Crux Mathematicorum

VOLUME 42, NO. 7 September / Septembre 2016

IN THIS ISSUE / DANS CE NUMÉRO

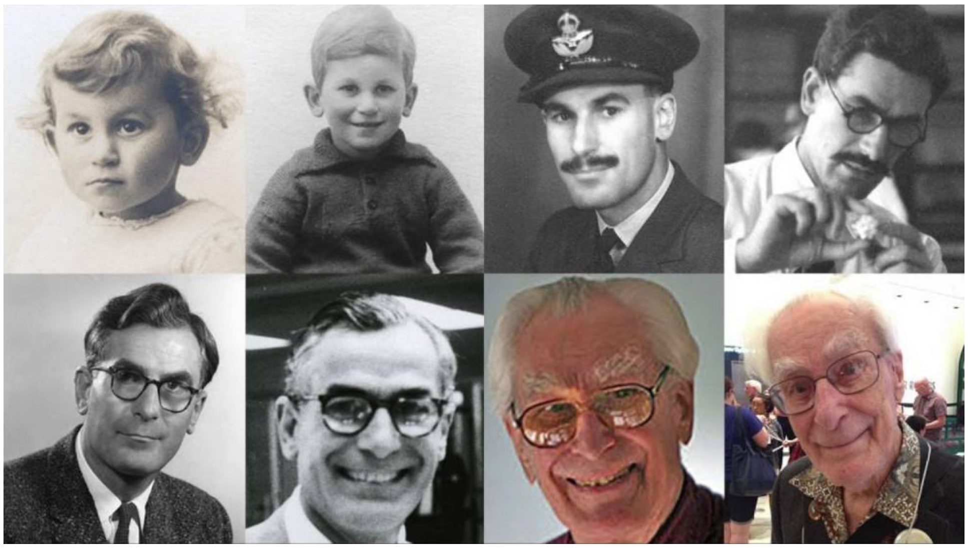
Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow,morla Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum
with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

Happy 100th Birthday



We are delighted to wish Richard Guy a Happy 100th Birthday, which he celebrated on September 30, 2016.

Richard Guy is a legend in the mathematical community, known for his work in number theory, combinatorics, geometry and recreational mathematics. Richard has a direct connection to ***Crux*** : he was on the very first Editorial Board in 1991, served as the Editor-at-Large until the end of 2003 and continues to take an interest in our publication.

Here are a couple of resources where you can learn more about Richard:

- University of Calgary's website `http://www.ucalgary.ca/richardguy100/` (Richard insists that he didn't retire in 1982, they just stopped paying him);
- his paper *Strong law of small numbers* ("There aren't enough small numbers to meet the many demands made of them");
- Richard's book *Unsolved Problems in Number Theory*, where many number theorist get their start and from where they draw their inspiration.

Happy Birthday Richard, and many more!

Honsberger Commemorative Issue

Ross Honsberger, a Canadian mathematician, passed away on April 3rd, 2016 at age 86. Ross was not only a familiar ***Crux*** contributor, he also used a lot of ***Crux*** materials and references in his own writing. To commemorate his memory, we are dedicating the April 2017 issue of ***Crux*** to Ross.

Readers are invited to contribute to the issue with problems and articles inspired by Ross's work or dedicated to him as well as tributes to Ross. Please forward all the correspondence to `crux-editors@cms.math.ca` by March 1st if possible.

THE CONTEST CORNER

No. 47

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **April 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC231. If $x^2 + y^2 = 6xy$ with $y > x > 0$, find $\frac{x+y}{x-y}$.

CC232. Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards if 100 students take these tests.

CC233. Let P be a point in the interior of the rectangle $ABCD$. Suppose that $PA = a, PB = b$ and $PC = c$, find, in terms of a, b, c, the length of the line segment PD.

CC234. Find B if

$$x = \frac{\log_{10} 16/3}{\log_{10} B}$$

is the solution to the exponential equation

$$2^{2x+4} + 3^{3x+2} = 4^{x+3}.$$

CC235. Find the area of a regular octagon formed by cutting equal isosceles triangles from the corners of a square with sides of one unit.

. . . .

CC231. Soit $x^2 + y^2 = 6xy$ où $y > x > 0$. Déterminer $\frac{x+y}{x-y}$.

CC232. Sept tests sont donnés à 100 élèves et il n'y a pas deux notes égales dans les résultats d'un même test. Chaque personne qui reçoit la plus haute note dans au moins un test ou qui reçoit une des six meilleures notes dans au moins quatre tests recevra un prix, mais chaque personne ne peut recevoir plus d'un prix. Déterminer le nombre maximum de personnes qui pourraient recevoir un prix.

CC233. Soit P un point à l'intérieur du rectangle $ABCD$ et soit $PA = a, PB = b$ et $PC = c$. Déterminer la longueur du segment PD en fonction de a, b et c.

CC234. Déterminer la valeur de B, sachant que l'équation exponentielle

$$2^{2x+4} + 3^{3x+2} = 4^{x+3}$$

a pour solution

$$x = \frac{\log_{10} 16/3}{\log_{10} B}.$$

CC235. Déterminer l'aire d'un octogone régulier formé en découpant un triangle isocèle de chaque coin d'un carré dont les côtés ont une longueur de 1.

Math Quotes

Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong; and if it is found to be wrong, there is a plenteous choice of amendments ready in the mathematicians' stock of formulae. Some verbal statements have not this merit; they are so vague that they could hardly be wrong, and are correspondingly useless.

Lewis Fry Richardson in "Mathematics of War and Foreign Politics."

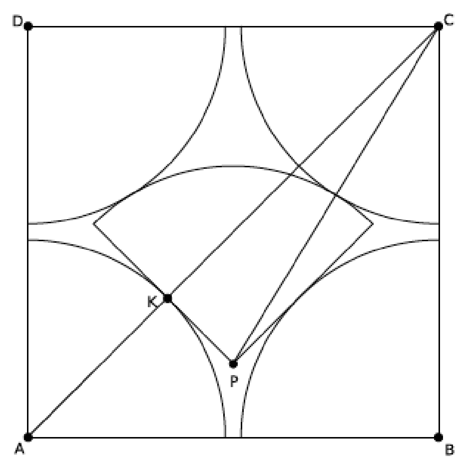
CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(7), p. 280–282.

CC181. A delivery boy decides to optimize the transportation of his pizzas. In his system, each box contains entirely and without any overlap 5 identical quarter pizzas (see below). The box is square, the drawing is symmetrical with respect to the center line and all contacts seen are mathematically perfect. The radius of a pizza is 16 cm. What is the minimal area of the bottom of the box rounded off to the nearest integer?

Originally problem 18 of the quarter-final of the 2012-13 Championnat International des Jeux Mathématiques et Logiques.

We received one correct solution by Ricard Peiró i Estruch which is presented below, modified by the editor.



Let $ABCD$ be the square forming the pizza box base with coordinates $A = (0, 0)$ and $C = (32+2x, 32+2x)$ where $2x$ is the distance between two of the corner pizza slices. Let K be the point where the bottom left pizza slice touches the central pizza slice and P be the point at the bottom of the central slice. Due to the symmetry of the slices and their right angle at the vertices, PK is perpendicular to AC and thus K lies on AC. We obtain $K = (8\sqrt{2}, 8\sqrt{2})$ and $P = (16 + x, 16\sqrt{2} - 16 - x)$. Finally $|PC| = 32$ and thus

$$32^2 = ((32 + 2x) - (16 + x))^2 + ((32 + 2x) - (16\sqrt{2} - 16 - x))^2.$$

We obtain

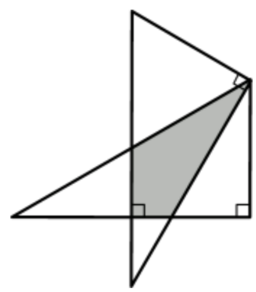
$$x = \frac{8\sqrt{38} + 24\sqrt{2} - 80}{5}$$

and

$$[ABCD] = (32 + 2x)^2 = \left(\frac{16\sqrt{38} + 48\sqrt{2}}{5}\right)^2 \approx 1109.$$

So the area of the pizza box is about 1109 cm^2.

CC182. We rearrange two halves of an equilateral triangle (cut along one of its altitudes) in the following way:

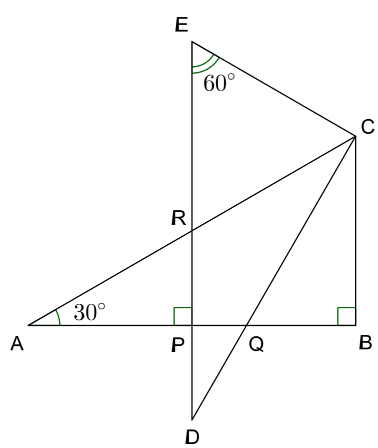


The area of the equilateral triangle was 600 cm^2. What is the area of the overlap region common to the 2 triangles?

Originally problem 14 of the 2012-13 quarter finals of the Championnat International des Jeux Mathématiques et Logiques.

We received five solutions. We present the solution by Ricard Peiró i Estruch.

Let $\triangle ABC$ and $\triangle DCE$ be our two right-angled triangles (with $\angle CAB = \angle EDC = 30°$, and $\angle ABC = \angle DCE = 90°$), as shown in the diagram below:



From the information given, the area of triangle $\triangle ABC$ (denoted $A_{\triangle ABC}$) is $\frac{1}{2} \cdot 600 = 300$. Denote the side BC by s; then, from standard equilateral triangle properties, $CE = s$, $AC = DE = 2s$, and $AB = DC = s\sqrt{3}$. Denote by $PQCR$

the quadrilateral formed by the intersection of the two triangles (that is, the quadrilateral whose area we want to calculate).

