

# *CruX Mathematicorum*

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## IN THIS ISSUE / DANS CE NUMÉRO

- 331 Editorial *Kseniya Garaschuk*
- 332 The Contest Corner: No. 48 *John McLoughlin*  
 332 Problems: CC236–CC240  
 334 Solutions: CC186–CC190
- 339 The Olympiad Corner: No. 346 *Carmen Bruni*  
 339 Problems: OC296–OC300  
 341 Solutions: OC236–OC240
- 345 Where did the complex plane come from?  
*Tom Archibald and Brenda Davidson*
- 350 One Theorem, Six Proofs *V. Dubrovsky*  
 355 Problems: 4171–4180  
 358 Solutions: 4071–4080
- 369 Solvers and proposers index

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## Cru $x$ Mathematicorum

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## Cru $x$ Mathematicorum with Mathematical Mayhem

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## EDITORIAL

Poincare said: “Mathematics is the art of giving the same name to different things.” I think this is a true strength of our discipline. Every new representation of the same object gives us access to a whole new set of mathematical tools from the corresponding area. We can then use the newfound machinery to discover new, previously unknown or unattainable properties of seemingly familiar objects.

In fact, many recent advances in research rely on bridging various mathematical areas. This previously intriguing feature of mathematics has now become somewhat of a necessity. Other sciences have also started looking into utilizing tools from all areas of mathematics, even the more exotic ones. For example, the latest 2016 Nobel prize in physics was awarded for applying topological tools to study states of matter.

But even just being able to recognize an object in its multiple representations is hard, not to mention work with it using tools from different subjects. Here is a smaller example:



What do you see? An old woman or a young one?

While not every image hides different meanings, in math you should always look for new representations. In problem solving in particular, we often develop the flexibility in working with familiar objects (take a function, for example: do we think of it analytically, graphically, numerically?). My challenge to you is to do this regularly with objects you are not yet comfortable with. An obscure connection or a concealed significance might just win you a Nobel prize.

Kseniya Garaschuk

# THE CONTEST CORNER

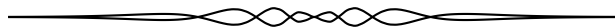
No. 48

John McLoughlin

*Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 1 mai 2017.*

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*



**CC236.** Une *loxodromie* est une courbe, sur la surface de la terre, qui suit une direction constante par rapport au nord (nord réel et non nord magnétique). Déterminer la longueur maximale d'une loxodromie dirigée en direction nord-est (à 45 degrés du nord), ou démontrer qu'elle peut être arbitrairement longue. Supposer que la terre est une sphère de circonférence 40 000 km.

**CC237.** Déterminer le volume de la portion d'un cube unitaire qui se trouve à une distance au moins  $\sqrt{2}$  d'un coin spécifié.

**CC238.** Pour le 8ième anniversaire de naissance de votre soeur, vous avez décidé de lui produire un gâteau en forme d'un octagone régulier. Puisque vous ne disposez pas d'un moule à gâteau de cette forme, vous avez décidé d'utiliser un moule rond de 8 pouces de diamètre, dont vous rognerez les côtés de façon à retenir le plus grand octagone régulier possible. La surface ainsi rognée afin de former ce gâteau octogonal est de la forme  $a\pi - b$  où  $a$  et  $b$  sont des nombres réels. Quel est  $\frac{b}{a}$ ?

**CC239.** Déterminer tous les entiers  $n > 1$  tels que  $4n + 9$  et  $9n + 4$  sont tous les deux des carrés parfaits.

**CC240.** Tit-Gus et Grand-Gus prennent leurs tours à choisir des entiers positifs comme coefficients du polynôme de degré seize

$$a_0 + a_1z + \cdots + a_{16}z^{16}.$$

Le même coefficient peut être utilisé plus d'une fois. Tit-Gus commence le jeu, et gagne si, à la fin, le polynôme a une racine multiple (réelle ou complexe) ou si le polynôme a deux racines distinctes  $\zeta_1$  et  $\zeta_2$ , réelles ou complexes, telles que

$|\zeta_1 - \zeta_2| \leq 1$ . Déterminer une stratégie gagnante pour Tit-Gus et démontrer qu'elle est bien gagnante.

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**CC236.** A *rhumb line* or *loxodrome* is a curve on the earth's surface that follows a constant direction relative to true (not magnetic) north. Find the maximum possible length of a rhumb line directed northeastward (a bearing of  $45^\circ$  true), or show that it can be arbitrarily long. You may assume the earth to be a sphere of circumference 40,000 km.

**CC237.** Find the volume of the portion of a unit cube that is at distance at most  $\sqrt{2}$  from a specified corner.

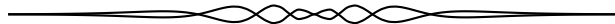
**CC238.** For your sister's 8th birthday, you decided to make her a cake in the shape of a regular octagon. Since you couldn't find a cake tin in this shape, you used an 8 inch diameter round cake tin and trimmed off the sides in such a way that you achieved the largest regular octagon possible. The area you cut off to form this octagonal cake is of the form  $a\pi - b$  where  $a$  and  $b$  are real numbers. What is  $\frac{b}{a}$ ?

**CC239.** Find all integers  $n > 1$  such that  $4n + 9$  and  $9n + 4$  are both perfect squares.

**CC240.** Big Sandy MacDonald and Little Sandy MacDonald take turns choosing positive integers to be the coefficients of a sixteenth degree polynomial

$$a_0 + a_1z + \cdots + a_{16}z^{16}.$$

The same coefficient may be used more than once. Little Sandy moves first, and wins the game if, at the end, the polynomial has a repeated root (real or complex), or two distinct real or complex roots  $\zeta_1, \zeta_2$  with  $|\zeta_1 - \zeta_2| \leq 1$ . Find a winning strategy for Little Sandy and show that it works.



# CONTEST CORNER SOLUTIONS

*Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(8), p. 326–328.*

**CC186.** Let  $n$  be a positive integer. Count the number of  $k \in \{0, 1, \dots, n\}$  for which  $\binom{n}{k}$  is odd. Prove that this number is a power of two, i.e. it is of the form  $2^p$  for some non-negative integer  $p$ .

*Originally problem 4 of the 1998 12th Nordic Mathematical Contest.*

*We received one submission. We present the solution by S. Muralidharan.*

We first prove the following:

**Lemma.** Let  $p$  be a prime and let  $m, n$  be natural numbers. Let  $n = lp + t$  and  $m = kp + s$ , where  $0 \leq t < p$  and  $0 \leq s < p$ . Then

$$\binom{n}{m} = \binom{l}{k} \binom{t}{s} \pmod{p} \quad (1)$$

**Proof.** Observe first that for  $0 < r < p$ ,  $\binom{p}{r}$  is divisible by  $p$ . Hence in

$$(1 + X)^p - (1 + X^p) = \binom{p}{1}X + \binom{p}{2}X^2 + \dots + \binom{p}{p-1}X^{p-1}$$

all the coefficients are divisible by  $p$ . Now, consider

$$\begin{aligned} P(X) &= (1 + X)^{lp+t} - (1 + X)^t(1 + X^p)^l \\ &= (1 + X)^t \{(1 + X)^{lp} - (1 + X^p)^l\} \\ &= (1 + X)^t \{(1 + X)^p - (1 + X^p)\} \{(1 + X)^{p(l-1)} + \dots + (1 + X^p)^{l-1}\} \end{aligned}$$

As observed before,  $P(X)$  is a multiple of  $p$ .

Consider the coefficient of  $X^{kp+s}$  in  $P(X)$ . This equals

$$\binom{lp+t}{kp+s} - \binom{t}{s} \binom{l}{k}$$

and hence all the coefficients are multiples of  $p$ . Note that this holds even when  $k = 0$  or  $l = 0$ . This completes the proof of the lemma.

Applying (1) repeatedly, it follows that if

$$\begin{aligned} n &= t_1 + t_2p + t_3p^2 + \dots + t_r p^{r-1} \\ m &= s_1 + s_2p + s_3p^2 + \dots + s_r p^{r-1} \end{aligned}$$

then

$$\binom{n}{m} = \binom{t_1}{s_1} \binom{t_2}{s_2} \cdots \binom{t_{r-1}}{s_{r-1}} \pmod{p}$$

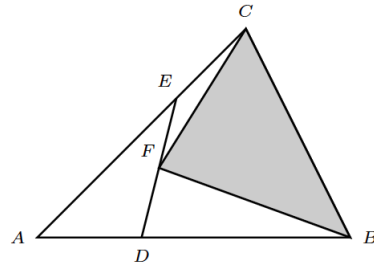
In the case  $p = 2$ , clearly,  $0 \leq t_i, s_i \leq 1$ . Thus  $\binom{n}{m}$  is odd if and only if  $\binom{t_i}{s_i} = 1$  for all  $i$ . Hence if  $t_i = 0$ , then  $s_i = 0$  and if  $t_i = 1$ ,  $s_i = 0, 1$ . Hence when  $t_i = 1$ , there are two possible values for  $s_i$  and when  $t_i = 0$ ,  $s_i$  also equals 0. Consequently, if the number of non zero coefficients in the binary representation of  $n$  is  $k$ , then the number of  $m$  such that  $\binom{n}{m}$  is odd is  $2^k$ .

The generalization of the problem is as follows: when  $p > 2$  is a prime and  $n = t_1 + t_2p + t_3p^2 + \cdots + t_rp^{r-1}$ , the number of  $m$  such that  $\binom{n}{m}$  is not divisible by  $p$  is

$$(t_1 + 1)(t_2 + 1) \cdots (t_r + 1).$$

This follows, since when  $t_i \neq 0$ , in the binary representation of  $m$ , the coefficient of  $p^{i-1}$  can take any of the values  $0, 1, \dots, t_i$  and when  $t_i = 0$ , the only possible coefficient of  $p^{i-1}$  in the binary representation of  $m$  is 0. Thus for each  $t_i$ , there are  $t_i + 1$  possibilities for the coefficient of  $p^{i-1}$  in the binary representation of  $m$ .

**CC187.** In the diagram the area of the triangle  $ABC$  is 1,  $\overline{AD} = \frac{1}{3}\overline{AB}$ ,  $\overline{EC} = \frac{1}{3}\overline{AC}$  and  $\overline{DF} = \overline{FE}$ . Find the area of the shaded triangle  $BFC$ .



*Originally problem 5 of the final round of the 2003 British Columbia Secondary School Mathematics Contest.*

*We received seven solutions. We present two of the solutions, one using barycentric coordinates and one using ratios.*

*Solution 1, by Andrea Fanchini.*

Use barycentric coordinates with respect to  $\triangle ABC$ . Points  $D$  and  $E$  have coordinates  $(\frac{2}{3}, \frac{1}{3}, 0)$  and  $(\frac{1}{3}, 0, \frac{2}{3})$  respectively, and so the midpoint  $F$  of  $DE$  has coordinates  $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ .

Therefore, the area of  $\triangle BFC$  is equal to

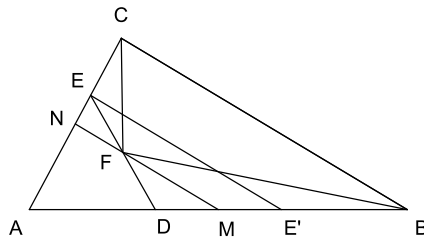
$$A_{\triangle BFC} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{vmatrix} \cdot A_{\triangle ABC} = \frac{1}{2} \cdot A_{\triangle ABC} = \frac{1}{2}.$$

*Solution 2, by Titu Zvonaru.*

Draw a line through  $E$  parallel to  $BC$ , and let it meet  $AB$  at  $E'$ . Then

$$E'B = \frac{1}{3}AB = AD.$$

Draw a line through  $F$  parallel to  $BC$  and denote by  $M$  and  $N$  the points where it intersects  $AB$  and respectively  $AC$ :



Then  $FM \parallel EE'$ , and since  $F$  is the midpoint of  $DE$  it follows that  $M$  is the midpoint of  $DE'$ . But  $AD = DE' = E'B$ , whence  $M$  is the midpoint of  $AB$ . Thus  $\overline{MN}$  is the line joining the midpoint  $M$  of  $AB$  to the midpoint  $N$  of  $BC$ , and since the vertex  $F$  is on the line  $MN$  the area of  $\triangle BFC$  is half of the area of  $\triangle ABC$  – that is,  $\frac{1}{2}$ .

