Crux Mathematicorum

VOLUME 44, NO. 6 June / Juin 2018

Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

A couple of weeks ago I got an email announcing the results of the 50th Canadian Mathematical Olympiad. Every year when I see the number in front, I am amazed at it. Canada, that itself turned only 150 years old last year, has had a national mathematical olympiad for half a century now. That is impressive. The history of the competition is also easy to track: Canadian Mathematical Society keeps careful records of the exams and you can find all of them (many with solutions) at `https://cms.math.ca/Competitions/CMO/`. In academia, we often talk about our weak students getting weaker, but little time is devoted to discussing the trends our strong students are following. No need for anecdotal evidence: check out the old olympiads, compare them to the more recent ones and make your own conclusions.

This year's CMO was also special because I recognized one of the winners' names. I'd like to congratulate a ***Crux*** regular Steven Chow for receiving an Honourable Mention at this competition. Well done!

Do you have what it takes to solve CMO problems? Turn the page to see this year's competition to find out.

Kseniya Garaschuk

The 2018 Canadian Mathematical Olympiad Exam
Official Problem Set

1. Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: Select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

   We say that an arrangement of n tokens is *collapsible* if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

2. Let five points on a circle be labelled A, B, C, D, and E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$. Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

3. Two positive integers a and b are *prime-related* if $a = pb$ or $b = pa$ for some prime p. Find all positive integers n, such that n has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related.

   Note that 1 and n are included as divisors.

4. Find all polynomials $p(x)$ with real coefficients that have the following property: There exists a polynomial $q(x)$ with real coefficients such that

   $$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

   for all positive integers n.

5. Let k be a given even positive integer. Sarah first picks a positive integer N greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor p of the current value of N, and multiplies the current N by $p^k - p^{-1}$ to produce the next value of N. Prove that there are infinitely many even positive integers k such that, no matter what choices Sarah makes, her number N will at some point be divisible by 2018.

THE CONTEST CORNER

No. 66

John McLoughlin

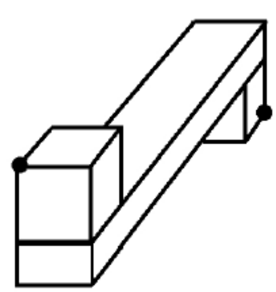
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er novembre 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC326. Étant donné un entier strictement positif n, soit a_n l'entier formé de n fois le chiffre 9 suivi des chiffres 488. Par exemple, $a_3 = 999488$ et $a_7 = 9999999488$. Pour toute valeur donnée de n, déterminer le plus grand entier $f(n)$ pour lequel $2^{f(n)}$ est un diviseur de a_n.

CC327. La figure suivante représente un conduit d'air de dimensions $1\times2\times10$ aux extrémités duquel il y a deux cubes de dimensions $2\times2\times2$. L'objet est vide et sa surface est entièrement en tôle. Une araignée marche à l'intérieur du conduit, d'un coin indiqué par un point jusqu'à l'autre coin indiqué par un point. Il existe des entiers positifs m et n de manière que le chemin le plus court que l'araignée peut emprunter ait pour longueur $\sqrt{m}+\sqrt{n}$. Déterminer $m+n$.



CC328. On dispose d'un bon nombre de blocs de bois identiques de forme cubique. On a aussi quatre couleurs de peinture. On peint chaque face de chaque bloc de manière que chaque bloc ait au moins une face peinte de chacune des quatre couleurs. Déterminer le nombre de façons distinguables de peindre les blocs. (Deux blocs sont distinguables s'il est impossible de faire subir une rotation à un des blocs de manière qu'il paraisse identique à l'autre bloc.)

CC329. Soit a, b, c et d des nombres réels tels que

$$a^2 + 3b^2 + \frac{c^2 + 3d^2}{2} = a + b + c + d - 1.$$

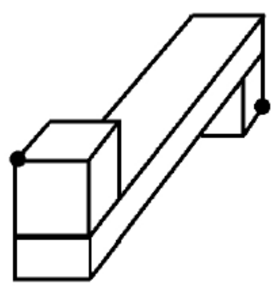
Déterminer $1000a + 100b + 10c + d$.

CC330. Six enfants attendent debout en ligne à l'extérieur de leur salle de classe. Lorsqu'ils entrent dans la salle, ils s'assoient en cercle dans un ordre aléatoire. Il existe deux entiers positifs premiers entre eux, m et n, de manière que $\frac{m}{n}$ représente la probabilité qu'il n'y ait pas deux enfants qui étaient debout l'un à côté de l'autre et qui finissent assis l'un à côté de l'autre en cercle. Déterminer $m + n$.

. . . .

CC326. For positive integer n, let a_n be the integer consisting of n digits of 9 followed by the digits 488. For example, $a_3 = 999488$ and $a_7 = 9999999488$. For each given n, determine the largest integer $f(n)$ such that $2^{f(n)}$ divides a_n.

CC327. The diagram below shows a $1 \times 2 \times 10$ duct with $2 \times 2 \times 2$ cubes attached to each end. The resulting object is empty, but the entire surface is solid sheet metal. A spider walks along the inside of the duct between the two marked corners. There are positive integers m and n so that the shortest path the spider could take has length $\sqrt{m} + \sqrt{n}$. Find $m + n$.



CC328. You have many identical cube-shaped wooden blocks. You have four colours of paint to use, and you paint each face of each block a solid colour so that each block has at least one face painted with each of the four colours. Find the number of distinguishable ways you could paint the blocks. (Two blocks are distinguishable if you cannot rotate one block so that it looks identical to the other block.)

CC329. Let a, b, c and d be real numbers such that

$$a^2 + 3b^2 + \frac{c^2 + 3d^2}{2} = a + b + c + d - 1.$$

Find $1000a + 100b + 10c + d$.

CC330. Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find $m + n$.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017: 43(6), p. 238–240.

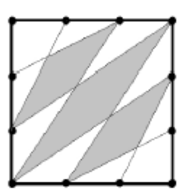
Problems in this Contest Corner came from Purple Comet! Math Meet.

Collaborative problem solving is more important than ever. The Purple Comet! Math Meet allows middle and high school students from across the globe to form teams and enjoy solving challenging problems with their peers during this online mathematics competition conducted annually since 2003.

The contest is free and last year was accessed by nearly 3,200 teams (over 12,000 students) from 59 countries and translated into 22 different languages. There is a ten-day window during which teams may compete choosing a start time most convenient for them. The problems range in difficulty from fairly easy to extremely challenging. To see past contests and information about how to register your own team for next year's competition, visit `https://purplecomet.org/`.

This contest is free due to the generosity of its sponsor, AwesomeMath, which is devoted to providing enriching experiences in mathematics for intellectually curious learners. For more information, visit `awesomemath.org`.

CC276. The figure below shows a 90×90 square with each side divided into three equal segments. Some of the endpoints of these segments are connected by straight lines. Find the area of the shaded region.



Originally problem 16 from Purple Comet! Math Meet, April 2013.

We received five submissions, of which four were correct. Konstantine Zelator and Ivko Dimitrić provided generalizations. We present the solution given by Andrea Fanchini.

We use cartesian coordinates. From the triangle EFG we have $30 : 60 = x : 60-x$, so $x = 20$ and hence the points A and D have coordinates $A(70, 20)$ and $D(20, 70)$.

Similarly from the triangle HOG we have $60 : 90 = y : 90 - y$, so $y = 36$ and the points B and C have coordinates $B(54, 36)$ and $C(36, 54)$.

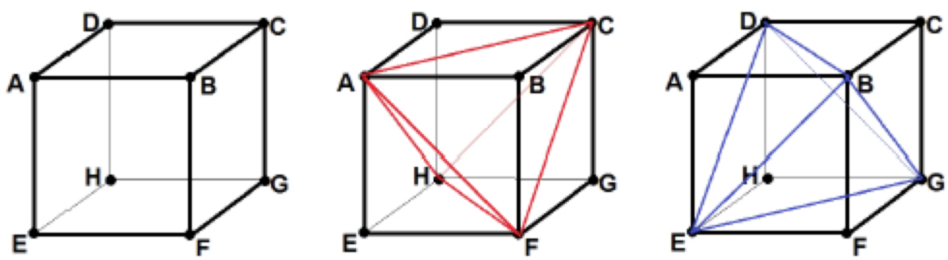
Then the triangles AFG and DIL have area $A_1 = \frac{60\cdot 20}{2} = 600$ and the triangles BFO and CIO have area $A_2 = \frac{30\cdot 36}{2} = 540$. Therefore, the entire white region has area

$$A_{white} = 4(A_1 + A_2) = 4560$$

and finally the area of the shaded region is

$$A_{shaded} = A_{square} - A_{white} = 8100 - 4560 = 3540.$$

CC277. Let A, B, C, D, E, F, G and H be the eight vertices of a $30 \times 30 \times 30$ cube as shown.



The two figures $ACFH$ and $BDEG$ are congruent regular tetrahedra. Find the volume of the intersection of these two tetrahedra.

Originally problem 28 from Purple Comet! Math Meet, April 2013.

We received one correct submission, by Ivko Dimitrić, and we present it here.

It is well-known that each convex polyhedron is the intersection of half-spaces of the planes determined by its faces. Conversely, a bounded intersection of a finite number of half-spaces is a convex polyhedron.

Let V and W be the centers of the top and the bottom faces of the cube, respectively, and let P, Q, R, S be the centers of the lateral faces of the cube in cyclic order, starting with P as the center of the face $AEHD$, followed by Q, the center of the face $AEFB$.

Since the edges of the two tetrahedra can be paired into six pairs (one edge from each tetrahedron in a pair) so that the two edges in each pair are the face diagonals of one of the six faces of the cube, the points P, Q, R, S, V and W belong to both tetrahedra, so the convex hull of these six points, which is a regular octahedron with its vertices being these six points, belongs to the (convex) intersection Π of the two tetrahedra.

We argue that the intersection Π is, in fact, that octahedron.

If O is the center of the cube, the segments joining pairs of opposite vertices of the octahedron meet at O perpendicularly and O bisects each of them. Thus the cube and the octahedron have the same center.

Each face of the octahedron is the medial triangle of one of the combined eight faces of the two tetrahedra, such as the face QRV being the medial triangle of ACF and similarly for others. Hence, the plane supporting QRV is the same as the plane ACF, and the octahedron is located on that side of the plane $(QRV) = (ACF)$ that contains O. A similar situation occurs for other planes determined by the faces of the octahedron. Thus, the intersection Π of the two tetrahedra, i. e. the intersection of eight half-spaces containing O, determined by the faces of these tetrahedra, is identical to the intersection of eight half-spaces containing O, determined by the planes supporting the faces of the octahedron, i.e. the solid Π is the octahedron.

The edge QR of this octahedron is the common midline of triangles BEG and AFC and its length is half the length of EG, i.e. equal to $\frac{1}{2}a\sqrt{2} = 30/\sqrt{2}$, where $a = 30$ is the edge length of the cube. The same is true of the lengths of other edges, each face of the octahedron being an equilateral triangle.

The octahedron is composed of two congruent pyramids sharing the same square base $PQRS$, each having height of $a/2 = 15$, the half-length of the edge of the cube. The area of the common base of the two pyramids is

$$QR^2 = \frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \frac{a^2}{2},$$

so the volume of the octahedron is

$$2 \cdot \frac{1}{3} \cdot \frac{a^2}{2} \cdot \frac{1}{2}a = \frac{1}{6}a^3 = \frac{1}{6}30^3 = 4500.$$

CC278. For positive integers m and n, the decimal representation for the fraction $\frac{m}{n}$ begins with 0.711 and is followed by other digits. Find the least possible value for n.

Originally problem 17 from Purple Comet! Math Meet, April 2013.

We received two correct solutions and one incomplete solution. We provide the solution by Titu Zvonaru.