Note that $\angle ARP = 60^\circ$, so $\angle ERC = 60^\circ$. It follows that $\triangle RCE$ is equilateral, so $RE = RC = s$, and also $AR = AC - RC = s$. Hence the similar triangles $\triangle APR$ and $\triangle ABC$ have side ratio $AR : AC = 1 : 2$. Therefore,

$$A_{\triangle APR} = \frac{1}{2^2} A_{\triangle ABC} = \frac{300}{4} = 75.$$

Since quadrilateral $PQCE$ is cyclic ($\angle EPQ = \angle ECQ = 90^\circ$), we have $\angle CQB = 60^\circ$. So $\triangle CQB$ is also similar to $\triangle ACB$ with side ratio $CB : AB = 1 : \sqrt{3}$. Therefore,

$$A_{\triangle CQB} = \frac{1}{\sqrt{3}^2} A_{\triangle ABC} = \frac{300}{3} = 100.$$

The area of quadrilateral $RPQC$ is the area of $\triangle ABC$ minus the areas of triangles $\triangle APR$ and $\triangle CQB$; that is,

$$A_{RPQC} = A_{\triangle ABC} - (A_{\triangle APR} + A_{\triangle CQB}) = 300 - (75 + 100) = 125 \text{ cm}^2.$$

CC183. We call a number *productive* if all the products of consecutive digits of the number can be found in its written form. 2013 and 1261 are examples of such numbers. Taking the first one as an example, we get the following consecutive products $2 \times 0 = 0, 0 \times 1 = 0$ and $1 \times 3 = 3$ which can all be found in the written form of 2013. For the second number, the products are $1 \times 2 = 2, 2 \times 6 = 12$ and $6 \times 1 = 6$ which can all be read in 1261. What is the smallest productive number which can be written using all the digits from 0 to 9?

Originally problem 17 of the semi-finals of the 2012-13 of the Championnat International des Jeux Mathmatiques et Logiques.

We received one correct submission. We present the solution by David Manes.

Starting with the numbers $10, 11, \ldots, 19, 20, 21, \ldots, 29, 30, 31$ and carefully trying to construct a productive number, but eliminating them for various and sundry reasons, we arrived at the number 3205486917, which easily satisfies the definition and is the smallest, hopefully, by construction. Some close, but non-productive numbers were 2463180795 and 3154207698, the last digit in each case causing all the problems.

CC184. We are looking for two positive integers such that the difference of their squares is a cube and the difference of their cubes is a square. What is the value of the greatest of the two given that it is smaller than 20?

Originally problem 16 of the 2012-13 semi-finals of the Championnat International des Jeux Mathématiques et Logiques.

We received two solutions. We present the solution by David Manes.

Consider values of x and y, $1 \le y < x \le 19$, such that $x^2 - y^2 = n^3$ for some integer n. Then $3^2 - 1^2 = 2^3$, $6^2 - 3^2 = 3^3$, $10^2 - 6^2 = 4^3$, $14^2 - 13^2 = 3^3$, $15^2 - 3^2 = 6^3$, $15^2 - 10^2 = 5^3$ and $17^2 - 15^2 = 4^3$.

We now consider values of x and y, $1 \le y < x \le 19$ such that $x^3 - y^3 = n^2$ for some integer n. Then $8^3 - 7^3 = 13^2$, $10^3 - 6^3 = 28^2$ and $14^3 - 7^3 = 7^2$.

Accordingly, the only solution to the problem of finding two positive integers smaller than 20 such that the difference of their squares is a cube and the difference of their cubes is a square is 10 and 6 since $10^2 - 6^2 = 4^3$ and $10^3 - 6^3 = 28^2$. The value of the greatest is 10.

CC185. Each asterisk in the following multiplication can only be replaced by a digit in the set $\{2, 3, 5, 7\}$. Complete the multiplication.

$$\begin{array}{rrrrr}
 & & * & * & * \\
 & \times & & * & * \\
\hline
 & * & * & * & * \\
* & * & * & * & \\
\hline
* & * & * & * & *
\end{array}$$

Originally problem 16 of the 2011-2012 quarter final of the Championnat International des Jeux Mathématiques et Logiques.

We received three correct answers, one of which provided justification. We give the main steps.

The answer is $775 \times 33 = 25575$ with both partial products equal to 2325.

By checking what feeds into the last two digits of the product, we find that either the three-digit multiplicand or two-digit multiplier ends in 5. If the multiplicand ends in 5, each partial product ends in 5 and a check of the last two digits of the first partial product reveals that the only possibilities of the two factors are $(*25, *3)$, $(*75, *3)$, $(**5, *5)$, $(*25, *7)$. Checking various first digits for the multiplicand forces the partial products to be either $775 \times 3 = 2325$, $555 \times 5 = 2775$ or $325 \times 7 = 2275$.

If the multiplier ends in 5, a brief analysis shows that the multiplicand ends in 5. Thus, the possible factor pairs are $(775, 33)$, $(325, 77)$ and $(555, 55)$ and only the first of these works.

THE OLYMPIAD CORNER

No. 345

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **April 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC291. Let $n \geq 2$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^n x_i = 1$. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \leq \frac{n}{2}.$$

OC292. Consider two points with integer coordinates on the graph of a polynomial function with integer coefficients. If the distance between them is an integer, prove that the segment that connects them is parallel to the horizontal axis.

OC293. You are given $N \geq 3$. A set of N points on a plane is *acceptable* if their abscissae are unique, and each of the points is coloured either red or blue. A graph of a polynomial function $P(x)$ *divides* a set of acceptable points if there are no red dots above the graph of $P(x)$ and no blue dots below, or if there are no blue dots above the graph of $P(x)$ and no red dots below. Keep in mind, dots of both colours can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary acceptable set of N points divisible by a polynomial of degree k?

OC294. In given triangle $\triangle ABC$, the difference between sizes of each pair of sides is at least $d > 0$. Let G and I be the centroid and incenter of $\triangle ABC$ and r be its inradius. Show that

$$[AIG] + [BIG] + [CIG] \geq \frac{2}{3}dr,$$

where $[XYZ]$ is the area of triangle $\triangle XYZ$.

OC295. Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function that gives a positive integer value, to every positive integer. Suppose that f satisfies the following conditions:

$$f(1) = 1 \quad \text{and} \quad f(a + b + ab) = a + b + f(ab).$$

Find the value of $f(2015)$.

. . . .

OC291. Soit $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ des réels strictement positifs tels que $\sum_{i=1}^{n} x_i = 1$. Démontrer que

$$\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \leq \frac{n}{2}.$$

OC292. Sur la représentation graphique d'une fonction polynôme à coefficients entiers, deux points sont choisis avec des entiers pour coordonnées. Démontrer que si la distance entre les points est un entier, alors le segment qui les joint est parallèle à l'axe horizontal.

OC293. Soit N un entier $(N \geq 3)$. Un ensemble de N points dans le plan est appelé *acceptable* si les abscisses des points sont distinctes et si chacun des points est coloré en bleu ou en rouge. On dit qu'un ensemble acceptable de points dans le plan est *divisible* par la courbe représentative d'une fonction polynôme s'il n'y a aucun point rouge au-dessus de la courbe et aucun point bleu au-dessous de la courbe ou bien s'il n'y a aucun point bleu au-dessus de la courbe et aucun point rouge au-dessous de la courbe. À noter que des points de chaque couleur peuvent être situés sur la courbe. Quelle est la plus petite valeur de k pour laquelle n'importe quel ensemble acceptable de N points est divisible par un polynôme de degré k?

OC294. On considère un triangle ABC dont la différence entre les longueurs de chaque paire de côtés est supérieure ou égale à d $(d > 0)$. Soit G le centre de gravité du triangle, I le centre du cercle inscrit dans le triangle et r le rayon de ce cercle. Démontrer que

$$[AIG] + [BIG] + [CIG] \geq \frac{2}{3} dr,$$

$[XYZ]$ étant l'aire du triangle XYZ.

OC295. Soit $\mathbb{N} = \{1, 2, 3, \ldots\}$ l'ensemble des entiers strictement positifs et soit $f : \mathbb{N} \to \mathbb{N}$ une fonction à valeurs entières strictement positives qui satisfait aux conditions suivantes:

$$f(1) = 1 \quad \text{et} \quad f(a + b + ab) = a + b + f(ab).$$

Déterminer la valeur de $f(2015)$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(5), p. 197–198.

OC231. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x+y) + xy. \tag{1}$$

for all $x, y \in \mathbb{R}$.

Originally problem 4 of the 2014 Balkan Mathematical Olympiad TST.

We received 8 correct submissions and 1 incorrect submission. We present the solution by Elnaz Hessami Pilehrood.

Substituting $x = 0$ in this equation, we get

$$f(0)f(y) = f(y) \qquad \text{or} \qquad f(y)(f(0) - 1) = 0.$$

Therefore, either $f(y) = 0$ or $f(0) = 1$.

We can see that $f = 0$ does not satisfy (1), so $f = 0$ cannot be such a function. Therefore, $f(0) = 1$.

When $f(0) = 1$, we can substitute $x = 1$ and $y = -1$ in (1) to get

$$f(1)f(-1) = f(0) - 1 = 0$$

and therefore, $f(1) = 0$ or $f(-1) = 0$.

If $f(1) = 0$, substitute $y = 1$ to get

$$f(x)f(1) = f(x+1) + x \qquad \text{or} \qquad 0 = f(x+1) + x,$$

which implies $f(x+1) = -x$ or $f(x) = 1 - x$. The function $f(x) = 1 - x$ satisfies all conditions, as $(1-x)(1-y) = 1 - x - y + xy$. If $f(-1) = 0$, substitute $y = -1$ to get

$$f(x)f(-1) = f(x-1) - x \qquad \text{or} \qquad f(x-1) = x.$$

The function $f(x) = x+1$ satisfies all conditions, as $(1+x)(1+y) = 1+x+y+xy$.

Therefore, all such functions are $f(x) = 1 - x$ and $f(x) = 1 + x$.

OC232. Given a positive integer m, prove that there exists a positive integer n_0 such that all first digits after the decimal points of $\sqrt{n^2 + 817n + m}$ in decimal representation are equal, for all integers $n > n_0$.

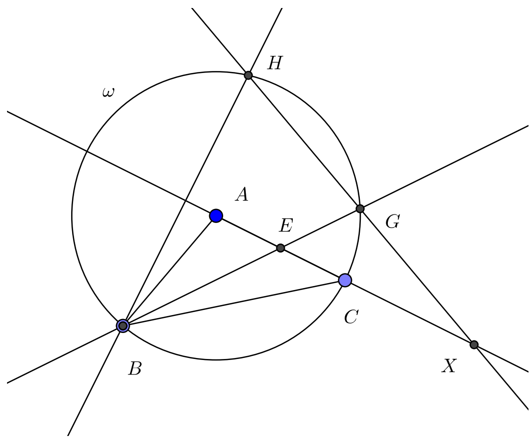
Originally problem 5 from day 2 of the 2014 China Western Mathematical Olympiad.