**CC188.** A plane divides space into two regions. Two planes that intersect in a line divide space into four regions. Now suppose that twelve planes are given in space so that three conditions are met:

- every two of them intersect in a line,
- every three of them intersect in a point, and
- no four of them have a common point.

Into how many regions is space divided? Justify your answer.

*Originally problem 4 of Part II of the University of Maryland Mathematics Competition.*

*We received one incorrect solution. We present a solution by one of the editors.*

We solve the more general question: Into how many regions is space divided by  $n$  planes with the properties as given in the question.



Let us first consider the simpler problem in one dimension lower. That is, into how many regions is the plane divided by  $r$  lines, any two of which intersect and no three of which intersect in a common point. Clearly 0 lines divide the plane into one region. If we add a line we get one extra region, since we cut one region into two. So suppose we already have  $k - 1$  lines in the plane and we add the  $k$ -th one. How many regions does this line cut into two? If we move along the line, we enter a new region every time we pass an intersection with another line and there are  $k - 1$  intersections, so we cut  $k$  regions into two. Thus the number of regions we get from  $r$  lines is

$$1 + \sum_{k=1}^r k = 1 + \binom{r+1}{2}.$$

Let's return to our original problem. We approach it the same way. If we already have  $k - 1$  planes in space and we add the  $k$ -th, how many regions are cut into two? Note that every other plane traces a line onto the new plane so on that plane we see  $k - 1$  lines dividing this plane into regions (by property a)). Furthermore, by property b) any two of the lines intersect in a point and by property c), no three intersect in a common point. Now we observe that the regions on this plane correspond exactly to the regions in space that are cut into two by this plane. But we already know that there are  $1 + \binom{k}{2}$  such regions. So overall, the number of regions space is divided into by  $n$  planes is

$$1 + \sum_{k=1}^n \left(1 + \binom{k}{2}\right) = 1 + n + \binom{n+1}{3}.$$

For  $n = 12$ , we obtain 299 regions.

*Note that the formula above can be written as*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3},$$

*which is equal to the number of  $k$ -subsets of  $n$  with  $k \leq 3$ . The reader is invited to find a solution using a bijection between the sets and the regions in space.*

*Editor's Comments.* The result of the problem is well-known and appears in various sources. For example, it is presented by George Pólya in the film "Let Us Teach Guessing".

**CC189.** Coins are placed on some of the 100 squares in a  $10 \times 10$  grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal.)

*Originally problem 4 of the 2009 Special K Contest, University of Waterloo.*

*We received no submissions to this problem.*

**CC190.** An arrangement of the letters from the word TRIANGLE is shown.

E  
 E L E  
 E L G L E  
 E L G N G L E  
 E L G N A N G L E  
 E L G N A I A N G L E  
 E L G N A I R I A N G L E  
 E L G N A I R T R I A N G L E

Find the number of ways that the word TRIANGLE can be spelled out, using adjacent letters, going up or left or right, in this arrangement.

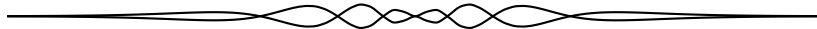
*Originally problem 1 of Part B of the final round of the 2003 British Columbia Secondary School Mathematics Contest.*

*We received three solutions of which two were correct and complete. We present the solution by S. Muralidharan.*

Consider the left half of the given picture:

E  
 E L  
 E L G  
 E L G N  
 E L G N A  
 E L G N A I  
 E L G N A I R  
 E L G N A I R T

Starting at the bottom  $T$ , at each stage we have two choices — either go to the left or go up. Hence there are  $2^7$  choices for spelling  $TRIANGLE$  from this half of the picture. Similarly, the other half has  $2^7$  choices. The central path (moving up at each stage starting at  $T$ ) is common to both the counting and hence the number of ways we can spell  $TRIANGLE$  is  $2^7 + 2^7 - 1 = 255$ .



# THE OLYMPIAD CORNER

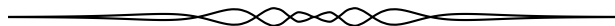
No. 346

Carmen Bruni

*Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 1 mai 2017.*

*La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.*



**OC296.** Soit  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Déterminer toutes les fonctions  $f : \mathbb{N} \rightarrow \mathbb{N}$  qui satisfont à  $(n-1)^2 < f(n)f(f(n)) < n^2 + n$  pour tout  $n$  dans  $\mathbb{N}$ .

**OC297.** Démontrer que

$$\frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(n+1)^2} < n \cdot \left(1 - \frac{1}{\sqrt[3]{2}}\right).$$

**OC298.** Soit  $ABC$  un triangle acutangle et  $H$  son orthocentre. Le cercle circonscrit au triangle  $ABH$  coupe la droite  $BC$  en  $D$ . Les droites  $DH$  et  $AC$  se coupent en  $P$ . Soit  $Q$  le centre du cercle circonscrit au triangle  $ADP$ . Démontrer que le centre du cercle circonscrit au triangle  $ABH$  est situé sur le cercle circonscrit au triangle  $BDQ$ .

**OC299.** Déterminer tous les entiers strictement positifs  $k$  pour lesquels

$$2^{(k-1)n+1}$$

n'est pas un diviseur de

$$\frac{(kn)!}{n!}$$

pour tout entier strictement positif  $n$ .

**OC300.** Les candidats d'un concours sont assis en  $n$  colonnes de manière à former une *bonne* configuration, c'est-à-dire une configuration dans laquelle on ne peut retrouver deux amis dans une même colonne. Dans une bonne configuration, il est impossible d'avoir tous les candidats assis en  $n-1$  colonnes. Démontrer qu'il est toujours possible de choisir des candidats  $M_1, M_2, \dots, M_n$  tels que  $M_i$  est assis

dans la  $i^{\text{ième}}$  colonne pour tout  $i = 1, 2, \dots, n$  et  $M_i$  est un ami de  $M_{i+1}$  pour tout  $i = 1, 2, \dots, n - 1$ .

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**OC296.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  be the set of positive integers. Determine all functions  $f$ , defined on  $\mathbb{N}$  and taking values in  $\mathbb{N}$ , such that  $(n-1)^2 < f(n)f(f(n)) < n^2 + n$  for every positive integer  $n$ .

**OC297.** Prove that

$$\frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(n+1)^2} < n \cdot \left(1 - \frac{1}{\sqrt[n]{2}}\right).$$

**OC298.** Triangle  $ABC$  is an acute triangle and its orthocenter is  $H$ . The circumcircle of  $\triangle ABH$  intersects line  $BC$  at  $D$ . Lines  $DH$  and  $AC$  meet at  $P$ , and the circumcenter of  $\triangle ADP$  is  $Q$ . Prove that the circumcenter of  $\triangle ABH$  lies on the circumcircle of  $\triangle BDQ$ .

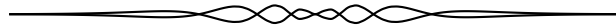
**OC299.** Find all positive integers  $k$  such that for any positive integer  $n$ ,

$$2^{(k-1)n+1}$$

does not divide

$$\frac{(kn)!}{n!}.$$

**OC300.** At a contest, all participants are sitting in  $n$  columns and are forming a “good” configuration. (We define one configuration as “good” when we do not have 2 friends sitting in the same column). It is impossible for all the students to sit in  $n - 1$  columns in a “good” configuration. Prove that we can always choose contestants  $M_1, M_2, \dots, M_n$  such that  $M_i$  is sitting in the  $i$ th column, for each  $i = 1, 2, \dots, n$  and  $M_i$  is a friend of  $M_{i+1}$  for each  $i = 1, 2, \dots, n - 1$ .



# OLYMPIAD SOLUTIONS

*Les énoncés des problèmes dans cette section paraissent initialement dans 2015 : 41(6), p. 240-241.*

**OC236.** Given a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x)^2 \leq f(y)$  for all  $x, y \in \mathbb{R}$ ,  $x > y$ , prove that  $f(x) \in [0, 1]$  for all  $x \in \mathbb{R}$ .

*Originally problem 2 from day 1 of the 2014 Grade 10 All Russian Math Olympiad.*

*We received 2 correct submissions. We present the solution by Michel Bataille.*

Let  $x \in \mathbb{R}$ . Pick some real  $a$  such that  $a > x$ . Then  $f(a)^2 \leq f(x)$  so that  $f(x) \geq 0$ . Thus  $f(x) \geq 0$  for all  $x \in \mathbb{R}$ .

Consider  $y < 0$  and, for each integer  $n \geq 2$ , choose  $y_1, y_2, \dots, y_{n-1}$  such that  $y > y_1 > y_2 > \dots > y_{n-1} > 2y$  (for example take  $y_k = y + \frac{ky}{n}$ ,  $k = 1, 2, \dots, n-1$ ). Then we have

$$f(y)^2 \leq f(y_1), f(y_1)^2 \leq f(y_2), \dots, f(y_{n-1})^2 \leq f(2y),$$

from which we successively deduce :

$$f(y)^4 \leq f(y_2), f(y)^8 \leq f(y_3), \dots, f(y)^{(2^n)} \leq f(2y).$$

Thus, the sequence  $(f(y)^{(2^n)})_{n \geq 2}$  is bounded above (by  $f(2y)$ ). It follows that  $f(y) \leq 1$  (otherwise  $\lim_{n \rightarrow \infty} (f(y))^{(2^n)} = \infty$ ). Thus  $f(y) \leq 1$  for all  $y < 0$ .

In addition, if  $x \geq 0$ , then  $x > b$  where  $b$  is a negative real number, therefore  $f(x)^2 \leq f(b) \leq 1$  and so  $f(x) \leq 1$  (since  $f(x) \geq 0$ ). We may conclude that  $f(x) \in [0, 1]$  for all  $x \in \mathbb{R}$ .

**OC237.** Let  $n$  be a positive integer, and let  $S$  be the set of all integers in  $\{1, 2, \dots, n\}$  which are relatively prime to  $n$ . Set  $S_1 = S \cap (0, \frac{n}{3}]$ ,  $S_2 = S \cap (\frac{n}{3}, \frac{2n}{3}]$ ,  $S_3 = S \cap (\frac{2n}{3}, n]$ . If the cardinality of  $S$  is a multiple of 3, prove that  $S_1, S_2, S_3$  have the same cardinality.

*Originally problem 8 from day 2 of the 2014 China Girls Math Olympiad.*

*No submitted solutions.*

**OC238.** Let  $a, b, c$  be positive reals such that  $a + b + c = 3$ . Prove :

$$\frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ca}} + \frac{c^2}{c + \sqrt[3]{ab}} \geq \frac{3}{2}$$

and determine when equality holds.

Originally problem 5 from day 2 of the 2014 Mexico National Olympiad.

We received 8 correct submissions. We present the solution by Ali Adnan.

From the AM-GM Inequality  $\sqrt[3]{bc} \leq \frac{b+c+1}{3}$  and from two other such inequalities, we get

$$\begin{aligned} \frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ca}} + \frac{c^2}{c + \sqrt[3]{ab}} &\geq \frac{a^2}{a + \frac{b+c+1}{3}} + \frac{b^2}{b + \frac{c+a+1}{3}} + \frac{c^2}{c + \frac{a+b+1}{3}} \\ &\geq \frac{(a+b+c)^2}{a+b+c + \frac{2(a+b+c)+3}{3}} = \frac{3}{2}, \end{aligned}$$

where the last inequality is the Cauchy-Schwarz Inequality. Equality holds iff  $a = b = c = 1$ .

*Editor's Note.* In this form, the Cauchy-Schwarz Inequality is also known as Titu's Lemma or alternatively Engel's form of the Cauchy-Schwarz Inequality.

**OC239.** In a school, there are  $n$  students and some of them are friends of each other. (Friendship is mutual.) Define  $a, b$  to be the minimum values which satisfy the following conditions :

1. We can divide students into  $a$  teams such that two students in the same team are always friends.
2. We can divide students into  $b$  teams such that two students in the same team are never friends.

