Crux Mathematicorum

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Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

THE CONTEST CORNER

No. 53

John McLoughlin

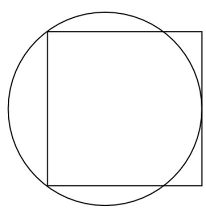
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **November 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC261. Peter is walking through a train tunnel when he hears a train approaching. He knows that on this section of track trains travel at 60 mph. The tunnel has equally spaced marker posts, with post 0 at one end and post 12 at the other end. Peter is by post 7 when he hears the train. He quickly works out that whether he runs to the nearer end or the further end of the tunnel as fast as he can (at constant speed) he will just exit the tunnel before the train reaches him. How fast can Peter run?

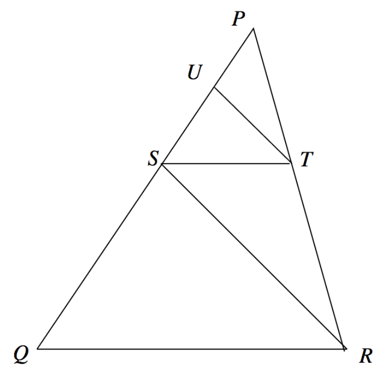
CC262. In the diagram, the square has two of its vertices on the circle of radius 1 unit and the other two vertices lie on a tangent to the circle. Find the area of the square.



CC263. An old fashioned tram starts from the station with a certain number of men and women on board. At the first stop, a third of the women get out and their places are taken by men. At the next stop, a third of the men get out and their places are taken by women. There are now two more women than men and as many men as there originally were women. How many men and women were there on board at the start?

CC264. Kirsty runs three times as fast as she walks. When going to school one day she walks for twice the time she runs and the journey takes 21 minutes. The next day she follows the same route but runs for twice the time she walks. How long does she take to get to school?

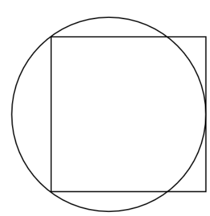
CC265. In the diagram, ST is parallel to QR, UT is parallel to SR, $PU = 4$ and $US = 6$. Find the length of SQ.



. . . .

CC261. Pierre marche dans un tunnel de chemin de fer lorsqu'il entend venir un train. Il sait que sur cette section du chemin de fer, le train se déplace à une vitesse de 60 km/h. Des bornes numériques équidistantes sont placées dans le tunnel, la borne 0 étant placée à l'entrée et la borne 12 étant à l'autre extrémité. Pierre est vis-à-vis la borne 7 lorsqu'il entend le train. Il calcule rapidement que s'il court à toute vitesse (constante) vers l'une ou l'autre extrémité du tunnel, il réussira à sortir du tunnel juste avant l'arrivée du train. À quelle vitesse Pierre peut-il courir?

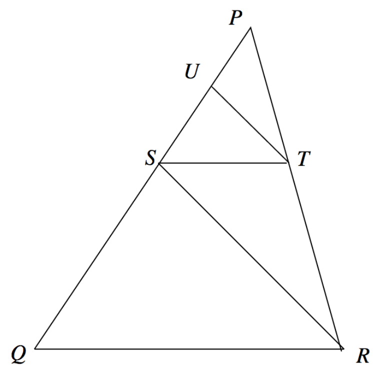
CC262. Dans la figure suivante, deux sommets du carré sont situés sur le cercle de rayon 1 et ses deux autres sommets sont situés sur une tangente au cercle. Déterminer l'aire du carré.



CC263. Un tramway quitte une station avec à bord un nombre quelconque d'hommes et de femmes. Au premier arrêt, un tiers des femmes descendent et leurs places sont prises par des hommes qui arrivent. Au deuxième arrêt, un tiers des hommes descendent et leurs places sont prises par des femmes qui arrivent. Il y a maintenant deux femmes de plus que d'hommes à bord. De plus, le nombre d'hommes est le même que le nombre initial de femmes à bord. Combien y avait-il d'hommes et de femmes à bord au départ?

CC264. Kim court trois fois plus vite qu'elle ne marche. Un jour, en se rendant à l'école, elle met deux fois plus de temps à marcher qu'à courir et complète le trajet en 21 minutes. Le lendemain, elle prend le même chemin, mais elle met deux fois plus de temps à courir qu'à marcher. Combien de temps met-elle pour se rendre à l'école?

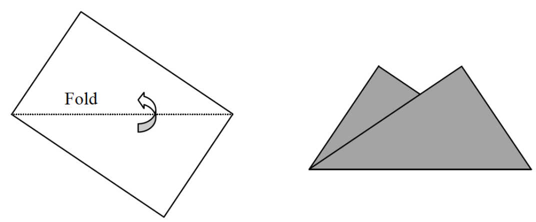
CC265. Dans la figure suivante, ST est parallèle à QR, UT est parallèle à SR, $PU = 4$ et $US = 6$. Déterminer la longueur de SQ.



CONTEST CORNER SOLUTIONS

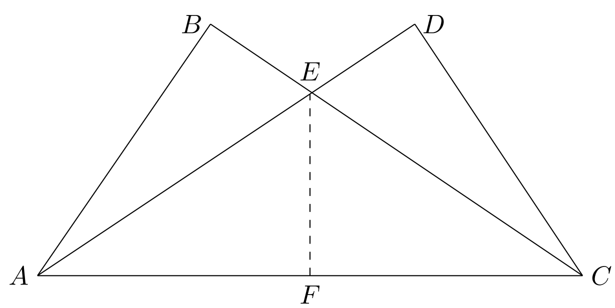
Statements of the problems in this section originally appear in 2016: 42(3), p. 96–98.

CC211. A rectangular sheet of paper whose dimensions are 12×18 is folded along a diagonal, which creates the M-shaped region drawn at the right. Find the area of the shaded region.



Originally question 7 of the 2016 University of North Colorado Math Contest (First Round).

We received eight correct solutions. We present the solution of Doddy Kastanya.



The area of the folded configuration is basically the area of the rectangle minus the area of triangle ACE since there is an overlap there. So, determining the area of the original rectangle is straight-forward. The area of the rectangle is 12×18 or 216. The base of triangle ACE (line AC) is simply the hypotenuse of triangle ABC (since $\angle ABC$ is right). So, the length of AC is $\sqrt{12^2 + 18^2}$ or $6\sqrt{13}$. The height of triangle ACE is EF which can be determined as $FC \times \tan \angle BAC$. The length of FC is $3\sqrt{13}$. Using triangle ABC, $\tan \angle BAC$ is equal to $\frac{12}{18} = \frac{2}{3}$. So, the area of triangle ACE is

$$\frac{1}{2} AC \times EF = \frac{1}{2}(6\sqrt{13}) \times (\frac{2}{3} 3\sqrt{13}) = 78.$$

So, the overall shaded area is $216 - 78$ or 138.

CC212. A cube that is one inch wide has had its eight corners shaved off. The cube's vertices have been replaced by eight congruent equilateral triangles, and the square faces have been replaced by six congruent octagons. If the combined area of the eight triangles equals the area of one of the octagons, what is that area? (Each octagonal face has two different edge lengths that occur in alternating order.)

Originally question 3 of the 2016 University of North Colorado Math Contest (Final Round).

We received three correct and complete solutions, out of which we present the one by John G. Heuver.

Let the edges of the base of a shaved off tetrahedron be x. Then the remaining faces of the tetrahedron are right angled isosceles triangles. Let the length of the legs of those triangles be p, so $2p^2 = x^2$. Thus the area of one of the octagons is

$$1 - 4 \cdot \frac{p^2}{2} = 1 - x^2 \qquad (1)$$

and the combined area of the eight triangles is

$$8 \cdot \frac{\sqrt{3}}{4} x^2 = 2\sqrt{3}\, x^2.$$

Hence $1 - x^2 = 2\sqrt{3}\, x^2$ or $x^2 = \frac{1}{1+2\sqrt{3}}$. Substituting this into (1) and rationalizing gives us that the area of one of the octagons in square inches is

$$\frac{12 - 2\sqrt{3}}{11}.$$

CC213. A pyramid is built from solid unit cubes that are stacked in square layers. The top layer has $1 \times 1 = 1$ cube, the second $3 \times 3 = 9$ cubes and the layer below that has $5 \times 5 = 25$ cubes, and so on, with each layer having two more cubes on a side than the layer above it. The pyramid has a total of 12 layers. Find the exposed surface area of this solid pyramid, including the bottom.