No submitted solutions.

OC233. Let ω be a circle with center A and radius R. On the circumference of ω four distinct points B, C, G, H are taken in that order in such a way that G lies on the extended B-median of the triangle ABC, and H lies on the extension of the altitude of ABC from B. Let X be the intersection of the straight lines AC and GH. Show that the segment AX has length $2R$.

Originally problem 4 of the 2014 Italy Mathematical Olympiad.

We received 8 correct submissions. We present the solution by Somasundaram Muralidharan.



There is no loss of generality in assuming that the circle ω is $|z| = 1$ in the complex plane. Let B, C, G, H be represented by the complex numbers b, c, g, h respectively. The slope of BG is given by $\dfrac{g-b}{\bar{g}-\bar{b}} = -bg$. The midpoint E of AC is $\dfrac{c}{2}$. Since G lies on the extension of BE, the slope of BE must be equal to the slope of BG. Hence, we have

$$-bg = \frac{b - \frac{c}{2}}{\bar{b} - \frac{\bar{c}}{2}} = \frac{\frac{2b-c}{2}}{\frac{1}{b} - \frac{1}{2c}} = bc\left(\frac{2b-c}{2c-b}\right).$$

Thus,

$$g = -c\left(\frac{2b-c}{2c-b}\right).$$

Now the slope of AC is $\dfrac{c}{\bar{c}} = c^2$ and since BH is perpendicular to AC, slope of BH is $-c^2$. Since the slope of BH is also given by $\dfrac{b-h}{\bar{b}-\bar{h}} = -bh$, we obtain $-c^2 = -bh$ or $h = \dfrac{c^2}{b}$.

Now, the equation of AC is $Z = c^2\overline{Z}$ and that of HG is $Z - h = -hg(\overline{Z} - \overline{h})$. The lines HG and AC meet at X. Solving for $\overline{Z}$, we obtain $\bar{x}$, the conjugate of the complex number x representing X. Thus,

$$\bar{x} = \frac{h+g}{c^2+hg} = \frac{\frac{c^2}{b} - c\left(\frac{2b-c}{2c-b}\right)}{c^2 - \frac{c^3}{b}\left(\frac{2b-c}{2c-b}\right)} = \frac{2c^2 - bc - 2b^2 + bc}{bc(2c-b) - c^2(2b-c)} = \frac{2(c^2-b^2)}{c(c^2-b^2)} = 2\bar{c}.$$

Hence, $x = 2c$ and $AX = 2AC$. Thus, AX is twice the radius of ω.

OC234. Let N be an integer, $N > 2$. Arnold and Bernold play the following game: there are initially N tokens on a pile. Arnold plays first and removes k tokens from the pile, $1 \leq k < N$. Then Bernold removes m tokens from the pile, $1 \leq m \leq 2k$ and so on, that is, each player, on its turn, removes a number of tokens from the pile that is between 1 and twice the number of tokens his opponent took last. The player that removes the last token wins.

For each value of N, find which player has a winning strategy and describe it.

Originally problem 3 from day 1 of the 2014 Brazil National Olympiad.

No submitted solutions.

OC235. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \ldots n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

Originally problem 6 from day 2 of the 2014 USA Mathematical Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

We prove the stronger bound

$$\min\{a, b\} > (cn)^n.$$

For $i, j \in \{0, 1, \ldots, n\}$, let p_{ij} be a prime such that $p_{ij} | \gcd(a+i, b+j)$. Then, for every prime number $p \in \{p_{ij} \mid i, j \in \{0, 1, \ldots, n\}\}$, the total number of pairs (i, j) such that $p = p_{ij}$, is not greater than $\left\lceil \frac{n+1}{p} \right\rceil^2 < \left(\frac{n+1}{p} + 1\right)^2$. Let n be a large integer and $N = (n+1)^2$. Then, the total number of pairs (i, j) such that $p_{ij} \leq N$, is not greater than

$$\sum_{p \leq N} \left(\frac{n+1}{p} + 1\right)^2 < (n+1)^2 \sum_{p \text{ prime}} \frac{1}{p^2} + 2(n+1) \sum_{p \leq N} \frac{1}{p} + \sum_{p \leq N} 1.$$

It is known that

$$\sum_{p \text{ prime}} \frac{1}{p^2} = 0.45\ldots < \frac{1}{2} \quad \text{and} \quad \sum_{p \leq N} \frac{1}{p} = O(\log \log N),$$

see [1] and [2], respectively. Also,

$$\sum_{p \leq N} 1 = O\left(\frac{N}{\log N}\right)$$

by the prime number theorem. Therefore, for every sufficiently large n, the total number of pairs (i,j) such that $p_{ij} \leq N$, is less than $(n+1)^2/2$.

By the pigeonhole principle there is an index $i \in \{0,1,\dots,n\}$ such that for more than half of the numbers $j \in \{0,1,\dots,n\}$ the respective prime p_{ij} is greater than N. Let i_0 denote such an index i. The primes $p_{i_0 j}$ that are greater than N are distinct and, therefore, are coprime divisors of $a+i_0$. A similar argument holds for the number b. We deduce that $\min\{a,b\} \geq N^{(n+1)/2} - n > n^n$ for every n exceeding some bound n_0. Let $c = 1/n_0$. For $n < n_0$ we have $\min\{a,b\} \geq 1 > (cn)^n$. For $n \geq n_0$ we obtain $\min\{a,b\} > n^n \geq (cn)^n$. Hence the result.

References

[1] The on-line encyclopedia of integer sequences, published electronically at https://oeis.org, sequence A085548.

[2] Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer 2010, theorem 4.12, page 90.

FOCUS ON...

No. 23

Michel Bataille

Vieta's Formulas

Introduction

For $k = 1, 2, \ldots, n$, the kth elementary symmetric polynomial $e_k(X_1, X_2, \ldots, X_n)$ in the indeterminates $X_1, X_2, \ldots, X_n$ is defined by

$$e_k(X_1, X_2, \ldots, X_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} X_{i_1} X_{i_2} \cdots X_{i_k},$$

a sum of $\binom{n}{k}$ terms.

In particular, $e_1(X_1, X_2, \ldots, X_n) = X_1 + X_2 + \cdots + X_n$ and $e_n(X_1, X_2, \ldots, X_n) = X_1 X_2 \cdots X_n$.

If $P = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ is a polynomial with complex coefficients and degree n ($a_n \neq 0$), Vieta's formulas establish a link between the n roots $z_1, z_2, \ldots, z_n$ of P (counted with multiplicity) and the coefficients of P. They can be summarized as follows

$$e_k(z_1, z_2, \ldots, z_n) = (-1)^k \frac{a_{n-k}}{a_n} \qquad (k = 1, 2, \ldots, n),$$

and are easily deduced by expanding $P = a_n(X - z_1)(X - z_2) \cdots (X - z_n)$.

Note that $e_1(z_1, z_2, \ldots, z_n) = -\dfrac{a_{n-1}}{a_n}$ and $e_n(z_1, z_2, \ldots, z_n) = (-1)^n \dfrac{a_0}{a_n}$ generalize the familiar formulas obtained when the degree of P is 2.

We will show these formulas at work first in a few algebraic problems and, in a second part, in combination with inequalities.

Three algebraic problems

Our first problem was set at the 1997 Mathematical Olympiad in Bosnia and Hercegovina [2000 : 325-6 ; 2002 : 485]:

> Solve the system of equations in $\mathbb{R}^3$:
>
> $$8(x^3+y^3+z^3) = 73, \qquad 2(x^2+y^2+z^2) = 3(xy+yz+zx), \qquad xyz = 1.$$

Here is a variant of the featured solution: suppose that (z_1, z_2, z_3) is a solution. Then z_1, z_2, z_3 are the roots of the polynomial $P = X^3 - aX^2 + bX - 1$, where $a = z_1 + z_2 + z_3$ and $b = z_1 z_2 + z_2 z_3 + z_3 z_1$. Adding the three relations

$$z_k^3 - a z_k^2 + b z_k - 1 = 0, \quad k = 1, 2, 3,$$

we obtain

$$\frac{73}{8} - a\cdot\left(\frac{3b}{2}\right) + ba - 3 = 0,$$

so that $ab = \frac{49}{4}$. Since $a^2 - 2b = z_1^2 + z_2^2 + z_3^2 = \frac{3b}{2}$, we have $a^2 = \frac{7b}{2}$, hence $a^3 = \frac{7ab}{2} = \frac{7^3}{2^3}$. We deduce $a = b = \frac{7}{2}$ and

$$P = X^3 - \frac{7}{2}X^2 + \frac{7}{2}X - 1 = (X-1)(X-2)(X-\frac{1}{2}).$$

Thus, (z_1, z_2, z_3) is a permutation of $(1, 2, \frac{1}{2})$. Conversely, it is readily checked that each of the six permutations of $(1, 2, \frac{1}{2})$ is a solution.

We now consider an example involving a polynomial of degree 4 :

> Suppose that a and b are two of the roots of the polynomial X^4+X^3-1. Find a polynomial of which ab is a root.

Let c, d denote the complex roots other than a, b of $X^4 + X^3 - 1$. Vieta's formulas give:

$$a+b+c+d = -1,\ ab+ac+ad+bc+bd+cd = 0,\ abc+abd+bcd+acd = 0,\ abcd = -1,$$

or, adopting the notations $p = a + b, q = ab, r = c + d, s = cd$,

$$p + r = -1, \quad q + s + pr = 0, \quad qr + sp = 0, \quad qs = -1.$$

Note that $q \neq 0$ since 0 is not a root of $X^4 + X^3 - 1$. The elimination of p, r, s will provide a condition on q. Specifically, substituting $s = -\frac{1}{q}$ and $r = -1 - p$ in the two central relations, we obtain conditions on p and q, namely:

$$q^2 - 1 - pq(p+1) = 0, \qquad p + q^2(p+1) = 0.$$

Lastly, substituting $p = -\dfrac{q^2}{q^2+1}$ (obtained from the latter) in the first equality yields, after some algebra, $q^6 + q^4 + q^3 - q^2 - 1 = 0$ so that ab is a root of $X^6 + X^4 + X^3 - X^2 - 1$.