Find the maximum value of  $N = a + b$  in terms of  $n$ .

Originally problem 3 of the 2014 Japan Mathematical Olympiad Finals.

We present the solution by Oliver Geupel. There were no other submissions.

The problem seems to ask for the maximum value of the sum of the chromatic numbers of two complementary graphs of order  $n$ . Two graphs  $A$  and  $B$  with the same set of vertices are called *complementary* if, for every two distinct vertices  $v$  and  $w$ , exactly one of the graphs  $A$  or  $B$  has an edge  $\{v, w\}$ . The chromatic number of a graph is the smallest number of colors needed to color its vertices so that no two adjacent vertices share the same color. We will show that the desired maximum value is  $n + 1$ . This is a well-known result (see E.A. Nordhaus, J.W. Gaddum, *On complementary graphs*, American Mathematical Monthly 63 (1956), 175-177). We reproduce the proof from the reference.

Denote the following assertion by  $P(n)$  : If  $A$  and  $B$  are two complementary graphs of order  $n$  with chromatic numbers  $a$  and  $b$ , then  $a + b \leq n + 1$ . We prove  $P(n)$  for all  $n \geq 1$  by mathematical induction.

The base case  $P(1)$  is clearly true. Let us assume  $P(n - 1)$  for some  $n \geq 2$ . We are going to prove  $P(n)$ . Withdraw one of the  $n$  vertices, say vertex  $v$ , and consider

the restrictions of graphs  $A$  and  $B$  to the remaining  $n - 1$  vertices. The sum of their chromatic numbers, say  $a'$  and  $b'$ , is not greater than  $n$ . Also  $a \leq a' + 1$  and  $b \leq b' + 1$ . As a consequence, we will have  $a + b > n + 1$  only if  $a' + b' = n$  and  $a = a' + 1$  and  $b = b' + 1$ . But if  $a = a' + 1$  and  $b = b' + 1$  then the order of vertex  $v$  in graph  $A$  is at least  $a'$  and the order of  $v$  in  $B$  is at least  $b'$ , so that  $a' + b' \leq n - 1$ . Hence  $a + b \leq n + 1$ , which proves  $P(n)$ .

The following example shows that the bound is sharp : If  $A$  is the complete graph of order  $n$ , then  $a = n$  and  $b = 1$ .

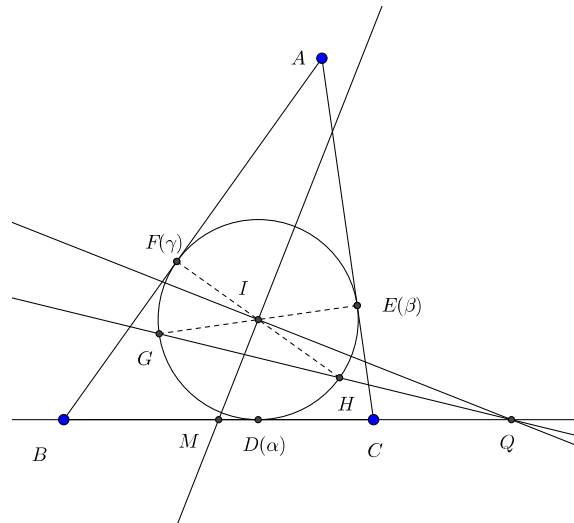
**OC240.** Let  $ABC$  be a triangle with incenter  $I$ , and suppose the incircle is tangent to  $CA$  and  $AB$  at  $E$  and  $F$ . Denote by  $G$  and  $H$  the reflections of  $E$  and  $F$  over  $I$ . Let  $Q$  be the intersection of  $BC$  with  $GH$ , and let  $M$  be the midpoint of  $BC$ . Prove that  $IQ$  and  $IM$  are perpendicular.

*Originally problem 3 from day 1 of the 2014 Taiwan Team Selection Test Round 1 Mock IMO.*

*We received 8 correct submissions. We present the solution by Somasundaram Muralidharan.*

Without loss of generality, we can assume that the incenter  $I$  is at the origin and the in-radius is 1. We write  $\bar{z}$  to denote the complex conjugate of the complex number  $z$ .

Let the points of tangency  $D, E, F$  of the incircle with the sides  $BC, CA, AB$  respectively be represented by the complex numbers  $\alpha, \beta, \gamma$ . The equation of  $BC$



is  $z + \alpha^2 \bar{z} = 2\alpha$ . Since  $G$  is the reflection of  $E$  in  $I$ , the origin, it follows that  $G$  is represented by the complex number  $-\beta$  and similarly  $H$  is represented by

the complex number  $-\gamma$ . Thus the equation of  $GH$  is  $z + \beta = -\beta\gamma(\bar{z} + \bar{\beta})$ . Let  $q$  denote the complex number representing  $Q$ . Solving the equations of  $BC$  and  $GH$  we obtain

$$\bar{q} = \frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta\gamma}$$

It is easy to see that  $B$  is represented by the complex number  $\frac{2\alpha\gamma}{\alpha + \gamma}$  and  $C$  by the complex number  $\frac{2\alpha\beta}{\alpha + \beta}$ . Thus the complex number  $m$  representing the mid point  $M$  of  $BC$  is given by

$$m = \frac{\alpha\gamma}{\alpha + \gamma} + \frac{\alpha\beta}{\alpha + \beta}$$

To show that  $IM$  and  $IQ$  are perpendicular, it is enough to show that  $I\bar{M}$  and  $I\bar{Q}$  are perpendicular where  $\bar{M}$  and  $\bar{Q}$  are the reflections of  $M$  and  $Q$  in the real axis. Note that the complex numbers representing  $\bar{M}$  and  $\bar{Q}$  are respectively  $\bar{m}$  and  $\bar{q}$ . The conjugate  $\bar{m}$  is given by

$$\bar{m} = \frac{\bar{\alpha}\bar{\gamma}}{\bar{\alpha} + \bar{\gamma}} + \frac{\bar{\alpha}\bar{\beta}}{\bar{\alpha} + \bar{\beta}} = \frac{\frac{1}{\alpha\gamma}}{\frac{1}{\alpha} + \frac{1}{\gamma}} + \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha} + \frac{1}{\beta}} = \frac{1}{\alpha + \gamma} + \frac{1}{\alpha + \beta}$$

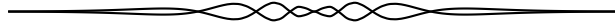
The slope of  $I\bar{M}$  is given by

$$u = \frac{\bar{m}}{m} = \frac{\frac{1}{\alpha + \gamma} + \frac{1}{\alpha + \beta}}{\frac{\alpha\gamma}{\alpha + \gamma} + \frac{\alpha\beta}{\alpha + \beta}} = \frac{2\alpha + \beta + \gamma}{\alpha(\alpha\beta + 2\beta\gamma + \gamma\alpha)}$$

Similarly the slope of  $I\bar{Q}$  is given by

$$v = \frac{\bar{q}}{q} = \frac{\frac{2\alpha + \beta + \gamma}{\alpha^2 - \beta\gamma}}{\frac{2\bar{\alpha} + \bar{\beta} + \bar{\gamma}}{\bar{\alpha}^2 - \bar{\beta}\bar{\gamma}}} = -\frac{2\alpha + \beta + \gamma}{\alpha(\alpha\beta + 2\beta\gamma + \gamma\alpha)}$$

Since  $u + v = 0$  it follows that  $I\bar{M}$  and  $I\bar{Q}$  are perpendicular.





# Where did the complex plane come from ?

Tom Archibald and Brenda Davison

Around 800 AD (184 AH) the first text on algebra was written by a man whose name tells us that he came from the area south of the Sea of Azov, currently Uzbekistan. Al-Khwarizmi's name eventually turned into the word "algorithm" and he appears to have invented the term algebra to describe the art of finding an unknown quantity by a key trick : you assume that you know what it is – we call it  $x$  or  $y$ , though he used a word that translates as thing – and then, so to speak, work backwards from what you do know about the unknown thing to determine, if possible, what its value is.

Al-Khwarizmi's text – which arrived in Europe about 400 years later – shows how to solve linear and quadratic equations. There were many differences between how he did it and how we do it now, but one important one was that in the old days there were no negative numbers allowed. As other people came to work with these techniques over the years, the number system gradually became extended to allow negative results. For a long time these were treated as a bit suspect, so that the solutions to  $x^2 + x - 6 = 0$  would be described as a "true" root,  $x = 2$ , and a "false" root,  $x = -3$ .

By the 1500s in Europe many experts had mastered these techniques, described in printed books and extended, for example including the solution of cubic equations. Cardano's *Ars magna* (Great Art), published in 1545, described how to solve quadratics, cubics, and quartics (degree 4 polynomial equations). In doing so, it emphasized that negative numbers were not the only inconvenient quantity that arises when algebra is done. We know already from the quadratic formula that the square roots of negative numbers come up routinely. Such numbers were worse than "false" : they were often called "impossible".

Negative numbers are relatively easy to reconcile with common sense. Often, for example, they are treated in monetary terms, as a debt (and written in red ink in accounting ledgers). They remain hard to interpret geometrically, though : what is meant by a negative area? Conventionally, it can be one you subtract from something bigger, and mathematical writers became used to such ideas, which go along with our ideas of the number line with a 0 in the middle (whatever the middle of an infinite line means).

The square roots of negative numbers don't easily make sense, though. Whether arithmetically or geometrically or as ratios they are hard to interpret until we get them into the right context. But even in the 1500s, Cardano and others saw that such quantities can indeed be used to calculate results that do make sense.

For example, Cardano's cubic formula for the case  $x^3 = cx + d$  has the form

$$x = \sqrt[3]{d/2 + \sqrt{(d/2)^2 - (c/3)^3}} + \sqrt[3]{d/2 - \sqrt{(d/2)^2 - (c/3)^3}}.$$

We won't derive the formula here, but as an exercise you can check that it will work. Note in passing that cubics always have at least one real root.

If you apply this to the equation  $x^3 = 15x + 4$ , you get an "impossible" number, one we would call complex :

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Now, it is easy to see that  $x = 4$  satisfies the given equation. Could it be that the complex number we found by the formula is really just a complicated way to write the number 4? A mathematician called Bombelli thought so, and gave an argument to support his idea. Writing  $a + \sqrt{-b} = \sqrt[3]{2 + \sqrt{-121}}$ , he then has  $a - \sqrt{-b} = \sqrt[3]{2 - \sqrt{-121}}$ , so that  $x = 2a$ . But it's easy to show that  $a^2 + b = 5$ ,  $a^3 - 3ab = 2$ , and so for  $a$  we need a number with square less than 5 and cube greater than 2. Now,  $a = 2$  is such a number, and the value  $b = 1$  satisfies the equations. Putting all this together,  $x = 4$ . This made it seem that there was some value of thinking of these numbers in the form  $a + \sqrt{b}$ .

Because of this idea that one could do good things mathematically with imaginary numbers, researchers kept trying to come up with interpretations that would make some kind of sense to the average educated person. Being able to see, or picture, what is going on in a mathematical situation often allows you to determine in which direction to move to proceed methodically towards a rigorous solution.

Into our picture of how to interpret imaginary numbers comes a French bookstore owner named Jean-Robert Argand (1768-1822). To see how powerful a good interpretation can be, consider the following :

Take a positive integer, for example 2. Now multiply this number by  $-1$ . The result, of course, is  $-2$ . But let's think carefully about this. The magnitude of the number hasn't changed – it is still 2 units from the origin but upon drawing a line from the origin to the number, we observe that we are pointing in the opposite direction. So, we could say that multiplying by  $-1$  flipped the number to the opposite direction or we could say that multiplying by  $-1$  rotated the line joining the origin to 2 by 180 degrees. This second idea is powerful.

Working in the early 1800s, Argand wanted to graph complex numbers, and he did so in a two-dimensional plane with the real part of the number on the  $x$ -axis and the imaginary part on the  $y$ -axis (this is now called the Argand plane or the complex plane) and he interpreted multiplying by  $i$  as a rotation of ninety degrees, where the imaginary unit  $i$  is the square root of  $-1$ . Multiplying by  $-1$  rotated by 180 degrees, so this was consistent with the algebraic situation. [Ed. Warning : the term "complex plane" is common, but one mathematician's plane is another mathematician's line.]