Originally question 8 of the 2016 University of North Colorado Math Contest (First Round).

We received four correct and complete solutions. We present the solution of Carlos Vega and Ángel Plaza.

Since each layer of the pyramid has two more cubes on a side than the layer above it, the n-th layer from the top contains $(2n-1) \times (2n-1)$ cubes. Therefore the bottom layer consists of 23×23 cubes. Thus from below and above one can see 23^2 squares. From each of the four sides, the number of squares one can see is

$$\sum_{k=1}^{12}(2n-1) = 12^2.$$

Hence the exposed surface area is $2 \times 23^2 + 4 \times 12^2 = 1634$ square units.

CC214. The points $(2,5)$ and $(6,5)$ are two of the vertices of a regular hexagon of side length two on a coordinate plane. There is a line L that goes through the point $(0,0)$ and cuts the hexagon into two pieces of equal area. What is the slope of line L?

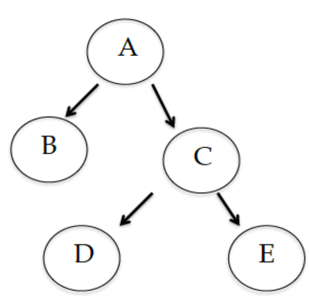
Originally question 6 of the 2016 University of North Colorado Math Contest (First Round).

We received seven submissions of which six were correct and complete. We present the solution by Fernando Ballesta Yagüe.

We have that $(2,5),(6,5)$ are two of the vertices of the hexagon. The segment AB has length 4. In a regular hexagon with side length 2, two vertices whose distance is 4 are opposed to the center. Therefore, the center of the hexagon will be the midpoint of A and B, which is $(4,5)$. Since a regular hexagon is a symmetric figure, a line that divides it in two pieces of equal area will pass through its center. As we know two points of this line (the center and the origin of coordinates), we can find out its slope: $m = \dfrac{5-0}{4-0} = \dfrac{5}{4}$.

CC215. Each circle in this tree diagram is to be assigned a value, chosen from a set S, in such a way that along every pathway down the tree the assigned values never increase. That is, $A \geq B, A \geq C, C \geq D, C \geq E$ and $A, B, C, D, E \in S$. (It is permissible for a value in S to appear more than once.) How many ways can the tree be so numbered using only values chosen from the set $S = \{1, \ldots, 6\}$?

(Optional extension: Generalize to a case with $S = \{1, 2, 3, , \ldots, n\}$ by finding an explicit algebraic expression for the number of ways the tree can be numbered.)



Originally question 8 of the 2016 University of North Colorado Math Contest (Final Round).

We received two correct solutions and one incomplete submission. We present the solution by Steven Chow.

A is any integer between 1 and n. If A is fixed, then B and C are any integers between 1 and A, which are A possibilities. If C is also fixed, then there are C possibilities for each of D and E. We can thus calculate the number of ways that the tree can be numbered as follows.

$$\begin{aligned}\sum_{A=1}^{n} A \sum_{C=1}^{A} C^2 &= \sum_{A=1}^{n} A\left(\frac{1}{6}A + \frac{1}{2}A^2 + \frac{1}{3}A^3\right)\\ &= \frac{1}{6}\sum_{A=1}^{n} A^2 + \frac{1}{2}\sum_{A=1}^{n} A^3 + \frac{1}{3}\sum_{A=1}^{n} A^4\\ &= \frac{1}{6}\left[\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}\right]\\ &\quad + \frac{1}{2}\left[\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}\right]\\ &\quad + \frac{1}{3}\left[\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}\right]\\ &= \binom{n}{1} + 9\binom{n}{2} + 23\binom{n}{3} + 23\binom{n}{4} + 8\binom{n}{5}.\end{aligned}$$

For the special case of $n = 6$, this comes to 994 ways to fill the tree.

Editor's comments. The formula for the sum of the k-th powers used in the calculation is

$$\sum_{k=0}^{n} k^m = \sum_{j=1}^{n} (j-1)! \left\{ {m+1 \atop j} \right\} \binom{n}{j},$$

where $\left\{ {m+1 \atop j} \right\}$ is a Stirling number of the second kind.

THE OLYMPIAD CORNER

No. 351

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **November 1, 2017***.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC321. Solve in positive integers

$$x^y y^x = (x+y)^z.$$

OC322. Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$. Prove that

$$a^2b + b^2c + c^2a \geq \sqrt{(a+b+c)(ab+bc+ca)}.$$

OC323. Let ABC be a triangle. M, and N points on BC, such that $BM = CN$, with M in the interior of BN. Let P and Q be points in AN and AM respectively such that $\angle PMC = \angle MAB$, and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

OC324. Given an integer $n > 1$ and its prime factorization $n = p_1^{\alpha 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, define a function

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} ... \alpha_k p_k^{\alpha_k - 1}.$$

Prove that there exist infinitely many integers n such that $f(n) = f(n-1) + 1$.

OC325. Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be coloured red or blue. (The subset itself is assigned a colour and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colourings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

. . . .

OC321. Déterminer les solutions entières strictement positives de l'équation

$$x^y y^x = (x+y)^z.$$

OC322. Soit a, b et c des réels strictement positifs tels que $abc = 1$. Démontrer que

$$a^2b + b^2c + c^2a \geq \sqrt{(a+b+c)(ab+bc+ca)}.$$

OC323. Soit un triangle ABC. Soit M et N des points sur BC tels que $BM = CN$ et que M soit sur le segment BN. Soit P et Q des points sur les segments respectifs AN et AM tels que $\angle PMC = \angle MAB$ et $\angle QNB = \angle NAC$. Démontrer que $\angle QBC = \angle PCB$.

OC324. Soit un entier $n, n > 1$, et sa factorisation première $n = p_1^{\alpha 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. On définit une fonction f comme suit:

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} ... \alpha_k p_k^{\alpha_k - 1}.$$

Démontrer qu'il existe une infinité de valeurs de n pour lesquelles

$$f(n) = f(n-1) + 1.$$

OC325. Soit $S = \{1, 2, \ldots, n\}$ $(n \geq 1)$. On veut colorer chacun des 2^n sous-ensembles de S en rouge ou en bleu. (Chaque sous-ensemble reçoit une couleur et non pas ses éléments.) Étant donné un sous-ensemble T de S, $f(T)$ représente le nombre de sous-ensembles de T qui sont bleus.

Déterminer le nombre de coloriages qui satisfont à la condition suivante:

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2)$$

pour tous sous-ensembles T_1 et T_2 de S.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(1), p. 11–12.

OC261. Show that there are no 2-tuples (x, y) of positive integers satisfying the equation $(x+1)(x+2)\cdots(x+2014) = (y+1)(y+2)\cdots(y+4028)$.

Originally problem 3 from day 1 of the 2014 China Team Selection Test.

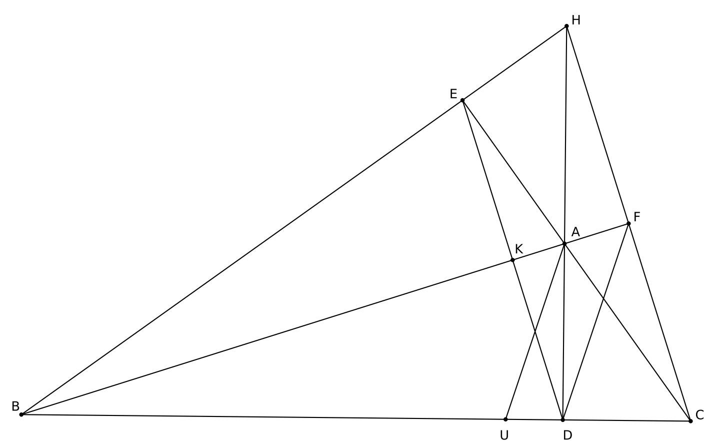
No submitted solutions.

OC262. In obtuse triangle ABC, with the obtuse angle at A, let D, E, F be the feet of the altitudes through A, B, C respectively. DE is parallel to CF, and DF is parallel to the angle bisector of $\angle BAC$. Find the angles of the triangle.

Originally problem 3 of the 2014 South Africa National Olympiad.

We received 5 correct submissions and 1 incorrect submission. We present the solution by Michel Bataille.

We show that $\angle A = \frac{3\pi}{5}$, $\angle B = \frac{\pi}{10}$, $\angle C = \frac{3\pi}{10}$.



Let $BC = a, CA = b, AB = c$, as usual. Since $DE \parallel CF$, the line DE intersects AB orthogonally, say at K. Let U be the foot of the internal bisector of $\angle BAC$. We know that $\frac{BU}{c} = \frac{UC}{b} = \frac{a}{b+c}$; in addition, since $AU \parallel DF$, we have $\frac{BD}{BF} = \frac{BU}{BA}$, hence $BD = \frac{a}{b+c}BF$.

Also $\frac{BD}{c} = \frac{BF}{a}(= \cos B)$ so that $BD = \frac{c}{a}BF$. This yields $\frac{a}{b+c} = \frac{c}{a}$, that is, $bc = a^2 - c^2$. It follows that $\sin B \sin C = \sin^2 A - \sin^2 C$, which successively rewrites as

$$\sin B \sin C = (\sin A - \sin C)(\sin A + \sin C),$$
$$\sin B \sin C = 2 \sin \frac{A-C}{2} \cos \frac{A+C}{2} \cdot 2 \sin \frac{A+C}{2} \cos \frac{A-C}{2},$$
$$\sin B \sin C = \sin(A - C) \sin(A + C).$$

Since $\sin(A+C) = \sin B$ and $A-C$ and C are acute, we obtain $A-C = C$. Thus, $A = 2C$ and $B = \pi - 3C$.

Now, let H be the orthocentre of ΔABC. Clearly, D and E are on the circle with diameter HC, hence the trapezoid $CHED$ is isosceles and $HE = DC$.

Then the right-angled triangles AEH and ADC, which obviously are similar, are congruent and so $AE = AD$. It follows that BA is the angle bisector of $\angle CBH$ and $\angle HBF = B = \pi - 3C$. Since we also have $\angle BAE = \pi - A = \pi - 2C$, we obtain that

$$\angle HBF = \angle EBA = \frac{\pi}{2} - (\pi - 2C) = 2C - \frac{\pi}{2}.$$

We conclude that $\pi - 3C = 2C - \frac{\pi}{2}$, hence

$$C = \frac{3\pi}{10}, \quad A = 2C = \frac{3\pi}{5}, \quad B = \pi - 3C = \frac{\pi}{10}.$$

OC263. An integer $n \geq 3$ is called *special* if it does not divide

$$(n-1)! \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right).$$

Find all special numbers n such that $10 \leq n \leq 100$.

Originally problem 5 from day 2 of the 2014 Argentine National Olympiad.

We received 2 correct submissions. We present the solution by Oliver Geupel.

We prove the following claims:

(1) Every odd number is not special.

(2) For every prime number p, the number $n = 2p$ is special.

(3) For every prime number $p \geq 3$, the number $n = 2p^2$ is not special.

(4) For every prime p and every integer $s \geq 3$, $n = 2p^s$ is not special.

(5) For any coprime integers $q \geq 2$ and $r \geq 2$, $n = 2qr$ is not special.

As a consequence, the desired numbers are

$$10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, \text{ and } 94.$$

Proofs. If n is odd, then

$$(n-1)!\sum_{k=1}^{n-1}\frac{1}{k}=\sum_{k=1}^{(n-1)/2}\frac{(n-1)!}{k(n-k)}\left((n-k)+k\right)=n\sum_{k=1}^{(n-1)/2}\frac{(n-1)!}{k(n-k)}$$

is divisible by n, which proves (1).

Next, let n be even, say, $n=2m$. Then,

$$(n-1)!\sum_{k=1}^{n-1}\frac{1}{k}=n\sum_{k=1}^{(n-1)/2}\frac{(n-1)!}{k(n-k)}+\frac{(n-1)!}{n/2};$$

whence n is special if and only if $2m^2\nmid(2m-1)!$. Since for every prime p it holds $p^2\nmid(2p-1)!$, the claim (2) follows.

For every prime $p\geq 3$, we have $p<2p<p^2<2p^2-1$; hence $2p^4\mid(2p^2-1)!$, which proves (3).

Let $p\geq 3$ be a prime number and $s\geq 3$ be a natural number. Then

$$2<p<p^2<\cdots<p^s\leq 2p^s-1.$$

Thus, $2p^{s(s+1)/2}=2p^{1+2+\cdots+s}$ divides $(2p^s-1)!$, so that

$$2p^{2s}\mid 2p^{s(s+1)/2}\mid(2p^s-1)!.$$

This proves (4) for $p\geq 3$.

For $s\geq 4$, we have $2<2^2<\cdots<2^s<2^{s+1}-1$, which implies

$$2^{2s+1}\mid 2^{s(s+1)/2}\mid(2^{s+1}-1)!.$$

Moreover, 16 is not special by inspection. We have proven (4) for $p=2$.

Let $q\geq 2$ and $r\geq 2$ be coprime numbers. Then, q, $2q$, r, and $2r$ are distinct numbers which are less than $2qr-1$. Consequently, $2q^2r^2\mid(2qr-1)!$. This proves (5) and completes the solution.

OC264. A positive integer is called beautiful if it can be represented in the form $\dfrac{x^2+y^2}{x+y}$ for two distinct positive integers x,y. A positive integer that is not beautiful is ugly.

1. Prove that 2014 is a product of a beautiful number and an ugly number.
2. Prove that the product of two ugly numbers is also ugly.

Originally problem 4 from day 2 of the 2014 Indonesia Mathematical Olympiad.

We received 2 correct submissions and 1 incorrect submission. We present the solution by Steven Chow.

From the Lemmas below, it follows that 2×19 is ugly and 53 is beautiful, so $2014 = (2)(19)(53)$ is the product of a beautiful number and an ugly number. Alternatively, note that

$$2014 = \frac{1330^2 + 2394^2}{1330 + 2394}$$

and thus 2014 is beautiful. Part 1 is then proven since 1 is ugly and $2014 = 1 \cdot 2014$.

Lemma 1 *For all integers $n \geq 1$, n is beautiful if and only if $2n$ is beautiful.*

Proof. If $n = \frac{x^2+y^2}{x+y}$ for some distinct integers $x, y \geq 1$, then $2n = \frac{(2x)^2+(2y)^2}{2x+2y}$.

If $2n = \frac{x^2+y^2}{x+y}$ for some distinct integers $x, y \geq 2$, then $x^2 + y^2 \equiv 0 \pmod 2$, so $(x, y) \in \{(0, 0), (1, 1)\} \pmod 2$.

If $(x, y) \equiv (1, 1) \pmod 2$, then $x^2 + y^2 \equiv 1 + 1 \pmod 4$ and $\frac{x^2+y^2}{x+y} \not\equiv 0 \pmod 2$, so $x \equiv y \equiv 0 \pmod 2$.

Therefore, for some distinct integers $x_1, y_1 \geq 1$, $2n = \frac{x^2+y^2}{x+y} = \frac{(2x_1)^2+(2y_1)^2}{2x_1+2y_1}$. Thus $n = \frac{x_1^2+y_1^2}{x_1+y_1}$. □

Lemma 2 *For all integers $n \geq 1$ such that $n \equiv 1 \pmod 2$, n is beautiful if and only if $2n^2$ is a sum of 2 distinct square numbers.*

Proof. If $n = \frac{x^2+y^2}{x+y}$ for some distinct integers $x, y \geq 1$, then $(2x-n)^2+(2y-n)^2 = 2n^2$.

If $2n^2 = a^2 + b^2$ for some positive integers $a \neq b$, then $a^2 + b^2 = 2n^2 \equiv 2 \pmod 4$. Thus, $a \equiv b \equiv 1 \pmod 2$, so $\frac{a+n}{2}$ and $\frac{b+n}{2}$ are distinct positive numbers and

$$n = \left(\left(\frac{a+n}{2}\right)^2 + \left(\frac{b+n}{2}\right)^2\right)\Big/\left(\frac{a+n}{2} + \frac{b+n}{2}\right).$$

□

Lemma 3 *Let $k \geq 1$ be any integer. Let $p_j \equiv 3 \pmod 4$ be any prime, for all integers $1 \leq j \leq k$. Then $2(\prod_{j=1}^k p_j)^2$ is not the sum of 2 distinct square numbers.*

Proof. Assume for the sake of contradiction that there exist positive integers $a \neq b$ such that $a^2 + b^2 = 2(\prod_{j=1}^k p_j)^2$.

If there does not exist an integer $1 \leq m \leq k$ such that $p_m \nmid a$ and $p_m \nmid b$, then $p_m \mid a$ or $p_m \mid b$ for all $1 \leq m \leq k$. Hence, $a = b = \prod_{j=1}^k p_j$ which is a contradiction.

Thus, there exists an integer $1 \leq m \leq k$ such that $p_m \nmid a$ and $p_m \nmid b$. This implies that there exists an integer $1 \leq u \leq p_m - 1$ such that $au \equiv b \pmod{p_m}$, so

$$0 \equiv a^2 + b^2 \equiv a^2(1 + u^2) \pmod{p_m} \implies u^2 \equiv -1 \pmod{p_m} \implies \left(\frac{-1}{p_m}\right) = 1.$$

But $p_m \equiv 3 \pmod 4$, so this is a contradiction. □

Lemma 4 *Let $c \geq 1$ be any integer such that $c \equiv 1 \pmod 2$. Let $p \equiv 1 \pmod 4$ be any prime. Then $2(cp)^2$ is the sum of two distinct square numbers.*

Proof. It is well known that p is the sum of two square numbers. Let $a > b \geq 1$ be the integers such that $a^2 + b^2 = p$. Since $(a+b)^2 + (a-b)^2 = 2(a^2+b^2) = 2p$, we have

$$\begin{aligned} 2p^2 = p(2p) &= (a^2+b^2)((a+b)^2+(a-b)^2) \\ &= (a(a+b)+b(a-b))^2 + (a(a-b)-b(a+b))^2 \\ &= (a^2+2ab-b^2)^2 + (a^2-2ab-b^2)^2. \end{aligned}$$

Thus $2(cp)^2 = ((a^2+2ab-b^2)c)^2 + ((a^2-2ab-b^2)c)^2$. □

The number 1 is an ugly number. From Lemmas 1–4, for all integers $n \geq 2$, n is beautiful if and only if n is divisible by a prime that is congruent to 1 (mod 4). Therefore, the product of two ugly numbers is also ugly.

OC265. Five airway companies operate in a country consisting of 36 cities. Between any pair of cities exactly one company operates two way flights. If some air company operates between cities A, B and B, C we say that the triple A, B, C is properly-connected. Determine the largest possible value of k such that no matter how these flights are arranged there are at least k properly-connected triples.

Originally problem 6 from day 2 of the 2014 Turkey Mathematical Olympiad.

No solutions were submitted.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **November 1, 2017**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

4221. *Proposed by Nguyen Viet Hung.*

Let a, b, c, p, q be distinct positive real numbers satisfying

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}=p,$$
$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}=q.$$

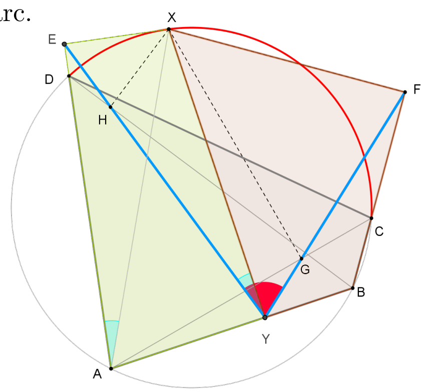
Evaluate

$$\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}$$

in terms of p and q.

4222. *Proposed by Mihaela Berindeanu.*

Let $ABCD$ be a quadrilateral inscribed in a circle and X a mobile point on the small arc CD. If E, F, G, H are X orthogonal projections on AD, BC, AC, BD show that the angle between EH and GF is always constant, regardless of the position of X on the arc.



4223. *Proposed by Leonard Giugiuc and Dorin Marghidanu.*

Let a, b and c be positive real numbers such that $a+b+c \leq 1$. Prove that

$$\sqrt[3]{(1-a^3)(1-b^3)(1-c^3)} \geq 26abc.$$

4224. *Proposed by Michel Bataille.*

Find the complex roots of the polynomial

$$16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1.$$

4225. *Proposed by Leonard Giugiuc, Daniel Dan and Daniel Sitaru.*

Prove that in any triangle ABC we have:

$$3(\cos^2 A + \cos^2 B + \cos^2 C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3.$$

4226. *Proposed by Daniel Sitaru.*

Prove that if $0 < a < b$ then:

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x} dx\right)^2 > (b-a)^2 + \ln^2\left(\frac{b}{a}\right).$$

4227. *Proposed by Dan Marinescu and Leonard Giugiuc.*

Let P be a point in the interior of an equilateral triangle ABC whose sides have length 1, and let R' and r' be the circumradius and inradius of the triangle whose sides are congruent to PA, PB and PC (which exists by Pompeiu's theorem). Prove that

$$3R' \geq 1 \geq 6r'.$$

4228. *Proposed by Mihály Bencze.*

Let $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$ such that $\sum_{k=1}^n z_k = \sum_{k=1}^n z_k^2 = 0$. Prove that

$$n \sum_{k=1}^n |z_k|^2 \leq (n-2) \left(\sum_{k=1}^n |z_k|\right)^2.$$

4229. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 2$ and let p be a prime number with $p > n$. Consider an $n \times n$ matrix X over $\mathbb{Z}_p$ with $X^p = I_n$. Prove that $(X - I_n)^n = O_n$.

4230. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle in which $\angle B = 2\angle C$ and let M be the midpoint of BC. The internal bisector of $\angle ACB$ intersects AM in D. Prove that $\angle CDM \leq 45°$ and find $\angle C$ for which the equality holds.

. . . .

4221. *Proposé par Nguyen Viet Hung.*

Soient a, b, c, p et q des nombres réels, positifs et distincts, tels que

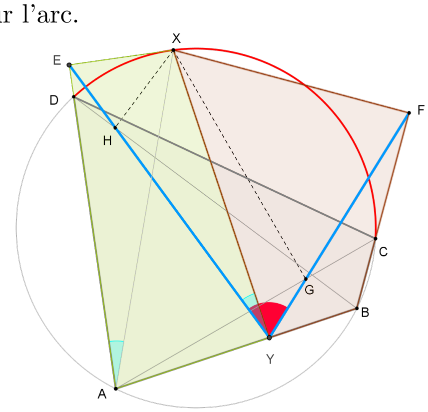
$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} = p,$$
$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2} = q.$$

Évaluer
$$\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2}$$

en termes de p et q.

4222. *Proposé par Mihaela Berindeanu.*

Soit $ABCD$ un quadrilatère inscrit dans un cercle et soit X un point situé sur le petit arc CD. Si E, F, G et H sont les projections orthogonales de X vers AD, BC, AC et BD, démontrer que l'angle entre EH et GF est constant, quel que soit le point X sur l'arc.



4223. *Proposé par Leonard Giugiuc et Dorin Marghidanu.*

Soient a, b et c des nombres réels positifs tels que $a + b + c \leq 1$. Démontrer que

$$\sqrt[3]{(1-a^3)(1-b^3)(1-c^3)} \geq 26abc.$$

4224. *Proposé par Michel Bataille.*

Déterminer les racines complexes du polynôme

$$16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1.$$

4225. *Proposé par Leonard Giugiuc, Daniel Dan et Daniel Sitaru.*

Soit un triangle ABC. Démontrer que

$$3(\cos^2 A + \cos^2 B + \cos^2 C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3.$$

4226. *Proposé par Daniel Sitaru.*

Démontrer que si $0 < a < b$, alors

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 > (b-a)^2 + \ln^2\left(\frac{b}{a}\right).$$

4227. *Proposé par Dan Marinescu et Leonard Giugiuc.*

Soit P un point à l'intérieur du triangle équilatéral ABC ayant des côtés de longueur 1 et soient R' et r' les rayons des cercles circonscrit et inscrit du triangle dont les côtés sont congrus à PA, PB et PC, où ce dernier triangle existe en raison du théorème de Pompeiu. Démontrer que

$$3R' \geq 1 \geq 6r'.$$

4228. *Proposé par Mihály Bencze.*

Soient $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$ tels que $\sum_{k=1}^n z_k = \sum_{k=1}^n z_k^2 = 0$. Démontrer que

$$n\sum_{k=1}^n |z_k|^2 \leq (n-2)\left(\sum_{k=1}^n |z_k|\right)^2.$$

4229. *Proposé par Leonard Giugiuc.*

Soit n un entier tel que $n \geq 2$ et soit p un nombre premier tel que $p > n$. Considérer une matrice $n \times n$ à valeurs dans $\mathbb{Z}_p$ telle que $X^p = I_n$. Démontrer que

$$(X - I_n)^n = O_n.$$

4230. *Proposé par Miguel Ochoa Sanchez et Leonard Giugiuc.*

Soit ABC un triangle tel que $\angle B = 2\angle C$ et soit M le mi point de BC. La bissectrice interne de $\angle ACB$ intersecte AM en D. Démontrer que $\angle CDM \leq 45°$ et déterminer $\angle C$ pour lequel l'égalité tient.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(3), p. 121–126.

4121. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let s be a fixed real number such that $s \geq 1$. Let a, b, c and d be non-negative numbers that satisfy $a+b+c+d = 4s$ and $ab+bc+cd+da+ac+bd = 6$. Express the minimum value of the product $abcd$ in terms of s.

The proposers provided a correct solution, given below. Two other submissions were incorrect.

We first establish that at least one of a, b, c, d can vanish if and only if $4s \geq 3\sqrt{2}$. Let $d = 0$, say. Since $(a+b+c)^2 \geq 3(ab+bc+ca)$, then $16s^2 \geq 18$. Conversely, consider the system of equations $u + 2v = 4s$, $2uv + v^2 = 6$. Eliminating u yields $3v^2 - 8sv + 6 = 0$, a quadratic equation with positive real solutions iff $64s^2 \geq 72$. With v the smaller one, the quadruple

$$\begin{aligned}(a,b,c,d) &= (u,v,v,0)\\ &= \left(\frac{4s+2\sqrt{16s^2-18}}{3}, \frac{4s-\sqrt{16s^2-18}}{3}, \frac{4s-\sqrt{16s^2-18}}{3}, 0\right)\end{aligned}$$

satisfies the conditions. Thus, the minimum value of $abcd$ is 0 when $s \geq 3\sqrt{2}/4$.

Henceforth, suppose that $1 \leq s < 3\sqrt{2}/4$. Let $abcd = p$, $abc+bcd+cda+dab = r$,

$$f(x) = \frac{1}{x}(x-a)(x-b)(x-c)(x-d) = x^3 - 4sx^2 + 6x - r + \frac{p}{x}.$$

and

$$g(x) = x^2 f'(x) = 3x^4 - 8sx^3 + 6x^2 - p.$$

Note that $p > 0$. Since $f(x)$ has four positive roots, by Rolle's theorem, $f'(x)$ has three positive roots (counting multiplicity). Since $g(0) < 0$, the quartic polynomial has one negative and three positive roots.

The polynomial $g'(x) = 12x(x^2 - 2sx + 1)$ has three roots, namely 0, $1/t$ and t, where

$$1 \leq t = s + \sqrt{s^2-1} < \sqrt{2}.$$

Since $g(x)$ has three positive roots (counting multiplicity) and $g(0) < 0$, we must have that $g(1/t) \geq 0$ and $g(t) \leq 0$. Hence

$$\begin{aligned}abcd = p \geq 3t^4 - 8st^3 + 6t^2 &= 3t^4 - 4(2st-1)t^2 + 2t^2\\ &= 3t^4 - 4t^4 + 2t^2 = 2t^2 - t^4\\ &= (s+\sqrt{s^2-1})^2[2 - (s+\sqrt{s^2-1})^2].\end{aligned}$$

Let $(a, b, c, d) = (t, t, t, 2t^{-1} - t)$. Then

$$a + b + c + d = 2(t + t^{-1}) = 4s,$$
$$ab + bc + ca + da + db + dc = 3t^2 + 6 - 3t^2 = 6, \text{ and}$$
$$abcd = 2t^2 - t^4.$$

Therefore, when $1 \le s < 3\sqrt{2}/4$, the minimum of $abcd$ is

$$(s + \sqrt{s^2 - 1})^2[2 - (s + \sqrt{s^2 - 1})^2],$$

and when $s \ge 3\sqrt{2}/4$, the minimum of $abcd$ is 0.

4122. *Proposed by Daniel Sitaru.*

Prove that for $n \in \mathbb{N}$, the following holds

$$\left(\frac{e^n - 1}{n}\right)^{2n+1} \le \frac{(e-1)(e^2-1)(e^3-1) \cdot \ldots \cdot (e^{2n}-1)}{(2n)!}.$$

We received six correct and complete solutions of which we present the one by Ángel Plaza, slightly modified by the editor.

Note that the inequality in the statement can be rewritten as

$$\left(\frac{e^n - 1}{n}\right)^{2n+1} \le \left(\frac{e-1}{1}\right)\left(\frac{e^2-1}{2}\right) \cdots \left(\frac{e^{2n}-1}{2n}\right). \tag{1}$$

Consider the function

$$f(x) = \ln\left(\frac{e^x - 1}{x}\right)$$

defined for $x > 0$ and set $f(0) = 0$. Then f is continuous for $x \ge 0$ and has second derivative

$$f''(x) = \frac{(e^x - 1)^2 - x^2 e^x}{x^2(e^x - 1)^2}.$$

To show that $f(x)$ is convex it suffices to prove that $(e^x - 1)^2 - x^2 e^x > 0$. This can be reformulated to $e^x(e^x + e^{-x} - (2 + x^2)) > 0$. But we have

$$e^x + e^{-x} = \sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!} > 2 + x^2.$$

Therefore, the second derivative of f is positive and $f(x)$ is convex for $x > 0$. Rephrasing inequality (1) by taking logarithms we obtain

$$(2n + 1)f(n) \le \sum_{k=0}^{2n} f(k),$$

which follows from Jensen's inequality.

4123. *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line ℓ is perpendicular to the plane of the acute triangle $A'B'C'$ at its orthocentre K. Let A, B, C be the midpoints of $B'C', C'A'$ and $A'B'$, respectively. Show that $BC > KA$ and if D on ℓ satisfies $KD = \sqrt{BC^2 - KA^2}$, that the tetrahedron $ABCD$ is isosceles. (A tetrahedron is called isosceles if its opposite edges are congruent.)

We received three correct submissions and feature the solution by Leonard Giugiuc.

Without loss of generality, we choose coordinates $A'(0,2,0), B'(-2u,0,0)$, and $C'(2v,0,0)$. Because A' is acute, we can assume that u and v are positive; moreover we have $u = \cot B'$ and $v = \cot C'$, whence (because the triangle is acute)

$$uv - 1 = (\cot B' + \cot C')\cot(B' + C') < 0.$$

Consequently, $A(v-u,0,0),\ B(v,1,0),\ C(-u,1,0),\ \text{ and }\ K(0,2uv,0)$. The line ℓ is therefore the set of points $\{2, 2uv, z\}$, and

$$BC^2 - KA^2 = (u+v)^2 - (u-v)^2 - 4u^2v^2 = 4uv(1-uv) > 0.$$

Thus, $BC > KA$, as claimed. We may now choose $D = (0, 2uv, 2\sqrt{uv(1-uv)})$. We conclude that

$$\begin{aligned} AD^2 &= (v-u)^2 + 4u^2v^2 + 4uv(1-uv) = (u+v)^2 = BC^2; \\ BD^2 &= v^2 + (2uv-1)^2 + 4uv(1-uv) = v^2 + 1 = CA^2; \\ CD^2 &= u^2 + (2uv-1)^2 + 4uv(1-uv) = u^2 + 1 = AB^2. \end{aligned}$$

The proof is complete.

Comment (by the proposer and the third solver John Heuver). Since $CD = AB = CA'$ and $BD = CA = BA'$, the point D is the image of A' under a suitable rotation about axis BC. Similarly, D is the image of B' and of C' under suitable rotations about axes CA and AB, respectively. Thus, from an acute triangle $A'B'C'$ and its midpoint triangle ABC drawn on cardboard, one can obtain an isosceles tetrahedron $DABC$ by folding along BC, CA, AB till A', B', C' coincide (and naming D the point of coincidence).

4124. *Proposed by George Apostolopoulos.*

Let A_1, B_1 and C_1 be points on the sides BC, CA and AB of a triangle ABC such that

$$\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = k.$$

Prove that

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{CA}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2 \left(\frac{2r}{R}\right)^4,$$

where R and r are the circumradius and the inradius of ABC, respectively.

We received three solutions. We present the solution by Titu Zvonaru, slightly modified by the editor.

Denote by a, b and c the sides of the triangle. Note that $\frac{A_1B}{A_1C} = k$ implies that
$$\frac{A_1B}{BC} = \frac{k}{k+1} \quad \text{and} \quad \frac{A_1C}{BC} = \frac{1}{k+1}.$$

Using Stewart's Theorem [see *Editor's Comments*] for the length of a cevian, and dividing by a^2, we get
$$\left(\frac{AA_1}{BC}\right)^2 = \frac{k}{k+1}\cdot\frac{b^2}{a^2} + \frac{1}{k+1}\cdot\frac{c^2}{a^2} - \frac{k}{(k+1)^2}.$$

Proceed similarly to get corresponding formulae for the remaining terms on the left hand side of the desired inequality to obtain
$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{AC}\right)^2 + \left(\frac{CC_1}{AB}\right)^2$$
$$= \frac{k}{k+1}\left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right) + \frac{1}{k+1}\left(\frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2}\right) - \frac{3k}{(k+1)^2}. \quad (1)$$

By the AM-GM inequality, $\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \geq 3$ and $\frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2} \geq 3$ (in both cases, with equality when $a = b = c$), so from (1) we get
$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{AC}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \frac{3k}{k+1} + \frac{3}{k+1} - \frac{3k}{(k+1)^2} = 3 - \frac{3k}{(k+1)^2}.$$

We now prove that $3 - \frac{3k}{(k+1)^2} \geq \left(\frac{3k}{k^2+1}\right)^2$, or equivalently $\frac{3k^2}{(k^2+1)^2} + \frac{k}{(k+1)^2} \leq 1$. Using AM-GM, we have
$$k^2 + 1 \geq 2k, \quad \text{which we can rearrange to} \quad \frac{3k^2}{(k^2+1)^2} \leq \frac{3}{4}, \quad \text{and}$$
$$k + 1 \geq 2\sqrt{k}, \quad \text{which we can rearrange to} \quad \frac{k}{(k+1)^2} \leq \frac{1}{4}.$$

It follows that $\frac{3k^2}{(k^2+1)^2} + \frac{k}{(k+1)^2} \leq 1$ (equality holds when $k = 1$). So far we have
$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{AC}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2$$

Note that Euler's inequality, $R \geq 2r$, implies $1 \geq \left(\frac{2r}{R}\right)^4$, which concludes the proof of the desired inequality. Equality holds when $a = b = c$ and $k = 1$.

Editor's Comments: Stewart's Theorem gives us a formula for calculating the length of a cevian in a triangle: given a cevian AA_1 with $\frac{A_1B}{BC} = m$ and $\frac{A_1C}{BC} = n$, we have
$$(AA_1)^2 = mb^2 + nc^2 - mna^2.$$

The formula can also be derived easily from the cosine law.

4125. *Proposed by Stephen Su and Cheng-Shyong Lee.*

Start with a triangle $A_1A_2A_3$ in the Euclidean plane and three nonzero real numbers ℓ_1, ℓ_2, ℓ_3. Define M_k and C_k to be points on the line $A_{k+1}A_{k+2}$ such that

$$\frac{A_{k+1}M_k}{M_kA_{k+2}} = \ell_k$$

and

$$C_kM_{k+1}||A_kA_{k+1}, \quad k = 1, 2, 3$$

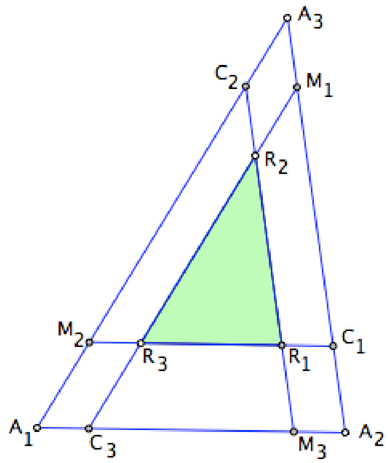
(with subscripts reduced modulo 3 and distances taken to be signed, so that M_k is between A_{k+1} and A_{k+2} precisely when ℓ_k is positive).

Denote by R_k the point where C_kM_{k+1} intersects $C_{k+1}M_{k+2}$, $k = 1, 2, 3$. Show that

$$\frac{[R_1R_2R_3]}{[A_1A_2A_3]} = \left(\frac{2 + \ell_1 + \ell_2 + \ell_3 - \ell_1\ell_2\ell_3}{(1 + \ell_1)(1 + \ell_2)(1 + \ell_3)}\right)^2,$$

where square brackets denote area.

We received three correct submissions and feature the solution by AN-anduud Problem Solving Group.



From $\triangle R_3R_1R_2 \sim \triangle A_1A_2A_3$ we get

$$\frac{[R_1R_2R_3]}{[A_1A_2A_3]} = \left(\frac{R_3R_1}{A_1A_2}\right)^2. \tag{1}$$

Of course,

$$\frac{R_3R_1}{A_1A_2} = \frac{M_2C_1 - M_2R_3 - R_1C_1}{A_1A_2} = \frac{M_2C_1}{A_1A_2} - \frac{M_2R_3}{A_1A_2} - \frac{R_1C_1}{A_1A_2}. \tag{2}$$

Because $\triangle A_1A_2A_3 \sim \triangle M_2C_1A_3$ we get

$$\frac{M_2C_1}{A_1A_2} = \frac{M_2A_3}{A_1A_3} = \frac{M_2A_3}{A_1M_2 + M_2A_3} = \frac{\frac{M_2A_3}{A_1M_2}}{1 + \frac{M_2A_3}{A_1M_2}} = \frac{\ell_2}{1 + \ell_2}. \tag{3}$$

From $M_2R_3 = A_1C_3$ and $A_1A_3 \parallel C_3M_1$ we have

$$\frac{M_2R_3}{A_1A_2} = \frac{A_1C_3}{A_1A_2} = \frac{A_1C_3}{A_1C_3 + C_3A_2} = \frac{1}{1 + \frac{C_3A_2}{A_1C_3}} = \frac{1}{1 + \frac{M_1A_2}{A_3M_1}} = \frac{1}{1+\ell_1}. \qquad (4)$$

Finally, $R_1C_1 = M_3A_2$, hence we get

$$\frac{R_1C_1}{A_1A_2} = \frac{M_3A_2}{A_1M_3 + M_3A_2} = \frac{1}{1 + \frac{A_1M_3}{M_3A_2}} = \frac{1}{1+\ell_3}. \qquad (5)$$

The desired result follows immediately from equations (1)–(5).