Vieta's formulas alone cannot provide the roots of the polynomial. However, accompanied with some extra information, they can be used in view of determining the roots. We give an example with a polynomial of degree 5 :

> Find the roots of $P = X^5 - 4X^4 + 9X^3 - 21X^2 + 20X - 5$, given that the product of two of the roots is 5.

Vieta's formulas are a bit more complicated in the case of the fifth degree than with polynomials of degree 3 or 4. However, if we denote by z_1, z_2, z_3, z_4, z_5 the roots of P and suppose without loss of generality that $z_1 z_2 = 5$, we are led to the following simplified form of Vieta's formulas:

$$\begin{aligned}
&z_3z_4z_5 = 1 \text{ (from } \ z_1z_2z_3z_4z_5 = 5),\\
&z_1 + z_2 + z_3 + z_4 + z_5 = 4,\\
&(z_1 + z_2)(z_3 + z_4 + z_5) + (z_3z_4 + z_4z_5 + z_3z_5) = 4,\\
&5(z_3 + z_4 + z_5) + (z_1 + z_2)(z_3z_4 + z_4z_5 + z_3z_5) = 20,\\
&z_1 + z_2 + 5(z_3z_4 + z_4z_5 + z_3z_5) = 20,
\end{aligned}$$

a clear invitation to set

$$a = z_1 + z_2, \ b = z_3z_4 + z_4z_5 + z_3z_5, \ c = z_3 + z_4 + z_5.$$

These numbers satisfy

$$a + c = 4, \ ac + b = 4, \ 5c + ab = 20, \ a + 5b = 20,$$

from which we successively deduce

$$b = 4 - a(4 - a) = a^2 - 4a + 4 = (a - 2)^2 \quad \text{and} \quad a + 5(a - 2)^2 = 20,$$

hence $a = 0$ or $a = \frac{19}{5}$. But the latter leads to $c = \frac{1}{5}$ (from $a+c = 4$), giving $b = 5$ (with $5c + ab = 20$), in contradiction with $b = (a - 2)^2 = \frac{81}{25}$. Thus, we must have $a = 0$ and then $b = 4 = c$.

To complete the solution, it just remains to remark that $z_1z_2 = 5$ and $z_1 + z_2 = 0$ yield the roots $i\sqrt{5}$ and $-i\sqrt{5}$, while z_3, z_4, z_5 are the roots of $X^3 - 4X^2 + 4X - 1$ (since $z_3 + z_4 + z_5 = 4, z_3z_4 + z_4z_5 + z_3z_5 = 4$ and $z_3z_4z_5 = 1$). The factorization

$$X^3 - 4X^2 + 4X - 1 = (X - 1)(X^2 - 3X + 1)$$

gives the three missing roots: 1, $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$.

Vieta's formulas and inequalities

Our first example, problem 766 of *The College Mathematics Journal*, offers a necessary condition for the roots of a polynomial of degree 3 in $\mathbb{R}[x]$ to be real and nonnegative.

> Suppose that the polynomial with real coefficients $A(z) = a_0 + a_1z + a_2z^2 + z^3$ has all its zeros real and nonnegative. Prove that
>
> $$9a_0^2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2.$$

Let x_1, x_2, x_3 denote the nonnegative zeros of $A(z)$, so that

$$a_2 = -(x_1 + x_2 + x_3), \quad a_1 = x_1x_2 + x_2x_3 + x_3x_1, \quad a_0 = -x_1x_2x_3.$$

The result is obvious if $x_1 = x_2 = x_3 = 0$. Otherwise, we have $x_1 + x_2 + x_3 > 0$ and, observing that $9a_0^2 + a_1^2a_2^2 - \frac{4}{3}a_1^3 - 6a_0a_1a_2$ is a homogeneous polynomial in

x_1, x_2, x_3, we may even suppose $x_1 + x_2 + x_3 = 1$. We are reduced to proving $9a_0^2 + a_1^2 - \frac{4}{3}a_1^3 + 6a_0a_1 \geq 0$ or

$$H^2 + 1 - \frac{4}{3}a_1 - 2H \geq 0 \qquad (*)$$

if we set $H = 3 \cdot \dfrac{x_1x_2x_3}{x_1x_2 + x_2x_3 + x_3x_1} \left(= \dfrac{-3a_0}{a_1}\right)$. Note that H is the harmonic mean of x_1, x_2, x_3.

The well-known inequality $x_1^2 + x_2^2 + x_3^2 \geq x_1x_2 + x_2x_3 + x_3x_1$ gives

$$1 = (x_1 + x_2 + x_3)^2 \geq 3(x_1x_2 + x_2x_3 + x_3x_1)$$

that is, $a_1 \leq \frac{1}{3}$, and therefore

$$H^2 + 1 - \frac{4}{3}a_1 - 2H \geq H^2 - 2H + \frac{5}{9} = \left(H - \frac{5}{3}\right)\left(H - \frac{1}{3}\right).$$

Now, $(*)$ follows from $H \leq \dfrac{x_1 + x_2 + x_3}{3} = \dfrac{1}{3}$.

In our next problem, again from *The College Mathematics Journal* (No 879), the advanced reader will recognize the Maclaurin inequalities: with the notation of our first part, if $z_1, z_2, \ldots, z_n$ are positive real numbers, then for $k = 1, 2, \ldots, n-1$,

$$\left(\frac{e_k(z_1, \ldots, z_n)}{\binom{n}{k}}\right)^{1/k} \geq \left(\frac{e_{k+1}(z_1, \ldots, z_n)}{\binom{n}{k+1}}\right)^{1/(k+1)},$$

a chain of inequalities from the arithmetic mean of $z_1, z_2, \ldots, z_n$ to their geometric mean.

The problem is stated as follows:

> Consider the polynomial $f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$, where a, b, c, and d are positive real numbers. Prove that if f has four positive distinct roots, then $a > b > c > d$.

Of course, the Maclaurin inequalities quickly give the answer. However the following direct solution offers a hint towards a general proof of the Maclaurin inequalities [for such a proof we refer the reader to **[1]** or **[2]**] .

Let x_1, x_2, x_3, x_4 be the four positive distinct roots of f with, $x_1 < x_2 < x_3 < x_4$. From Rolle's Theorem, the derivative $f'(x) = 4x^3 - 12ax^2 + 12b^2x - 4c^3$ has three positive distinct roots y_1, y_2, y_3 (with $x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < x_4$) and the second derivative $f''(x) = 12x^2 - 24ax + 12b^2$ has two positive distinct roots z_1, z_2.

Now, using Vieta's formulas and the arithmetic mean - geometric mean inequality, we successively obtain

$$a = \frac{z_1 + z_2}{2} > \sqrt{z_1 z_2} = b,$$
$$b = \left(\frac{y_1y_2 + y_2y_3 + y_3y_1}{3}\right)^{1/2} > \left(\sqrt[3]{y_1^2 y_2^2 y_3^2}\right)^{1/2} = \sqrt[3]{y_1y_2y_3} = c,$$
$$c = \left(\frac{x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4}{4}\right)^{\frac{1}{3}} > \left(\sqrt[4]{x_1^3x_2^3x_3^3x_4^3}\right)^{\frac{1}{3}} = \sqrt[4]{x_1x_2x_3x_4} = d,$$

so that $a > b > c > d$.

We complete this number with a couple of exercises.

Exercises

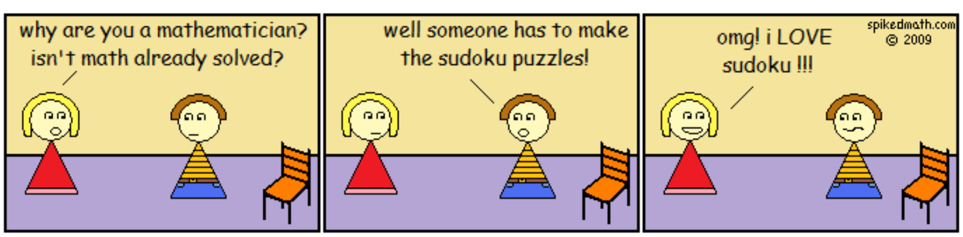
1. Given that the polynomial $X^3 - 5X + m$ has two roots z_1, z_2 such that $z_1 + z_2 = 2z_1z_2$, find the value of m and all the roots.

2. Let $Q(x) \in \mathbb{R}[x]$ and $P(x) = a + bx + cx^2 + x^3Q(x)$ where a, b, c are real numbers and $ac \neq 0$. Prove that if all the roots of P are real, then $b^2 > 2ac$. (Hint: if n is the degree of P, consider $x^nP(1/x)$.)

References

[1] R. Mosier, J. Nieto, Solution to problem 879, *The College Mathematics Journal*, Vol. 40, No 3, May 2009, p. 219.

[2] J. M. Steele, *The Cauchy-Schwarz Master Class*, Cambridge U. Press, 2006, chapter 12.



Latin squares and Sudoku

Peter J. Dukes

By now, virtually everyone has heard of Sudoku puzzles. In the standard puzzle, a 9×9 partially filled array must be completed with digits 1 to 9 so that every row, every column, and each of nine disjoint 3×3 'boxes' contains every digit exactly once. There are many ways to vary the puzzle, including different sizes (e.g. 4×4 mini-Sudoku or the 100×100 'Sudoku-zilla'), different boxes (6×6 with rectangular boxes, jigsaw Sudoku), puzzles with arithmetic constraints (Kakuro), and crossword-Sudoku hybrids using letters.

Here, we are interested in a relaxation which drops the condition on boxes. A *latin square* is an $n \times n$ array of n symbols (we often use the first n positive integers) such that every row and every column contains every symbol exactly once. For example, if our first row is filled $\boxed{1}\boxed{2}\boxed{3}\ldots\boxed{9}$ and we form eight more rows from different cyclic shifts, then the result is a 9×9 latin square. Note that it is not necessarily a filled Sudoku square unless we carefully arrange the shifted rows.