The answer is $n = 45$.

Since $32/45 = 0.7\overline{1}$, the least possible value for n cannot exceed 45. The number m/n begins with 0.711 if and only if $711n \leq 1000 < 712n$. It is straightforward to check that no multiple of 1000 lies in the interval $[711n, 712n)$ for $1 \leq n \leq 44$.

CC279. There is a pile of eggs. Joan counted the eggs, but her count was off by 1 in the 1's place. Tom counted the eggs, but his count was off by 1 in the

10's place. Raoul counted the eggs, but his count was off by 1 in the 100's place. Sasha, Jose, Peter, and Morris all counted the eggs and got the correct count. When these seven people added their counts together, the sum was 3162. How many eggs were in the pile?

Originally problem 19 from Purple Comet! Math Meet, April 2013.

We received 5 submissions, of which 2 were correct and complete and 3 were incomplete. We present the solution by Ivko Dimitrić.

We do not consider changing digit 9 to 0 or 0 to 9 as being off by 1, since in this case one would be off by 9. So for any digit d, 1 through 8, being off by 1 means that that digit is given as $d \pm 1$, whereas when $d = 9$ being off by 1 means the digit is changed to $d - 1 = 8$ and when $d = 0$ being off by 1 means the digit is changed to $d + 1 = 1$. Since the average of the seven counts is about 451.7 and the error is at most ± 111, the number of eggs is some 3-digit number

$$N = \overline{abc} = 100a + 10b + c,$$

with a being the digit of hundreds, b the digit of tens and c the digit of ones. Then Joan's, Tom's and Raoul's counts are respectively

$$J = 100a + 10b + c \pm 1, \quad T = 100a + 10(b \pm 1) + c, \quad R = 100(a \pm 1) + 10b + c,$$

with the proviso that if one of the digits is 0 or 9 the error can go only one way. Then adding these three together with the four correct counts yields

$$4N + 3(100a + 10b + c) \pm 100 \pm 10 \pm 1 = 3162.$$

Therefore,

$$7N \pm 100 \pm 10 \pm 1 = 3162 = 7 \cdot 452 - 2.$$

It follows that

$$2 \pm 100 \pm 10 \pm 1 = 7(452 - N)$$

is divisible by 7. Since $10 \equiv 3 \pmod 7$ and $100 \equiv 2 \pmod 7$, that means that $2 \pm 2 \pm 3 \pm 1$ must be divisible by 7 with an appropriate choice of signs.

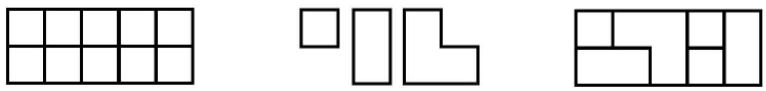
Considering all possible choices for $+$ and $-$ signs we see that this is possible only for $2 + 2 - 3 - 1 = 0$ combination. With this choice of signs, Joan's count was off by 1 by under-counting by 1 the units, Tom's count was off by 1 in tens by under-counting by 10. and Raoul's count was off by 1 in hundreds by over-counting by 100. Then

$$7N + 100 - 10 - 1 = 3162,$$

from where $N = 439$ is the true number of eggs in the pile, whereas Joan's, Tom's and Raoul's counts were respectively

$$J = 438, \quad T = 429 \quad \text{and} \quad R = 539.$$

CC280. You can tile a 2×5 grid of squares using any combination of three types of tiles: single unit squares, two side by side unit squares, and three unit squares in the shape of an L. The diagram below shows the grid, the available tile shapes, and one way to tile the grid.



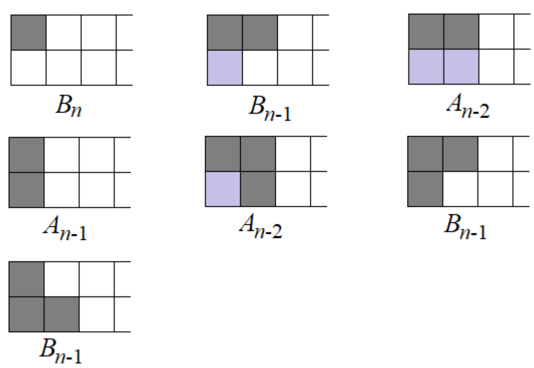
In how many ways can the grid be tiled?

Originally problem 29 from Purple Comet! Math Meet, April 2013.

We received two solutions, one of which was correct. We present the solution by Missouri State University Problem Solving Group, modified by the editor.

More generally, we find a closed form for the number of tilings of a $2 \times n$ grid. To do so we develop a recurrence relation. Let A_n denote the number of ways of tiling a $2 \times n$ grid; let B_n denote the number of ways of tiling a $2 \times n$ grid with the lower-left square removed.

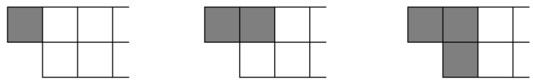
Consider the upper-left square of a $2 \times n$ grid. It can be covered with a 1×1 tile, resulting in B_n ways of tiling the remainder of the grid. It can be covered with a horizontal 1×2 tile, resulting in two possibilities for covering the lower-left square. Using a 1×1 tile results in B_{n-1} ways of tiling the remainder of the grid. Using a horizontal 1×2 tile results in A_{n-2} ways of tiling the remainder of the grid. It can be covered with a vertical 1×2 tile, resulting in A_{n-1} ways of tiling the remainder of the grid. Finally, there are three ways of covering the square with an L-tromino. One forces the lower-left square to be covered with a 1×1 tile, resulting in A_{n-2} ways of tiling the remainder of the grid. The other two give B_{n-1} ways of tiling the remainder of the grid. Each of these cases is illustrated below.



In summary, we have the following recurrence relation for A_n:

$$A_n = A_{n-1} + 2A_{n-2} + B_n + 3B_{n-1}.$$

We now find a recurrence relation for B_n. We can place a 1×1 tile, 1×2 tile (horizontally), or an L-tromino on the upper-left square giving, respectively, A_{n-1}, B_{n-1}, or A_{n-2} ways of tiling the remainder. These cases are shown below:



Therefore, we have the following recurrence relation for B_n

$$B_n = A_{n-1} + A_{n-2} + B_{n-1}.$$

Substituting this expression into our recurrence relation for A_n yields

$$A_n = 2A_{n-1} + 3A_{n-2} + 4B_{n-1}.$$

One easily verifies that $A_1 = 2, A_2 = 11$, and $B_2 = 4$.

Applying our double recurrence yields

$$A_3 = 44, B_3 = 17; \quad A_4 = 189, B_4 = 72; \quad \text{and} \quad A_5 = 798, B_5 = 305.$$

Therefore, the answer to the original question is 798 tilings.

We now derive a closed form for A_n. Our recurrence relation can be rewritten in matrix form as

$$\begin{bmatrix} A_n \\ A_{n-1} \\ B_n \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A_{n-1} \\ A_{n-2} \\ B_{n-1} \end{bmatrix}.$$

It is well-known that A_n can be expressed as a linear combination of powers of the eigenvalues of the multiplying matrix. In this case, the characteristic polynomial is

$$\lambda^3 - 3\lambda^2 - 5\lambda - 1$$

with roots $\lambda = -1, 2 \pm \sqrt{5}$. Therefore

$$A_n = c_1(-1)^n + c_2\left(2+\sqrt{5}\right)^n + c_3\left(2-\sqrt{5}\right)^n.$$

Using the known values of A_1, A_2, and A_3 and solving, we find

$$c_1 = \frac{1}{2}, c_2 = \frac{5+3\sqrt{5}}{20}, c_3 = \frac{5-3\sqrt{5}}{20}.$$

In general we have that

$$A_n = \frac{10(-1)^n + \left(5+3\sqrt{5}\right)\left(2+\sqrt{5}\right)^n + \left(5-3\sqrt{5}\right)\left(2-\sqrt{5}\right)^n}{20}.$$

THE OLYMPIAD CORNER

No. 364

Alessandro Ventullo

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er novembre 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC386. Déterminer tous les polynômes unitaires non constants P et Q, à coefficients réels, qui vérifient

$$2P(x) = Q\left(\frac{(x+1)^2}{2}\right) - Q\left(\frac{(x-1)^2}{2}\right)$$

et $P(1) = 1$ pour tous réels x.

OC387. Soit $X_1, X_2, \ldots, X_{100}$ une suite de sous-ensembles distincts non vides d'un ensemble S. On sait que n'importe quels deux sous-ensembles X_i et X_{i+1} sont toujours disjoints et que leur réunion n'est jamais égale à l'ensemble S, c'est-à-dire que $X_i \cap X_{i+1} = \emptyset$ et $X_i \cup X_{i+1} \neq S$, pour tous $i \in \{1, \ldots, 99\}$. Déterminer le plus petit nombre possible d'éléments dans S.

OC388. Soit $ABCD$ un quadrilatère inscriptible avec $\angle BAC = \angle DAC$. Soit I_1 et I_2 les cercles inscrits dans les triangles respectifs ABD et ADC. Démontrer qu'une des tangentes externes communes aux cercles I_1 et I_2 est parallèle à BD.

OC389. Soit n un entier strictement positif. Dans un royaume, il y a 2^n citoyens et un roi. L'argent du royaume consiste en billets d'une valeur de 2^n et de pièces de valeurs $2^a (a = 0, 1, \ldots, n-1)$. Chaque citoyen a une quantité infinie de billets. Soit S le nombre total de pièces d'argent dans le royaume. Un bon jour, le roi émet un édit selon lequel, chaque soir :

- chaque citoyen doit choisir une quantité finie d'argent à partir des pièces en sa possession et remettre cet argent à un autre citoyen ou au roi;
- la quantité d'argent que chaque citoyen remet doit être 1 de plus que la quantité d'argent qu'il reçoit des autres citoyens.

Déterminer la valeur minimale de S de manière que le roi puisse recevoir de l'argent chaque soir pour l'éternité.

OC390. Soit n un entier ($n \geq 2$). Déterminer la plus petite valeur de γ pour laquelle l'inégalité

$$x_1 x_2 \cdots x_n \leq \gamma \left(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \right)$$

est satisfaite pour n'importe quels réels strictement positifs $x_1, x_2, \ldots, x_n$ (tels que $x_1 + x_2 + \ldots + x_n = 1$) et n'importe quels réels $y_1, y_2, \ldots, y_n$ (tels que $y_1 + y_2 + \ldots + y_n = 1$ et $0 \leq y_1, y_2, \ldots, y_n \leq \frac{1}{2}$).

. . . .

OC386. Find all monic polynomials P, Q which are non-constant, have real coefficients and satisfy

$$2P(x) = Q\left(\frac{(x+1)^2}{2}\right) - Q\left(\frac{(x-1)^2}{2}\right)$$

and $P(1) = 1$ for all real x.

OC387. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

OC388. Let $ABCD$ be a cyclic quadrilateral with $\angle BAC = \angle DAC$. Suppose I_1 and I_2 are the incircles of $\triangle ABD$ and $\triangle ADC$ respectively. Prove that one of the common external tangents of I_1 and I_2 is parallel to BD.

OC389. Let n be a positive integer. In a kingdom there are 2^n citizens and a king. In terms of currency, the kingdom uses paper bills with value 2^n and coins with value 2^a with $a = 0, 1 \ldots, n-1$. Every citizen has infinitely many paper bills. Let the total number of coins in the kingdom be S. One fine day, the king decided to implement a policy which is to be carried out every night:

- each citizen must decide on a finite amount of money based on the coins that he/she currently has, and he/she must pass that amount to either another citizen or the king;
- each citizen must pass exactly 1 more than the amount he/she received from other citizens.

Find the minimum value of S which will guarantee that the king will be able to collect money every night eternally.