Without further knowledge, ask yourself : what is the square root of  $i$ ? This seems very difficult but, using the geometrical interpretation of Argand, we should conclude that can be found from  $+1$  by a rotation of 45 degrees. Take the unit vector  $(1,0)$  and rotate it 45 degrees – where are you in terms of the  $x$  and  $y$

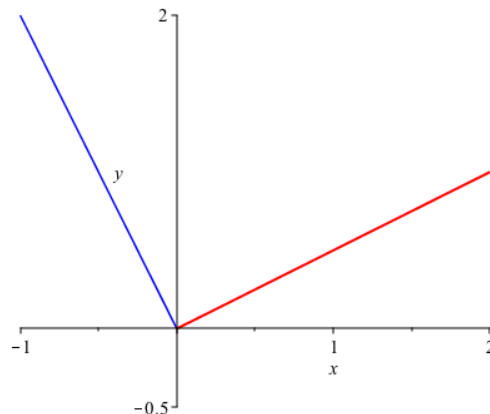


FIGURE 1: A rotation from  $2+i$  (red) to  $-1+2i$  (blue) through multiplication by  $i$ .

coordinates? Think trigonometry and the unit circle. You are at  $\sin(45^\circ)$  on the  $y$ -axis and  $\cos(45^\circ)$  on the  $x$ -axis. This would mean that :

$$\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

No algebra at all! Just Argand's geometrical insight.

But we can use algebra to check this. Before you read on, try it yourself, remembering to use the idea that  $i^2 = -1$  to simplify when possible.

Tried it? We have

$$\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + 2i \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{i^2}{2} = i.$$

In fact, Argand was not the first person to come up with the idea of representing a complex number as a point in a two-dimensional plane. He was preceded by a Norwegian land surveyor, Caspar Wessel. But Mr. Wessel wrote in Danish in a journal not much read and, though his work was published 9 years before Argand's, it went unnoticed.

This geometrical interpretation of complex numbers was independently discovered for a third time by a titan of mathematics – a figure who towers above both Wessel and Argand in mathematical creativity. In 1831, Carl Friedrich Gauss (1777-1855) came up with the idea again. In Germany, in fact, the complex plane, where the point  $(a, b)$  is associated with the number  $a + ib$ , is called the Gaussian plane.

Now the idea of taking roots in the complex plane is tied to whether or not you can construct a regular polygon with a given number of sides using simply a straight edge and compass. Try, for example, to inscribe an equilateral triangle or a hexagon inside the unit circle using only a compass and ruler (but the ruler can only be used to draw straight lines – it cannot be used to measure anything).

It turns out that you cannot always do this – it will only work for certain regular polygons. It is not easy to figure out which ones, but that is exactly what Gauss did at age 19(!) (The proof was actually completed later by Wantzel.)

Gauss and Wantzel showed it is possible to construct the regular polygon using only compass and straightedge exactly when the number of sides is (a) a power of 2 (greater than 1) or (b) the product of a power of 2 (possibly 0) and distinct Fermat primes. These are prime numbers of the form  $2^{2^n} + 1$ . Of numbers having this form, these 5 are known to be prime : 3, 5, 17, 257, 65537.

While it was then known that it is possible to do so, it doesn't mean that anyone knew how to use the compass and straightedge to do so for any but the smallest number of sides. Gauss, in fact, figured out how to do this for the 17-gon. He was so proud of this that he requested that the 17-sided regular polygon be inscribed on his tombstone. The stone mason declined. Too hard to do? Or would it just look like a circle given the precision of working in stone? Both the others have since been constructed : in 1894 J. G. Hermes completed the 65537-gon construction in a 200-page manuscript.

Let's conclude by looking at the relationship between the regular pentagon and a plot in the complex plane. This turns out to be related to solving the (complex) equation  $z^5 = 1$ . Here we will use the polar form of the complex number :

$$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

where  $r$  is the *modulus* of the complex number (the length of the vector joining the point to the origin) and  $\theta$  is the angle between the positive real axis (the  $x$ -axis) and the same vector. It is easy to see that  $r^2 = x^2 + y^2$ , and you can see also that  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ . The relationship between the trigonometric functions and the exponential function can be taken as a definition, though there is a good reason for it.

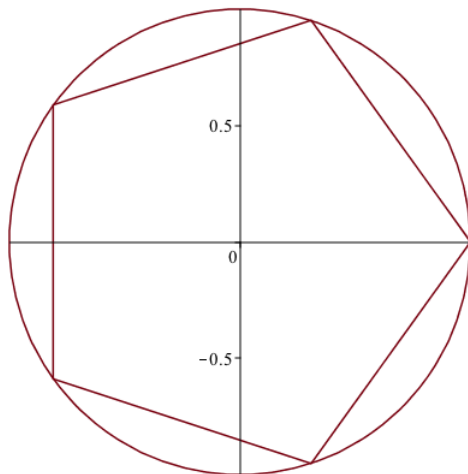


FIGURE 2: Fifth roots of unity and the regular pentagon

Then we have

$$z^5 = r^5 e^{5i\theta} = r^5 (\cos(5\theta) + i \sin(5\theta)) = 1 + 0i$$

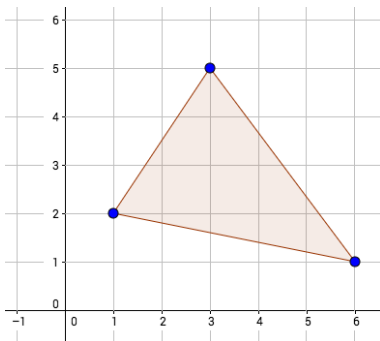
meaning that  $r = 1, \cos(5\theta) = 1$ , and  $\sin(5\theta) = 0$ , since we equate real and imaginary parts to solve. This means that  $5\theta = 0, 2\pi, 4\pi, \dots$  so that we get a complete set of points on the graph if we have  $\theta = 2k\pi/5$ , where  $k = 0, 1, 2, 3, 4$ . If you plot these points you see that we have the vertices of a regular pentagon consisting of points on the unit circle at intervals of  $72^\circ$ . Gauss' method was called *cyclotomy*, which means "circle-cutting", so this shows us where that term comes from and what it has to do with the Gaussian plane.

**Exercise 1**

Use geometric reasoning similar to how we found  $\sqrt{i}$  to find a  $4^{th}$  root of  $i$ .

**Exercise 2**

Consider the 2-dimensional domain shown in the graph below :



Find the image of this domain under the function  $f(z) = 2iz$ . (Suggestion : Draw another copy of the complex plane to plot the image).

**Exercise 3**

Using the fact that  $\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ , find the coefficients of the  $2 \times 2$  matrix that will rotate any given vector by 45 degrees.

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# One Theorem, Six Proofs

V. Dubrovsky

It is often more useful to acquaint yourself with many proofs of the same theorem rather than with similar proofs of numerous results. The theorem about the medians of a triangle is a result that has several insightful proofs.

**Theorem.** The medians  $AA_1, BB_1$  and  $CC_1$  of a triangle  $ABC$  intersect in a single point  $M$ . Furthermore, point  $M$  divides each median into segments with a  $2 : 1$  ratio from the vertex, that is,

$$\frac{AM}{MA_1} = \frac{BM}{MB_1} = \frac{CM}{MC_1} = 2. \quad (1)$$

For the first five proofs we define  $M$  to be the point that divides the median  $AA_1$  in the ratio  $2 : 1$  and prove that the median  $BB_1$  goes through  $M$ . The theorem then follows by applying the same argument to the median  $CC_1$ .

**Proof 1.** Let  $K$  be the midpoint of  $AM$  and let  $B'$  be the intersection point of the line  $BM$  and the side  $AC$ . It suffices to show that  $AB' = B'C$ . Draw the line segments  $KL$  and  $A_1N$  parallel to  $BB'$  (Figure 1).

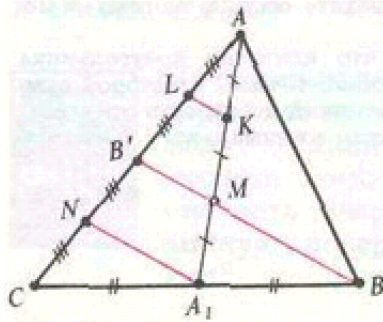


FIGURE 1: Construction for Proof 1.

Since  $AK = KM = MA_1$  and  $CA_1 = A_1B$ , by the intercept theorem (sometimes attributed to Thales) we get  $AL = LB' = B'N = NC$ . Therefore,  $AB' = B'C$ .  $\square$

**Proof 2.** Consider a homothety with center at  $M$  and ratio of  $-1/2$ . Under this homothety, the point  $A$  transforms into the point  $A_1$ . Suppose the point  $B$  transforms into the point  $B'$  (Figure 2). Then  $\overrightarrow{A_1B'} = -\frac{1}{2}\overrightarrow{AB}$ .

On the other hand, the side  $BA$  transforms into the segment  $A_1B_1$  in the homothety with center  $C$  and coefficient of  $1/2$  so  $\overrightarrow{A_1B_1} = \frac{1}{2}\overrightarrow{BA} = -\frac{1}{2}\overrightarrow{AB}$ . Therefore,  $\overrightarrow{A_1B'} = \overrightarrow{A_1B_1}$  and hence  $B' = B_1$ . So triangles  $ABC$  and  $A_1B_1C_1$  are homothetic.

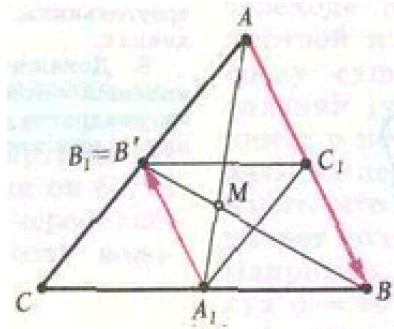


FIGURE 2: Construction for Proof 2.

tic under our homothety. Since a homothety preserves lines, the points  $B, M$  and  $B' = B_1$  are collinear.  $\square$

**Proof 3.** Consider triangles  $MAC$  and  $MA_1C$  (Figure 3). Their altitudes from vertex  $C$  coincide, while  $AM = 2A_1M$ . Hence,  $\text{Area}(AMC) = 2 \cdot \text{Area}(A_1MC)$ . Similarly,  $\text{Area}(AMB) = 2 \cdot \text{Area}(A_1MB)$ . But,  $\text{Area}(A_1MC) = \text{Area}(A_1MB)$ . Hence,  $\text{Area}(AMC) = \text{Area}(AMB) = \text{Area}(BMC)$ . Therefore, triangles  $MAB, MBC$  and  $MCA$  have equal areas. Let  $B'$  be the intersection point of lines  $BM$  and  $AC$ . We will show that  $AB' = B'C$ .

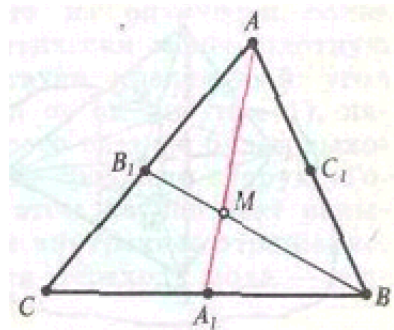


FIGURE 3: Construction for Proof 3.