Problem 1 *On $\triangle ABC$ are given distinct points $L_1, \ldots, L_{10}$ on BC, $M_1, \ldots, M_{10}$ on AC, and $N_1, \ldots, N_{10}$ on AB. Show that there are 100 triangles $\triangle L_i M_j N_k$, no two of which share a common edge. Use a latin square.*

Latin squares have a rich history, appearing in art, agriculture, and statistics. Today, they are used in information-theoretic settings such as network routing, hash functions, and pseudo-random number generation. Research on the mathematics behind latin squares began with Euler in the 18th century and is still ongoing. Typical research involves counting or generating them, their existence (or non-existence) with various extra structure, and connections to other topics in combinatorics.

Regarding enumeration, the number $f(n)$ of $n \times n$ latin squares has been exactly determined for n up to 11; these values can be found at `http://oeis.org/A002860`. Asymptotically, it is known that $f(n)^{1/n^2} \sim e^{-2}n$.

As an example of extra structure, a *transversal* in a latin square is a set of cells in distinct rows and columns, and filled with distinct symbols; see Figure 1. Ryser's conjecture asserts that, for n odd, every $n \times n$ latin square has a transversal. This is still open, however.

1	5	3	4	**2**
3	1	2	**5**	4
5	**3**	4	2	1
2	4	**1**	3	5
4	2	5	1	3

Figure 1: A transversal

Problem 2 *Suppose* $a_1, a_2, \ldots, a_n$ *is a permutation of* $\{1, 2, \ldots, n\}$*, where* n *is even. For each* i*, put* $b_i \equiv a_i + i \pmod{n}$ *so that* $1 \leq b_i \leq n$*. Prove that* $b_1, b_2, \ldots, b_n$ *is not a permutation. Conclude that there is no transversal in a latin square of even size if its rows are cyclic shifts of each other.*

Two $n \times n$ latin squares are said to be *orthogonal* if, when superimposed, each of the n^2 possible pairings of symbols appears exactly once. A cute example for $n = 4$ is shown in Figure 2.

A♠	J♡	Q♣	K♢
J♣	A♢	K♠	Q♡
Q♢	K♣	A♡	J♠
K♡	Q♠	J♢	A♣

Figure 2: Orthogonal latin squares

It is easy to see that orthogonal latin squares each possess n disjoint transversals. Indeed, the transversals of one square are determined by the cells on which its orthogonal mate is constant. Orthogonal latin squares (more generally, sets of mutually orthogonal latin squares) are useful in the design of statistical experiments and are connected with other areas of discrete mathematics, such as finite geometries and extremal graph theory.

It is also possible to construct magic squares using orthogonal latin squares. Letting $N(n)$ denote the maximum size of a family of mutually orthogonal $n \times n$ latin squares, it is known that $N(n) \geq n^{1/14.8}$ for large n. It is presently a challenging open problem to find any improvement on the exponent.

Problem 3 *Show that* $N(n) \leq n - 1$ *for* $n \geq 2$*, and that equality holds when* n *is prime.*

There are many additional topics worth exploring from here, but let us now return to where we started and focus on the Sudoku puzzle, at least in spirit. Define a *partial latin square* as an $n \times n$ array, each of whose cells is either empty or contains one of n symbols, and such that no symbol appears twice in any row or column. See Figure 3.

1		4	
	3		
4		2	

Figure 3: A partial latin square

Naturally, a *completion* of a partial latin square P is a latin square L where P agrees with L on its nonblank cells. Figure 4 shows that even very sparsely filled partial latin squares may admit no completion.

2				
	1			
		1		
			$\ddots$	
				1

Figure 4: One with no completion

In studying the completion question, it is worth beginning with some special partial latin squares. For $1 \le k \le n$, a $k \times n$ *latin rectangle* is a $k \times n$ array of symbols from an n-element set such that every row contains all symbols exactly once, and every column contains k distinct symbols. An example 3×6 latin rectangle is shown on the left of Figure 5.

3	4	5	6	1	2
6	1	4	5	2	3
2	6	1	4	3	5

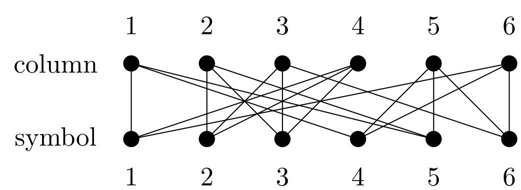


Figure 5: A latin rectangle and its graph of unused symbols

Theorem 1 *For $1 \le k \le n$, every $k \times n$ latin rectangle admits a completion to an $n \times n$ latin square.*

To prove Theorem 1, it is enough to show that a new row can be added when $k < n$. This latter fact comes from a special case of Phillip Hall's so-called 'marriage theorem', a major result in the early development of combinatorics.

In the language of graph theory, this theorem guarantees a perfect matching in any k-regular bipartite graph on $2n$ vertices. An example of such a graph with $k = 3$ and $n = 6$ is shown in the diagram. Lines, or 'edges' are drawn between each column and the symbols available for that column. The interested reader can find a perfect matching in the graph and use it to extend the given latin rectangle. More details on latin rectangles can be found in [6].

Problem 4 *Show that the number of different ways to complete an $(n-2) \times n$ latin rectangle is a power of two.*

It is also possible to complete smaller sub-rectangles under a mild hypothesis.

Theorem 2 (Ryser, 1956) *Let P be an $n \times n$ partial latin square in which the filled cells are precisely those of an $r \times s$ sub-array. Suppose that, in P, every symbol appears at least $r + s - n$ times. Then P admits a completion to an $n \times n$ latin square.*

The above can be reduced to the latin rectangle case by appending an $r \times (n - s)$ sub-array to the right of P.

Problem 5 *Let $n > m$ be positive integers. Use Theorem 1 to prove that an $m \times m$ latin square can be extended to an $n \times n$ latin square if and only if $n \geq 2m$.*

Problem 6 *Let P be any partial latin square. Prove that it is possible to erase at most three-quarters of the cells of P so that the resulting partial latin square P' admits a completion. Can the fraction of erased cells be lowered?*

We have seen that partial latin squares with as few as n filled cells can fail to admit completions. On the other hand, a celebrated result, previously known as Evans' conjecture, asserts that $n - 1$ or fewer filled cells can always be completed.

Theorem 3 (Smetaniuk, 1981) *Every $n \times n$ partial latin square with at most $n - 1$ filled cells admits a completion.*

The proof is a beautiful and surprisingly simple use of mathematical induction.

Problem 7 *Using Theorem 2, obtain an easy proof of the following weakening of Theorem 3: Every $n \times n$ partial latin square with at most $n/2$ filled cells admits a completion.*

As a sort of middle ground between Theorems 2 and 3, we may want to complete typical 'sparse' partial latin squares. For a positive real number ϵ, let us call an $n \times n$ partial latin square ϵ-*dense* if every row, column, and symbol is used at most ϵn times. It is helpful to think of n as very large. Recall that our partial latin square with no completion in Figure 4 over-used one symbol; this sort of thing is excluded by our sparseness condition, and the completion question is back in play.

Thresholds on ϵ for the completion of ϵ-dense latin squares have been a topic of interest over the past few decades. Daykin and Häggkvist conjectured in [4] that all 1/4-dense partial latin squares can be completed. The first serious progress toward this conjecture was by Chetwynd and Häggkvist, who showed in [3] that, for sufficiently large even integers n, $\epsilon = 10^{-5}$ suffices to guarantee a completion. Gustavsson [5] obtained the threshold $\epsilon = 10^{-7}$ for all n. These technical proofs required long chains of substitutions. Recently, Bartlett [2] obtained completions with $\epsilon = 10^{-4}$ for large n using a neat idea involving 'negatively occurring' symbols. In fact, he showed that completion is possible for densities near 1/12, but under a strong additional assumption on the total number of filled cells.

Problem 8 *Justify why $\epsilon = 1/4$ is a barrier for the sparse completion problem.*

Very recently, work of the author, along with a key theorem in [1], has shown that all large partial latin squares which are about 4%-dense have a completion. The proofs are constructive in principle, yet very technical. And we are still a long way from the barrier of 25%. What's worse, it is more honest to say that the squares to which it applies are not just large, but colossal beyond comprehension! Still, it represents a step toward a better understanding of the completion problem. The work actually addresses a series of more general questions in graph theory, drawing upon techniques from different areas of mathematics, including probability and linear algebra.

Mathematics is truly exciting when insightful solutions meet natural or universal questions, especially those that might be casually pondered by a curious non-mathematician. Science and technology have fed a steady diet of such questions to the mathematician. But it is fair to say that puzzles such as Sudoku, with their simplicity and broad appeal, offer another source of 'natural' questions. In some cases, even seemingly innocent puzzles are pushing the frontiers of mathematics.

References

[1] B. Barber, D. Kühn, A. Lo, D. Osthus and A. Taylor, Clique decompositions of multipartite graphs and completion of Latin squares. Preprint at `http://arxiv.org/abs/1603.01043`.

[2] P. Bartlett, Completions of ϵ-dense partial latin squares, *J. Combin. Des.* 21 (2013), 447–463.

[3] A.G. Chetwynd and R. Häggkvist, Completing partial $n \times n$ latin squares where each row, column and symbol is used at most cn times. *Reports, Dept. of Mathematics, University of Stockholm*, 1985.

[4] D.E. Daykin and R. Häggkvist, Completion of sparse partial latin squares, in: Graph Theory and Combinatorics: Proceedings of the Cambridge Conference in Honor of Paul Erdos. Academic Press, London, pp. 127–132, 1984.

[5] T. Gustavsson, Decompositions of large graphs and digraphs with high minimum degree. Doctoral Dissertation, Department of Mathematics, Stockholm University, 1991.

[6] J.H. van Lint and R.M. Wilson, A course in combinatorics, 2nd ed. Cambridge University Press, Cambridge, 2001.

Peter J. Dukes
Department of Mathematics and Statistics
University of Victoria
Victoria, BC, Canada
dukes@uvic.ca

PROBLEMS

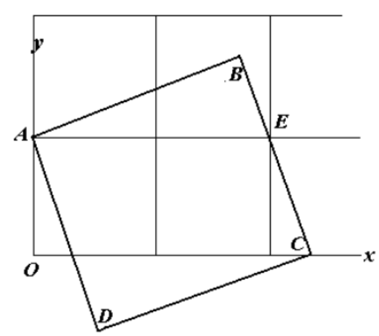
Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **April 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

4161. *Proposed by Peter Y. Woo.*

A high-school math teacher discovered some geometry problems while sliding a rug under his feet, over a floor with square tiles of length 1 unit. He chose x and y axes along two edges of some arbitrary tile. Today, he moved the square rug $ABCD$ of length between 1 and 2 units, so that A is on $(0,1)$ and C is on $(c,0)$ for some $c>2$. Then surprise! He noticed that the edge BC goes through the point $(2,1)$. Can you find $\angle BAE$?