OC390. Let $n \geq 2$ an integer. Find the least value of γ which satisfies the inequality

$$x_1 x_2 \cdots x_n \leq \gamma \left(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \right)$$

for any positive real numbers $x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \ldots + x_n = 1$ and any real numbers $y_1, y_2, \ldots, y_n$ with $y_1 + y_2 + \ldots + y_n = 1$ and $0 \leq y_1, y_2, \ldots, y_n \leq \frac{1}{2}$.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017: 43(4), p. 135–136.

OC326. Let n be a positive integer. Mary writes the n^3 triples of not necessarily distinct integers, each between 1 and n inclusive on a board. Afterwards, she finds the greatest number (possibly more than one) in each triple, and erases the rest. For example, in the triple $(1, 3, 4)$ she erases the numbers 1 and 3, and in the triple $(1, 2, 2)$ she erases only the number 1. Show after finishing this process, the amount of remaining numbers on the board cannot be a perfect square.

Originally 2015 Mexico National Olympiad Day 2 Problem 4.

We received 2 solutions. We present the solution by Steven Chow.

Each triple has either 3 distinct numbers, or 2 equal numbers and 1 other number (by the obvious bijection, $\frac{1}{2}$ of these triples have exactly 1 greatest number and $\frac{1}{2}$ of them have exactly 2 greatest numbers), or 3 equal numbers.

By basic counting, therefore the amount of remaining numbers on the board is

$$\begin{aligned} & 1 \cdot n\,(n-1)\,(n-2) + 2 \cdot \frac{3}{2} \cdot n\,(n-1) + 1 \cdot \frac{3}{2} \cdot n\,(n-1) + 3 \cdot n \\ &= \frac{1}{2} n\,(n+1)\,(2n+1) \\ &= \binom{n+1}{2}(2n+1)\,. \end{aligned}$$

If that number is a perfect square, since $\binom{n+1}{2}$ and $2n+1$ are coprime, then $\binom{n+1}{2}$ and $2n+1$ are both perfect squares, so for some integer $a \geq 1$, we have

$$2n + 1 = (2a+1)^2 \quad \Longrightarrow \quad n = 2a^2 + 2a,$$

so

$$\binom{n+1}{2} = a\,(a+1)\left(2a^2 + 2a + 1\right).$$

Since a, $a+1$ and $2a^2 + 2a + 1$ are pairwise coprime, then a and $a+1$ are perfect squares, which is a contradiction since $a \geq 1$. Therefore the amount of remaining numbers on the board cannot be a perfect square.

OC327. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Originally 2015 USAMO Day 1 Problem 2.

We received 1 solution. We present the solution by Steven Chow.

The following proof also proves that the problem is true without the condition that $AQ < BP$ and without the condition that T is on the arc AQB (as long as $T \in \omega$).

Let the unit circle be ω such that 1 is on A. Let the corresponding lowercase letter of the name of a point be the complex number of the point.

Since $\angle APB = \frac{1}{2}\pi$, $\overline{AB}$ is a diameter of ω. Since $AP = AQ$, $q = \overline{p} = \frac{1}{p}$. Since $X \in \overline{PQ}$, therefore $x + \overline{x} = p + \frac{1}{p}$.

Since $s \neq 1$ and $\overline{s} = \frac{1}{s}$ and A, S, and X are collinear,

$$\frac{s-1}{x-1} = \frac{\frac{1}{s}-1}{\overline{x}-1} = \frac{-s+1}{s\left(\overline{x}-1\right)} \implies s = \frac{-x+1}{\overline{x}-1}.$$

Since $\overline{t} = \frac{1}{t}$ and $\angle AXT = \frac{1}{2}\pi$,

$$0 = \frac{t-x}{x-1} + \frac{\frac{1}{t}-\overline{x}}{\overline{x}-1} \implies \frac{t}{-x+1} + \frac{1}{t\left(-\overline{x}+1\right)} = \frac{x+\overline{x}-2x\overline{x}}{\left(x-1\right)\left(\overline{x}-1\right)}.$$

Let C be the point at $\frac{1}{2}$. It shall be proved that M is on the circle with centre C and radius CP. It suffices to prove that $CM = CP$, which is equivalent to

$$\begin{aligned}
\iff 0 &= CM^2 - CP^2 \\
&= \left(m-c\right)\left(\overline{m}-\overline{c}\right) - \left(p-c\right)\left(\overline{p}-\overline{c}\right) \\
&= \left(\frac{s+t}{2}-\frac{1}{2}\right)\left(\frac{\frac{1}{s}+\frac{1}{t}}{2}-\frac{1}{2}\right) - \left(p-\frac{1}{2}\right)\left(\frac{1}{p}-\frac{1}{2}\right) \\
\iff 0 &= \left(s+t-1\right)\left(\frac{1}{s}+\frac{1}{t}-1\right) - \left(2p-1\right)\left(\frac{2}{p}-1\right) \\
&= t\left(\frac{\overline{x}-1}{-x+1}-1\right) + \frac{1}{t}\left(\frac{-x+1}{\overline{x}-1}-1\right) - \frac{-x+1}{\overline{x}-1} - \frac{\overline{x}-1}{-x+1} + 2p + \frac{2}{p} - 2 \\
&= \left(x+\overline{x}-2\right)\left(\frac{t}{-x+1} + \frac{1}{t\left(-\overline{x}+1\right)}\right) + \frac{\left(x-1\right)^2 + \left(\overline{x}-1\right)^2}{\left(x-1\right)\left(\overline{x}-1\right)} + 2p + \frac{2}{p} - 2 \\
&= \left(x+\overline{x}-2\right)\cdot\frac{x+\overline{x}-2x\overline{x}}{\left(x-1\right)\left(\overline{x}-1\right)} + \frac{\left(x-1\right)^2 + \left(\overline{x}-1\right)^2}{\left(x-1\right)\left(\overline{x}-1\right)} + 2p + \frac{2}{p} - 2 \\
&= \frac{-2x^2\overline{x} - 2x\overline{x}^2 + 2x^2 + 6x\overline{x} + 2\overline{x}^2 - 4x - 4\overline{x} + 2}{\left(x-1\right)\left(\overline{x}-1\right)} + 2p + \frac{2}{p} - 2 \\
&= 2\left(-x-\overline{x}+1\right) + 2p + \frac{2}{p} - 2 \\
&= 0.
\end{aligned}$$

Therefore as X varies on $\overline{PQ}$, M moves along a circle.

OC328. We call a divisor d of a positive integer n special if $d+1$ is also a divisor of n. Prove: at most half the positive divisors of a positive integer can be special. Determine all positive integers for which exactly half the positive divisors are special.

Originally from the 2015 South Africa National Olympiad.

We received 2 solutions. We present the solution by Mohammed Aassila.

We prove that no positive divisor d of n that is greater or equal to $\sqrt{n}$ can be special: if d is special, then $d+1$ is also a divisor, so $\frac{n}{d}$ and $\frac{n}{d+1}$ are both integers, which means that their difference is at least 1. Thus

$$\frac{n}{d} \geq \frac{n}{d+1} + 1,$$

which is equivalent to $n \geq d(d+1)$. But since $d(d+1) > d^2 \geq n$, this is a contradiction. Thus only divisors less than $\sqrt{n}$ can be special.

Since divisors come in pairs (a and n/a) such that one of them is less than $\sqrt{n}$ and one greater than $\sqrt{n}$ (when n is a square, $\sqrt{n}$ is paired with itself), this means that at most half the divisors can be special.

If precisely half the divisors are special, then n cannot be a square and every divisor less than $\sqrt{n}$ has to be special. Thus 1 has to be a special divisor, meaning that 2 is a divisor (and thus also special), so 3 is a divisor and so on, up to the greatest integer k that is less than $\sqrt{n}$.

Finally, k is special, so $k+1$ has to be a divisor as well. Since k is the greatest divisor less than $\sqrt{n}$ and $k+1$ the least divisor greater than $\sqrt{n}$, their product must be n, so

$$n = k(k+1) = k^2 + k.$$

Moreover, $k-1$ is also a divisor of $n = k^2 + k$ (unless $k = 1$), so it also divides

$$n - (k-1)(k+2) = k^2 + k - (k^2 + k - 2) = 2.$$

This leaves us with $k = 1$, $k = 2$ and $k = 3$ as the only possibilities, giving us $n = 2$, $n = 6$ or $n = 12$.

OC329. Let $n \geq 5$ be a positive integer and let A and B be sets of integers satisfying the following conditions:

1. $|A| = n$, $|B| = m$ and A is a subset of B
2. For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of m.

Originally 2015 China National Olympiad Day 1 Problem 3.

No solutions were submitted.

OC330. Solve the following equation in nonnegative integers:

$$(2^{2015}+1)^x + 2^{2015} = 2^y + 1$$

Originally 2015 Serbian National Mathematics Olympiad Day 2 Problem 6.

We received 2 solutions. We present the solution by José Luis Díaz-Barrero.

If $x = 0$ then we get $y = 2015$ and if $x = 1$ then we obtain $y = 2016$. Suppose that $x > 1$. Since

$$2^{2015} + 1 \equiv (-1)^{2015} + 1 \equiv 0 \pmod{3},$$

we have

$$2^y + 1 = (2^{2015}+1)^x + 2^{2015} \equiv 2^{2015} \equiv 5 \pmod{9}$$

from which $2^y \equiv 4 \pmod{9}$. Since $2^6 \equiv 1 \pmod{9}$, then $2^y \equiv 4 \pmod{9}$ gives $y = 6k+2$ for some positive integer k. Working modulo 13, we have

$$2^y + 1 = (2^6)^k \cdot 2^2 + 1 \equiv \pm 4 + 1 \pmod{13},$$

that is

$$2^y + 1 \equiv 5 \pmod{13} \quad \text{or} \quad 2^y + 1 \equiv -3 \pmod{13}.$$

On the other hand, since $2^{2015} \equiv 7 \pmod{13}$ then

$$8^x + 7 \equiv 5 \pmod{13} \quad \text{or} \quad 8^x + 7 \equiv -3 \pmod{13}.$$

Both cases are impossible because the remainders of 8^x modulo 13 are $1, 5, 8$ and 12 respectively. Finally, we conclude that the only solutions to the given equation are $(0, 2015)$ and $(1, 2016)$.

Editor's comments. Konstantine Zelator generalized the problem and proved that the equation

$$(2^n+1)^x + 2^n = 2^y + 1,$$

where n is a positive integer, $n \equiv 5 \pmod{6}$, has two nonnegative integer solutions $(0, n)$ and $(1, n+1)$. So, the proposed problem is the case $n = 2015$.

Steep and Shallow Functions

Aditya Guha Roy

1 Introduction

We first introduce the following definition for real numbers a, b, c and d :

- (a, b) and (c, d) are *similarly sorted* if $(a-b)(c-d) \geq 0$,
- (a, b) and (c, d) are *oppositely sorted* if $(a-b)(c-d) \leq 0$.

Definition 1 *A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is steep if the pairs $(f(x) \cdot x, f(y) \cdot y)$ and $\left(\frac{f(x)}{x}, \frac{f(y)}{y}\right)$ are always similarly sorted.*

Definition 2 *A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is shallow if the pairs $(f(x) \cdot x, f(y) \cdot y)$ and $\left(\frac{f(x)}{x}, \frac{f(y)}{y}\right)$ are always oppositely sorted.*

Geometrically, if the graph of f passes through the intersection of a radial line $y = kx$ and the right hyperbola $xy = c$ (which divide the positive quadrant into four sections), then if f is shallow, it always passes from the left section to the right section, whereas if f is steep it always passes between the upper and lower sections.

For differentiable functions we have the following equivalent criteria :

- A differentiable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is steep if and only if $\forall x \in \mathbb{R}^+$, $|f'(x)| > \frac{f(x)}{x}$
- A differentiable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is shallow if and only if $\forall x \in \mathbb{R}^+$, $|f'(x)| < \frac{f(x)}{x}$.

