}} + \frac{z^2}{\sqrt[4]{z^3+8}} \geq \sqrt{3}.$$

Thus

$$\frac{x^3}{y \cdot \sqrt{x^3+8}} + \frac{y^3}{z \cdot \sqrt{y^3+8}} + \frac{z^3}{x \cdot \sqrt{z^3+8}} \geq 1.$$

Note that if $x = y = z = 1$, then the left-hand side of the last inequality equals 1. In conclusion,

$$\min\left(\frac{x^3}{y \cdot \sqrt{x^3+8}} + \frac{y^3}{z \cdot \sqrt{y^3+8}} + \frac{z^3}{x \cdot \sqrt{z^3+8}}\right) = 1.$$

4350. *Proposed by Leonard Giugiuc.*

Let $f:[0,1]\mapsto\mathbb{R}$ be a decreasing, differentiable and concave function. Prove that

$$f(a)+f(b)+f(c)+f(d)\le 3f(0)+f(d-c+b-a),$$

for any real numbers a,b,c,d such that $0\le a\le b\le c\le d\le 1$.

We received 5 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Since $f(x)$ is decreasing, we have

$$f(a)\le f(0),\ f(b)\le f(0),\ f(c)\le f(0), \tag{1}$$

and

$$0\le(d-c)+(b-a)=d-(c-b)-a\le d\le 1,$$

so that

$$f(d)\le f(d-c+b-a). \tag{2}$$

From (1) and (2), we get

$$f(a)+f(b)+f(c)+f(d)\le 3f(0)+f(d-c+b-a).$$

Editor's comments. Note that all of the solvers, other than the proposer, explicitly or implicitly noted that neither concavity nor differentiability were needed.