4162. *Proposed by George Apostolopoulos.*

Let ABC be a triangle such that $\angle B = 2\angle C$. We extend the side BC by a segment CD equal to $\frac{1}{3}BC$. Prove that

$$Area(ABC) = \frac{1}{4}|BC|^2 \cdot \cot\frac{\theta}{2},$$

where $\theta = \angle BAD$.

4163. *Proposed by Leonard Giugiuc.*

Let a, b be real numbers with $0 < a < b$ and consider a positive sequence x_n such that

$$\lim_{n\to\infty}\left(ax_n + \frac{b}{x_n}\right) = 2\sqrt{ab}.$$

Find $\lim_{n\to\infty} x_n$ or show that it does not exist.

4164. *Proposed by G. Di Bona, A. Fiorentino, A. Moscariello, and G. G. N. Angilella.*

In an election, N voters are to elect k representatives. Each voter must indicate exactly m distinct preferences, with $m \leq k < N$. Every voter is a candidate themselves, and all candidates have a distinct age. The candidates are then ranked according to the number of votes received, and the k candidates who receive the largest number of votes are elected. In case of degeneracies, the eldest candidate is elected.

What is the minimum number of votes that a candidate should receive, in order to be sure to get elected?

4165. *Proposed by Daniel Sitaru.*

Prove that for all real numbers x_1, x_2, x_3 and x_4, we have,

$$|x_1 + x_2 + x_3 + x_4| + 2(|x_1| + |x_2| + |x_3| + |x_4|) \geq 6 \sqrt[6]{\prod_{1 \leq i < j \leq 4} |x_i + x_j|}.$$

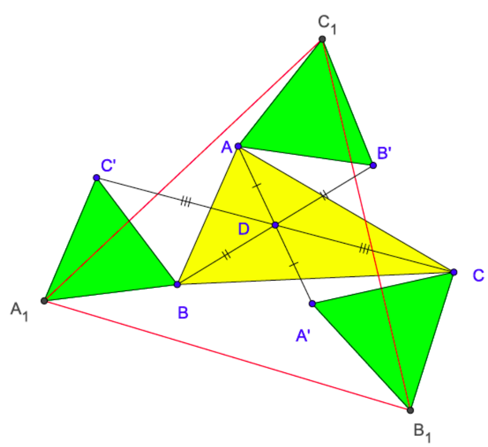
4166. *Proposed by Mihaela Berindeanu.*

Show that for all real numbers x, y and z, we have:

$$2^{4x-y} + 2^{4y-z} + 2^{4z-x} \geq 2^{x+2y} + 2^{y+2z} + 2^{z+2x}.$$

4167. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Consider triangle ABC and let D be any point in the plane. Let points A', B', C' be reflections of points A, B, C in D, respectively. Construct the 3 triangles $AB'C_1$, $CA'B_1$ and $BC'A_1$ outwardly as the given diagram indicates:



Show that $A_1B_1C_1$ is an equilateral triangle.

4168. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 18$ and $abc = 4$. Prove that

$$6 \leq a + b + c \leq 2\sqrt{2\sqrt{6} + 4} + \sqrt{6} - 2.$$

When does equality hold?

4169. *Proposed by Michel Bataille.*

Let a, b, c be positive real numbers. Prove that

$$\left(a\sqrt{\frac{b}{a+b}} + b\sqrt{\frac{c}{b+c}} + c\sqrt{\frac{a}{c+a}}\right)\left(b\sqrt{\frac{a+b}{b}} + c\sqrt{\frac{b+c}{c}} + a\sqrt{\frac{c+a}{a}}\right) \leq (a+b+c)^2.$$

4170. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

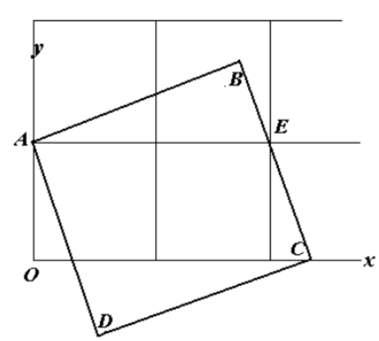
Let $ABCD$ be a circumscribed quadrilateral (that is, a quadrilateral for which an incircle can be constructed) and let P be the intersection point of AC and BD. Let h_a, h_b, h_c and h_d denote the distances from P to AB, BC, CD and DA, respectively. Prove that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{h_b + h_d}{h_a + h_c}.$$

. . . .

4161. *Proposé par Peter Y. Woo.*

Une enseignante du secondaire a pondu un problème de géométrie pendant qu'elle remuait un tapis sur un parquet avec ses pieds. Le parquet est recouvert de tuiles carrées avec des côtés de longueur 1. Elle choisit arbitrairement les axes des abscisses et des ordonnées le long de deux côtés d'une tuile particulière. Son tapis $ABCD$ est de forme carrée dont les côtés mesurent entre 1 et 2 unités. Elle place le coin A au point $(0, 1)$ et le coin C sur un point $(c, 0)$, où $c > 2$. Surprise ! Elle constate que le côté BC passe au point $(2, 1)$. Quelle est la mesure de l'angle BAE ?



4162. *Proposé par George Apostolopoulos.*

Soit ABC un triangle pour lequel $\angle B = 2\angle C$. On prolonge le côté BC jusqu'au point D de manière que $CD = \frac{1}{3}BC$. Démontrer que

$$Aire(ABC) = \frac{1}{4}|BC|^2 \cdot \cot\frac{\theta}{2},$$

où $\theta = \angle BAD$.

4163. *Proposé par Leonard Giugiuc.*

Soit a et b des réels tels que $0 < a < b$ et soit une suite x_n de réels strictement positifs telle que

$$\lim_{n\to\infty}\left(ax_n + \frac{b}{x_n}\right) = 2\sqrt{ab}.$$

Déterminer $\lim_{n\to\infty} x_n$ ou démontrer qu'elle n'existe pas.

4164. *Proposé par G. Di Bona, A. Fiorentino, A. Moscariello et G. G. N. Angilella.*

Dans une élection, N électeurs doivent choisir k représentants. Chaque électeur doit indiquer exactement m choix distincts, où $m \leq k < N$. Chaque électeur est aussi un candidat et chaque candidat a un âge distinct. Les candidats sont classés selon le nombre de votes qu'ils ont reçus et les k candidats qui ont reçu le plus grand nombre de votes sont élus. Dans le cas de dégénérescences, le candidat le plus âgé est élu.

Quel est le nombre minimal de votes qu'un candidat doit recevoir pour s'assurer d'être élu ?

4165. *Proposé par Daniel Sitaru.*

Démontrer que pour tous réels x_1, x_2, x_3 et x_4, on a

$$|x_1 + x_2 + x_3 + x_4| + 2(|x_1| + |x_2| + |x_3| + |x_4|) \geq 6\sqrt[6]{\prod_{1\leq i<j\leq 4}|x_i + x_j|}.$$

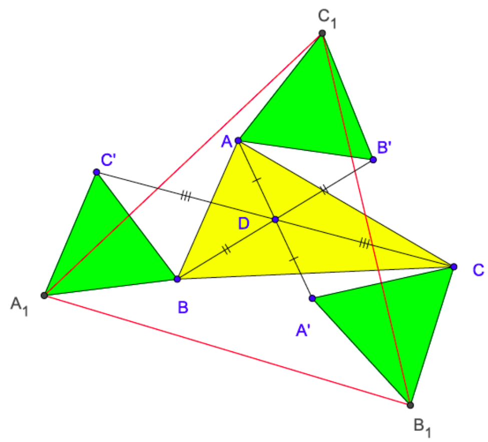
4166. *Proposé par Mihaela Berindeanu.*

Démontrer que pour tous réels x, y et z, on a

$$2^{4x-y} + 2^{4y-z} + 2^{4z-x} \geq 2^{x+2y} + 2^{y+2z} + 2^{z+2x}.$$

4167. *Proposé par Dao Thanh Oai and Leonard Giugiuc.*

On considère un triangle ABC et un point quelconque D dans le plan. Soit A', B' et C' les images respectives des points A, B et C par une réflexion par rapport à D. On construit les triangles $AB'C_1$, $CA'B_1$ et $BC'A_1$ vers l'extérieur, comme dans la figure suivante :



Démontrer que le triangle $A_1B_1C_1$ est équilatéral.

4168. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit a, b et c des réels positifs tels que $a^2+b^2+c^2 = 18$ et $abc = 4$. Démontrer que

$$6 \leq a+b+c \leq 2\sqrt{2\sqrt{6}+4}+\sqrt{6}-2.$$

Quand y a-t-il égalité ?

4169. *Proposé par Michel Bataille.*

Soit a, b et c des réels strictement positifs. Démontrer que

$$\left(a\sqrt{\frac{b}{a+b}}+b\sqrt{\frac{c}{b+c}}+c\sqrt{\frac{a}{c+a}}\right)\left(b\sqrt{\frac{a+b}{b}}+c\sqrt{\frac{b+c}{c}}+a\sqrt{\frac{c+a}{a}}\right) \leq (a+b+c)^2.$$

4170. *Proposé par Miguel Ochoa Sanchez et Leonard Giugiuc.*

Soit $ABCD$ un quadrilatère circonscriptible (c.-à-d. qui admet un cercle inscrit) et soit P le point d'intersection de AC et de BD. Soit h_a, h_b, h_c et h_d les distances respectives de P à AB, BC, CD et DA. Démontrer que

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{h_b+h_d}{h_a+h_c}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015 : 41(7), p. 302–305.

4061. *Proposed by Leonard Giugiuc.*

Let ABC be a non-obtuse triangle none of whose angles are less than $\frac{\pi}{4}$. Find the minimum value of $\sin A \sin B \sin C$.

We received twelve submissions, of which 10 were correct and one was faulty. We present two solutions.

Solution 1, by Adnan Ali.

Let us first fix angle A and determine the values of B and C for which the product $\sin A \sin B \sin C$ is smallest. Since A is fixed, this product is minimized if and only if

$$\sin B \sin C = \frac{\cos(B-C) - \cos(B+C)}{2} = \frac{\cos(B-C) - \cos(\pi - A)}{2}$$

is minimized. But $\cos(\pi - A)$ is fixed and so it is enough to minimize $\cos(B-C)$. Because the triangle is not obtuse and all angles are not less than $\pi/4$, we have $|B - C| \leq \pi/4$. Since the cosine function is decreasing over $[0, \pi/2]$, $\cos(B - C)$ is minimized if $|B - C|$ is maximum, and that happens when $B = \pi/4$ and $C = 3\pi/4 - A$, or vice-versa. So now we have reduced the problem of finding the minimum value of the given product to finding the minimum value of

$$\sin A \sin(\pi/4) \sin(3\pi/4 - A), \quad \pi/4 \leq A \leq \pi/2.$$

This is quickly done by minimizing $\sin A \sin(3\pi/4 - A) = \frac{\cos(3\pi/4-2A) - \cos(3\pi/4)}{2}$, which is same as minimizing $\cos(3\pi/4 - 2A)$, where $\pi/4 \leq A \leq \pi/2$. The bounds on A imply that $-\pi/4 \leq 3\pi/4 - 2A \leq \pi/4$, and so the minimum value of $\cos(3\pi/4 - 2A)$ is achieved for $3\pi/4 - 2A = -\pi/4$ or $\pi/4$; each of these values leads to an isosceles right triangle. Thus, the minimum value of $\sin A \sin B \sin C$ is $1/2$, achieved for an isosceles right triangle ABC.

Solution 2, by Daniel Dan.

We use the identity $\sin A \sin B \sin C = \frac{1}{4}(\sin 2A + \sin 2B + \sin 2C)$. Define

$$f(x) : \left[\frac{\pi}{2}, \pi\right] \to [0,1], \quad f(x) = \sin x,$$

and note that the function is concave; in particular, every point of the graph of $f(x)$ except for its end points, namely $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$, lies above the line

$$g(x) = -\frac{2}{\pi}x + 2$$

that joins those end points. Consequently, we have

$$\begin{aligned}\frac{1}{4}\left((f(2A)+f(2B)+f(2C)\right) &\geq \frac{1}{4}\left((g(2A)+g(2B)+g(2C)\right)\\ &= \frac{1}{4}\left(-\frac{2(2A+2B+2C)}{\pi}+6\right)=\frac{1}{2}.\end{aligned}$$

We conclude that the product $\sin A \sin B \sin C$ cannot be less than $\frac{1}{2}$ when all three angles are restricted to the domain $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. The minimum is achieved if and only if $f(x) = g(x)$ for x equal to $2A, 2B$, and $2C$; because $A + B + C = \pi$, this is possible only if one of the angles is $\frac{\pi}{2}$ while the other two are $\frac{\pi}{4}$.

4062. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let L_n denote the nth Lucas number defined by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Prove that

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n} \geq \frac{2}{3}L_{n+4}^2.$$

We received ten correct and complete solutions. We present the solutions of Arkady Alt, who like most submitters used standard inequalities for a simple proof, and a slightly modified version of the solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, who made heavier use of the given recursion to find a stronger bound.

Solution 1, by Arkady Alt.

Since $a^4 + b^4 \geq ab(a^2 + b^2)$ (as this can be rewritten as $(a^2 + ab + b^2)(a - b)^2 \geq 0$) and $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$ for all $a, b, c \in \mathbb{R}$, we have

$$\begin{aligned}&\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\\ &\geq \frac{L_nL_{n+1}(L_n^2+L_{n+1}^2)}{L_nL_{n+1}}+\frac{L_{n+1}L_{n+3}(L_{n+1}^2+L_{n+3}^2)}{L_{n+1}L_{n+3}}+\frac{L_{n+3}L_n(L_{n+3}^2+L_n^2)}{L_{n+3}L_n}\\ &= 2(L_n^2+L_{n+1}^2+L_{n+3}^2)\\ &\geq 2\frac{(L_n+L_{n+1}+L_{n+3})^2}{3}=\frac{2(L_{n+2}+L_{n+3})^2}{3}\\ &= \frac{2L_{n+4}^2}{3}\end{aligned}$$

Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

More generally, we will show that for all $n \geq 0$,

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n} > 2L_{n+4}^2.$$

We can check by hand that this holds for $n \leq 2$.

For $n \geq 3$ we first use the Arithmetic Mean - Geometric Mean inequality to obtain

$$\begin{aligned} x^4 + y^4 &= 2x^2y^2 + (x^2 - y^2)^2 \\ &= 2x^2y^2 + (x+y)^2(x-y)^2 \\ &\geq 2x^2y^2 + 4xy(x-y)^2 \\ &= xy(2xy + 4(x-y)^2) \end{aligned}$$

and hence

$$\frac{x^4 + y^4}{xy} \geq 2xy + 4(x-y)^2.$$

Using this property and the recursion for the Lucas numbers (multiple times, when necessary), we get

$$\frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} \geq 2L_n L_{n+1} + 4(L_{n+1} - L_n)^2 = 4L_{n+1}^2 - 6L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} \geq 2L_{n+1} L_{n+3} + 4(L_{n+3} - L_{n+1})^2 = 8L_{n+1}^2 + 10L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} \geq 2L_{n+3} L_n + 4(L_{n+3} - L_n)^2 = 16L_{n+1}^2 + 4L_{n+1}L_n + 2L_n^2,$$

and

$$2L_{n+4}^2 = 18L_{n+1}^2 + 24L_{n+1}L_n + 8L_n^2.$$

Combining these, we obtain

$$\begin{aligned} \frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} &\geq 28L_{n+1}^2 + 8L_{n+1}L_n + 10L_n^2 \\ &= 2L_{n+4}^2 + 10L_{n+1}^2 - 16L_{n+1}L_n + 2L_n^2 \\ &= 2L_{n+4}^2 - 6L_{n+1}L_n + 10L_{n+1}L_{n-1} + 2L_n^2 \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 6L_{n+1}L_{n-2} + 2L_n^2 \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 4L_{n+1}L_{n-2} + 2L_nL_{n-1} - 2L_{n-1}L_{n-2} \\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-3} + L_{n-1}^2 \\ &> 2L_{n+4}^2. \end{aligned}$$

4063. *Proposed by Marcel Chiriţă.*

Let a, b, c be real numbers greater than or equal to 3. Show that

$$\min\left(\frac{a^2b^2 + 3b^2}{b^2 + 27}, \frac{b^2c^2 + 3c^2}{c^2 + 27}, \frac{a^2c^2 + 3a^2}{a^2 + 27}\right) \leq \frac{abc}{9}.$$

We received six submissions all of which were correct. We present a composite of the similar solutions by Arkady Alt and Leonard Giugiuc.

Suppose to the contrary that

$$\min\left(\frac{a^2b^2+3b^2}{b^2+27}, \frac{b^2c^2+3c^2}{c^2+27}, \frac{a^2c^2+3a^2}{a^2+27}\right) > \frac{abc}{9}.$$

Then we have $\prod_{cyc} \dfrac{a^2b^2+3b^2}{b^2+27} > \dfrac{a^3b^3c^3}{9^3}$, so $\prod_{cyc} \dfrac{a^2+3}{a^2+27} > \dfrac{abc}{9^3}$.

But since $\frac{a}{9} - \frac{a^2+3}{a^2+27} = \frac{a^3-9a^2+27a-27}{9(a^2+27)} = \frac{(a-3)^2}{9(a^2+27)} \geq 0$, we have $\frac{a^2+3}{a^2+27} \leq \frac{a}{9}$.

Similarly, $\frac{b^2+3}{b^2+27} \leq \frac{b}{9}$ and $\frac{c^2+3}{c^2+27} \leq \frac{c}{9}$.

Hence, $\prod_{cyc} \dfrac{a^2+3}{a^2+27} > \dfrac{abc}{9^3}$ is a contradiction.

4064. *Proposed by Michel Bataille.*

In the plane of a triangle ABC, let Γ be a circle whose centre O is not on the sidelines AB, BC, CA. Let A', B', C' be the poles of the lines BC, CA, AB with respect to Γ, respectively. Prove that

$$\frac{OA' \cdot B'C'}{OA \cdot BC} = \frac{OB' \cdot C'A'}{OB \cdot CA} = \frac{OC' \cdot A'B'}{OC \cdot AB}.$$

We received five solutions, all correct, and present the solution by Joel Schlosberg, slightly modified by the editor.

One way to define the pole A' of the line BC with respect to the circle Γ is by reciprocation, namely A' is the inverse in Γ of the foot of the perpendicular from O to BC. [See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (The Mathematical Association of America, 1967), Section 6.1.] Conversely, if D is the inverse of A in Γ, then the polar of A, namely $B'C'$, is the line through D that is perpendicular to OA. We shall use three immediate consequences of this definition. If r is the radius of Γ, then $OA \cdot OD = r^2$, or

$$OA = \frac{r^2}{OD}. \tag{1}$$

Since $OB' \perp CA$ and $OC' \perp AB$, $\angle B'OC'$ is equal to or supplementary to $\angle BAC$. Let R be the circumradius of $\triangle ABC$. By the law of sines,

$$BC = 2R\sin\angle BAC = 2R\sin\angle B'OC'. \tag{2}$$

Finally, since each is the area of $\triangle OB'C'$,

$$\frac{1}{2}B'C' \cdot OD = \frac{1}{2}OB' \cdot OC' \sin\angle B'OC'. \tag{3}$$

Using in turn (1) and (2), then (3), we get

$$\frac{OA'\cdot B'C'}{OA\cdot BC}=\frac{OA'\cdot B'C'}{(r^2/OD)\cdot 2R\sin\angle B'OC'}=\frac{OA'}{2Rr^2}\cdot\frac{B'C'\cdot OD}{\sin\angle B'OC'}=\frac{OA'\cdot OB'\cdot OC'}{2Rr^2}.$$

The same reasoning shows that $\dfrac{OB'\cdot C'A'}{OB\cdot CA}$ and $\dfrac{OC'\cdot A'B'}{OC\cdot AB}$ are also equal to $\dfrac{OA'\cdot OB'\cdot OC'}{2Rr^2}$.