The proofs are left as an exercise for the readers.

2 Inequalities for steep and shallow functions

Lemma 1 *Let a, b, c, d be real numbers such that the pairs (a, b) and (c, d) are similarly sorted. Then we must have $ac + bd \geq ad + bc$.*

Proof: Since the pairs (a, b) and (c, d) are similarly sorted, it follows that

$$(a-b)(c-d) \geq 0, \iff (ac+bd) - (ad+bc) \geq 0.$$

□

Now using this lemma we prove some interesting results about steep and shallow functions.

Proposition 1 *If $g : \mathbb{R}^+ \to \mathbb{R}^+$ is shallow then, for all real numbers x, y we have*

$$2 \le \frac{g(x)}{g(y)} + \frac{g(y)}{g(x)} \le \frac{x}{y} + \frac{y}{x} \; .$$

Proof: As $xy > 0$, the pairs $(g(x) \cdot x^2 y, g(y) \cdot y^2 x)$ and $(g(x) \cdot x, g(y) \cdot y)$ must be similarly sorted. Furthermore since g is shallow the pairs $\left(\frac{g(x)}{x}, \frac{g(y)}{y}\right)$ and $(g(x) \cdot x, g(y) \cdot y)$ are oppositely sorted, as are the pairs

$$(g(x) \cdot x^2 y, g(y) \cdot y^2 x) \text{ and } \left(\frac{g(x)}{x}, \frac{g(y)}{y}\right) \; .$$

Thus by (1), we have

$$g(x) \cdot x^2 y \cdot \frac{g(x)}{x} + g(y) \cdot y^2 x \cdot \frac{g(y)}{y} \le g(x) \cdot x^2 y \cdot \frac{g(y)}{y} + g(y) \cdot y^2 x \cdot \frac{g(x)}{x} \; ,$$

and thus we get

$$xy \cdot ((g(x))^2 + (g(y))^2) \le g(x)g(y) \cdot (x^2 + y^2) \; .$$

Now dividing both sides by the positive real number $g(x)g(y)xy$ yields the required inequality. □

Proposition 2 *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is steep then for all real numbers x, y*

$$\frac{x}{y} + \frac{y}{x} \le \frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}$$

(The proof is similar to that of Proposition 1.)

3 Problems

Problem 1 *For all positive reals a, b and all non-negative real numbers c prove:*

$$\frac{a+c}{b+c} + \frac{b+c}{a+c} \le \frac{a}{b} + \frac{b}{a} \; .$$

Problem 2 *For all positive a, b and all nonnegative c prove that*

$$\frac{a^{1+c}}{b^{1+c}} + \frac{b^{1+c}}{a^{1+c}} \ge \frac{a}{b} + \frac{b}{a} \; .$$

Problem 3 *For all acute angles A, B prove that*

$$\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A} \le \frac{A}{B} + \frac{B}{A} \le \frac{\tan A}{\tan B} + \frac{\tan B}{\tan A} \; .$$

Problem 4 *For all $x > 1, y > 1$ show that*

$$\frac{x^x}{y^y} + \frac{y^y}{x^x} \ge \frac{x}{y} + \frac{y}{x} \; .$$

Hints

Hint for Problem 1 *Note that the function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by* $f(x) = x+c$ *is* shallow *over the set of all positive real numbers.*

Hint for Problem 2 *Note that the function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by* $f(x) = x^{1+c}$ *is* steep *over the set of all positive real numbers.*

Hint for Problem 3 *Consider the steepness and shallowness of the functions* $\sin(x)$ *and* $\tan(x)$. *(See [1] for the full solution.)*

Hint for Problem 4 *Consider the function* $f(x) = x^x$.

References

[1] http://www.artofproblemsolving.com/community/c6h1299029p6910573

. . . .

Aditya Guha Roy
Indian Statistical Institute
Kolkata, India

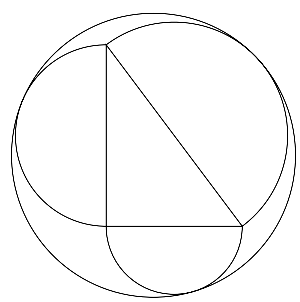
PROBLEM SOLVING 101

No. 5

Shawn Godin

This month, we will look at problem **A6** from the 2017 Canadian Senior Mathematics Contest, hosted by the The Centre for Education in Mathematics and Computing at the University of Waterloo. You can check out the contest, and past contests on the CEMC website at `www.cemc.uwaterloo.ca`.

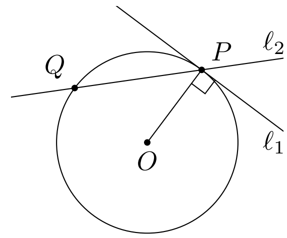
> **A6.** In the diagram, the triangle has side lengths 6, 8 and 10. Three semi-circles are drawn using the sides of the triangle as diameters. A large circle is drawn so that it just touches each of the three semi-circles. What is the radius of the large circle?



Over the years, in Ontario at least, the amount of geometry in the high school curriculum has decreased dramatically. All that remains are a few elementary angle properties, some explorations of geometric properties from an analytic geometry point of view and a brief mention of congruent and similar triangles on the way to defining the trigonometric ratios. There remains no study of the geometry of the circle or deductive proofs. In this day where there exists many dynamic geometry software packages like *the Geometer's Sketchpad* and *Geogebra*, where students could discover, better understand and delve deeper into geometric thinking, it seems like a lost opportunity.

I will now get down off my soapbox, and get down to business. We will need a few geometric definitions and theorems before we begin.

First we need to define a *tangent* to a circle as a line that just touches the circle at a single point, called the *point of tangency.* Any other line that passes through the point of tangency will intersect the circle at another point. In the diagram below, line ℓ_1 is tangent to the circle at P, while ℓ_2 is not, and thus passes through a second point Q on the circle.



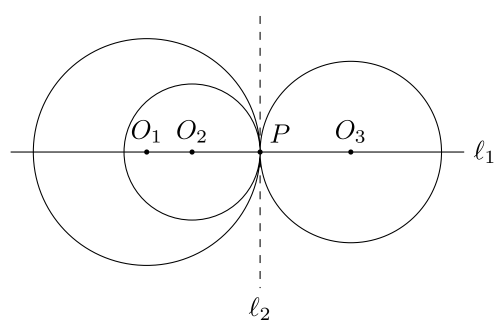
A useful property of tangents to circles is given in the following theorem.

Theorem 1: Any line ℓ tangent to a circle is perpendicular to the radius at the point of tangency.

In the diagram above, radius OP is perpendicular to the tangent at P, ℓ_1. We can extend this idea and talk about two circles being tangent if they too just touch at one point. They can be internally tangent if one circle is inside the other or externally tangent otherwise. The fact that lines tangent to circles are perpendicular to the radius at the point of tangency leads to the fact that tangent circles will share a tangent line at their point of tangency. This leads to the following theorem that we will need to solve our problem.

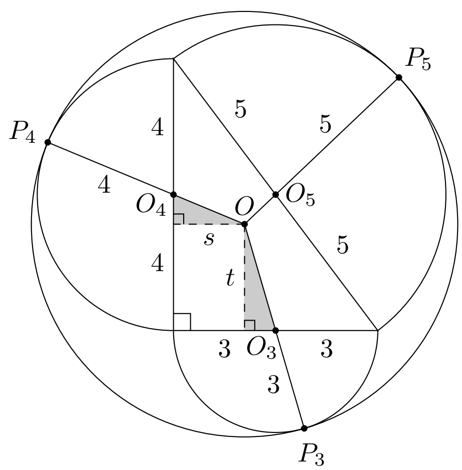
Theorem 2: If two circles with centres O_1 and O_2 are tangent to each other at a point P, then O_1, O_2 and P are collinear.

As an example, in the diagram below circles with centres O_1 and O_2 are internally tangent at P while circles with centres O_1 and O_3 are externally tangent at P. Note that all the centres and the common point of tangency lie on ℓ_1 while ℓ_2 is a common tangent for the three circles.



Now on to the solution of our problem. First, since the triangle has sides 6, 8 and 10 units, the Pythagorean theorem shows that this is a right triangle.

In the diagram, O_r and P_r represent the centre of the semicircle with radius r and its point of tangency with the outer circle, respectively, while O represents the centre of the outer circle. For $r = 3, 4, 5$: P_r, O_r, and O are collinear, by theorem 2. Thus we can draw in radii for the outer circle to the points of tangency with the semicircles.

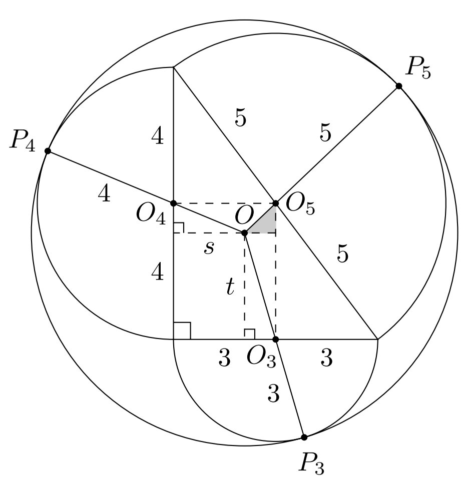


Let r represent the desired radius and s and t represent the distance from the centre of the large circle to the legs of the triangle, as indicated in the diagram. From the two shaded right triangles we get

$$(3-s)^2 + t^2 = (r-3)^2 \quad (1)$$
$$s^2 + (4-t)^2 = (r-4)^2. \quad (2)$$

We need the following property of triangles: when the midpoints of two sides of a triangle are joined, the resulting segment is parallel to, and half the length, of the third side of the triangle. Thus, if we join O_3 to O_5 and O_4 to O_5 the resulting segments are perpendicular. We can then create another right angled triangle (shaded in the diagram below) which yields

$$(3-s)^2 + (4-t)^2 = (r-5)^2. \quad (3)$$



Subtracting (3) from (1) yields

$$\begin{aligned} t^2 - (16 - 8t + t^2) &= (r^2 - 6t + 9) - (r^2 - 10t + 25) \\ 8t - 16 &= 4r - 16 \\ t &= \frac{r}{2}. \end{aligned} \tag{4}$$

Similarly, subtracting (3) from (2) will eventually give us

$$s = \frac{r}{3}. \tag{5}$$

Substituting (4) and (5) into (1) to yields

$$\begin{aligned} \left(3 - \frac{r}{3}\right) + \left(\frac{r}{2}\right) &= (r-3)^2 \\ 9 - 2r + \frac{r^2}{9} + \frac{r^2}{4} &= r^2 - 6r + 9 \\ 4r - \frac{23}{36}r^2 &= 0 \\ \frac{23}{36}r\left(\frac{144}{23} - r\right) &= 0 \end{aligned}$$

which yields $r = 0$ and $r = \frac{144}{23}$. Clearly $r > 0$, hence $r = 0$ is impossible, thus the desired radius is $\frac{144}{23}$.

The two theorems stated in this article come from book III of Euclid's Elements. Theorem 1 is Proposition 18, the converse to Theorem 1 is Proposition 19 and Theorem 2 is Proposition 11. Professor David E. Joyce of the Department of Mathematics and Computer Science at Clark University in Worcester, Massachusetts has created an online, interactive version of Euclid's Elements. You can access his work at `https://mathcs.clarku.edu/ djoyce/java/elements/elements.html`. My thanks to editor Chris Fisher for pointing out the references and making suggestions that improved this column from its initial draft.

Circles are such simple figures, but they have many interesting properties. We will explore other properties of circles in a future column.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er novembre 2018**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4351. *Proposé par Miguel Ochoa Sanchez et Leonard Giugiuc.*

Soit ABC un triangle avec cercle inscrit ω. Soient E et F les points de tangence de ω avec les côtés AC et AB respectivement. Soit G le deuxième point d'intersection de ω et BE, puis D le deuxième point d'intersection de ω et CF. Démontrer que

$$\frac{FE \cdot GD}{FG \cdot ED} = 3.$$

4352. *Proposé par Thanos Kalogerakis.*

Expliquer comment situer six points A, B, C, D, E, F, dans cet ordre, sur la circonférence d'un cerle, de façon à ce que l'hexagone ainsi obtenu possède un cercle inscrit, sans être régulier.

4353. *Proposé par Michel Bataille.*

Évaluer

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \sum_{j=1}^{n} \frac{1}{k\binom{j+k-1}{j}}.$$

4354. *Proposé par Ruben Dario Auqui et Leonard Giugiuc.*

Soit $ABCD$ un carré de côtés de longueur 1. Soient $M \in AB$, $N \in BC$ et $P \in CA$ tels que les triangles BMN et PMN sont congrus. Démontrer que

$$\frac{1}{MB} + \frac{1}{BN} = 2 + \frac{2}{MB + BN}.$$

4355. *Proposé par Mihaela Berindeanu.*

Soit H l'orthocentre du triangle ABC et soient P le mi point de AB, puis Q le mi point de AC. Si WY est la ligne perpendiculaire à HP, où $W \in AC$ et $Y \in BC$, tandis que XZ est perpendiculaire à HQ, où $X \in AB$ et $Z \in BC$, démontrer que le quadrilatère $WXYZ$ est un parallèlogramme.

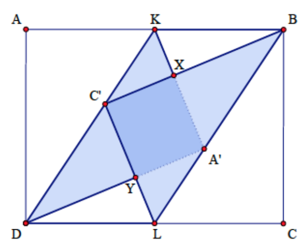
4356. *Proposé par Leonard Giugiuc et Diana Trailescu.*

Résoudre le système qui suit dans l'ensemble des réels:

$$\begin{cases} a+b+c+d=6, \\ a^2+b^2+c^2+d^2=12, \\ abc+abd+acd+bcd=8+abcd. \end{cases}$$

4357. *Proposé par Arsalan Wares.*

Supposer que $ABCD$ représente une feuille de papier rectangulaire de taille 9 par 12. Les points K et L sont les mi points de AB et DC respectivement. Le côté AD est replié en DK, puis le côté BC est replié en BL. Les côtés ainsi repliés se chevauchent et forment la région polygonale $C'XA'Y$ illustrée ci-bas.



Déterminer la surface de ce polygone $C'XA'Y$.

4358. *Proposé par George Stoica.*

Soit $f:[0,\infty)\to[0,\infty)$ une fonction non croissante telle que $\displaystyle\int_0^\infty f(x)\sin(2\pi x)dx=0$. Démontrer que f est constante sur chacun des intervalles $(n,n+1)$, $n\in\mathbb{N}$.

4359. *Proposé par Daniel Sitaru.*

Soient a,b et c des nombres reels. Démontrer que

$$3\ln(a^b+b^c+c^a)+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\geq a+b+c+\ln 27.$$

4360. *Proposé par H. A. ShahAli.*

Soient a,b,c des nombres réels non négatifs tels que $a+b+c=1$. Déterminer les valeurs minimale et maximale de l'expression

$$\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac}.$$

Où ces valeurs extrêmes ont-elles lieu ?

. . . .

4351. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle and let ω be its incircle. Let E and F be the tangency points of ω and the sides AC and AB, respectively. Let G be the second intersection point of ω and BE and, similarly, let D be the second intersection point of ω and CF. Prove that

$$\frac{FE \cdot GD}{FG \cdot ED} = 3.$$

4352. *Proposed by Thanos Kalogerakis.*

Explain how to locate six points A, B, C, D, E, F in that order about the circumference of a circle so that the resulting convex hexagon has an incircle, yet is not regular.

4353. *Proposed by Michel Bataille.*

Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \sum_{j=1}^{n} \frac{1}{k\binom{j+k-1}{j}}.$$

4354. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square with side length 1. Consider points $M \in AB$, $N \in BC$ and $P \in CA$ such that the triangles BMN and PMN are congruent. Prove that

$$\frac{1}{MB} + \frac{1}{BN} = 2 + \frac{2}{MB + BN}.$$

4355. *Proposed by Mihaela Berindeanu.*

Let H be the orthocenter of triangle ABC with P the midpoint of AB and Q the midpoint of AC. If WY is the line perpendicular to HP with $W \in AC$ and $Y \in BC$, while XZ is the perpendicular to HQ with $X \in AB$ and $Z \in BC$, prove that the quadrilateral $WXYZ$ is a parallelogram.

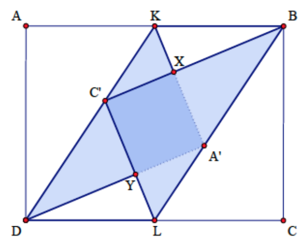
4356. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Solve the following system over reals:

$$\begin{cases} a+b+c+d = 6, \\ a^2+b^2+c^2+d^2 = 12, \\ abc+abd+acd+bcd = 8+abcd. \end{cases}$$

4357. *Proposed by Arsalan Wares.*

Suppose $ABCD$ represents a 9 by 12 rectangular sheet. Points K and L are midpoints of sides AB and DC, respectively. First, edge AD, of the rectangular sheet $ABCD$, is folded over by making a crease along DK. Then edge BC is folded over by making a crease along BL. Folded corners of the sheet overlap over a polygonal region $C'XA'Y$ as shown.



Find the area of the overlapping polygon $C'XA'Y$.

4358. *Proposed by George Stoica.*

Let $f : [0, \infty) \to [0, \infty)$ be a non-increasing function such that $\int_0^\infty f(x)\sin(2\pi x)dx = 0$. Prove that f is constant on each of the intervals $(n, n+1)$, $n \in \mathbb{N}$.

4359. *Proposed by Daniel Sitaru.*

Let a, b and c be positive real numbers. Prove that

$$3\ln(a^b + b^c + c^a) + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c + \ln 27.$$

4360. *Proposed by H. A. ShahAli.*

Let a, b, c be non-negative real numbers such that $a+b+c = 1$. Find the minimum and maximum values of the expression

$$\frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac}.$$

When do those extreme values occur?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(6), p. 262–265.

4251. *Proposed by Paolo Perfetti.*

Let $8x = y^2$ and $z = 0$ be the equations of a parabola in the (x, y) plane of $\mathbb{R}^3$. Let C be the cone of vertex in $P = (0, 0, 4)$ and generate the segments from P to the points of the parabola. Let S be the sphere of equation $x^2 + y^2 + z^2 - 4z = 0$.

a) Calculate the area of that portion of C inside S.

b) Calculate the area of that portion of S inside C.

We received 2 correct solutions. We present the solution by the proposer. Note that the proposer solves the slightly more general problem in which the equation of the parabola is $2ax = y^2$.

a) The parametric equations of the cone are

$$(x, y, z) \doteq (\varphi_1, \varphi_2, \varphi_3)(t, v),$$

where

$$(\varphi_1, \varphi_2, \varphi_3)(t, v) = (0, 0, a) + t\left[\left(\frac{v^2}{2a}, v, 0\right) - (0, 0, a)\right] = \left(\frac{tv^2}{2a}, tv, a - ta\right),$$

where $0 \le t \le 1$, $-\infty < v < +\infty$. The point (x, y, z) of the cone lies inside S iff

$$\frac{t^2v^4}{4a^2} + v^2t^2 + a^2(1-t)^2 \le a^2(1-t) \quad\iff\quad t \le \frac{a^2}{\left(\frac{v^2}{2a} + a\right)^2} < 1.$$

Now,

$$\left(\frac{\partial\varphi_1}{\partial t}, \frac{\partial\varphi_2}{\partial t}, \frac{\partial\varphi_3}{\partial t}\right) \times \left(\frac{\partial\varphi_1}{\partial v}, \frac{\partial\varphi_2}{\partial v}, \frac{\partial\varphi_3}{\partial v}\right) = at\mathbf{e}_1 - tv\mathbf{e}_2 - \frac{v^2t}{2a}\mathbf{e}_3,$$

where $\mathbf{e}_k$ is the kth standard unit vector in $\mathbf{R}^3$. Thus

$$\|(\varphi_1, \varphi_2, \varphi_3)_t \times (\varphi_1, \varphi_2, \varphi_3)_v\| = \sqrt{a^2t^2 + t^2v^2 + \frac{v^4t^2}{4a^2}} = \frac{v^2t}{2a} + ta.$$

The area of C inside S is given by the integral

$$\int_{-\infty}^{\infty} dv \int_0^{\frac{a^2}{\left(\frac{v^2}{2a}+a\right)^2}} \left(\frac{v^2}{2a} + a\right) t\,dt = \frac{8a^7}{2}\int_{-\infty}^{\infty} \frac{dv}{(v^2 + 2a^2)^3} = \frac{3\pi\sqrt{2}a^2}{16}.$$

b) We adopt stereographic parametric equations for the sphere

$$x \doteq \varphi_1(u,v) = \frac{a^2 u}{u^2+v^2+a^2}, \quad y \doteq \varphi_2(u,v) = \frac{a^2 v}{u^2+v^2+a^2},$$
$$z \doteq \varphi_3(u,v) = \frac{a(u^2+v^2)}{u^2+v^2+a^2}.$$

The point (x, y, z) of the sphere lies inside the cone C if and only if $v^2 \leq 2au$ with $u \in (-\infty, \infty)$.

$$\left(\frac{\partial\varphi_1}{\partial u}, \frac{\partial\varphi_2}{\partial u}, \frac{\partial\varphi_3}{\partial u}\right) \times \left(\frac{\partial\varphi_1}{\partial v}, \frac{\partial\varphi_2}{\partial v}, \frac{\partial\varphi_3}{\partial v}\right) =$$
$$= \left[\frac{-2a^6 u}{(u^2+v^2+a^2)^2}\right]\mathbf{e}_1 + \left[\frac{2a^6 v}{(u^2+v^2+a^2)^2}\right]\mathbf{e}_2 + \left[\frac{a^4(a^2-u^2-v^2)}{(a^2+v^2+u^2)^3}\right]\mathbf{e}_3,$$

so that

$$\|(\varphi_1,\varphi_2,\varphi_3)_u \times (\varphi_1,\varphi_2,\varphi_3)_v\| = \frac{a^4}{(u^2+v^2+a^2)^2}.$$

The area of S inside C is

$$a^4 \int_{-\infty}^{\infty} dv \int_{\frac{v^2}{2a}}^{\infty} \frac{du}{(u^2+v^2+a^2)^2},$$

which we claim is $\frac{\pi a^2\sqrt{2}}{4}$. To prove it we introduce polar coordinates in the plane (u, v). We have $u = r\cos\vartheta$, $v = r\sin\vartheta$, $-\pi/2 \leq \vartheta \leq \pi/2$, and $v^2 \leq 2au$ becomes $r \leq 2a\cos\vartheta/(\sin\vartheta)^2$. Thus the integral is

$$a^4 \int_{-\pi/2}^{\pi/2} d\vartheta \int_0^{\frac{2a\cos\vartheta}{\sin^2\vartheta}} \frac{r}{(r^2+a^2)^2}\,dr = a^4 \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{\frac{4a^2\cos^2\vartheta}{\sin^4\vartheta}+a^2}\right)d\vartheta;$$

that is,

$$\frac{a^2\pi}{2} - \frac{a^2}{2}\int_{-\pi/2}^{\pi/2} \frac{1}{\frac{4\cos^2\vartheta}{\sin^4\vartheta}+1}\,d\vartheta.$$

Evaluating this integral yields

$$a^2 \cdot \frac{\pi}{2} - a^2\left(\frac{\pi}{2} - \frac{\pi}{4}\sqrt{2}\right) = \frac{a^2\pi\sqrt{2}}{4}.$$

4252. *Proposed by Leonard Giugiuc and Marian Cucoanes.*

Let $ABCD$ be a tetrahedron with $\angle BAC = \angle CAD = \angle DAB = 60°$. Denote by R_a, R_b, R_c the circumradii of the triangles BAC, CAD and DAB, respectively. Prove that

$$R_a + R_b + R_c \geq \sqrt{AB^2 + AC^2 + BC^2}.$$

We received 2 correct solutions. We present the solution by Oliver Geupel.

Editor's comment. The statement above contain a typo that was corrected in the proposers' solution. The intended result (proven below) is as following:

Let $ABCD$ be a tetrahedron with $\angle BAC = \angle CAD = \angle DAB = 60°$. Denote by R_b, R_c, and R_d the circumradii of the triangles CAD, DAB, and BAC, respectively. Then,

$$R_b + R_c + R_d \geq \sqrt{AB^2 + AC^2 + AD^2}.$$

Let $b = AB$, $c = AC$, and $d = AD$. By the law of cosines we have

$$R_b = \frac{CD}{2\sin\angle CAD} = \frac{\sqrt{c^2 + d^2 - 2cd\cos\angle CAD}}{2\sin\angle CAD} = \sqrt{\frac{c^2 - cd + d^2}{3}}$$

with similar identities for the other circumradii. Applying the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} 3R_bR_c &= \sqrt{\left(d - \frac{c}{2}\right)^2 + \frac{3}{4}c^2} \cdot \sqrt{\left(d - \frac{b}{2}\right)^2 + \frac{3}{4}b^2} \\ &\geq \left(d - \frac{c}{2}\right)\left(d - \frac{b}{2}\right) + \frac{3}{4}bc \\ &= d^2 + bc - \frac{1}{2}cd - \frac{1}{2}db, \end{aligned}$$

and two similar inequalities.

Finally,

$$\begin{aligned} (R_b + R_c + R_d)^2 &\geq 3(R_bR_c + R_cR_d + R_dR_b) \\ &\geq \sum_{\text{cyc}} \left(d^2 + bc - \frac{1}{2}cd - \frac{1}{2}db\right) \\ &= b^2 + c^2 + d^2. \end{aligned}$$

Hence the result.

4253. *Proposed by Titu Zvonaru.*

Let ABC be a triangle with $A = 90°$ and $45° < C < 60°$. Let M be the midpoint of BC. The perpendicular from C to AM intersects the leg AB at D. On the side AC we take a point E and let K be the intersection of the lines CD and BE. If $BK = 2AE$, then prove that the triangle CEK is isosceles.

We received seven submissions, all correct. We present the solution by Steven Chow.

Let $(0,0) = A$, $(b,0) = B$, $(0,c) = C$ and $(0,e) = E$ such that $b, c, e > 0$. Then $\overleftrightarrow{CD}$ is $y = -\frac{b}{c}x + c$ and $\overleftrightarrow{BE}$ is $y = -\frac{e}{b}(x - b)$. Since K is $\overleftrightarrow{CD} \cap \overleftrightarrow{BE}$, by solving the previous system of equations we get the coordinates of the point K:

$$\left(\frac{bc\,(c-e)}{b^2 - ce}, \frac{(b+c)\,(b-c)\,e}{b^2 - ce}\right).$$

Therefore,

$$(2e)^2 = (2AE)^2 = BK^2 = \left(\frac{b\,(b+c)\,(b-c)}{b^2-ce}\right)^2 + \left(\frac{(b+c)\,(b-c)\,e}{b^2-ce}\right)^2,$$

and expanding, simplifying, and factoring reduces the equation to

$$\left(2ce - b^2 + c^2\right)\left(2ce^3 - \left(3b^2+c^2\right)e^2 + 2b^2ce + b^4 - b^2c^2\right) = 0.$$

If $2ce - b^2 + c^2 = 0$, and $2AE = BK$, from here we have $CK = EK$.

Let

$$f\,(x) = 2cx^3 - \left(3b^2+c^2\right)x^2 + 2b^2cx + b^4 - b^2c^2$$

for all x. Since $C > 45^\circ$, $b > c > 0$ so $0 < b^4 - b^2c^2 = f\,(0)$, and as $x \to -\infty$, $f\,(x) \to -\infty$, so $f\,(x)$ has a root less than 0. Since $f\,(c) = \left(b^2-c^2\right)^2 > 0$ and $f\,(b) = -2b^2\,(b-c)^2 < 0$, and as $x \to \infty$, $f\,(x) \to \infty$, $f\,(x)$ cannot have a positive root less than c.

4254. *Proposed by George Apostolopoulos.*

Let ABC be a triangle. Prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum_{\text{cyc}}\frac{(\sin A+\sin B)^2}{\sin C} \le \frac{3\sqrt{3}}{4}.$$

We received 10 submissions, all correct, and we present the solution by Michel Bataille.

Let S denote the left side of the given inequality.

From $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} = 2\cos\frac{C}{2}\cos\frac{A-B}{2}$ and $\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$, we deduce that

$$\frac{(\sin A+\sin B)^2}{\sin C} = 2\cot\frac{C}{2}\cos^2\frac{A-B}{2} \le 2\cot\frac{C}{2},$$

so

$$\sum_{\text{cyclic}}\frac{(\sin A+\sin B)^2}{\sin C} \le 2\sum_{\text{cyclic}}\frac{1}{\tan\frac{A}{2}} = \frac{2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

since it is well known that $\sum_{\text{cyclic}}\tan\frac{A}{2}\tan\frac{B}{2} = 1$. Since it is also well known that $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$ [*Ed.*: cf e.g., item 2.28 on p.26 of the book Geometric Inequalities by O. Bottema et al.], it follows that

$$S \le 2\left(\frac{3\sqrt{3}}{8}\right) = \frac{3\sqrt{3}}{4},$$

completing the proof.

Editor's comments. By using the concavity of $\sin x$ and Jensen's Inequality, Bailey, Campbell, and Diminnie (jointly) gave a slight improvement to the inequality by proving that

$$S \leq \frac{1}{2}(\sin A + \sin B + \sin C),$$

which implies the given inequality since it is well known [see item 2.1 on p. 18 of the book mentioned in the solution presented above] that $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

4255. *Proposed by Michel Bataille.*

Let ABC be an isosceles triangle with $AB = AC$ and P a point of its circumcircle (with $P \neq A$). The reflection about AP of the circle with diameter AB intersects the circle with diameter AP at A and Q. Prove that AQ and QC are perpendicular.

Five complete and four incomplete solutions were obtained. One other submission was incorrect. One of the solvers used barycentric coordinates. We present two solutions.

Solution 1, by Ivko Dimitrić.

Let A, B, C be the angles of triangle ABC at the eponymous vertices. Let Γ_1 be the circle with AB as diameter, Γ_1' its reflection about AP and Γ_2 the circle with diameter AP. Let R be the point of intersection of Γ_2 and Γ_1 other than A. Then $\angle ARP = 90^\circ = \angle ARB$, so that B, P, R are collinear. Since Γ_1 and Γ_1' are reflections of each other about AP and Γ_2 is its own reflection, Q and R are reflected images so that $\angle APQ = \angle APR$.

We note that P should be distinct not only from A but from B as well. When $P = B$, the three circles Γ_1, Γ_1' and Γ_2 coincide and Q is indeterminate. If $P = C$, then $QC = QP$ is perpendicular to AQ.

If P belongs to one of the open arcs AC or BC that does not contain the third vertex of the triangle, then B and R are on the same side of AP (and Q is on the other side). Thus, $\angle APR = \angle APB = C$ and $\angle APQ = \angle APR = C = B$, so that in the cyclic kite $ARPQ$,

$$\angle QAR = 180^\circ - \angle QPR = 180^\circ - (B + C) = A.$$

If P belongs to the open arc AB, then B and Q are on one side of AP and R is on the other, so that P is between B and R. Then $\angle APR$ is an exterior angle of triangle ABP so that

$$\angle APR = \angle ABP + \angle PAB = \angle ACP + \angle PCB = \angle ACB = C.$$

Then $\angle APQ = C = B$ and hence $\angle QAR = 180^\circ - (B + C) = A$, again.

In all cases, the counterclockwise rotation about A through angle A takes the points A, R, B to the points A, Q, C Thus $\angle AQC = \angle ARB = 90^\circ$, so that $AQ \perp QC$. We note that, since $\angle AQC = \angle AQP = 90^\circ$, the points Q, P and C are collinear.

Solution 2, by Steven Chow.

There are essentially three configurations according as P lies on one of the short arcs AB, BC and CA of the circumcircle of triangle ABC. To unify their treatment, we use directed angles with addition modulo 180°. Thus $\angle UVW$ is equal to $-\angle WVU$ or $180^\circ - \angle WVU$, depending on the situation. Thus, when U, V, X, Y are concyclic, we can always write $\angle UXV = \angle UYV$ even if X and Y are on opposite sides of UV (and the two angles are measured in opposite senses).

The reflection Q' of Q about AP lies on the circles with diameters AB and AP, so that

$$\angle BQ'P = \angle BQ'A + \angle AQ'P = 90^\circ + 90^\circ = 0^\circ$$

so that B, P and Q' are collinear. Thus

$$\angle APQ = \angle Q'PA = \angle BPA = \angle BCA.$$

Therefore

$$\begin{aligned}\angle CPQ &= \angle CPB + \angle BPA + \angle APQ\\ &= \angle CAB + \angle BCA + \angle BCA\\ &= \angle CAB + \angle BCA + \angle ABC = 0^\circ,\end{aligned}$$

so that C, P and Q are collinear. Since $AQ \perp QP$, then $AQ \perp QC$.

Editor's comments. The reflection in AP of the circle with diameter AB coincides with the reflection about AQ of circle with diameter AC. Is there a nice way of seeing this?

4256. *Proposed by Daniel Sitaru.*

Let $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Prove that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 4.$$

We received 13 solutions, 12 of which were correct and complete. We present 3 solutions.

Solution 1, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

We will prove the slight improvement that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 3e^{\frac{1}{3}} > 4$$

for distinct $a, b, c \in \mathbb{R}$, which satisfy the condition $a + b + c = 1$.

Note first that the last inequality follows from the fact that

$$\left(\frac{4}{3}\right)^3 = \frac{64}{27} = 2.\overline{370} < e. \tag{1}$$

For the remainder of our solution, we will utilize Hadamard's Inequality which states that if $f(x)$ is continuous and convex on $[p, q]$, then

$$\frac{1}{q-p}\int_p^q f(x)\,dx \geq f\left(\frac{p+q}{2}\right). \tag{2}$$

A proof of this result can be found in R. P. Boas, Jr., *A Primer of Real Functions (3rd. ed.),* Carus Mathematical Monograph No. 13, The Mathematical Association of America, 1981, pg. 174.

Since a and b must be distinct and

$$\frac{e^b - e^a}{b-a} = \frac{e^a - e^b}{a-b},$$

we may assume without loss of generality that $a < b$. Then, since $f(x) = e^x$ is continuous and convex on $\mathbb{R}$, (2) implies that

$$\frac{e^b - e^a}{b-a} = \frac{1}{b-a}\int_a^b e^x\,dx \geq e^{\frac{a+b}{2}}. \tag{3}$$

Similar arguments show that

$$\frac{e^c - e^b}{c-b} \geq e^{\frac{b+c}{2}} \quad \text{and} \quad \frac{e^a - e^c}{a-c} \geq e^{\frac{a+c}{2}}. \tag{4}$$

Further, because $f(x) = e^x$ is strictly convex on $\mathbb{R}$, Jensen's Theorem and the distinct values of a, b, and c imply that

$$e^{\frac{a+b}{2}} + e^{\frac{b+c}{2}} + e^{\frac{a+c}{2}} > 3e^{\frac{1}{3}\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{a+c}{2}\right)} = 3e^{\frac{a+b+c}{3}} = 3e^{\frac{1}{3}}. \tag{5}$$

Finally, it follows from (1), (3), (4), and (5) that

$$\frac{e^b - e^a}{b-a} + \frac{e^c - e^b}{c-b} + \frac{e^a - e^c}{a-c} \geq e^{\frac{a+b}{2}} + e^{\frac{b+c}{2}} + e^{\frac{a+c}{2}} > 3e^{\frac{1}{3}} > 3\left(\frac{4}{3}\right) = 4.$$

Solution 2, by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.

We prove a more general result.

Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = 1$, then

$$\sum_{k=1}^{n} \frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq ne^{1/n}, \tag{1}$$

with $a_{n+1} = a_1$. Moreover, the equality holds if and only if $a_i = \frac{1}{n}$, for $i = 1, \ldots, n$ (in this case the left hand side has to be understood as a limit).

The proposed inequality follows taking $n = 3$, $a_1 = a$, $a_2 = b$, and $a_3 = c$ and using that $3e^{1/3} = 4.186837 > 4$.

Let us prove (1). From the inequality $\frac{\sinh x}{x} \geq 1$, for $x \in \mathbb{R}$, with equality for $x = 0$ only, taking $x = (a_{k+1} - a_k)/2$, we deduce that

$$\frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq e^{(a_{k+1}+a_k)/2},$$

with equality when $a_{k+1} = a_k$. In this way, applying the AM-GM inequality, we have

$$\sum_{k=1}^{n} \frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq \sum_{k=1}^{n} e^{(a_{k+1}+a_k)/2} \geq ne^{(a_1+\cdots+a_n)/n} = ne^{1/n}$$

and the equality holds when $a_i = 1/n$, for $i = 1, \ldots, n$, only.

Solution 3, by Paul Bracken.

By Taylor's theorem, we have the expansion with remainder

$$e^b = e^a + e^a(b-a) + \frac{1}{2}e^a(b-a)^2 + \frac{e^{\tau_1}}{6}(b-a)^3,$$

where τ_1 in the remainder is between a and b. This implies that

$$\frac{e^b - e^a}{b-a} = e^a + \frac{1}{2}e^a(b-a) + \frac{e^{\tau_1}}{6}(b-a)^2 \geq e^a + \frac{1}{2}e^a(b-a),$$

since $e^{\tau_1} > 0$ and $(b-a)^2 \geq 0$ always holds. In exactly the same way, we obtain the inequalities

$$\frac{e^c - e^b}{c-b} = e^b + \frac{1}{2}e^b(c-b) + \frac{e^{\tau_2}}{6}(c-b)^2 \geq e^b + \frac{1}{2}e^b(c-b),$$

$$\frac{e^a - e^c}{a-c} = e^c + \frac{1}{2}e^c(a-c) + \frac{e^{\tau_3}}{6}(a-c)^2 \geq e^c + \frac{1}{2}e^c(a-c).$$

Adding these three results, the following lower bound for the function in (1) is obtained,

$$h(a,b,c) = \frac{e^b - e^a}{b-a} + \frac{e^c - e^b}{c-b} + \frac{e^a - e^c}{a-c} \geq e^a + e^b + e^c + \frac{1}{2}(e^a(b-a) + e^b(c-b) + e^c(a-c)). \quad (2)$$

This result holds for all a, b, c and is independent of the constraint which has not been used.

Let us minimize the function on the right of (2),

$$f(a,b,c) = e^a + e^b + e^c + \frac{1}{2}(e^b(c-b) + e^b(c-b) + e^c(a-c)),$$

by introducing a Lagrange multiplier λ

$$\mathcal{L} = f(a,b,c) - \lambda(a+b+c-1).$$

Differentiating $\mathcal{L}$ with respect to a, b, c and λ , the following nonlinear system results,

$$\begin{aligned} e^a + e^c + e^a(b-a) - 2\lambda &= 0, \\ e^b + e^a + e^b(c-b) - 2\lambda &= 0, \\ e^c + e^b + e^c(a-c) - 2\lambda &= 0, \\ a + b + c - 1 &= 0. \end{aligned} \tag{3}$$

This set of equations maps into itself under a cyclic permutation of the variables. The first three equations of (3) can be put in the form,

$$1 + b - a + e^{c-a} = 2\lambda e^{-a}, \quad 1 + c - b + e^{a-b} = 2\lambda e^{-b}, \quad 1 + a - c + e^{c-b} = 2\lambda e^{-c}.$$

For example, adding these three equations, an expression for λ results,

$$\lambda = \frac{e^{a-b} + e^{c-a} + e^{c-b} + 3}{2(e^{-a} + e^{-b} + e^{-c})}.$$

In fact, the solution to the system (3) is given by

$$a = b = c = \frac{1}{3}, \qquad \lambda = e^{1/3}.$$

The minimum value of f is found to be

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3e^{1/3} > 4. \tag{4}$$

This will correspond to a minimum since a maximum is not expected. Take for example $a = N$, $b = -N + 1$ and $c = 0$, then $e^N \to \infty$ as $N \to \infty$, so h can be made as large as we please. Combining (2) and (4), these imply (1).

Letting $c \to b$ and then $b \to a$ in h and the constraint, or using Taylor's formula, it can be seen that h reduces to $3e^{1/3}$ which matches the minimum (4). Thus the absolute minimum of h under the constraint is $3e^{1/3}$.

4257. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Calculate the following limit

$$\lim_{n\to\infty} \left(\frac{\sqrt[n(n+1)]{1! \cdot 2! \cdot \ldots n!}}{\sqrt{n}} \right).$$

We received 9 submissions, all correct, and we present the solution by the Missouri State University Problem Solving Group, modified slightly by the editor.

Let

$$A_n = \frac{\sqrt[n(n+1)]{1!2! \cdots n!}}{\sqrt{n}}.$$

Then

$$\begin{aligned}
\log A_n &= \frac{1}{n(n+1)}\sum_{k=1}^{n}\log k! - \frac{1}{2}\log n \\
&= \frac{1}{n(n+1)}\sum_{k=1}^{n}(n-k+1)\log k - \frac{1}{2}\log n \\
&= \frac{1}{n(n+1)}\left(\sum_{k=1}^{n}(n-k+1)\log k - \frac{n(n+1)}{2}\log n\right) \\
&= \frac{1}{n(n+1)}\left(\sum_{k=1}^{n}(n-k+1)\log k - \left(\sum_{k=1}^{n}k\right)\cdot\log n\right) \\
&= \frac{1}{n(n+1)}\left(\sum_{k=1}^{n}(n-k+1)\log k - \sum_{k=1}^{n}(n-k+1)\log n\right) \\
&= \frac{1}{n(n+1)}\left(\sum_{k=1}^{n}(n-k+1)\log\frac{k}{n}\right) \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n}\frac{n-k+1}{n}\log\frac{k}{n}\right) \\
&= \frac{1}{n+1}\sum_{k=1}^{n}\left(1-\frac{k-1}{n}\right)\log\frac{k}{n} \\
&= \frac{1}{n+1}\sum_{k=1}^{n}\left(\left(1-\frac{k}{n}\right)\log\frac{k}{n} + \frac{1}{n}\log\frac{k}{n}\right) \\
&= \frac{1}{n+1}B_n + \frac{1}{n+1}C_n \qquad (1)
\end{aligned}$$

where

$$B_n = \sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\log\frac{k}{n}$$

and

$$C_n = \frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{n}.$$

Note that for all n, we have $C_n \le 0$ and

$$C_n > \frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{n} = \log\frac{1}{n} = -\log n,$$

so

$$-\frac{\log n}{n+1} \le \frac{1}{n+1}C_n \le 0.$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n+1}C_n = 0. \qquad (2)$$

On the other hand,

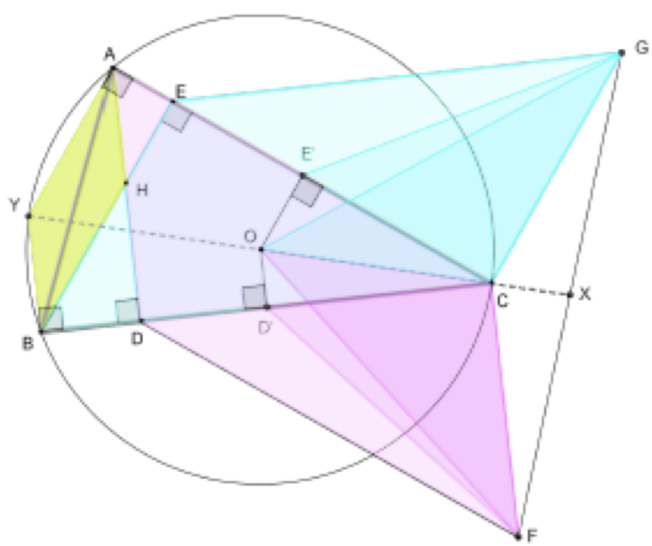
$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n+1}B_n &= \lim_{n\to\infty}\frac{n}{n+1}\frac{1}{n}B_n\\ &= \lim_{n\to\infty}\frac{1}{n}B_n\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\log\frac{k}{n}\left(\frac{1}{n}\right)\\ &= \int_0^1(1-x)\log x dx = -\frac{3}{4}. \qquad (3)\end{aligned}$$

From (1) to (3) we then obtain

$$\begin{aligned}\lim_{n\to\infty}A_n &= \exp\left(\lim_{n\to\infty}\frac{1}{n+1}B_n+\lim_{n\to\infty}\frac{1}{n+1}C_n\right)\\ &= e^{-\frac{3}{4}}.\end{aligned}$$

4258. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle O, orthocentre H, $D \in BC, AD \perp BC, E \in AC, BE \perp AC$. Define points F and G to be the fourth vertices of parallelograms $CADF$ and $CBEG$. If X is the midpoint of FG, and Y is the point where XC intersects the circumcircle again, prove that $AHBY$ is a parallelogram.



All ten submissions were complete and correct. They came in two varieties, so we feature an example of each.

Solution 1, by Oliver Geupel.

If the points C, O, and X are collinear then the line segment CY is a diameter of the circumcircle (O) of triangle ABC, and by Thales' Theorem the angles $\angle YAC$ and $\angle CBY$ are right angles. This implies that $AY \parallel EB$ and $BY \parallel DA$; that is, the quadrilateral $AHBY$ is a parallelogram. So all we have to do is to prove that the points C, O, and X are collinear.

Consider the problem in the plane of complex numbers where (O) is the unit circle, and capital letters serve as the complex numbers that represent their corresponding points. It is an easily verifiable fact that the foot of the perpendicular from an arbitrary point Z to the chord through points P and Q of the unit circle is the point $(P + Q + Z - PQ\bar{Z})/2$. Hence,

$$D = \frac{1}{2}\left(A + B + C - \frac{BC}{A}\right)$$

and

$$F = C + D - A = \frac{1}{2}\left(-A + B + 3C - \frac{BC}{A}\right).$$

Similarly,

$$G = \frac{1}{2}\left(A - B + 3C - \frac{AC}{B}\right).$$

Thus,

$$X = \frac{1}{2}(F + G) = \frac{1}{4}\left(6 - \frac{B}{A} - \frac{A}{B}\right)C.$$

The complex number $\frac{B}{A} + \frac{A}{B}$ is a real number because it is equal to its complex conjugate. Hence the complex number X is a real multiple of the complex number C, which proves that the points C, O, and X are collinear.

Solution 2, by Titu Zvonaru.

As usual, we let $a = BC, b = CA, c = AB, h_a = AD, h_b = BE$, and we use square brackets to represent areas. Suppose that OC intersects FG at X'. Since $\angle ACG = 90^\circ$ and $\angle AOC = \angle 2B$, we have $\angle OCA = 90^\circ - \angle B$ and $\angle X'CG = \angle B$. Similarly, $\angle FCX' = \angle A$. It follows that

$$\frac{[GCX']}{[CFX']} = \frac{CG \cdot CX' \sin\angle X'CG}{CF \cdot CX' \sin\angle FCX'} = \frac{h_b \sin\angle B}{h_a \sin\angle A} = \frac{bh_b}{ah_a} = 1.$$

We deduce that the point X' coincides with X, the midpoint of FG, whence CY is a diameter of the circumcircle. Consequently, $YA \perp AC$, implying that $YA||BH$; and $YB \perp BC$, implying that $YB||AH$; hence the quadrilateral $AHBY$ is a parallelogram.

Editor's comments. Muralidharan observed that there is no need to place restrictions on the angles of the given triangle: no matter what the angles of ΔABC

might be, $AHBY$ will be a parallelogram, although if there is a right angle at A or B then the resulting parallelogram will degenerate into the line segment AB. Either of the featured solutions apply to an arbitrary triangle (although in Solution 2, angles should be interpreted as directed angles).

4259. *Proposed by Mihály Bencze.*

Prove that

$$\prod_{k=1}^{n}\left(\frac{\sum_{p=1}^{k}\frac{1}{2p-1}}{\sum_{p=1}^{k}\frac{1}{p}}\right) \geq \frac{n+1}{2^n}.$$

We received 11 correct solutions. We present the solution by AN-anduud Problem Solving Group.

The result is clear for $n = 1$. Let $n > 1$. We have

$$\begin{aligned} A_k &= \frac{\sum_{p=1}^{k}\frac{1}{2p-1}}{\sum_{p=1}^{k}\frac{1}{p}} \\ &= \frac{1+\sum_{p=2}^{k}\frac{1}{2p-1}}{\sum_{p=1}^{k}\frac{1}{p}} \\ &> \frac{\frac{1}{2}+\frac{1}{2}+\sum_{p=2}^{k}\frac{1}{2p}}{\sum_{p=1}^{k}\frac{1}{p}} \\ &= \frac{1}{2}\cdot\frac{1}{\sum_{p=1}^{k}\frac{1}{p}}+\frac{1}{2}\cdot\frac{1+\sum_{p=2}^{k}\frac{1}{p}}{\sum_{p=1}^{k}\frac{1}{p}} \\ &> \frac{1}{2}\cdot\frac{1}{k}+\frac{1}{2} = \frac{k+1}{2k}. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} A_n &> \frac{n+1}{2n}, \\ A_{n-1} &> \frac{n}{2(n-1)}, \\ \cdots &\quad \cdots \quad \cdots \\ A_3 &> \frac{4}{2\cdot 3}, \\ A_2 &> \frac{3}{2\cdot 2}, \\ A_1 &= 1, \end{aligned}$$

which implies that

$$\prod_{k=1}^{n} A_k > \frac{n+1}{2^n}.$$

4260. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let $a_i, i = 1, \ldots, 6$ be positive numbers such that

$$a_1 + a_2 + a_3 - a_4 - a_5 - a_6 = 3 \quad \text{and} \quad a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 9.$$

Prove that

$$a_1 a_2 a_3 a_4 a_5 a_6 \leq 1.$$

We received 8 solutions. We present the solution by the AN-anduud Problem Solving Group.

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 3 + a_4 + a_5 + a_6 = a_1 + a_2 + a_3 &\leq \sqrt{(1^2 + 1^2 + 1^2)(a_1^2 + a_2^2 + a_3^2)} \\ &= \sqrt{3(a_1^2 + a_2^2 + a_3^2)} \\ &= \sqrt{3(9 - (a_4^2 + a_5^2 + a_6^2))} \\ &= \sqrt{27 - (1^2 + 1^2 + 1^2)(a_4^2 + a_5^2 + a_6^2)} \\ &\leq \sqrt{27 - (a_4 + a_5 + a_6)^2}. \end{aligned}$$

Hence

$$\begin{aligned} &(3 + (a_4 + a_5 + a_6))^2 \leq 27 - (a_4 + a_5 + a_6)^2 \\ &\Leftrightarrow \ (a_4 + a_5 + a_6)^2 + 3(a_4 + a_5 + a_6) - 9 \leq 0 \\ &\Rightarrow \ a_4 + a_5 + a_6 \leq \frac{-3 + 3\sqrt{5}}{2} \qquad (1) \\ &\Rightarrow \ \frac{a_4 + a_5 + a_6}{3} \leq \frac{-1 + \sqrt{5}}{2}. \end{aligned}$$

Applying the AM-GM inequality, we have

$$\sqrt[3]{a_3 a_4 a_5} \leq \frac{a_4 + a_5 + a_6}{3} \leq \frac{-1 + \sqrt{5}}{2}. \qquad (2)$$

From (1), we get

$$a_1 + a_2 + a_3 = 3 + (a_4 + a_5 + a_6) \leq 3 + \frac{-3 + 3\sqrt{5}}{2} = \frac{3 + 3\sqrt{5}}{2},$$

which implies that

$$\frac{a_1 + a_2 + a_3}{3} \leq \frac{1 + \sqrt{5}}{2}.$$

Using the AM-GM inequality, we get

$$\sqrt[3]{a_1 a_2 a_3} \leq \frac{a_1 + a_2 + a_3}{3} \leq \frac{1 + \sqrt{5}}{2}. \qquad (3)$$

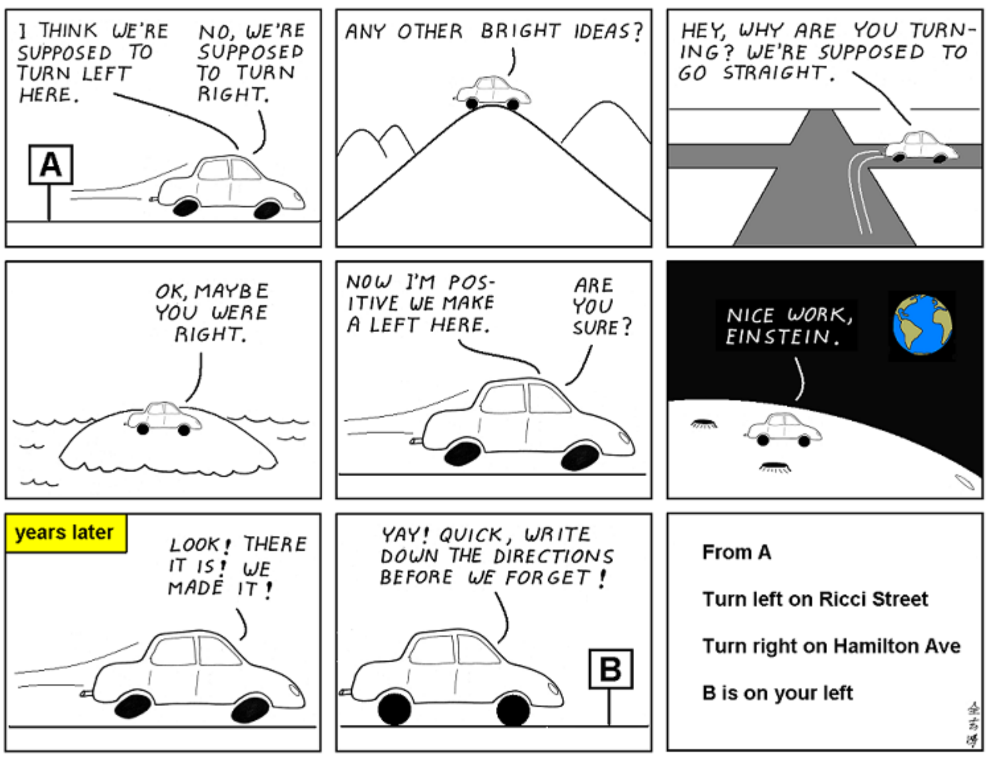
From (2) and (3), we have

$$\sqrt[3]{a_1a_2a_3} \cdot \sqrt[3]{a_4a_5a_6} \leq \frac{-1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} = 1$$

or

$$a_1a_2a_3a_4a_5a_6 \leq 1.$$

Equality holds only when $a_1 = a_2 = a_3 = \frac{1+\sqrt{5}}{2}$, $a_4 = a_5 = a_6 = \frac{-1+\sqrt{5}}{2}$.



This is how most mathematical proofs are written.

AUTHORS' INDEX

Solvers and proposers appearing in issues 5 and 6

(Bold font indicates featured solution.)

Proposers

Michel Bataille, Rouen, France: 4344, 4353
Mihaela Berindeanu, Bucharest, Romania : 4343, 4355
Ruben Dario Auqui, Peru, and Leonard Giugiuc, Romania : 4354
Marius Drăgan, "Mircea cel Batran" Technical College, Bucharest, Romania : 4348
J. Chris Fisher, Cleveland Heights, OH, USA : 4347
Leonard Giugiuc, Romania : 4340
Leonard Giugiuc and Diana Trailescu, Romania : 4356
Thanos Kalogerakis, Kiato, Greece : 4352
Hoang Le Nhat Tung, Hanoi, Vietnam : 4349
Mihai Miculiţa and Titu Zvonaru, Romania : 4345
Miguel Ochoa Sanchez, Peru, and Leonard Giugiuc, Romania : 4351
H. A. ShahAli, Tehran, Iran : 4360
Daniel Sitaru, Romania: 4359
Daniel Sitaru and Leonard Giugiuc, Romania : 4341
George Stoica, Saint John, New Brunswick, Canada : 4358
Oai Thanh Dao, Vietnam, and Leonard Giugiuc, Romania : 4342
Arsalan Wares, Valdosta State University, Valdosta, USA : 4357

Solvers - individuals

Mohammed Aassila, Strasbourg, France : **OC323**, OC326, **OC328**, **4244**
Arkady Alt, San Jose, CA, USA : 4243, **4248**, **4249**, 4254, 4256, 4259
George Apostolopoulos, Messolonghi, Greece : 4254
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : CC276, 4254, 4256, 4257
Roy Barbara, Lebanese University, Fanar, Lebanon : **4241**, 4242, 4250, 4253
Michel Bataille, Rouen, France : 4243, **4244**, 4245, **4247**, 4248, 4250, 4253, **4254**, 4255, 4256, 4257, 4258, OC322
Brian D. Beasley, Presbyterian College, Clinton, USA : **4241**, 4242
Mihaly Bencze, Ady Endre Highschool, Bucharest, Romania : 4242, 4259
Mihaela Berindeanu, Bucharest, Romania: 4245, 4258
Paul Bracken, University of Texas, Edinburg, TX : **4248**, **4256**, 4259
Mihai Bunget, Taru Jiu, Romania : 4241, 4242, 4243, 4246, 4247, 4249, 4250
Steven Chow, Albert Campbell Collegiate Institute, Scarborough, ON : 4241, 4242, 4244, 4245, 4247, 4249, **4250**, **4253**, **4255**, **OC321**, OC322, OC323, **OC326**, **OC327**, OC328, 4258, 4259
Marian Cucoanes, Marasesti, Romania : 4250
Dan Daniel, National College Gheorghe Titeica, Romania : 4249, 4250, 4260, OC322
José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain : 4259, **OC330**
Ivko Dimitrić, Pennsylvania State University, Fayette, PA: 4241, 4247, **4248**, **4255**, 4253, 4258, CC271, **CC272**, **CC273**, **CC274**, **CC275**, CC276, **CC277**, CC278, **CC279**

Solvers - collaborations