We have

$$\frac{AB'}{B'C} = \frac{\text{Area}(AB'M)}{\text{Area}(CB'M)} \quad \text{and} \quad \frac{AB'}{B'C} = \frac{\text{Area}(AB'B)}{\text{Area}(CB'B)}$$

Algebraic manipulations yield

$$\frac{AB'}{B'C} = \frac{\text{Area}(AB'B) - \text{Area}(AB'M)}{\text{Area}(CB'B) - \text{Area}(CB'M)} = \frac{\text{Area}(ABM)}{\text{Area}(CBM)} = 1.$$

Therefore,  $AB' = B'C$ .  $\square$

**Proof 4.** From Figure 4, we have :

$$\begin{aligned}
 \overrightarrow{BM} &= \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AM} \\
 &= \overrightarrow{BC} + \overrightarrow{CA} + \frac{2}{3}\overrightarrow{AA_1} \\
 &= \overrightarrow{BC} + \overrightarrow{CA} + \frac{2}{3}(\overrightarrow{AC} + \overrightarrow{CA_1}) \\
 &= \overrightarrow{BC} + \overrightarrow{CA} + \frac{2}{3}\left(\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB}\right) \\
 &= \frac{2}{3}\left(\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA}\right) \\
 &= \frac{2}{3}(\overrightarrow{BC} + \overrightarrow{CB_1}) \\
 &= \frac{2}{3}\overrightarrow{BB_1}.
 \end{aligned}$$

Therefore, the point  $M$  lies on the median  $BB_1$ . □

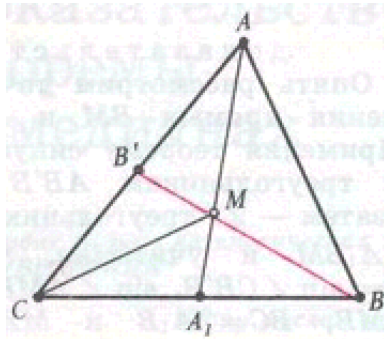


FIGURE 4: Construction for Proof 4.

**Proof 5.** Consider again the intersection point  $B'$  of the lines  $BM$  and  $AC$  (see Figure 3). Apply the Law of Sines to triangles  $AB'B$  and  $CB'B$  and then to triangles  $ABM$  and  $AB_1M$ . Since

$$\sin \angle AB'B = \sin \angle CB'B \quad \text{and} \quad BC = 2A_1B,$$

while

$$\sin \angle AMB = \sin \angle A_1MB \quad \text{and} \quad MA = 2MA_1,$$

we get

$$\begin{aligned}
 \frac{AB'}{B'C} &= \frac{AB \sin \angle ABM}{\sin \angle AB'B} : \frac{BC \sin \angle A_1BM}{\sin \angle CB'B} \\
 &= AB \sin \angle ABM : 2A_1B \sin \angle A_1BM = \frac{MA \sin \angle AMB}{2MA_1 \sin \angle A_1MB} = 1.
 \end{aligned}$$

Therefore,  $AB' = B'C$ . □



**Proof 6.** Let  $\alpha$  be a plane through the points  $A$  and  $B$ , which does not contain the point  $C$ . In this plane, construct the equilateral triangle  $ABC'$  (Figure 5).

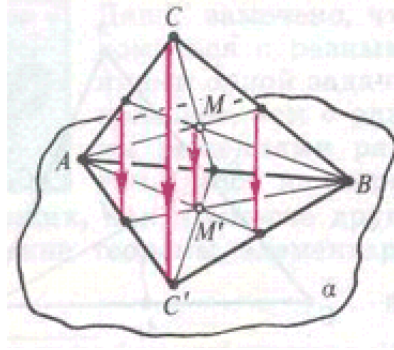


FIGURE 5: Construction for Proof 6.

The parallel projection along the line  $CC'$  that takes the original plane to  $\alpha$  transforms triangle  $ABC$  and its medians to triangle  $ABC'$  and its medians. However, in an equilateral triangle, the medians are also angle bisectors and therefore they intersect in one point. It is also easy to show that (1) holds for triangle  $ABC'$ . Therefore, the theorem also holds for triangle  $ABC$ .  $\square$

Let us mention another, possibly most natural, proof of this theorem. Place equal masses onto the vertices of the triangle. By grouping them in pairs, we can see that the center of mass lies on each of the medians. This is why the point of intersection of the medians is called the centroid of the triangle.

We conclude with some exercises.

### Exercises.

1. Given a trapezoid, prove that the point of intersection of its diagonals, the point of intersection of the extensions of its nonparallel sides and the midpoints of its bases are all collinear. Use this result to prove the theorem about the medians.
2. Given a triangle  $ABC$ , find all points  $P$  such that  $\text{Area}(PAB) = \text{Area}(PBC) = \text{Area}(PCA)$ .
3. In a given pentagon, each vertex is connected to the midpoint of the opposite side. Prove that if four of the resulting lines intersect in one point, then the fifth line also goes through this point.
4. Consider a tetrahedron  $ABCD$ . Construct three planes, one through each edge from  $A$  and the bisector of  $\angle A$  on the opposite face. Prove that these three planes intersect in one line.
5. Suppose the points  $A_1$ ,  $B_1$  and  $C_1$  lie, respectively, on the sides  $BC$ ,  $CA$  and  $AB$  of triangle  $ABC$ . Suppose further that segments  $AA_1$ ,  $BB_1$  and  $CC_1$  intersect

at a point  $P$  such that

$$\frac{AP}{PA_1} = \frac{BP}{PB_1} = \frac{CP}{PC_1}.$$

Prove that  $P$  is the centroid of  $ABC$ .

**6.** Consider seven lines in a tetrahedron : the four lines that join a vertex to the centroid of the opposite face and the three lines that connect the midpoints of the opposite edges. Prove that these seven lines all intersect at one point. With what ratio does this point of intersection divide the given line segments ?

**7.** Prove that the point  $M$  in the article

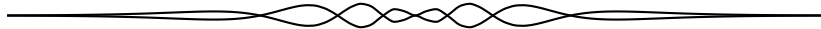
- a) is the unique point in a triangle such that  $\vec{MA} + \vec{MB} + \vec{MC} = \vec{0}$  ;
- b) has coordinates (in any fixed coordinate system) that are the arithmetic means of the coordinates of the vertices of the triangle.

Use these results to prove our theorem about the medians.

**8.** Prove that the centroid of a triangle is the unique point that minimizes the sum of squares of distances to the vertices.

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*This article appeared in Russian in Kvant, 1990(1), p. 54–56. It has been translated and adapted with permission.*



# PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2017**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

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**4171.** *Proposé par J. Chris Fisher.*

Dans un triangle  $ABC$ , soient  $L$  et  $M$  les pieds des perpendiculaires émanant de  $A$  vers les bissectrices internes et externes de l'angle à  $B$  et soient  $L'$  et  $M'$  les pieds des perpendiculaires émanant de l'orthocentre vers ces mêmes lignes. Démontrer que les lignes  $LM$  et  $L'M'$  intersectent à un point sur  $AC$ .

**4172.** *Proposé par Nathan Soedjak.*

Soit  $S$  l'ensemble de tous les  $n$ -tuples  $a_1, \dots, a_n \geq 0$  tels que  $\sum_{i=1}^n a_i i = n$ . Démontrer que

$$\sum_{(a_1, \dots, a_n) \in S} \frac{1}{1^{a_1} 2^{a_2} \dots n^{a_n} (a_1)! (a_2)! \dots (a_n)!} = 1.$$

**4173.** *Proposé par Dao Thanh Oai, Leonard Giugiuc et Daniel Sitaru.*

Soit  $ABC$  un triangle équilatéral avec centroïde  $O$ , où  $R$  dénote le rayon du cercle circonscrit. Soit  $P$  un point arbitraire à l'intérieur de  $ABC$  et soient  $P_a$ ,  $P_b$  et  $P_c$  les pieds des perpendiculaires émanant de  $P$  vers les côtés  $BC$ ,  $CA$  et  $AB$  respectivement. Enfin,  $OA = R$  et  $OP = d$ . Démontrer que

$$\frac{2}{PP_a + r} + \frac{2}{PP_b + r} + \frac{2}{PP_c + r} \leq \frac{3}{R - d} + \frac{3}{R + d} \leq \frac{1}{PP_a} + \frac{1}{PP_b} + \frac{1}{PP_c}.$$

**4174.** *Proposé par Mihaela Berindeanu.*

Soient  $a$ ,  $b$  et  $c$  des nombres réels positifs tels que  $a + b + c = 3$ . Démontrer que

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a^2 + b^2 + c^2.$$

**4175.** *Proposé par George Apostolopoulos.*

Déterminer toutes les paires d'entiers positifs  $x$  et  $y$  satisfaisant à l'équation

$$4x^2 + 3y^2 - 7xy - 6x + 5y = 0.$$

**4176.** *Proposé par Michel Bataille.*

Pour tout entier  $n \geq 2$ , soit  $\mathcal{P}_n$  l'ensemble des polynômes de degré  $n$  à coefficients rationnels, ayant une racine (pas nécessairement réelle) de multiplicité plus grande que 1. Pour quels  $n$  est-il vrai que tout élément de  $\mathcal{P}_n$  a une racine rationnelle ?

**4177.** *Proposé par Daniel Sitaru.*

Si  $a, b, c$  et  $d$  sont des nombres réels non négatifs tels que  $(a^2 + b^2)(c^2 + d^2) \neq 0$ , démontrer que

$$4\left((ac + bd)^6 + (ad - bc)^6\right) \geq (a^2 + b^2)^3(c^2 + d^2)^3.$$

**4178.** *Proposé par Leonard Giugiuc, Dan Marinescu et Marian Cucoanes.*

Pour  $0 \leq a < \frac{\pi}{2}$ , démontrer que  $\lim_{n \rightarrow \infty} \int_a^{\pi-a} x^n \cos x dx = -\infty$ .

**4179.** *Proposé par D. M. Bătinețu-Giurgiu et Neculai Stanciu.*

Considérer la suite  $(a_n)$  for  $n \geq 1$  définie par  $a_n = \prod_{k=1}^n (k!)^2$ . Calculer  $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n^2]{a_n}}$ .

**4180.** *Proposé par Leonard Giugiuc.*

Soit  $ABC$  un triangle tel que  $\angle BAC \geq \frac{2\pi}{3}$ . Démontrer que  $\frac{r}{R} \leq \frac{2\sqrt{3}-3}{2}$ .

.....

**4171.** *Proposed by J. Chris Fisher.*

In triangle  $ABC$ , if  $L$  and  $M$  are the feet of the perpendiculars dropped from vertex  $A$  to the internal and external bisectors of the angle at  $B$ , while  $L'$  and  $M'$  are the feet of the perpendiculars from the orthocenter to those lines, prove that the lines  $LM$  and  $L'M'$  intersect at a point of  $AC$ .

**4172.** *Proposed by Nathan Soedjak.*

Let  $S$  be the set of all  $n$ -tuples  $a_1, \dots, a_n \geq 0$  such that  $\sum_{i=1}^n a_i i = n$ . Prove that

$$\sum_{(a_1, \dots, a_n) \in S} \frac{1}{1^{a_1} 2^{a_2} \dots n^{a_n} (a_1)! (a_2)! \dots (a_n)!} = 1.$$

**4173.** *Proposed by Dao Thanh Oai, Leonard Giugiuc and Daniel Sitaru.*

Let  $ABC$  be an equilateral triangle with circumradius  $R$  and centroid  $O$ . Let  $P$  be an arbitrary point inside  $ABC$  and let  $P_a, P_b$  and  $P_c$  be the feet of the perpendiculars dropped from  $P$  onto the sides  $BC, CA$  and  $AB$ , respectively. Finally, let

$OA = R$  and  $OP = d$ . Prove that

$$\frac{2}{PP_a + r} + \frac{2}{PP_b + r} + \frac{2}{PP_c + r} \leq \frac{3}{R - d} + \frac{3}{R + d} \leq \frac{1}{PP_a} + \frac{1}{PP_b} + \frac{1}{PP_c}.$$

**4174.** *Proposed by Mihaela Berindeanu.*

Let  $a, b, c$  be positive real numbers such that  $a + b + c = 3$ . Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a^2 + b^2 + c^2.$$

**4175.** *Proposed by George Apostolopoulos.*

Find all pairs of positive integers  $x, y$  satisfying the equation

$$4x^2 + 3y^2 - 7xy - 6x + 5y = 0.$$

**4176.** *Proposed by Michel Bataille.*

For each integer  $n$  with  $n \geq 2$ , let  $\mathcal{P}_n$  be the set of all polynomials of degree  $n$ , with rational coefficients, having a root (not necessarily real) of multiplicity greater than 1. For which  $n$  is it true that every element of  $\mathcal{P}_n$  has a rational root?