4065. *Proposed by Martin Lukarevski.*

Let ABC be a triangle with a, b, c as lengths of its sides and let R, r, s denote the circumradius, inradius and semiperimeter, respectively. Prove that

$$\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\geq\frac{2}{r}\left(\frac{1}{r}-\frac{1}{R}\right).$$

We received ten correct and complete solutions. We present the solution by the proposer.

We use the Garfunkel-Bankoff inequality (Problem 825, proposed by J. Garfunkel, solution by L. Bankoff, ***Crux*** 9 (1983), p.79 and 10 (1984), p.168) :

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\qquad(1)$$

which by the well-known identity

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}\qquad(2)$$

is equivalent to

$$\tan^2\frac{A}{2}+\tan^2\frac{A}{2}+\tan^2\frac{A}{2}\geq 2-\frac{2r}{R}.$$

By another well-known identity, which states that

$$\frac{1}{s-a}=\frac{1}{r}\tan\frac{A}{2},\qquad(3)$$

we have that

$$\begin{aligned}\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}&=\frac{1}{r^2}\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)\\&\geq\frac{1}{r^2}\left(2-\frac{2r}{R}\right)\\&=\frac{2}{r}\left(\frac{1}{r}-\frac{1}{R}\right),\end{aligned}$$

with equality, as in (1), only for the equilateral triangle.

4066. *Proposed by Mihaela Berindeanu.*

Prove that for $a, b, c > 0$ and $ab + ac + bc = 2016$,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{2019^2}{2016}.$$

We received 19 solutions all of which are correct. We present a composite of nearly identical solutions by Andrea Fanchini and Titu Zvonaru.

We prove the more general result that if $a, b, c > 0$ such that $ab+bc+ca = k$, then

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{(k+3)^2}{k}.$$

Note first that the trivial inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ implies

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}. \tag{1}$$

Using (1) together with AM-GM and AM-HM inequalities we then have

$$(a+\frac{1}{b})^2+(b+\frac{1}{c})^2+(c+\frac{1}{a})^2 = a^2+b^2+c^2+2(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$$\begin{aligned}
&\geq ab+bc+ca+2\cdot 3+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\\
&\geq k+6+\frac{9}{ab+bc+ca}\\
&= \frac{(k+3)^2}{k}
\end{aligned}$$

Editor's comments. Most of the other solutions used AM-GM, AM-QM and/or Cauchy Schwarz Inequalities. It is trivial to see that equality holds if and only if $a = b = c = \frac{\sqrt{3k}}{3}$.

4067. *Proposed by Mehtaab Sawhney.*

Consider a graph G such that between any three vertices in G there are either 0 or 2 edges. Classify all such graphs G.

We received seven correct and complete solutions. We present the solution by Joel Schlosberg.

We claim that a graph G satisfies the condition if and only if it is either an edgeless graph or a complete bipartite graph.

If G is edgeless, any three vertices have zero edges between them, so G trivially satisfies the condition. If G is a complete bipartite graph, the vertices of G can be partitioned into two sets S_1, S_2, such that two vertices are adjacent if and only if they are in different sets. If three vertices are in the same set, they have zero edges between them ; otherwise, two of them are in one set and one is in the other, leading to two edges between them. Thus G satisfies the condition.

Conversely, suppose that G satisfies the condition. If G is not edgeless, then there exist two vertices v_1, v_2 with an edge between them. For $k = 1, 2$, let S_k be the set of vertices that share an edge with v_k. Clearly v_1 is in S_2 but not S_1 and v_2 in S_1 but not S_2. If v is a vertex of G different from v_1 and v_2, then the three vertices v, v_1, v_2 must have exactly two edges between them, since they cannot have zero. Thus v is in exactly one of S_1 or S_2. Therefore S_1 and S_2 form a partition of the vertices of G.

Suppose v, w are vertices in the same set, say S_1. Then there is no edge between v and w, as otherwise we would have three edges between v, w and v_1. Now suppose $v \in S_1$ and $w \in S_2$. If $v = v_2$ then there is an edge between v and w by the definition of S_2. Otherwise consider the three vertices v, w and v_2. There is an edge between w and v_2 by the definition of S_2 and no edge between v and v_2, as just shown. Therefore there must be an edge between v and w. Thus we have proven that G is a complete bipartite graph.

4068. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{2a+3b+c} + \frac{b+2c}{a+2b+3c} + \frac{c+2a}{3a+b+2c} \leq \frac{3}{2}.$$

Editor's comments. We received 25 submissions all of which are correct. However, it was pointed out by Michael Bataille, and Dionne Bailey, Elsie Campbell, and Charles Diminnie that this problem is the same as Crux problem #4016 (by the same proposer) which appeared on p. 74 of ***Crux*** 41 (2). The only difference being that in #4016, it was asked to find the maximum value of the given expression while in #4068, it becomes a proof question with the maximum value given. So, it can not be viewed as a "variation". Two different solutions to #4016 given by Arkady Alt and Šefket Arslanagić have appeared on pp. 85-86 of ***Crux*** 42 (2).

4069. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(u_n)_{n\geq 0}$ be an arithmetic progression with a positive common difference d and with $u_1 > 0$. Let $(x_n)_{n\geq 0}$ be a sequence with $x_0 = 0, x_1 = x_2 = 1$ and

$$\sum_{k=1}^{n} u_k x_k = u_n x_{n+2} + d(x_4 - x_{n+3}) - x_2 u_1,\ \forall n \geq 0.$$

Prove that $(x_n)_{n\geq 0}$ is the Fibonacci sequence.

There were twelve correct solutions. All were variants of the ones below, with three using the closed form of the Fibonacci sums in Solution 2.

Solution 1.

When $n = 0$, the condition is that

$$0 = u_0 x_2 + d(x_4 - x_3) - u_1 x_2 = d(x_4 - x_3 - x_2),$$

whence $x_4 = x_3 + x_2$. When $n = 1$, we have that $u_1 x_1 = u_1 x_3 - u_1 x_2$, whence $x_3 = x_1 + x_2$. Since $x_0 = 0 = F_0$ and $x_1 = x_2 = 1 = F_1 = F_2$, then $x_3 = F_3$ and $x_4 = F_4$.

For $n \geq 1$, we have that

$$\begin{aligned} u_{n+1}x_{n+1} &= [u_{n+1}x_{n+3} + d(x_4 - x_{n+4}) - u_1 x_2] - [u_n x_{n+2} + d(x_4 - x_{n+3}) - u_1 x_2] \\ &= -dx_{n+4} + (u_{n+1} + d)x_{n+3} - (u_{n+1} - d)x_{n+2}, \end{aligned}$$

so that

$$dx_{n+4} = d(x_{n+3} + x_{n+2}) + u_{n+1}(x_{n+3} - x_{n+2} - x_{n+1}).$$

We establish the result by induction. Suppose that $x_k = F_k$ for $0 \leq k \leq n+3$. This is true for $n = 1$. The foregoing equation establishes that if $x_k = F_k$ for $k = n+1, n+2, n+3$, then $dx_{n+4} = dF_{n+4} + u_{n+1}(0)$ and $x_{n+4} = F_{n+4}$.

Solution 2.

The following Fibonacci relationships are easily established by induction for $n \geq 1$:

$$F_1 + F_2, + \cdots + F_n = F_{n+2} - 1 \quad \text{and} \quad \sum_{k=1}^{n} (k-1)F_k = (n-1)F_{n+2} - F_{n+3} + 3.$$

As in Solution 1, we show that $x_k = F_k$ for $0 \leq k \leq 4$. Suppose, as an induction hypothesis, this holds for $1 \leq k \leq n+2$. By the foregoing relationships, we have that

$$\begin{aligned} \sum_{k=1}^{n} u_k x_k &= \sum_{k=1}^{n} [u_1 + (k-1)d]F_k \\ &= u_1 \sum_{k=1}^{n} F_k + d \sum_{k=1}^{n} (k-1)F_k \\ &= u_1[F_{n+2} - 1] + d(n-1)F_{n+2} - dF_{n+3} + 3d \\ &= F_{n+2}[u_1 + (n-1)d] + d(3 - F_{n+3}) - u_1 \\ &= u_n F_{n+2} + d(F_4 - F_{n+3}) - u_1 F_2. \end{aligned}$$

However, the given condition provides that

$$\sum_{k=1}^{n} u_k x_k = u_n F_{n+2} + d(F_4 - x_{n+3}) - u_1 F_2.$$

Therefore $x_{n+3} = F_{n+3}$, and the result holds.

4070. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Compute

$$\lim_{n\to\infty}\left[\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan\,(n-1)}{2}+\arctan n\right)\right].$$

We received six correct and complete solutions. We present the solution by Joel Schlosberg.

Let $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Since $\arctan x$ is an increasing function,

$$\begin{aligned}&\frac{1}{\ln n}\left[\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right]\\ &\le \frac{1}{\ln n}\left(\frac{\arctan n}{n}+\frac{\arctan n}{n-1}+\cdots+\frac{\arctan n}{2}+\arctan n\right)\\ &= \frac{H_n}{\ln n}\cdot\arctan n;\end{aligned}$$

and for any positive integer m, if $n\ge m$,

$$\begin{aligned}&\frac{1}{\ln n}\left[\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right]\\ &\ge \frac{1}{\ln n}\left(\frac{\arctan m}{n-m+1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)\\ &\ge \frac{1}{\ln n}\left(\frac{\arctan m}{n-m+1}+\cdots+\frac{\arctan m}{2}+\arctan m\right)\\ &= \frac{H_{n-m+1}}{\ln n}\cdot\arctan m.\end{aligned}$$

It is well known that H_n is asymptotic to $\ln n$ and $\lim_{x\to\infty}\arctan x=\pi/2$. Hence,

$$\lim_{n\to\infty}\frac{H_n}{\ln n}\cdot\arctan n=\frac{\pi}{2}$$

and

$$\begin{aligned}&\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{H_{n-m+1}}{\ln n}\cdot\arctan m\right)\\ &=\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{\ln(n-m+1)}{\ln n}\cdot\arctan m\right)=\lim_{m\to\infty}\arctan m=\frac{\pi}{2}\end{aligned}$$

so by the squeeze theorem,

$$\lim_{n\to\infty}\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)=\frac{\pi}{2}.$$

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations