Crux Mathematicorum

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Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

YEAR-END FINALE

You are reading this Volume 43 Year-End Finale of year 2017 in year 2017. When I first stepped into the Editor-in-Chief position, I was faced with the task of clearing a year-long backlog. The majority of people I talked to considered the backlog problem to be essentially unsolvable, yet it took less time to work out than my PhD (significantly less, I might add). And that is because I wasn't alone in it: the entire Editorial Board greatly contributed to this amazing achievement and I am immensely thankful for their efforts and their enthusiasm. Not only did they believe in the mission, they rather happily adjusted to the increased workload and faster deadlines. The entire ***Crux*** team together with the CMS staff, we collectively dealt with the influx of new proposals like we have never seen before and the nap-time-only schedule during my maternity leave helped to ensure that it did not affect our progress. We succeeded.

This Volume, like any other past Volume of ***Crux***, is now a Cheshire Cat: just like the cat disappears but leaves his grin, the year 2017 disappears but leaves Volume 43 behind. Volume 44 is a house cat: it patiently waits for you to come home so you can chase mice/problems together. Volume 45, is a Schrödinger's Cat: just like the cat may or may not be alive, Volume 45+ may or may not happen.

As I am writing this, I am headed to the Canadian Mathematical Society meeting where the future of ***Crux*** will be discussed yet again: do we go full electronic or stay in print, do we aim for open access or keep the subscription fees, do we push for fundraising, do we eliminate some sections, and so on. If you have ideas for the future of ***Crux***, drop me a line at `crux-editors@cms.math.ca`.

Kseniya Garaschuk

EDITORIAL

To get to know, to discover, to publish – this is the destiny of a scientist.
François Arago

We live in a pragmatic world, where search for the truth is sometimes overshadowed by the desire, or even the need, to publish the discovered. While the pursuit of knowledge is a noble endeavour on its own, it is often the publication of said knowledge that pays dividends (figuratively or literally): being a published author can be used for a variety of purposes, from supporting a grant application to bragging rights.

We have recently come across an unpleasant issue of a ***Crux*** problem appearing somewhere else in print at approximately the same time (see comments on problems 4191 and 4192). It is unacceptable and unethical for a repeated **proposal or solution** to be submitted when it can already be referenced and found somewhere else. It wastes time and space that could be used for consideration and publication of original materials. Of course some potential repetition is acceptable (such as the problem that appears in another language or in materials not easily available to the general audience), but the original source must be acknowledged in the proposal with a proper reference.

As authors, we don't often think about how much effort a journal spends on our submission. At ***Crux***, each problem proposal gets reviewed by the entire Editorial Board. In accepting or rejecting a proposal, we consider many factors and regularly have lengthy discussions about them. We do not take these decisions lightly and there are a lot of decisions to be made! Last year, we received 780 problem proposals in total and so far this year we are on track to beat 2016 statistics. To me, editors' time is precious. And not just because it is finite, but because it is voluntarily donated. ***Crux*** runs on the good will of its editors and they are its most valuable resource. So I ask you – our authors, solvers and proposers – to be mindful and respectful of our time and of the peer review system as a whole.

Since the Editorial Board cannot possibly check all the potential other sources, we ask our proposers to be honest about the originality of the material and our solvers to be vigilant and inform us if the material is not new (otherwise, you become an accessory to the crime).

To err is human, so if you've submitted a repeat proposal recently, I kindly ask you to withdraw it from ***Crux*** by emailing me.

Kseniya Garaschuk

THE CONTEST CORNER

No. 60

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er mai 2018**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

CC296. Déterminer le nombre d'entiers positifs k, $10000 \leq k \leq 99999$, tels que le troisième chiffre est la moyenne des premier et cinquième chiffres.

CC297. Six dames noires sont placées en permanence sur un échiquier 6 par 6, aux positions indiquées en Figure 1. Une dame blanche commence un trajet au coin à gauche et en bas de l'échiquier, dénoté A en Figure 1, et se faufile de carré en carré, pour arriver au carré à droite et en haut de l'échiquier, dénoté B. Déterminer le nombre de tels trajets de A vers B si à chaque étape la dame blanche peut bouger un carré à droite, un carré vers le haut ou un carré en diagonale (vers la droite et vers le haut), tout en évitant les carrés occupés par les dames noires. Un tel trajet est indiqué en Figure 2.

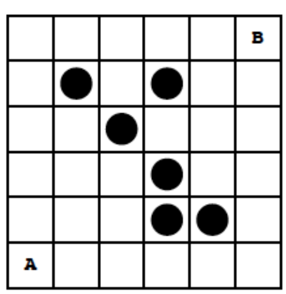


Figure 1

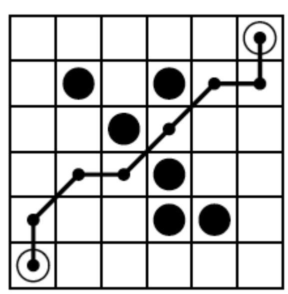


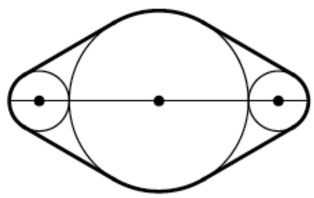
Figure 2

CC298. Soit ABC le triangle formé par les sommets $(0, 0), (4, 0)$ et $(2, 3)$. Déterminer les coordonnées du point P qui est équidistant de A, B et C.

CC299. Déterminer l'aire de la région bornée par les graphiques de

$$\begin{cases} x + y + |x| = 10, \\ x + y - |x| = -8. \end{cases}$$

CC300. Trois poteaux de coupes transversales circulaires sont liés ensemble à l'aide de fil de fer. Les rayons de ces coupes transversales sont 1, 3 et 1 pouce. Les centres se trouvent sur une même ligne, comme indiqué ci-bas. Si la longueur du fil de fer est donnée en forme $a\sqrt{3} + b\pi$, où a et b sont des nombres rationnels, déterminer $a + b$. Supposer que le fil de fer est d'épaisseur négligeable.



. . . .

CC296. Find the number of positive integers k with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

CC297. Six black checkers are placed on squares of a 6 by 6 checkerboard in the positions shown in Figure 1 and are left in place. A white checker begins on the square at the lower left corner of the board (marked A in Figure 1) and follows a path from square to square across the board, ending in the upper right corner of the board (marked B). How many different paths are there from A to B if at each step the white checker can move one square to the right, one square up or one square diagonally upward to the right and may not pass though any square occupied by a black checker? One such path is shown in Figure 2.

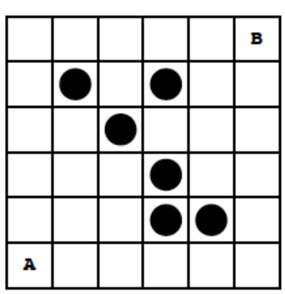


Figure 1

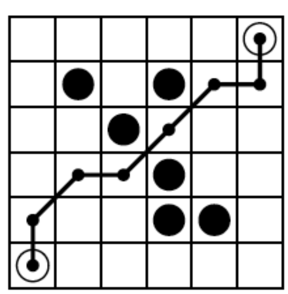


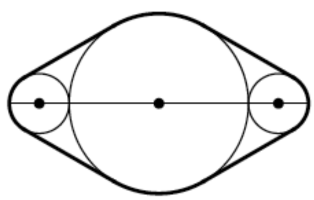
Figure 2

CC298. Let ABC be the triangle with vertices $(0,0),(4,0)$ and $(2,3)$. Find the coordinates of the point P that is equidistant from A,B and C.

CC299. Find the area of the region bounded by the graphs of

$$\begin{cases} x+y+|x|=10, \\ x+y-|x|=-8. \end{cases}$$

CC300. Three poles with circular cross sections are to be bound together with a wire. The radii of the circular cross sections are 1, 3 and 1 inches. The centers of the circles are on the same straight line as indicated in the sketch. If the length of the wire is written in the form $a\sqrt{3}+b\pi$, where a and b are rational numbers, find $a+b$. Assume that the wire has negligible thickness.



CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(10), p. 420–421.

Problems in this Contest Corner came from Math Day at the Beach.

Hosted by the College of Natural Sciences and Mathematics, Math Day at the Beach is an annual high school contest open to 6-student teams from high schools in the greater Los Angeles metropolitan area. This competition was founded in 2000 and has been held every year since then.

There is an individual round (consisting of multiple choice and short answer questions), a team round, a relay round and a Faceoff Round. The Faceoff Round, the most exciting of all rounds, is patterned after the Mathcounts Countdown round, but with longer questions: the top 4 students on the individual round come up two at a time and get a chance to answer questions posed to them. Whoever rings in first gets a chance to answer the question; if that student is wrong, then the other student gets a limited time to answer. A real public spectacle, Faceoff Round draws the intense attention of all the students.

Despite the name, the contest is held indoors. The name is a play on the use of "the Beach" as a nickname for the California State University, Long Beach.

More information on the contest and past contest archives can be found here: `http://web.csulb.edu/web/depts/math/?q=node/32`

CC246. Place the numbers $1, 2, \ldots, 9$ at random so that they fill a 3×3 grid. What is the probability that each of the row sums and each of the column sums is odd?

Originally Question 17, Individual Response Round, Math Day at the Beach 2013.

We received seven correct solutions. We present the solution of Missouri State Problem Solving Group.

In order for the sum of three numbers to be odd, they must all be odd or exactly one number is odd. There are five odd numbers available, so one row must have three odd numbers and the other two rows have exactly one odd number with the analogous result for columns.

Once we choose a row with three odd numbers and a column with three odd numbers, we have used up all five available odd numbers. There are $3 \times 3 = 9$ ways of choosing the row and column, 5! ways of determining where the odd numbers are, and 4! ways of determining where the even numbers are. This gives a total of $9 \times 5! \times 4!$ matrices with the desired property.

There are a total of 9! matrices with entries from $1, 2, \ldots, 9$, so the probability we seek is

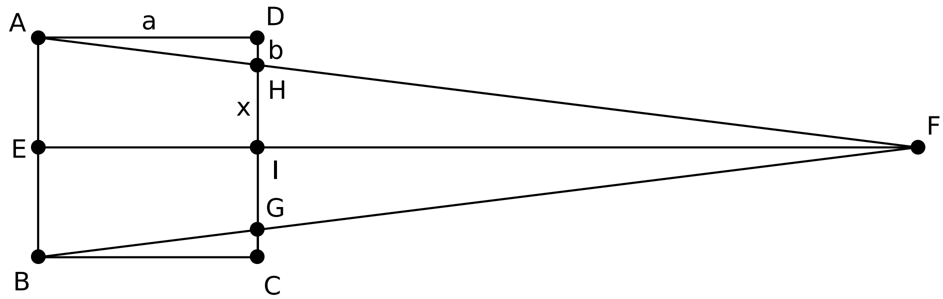
$$\frac{9 \times 5! \times 4!}{9!} = \frac{1}{14}.$$

CC247. An isosceles triangle and a square share the same base. The area of the triangle is twice the area of the square. The square splits the larger triangle into a smaller triangle and a trapezoid. What is the ratio of the area of that smaller triangle to the area of the trapezoid?

Originally Question 2, Team Round, Math Day at the Beach 2013.

We received twelve correct submissions. We present the solution by Catherine Doan and a picture proof by Joel Schlosberg.

Solution by Catherine Doan.



Let $ABCD$ be the square of side length a and ABF be the triangle with altitude EF intersecting CD at I. By definition of the triangle we have for its area that

$$\frac{a \cdot EF}{2} = 2a^2,$$

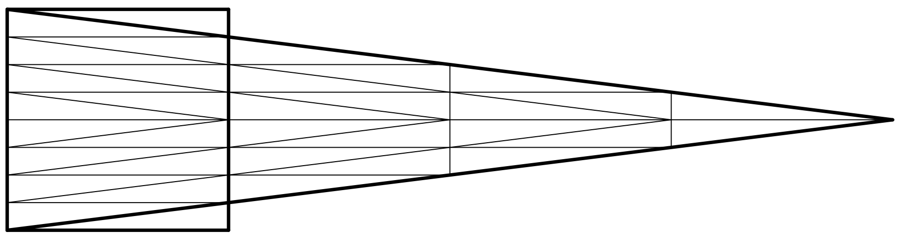
giving us $EF = 4a$ and $IF = 3a$. Let $b = CG = DH$ and $x = GI = HI$. Since the triangles ADH and FIH are similar,

$$\frac{a}{b} = \frac{3a}{x} \Rightarrow x = 3b.$$

Furthermore $a = 2x + 2b$, so $a = 8b$. Then

$$\frac{[FGH]}{[ABGH]} = \frac{\frac{1}{2} \cdot 3a \cdot 2x}{a(\frac{a+2x}{2})} = \frac{\frac{3}{2} \cdot 8b \cdot 6b}{8b(\frac{8b+6b}{2})} = \frac{9}{7}.$$

Picture by Joel Schlosberg, all explanation removed by the editor.



CC248. Nine points are arranged in $\mathbb{R}^8$ so that each pair of points is distance 1 apart. (That is, this is a regular simplex of edge length 1.) Find the radius of the smallest hypersphere that contains all 9 points.

Originally from the Face-off Round, Math Day at the Beach 2015.

We received two correct solutions. We present the solution of Ivko Dimitrić.

More generally, we find the circumradius of a regular n-simplex, which is the convex hull of $n+1$ points in $\mathbb{R}^n$ with unit distance between any two of its vertices.

By an inductive argument on the dimension, it can be readily shown that the smallest sphere containing vertices of a regular simplex in $\mathbb{R}^n$ is unique and coincides with the circumsphere of that simplex. Namely, any sphere that contains all of the vertices is the sphere containing one selected vertex and the circumsphere of the $(n-1)$-simplex in $\mathbb{R}^{n-1}$ formed by the face opposite the selected vertex. Its center lies on the segment joining the selected vertex and the center of the circumsphere of the opposite face and is the unique point on that segment for which the distances from that point to the selected vertex and from that point to any point of the circumsphere of the opposite face are equal.

Let R_n be the circumradius of a regular n-simplex of unit edgelength. Because of the symmetry of the simplex, the circumcenter is at the centroid of vertices of that simplex. If we load each vertex with unit mass then the center of mass of the n-simplex belongs to a solid median of the simplex joining the selected vertex with the centroid of the opposite face (the circumcenter of that face) and divides it in the ratio $n:1$. Consequently,

$$R_n = \frac{n}{n+1} m_n = \frac{n}{n+1}\sqrt{1^2 - R_{n-1}^2}, \tag{1}$$

where $m_n = \sqrt{1 - R_{n-1}^2}$ is the length of the solid median of a unit n-simplex. Then using the mathematical induction we prove

$$R_n = \frac{n}{n+1}\sqrt{\frac{n+1}{2n}} = \sqrt{\frac{n}{2(n+1)}}. \tag{2}$$

Indeed, the formula is clearly true for $n = 1$ and $n = 2$ (the circumradius of a unit 1-simplex is $1/2$ and the circumradius of a unit equilateral triangle is $1/\sqrt{3}$).

Assume the formula is true for $n-1$. Then from (1) and (2):

$$R_n = \frac{n}{n+1}\sqrt{1-R_{n-1}^2} = \frac{n}{n+1}\sqrt{1-\frac{n-1}{2n}} = \frac{n}{n+1}\sqrt{\frac{n+1}{2n}} = \sqrt{\frac{n}{2(n+1)}},$$

which proves the claim. Formula (2) thus gives the answer for a circumradius of a unit n-simplex in $\mathbb{R}^n$. In particular, when $n=8$, $R_8 = \sqrt{\frac{8}{2\cdot 9}} = \frac{2}{3}$.

CC249. Let S be a set of integers. The set of all possible sums of two different elements of S is $\{7, 8, 10, 11, 13, 14, 16, 19, 20, 22\}$. Each of these sums happens in only one way. If X is the mean of the set S and Y is the median of the set S, find $X+Y$.

Originally Question 2, Team Round, Math Day at the Beach, 2012.

There were seven solutions submitted by six solvers. We present the solution due, independently, to Catherine Doan, Mohsen Rahmani, and Joel Schlosberg.

Since there are $\binom{5}{2} = 10$ distinct pairwise sums, the set S has 5 distinct elements, a, b, c, d, e in increasing order. If we add all the pairwise sums of S, each element of S will be counted four times. Hence

$$a+b+c+d+e = \frac{1}{4}(7+8+10+11+13+14+16+19+20+22) = 35$$

and so $X = \frac{1}{5}(35) = 7$. Since

$$Y = c = 35-(a+b)-(d+e) = 35-7-22 = 6,$$

then $X+Y = 13$. It can be shown that $\{a, b, c, d, e\} = \{2, 5, 6, 8, 14\}$.

CC250. Two 9th graders and n 10th graders play a chess tournament. Every student plays every other student once. A student scores one point for winning a match, one half of a point for drawing a match, and zero points for losing a match. The total number of points scored by the two 9th graders was 8. Each 10th grader scored the same number of points as each other. The two 9th grade students each had scores lower than any 10th grader. How many 10th grade students were there?

Originally Question 20, Individual Free Response Round, Math Day at the Beach, 2011.

Editor's Note. According to contest organizers, this question was answered correctly only by 17 out of 208 Math Day at the Beach competitors. This was the most difficult individual question that year, but one of the easiest "most difficult" questions over a span of several years.

We received 5 solutions, out of which we present the one by Dimitrić Ivko.

Since there were $n+2$ contestants, the total number of matches played was $\frac{(n+2)(n+1)}{2}$. During each match the total number of points earned by the two players was 1, so the total number of points of all contestants at the end of the tournament was also $\frac{(n+2)(n+1)}{2}$.

Let k be the number of points earned by each 10th grader. Then the 10th graders earned a total of nk points, to which we add the total of 8 points earned by the 9th graders to form an equation in which the total number of points is computed in two different ways:

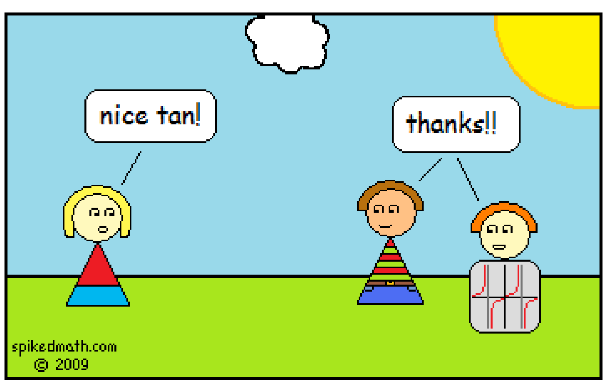
$$\frac{(n+2)(n+1)}{2} = nk + 8.$$

Clearing fractions, multiplying out, and simplifying, we get

$$n(n+3-2k) = 14, \tag{1}$$

where 14 is represented as a product of two (positive) integers. Hence we must have $n = 1$, $n = 2$, $n = 7$ or $n = 14$.

Given that the two 9th graders had earned a total of 8 points, at least one of them earned 4 or more points. Since each 9th grader earned less than k points, we have $k \geq 5$. This rules out the cases $n = 1$, $n = 2$ and $n = 7$ (since the second factor $n + 3 - 2k$ on the left hand side of (1) would not be positive). That leaves us with the case $n = 14$ and $k = 8$ as the only possibility; therefore, there were fourteen 10th graders in the tournament.



THE OLYMPIAD CORNER

No. 358

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er mai 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC356. On suppose que 2016 points sur un cercle sont coloriés en rouge et que tous les autres points du cercle sont coloriés en bleu. Étant donné un entier n ($n \geq 3$), démontrer qu'il existe un polygone régulier de n côtés dont tous les sommets sont bleus.

OC357. Soit un triangle AEF et deux points B et D sur les côtés respectifs AE et AF. Soit C le point d'intersection des segments ED et FB. On définit les points K, L, M et N sur les segments respectifs AB, BC, CD et DA tels que $\frac{AK}{KB} = \frac{AD}{BC}$ et les égalités équivalentes en procédant de façon cyclique. Le cercle inscrit dans le triangle AEF touche AE et AF aux points respectifs S et T, tandis que le cercle inscrit dans le triangle CEF touche CE et CF aux points respectifs U et V. Démontrer que si les points K, L, M et N sont cocycliques, les points S, T, U et V le sont aussi.

OC358. Démontrer que si n est un nombre parfait impair, alors n est de la forme

$$n = p^s m^2,$$

p étant un nombre premier de la forme $4k+1$, s étant un entier positif de la forme $4h+1$, et $m \in \mathbb{Z}^+$, m n'étant pas divisible par p. De plus, déterminer tous les entiers n ($n > 1$) tels que $n-1$ et $\frac{n(n+1)}{2}$ soient des nombres parfaits.

OC359. Soit a, b, c et d des nombres strictement positifs tels que $a+b+c+d = 3$. Démontrer que

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leq \frac{1}{a^3b^3c^3d^3}.$$

OC360. Soit A, B et F des nombres entiers strictement positifs choisis de manière que $A < B < 2A$. Une puce est située sur le nombre 0 de la droite numérique. La puce se déplace en faisant des sauts de longueur A ou B vers la droite. Avant que la puce ne commence à sauter, Vulcain choisit un nombre fini d'intervalles $\{m+1, m+2, \ldots, m+A\}$ contenant A entiers consécutifs strictement positifs, et met de la lave sur tous les entiers dans ces intervalles. Les intervalles doivent être choisis de manière que:

1. n'importe quels deux intervalles distincts ne soient ni superposés, ni adjacents;

2. il y ait au moins F entiers sans lave entre chaque paire d'intervalles; et

3. il n'y ait aucune lave sur les entiers inférieurs à F.

Démontrer que la plus petite valeur de F pour laquelle la puce peut traverser les intervalles sans toucher à la lave, peu importe les choix de Vulcain, est $F = (n-1)A + B$, n étant l'entier positif tel que $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

. . . .

OC356. Suppose 2016 points of the circumference of a circle are colored red and the remaining points are colored blue. Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

OC357. In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C. Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively. Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

OC358. Prove that if n is an odd perfect number then n has the following form

$$n = p^s m^2$$

where p is prime of the form $4k+1$, s is a positive integer of the form $4h+1$, and $m \in \mathbb{Z}^+$, m is not divisible by p. Also, find all $n \in \mathbb{Z}^+$, $n > 1$ such that $n-1$ and $\frac{n(n+1)}{2}$ is a perfect number.

OC359. Let a, b, c, d be positive numbers such that $a+b+c+d = 3$. Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leq \frac{1}{a^3b^3c^3d^3}.$$

OC360. Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

1. any two distinct intervals are disjoint and not adjacent;
2. there are at least F positive integers with no lava between any two intervals; and
3. no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 43(8), p. 339–340.

OC296. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2+n$ for every positive integer n.

Originally problem 1 of the 2015 Canadian Mathematical Olympiad.

We received 4 correct submissions. We present the solution by Joseph Ling.

It is easy to see that the function $f(n) = n$ satisfies the inequalities

$$(n-1)^2 < f(n) f(f(n)) < n^2 + n$$

for every positive integer n. We claim that it is the only function with this property.

Suppose that for some value of n, $f(f(n)) \leq f(n)$. Then we have both

$$\begin{aligned}(n-1)^2 < f(n) f(f(n)) \leq f(n)^2 &\Longrightarrow n-1 < f(n) \\ &\Longrightarrow n \leq f(n)\end{aligned}$$

and

$$\begin{aligned}f(f(n))^2 \leq f(n) f(f(n)) < n^2 + n < (n+1)^2 &\Longrightarrow f(f(n)) < n+1 \\ &\Longrightarrow f(f(n)) \leq n.\end{aligned}$$

This shows that n lies (weakly) between $f(n)$ and $f(f(n))$. Similarly, n must lie (weakly) between $f(n)$ and $f(f(n))$ also in the case where $f(n) \leq f(f(n))$.

Now, for any positive integer n, consider the sequence $\{a_k\}_{k=1}^{\infty}$ defined by $a_0 = n$ and $a_{k+1} = f(a_k)$ for all non-negative integers k. Consider the case where $a_0 \leq a_1$. The argument for the case with $a_0 > a_1$ is similar. By the above analysis, we see that a_k always lies between a_{k+1} and a_{k+2}. Therefore, an inductive argument yields the following relative ordering of the terms of the sequence:

$$\ldots \leq a_{2k+2} \leq a_{2k} \leq \ldots \leq a_2 \leq a_0 \leq a_1 \leq a_3 \leq \ldots \leq a_{2k-1} \leq a_{2k+1} \leq \ldots$$

By the well-ordering principle, the even-index subsequence $\{a_{2k}\}_{k=1}^{\infty}$ must eventually take a constant value, say L_o. It follows that the odd-index subsequence $\{a_{2k-1}\}_{k=1}^{\infty}$ is also eventually a constant, say L_e, where $L_o \leq L_e$. As well, $f(L_o) = L_e$ and $f(L_e) = L_o$. It suffices to show that $L_o = L_e$. For this will imply that $\{a_k\}_{k=1}^{\infty}$ is a constant sequence, and in particular, $f(n) = n$.

Suppose the contrary, i.e., suppose that $L_o < L_e$. Since

$$L_o L_e = f(L_o) f(f(L_o)) = f(L_e) f(f(L_e)),$$

this common value satisfies both

$$(L_o - 1)^2 < L_o L_e < L_0^2 + L_o \text{ and } (L_e - 1)^2 < L_o L_e < L_e^2 + L_e.$$

Since $L_0 < L_e$, this can happen only if $(L_e - 1)^2 < L_o^2 + L_o$, which implies that $L_e - 1 < L_o + 1$. It follows that $L_e = L_o + 1$. But then, $L_o L_e = L_o^2 + L_o$, a contradiction. Our proof is complete.

OC297. Prove that $\dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{(n+1)^2} < n \cdot \left(1 - \dfrac{1}{\sqrt[n]{2}}\right)$.

Originally problem 1 from day 1 of the 2015 Kazakhstan National Olympiad.

We received 6 correct submissions. We present the solution by Michel Bataille.

The inequality rewrites as $X > n/\sqrt[n]{2}$, where

$$X = \left(1 - \frac{1}{2^2}\right) + \left(1 - \frac{1}{3^2}\right) + \cdots + \left(1 - \frac{1}{(n+1)^2}\right) = \frac{1 \cdot 3}{2^2} + \frac{2 \cdot 4}{3^2} + \cdots + \frac{n(n+2)}{(n+1)^2}.$$

Now, by AM-GM, we obtain $X > Y$ where

$$\begin{aligned} Y &= n \cdot \sqrt[n]{\frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \cdots \cdot \frac{n(n+2)}{(n+1)^2}} \\ &= n \cdot \sqrt[n]{(n!) \cdot \frac{(n+2)!}{2} \cdot \frac{1}{((n+1)!)^2}} \\ &= \frac{n}{\sqrt[n]{2}} \left(\frac{n!(n+2)!}{(n+1)!(n+1)!}\right)^{1/n} \\ &= \frac{n}{\sqrt[n]{2}} \cdot \left(\frac{n+2}{n+1}\right)^{1/n}. \end{aligned}$$

Since $\dfrac{n+2}{n+1} > 1$, we have $\left(\dfrac{n+2}{n+1}\right)^{1/n} > 1$, hence $Y > \dfrac{n}{\sqrt[n]{2}}$. Thus, $X > Y > \dfrac{n}{\sqrt[n]{2}}$ and the desired inequality $X > \dfrac{n}{\sqrt[n]{2}}$ follows.

OC298. Triangle ABC is an acute triangle and its orthocenter is H. The circumcircle of $\triangle ABH$ intersects line BC at D. Lines DH and AC meets at P, and the circumcenter of $\triangle ADP$ is Q. Prove that the circumcenter of $\triangle ABH$ lies on the circumcircle of $\triangle BDQ$.

Originally problem 1 from day 2 of the 2015 Final Round of the Korean National Olympiad.

We received 4 correct submissions. We present the solution by Steven Chow.

Directed angles are used $\pmod{\pi}$. Let $B = \measuredangle CBA$ and $C = \measuredangle ACB$ for short. Let E be the circumcenter of $\triangle ABH$.

Since A, B, D, and H are concyclic and H is the orthocentre of acute $\triangle ABC$, $\measuredangle BDA = \measuredangle BHA = C$.

Since Q is the circumcenter of $\triangle ADP$,

$$\measuredangle QAC = \measuredangle QAP = \frac{1}{2}\pi - \measuredangle PDA = \frac{1}{2}\pi - \measuredangle HDA = \frac{1}{2}\pi - \measuredangle HBA = \measuredangle BAC,$$

so Q is on $\overleftrightarrow{AB}$.

Therefore

$$\measuredangle QDA = \measuredangle DAQ = \measuredangle DAB = \measuredangle CBA - \measuredangle BDA = B - C,$$

so

$$\measuredangle BDQ = \measuredangle BDA - \measuredangle QDA = C - (B - C) = -B + 2C.$$

Since E is the circumcenter of $\triangle ABD$,

$$\measuredangle QBE = \measuredangle ABE = \frac{1}{2}\pi - \measuredangle BDA = \frac{1}{2}\pi - C.$$

The radical axis of circles (ADP) and $(ABDH)$ is $\overleftrightarrow{AD}$, so $\overleftrightarrow{AD} \perp \overleftrightarrow{EQ}$, so

$$\measuredangle EQB = \frac{1}{2}\pi + \measuredangle DAQ = \frac{1}{2}\pi + B - C.$$

Therefore $\measuredangle BEQ = \pi - \measuredangle QBE - \measuredangle EQB = -B + 2C = \measuredangle BDQ$, so B, D, E, and Q are concyclic.

Therefore the circumcenter of $\triangle ABH$ lies on the circumcircle of $\triangle BDQ$.

OC299. Find all positive integers k such that for any positive integer n, $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$.

Originally problem 4 from part A of the 2015 China Second Round National Olympiad.

We present the solution by Mohammed Aassila. There were no other submissions.

We will prove that the set of positive integers k such that for any positive integer n, the integer $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$ is the set of all powers of 2. By Legendre's Formula, we know that

$$v_p(n!) = \frac{n - S_p(n)}{p-1}$$

where $S_p(n)$ is the sum of the digits of n when written in base p and v_p is the usual p-adic valuation. We have

$$v_2\left(\frac{(kn)!}{n!}\right) = (k-1)n + (S_2(n) - s_2(kn)).$$

Thus, the condition of the problem is equivalent to $S_2(kn) \geq S_2(n)$.

If $k = 1$, then this is trivially true. Notice that this is satisfied if and only if $2k$ satisfies the condition as well. Therefore, it suffices to show that there is no such $k > 1$ for which $2^{(k-1)n+1}$ does not divide $\frac{(kn)!}{n!}$ for all n.

Let m be a multiple of k with a minimal number of 1 digits (say v many) and take m to be odd. Then, let $m = 2^a + b$ with $2^a > b$. Notice that the number

$$w = 2^{a+s\varphi(k)} + b$$

is also a multiple of k with v ones for every $s \in \mathbb{N}$ where $\varphi(k)$ is Euler's phi function.

We now prove that w/k must have more than v digits that are 1 for sufficiently large s and so does not satisfy the condition of the question. Assume otherwise and pick s so that w/k has at least z consecutive zeroes where z is the length of the binary representation of k. Then for sufficiently large s, the number of one digits in the binary representation of $w/k \cdot k$ is greater than v, a contradiction.

OC300. All contestants at one contest are sitting in n columns and are forming a "good" configuration. (We define one configuration as "good" when we don't have 2 friends sitting in the same column). It's impossible for all the students to sit in $n-1$ columns in a "good" configuration. Prove that we can always choose contestants $M_1, M_2, \ldots, M_n$ such that M_i is sitting in the $i-th$ column, for each $i = 1, 2, \ldots, n$ and M_i is friend of M_{i+1} for each $i = 1, 2, \ldots, n-1$.

Originally problem 3 of the 2015 Macedonia National Olympiad.

We received 2 correct submissions. We present the solution by Oliver Guepel.

Our proof is by contradiction. Suppose that the result is false. We will construct a good configuration consisting of at most $n-1$ columns, which contradicts the hypothesis of the problem.

Let $\mathcal{C}_i$ denote the set of contestants sitting in the ith column, $1 \leq i \leq n$. For contestants M and N, we write $M \rightarrow N$ when M and N are friends and $M \in \mathcal{C}_i$, $N \in \mathcal{C}_{i+1}$, for some $\in \{1, \ldots, n-1\}$. Let us say that contestant M is *reachable* if there is a chain $M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_i$ such that $M_1 \in \mathcal{C}_1$ and $M_i = M$ for some $i \in \{1, \ldots, n\}$. (The members of $\mathcal{C}_1$ are reachable.) A contestant is called *unreachable* if he or she is not reachable. By hypothesis, all the students in the nth column are unreachable.

Let Γ_1 denote the given good configuration. We construct a new good configuration Γ_2 as follows. Choose an unreachable participant $M \in \mathcal{C}_i$ such that the number $i \geq 2$ is minimal. Contestant M has no friends in $\mathcal{C}_{i-1}$ since the members of $\mathcal{C}_{i-1}$ are reachable but M is not. As a consequence, we may shift M to column $\mathcal{C}_{i-1}$ to obtain our desired good configuration Γ_2. It is crucial to observe that Γ_2 has not more than n columns and the remaining members of column $\mathcal{C}_n$ remain unreachable in Γ_2.

We iterate the same procedure as long as an unreachable contestant can be found. The process is finite because a student is shifted to a "lesser" column in each step. After a finite number of steps we arrive at a good configuration where all contestants are reachable. Consequently, column $\mathcal{C}_n$ is empty in this final configuration. The proof is complete.

Polynomial Division in Number Theory

James Rickards

1 Introduction

Polynomial division is a useful technique in mathematics, which can be used to solve many algebra problems. Lesser known however is its power to solve difficult number theoretic problems, where the normally tricky ideas now naturally appear as a result of the algebra. The general setup is when you are given an expression involving a quotient of two polynomials, and the assertion that the quotient is an integer. The question will typically be along the lines of “find all solutions to this assertion”, or “prove some property of the quotient”. The method of solution is to polynomially divide and find an approximation to what the quotient should be, ideally involving “good” terms and a “bad/ugly” term, where the bad term is small. We then impose the conditions of the quotient being an integer to deduce what the bad term should be, from which we get nice algebraic equations to work with. Let’s call this method the “division method” for this article.

A classical example looks like the following:

Problem 1: Find all integers x such that

$$Q := \frac{3x^3 - 5x + 1}{2x - 1}$$

is an integer.

For this problem, it’s best to multiply by 8 to get

$$\begin{aligned} 8Q &= \frac{24x^3 - 40x + 8}{2x - 1} \\ &= 12x^2 + \frac{12x^2 - 40x + 8}{2x - 1} \\ &= 12x^2 + 6x + \frac{-34x + 8}{2x - 1} \\ &= 12x^2 + 6x - 17 + \frac{-9}{2x - 1}. \end{aligned}$$

Here, $12x^2 + 6x - 17$ is the good term and $\frac{-9}{2x-1}$ is the bad term. Since Q, x are integers, we see that $\frac{-9}{2x-1}$ is an integer, which reduces us to finitely many cases: $2x - 1 = -9, -3, -1, 1, 3, 9$, so $x = -4, -1, 0, 1, 2, 5$. In each of these cases we have shown that $8Q \in \mathbb{Z}$, but since the denominator of Q in lowest terms is a divisor of the odd number $2x - 1$, this implies that $Q \in \mathbb{Z}$. Thus $x = -4, -1, 0, 1, 2, 5$ is the set of solutions.

2 A Typical Example

Let's go on to a more difficult example: the famous IMO 1988 problem 6.

Problem 2: Let a, b be two positive integers such that $ab + 1 \mid a^2 + b^2$. Prove that

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.

The normal technique used to solve this problem is *Vieta jumping*. While this is a very standard trick now, in 1988 it wasn't well known, and this was a very difficult problem. It even stumped future Fields Medalist Terence Tao! The basic idea behind Vieta jumping is you start with the "smallest" solution disproving the given assertion, and manipulate the assertion into a quadratic equation in some variable. Using Vieta's formulas, you construct a smaller solution, giving a contradiction. The reader seeking a longer exposition on Vieta jumping can start with the article by Yimin Ge, found on his website (http://www.yimin-ge.com/doc/VietaJumping.pdf). The solution we give here will be somewhat similar, but where the clever ideas naturally pop out of the algebra.

To start off, suppose without loss of generality that $b \geq a$. Thus the b term is the "dominant term", and to get a small term in the numerator we wish to eliminate that. We treat the expressions as polynomials in b, and write

$$Q := \frac{a^2 + b^2}{ab + 1} = \frac{b}{a} + \frac{-\frac{b}{a} + a^2}{ab + 1} = \frac{b}{a} + \frac{a^3 - b}{a^2 b + a}.$$

Since $a \leq b$, we see that either

$$0 \leq a^3 - b < a^3 \leq a^2 b < a^2 b + a \quad \text{or} \quad 0 \leq b - a^3 < b < a^2 b + a.$$

In any case, we have a nice term of $\frac{b}{a}$ and a bad term of $\frac{a^3 - b}{a^2 b + a}$ which satisfies

$$\left| \frac{a^3 - b}{a^2 b + a} \right| < 1.$$

In particular, $Q \approx \frac{b}{a}$; it is the ceiling or floor of $\frac{b}{a}$ if $a^3 - b \geq 0$ or $a^3 - b \leq 0$ respectively. Thus it is natural to write $b = an + r$, where $n \in \mathbb{Z}^+$, $r \in \mathbb{Z}$, $0 \leq r < a$. We must get $Q = n$ or $Q = n + 1$, and plugging in the expression for b yields

$$\begin{aligned} Q = n + \frac{r}{a} + \frac{a^3 - an - r}{a(na^2 + ra + 1)} &= n + \frac{(nra^2 + r^2 a + r) + (a^3 - an - r)}{a(na^2 + ra + 1)} \\ &= n + \frac{a^2 + nra + r^2 - n}{na^2 + ra + 1}. \end{aligned}$$

Therefore $\frac{a^2 + nra + r^2 - n}{na^2 + ra + 1} = 0, 1$.

If it is 0, then note our expression is linear in n and we solve to get $n = \frac{a^2+r^2}{1-ar}$. However $n > 0$, whence $1 \geq 1 - ar > 0$, so $ar = 0$ and thus $r = 0$. This gives $n = a^2$ and $Q = n = a^2$ is a perfect square (this is the solution $(a, b) = (a, a^3)$).

The other case is $\frac{a^2+nra+r^2-n}{na^2+ra+1} = 1$. When we multiply out, it is again linear in n, and we solve to get

$$Q = n + 1 = \frac{a^2 + r^2 - ra - 1}{a^2 - ra + 1} + 1 = \frac{2a^2 + r^2 - 2ar}{a^2 - ra + 1} = \frac{(a - r)^2 + a^2}{(a - r)a + 1}$$

where $a, a - r \in \mathbb{Z}^+$. In particular, we started with the pair $(a, b) = (a, an + r)$, and found that $(a - r, a)$ gave the same quotient. If $a = b$, we can do an easy polynomial division to check that $(1, 1)$ is they only such possibility, and it gives the quotient of $1 = 1^2$. Otherwise, this decreases the sum $a+b$, whence we can only do this finitely many times, whereupon we must be in the first case or $a = b = 1$. In each case the quotient was a square, so we are done. Note that our solution also describes how to recursively find all possible positive integer pairs (a, b) such that $ab + 1 \mid a^2 + b^2$.

3 A Difficult Example

Recognizing problems where the division method is applicable is normally fairly easy, as they often are similar to the example given in the previous section. However you have to keep an open mind, as there are many more problems for which it is applicable. A fantastic example is question 2 of IMO 2015:

Problem 3: Determine all triples (a, b, c) of positive integers such that each of the numbers $ab - c, bc - a, ca - b$ is a power of 2.

This question was a controversial choice for the exam, and was a pain to mark. There turn out to be 4 different triples which worked (up to permutation), and thus some casework is expected. The main problem was many incomplete solutions had cases upon cases, and it was unclear if the casework would ever terminate. It was this problem where I first realized how useful the division method could be: using it we can produce a straightforward solution without mounds of endless casework.

The first question to ask is: what's special about these numbers being powers of 2? To me, the answer was that they have to divide each other. Since the expressions we are given are polynomial in nature, this seemed like quite a strong requirement. To get started, assume that $0 < a \leq b \leq c$ and let

$$ab - c = 2^x, \ ac - b = 2^y, \ bc - a = 2^z.$$

Note that $b \leq c$ implies $b(a + 1) \leq c(a + 1)$, and thus $ab - c \leq ac - b$. Similarly we get $ac - b \leq bc - a$, so we have $x \leq y \leq z$ (also $x = y$ implies $b = c$, and $y = z$ implies $a = b$). Furthermore, $a > 1$ as $a = 1$ implies $2^x = b - c \leq 0$.

Let's eliminate c by using $c = ab - 2^x$, and then our division is $ac - b \mid bc - a$, i.e.

$$2^{z-y} = \frac{bc-a}{ac-b} = \frac{ab^2 - 2^x b - a}{(a^2-1)b - 2^x a}$$

Now, $2^x = ab - c < ab$, so the dominant terms on the top and bottom are the ab^2 and $(a^2-1)b$ respectively; let's bring them out. We get

$$2^{z-y} = \frac{ab}{a^2-1} + \frac{-2^x b - a + \frac{2^x a^2 b}{a^2-1}}{(a^2-1)b - 2^x a} = \frac{ab}{a^2-1} + \frac{1}{a^2-1}\frac{-a^3 + 2^x b + a}{(a^2-1)b - 2^x a},$$

and so

$$(a^2-1)2^{z-y} - ab = \frac{-a^3 + 2^x b + a}{(a^2-1)b - 2^x a} := \epsilon, \tag{1}$$

where $\epsilon \in \mathbb{Z}$ necessarily. This is only useful if we have a reasonable bound on ϵ, which seems likely considering both $-a^3 + a$ and $2^x b$ are dominated by $a^2 b$.

Recall that the denominator of ϵ is $ac - b > 0$, hence

$$\begin{aligned} &\epsilon \geq -1 \\ &\Leftrightarrow -a^3 + 2^x b + a \geq (1-a^2)b + 2^x a \\ &\Leftrightarrow (b-a)(a^2 + 2^x - 1) \geq 0, \end{aligned}$$

which is true, and equality is equivalent to $a = b$ (since $a^2 + 2^x - 1 \geq a^2 > 0$). Next,

$$\begin{aligned} &\epsilon \leq 1 \\ &\Leftrightarrow -a^3 + 2^x b + a \leq (a^2-1)b - 2^x a \\ &\Leftrightarrow (a+b)(a^2 - 2^x - 1) \geq 0, \end{aligned}$$

which is not as clear. If $x = 0$, this is implied by $a > 1$; otherwise $x \geq 1$ and:

$$ab \equiv c \pmod{2^x}, \quad ac \equiv b \pmod{2^x}, \quad bc \equiv a \pmod{2^x}$$

whence $b \equiv ac \equiv a^2 b \pmod{2^x}$, so $2^x \mid (a^2-1)b$. In particular, if b is odd, then $2^x \mid a^2 - 1$, so $a > 1$ implies that $a^2 - 2^x - 1 \geq 0$. Otherwise, b is even, and $a \equiv bc \equiv ab^2 \pmod{2^x}$, so $2^x \mid (b^2-1)a$, and thus $2^x \mid a$ since $b^2 - 1$ is odd. Now we have $a^2 - 2^x - 1 \geq 2^{2x} - 2^x - 1 \geq 2^{2x-1} - 1 \geq 1 > 0$ as desired.

To recap, we have $-1 \leq \epsilon \leq 1$, where $\epsilon = -1$ is equivalent to $a = b$, and $\epsilon = 1$ is equivalent to $a^2 = 2^x + 1$. Since ϵ was an integer, this gives three clear cases.

Case 1: $\epsilon = -1$. Thus $a = b$, and so $y = z$. We have $2^x = ab - c = a^2 - c$, and $2^y = ac - b = a(c-1)$, whence $a = 2^u$, $c = 2^v + 1$ for some $u, v \in \mathbb{Z}^{\geq 0}$ with $u + v = y$. Plugging this into the first equation gives

$$2^{2u} = 2^x + 2^v + 1.$$

It is now obvious what happens when you consider binary expansions. The binary expression on the left is a single term 2^{2u}, whence the 3 powers of 2 on the right must combine into one, i.e. we necessarily have $x = 0, v = 1, u = 1$ or $x = 1, v = 0, u = 1$. This leads us to the solutions $(2, 2, 2)$ and $(2, 2, 3)$ (as well as permutations).

Case 2: $\epsilon = 0$**.** The equation $\epsilon = 0$ gives us $-a^3 + 2^x b + a = 0$, which implies that $b = \dfrac{a^3 - a}{2^x}$, so combining with equation 1 we get $2^{z-y} = \dfrac{ab + \epsilon}{a^2 - 1} = \dfrac{a^2}{2^x}$. Thus

$$\begin{aligned}
a &= 2^{\frac{x+z-y}{2}} := 2^r \\
b &= \frac{a^3 - a}{2^x} = 2^{3r-x} - 2^{r-x} \\
c &= ab - 2^x = 2^{4r-x} - 2^{2r-x} - 2^x \\
2^y &= ac - b = 2^{5r-x} - 2^{3r-x} - 2^{x+r} - 2^{3r-x} + 2^{r-x} \\
&= 2^{5r-x} - 2^{3r-x+1} - 2^{x+r} + 2^{r-x},
\end{aligned}$$

where $r \in \mathbb{Z}^+$ necessarily. Upon rearrangement,

$$2^y + 2^{3r-x+1} + 2^{x+r} = 2^{5r-x} + 2^{r-x}.$$

Since $5r - x > r - x$, the right hand side is a valid binary representation. Since $2^{5r-x} > 2^{3r-x+1} > 2^{r-x}$, we necessarily have 2^{3r-x+1} and one of $2^y, 2^{r+x}$ must combine to give 2^{5r-x}, with the remaining term being 2^{r-x}. The two cases become

$$3r - x + 1 = x + r = 5r - x - 1, \text{ and } y = r - x$$
$$3r - x + 1 = y = 5r - x - 1, \text{ and } x + r = r - x.$$

We always have $3r - x + 1 = 5r - x - 1$, whence $r = 1$. The first case gives $x = \frac{3}{2}$, contradiction, and the second case gives $x = 0, y = 4$. Plugging this back in we get the valid triple $(a, b, c) = (2, 6, 11)$.

Case 3: $\epsilon = 1$**.** Equality implied $a^2 = 2^x + 1$, and from (1) we have $2^{z-y} = \frac{ab+\epsilon}{a^2-1} = \frac{ab+1}{2^x}$. We factorize $(a - 1)(a + 1) = 2^x$, and since $\gcd(a - 1, a + 1) \leq 2$, if $a-1 > 2$ then one of these two factors must have a prime factor other than 2. Thus $a - 1 \leq 2$, and the only solution is $a = 3, x = 3$. So $2^{z-y} = \frac{3b+1}{8}$ or $b = \frac{2^r-1}{3}$ with $r = z - y + 3$. From $8 = 2^x = ab - c = 3b - c$, we get $c = 3b - 8 = 2^r - 9$. Finally, we plug this back into the equation for 2^y to get $2^y = ac - b = 3 \cdot 2^r - 27 - \frac{2^r-1}{3}$, whence

$$3 \cdot 2^y = 2^{r+3} - 80 \Rightarrow 2^{r+3} = 2^{y+1} + 2^y + 2^6 + 2^4.$$

This forces $y = 4$, $r+3 = 7$, and so $r = 4$ which gives the solution $(a, b, c) = (3, 5, 7)$.

To summarize, the only triples which work are

$$(a, b, c) = (2, 2, 2), (2, 2, 3), (2, 6, 11), (3, 5, 7)$$

and permutations. To get this result, we translated the power of two condition into a divisibility assertion and applied the division method. Bounding the bad term yielded 3 cases, which all reduced to equations involving sums of powers of 2, which were straightforward to solve.

4 More Problems

For the intrepid reader, here is a short list of problems solvable with the division method:

1. (APMO 2002) Find all pairs of positive integers (a, b) such that

$$\frac{a^2+b}{b^2-a}, \frac{b^2+a}{a^2-b} \in \mathbb{Z}$$

2. (IMO 1988 variant) Let x, y be integers such that $xy+1 \mid x^2+y^2$. Prove that if

$$N := \frac{x^2+y^2}{xy+1} < 0,$$

then $N = -5$.

3. (IMO 1998) Find all pairs of positive integers (a, b) such that

$$a^2b+b+7 \mid ab^2+a+b$$

4. (Russia 2001) Find all positive integers which can be represented uniquely as

$$\frac{x^2+y}{xy+1},$$

for x, y positive integers.

5. (IMO 1994) Find all pairs of positive integers (m, n) such that

$$\frac{n^3+1}{mn-1} \in \mathbb{Z}.$$

6. (IMO 2003) Find all pairs of integers (a, b) such that

$$\frac{a^2}{2ab^2-b^3+1} \in \mathbb{Z}^+.$$

7. (IMO 2015 variant) Prove that there are no quadruples (p, a, b, c) where p is an odd prime and $a, b, c \in \mathbb{Z}^+$ such that each of the numbers

$$ab-c, bc-a, ca-b$$

is a power of p.

8. (IMO 2015 variant) Let $a \leq b \leq c$ be positive integers and p a prime such that each of the numbers

$$-(ab-c), bc-a, ca-b$$

is a power of p. Prove that either $(a, b, c) = (p^u, p^u, p^{2u}+1)$ for some non-negative integer u, or $a = 1$.

PROBLEM SOLVING 101

No. 2

Shawn Godin

This month, we will look at assignment #3 from the course C & O 380 that I took back in 1986. You can check out the first two assignments in earlier issues [2017: 43(4), p. 151 - 153] and [2017: 43(8), p. 344 - 346].

In the Ross Honsberger Commemorative issue, I presented the problems, and a solution to problem #5, from Assignment #1 of the course C & O 380 that I took with Professor Honsberger back in 1986 (***Crux*** 43(4), p. 151–153). This month, we will look at Assignment #2.

C&O 380	**Assignment #2**	**Due: February 26, 1986**

#1. P is an arbitrary point on the circumcircle of equilateral triangle ABC. Prove that the smaller two of the lengths among AP, BP and CP add up to the biggest of them.

#2. What are the last 2 digits in the decimal representation of

$$21^{19^{18^{17}}} \quad \text{(i.e. } 21^{\left[19^{\left(18^{17}\right)}\right]}\text{)}.$$

#3. Prove that the first 1000 digits after the decimal point in the decimal expansion of $(6+\sqrt{35})^{1986}$ are all 9's.

#4. A circle of radius $\frac{1}{2}$ is tossed at random onto a coordinate plane. What is the probability that it covers a lattice point?

#5. A positive integer, of fewer than 25 digits, begins with the digits 15. The effect of multiplying this integer by 5 is merely to re-locate these first two digits to the end:

$$5(15abc\dots k) = abc\dots k15.$$

What is the integer?

#6. A and B take turns striking out a single number from a string of n consecutive positive integers, A going first. In order to eliminate special cases, suppose $n \geq 12$. The game ends when there are just two numbers left in the string. A wins if the two numbers are relatively prime and B wins if they are not. Devise

(i) a winning strategy for A in the case of n odd, and

(ii) a winning strategy for B in the case when n is even.

We will look at problem #2. These types of problems show up all the time in mathematics contests and introductory number theory courses. We have several paths of attack, each with its own merits.

First off, we can actually evaluate a few powers of 21 to see what we get

$21^0 = 1$	$21^2 = 441$	$21^4 = 194\,481$	$21^6 = 85\,766\,121$
$21^1 = 21$	$21^3 = 9261$	$21^5 = 4\,084\,101$	$21^7 = 1\,801\,088\,541$

We are only interested in the last two digits and they seem to fall into a pattern $01, 21, 41, 61, 81, 01, 21, \ldots$. Does this pattern continue? If a number, N, last two digits 01, 21, 41, 61 or 81, we can write it in the form $N = 100m + 20n + 1$, where m and n are nonnegative integers and we can assume, without loss of generality, that $0 \leq n \leq 4$. Then, if we multiply by 21 we get

$$\begin{aligned} 21N &= 21(100m + 20n + 1) \\ &= 2100m + 420n + 21 \\ &= 2100m + 400n + 20n + 20 + 1 \\ &= 100(21m + 4n) + 20(n + 1) + 1 \end{aligned}$$

If $n = 4$, then $20(n + 1) = 100$ so $21N = 100(21m + 4n + 1) + 1$, so the last digits of 21^n follow the cycle $21 \to 41 \to 61 \to 81 \to 01 \to 21 \to \ldots$ as n goes through the positive integers.

Next, we have to find out where in the cycle we will be when we evaluate the expression. We can build up an argument, but there is a tool that will make the process much easier: *modular arithmetic*. We will say $a \equiv b \pmod{n}$, read a is congruent to b modulo n, if $n \mid (a - b)$ (that is n divides evenly into $a - b$). This happens if a and b have the same remainder when divided by n (called the *modulus*). As a result, all integers are grouped into n groups, called *equivalence classes*. For example if we look at the integers modulo 6, we can write them as

$\cdots$	-11	-10	-9	-8	-7
-6	-5	-4	-3	-2	-1
0	**1**	**2**	**3**	**4**	**5**
6	7	8	9	10	11
12	13	14	15	16	$\cdots$

Then each column represents an equivalence class and all the numbers in that equivalence class are equivalent. Thus $-10 \equiv 8 \pmod{6}$ since -10 and 8 are in the same column and $6 \mid (-10 - 8)$.

The equivalence relationship, congruence modulo n, has several important properties. If $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$ then:

1. $a + b \equiv c + d \pmod{n}$,
2. $ab \equiv cd \pmod{n}$,
3. $a^m \equiv c^m \pmod{n}$ for any non-negative integer m.

This becomes useful because we can *reduce* numbers modulo n and do the arithmetic with smaller numbers. In our case, we know that the last two digits of our expression will be 01, 21, 41, 61 or 81 corresponding to the exponent of 21 being congruent to 0, 1, 2, 3 or 4 modulo 5. Since 19 leaves a remainder of 4 when divided by 5, and 4 and -1 are "in the same column" (i.e. being 4 above a multiple of 5 is the same as being 1 below a [different] multiple of 5), we can write

$$19^{18^{17}} \equiv 4^{18^{17}} \pmod 5 \equiv (-1)^{18^{17}} \pmod 5 \equiv 1 \pmod 5$$

since 18^{17} is clearly even. This means that $21^{19^{18^{17}}}$ has the same last two digits as $21^1 = 21$, so the last two digits are 21.

Now that we have modular arithmetic, we could have actually solved the problem another way. If we utilize the binomial theorem and the properties of equivalence modulo n we would get

$$\begin{aligned}
21^{19^{18^{17}}} &= (20+1)^{19^{18^{17}}} \\
&= \sum_{i=0}^{19^{18^{17}}} \binom{19^{18^{17}}}{i} 20^i \\
&= \sum_{i=0}^{19^{18^{17}}} \binom{19^{18^{17}}}{i} 2^i \cdot 10^i \\
&= \sum_{i=2}^{19^{18^{17}}} \binom{19^{18^{17}}}{i} 2^i \cdot 10^i + \binom{19^{18^{17}}}{1} 20 + \binom{19^{18^{17}}}{0} \\
&\equiv \binom{19^{18^{17}}}{1} 20 + \binom{19^{18^{17}}}{0} \pmod{100} \text{ since } 10^k \equiv 0 \pmod{100} \text{ for } k \geq 2 \\
&\equiv 20 \cdot 19^{18^{17}} + 1 \pmod{100}
\end{aligned}$$

Since $20 \times 5 \equiv 0 \pmod{100}$, we use that $19^{18^{17}} \equiv 1 \pmod 5$ to get our result.

Modular arithmetic comes in handy many places. Search through some old math contests to find similar problems that ask for the last few digits of some expression that would be next to impossible to evaluate by hand. It is also the basis for a number of divisibility tests. For example, since $10 \equiv 1 \pmod 3$, the well known test for divisibility by 3 pops out. Since we can write the n digit number as $\overline{a_{n-1}a_{n-2}\cdots a_2a_1a_0} = \sum_{i=0}^{n-1} a_i 10^i$, we get

$$\begin{aligned}
\overline{a_{n-1}a_{n-2}\cdots a_2a_1a_0} &\equiv \sum_{i=0}^{n-1} a_i 1^i \pmod 3 \\
&\equiv a_{n-1} + a_{n-2} + \cdots + a_2 + a_1 + a_0 \pmod 3 .
\end{aligned}$$

Keep your eye open for where this powerful tool can be used.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er mai 2018**.

Un astérisque ($\star$) signale un problème proposé sans solution.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4291. *Proposé par George Stoica.*

i) Déterminer le nombre de permutations de n entiers distincts choisis dans l'ensemble d'entiers $1, 2, \ldots, N$, de façon à ne pas inclure d'entiers consécutifs.

ii) Déterminer le nombre de permutations de $n+1$ entiers distincts choisis dans l'ensemble d'entiers $1, 2, \ldots, N$, de façon à ne pas inclure d'entiers consécutifs parmi les n premiers entiers de la permutation, tout en faisant en sorte que le $(n+1)^e$ entier est consécutif à un des n premiers entiers.

4292. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle aigu et soient $A_1 \in BC, B_1 \in CA, C_1 \in AB$ les pieds de ses altitudes. De plus, supposer que X, Y et Z sont les centres des cercles inscrits des triangles $AC_1B_1, BA_1C_1, CB_1A_1$ respectivement. Démontrer que ABC est équilatéral si et seulement si $\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ} = \overrightarrow{0}$.

4293. *Proposé par Eugen Ionascu.*

Soit ϕ le nombre d'or. Démontrer qu'il existe un nombre infini de suites $0-1$ $(x_n)_{n\geq 1}$ telles que

$$\sum_{n=1}^{\infty} \frac{x_n}{\phi^n} = 1.$$

4294. *Proposé par Miguel Ochoa Sanchez and Leonard Giugiuc.*

Soit ABC un triangle isocèle tel que $AB = AC$. Soit D un point sur le côté AC tel que $CD = 2DA$ et soit P un point sur BD tel que $PA \perp PC$. Démontrer que

$$\frac{BP}{PD} = \frac{3BC^2}{4AC^2}.$$

4295. *Proposé par Khang Nguyen Thanh.*

Soient a, b et c des nombres réels non nuls et distincts tels que $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=3$. Déterminer la valeur maximale de

$$P=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

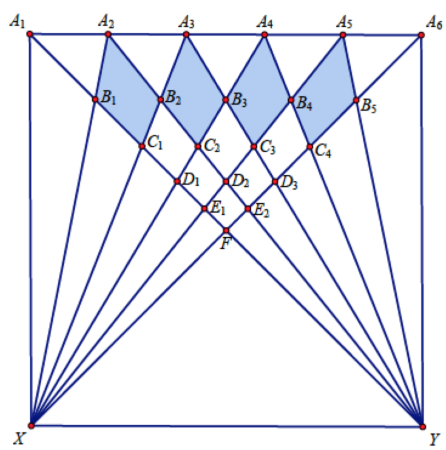
4296. *Proposé par Marius Drăgan.*

Démontrer l'inégalité suivante, valide pour tout triangle ABC:

$$5\sum_{cyclique}\tan^4\frac{A}{2}\tan^4\frac{B}{2}-4\sum_{cyclique}\tan^5\frac{A}{2}\tan^5\frac{B}{2}\geq\frac{11}{81}.$$

4297. *Proposé par Arsalan Wares.*

Le polygone A_1A_6YX est un carré. Son côté A_1A_6 est découpé en cinq parties égales à l'aide des points A_2, A_3, A_4 et A_5. Le point X est relié aux points A_2, A_3, A_4, A_5 et A_6, et puis le point Y est relié aux points A_1, A_2, A_3, A_4 et A_5. Les points B_i, C_i, D_i et F sont les points d'intersection de segments, comme indiqué ci-bas.



Déterminer le ratio de la somme des surfaces des quadrilatères ombragés (notamment les quadrilatères $B_1A_2B_2C_1, B_2A_3B_3C_2, B_3A_4B_4C_3$ et $B_4A_5B_5C_4$) par rapport à la surface du carré A_1A_6YX.

4298. *Proposé par Daniel Sitaru.*

Calculer

$$L=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}.$$

4299. *Proposé par Michel Bataille.*

Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f\left(xy + f(x+y)\right) = f\left(f(xy)\right) + x + y$$

pour tout $x, y \in \mathbb{R}$.

4300$\star$. *Proposé par Leonard Giugiuc.*

Soient a, b et c des nombres réels positifs tels que $a + b + c = ab + bc + ca > 0$. Démontrer ou réfuter l'inégalité

$$\sqrt{24ab + 25} + \sqrt{24bc + 25} + \sqrt{24ca + 25} \geq 21.$$

. . . .

4291. *Proposed by George Stoica.*

i) Find the number of permutations of n distinct integers from the set of integers $1, 2, \ldots, N$ so that no two integers in a permutation are consecutive.

ii) Find the number of permutations of $n + 1$ distinct integers from the set of integers $1, 2, \ldots, N$ so that no two of the first n integers in a permutation are consecutive, but the $(n + 1)^{th}$ is consecutive with one of the first n.

4292. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and let $A_1 \in BC, B_1 \in CA, C_1 \in AB$ be the feet of its altitudes. Suppose further that X, Y, Z are the incenters of triangles AC_1B_1, BA_1C_1 and CB_1A_1, respectively. Show that the given triangle is equilateral if and only if $\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ} = \overrightarrow{0}$.

4293. *Proposed by Eugen Ionascu.*

Let ϕ be the golden ratio. Prove that there exist infinitely many $0 - 1$ sequences $(x_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} \frac{x_n}{\phi^n} = 1.$$

4294. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on the side AC such that $CD = 2DA$. Let P be a point on BD such that $PA \perp PC$. Prove that

$$\frac{BP}{PD} = \frac{3BC^2}{4AC^2}.$$

4295. *Proposed by Khang Nguyen Thanh.*

Let a, b and c be distinct non-zero real numbers such that $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=3$. Find the maximum value of

$$P=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

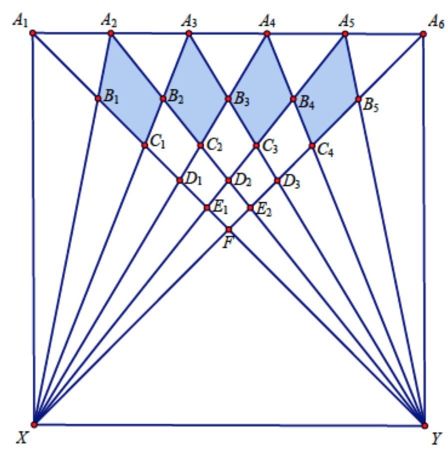
4296. *Proposed by Marius Drăgan.*

Prove that the following inequality holds for every triangle ABC:

$$5\sum_{cyclic}\tan^4\frac{A}{2}\tan^4\frac{B}{2}-4\sum_{cyclic}\tan^5\frac{A}{2}\tan^5\frac{B}{2}\geq\frac{11}{81}.$$

4297. *Proposed by Arsalan Wares.*

Suppose polygon A_1A_6YX is a square. Points A_2, A_3, A_4 and A_5 divide side A_1A_6 into five equal parts. Point X is connected to points A_2, A_3, A_4, A_5 and A_6, and point Y is connected to points A_1, A_2, A_3, A_4 and A_5. Points B_i, C_i, D_i, E_i and F are points of intersections of line segments shown in the figure.



Find the ratio of the sum of the areas of the shaded quadrilaterals (namely, quadrilaterals $B_1A_2B_2C_1, B_2A_3B_3C_2, B_3A_4B_4C_3$ and $B_4A_5B_5C_4$) to the area of square A_1A_6YX.

4298. *Proposed by Daniel Sitaru.*

Compute:

$$L=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}.$$

4299. *Proposed by Michel Bataille.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(xy + f(x+y)\right) = f\left(f(xy)\right) + x + y$$

for all $x, y \in \mathbb{R}$.

4300$\star$. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers with $a + b + c = ab + bc + ca > 0$. Prove or disprove that

$$\sqrt{24ab + 25} + \sqrt{24bc + 25} + \sqrt{24ca + 25} \geq 21.$$

Math Quotes

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange when he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others.

Karl Friedrich Gauss in a letter to Bolyai, 1808.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(10), p. 442–446.

4191. *Proposed by Mehmet Berke Işler.*

Let a, b, c be positive real numbers such that $a + b + c \geq 3$. Show that

$$abc + 2 \geq \frac{9}{a^3 + b^3 + c^3}.$$

We received ten solutions, all of which are correct. However, as pointed out by Michel Bataille, the same problem (by the same proposer) and a full solution have appeared in Mathematical Reflections (2016, No. 3) as problem J374 with solution on page 2 of the "Solution Part." But we will present a solution by Adnan Ali which actually proves the following stronger inequality:

$$abc + 2 \geq \frac{9}{a^2 + b^2 + c^2} \tag{1}$$

Solution by Adnan Ali.

We first show that (1) is stronger than the proposed inequality by proving that if $a, b, c > 0$ such that $a + b + c \geq 3$, then

$$a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3. \tag{2}$$

To prove (2), it suffices to show that

$$3(a^3 + b^3 + c^3) \geq (a + b + c)(a^2 + b^2 + c^2). \tag{3}$$

Without loss of generality, we assume $a \geq b \geq c$. Then $a^2 \geq b^2 \geq c^2$. So $3(a^3 + b^3 + c^3) = 3(a \cdot a^2 + b \cdot b^2 + c \cdot c^2) \geq (a + b + c)(a^2 + b^2 + c^2)$, by Chebyshev's Inequality. Hence, (3) holds.

To prove (1), note first that $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \geq 9$ by the weighted mean inequality. Hence we have

$$\begin{aligned}(abc + 2)(a^2 + b^2 + c^2) &= 2(a^2 + b^2 + c^2) + abc(a^2 + b^2 + c^2) \\ &\geq 2(a^2 + b^2 + c^2) + 3abc \geq 2(a^2 + b^2 + c^2) + \frac{9abc}{a + b + c} \end{aligned} \tag{4}$$

Next, we prove that

$$2(a^2 + b^2 + c^2) + \frac{9abc}{a + b + c} \geq (a + b + c)^2 \tag{5}$$

which is equivalent, in succession, to

$$2(a^2+b^2+c^2)(a+b+c)+9abc \geq (a+b+c)^3$$

$$2\left(a^3+b^3+c^3+\sum_{cyc} ab(a+b)\right)+9abc \geq a^3+b^3+c^3+3\left(\sum_{cyc} ab(a+b)\right)+6abc$$

$$a^3+b^3+c^3+3abc \geq \sum_{cyc} ab(a+b)$$

$$\sum_{cyc} a(a-b)(a-c) \geq 0,$$

which is Schur's Inequality of degree 3. From (4) and (5), it follows that

$$2(a^2+b^2+c^2)+\frac{9abc}{a+b+c} \geq 9,$$

or $(abc+2)(a^2+b^2+c^2) \geq 9$. Hence, $abc+2 \geq \dfrac{9}{a^2+b^2+c^2}$, establishing (1) and completing the proof. Note finally that equality holds if and only if $a=b=c=1$.

4192. *Proposed by Florin Stanescu.*

Consider a polynomial $P = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{C}[x]$ such that all roots of P are equal in modulus to 1. If $\sum_{k=0}^{n} a_k \neq 0$, show that

$$Re\left(\frac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n}\right)=\frac{n}{2},$$

where $Re(z)$ represents the real part of $z \in \mathbb{C}$.

Michel Bataille pointed out that virtually the same problem by the same proposer has appeared in Mathematical Reflections (2016, No. 6) as problem S396 with two solutions given on page 12 of the "Solution Part". That problem asks to show that under the same assumptions, we have

$$|a_1+a_2+\cdots+a_n| \leq \frac{2}{n}|a_1+2a_2+\cdots+na_n|.$$

We received nine submissions, all correct. Since the proofs given are very similar, we present a composite solution.

Let $z_k = e^{i\theta_k}$, where $0 \leq \theta_k \leq 2\pi$ and $k = 1, 2, \ldots, n$, be the roots of $P(x)$. Then

$$\frac{P'(x)}{P(x)} = \sum_{k=1}^{n} \frac{1}{1-z_k},$$

so

$$Re\left(\frac{a_1+2a_2+\cdots+na_n}{a_1+a_2+\cdots+a_n}\right) = Re\left(\frac{P'(1)}{P(1)}\right) = Re\left(\sum_{k=1}^{n}\frac{1}{1-z_k}\right) = \sum_{k=1}^{n} Re\left(\frac{1}{1-z_k}\right).$$

But then

$$Re\left(\frac{1}{1-z_k}\right) = Re\left(\frac{1-e^{-i\theta_k}}{|1-e^{i\theta_k}|^2}\right) = Re\left(\frac{1-\cos\theta_k - i\sin\theta_k}{(1-\cos\theta_k)^2+\sin^2\theta_k}\right) = \frac{1-\cos\theta_k}{2-2\cos\theta_k} = \frac{1}{2}.$$

Substituting this expression into the first, the result follows.

4193. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let $f:[0,\infty)\mapsto\mathbb{R}$ be a differentiable function such that its derivative f' is convex and $f(0)=0$. Prove that for any nonnegative numbers x,y,z, we have

$$f(x)+f(y)+f(z)+f(x+y+z)\geq f(x+y)+f(x+z)+f(y+z).$$

We received 5 solutions. We present the solution by Adnan Ali.

For any $t\geq 0$ we have that

$$(t+y+z)+t=(t+y)+(t+z) \text{ and}$$
$$t+y+z\geq \max(t+y,t+z);$$

hence $(t+y+z,t)$ majorizes $(t+y,t+z)$. Applying Karamata's (majorization) inequality to the convex function f' yields

$$f'(t+y+z)+f'(t)\geq f'(t+y)+f'(t+z) \text{ for all } t,y,z\geq 0.$$

Integrate both sides of the above inequality, with respect to t, from 0 to x:

$$\int_0^x f'(t+y+z)\,dt+\int_0^x f'(t)\,dt\geq \int_0^x f'(t+y)\,dt+\int_0^x f'(t+z)\,dt.$$

Finally, apply the fundamental theorem of calculus and use $f(0)=0$ to get the desired result:

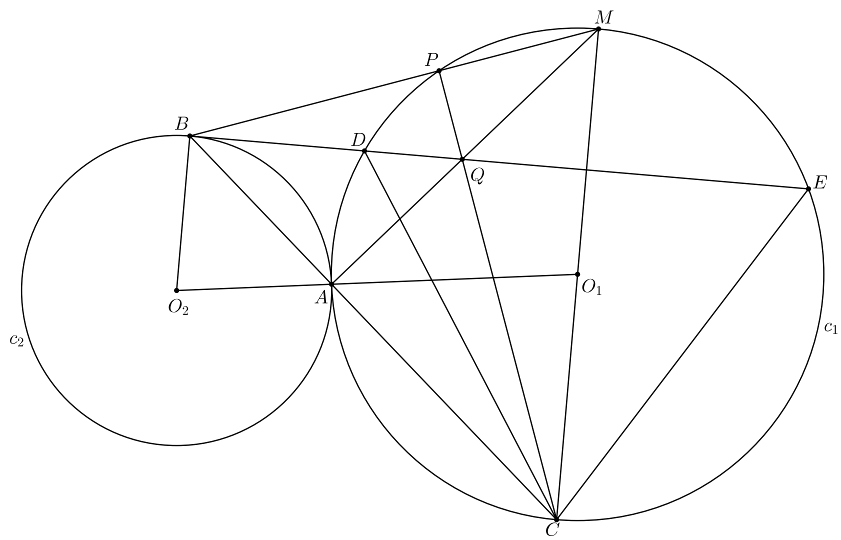
$$f(x+y+z)-f(y+z)+f(x)-f(0)\geq f(x+y)-f(y)+f(x+z)-f(z)$$
$$\text{if and only if}$$
$$f(x)+f(y)+f(z)+f(x+y+z)\geq f(x+y)+f(x+z)+f(y+z).$$

4194. *Proposed by Mihaela Berindeanu.*

Given circles c_1 and c_2 that are externally tangent at A, let the tangent to c_2 at B intersect c_1 at D and E. Furthermore, let c_1 intersect AB again at C and the bisector of $\angle DCE$ at M, and define Q and P to be the points where AM intersects BE and BM intersects CQ. Show that

$$\frac{BP}{BC}=\frac{BA}{BM}.$$

We received six submissions, all correct, and feature two of the various approaches.



Solution 1 is a composite of the similar solutions from Adnan Ali, Steven Chow, and Peter Woo.

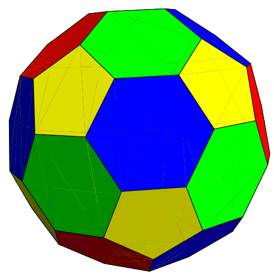
We shall see that the result also holds when c_2 is internally tangent to c_1, (with c_2 inside c_1 so that the points D and E are defined). Let O_1 and O_2 be the centres of c_1 and c_2, respectively. Because the isosceles triangles AO_2B and AO_1C have equal angles at A, they are similar. Consequently BO_2 and CO_1 are parallel and, therefore, they are perpendicular to the line BDE. But the line through the centre that is perpendicular to the chord DE must be its perpendicular bisector, from which we deduce that CO_1 bisects $\angle DCE$, whence the chord CM passes through O_1 and is thus a diameter of c_1. Consequently $\angle CAM = 90°$, which implies that AM is an altitude of ΔBCM; since $Q \in AM$ and $BQ \perp CM$, Q must be the orthocentre of ΔBCM. Since P is the intersection of BM with CQ, this implies that $CP \perp MP$, whence P lies on the circle c_1. The power of B with respect to c_1 is therefore $BP \cdot BM = BA \cdot BC$, from which $\frac{BP}{BC} = \frac{BA}{BM}$, as desired.

Solution 2, by Michel Bataille.

Let O_1, O_2 be the centres of c_1, c_2, respectively, and let h_A be the homothety with centre A transforming O_2 into O_1. Then $\mathrm{h}_A(c_2) = c_1$ so that $\mathrm{h}_A(B) = C$. It follows that CO_1 is parallel to BO_2, hence is perpendicular to DE. In addition $O_1D = O_1E$, hence CO_1 is the perpendicular bisector of DE and so $CD = CE$. Thus, the bisector of $\angle DCE$ coincides with the line CO_1 and therefore CM is a diameter of c_1. Consequently, BD is perpendicular to CM and MA is perpendicular to BC, hence Q is the orthocenter of ΔBMC. It follows that CQ is perpendicular to BM, hence $\angle CPM = 90°$ and therefore P lies on the circle c_1. Now, consider the right-angled triangles BAM and BPC. The acute angles $\angle BMA = \angle PMA$ and $\angle BCP = \angle ACP$ subtend the same arc AP of the circle c_1, hence $\angle BMA = \angle BCP$ so that $\sin(\angle BMA) = \sin(\angle BCP)$, that is, $\frac{BA}{BM} = \frac{BP}{BC}$.

4195. *Proposed by Eugen Ionascu.*

On the faces of a regular truncated icosahedron (12 faces are regular pentagons, and 20 faces are regular hexagons, see figure below), a positive integer is written such that the sum of the numbers on the hexagons is 39 and the sum of the numbers on the pentagons is 25. Show that there are two faces that share a vertex and have the same integer written on them.



We received two solutions to the problem. We present the one by Joseph DiMuro.

Number the vertices of the polyhedron from 1 to 60. Let s_n be the sum of the numbers of the three faces adjacent to the n-th vertex, and let

$$S = \sum_{n=1}^{60} s_n.$$

In the sum, each number on a pentagon gets counted five times and each number on a hexagon six times. Thus $S = 5 \cdot 25 + 6 \cdot 39 = 359$. Now assume that there exists a labeling of the faces with positive integers such that no two faces sharing a vertex have the same integer on them. Then $s_n \geq 1 + 2 + 3 = 6$ for all n. But then $S \geq 6 \cdot 60 = 360$, a contradiction. So there must be two faces sharing a vertex that have the same integer.

4196. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Show that for all positive real numbers a, b and c, we have

$$1 \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq 2.$$

We received 15 correct solutions and will feature the one by Adnan Ali.

For the lower bound: $\dfrac{a}{a+b}+\dfrac{b}{b+c}+\dfrac{c}{c+a} \geq \dfrac{a}{a+b+c}+\dfrac{b}{b+c+a}+\dfrac{c}{c+a+b} = 1.$

For the upper bound: $\dfrac{a}{a+b}+\dfrac{b}{b+c}+\dfrac{c}{c+a} \leq \dfrac{a+c}{a+b+c}+\dfrac{b+a}{b+c+a}+\dfrac{c+b}{c+a+b} = 2,$ since $\frac{X}{Y} \leq \frac{X+a}{Y+a}$ if $X \leq Y$ and $a \geq 0$.

It is easily noticed that for positive values of a, b and c, both of the bounds above are strict.

Remark. It is remarkable to see that the left-hand-side inequality is equivalent to the right-hand -side inequality as

$$\text{RHS} \Leftrightarrow \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{b}{a+b},$$

and

$$\text{LHS} \Leftrightarrow \frac{c}{b+c} + \frac{a}{c+a} \geq \frac{a}{a+b}.$$

The above inequalities differ only by a cyclic shift. To verify this we can see that

$$\text{RHS} \Leftrightarrow (a+b)(b(c+a)+c(b+c)) - b(b+c)(c+a) > 0 \Leftrightarrow a^2b+b^2c+c^2a+abc > 0$$

and

$$\text{LHS} \Leftrightarrow (a+b)(c(c+a)+a(b+c)) - a(b+c)(c+a) > 0 \Leftrightarrow a^2b+b^2c+c^2a+abc > 0.$$

4197. *Proposed by Michel Bataille.*

Let x, y, z be positive real numbers such that $xy + yz + zx + 2xyz = 1$. Prove that

(a) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 6,$

(b) $x + y + z \geq \dfrac{3}{2}.$

We received 19 submissions, all of which are correct. We present a composite of several very similar solutions which are elementary and representative. Solution by Prithwijit De; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly); Salem Malikic; and Kevin Soto Palacios.

By the AM-GM Inequality, we have

$$1 = xy + yz + zx + 2xyz \geq 4\sqrt[4]{2(xyz)^3},$$

so $xyz \leq \frac{1}{8}$. Hence,

(a) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 3\sqrt[3]{\frac{1}{xyz}} \geq 3\sqrt[3]{8} = 6.$

(b) Since $(x+y+z)^2 \geq 3(xy+yz+zx)$, we have
$x + y + z \geq \sqrt{3(xy+yz+zx)} = \sqrt{3(1-2xyz)} \geq \sqrt{3(1-\frac{1}{4})} = \dfrac{3}{2}.$

It is easy to see that in both (a) and (b) equality holds if and only if $x = y = z = \frac{1}{2}$.

Editor's comments: Ali, Giugiuc and Zvonaru gave a proof based on the fact that under the given condition, there exist $a, b, c > 0$ such that $x = \frac{a}{b+c}$, $y = \frac{b}{c+a}$, and $z = \frac{c}{a+b}$ (see p. 3–6 of T. Andreescu, G. Dospinescu, *Problems from the Book*, XYZ Press, 2008; thanks to Zvonaru for providing the reference.).

4198. *Proposed by Leonard Giugiuc.*

In a triangle ABC, we have that $\sin A \sin B \sin C = \frac{2+\sqrt{3}}{8}$. Find the maximum possible value of $\cos A \cos B \cos C$.

We received three correct solutions and one incorrect solution. We present the solution by Arkady Alt.

Since

$$\sin^2 75^\circ = \frac{1-\cos 150^\circ}{2} = \frac{1+\cos 30^\circ}{2} = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4}$$

and

$$\cos^2 75^\circ = 1 - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4},$$

then for $A = 30^\circ$ and $B = C = 75^\circ$ we have

$$\sin A \sin B \sin C = \sin 30^\circ \sin^2 75^\circ = \frac{2+\sqrt{3}}{8}$$

and

$$\cos A \cos B \cos C = \cos 30^\circ \cos^2 75^\circ = \frac{2-\sqrt{3}}{4} \cdot \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}-3}{8}.$$

We will prove that for any $A, B, C > 0$ such that $\sin A \sin B \sin C = \frac{2+\sqrt{3}}{8}$ and $A + B + C = \pi$, we have

$$\cos A \cos B \cos C \le \frac{2\sqrt{3}-3}{8}, \tag{1}$$

with equality if $A = 30^\circ$ and $B = C = 75^\circ$ and permutations of this.

By the symmetry of inequality (1), we can assume that $C = \max\{A, B, C\}$, and hence that $C \ge \frac{\pi}{3}$.

We can also assume that $C < \frac{\pi}{2}$ since otherwise $\cos A \cos B \cos C \le 0$. Since

$$\cos A \cos B - \sin A \sin B = \cos(A+B) = -\cos C$$

or, equivalently

$$\cos A \cos B = \sin A \sin B - \cos C$$

and

$$\sin A \sin B = \frac{2+\sqrt{3}}{8},$$

then

$$\cos A \cos B \cos C = \left(\frac{2+\sqrt{3}}{8\sin C} - \cos C\right)\cos C = \frac{2+\sqrt{3}}{8t} - \frac{1}{1+t^2},$$

where $t = \tan C \geq \sqrt{3}$. Inequality (1) becomes

$$\frac{2+\sqrt{3}}{8t} - \frac{1}{1+t^2} \leq \frac{2\sqrt{3}-3}{8}. \tag{2}$$

Let $k = 2+\sqrt{3}$ and $h(t) = \frac{k}{8t} - \frac{1}{1+t^2}$. Then $k^2 - 4k + 1 = 0$ and

$$h(k) = \frac{1}{8} - \frac{1}{1+k^2} = \frac{1}{8} - \frac{1}{4k} = \frac{k-2}{8k} = \frac{\sqrt{3}}{8(2+\sqrt{3})} = \frac{2\sqrt{3}-3}{8}.$$

Thus inequality (2) can be rewritten as $h(t) \leq h(k)$ for $t \geq \sqrt{3}$. We have

$$\begin{aligned} h(k) - h(t) &= \frac{1}{8} - \frac{1}{1+k^2} - \frac{k}{8t} + \frac{1}{1+t^2} = \frac{t-k}{8t} - \frac{t^2-k^2}{(k^2+1)(t^2+1)} \\ &= (t-k)\left(\frac{1}{8t} - \frac{t+k}{4k(t^2+1)}\right) = \frac{(t-k)[k(t-1)^2 - 2t^2]}{8kt(t^2+1)} \\ &= \frac{(t-k)^2[t(k-2)-1]}{8kt(t^2+1)} \geq 0 \end{aligned}$$

because $k - 2 = \sqrt{3}$ and $t \geq \sqrt{3}$. Hence,

$$\max \cos A \cos B \cos C = \max_{t>\sqrt{3}} h(t) = h(k) = \frac{2\sqrt{3}-3}{8}.$$

4199. *Proposed by Michel Bataille.*

Let two circles Γ_1, Γ_2, with respective centres O_1, O_2, intersect at A and B and let ℓ be the internal bissector of $\angle O_1AO_2$. Let $M_1, M_2 \neq A, B$ be points on Γ_2. For $k = 1, 2$, the line BM_k and the reflection of AM_k in ℓ intersect Γ_1 again at N_k and P_k , respectively. Prove that $N_1P_2 = N_2P_1$.

All three submissions were correct, but only Steven Chow, whose solution we feature, established the generalization.

The restriction on the line ℓ can be relaxed; specifically, the following argument shows that the result holds for an arbitrary line ℓ through A. We use directed angles (mod π).

Note that AM_1 and AP_1 are symmetric about ℓ, as are AM_2 and AP_2. Therefore, $\angle P_2AP_1 = \angle M_1AM_2$. Because $\angle M_1AM_2 = \angle M_1BM_2$, we therefore have

$$\angle P_2AP_1 = \angle M_1BM_2 = \angle N_1BN_2.$$

Consequently,

$$\angle N_2AP_1 = \angle N_2AP_2 + \angle P_2AP_1 = \angle N_2BP_2 + \angle N_1BN_2 = \angle N_1BP_2.$$

It follows that $\angle N_1BP_2$ and $\angle N_2AP_1$ are subtended by equal chords, namely $N_1P_2 = N_2P_1$.

4200. *Proposed by Van Khea and Leonard Giugiuc.*

Let the cevians AD, BE, and CF of triangle ABC intersect at Q. Let points M and N lie on sides AB and AC, respectively, and let P be the point where MN intersects AD. Prove that

$$\frac{BM}{MA}\cdot\frac{AF}{FB}+\frac{CN}{NA}\cdot\frac{AE}{EC}=\frac{AQ}{QD}\cdot\frac{DP}{PA}.$$

All three submitted solutions were correct; we feature the solution by Steven Chow.

We use barycentric coordinates

$$A=(1,0,0),\quad B=(0,1,0),\quad C=(0,0,1).$$

We set

$$M=(m,1-m,0),\quad N=(n,0,1-n),\quad \text{and}\quad Q=(q_1,q_2,q_3),$$

where $q_1+q_2+q_3=1$. Since D, E, F are the feet of the cevians through Q, their homogeneous coordinates must be

$$D=(0:q_2:q_3),\quad E=(q_1:0:q_3),\quad F=(q_1:q_2:0).$$

Therefore,

$$\frac{BM}{MA}=\frac{m}{1-m},\quad \frac{AF}{FB}=\frac{q_2}{q_1},\quad \frac{CN}{NA}=\frac{n}{1-n},\quad \frac{AE}{EC}=\frac{q_3}{q_1}$$

$$\text{and}\quad \frac{AQ}{QD}=\frac{1-q_1}{q_1}=\frac{q_2+q_3}{q_1}.$$

Since P is the point where AD (with equation q_3y-q_2z) intersects MN (with equation $(1-m)(1-n)x-m(1-n)y-(1-m)nz=0$), its homogeneous coordinates must be

$$P=\left(\frac{m}{1-m}q_2+\frac{n}{1-n}q_3:q_2:q_3\right),$$

so that

$$\frac{DP}{PA}=\frac{\frac{m}{1-m}q_2+\frac{n}{1-n}q_3}{q_2+q_3}.$$

Therefore

$$\begin{aligned}\frac{BM}{MA}\cdot\frac{AF}{FB}+\frac{CN}{NA}\cdot\frac{AE}{EC}&=\frac{m}{1-m}\cdot\frac{q_2}{q_1}+\frac{n}{1-n}\cdot\frac{q_3}{q_1}\\&=\frac{q_2+q_3}{q_1}\cdot\frac{\frac{m}{1-m}q_2+\frac{n}{1-n}q_3}{q_2+q_3}\\&=\frac{AQ}{QD}\cdot\frac{DP}{PA}\end{aligned}$$

as required.

3500. *Proposed by Paul Bracken.*

Let $\beta = -f(1) + \frac{1}{4}f(\frac{1}{2}) - \frac{1}{4}f(-\frac{1}{2})$, where the function f is defined as follows:

$$f(a) = \sum_{k=1}^{\infty} \frac{\log{(k)}}{k(k+a)}, \quad a \in (-1, \infty).$$

Show that

$$\prod_{k=1}^{\infty} \frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}} = 2^{-\frac{3}{2}\log{(2)}+1-\gamma} \cdot e^{\beta},$$

where γ is Euler's constant.

We received three correct solutions. We present the solution by Perfetti Paolo.

We need the well-known result

$$\sum_{k=1}^{n} \frac{1}{2k-1} = \frac{\ln n}{2} + \ln 2 + \frac{\gamma}{2} + o(1). \tag{1}$$

Taking the logarithm of the left-hand side of the claimed identity gives

$$\ln\left[\prod_{k=1}^{\infty} \frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}}\right] = \sum_{k=1}^{n} \frac{\ln(2k-1)}{2k} - \sum_{k=1}^{n} \frac{\ln(2k)}{2k-1}.$$

We have

$$\begin{aligned} \sum_{k=1}^{n} \frac{\ln(2k-1)}{2k} &= \sum_{k=1}^{2n} \frac{\ln k}{k+1} - \sum_{k=1}^{n} \frac{\ln(2k)}{2k+1} \\ &= \sum_{k=1}^{2n} \frac{\ln k}{k+1} - \sum_{k=1}^{n} \frac{\ln 2}{2k+1} - \sum_{k=1}^{n} \frac{\ln k}{2k+1} \end{aligned} \tag{2}$$

and

$$-\sum_{k=1}^{n} \frac{\ln(2k)}{2k-1} = -\sum_{k=1}^{n} \frac{\ln 2}{2k-1} - \sum_{k=1}^{n} \frac{\ln k}{2k-1}. \tag{3}$$

The sum of (2) and (3) yields

$$\sum_{k=1}^{2n} \frac{\ln k}{k+1} - \sum_{k=1}^{n} \frac{\ln k}{2k+1} - \sum_{k=1}^{n} \frac{\ln k}{2k-1} - \sum_{k=1}^{n} \frac{2\ln 2}{2k-1} + \ln 2 - \frac{\ln 2}{n+1},$$

which we rewrite as

$$\begin{aligned} &\sum_{k=1}^{n} \left(\frac{\ln k}{k+1} - \frac{\ln k}{k}\right) + \sum_{k=1}^{n} \left(\frac{\ln k}{2k} - \frac{\ln k}{2k+1}\right) + \sum_{k=1}^{n} \left(\frac{\ln k}{2k} - \frac{\ln k}{2k-1}\right) \\ &\qquad + \sum_{k=n+1}^{2n} \frac{\ln k}{k+1} - \sum_{k=1}^{2n} \frac{2\ln 2}{2k-1} + \ln 2 + o(1); \end{aligned} \tag{4}$$

that is

$$\begin{aligned}
&-f(1)+\frac{1}{4}f\left(\frac{1}{2}\right)-\frac{1}{4}f\left(-\frac{1}{2}\right)+\sum_{k=n+1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{2n}\frac{2\ln 2}{2k-1}+\ln 2+o(1)\\
&\quad=\beta+\sum_{k=n+1}^{2n}\frac{\ln k}{k+1}-\sum_{k=1}^{2n}\frac{2\ln 2}{2k-1}+\ln 2+o(1).
\end{aligned} \tag{5}$$

The asymptotic behavior of monotonic series allows us to write

$$\sum_{k=n+1}^{2n}\frac{\ln k}{k+1}=\int_{n+1}^{2n}\frac{\ln x}{x+1}dx+o(1).$$

Now

$$\int_{n+1}^{2n}\frac{\ln x}{x+1}dx=\int_{n+1}^{2n}\ln x\left(\frac{1}{x+1}-\frac{1}{x}\right)dx+\int_{n+1}^{2n}\frac{\ln x}{x}\,dx.$$

Since $\left|\ln x\left(\frac{1}{x+1}-\frac{1}{x}\right)\right|\le Cx^{-3/2}$, the first integral goes to zero as $n\to\infty$. We have

$$\begin{aligned}
\int_{n+1}^{2n}\frac{\ln x}{x}\,dx&=\frac{1}{2}\ln^2 x\Big|_{n+1}^{2n}=\frac{1}{2}\left[\ln^2(2n)-\ln^2(n+1)\right]\\
&=\frac{1}{2}\left[\ln(2n)-\ln(n+1)\right]\left[\ln(2n)+\ln(n+1)\right]\\
&=\frac{1}{2}(\ln 2)(\ln 2+2\ln n)+o(1).
\end{aligned}$$

By inserting it into (5) and taking into account (1) we have

$$\begin{aligned}
&\beta+\frac{1}{2}\ln 2(\ln 2+2\ln n)-2\ln 2\left(\frac{1}{2}\ln n+\ln 2+\frac{\gamma}{2}\right)+\ln 2+o(1)\\
&=\beta-\frac{3}{2}\ln^2 2-\gamma\ln 2+\ln 2\\
&=\beta+\ln 2\left(-\frac{3}{2}\ln 2-\gamma+1\right).
\end{aligned}$$

Exponentiating, we get

$$e^{\beta}\cdot 2^{-\frac{3}{2}\ln 2-\gamma+1},$$

concluding the proof.

Editor's comment. Problem 3500 above was originally published in ***Crux*** 35(8), p. 519, in December 2009. Its solution was supposed to appear in ***Crux*** 36(8) in December 2010. However, there is no mentioning of the problem in that issue, so it recently reappeared in ***Crux*** 42(10).

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