**4177.** *Proposed by Daniel Sitaru.*

Prove that if  $a, b, c$  and  $d$  are nonnegative real numbers such that  $(a^2 + b^2)(c^2 + d^2) \neq 0$ , then

$$4((ac + bd)^6 + (ad - bc)^6) \geq (a^2 + b^2)^3(c^2 + d^2)^3.$$

**4178.** *Proposed by Leonard Giugiuc, Dan Marinescu and Marian Cucoanes.*

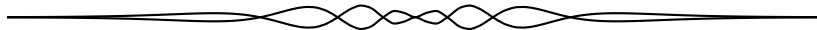
For  $0 \leq a < \frac{\pi}{2}$ , prove that  $\lim_{n \rightarrow \infty} \int_a^{\pi - a} x^n \cos x dx = -\infty$ .

**4179.** *Proposed by D. M. Băţineţu-Giurgiu and Neculai Stanciu.*

Consider the sequence  $(a_n)$ ,  $n \geq 1$ , such that  $a_n = \prod_{k=1}^n (k!)^2$ . Find  $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n^2]{a_n}}$ .

**4180.** *Proposed by Leonard Giugiuc.*

Let  $ABC$  be a triangle such that  $\angle BAC \geq \frac{2\pi}{3}$ . Prove that  $\frac{r}{R} \leq \frac{2\sqrt{3} - 3}{2}$ .



# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2015 : 41(8), p. 352–354.*



**4071.** *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that if  $a, b, c \in (0, 1)$ , then  $a^{a+1}b^{b+1}c^{c+1} < e^{2(a+b+c)-6}$ .

*There were 14 correct solutions and one incorrect submission. We present the solution submitted by various solvers.*

Let

$$f(x) = 2(x-1) - (x+1)\ln x = 2(x-1) - x\ln x - \ln x$$

for  $0 < x < 1$ . The derivative

$$f'(x) = 1 - x^{-1} - \ln x = \ln(x^{-1}) - (x^{-1} - 1)$$

is negative and  $f(1) = 0$ , so that  $f(x) > 0$  on  $(0, 1)$ . Therefore  $(x+1)\ln x < 2(x-1)$  for  $0 < x < 1$ .

Thus

$$(a+1)\ln a + (b+1)\ln b + (c+1)\ln c < 2(a+b+c) - 6.$$

Exponentiating yields the result.

*Editor's Comments.* Several solvers used the function in the solution; another popular function studied was  $\ln x - 2(x-1)(x+1)^{-1} = \ln x + 4(x+1)^{-1} - 2$  or a close relative. Three solvers noted that the function  $(x+1)\ln x$  was concave and applied Jensen's inequality. One respondent took a stroll down the garden path with the following argument.

Let  $L = a^{a+1}b^{b+1}c^{c+1}$ . Since  $\ln x < x-1$  for  $0 < x < 1$ ,

$$\begin{aligned} \ln L &= (a+1)\ln a + (b+1)\ln b + (c+1)\ln c < 2(\ln a + \ln b + \ln c) \\ &< 2((a-1) + (b-1) + (c-1)) = 2(a+b+c) - 6. \end{aligned}$$

Exponentiating yields the desired result.

**4072.** *Proposed by Michel Bataille.*

Let  $a, b$  be distinct positive real numbers and  $A = \frac{a+b}{2}$ ,  $G = \sqrt{ab}$ ,  $L = \frac{a-b}{\ln a - \ln b}$ . Prove that

$$\frac{L}{G} > \frac{4A+5G}{A+8G}.$$

*There were five correct solutions, of which we present two.*

*Solution 1, by the proposer.*

This inequality is a refinement of the known inequality  $L > G$ . Wolog, let  $a > b$  and set  $x = \sqrt{a}$ ,  $y = \sqrt{b}$ . The inequality can be rewritten as

$$\frac{\ln x - \ln y}{x^2 - y^2} < \frac{1}{4xy} \cdot \frac{x^2 + y^2 + 16xy}{2x^2 + 2y^2 + 5xy}.$$

Setting  $t = x/y$  converts it to

$$\frac{4 \ln t}{t^2 - 1} < \frac{1}{t} \cdot \frac{t^2 + 16t + 1}{2t^2 + 5t + 2}.$$

Since

$$\frac{1}{t} \cdot \frac{t^2 + 16t + 1}{2t^2 + 5t + 2} = \frac{1}{2} \left( \frac{1}{t} + \frac{27}{2t^2 + 5t + 2} \right),$$

it all boils down to proving for  $t > 1$  the inequality

$$\frac{8 \ln t}{t^2 - 1} < \frac{1}{t} + \frac{27}{2t^2 + 5t + 2}.$$

Recall Simpson's  $\frac{3}{8}$  Rule for numerical integration of a  $C^4$ -function  $f$  on a closed interval  $[u, v]$ :

$$\int_u^v f(s) ds = \frac{v-u}{8} \left( f(u) + 3f\left(\frac{2u+v}{3}\right) + 3f\left(\frac{u+2v}{3}\right) + f(v) \right) - \frac{(v-u)^5}{6480} f^{(4)}(\xi)$$

for some  $\xi \in (u, v)$ . Taking  $f(s) = 1/s$ ,  $u = 1$  and  $v = t > 1$ , we obtain

$$\ln t = \frac{t-1}{8} \left( 1 + \frac{9}{2+t} + \frac{9}{1+2t} + \frac{1}{t} \right) - \frac{24(t-1)^5}{6480\xi^2}$$

and so

$$\ln t < \frac{t-1}{8} \left( 1 + \frac{1}{t} + \frac{27(t+1)}{2t^2 + 5t + 2} \right) = \frac{t^2 - 1}{8} \left( \frac{1}{t} + \frac{27}{2t^2 + 5t + 2} \right).$$

This leads to the desired inequality.

*Solution 2, by Arkady Alt.*

Assume  $a > b$  and let  $t = \sqrt{a/b}$ . Then as in the previous solution, we have to establish that

$$4 \ln t < \frac{(t^2 - 1)(t^2 + 16t + 1)}{t(t+2)(2t+1)}.$$

Let

$$\begin{aligned} h(t) &= \frac{(t^2 - 1)(t^2 + 16t + 1)}{t(t+2)(2t+1)} - 4 \ln t \\ &= \frac{t}{2} - \frac{27}{2(t+2)} - \frac{27}{4(2t+1)} - \frac{1}{2t} + \frac{27}{4} - 4 \ln t. \end{aligned}$$

Then

$$\begin{aligned} h'(t) &= \frac{1}{2} + \frac{27}{2(t+2)^2} + \frac{27}{2(2t+1)^2} + \frac{1}{2t^2} - \frac{4}{t} \\ &= \frac{2(t^2+t+1)(t-1)^4}{t^2(t+2)^2(2t+1)^2} > 0 \end{aligned}$$

for  $t > 1$ . Since  $h(t) > h(1) = 0$  for  $t > 1$ , the inequality follows.

**4073.** *Proposed by Daniel Sitaru.*

Solve the following system :

$$\begin{cases} \sin 2x + \cos 3y = -1, \\ \sqrt{\sin^2 x + \sin^2 y} + \sqrt{\cos^2 x + \cos^2 y} = 1 + \sin(x+y). \end{cases}$$

*The solution from Michel Bataille was the only one of the 2 submissions that was complete and correct. We present his solution.*

We first show that the second equation is equivalent to  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ .

If  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ , then  $\sin^2 y = \cos^2 x$  and  $\cos^2 y = \sin^2 x$ . It immediately follows that both sides of the equation equal 2. Conversely, if the equation holds, then squaring gives

$$2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x+y)} = 2\sin(x+y) - (1 - \sin^2(x+y)),$$

and therefore

$$\begin{aligned} 2\sin(x+y) \leq 2\sqrt{\sin^2(x+y)} &\leq 2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x+y)} \\ &= 2\sin(x+y) - (1 - \sin^2(x+y)) \leq 2\sin(x+y). \end{aligned}$$

Thus, equality must hold throughout and in particular  $\sin(x+y) \geq 0$  and  $\sin^2(x+y) = 1$ . We deduce that  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ .

Since  $\cos 3\left(\frac{\pi}{2} - x\right) = -\sin 3x$ , we are led to seek the solutions to the equation  $f(x) = 1$  where  $f(x) = \sin 3x - \sin 2x$ . Note that  $f\left(-\frac{\pi}{2}\right) = 1$  so that the numbers  $-\frac{\pi}{2} + 2k\pi$  ( $k \in \mathbb{Z}$ ) are solutions. For other solutions note that  $f$  is odd and  $2\pi$ -periodic; consequently, we may restrict the study of  $f$  to the interval  $[0, \pi]$  and look for  $x$  satisfying either  $f(x) = 1$  or  $f(x) = -1$  (the latter since then  $f(-x) = 1$ ). Consider first the interval  $[0, \frac{\pi}{2})$ . We have  $f(0) = 0$  and if  $x \in (0, \frac{\pi}{2})$ , then  $\sin 2x > 0$  and so  $f(x) < 1$ .

- $x \in (0, \frac{\pi}{3}]$  :  $\sin 3x > 0$  for  $x$  between 0 and  $\frac{\pi}{3}$ , hence  $f(x) > -1$ ; since  $f\left(\frac{\pi}{3}\right) > -1$ , there is no  $x \in (0, \frac{\pi}{3}]$  such that  $f(x) = -1$ .
- $x \in (\frac{\pi}{3}, \frac{\pi}{2})$  :  $f''(x) = 4\sin 2x - 9\sin 3x > 0$ , hence  $f'(x) = 3\cos 3x - 2\cos 2x$  is nondecreasing on the interval  $(\frac{\pi}{3}, \frac{\pi}{2})$ . For some  $x_1 \in (\frac{\pi}{3}, \frac{\pi}{2})$ , we have  $f'(x) \leq 0$



for  $x \in (\frac{\pi}{3}, x_1]$  and  $f'(x) > 0$  for  $x \in (x_1, \frac{\pi}{2})$ . Since  $f(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$  and  $f(\frac{\pi}{2}) = -1$ , we have  $f(x_1) < -1$  and  $f(\alpha) = -1$  for a unique  $\alpha$  in  $(\frac{\pi}{3}, \frac{\pi}{2})$ .

In a similar way we treat the interval  $(\frac{\pi}{2}, \pi]$ . We have  $f(\pi) = 0$  and if  $x \in (\frac{\pi}{2}, \pi)$ , then  $\sin 2x < 0$ , hence  $f(x) > -1$ .

- $x \in (\frac{\pi}{2}, \frac{3\pi}{4})$  :  $f'(x) > 0$  and so  $f$  is increasing from  $-1$  to  $1 + \frac{\sqrt{2}}{2}$ . Thus,  $f(\beta) = 1$  for a unique  $\beta \in (\frac{\pi}{2}, \frac{3\pi}{4})$ .
- $x \in (\frac{5\pi}{6}, \pi)$  :  $f$  is decreasing from  $1 + \frac{\sqrt{3}}{2}$  to  $0$ , hence  $f(\gamma) = 1$  for a unique  $\gamma$  of  $(\frac{5\pi}{6}, \pi)$ .
- $x \in [\frac{3\pi}{4}, \frac{5\pi}{6}]$  : Resorting to  $f''(x)$ , we see that  $f'(x)$  decreases from positive to negative so that  $f(x) > 1$ .

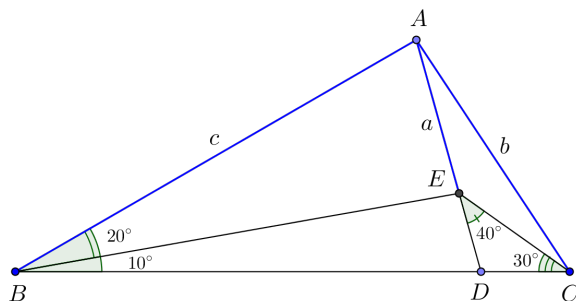
In conclusion, on the interval  $[-\pi, \pi]$  the solutions  $(x, y)$  of the system are the pairs

$$\left(-\frac{\pi}{2}, \pi\right), \left(-\alpha, \frac{\pi}{2} + \alpha\right), \left(\beta, \frac{\pi}{2} - \beta\right), \quad \text{and} \quad \left(\gamma, \frac{\pi}{2} - \gamma\right).$$

All other solutions are obtained by adding multiples of  $2\pi$  to  $x$  or  $y$ .

#### 4074. Proposed by Abdilkadir Altınbaş.

Consider the triangle  $ABC$  with the following measures :



Show that  $a + b = c$ ; that is,  $|AE| + |AC| = |AB|$ .

We received 16 solutions, all correct. We feature the solution of C.R. Pranesachar that is typical of the approach used by most solvers.

Because the angles of  $\triangle BCE$  sum to  $180^\circ$ , we see that  $\angle BED = 100^\circ$  and its supplement  $\angle BEA = 80^\circ$ . It follows that in  $\triangle BEA$  we also have  $\angle BAE = 80^\circ$ , so that  $BE = AB = c$  and

$$\frac{AE/2}{AB} = \sin \frac{20^\circ}{2},$$

or

$$a = 2c \sin 10^\circ.$$

Further, by the Sine Rule applied to triangle  $BCE$ ,

$$CE = BE \frac{\sin 10^\circ}{\sin 30^\circ} = 2c \sin 10^\circ = a.$$

This means that  $\triangle AEC$  is also isosceles (with  $EA = EC = a$ ), and because its exterior angle at  $E$  equals  $40^\circ$ , its interior angles at  $A$  and  $C$  must each be  $20^\circ$ ; thus

$$\frac{b/2}{a} = \cos 20^\circ \quad \text{or} \quad b = 2(2c \sin 10^\circ) \cdot \cos 20^\circ.$$

The given relation  $a + b = c$  now translates to

$$2 \sin 10^\circ + 4 \sin 10^\circ \cos 20^\circ = 1.$$

This is easy to prove :

$$\text{lhs} = 2 \sin 10^\circ + 2(\sin 30^\circ - \sin 10^\circ) = 1 = \text{rhs}.$$

**4075.** *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that in any triangle  $ABC$  with  $BC = a$ ,  $CA = b$ ,  $AB = c$  the following inequality holds :

$$\sqrt[3]{abc} \cdot \sqrt{a^2 + b^2 + c^2} \geq 4[ABC],$$

where  $[ABC]$  is the area of triangle  $ABC$ .

*We received 16 correct solutions and we present the solution by Martin Lukarevski.*

The inequality can be sharpened to

$$\sqrt[3]{abc} \cdot \sqrt{ab + bc + ca} \geq 4[ABC].$$

We use the inequality

$$\sqrt[3]{abc} \geq \sqrt{4[ABC]/\sqrt{3}},$$

which is equivalent to the well-known inequality

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8},$$

and Gordon's inequality (4.5 in O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinovic, P. M. Vasić, *Geometric inequalities*, Groningen, Wolters-Noordhoff, 1969)

$$\sqrt{ab + bc + ca} \geq \sqrt{4[ABC]\sqrt{3}}.$$

Hence

$$\sqrt[3]{abc} \cdot \sqrt{a^2 + b^2 + c^2} \geq \sqrt[3]{abc} \cdot \sqrt{ab + bc + ca} \geq 4[ABC].$$

*Editor's Comments.* Many of the solutions were rather similar in nature as most verifications were the result of combining existing inequalities. In fact, Martin Lukarevski submitted two solutions of which his second is presented above.

**4076.** *Proposed by Mehtaab Sawhney.*

Prove that  $(x^2 + y^2 + z^2)^3 \geq (x^3 + y^3 + z^3 + 3(\sqrt{3} - 1)xyz)^2$  for all nonnegative reals  $x, y$ , and  $z$ .

*We received seven submissions, six of which were correct. We present the solution by Michel Bataille, expanded slightly by the editor.*

Note first that equality holds if  $x = y = z = 0$ . Now suppose  $x + y + z > 0$ . Then by homogeneity we may assume that  $x + y + z = 1$ . Let  $m = xy + yz + zx$  and  $k = xyz$ . Then  $x^2 + y^2 + z^2 = 1 - 2m$  and

$$x^3 + y^3 + z^3 = 3xyz + (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 1 - 3m + 3k.$$

The given inequality is equivalent, in succession, to

$$\begin{aligned} (1 - 2m)^3 &\geq (1 - 3m + 3\sqrt{3}k)^2 \\ 1 - 6m + 12m^2 - 8m^3 &\geq 1 + 9m^2 + 27k^2 - 6m + 6\sqrt{3}k - 18\sqrt{3}mk \\ 3m^2 + 18\sqrt{3}mk - 8m^3 - 27k^2 - 6\sqrt{3}k &\geq 0. \end{aligned} \quad (1)$$

Note that  $1 - 3m \geq 0$  since  $x^2 + y^2 + z^2 \geq xy + yz + zx$ . We set  $u = \sqrt{1 - 3m}$  so  $m = \frac{1}{3}(1 - u^2)$  and then (1) becomes

$$\begin{aligned} \frac{8}{27}(1 - u^2)^3 + 27k^2 + 6\sqrt{3}k &\leq \frac{1}{3}(1 - u^2)^2 + 6\sqrt{3}(1 - u^2)k \\ \frac{8}{27}(1 - 3u^2 + 3u^4 - u^6) + 27k^2 + 6\sqrt{3}k &\leq \frac{1}{3}(1 - 2u^2 + u^4) + 6\sqrt{3}k - 6\sqrt{3}ku^2 \\ 27 \cdot 6\sqrt{3}ku^2 + (27k)^2 &\leq 1 + 6u^2 - 15u^4 + 8u^6. \end{aligned} \quad (2)$$

We now apply the following known result :

$$27k \leq (1 - u)^2(1 + 2u) = 1 - 3u^2 + 2u^3. \quad (3)$$

(See the article ‘‘On a class of three-variable Inequalities’’ by Vo Quoc Ba Can, *Mathematical Reflections*, 2007, issue 2, and the proof by Paolo Perfetti in his solution to **CruX** Problem 3663, 38 (7), pp. 291-292.)

Using (3), we see that in order to establish (2), it suffices to prove the inequality :

$$6\sqrt{3}u^2(1 - u)^2(1 + 2u) + (1 - u)^4(1 + 2u)^2 \leq 1 + 6u^2 - 15u^4 + 8u^6. \quad (4)$$

After straightforward computations, we see that (4) is successively equivalent to :

$$\begin{aligned} (1 - u)^2(6\sqrt{3}u^2(1 + 2u) + (1 - u)^2(1 + 2u)^2) &\leq (1 - u)^2(1 + 2u + 9u^2 + 16u^3 + 8u^4) \\ 1 + 2u + 9u^2 + 16u^3 + 8u^4 - (6\sqrt{3}u^2 + 12\sqrt{3}u^3 + 1 + 2u - 3u^2 - 4u^3 + 4u^4) &\geq 0 \\ ((12 - 6\sqrt{3})u^2 + (20 - 12\sqrt{3})u^3 + 4u^4) &\geq 0 \\ 2u^2 + (10 - 6\sqrt{3})u + 6 - 3\sqrt{3} &\geq 0, \end{aligned}$$

which is true since  $(10 - 6\sqrt{3})^2 - 8(6 - 3\sqrt{3}) = 32(5 - 3\sqrt{3}) < 0$ . Hence (4) is true and our proof is complete.

**4077.** *Proposed by George Apostolopoulos.*

Let  $ABC$  be a triangle. Prove that

$$\sin \frac{A}{2} \cdot \sin B \cdot \sin C + \sin A \cdot \sin \frac{B}{2} \cdot \sin C + \sin A \cdot \sin B \cdot \sin \frac{C}{2} \leq \frac{9}{8}.$$

*We received 15 submissions, of which 14 were correct and complete. We present the solution by Phil McCartney.*

Let  $a = \frac{\pi-A}{2}$ ,  $b = \frac{\pi-B}{2}$ ,  $c = \frac{\pi-C}{2}$ . Then  $a + b + c = \pi$  and  $a$ ,  $b$  and  $c$  are in  $(0, \frac{\pi}{2})$ . We have

$$\begin{aligned} \sum_{cyc} \sin \frac{A}{2} \sin B \sin C &= \sum_{cyc} \cos a \sin(\pi - 2b) \sin(\pi - 2c) \\ &= \sum_{cyc} \cos a \sin(2b) \sin(2c) \\ &= 4 \cos a \cos b \cos c \cdot \sum_{cyc} \sin a \sin b \end{aligned}$$

Hence it suffices to show that

$$\cos a \cos b \cos c \leq \frac{1}{8} \quad \text{and} \quad \sum_{cyc} \sin a \sin b \leq \frac{9}{4}. \quad (\dagger)$$

Since  $\cos(t)$  is a concave function on  $(0, \frac{\pi}{2})$ , the AM-GM inequality followed by Jensen's inequality yields :

$$\cos a \cos b \cos c \leq \left( \frac{\cos a + \cos b + \cos c}{3} \right)^3 \leq \cos^3 \left( \frac{a + b + c}{3} \right) = \frac{1}{8},$$

proving the first of the two inequalities in  $(\dagger)$ .

By Cauchy's inequality,  $\sum_{cyc} \sin a \sin b \leq \sum_{cyc} \sin^2 a$ ; so, in order to conclude the second inequality also holds, it suffices to prove that  $\sum_{cyc} \sin^2 a \leq \frac{9}{4}$ . However, one can show that  $\sum_{cyc} \sin^2 a = 2(1 + \cos a \cos b \cos c)$  : using the fact that  $a + b + c = \pi$ , and hence  $\sin(a) = \sin(b + c)$ , we have

$$\begin{aligned} \sin^2 a &= (\sin b \cos c + \cos b \sin c)^2 \\ &= \sin^2 b \cos^2 c + 2 \sin b \cos b \cos c \sin c + \cos^2 b \sin^2 c \\ &= (1 - \cos^2 b) \cos^2 c + 2 \sin b \cos b \cos c \sin c + \cos^2 b (1 - \cos^2 c) \\ &= \cos^2 b + \cos^2 c - 2 \cos^2 b \cos^2 c + 2 \cos b \cos c \sin b \sin c \\ &= \cos^2 b + \cos^2 c - 2 \cos b \cos c \cos(b + c) \\ &= 1 - \sin^2 b + 1 - \sin^2 c + 2 \cos b \cos c \cos a, \end{aligned}$$

which can be rearranged to  $\sin^2 a + \sin^2 b + \sin^2 c = 2 + 2 \cos a \cos b \cos c$  as claimed.

Finally, using the first inequality in (†), we can conclude that  $\sum_{cyc} \sin^2 a \leq 2 + \frac{1}{4} = \frac{9}{8}$ , showing the second inequality in (†).

*Editor's Comments.* The inequalities in (†) are known and can be found in O. Bottema et al., *Geometric inequalities*, Groningen, Wolters-Noordhoff, 1969.

**4078.** *Proposed by Michel Bataille.*

Given  $\theta$  such that  $\frac{\pi}{3} \leq \theta \leq \frac{5\pi}{3}$ , let  $M_0$  be a point of a circle with centre  $O$  and radius  $R$  and  $M_k$  its image under the counterclockwise rotation with centre  $O$  and angle  $k\theta$ . If  $M$  is the point diametrically opposite to  $M_0$  and  $n$  is a positive integer, show that

$$\sum_{k=0}^n MM_k \geq (2n+1) \cdot \frac{R}{2}.$$

*We received two submissions, both correct, and feature the solution by AN-anduud Problem Solving Group.*

We can assume that  $R = 1$ ,  $M_0 = e^0 = 1$ , and  $M = -1$ ; then  $M_k = e^{ik\theta}$ ,  $k = 1, 2, \dots, n$ . Let  $e^{i\theta} = z$ , so that  $M_k = z^k$ , and

$$MM_k = |-1 - e^{ik\theta}| = |1 + z^k| \leq 1 + |z|^k = 2.$$

Thus we have

$$\begin{aligned} \sum_{k=0}^n MM_k &= \sum_{k=0}^n |1 + z^k| = \frac{1}{2} \sum_{k=0}^n 2 \cdot |1 + z^k| \\ &\geq \frac{1}{2} \sum_{k=0}^n |1 + z^k|^2 = \frac{1}{2} \sum_{k=0}^n (1 + z^k)(\overline{1 + z^k}) \\ &= \frac{1}{2} \sum_{k=0}^n (1 + z^k)(1 + z^{-k}) = \frac{1}{2} \sum_{k=0}^n (2 + (z^k + z^{-k})) \\ &= \sum_{k=0}^n \left( 1 + \frac{z^k + z^{-k}}{2} \right) = \sum_{k=0}^n (1 + \cos k\theta) \\ &= n + \frac{1}{2} + \left( 1 + \frac{1}{2} + \sum_{k=1}^n \cos k\theta \right) \\ &= \frac{2n+1}{2} + \left( 1 + \frac{\sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}} \right) \\ &= \frac{2n+1}{2} + \frac{2 \sin \frac{\theta}{2} + \sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}} \\ &\quad \left( \frac{\pi}{6} \leq \frac{\theta}{2} \leq \frac{5\pi}{6} \Rightarrow \sin \frac{\theta}{2} \geq \frac{1}{2} \Rightarrow 2 \sin \frac{\theta}{2} \geq 1 \right) \\ &\geq \frac{2n+1}{2} + \frac{1 + \sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}} \geq \frac{2n+1}{2}. \end{aligned}$$

**4079.** *Proposed by Mihaela Berindeanu.*

Let  $x, y, z > 0$  and  $x + y + z = 2016$ . Prove that :

$$x\sqrt{\frac{yz}{y+2015z}} + y\sqrt{\frac{xz}{z+2015x}} + z\sqrt{\frac{xy}{x+2015y}} \leq \frac{2016}{\sqrt{3}}.$$

*We received eleven solutions. We present 2 solutions.*

*Solution 1, by Titu Zvonaru.*

We prove the general case. Let  $x + y + z = t$ , where  $t \geq 1$ . We have

$$\frac{yz}{y+(t-1)z} \leq \frac{(t-1)y+z}{t^2}. \quad (1)$$

Indeed, the inequality (1) is equivalent to

$$(t-1)y^2 + (t-1)z^2 + ((t-1)^2 + 1)yz \geq t^2yz \iff (t-1)(y-z)^2 \geq 0.$$

Using (1), the Cauchy-Schwarz Inequality and the known inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx),$$

we obtain

$$\begin{aligned} & x\sqrt{\frac{yz}{y+(t-1)z}} + y\sqrt{\frac{xz}{z+(t-1)x}} + z\sqrt{\frac{xy}{x+(t-1)z}} \\ & \leq \frac{x}{t}\sqrt{(t-1)y+z} + \frac{y}{t}\sqrt{(t-1)z+x} + \frac{z}{t}\sqrt{(t-1)x+y} \\ & = \frac{1}{t}(\sqrt{x}\sqrt{(t-1)xy+zx} + \sqrt{y}\sqrt{(t-1)yz+xy} + \sqrt{z}\sqrt{(t-1)zx+yz}) \\ & \leq \frac{1}{t}\sqrt{(x+y+z)[(t-1)xy+zx + (t-1)yz+xy + (t-1)zx+yz]} \\ & = \frac{1}{t}\sqrt{t^2(xy+yz+zx)} \\ & \leq \frac{1}{t} \cdot t\sqrt{\frac{(x+y+z)^2}{3}} = \frac{t}{\sqrt{3}} \end{aligned}$$

The equality holds if and only if  $x = y = z = t/3$ .

*Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.*

We will prove the following slight generalization of the given problem :

If  $x, y, z > 0$  and  $x + y + z = S + 1$ , then

$$x\sqrt{\frac{yz}{y+Sz}} + y\sqrt{\frac{zx}{z+Sx}} + z\sqrt{\frac{xy}{x+Sy}} \leq \frac{S+1}{\sqrt{3}}.$$

To begin, we invoke the general form of the Arithmetic - Geometric Mean Inequality which states that if  $a, b, \alpha, \beta > 0$  and  $\alpha + \beta = 1$ , then

$$a^\alpha \cdot b^\beta \leq \alpha a + \beta b, \quad (2)$$

with equality if and only if  $a = b$ . It follows from (2) that

$$\begin{aligned} (y + Sz)(Sy + z) &= (S + 1)^2 \left( \frac{1}{S + 1}y + \frac{S}{S + 1}z \right) \left( \frac{S}{S + 1}y + \frac{1}{S + 1}z \right) \\ &\geq (S + 1)^2 \left( y^{\frac{1}{S+1}} z^{\frac{S}{S+1}} \right) \left( y^{\frac{S}{S+1}} z^{\frac{1}{S+1}} \right) \\ &= (S + 1)^2 yz, \end{aligned}$$

and hence,

$$x \sqrt{\frac{yz}{y + Sz}} \leq \frac{x}{S + 1} \sqrt{Sy + z}. \quad (3)$$

Further, equality is attained in (3) if and only if  $y = z$ .

Similar arguments show that

$$y \sqrt{\frac{zx}{z + Sx}} \leq \frac{y}{S + 1} \sqrt{Sz + x}, \quad (4)$$

with equality if and only if  $z = x$ , and

$$z \sqrt{\frac{xy}{x + Sy}} \leq \frac{z}{S + 1} \sqrt{Sx + y}, \quad (5)$$

with equality if and only if  $x = y$ .

Since  $f(t) = \sqrt{t}$  is strictly concave on  $(0, \infty)$ , we utilize conditions (3), (4), (5), the constraint equation  $x + y + z = S + 1$ , and Jensen's Inequality to obtain

$$\begin{aligned} &x \sqrt{\frac{yz}{y + Sz}} + y \sqrt{\frac{zx}{z + Sx}} + z \sqrt{\frac{xy}{x + Sy}} \\ &\leq \frac{x}{S + 1} \sqrt{Sy + z} + \frac{y}{S + 1} \sqrt{Sz + x} + \frac{z}{S + 1} \sqrt{Sx + y} \\ &\leq \sqrt{\frac{x(Sy + z) + y(Sz + x) + z(Sx + y)}{S + 1}} \\ &= \sqrt{xy + yz + zx} \\ &\leq \sqrt{\frac{(x + y + z)^2}{3}} \\ &= \frac{S + 1}{\sqrt{3}}, \end{aligned}$$

with equality if and only if  $x = y = z = \frac{S + 1}{3}$ .

**4080.** *Proposed by Alina Sîntămărian and Ovidiu Furdui.*

Let  $a, b \in \mathbb{R}$ , with  $ab > 0$ . Calculate

$$\int_0^\infty x^2 e^{-(ax - \frac{b}{x})^2} dx.$$

*We received nine submissions of which five were correct and complete solutions. We present the solution by Michel Bataille.*

We show that the value of the given integral  $I$  is

$$I = \frac{\sqrt{\pi}(1+2ab)}{4|a|^3}.$$

Recall that  $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ . For later use, we also calculate  $J = \int_0^\infty t^2 e^{-t^2} dt$ . For  $X > 0$ , integrating by parts, we obtain

$$\int_0^X t^2 e^{-t^2} dt = \frac{1}{2} \left( [-te^{-t^2}]_0^X + \int_0^X e^{-t^2} dt \right) = \frac{1}{2} \left( -Xe^{-X^2} + \int_0^X e^{-t^2} dt \right)$$

and letting  $X \rightarrow \infty$ ,

$$J = \frac{1}{2} \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{4}.$$

The equation  $ax - \frac{b}{x} = y$  has a unique positive solution for  $x = \frac{1}{2a}(y + \varepsilon\sqrt{y^2 + 4ab})$ , where  $\varepsilon = 1$  if  $a, b > 0$  and  $\varepsilon = -1$  if  $a, b < 0$ . The change of variables

$$x = \frac{1}{2a}(y + \varepsilon\sqrt{y^2 + 4ab}), \quad dx = \frac{\varepsilon}{2a} \cdot \frac{y + \varepsilon\sqrt{y^2 + 4ab}}{\sqrt{y^2 + 4ab}} dy$$

yields

$$I = \frac{1}{8a^3} \int_{-\infty}^\infty \frac{(y + \varepsilon\sqrt{y^2 + 4ab})^3}{\sqrt{y^2 + 4ab}} e^{-y^2} dy.$$

We expand the non-exponential factor in the integrand as

$$\frac{(y + \varepsilon\sqrt{y^2 + 4ab})^3}{\sqrt{y^2 + 4ab}} = \frac{y^3}{\sqrt{y^2 + 4ab}} + 3\varepsilon y^2 + 3y\sqrt{y^2 + 4ab} + \varepsilon(y^2 + 4ab).$$

Note the behaviour of the first and third terms on the right-hand side :

$$|y|\sqrt{y^2 + 4ab}e^{-y^2} \sim \frac{|y|^3}{\sqrt{y^2 + 4ab}} e^{-y^2} \sim y^2 e^{-y^2} \quad \text{as } y \rightarrow \infty$$

It follows that the integrals

$$\int_{-\infty}^\infty \frac{y^3}{\sqrt{y^2 + 4ab}} e^{-y^2} dy \quad \text{and} \quad \int_{-\infty}^\infty 3y\sqrt{y^2 + 4ab} e^{-y^2} dy$$

exist and are therefore zero, since each integrand is odd. Thus,

$$I = \frac{1}{8a^3} \left( 6\varepsilon J + 2\varepsilon J + 8\varepsilon ab \int_0^\infty e^{-y^2} dy \right) = \frac{\sqrt{\pi}(1+2ab)}{4|a|^3}.$$



# AUTHORS' INDEX

Solvers and proposers appearing in this issue  
(Bold font indicates featured solution.)

## Proposers

George Apostolopoulos, Messolonghi, Greece : 4175  
 Michel Bataille, Rouen, France : 4176  
 D. M. Băţineţu-Giurgiu and Neculai Stanciu, Romania : 4179  
 Mihaela Berindeanu, Bucharest, Romania : 4174  
 J. Chris Fisher, Cleveland Heights, OH, USA : 4171  
 Leonard Giugiuc, Romania : 4180  
 Leonard Giugiuc, Dan Marinescu and Marian Cucoanes, Romania : 4178  
 Dao Thanh Oai, Vietnam, Leonard Giugiuc, Romania, and Daniel Sitaru, Romania :  
 4173  
 Daniel Sitaru, Drobeta Turnu Severin, Romania : 4177  
 Nathan Soedjak, Rogersville, MO, USA : 4172

## Solvers - individuals

Ali Adnan, Atomic Energy Central School-4, Mumbai, India : OC238  
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 Abdikadir Altıntaş, Emirdağ, Turkey : 4074  
 George Apostolopoulos, Messolonghi, Greece : 4071, 4077  
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 4074, 4075, 4077  
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 Michel Bataille, Rouen, France : OC236, OC238, OC240, 4071, **4072**, **4073**, 4074, 4075,  
**4076**, 4077, 4078, 4079, **4080**  
 Mihaela Berindeanu, Bucharest, Romania : 4079  
 Paul Bracken, University of Texas, Edinburg, TX, USA : 4071, 4072, 4074  
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 4074, 4077, 4079, 4080  
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 Leonard Giugiuc, Romania : 4079, 4080  
 John G. Heuver, Grande Prairie, AB : CC187, 4077  
 Dag Jonsson, Uppsala, Sweden : 4074  
 Doddy Kastanya, Toronto, ON : CC187  
 Václav Konečný, Big Rapids, MI, USA : 4074, 4075, 4080  
 Kee-Wai Lau, Hong Kong, China : 4071, 4074, 4075, 4076, 4077, 4079  
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**4077**  
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Joel Schlosberg, Bayside, NY, USA : 4071  
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OC238, 4071, 4072, 4074, 4075, 4077, 4079  
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4072  
Ovidiu Furdui and Alina Șintămărian, Cluj-Napoca, Romania : 4080  
Leonard Giugiuc and Daniel Sitaru, Romania : 4071, 4075  
D. M. Băținețu-Giurgiu and Neculai Stanciu, Romania : 4075