4126. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute-angled triangle. Prove that

$$\sum_{\text{cyc}} \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}} \geq \sqrt{\frac{3}{2}}.$$

We received 22 solutions, all correct. Titu Zvonaru contributed five of them, and Arkady Alt supplied another two. Of the many different approaches, the solutions that exploited Jensen's inequality were the shortest and most popular. We feature the solution from Prithwijit De, which is typical of that approach.

Our argument is valid for all proper triangles, so we drop the restriction to acute angles.

We know that for all triangles ABC,

$$\tan(A/2)\tan(B/2) + \tan(B/2)\tan(C/2) + \tan(C/2)\tan(A/2) = 1.$$

Let

$$\begin{aligned} x &= \tan(A/2)\tan(B/2),\\ y &= \tan(B/2)\tan(C/2),\\ z &= \tan(C/2)\tan(A/2). \end{aligned}$$

Then we have to show that

$$\sum_{\text{cyc}} \frac{x}{\sqrt{1-x}} \geq \sqrt{\frac{3}{2}},$$

subject to the conditions $x + y + z = 1$ and $0 < x, y, z < 1$. Now

$$\frac{x}{\sqrt{1-x}} = \frac{1}{\sqrt{1-x}} - \sqrt{1-x},$$

and the function $f(x) = \sqrt{1-x}$ is continuous and concave on $(0, 1)$. Therefore the function

$$h(x) = \frac{1}{\sqrt{1-x}} - \sqrt{1-x}$$

is continuous and convex on $(0, 1)$. Thus by Jensen's inequality we obtain

$$\sum_{\text{cyc}} \frac{x}{\sqrt{1-x}} \geq 3\left(\frac{\frac{x+y+z}{3}}{\sqrt{1-\frac{x+y+z}{3}}}\right) = \sqrt{\frac{3}{2}},$$

as desired.

Equality holds if and only if $x = y = z$, which is equivalent to ΔABC being equilateral.

4127. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Calculate

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx,$$

where $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function.

We received 8 solutions. We present the solution by Leonard Giugiuc, slightly modified by the editor.

Perform the substitution $t = \frac{x}{n}$ to get

$$\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right)\, dx = n \cdot \int_{\frac{\sqrt[n]{n!}}{n}}^{\frac{\sqrt[n+1]{(n+1)!}}{n}} f(t)\, dt.$$

By the mean value theorem, there exists a c_n with

$$\frac{\sqrt[n]{n!}}{n} < c_n < \frac{\sqrt[n+1]{(n+1)!}}{n},$$

such that

$$\int_{\frac{\sqrt[n]{n!}}{n}}^{\frac{\sqrt[n+1]{(n+1)!}}{n}} f(t)\, dt = \left(\frac{\sqrt[n+1]{(n+1)!}}{n} - \frac{\sqrt[n]{n!}}{n}\right) f(c_n).$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right)\, dx &= \lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{(n+1)!}}{n} - \frac{\sqrt[n]{n!}}{n}\right) f(c_n)\\ &= \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) \cdot \lim_{n\to\infty} f(c_n), \qquad (1)\end{aligned}$$

assuming we can show that both limits exist, which we now proceed to do.

From Stirling's approximation for factorials we know that for all positive integers n

$$\sqrt{2\pi n} \cdot n^n \cdot e^{-n} \leq n! \leq \sqrt{2\pi n} \cdot n^n \cdot e^{-n} \cdot e^{\frac{1}{12n}}.$$

This enables us to show (using the squeeze theorem and standard limit techniques) that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

It follows that

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}=\frac{1}{e}$$

as well, whence by the squeeze theorem applied to the interval of c_n we conclude that $\lim\limits_{n\to\infty} c_n = \frac{1}{e}$. Hence, since f is continuous, $\lim\limits_{n\to\infty} f(c_n) = f\left(\frac{1}{e}\right)$.

Using Stirling's approximation again, one can also show that

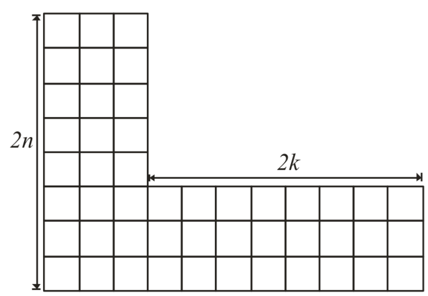
$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}.$$

Therefore, from (1), it follows that

$$\lim_{n\to\infty}\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right)\,dx=\frac{1}{e}\cdot f\left(\frac{1}{e}\right).$$

4128. *Proposed by Valcho Milchev and Tsvetelina Karamfilova.*

Let A_n be the number of domino tilings of a rectangular $3\times 2n$ grid. Let $L(2n,2k)$ be the number of domino tilings of the grid composed of two rectangular grids of dimensions $3\times 2n$ and $3\times 2k$ with $n\geq 2$ and $k\geq 1$ (depicted below):



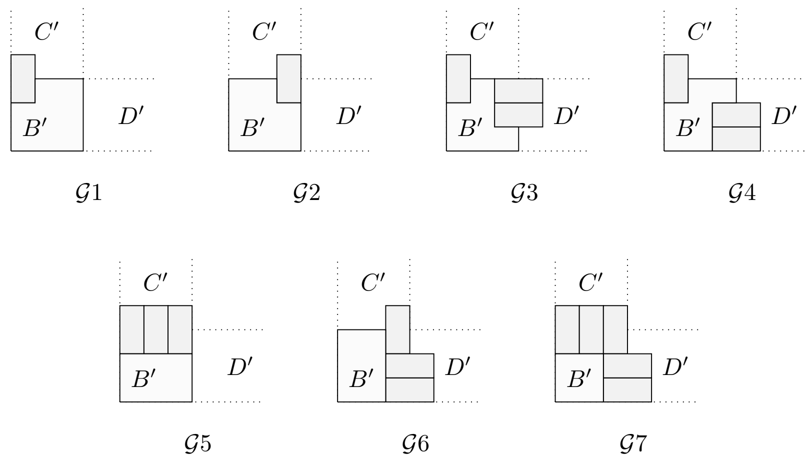
Prove that $L(2n,2n)=A_{2n}$.

We received three correct solutions. We present the solution by Oliver Geupel.

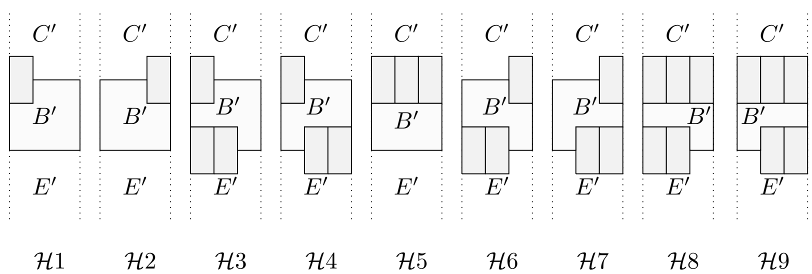
Let $\mathcal{G}$ be a grid composed of a 3×3 grid B, a $3\times(2n-3)$ grid C above B and a $2n\times 3$ grid D to the right of B. Also, let $\mathcal{H}$ be a grid composed of grids B and C and a $3\times 2n$ grid E below B. We have to show that $\mathcal{G}$ and $\mathcal{H}$ admit the same number of domino tilings.

Suppose we have a domino tiling of $\mathcal{G}$. Consider the dominoes that cover a cell in both B and C. As the number of cells in C is odd, there have to be one or three such dominoes. Furthermore, if there is only one such domino then it cannot be in the middle column, as can be seen by a chessboard colouring argument (colour the cells of C minus the bottom cell of the middle column alternatingly white and black; the number of white cells differs from the number of black cells by two, but any domino in a tiling covers one white and one black cell). Similarly we consider dominoes of the tiling that cover a cell in both B and D. As the number of cells in D is even, there must be zero or two such dominoes and, again by a chessboard colouring argument, if there are two such dominoes then one must be in the middle column.

We obtain the following cases for tilings of $\mathcal{G}$, where B', C', and D' are the subgrids of $\mathcal{G}$ formed by the dominoes that only cover cells of B, C, or D, respectively.



For domino tilings of $\mathcal{H}$, we can make the same argument as above to obtain the following cases in terms of B', C, and E', the subgrids of $\mathcal{H}$ formed by the dominoes that only cover cells of B, C, and E, respectively.



Comparing shapes of B', C', and D' for $\mathcal{G}$ with the shapes of B', C', and E' for $\mathcal{H}$ we see that cases $\mathcal{G}1$ and $\mathcal{H}1$ admit the same number of tilings, as do cases $\mathcal{G}2$-$\mathcal{G}5$ and $\mathcal{H}2$-$\mathcal{H}5$.

In the case $\mathcal{G}6$ the grid B' admits three tilings, which is the same as the number of tilings of B' in cases $\mathcal{H}6$ and $\mathcal{H}7$ combined. Hence the number of tilings in case $\mathcal{G}6$ equals those of cases $\mathcal{H}6$ and $\mathcal{H}7$ combined.

Similarly, case $\mathcal{G}7$ admits the same number of tilings as cases $\mathcal{H}8$ and $\mathcal{H}9$ combined.

In conclusion, we obtain that the number of tilings of $\mathcal{G}$ is the same as the number of tilings of $\mathcal{H}$.

4129. *Proposed by Lorean Saceanu.*

Let ABC be an acute-angle triangle and let $\gamma = 3(2-\sqrt{3})$. Prove that

$$\sec A + \sec B + \sec C \geq \gamma + \tan A + \tan B + \tan C.$$

We received 12 submissions all of which were correct. Except for one, all of them are very similar to one another so we will present a composite of these solutions.

Let $f(x) = \sec x - \tan x$, $x \in (0, \frac{\pi}{2})$. Then we have $f'(x) = \sec x \tan x - \sec^2 x$ and

$$f''(x) = \sec^3 x + \sec x \tan^2 x - 2\sec^2 x \tan x = (\sec x)(\sec x - \tan x)^2 > 0,$$

so $f(x)$ is strictly convex on $(0, \frac{\pi}{2})$.

Since $A, B, C \in (0, \frac{\pi}{2})$ such that $A + B + C = \pi$, we have, by Jensen's Inequality that

$$\begin{aligned}
&\sec A + \sec B + \sec C - \tan A - \tan B - \tan C \\
&= f(A) + f(B) + f(C) \\
&\geq 3f(\frac{A+B+C}{3}) = 3f(\frac{\pi}{3}) \\
&= 3(\sec\frac{\pi}{3} - \tan\frac{\pi}{3}) = 3(2-\sqrt{3}) \\
&= \gamma.
\end{aligned}$$

This completes the proof. It is easy to see that equality holds if and only if $A = B = C = \frac{\pi}{3}$; i.e. if and only if $\triangle ABC$ is equilateral.

4130. *Proposed by Leonard Giugiuc.*

Let a, b and c be nonnegative real numbers such that $a + b + c = ab + bc + ac > 0$. Prove that

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2\sqrt[n]{2}$$

for any integer $n \geq 3$ and determine the case for equality to hold.

Correct solutions were received from the proposer and Ardak Mirzakhmedov. There was one incorrect submission. We present the solution of Mirzakhmedov.

Solution. The condition implies that at most one of a, b, c can vanish. Since

$$2 = \frac{2(ab+bc+ca)}{a+b+c} \leq \frac{2ab}{a+b} + \frac{2bc}{b+c} + \frac{2ca}{c+a} \leq \sqrt{ab} + \sqrt{bc} + \sqrt{ca},$$

with equality iff $(a, b, c) = (0, 2, 2), (2, 0, 2), (2, 2, 0)$, it is enough to prove that

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2 \cdot \sqrt[n]{\sqrt{ab} + \sqrt{bc} + \sqrt{ca}}.$$

Let $(a, b, c) = (u^{2n}, v^{2n}, w^{2n})$. The desired inequality becomes

$$(u^2 + v^2 + w^2)^n \geq 2^n(u^n v^n + v^n w^n + w^n u^n).$$

We establish this by induction. When $n = 3$, the inequality follows from

$$\begin{aligned}
&(u^2+v^2+w^2)^3 - 8(u^3v^3+v^3w^3+w^3u^3) \\
&= [u^2(u^2-v^2)(u^2-w^2) + v^2(v^2-w^2)(v^2-u^2) + w^2(w^2-u^2)(w^2-v^2)] \\
&\quad + 4[u^2v^2(u-v)^2 + v^2w^2(v-w)^2 + w^2u^2(w-u)^2] \\
&\quad + 3u^2v^2w^2 \ge 0,
\end{aligned}$$

the first term in square brackets being nonnegative by Schur's inequality.

Suppose that the inequality holds for $n \ge 3$. Then

$$\begin{aligned}
(u^2+v^2+w^2)^{n+1} &\ge 2^n(u^2+v^2+w^2)(u^nv^n+v^nw^n+w^nu^n) \\
&\ge 2^n[(u^{n+2}v^n+u^nv^{n+2}) + (v^{n+2}w^n+v^nw^{n+2}) \\
&\qquad + (w^{n+2}u^n+w^nu^{n+2})] \\
&\ge 2^{n+1}(u^{n+1}v^{n+1}+v^{n+1}w^{n+1}+w^{n+1}u^{n+1}),
\end{aligned}$$

by the arithmetic-geometric means inequality.

Editor's Comment. The proposer has pointed out that this technique can be used for other problems, such as in proofs of the following inequalities for $n \ge 3$ and $m > 0$:

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \ge 2 \cdot \sqrt[n]{2};$$
$$\sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca} \ge \sqrt[n]{4};$$
$$\sqrt[4]{ab} + \sqrt[4]{bc} + \sqrt[4]{ca} \ge \sqrt{2};$$
$$(ab)^m + (bc)^m + (ca)^m \ge \min\{4^m, 3\}.$$

In each case we begin in the same way. Wolog, let $a \le b \le c$. Since we have that $(a+b+c)^2 \ge 3(ab+bc+ca)$, then $a+b+c = ab+bc+ca \ge 3$ and $bc \ge 1$. Let $b+c = 2s$ and $bc = p^2$, so that $s \ge p \ge 1$ and $a = (2s-p^2)(2s-1)^{-1}$. From the arithmetic-geometric means inequality, we have that $b^m + c^m \ge 2p^m$, in particular for $m = 1/n$.

If $p \ge 2$, then all four inequalities hold easily, with equality when $(a,b,c) = (0,2,2)$. The hard part is the case $1 \le p < 2$. Since

$$(2s-p^2)(2s-1)^{-1} \ge (2p-p^2)(2p-1)^{-1},$$

it is enough to establish the inequalities with a replaced by $(2p-p^2)(2p-1)^{-1}$ and $b^m + c^m$ replaced by $2p^m$. At this point, the computations become rather complicated. We feel that there ought to be a more elegant way to end these proofs! Can any reader supply one?

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations