Crux Mathematicorum

VOLUME 42, NO. 2 February / Février 2016

Editorial Board

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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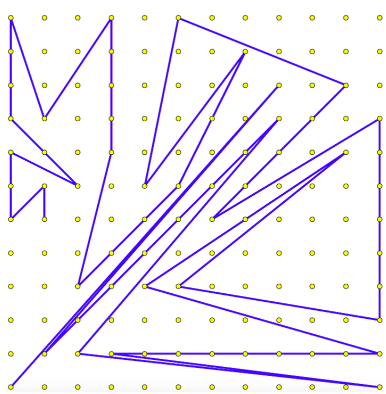
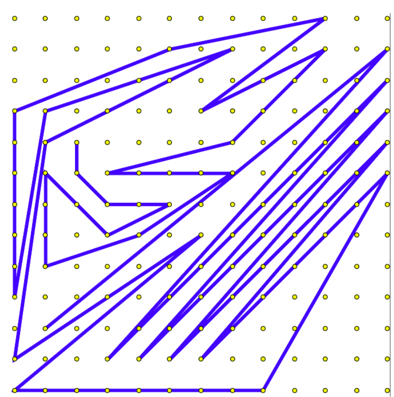
EDITORIAL

Often problems that are easy to state turn out to be hard to solve. But the less background required to understand the problem, the more tempting it is to try to break it: after all, it seems anyone should be able to do it. I see many examples of such problems in my outreach ventures and these puzzles often inspire my university classroom exercises. One particular puzzle I mentioned in my Editorial in ***Crux*** 40(8), called Ariadne's String (see `http://mathpickle.com`), goes even beyond that: it is an open problem and hence can be of interest for a problem solver of any level. In fact, my editorial prompted some work on the problem and the known results have been extended further. So here is the problem again:

Suppose you have an $n \times n$ grid with vertices at lattice points. The rules of the game are as follows: draw a continuous zigzag line, where each line segment starts and ends at lattice points; line segments cannot touch (even at a vertex) and each subsequent line segment must be longer than the previous one. The goal: get as many line segments in as possible. Try it out for yourself on small grids.

At the time of my editorial, the largest known result was for the 9×9 grid. Through correspondence with Stan Wagon, we now have the following extensions: Joseph DeVincentis has solved the problem for the 10×10, 11×11 and 12×12 grids, while Charles Greathouse computed the result for the 13×13 grid. Here are the pictures for the two largest grids:





So we have the following results for all square grids from 1×1 to 13×13:

$$0, 2, 4, 7, 9, 12, 15, 17, 20, 24, 27, 29, 33.$$

It is now sequence A226595 in *The On-Line Encyclopedia of Integer Sequences*. The next value is still unknown. Also unknown is the general formula or algorithm. I could probably figure it out, but, much like in Fermat's case, these margins are a bit too small to contain the proof.

Kseniya Garaschuk

THE CONTEST CORNER

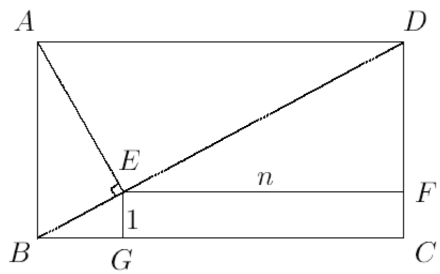
No. 42

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

CC206. Un rectangle $ABCD$ a une diagonale de longeur d. On abaisse une perpendiculaire AE à la diagonale BD. Le rectangle $EFCG$ a des côtés de longueurs n et 1. Démontrer que $d^{2/3} = n^{2/3} + 1$.



CC207. On considère les dix nombres $ar, ar^2, \ldots, ar^{10}$. Déterminer leur produit, sachant que leur somme est égale à 18 et que la somme de leurs inverses est égale à 6.

CC208.

a) Soit deux chiffres A et B. (A et B sont donc des symboles de 0 à 9 utilisés pour écrire les entiers.) Sachant que le produit des deux nombres de trois chiffres, $2A5$ et $13B$, est divisible par 36, déterminer les *quatre* couples (A, B) possibles. Justifier sa réponse.

b) Un entier n est un multiple de 7 si $n = 7k$ pour un entier quelconque k.

   i) Si a et b sont des entiers tels que $10a+b = 7m$ pour un entier quelconque m, démontrer que $a - 2b$ est un multiple de 7.

   ii) Si c et d sont des entiers tels que $5c+4d$ est un multiple de 7, démontrer que $4c - d$ est aussi un multiple de 7.

CC209.

a) Déterminer les deux valeurs de x qui vérifient $x^2 - 4x - 12 = 0$.

b) Déterminer *la* valeur de x qui vérifie $x - \sqrt{4x + 12} = 0$. Justifier sa réponse.

c) Déterminer toutes les valeurs réelles de c pour lesquelles l'équation

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

admet exactement deux racines réelles distinctes.

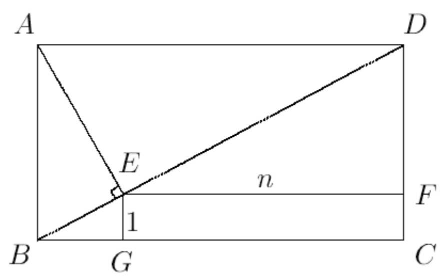
CC210. Il existe un unique triplet d'entiers strictement positifs (a, b, c) tel que $a \leq b \leq c$ et

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Déterminer la valeur $a + b + c$.

. . . .

CC206. A rectangle $ABCD$ has diagonal of length d. The line AE is drawn perpendicular to the diagonal BD. The sides of the rectangle $EFCG$ have lengths n and 1. Prove that $d^{2/3} = n^{2/3} + 1$.



CC207. Consider the ten numbers $ar, ar^2, \ldots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6, determine their product.

CC208.

a) Let A and B be digits (that is, A and B are integers between 0 and 9 inclusive). If the product of the three-digit integers $2A5$ and $13B$ is divisible by 36, determine with justification the *four* possible ordered pairs (A, B).

b) An integer n is said to be a multiple of 7 if $n = 7k$ for some integer k.

   i) If a and b are integers and $10a + b = 7m$ for some integer m, prove that $a - 2b$ is a multiple of 7.

ii) If c and d are integers and $5c + 4d$ is a multiple of 7, prove that $4c - d$ is also a multiple of 7.

CC209.

a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.

b) Determine the *one* value of x such that $x - \sqrt{4x + 12} = 0$. Justify your answer.

c) Determine all real values of c such that

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

has precisely two distinct real solutions for x.

CC210. There is a unique triplet of positive integers (a, b, c) such that $a \leq b \leq c$ and

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Determine $a + b + c$.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015 : 41(2), p. 48–50.

CC156. Describe and accurately sketch the region

$$\{(x, y, z) : |x| + |y| \leq 1, |y| + |z| \leq 1, |z| + |x| \leq 1\}.$$

Originally problem 2 of the 2014 Science Atlantic Math Contest.

We received two incorrect submissions and no correct solutions.

CC157. Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 submatrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Originally problem 3 of the 2014 Science Atlantic Math Contest.

We received four correct submissions. We present the solution given by Kathleen Lewis.

Suppose that a 5×5 matrix of zeroes and ones contains no 2×2 submatrix of zeroes. We will show that it must contain a submatrix of ones. Let A be the column with the fewest ones, or one of those columns if there are several with the same smallest number of ones, and let n be the number of ones in column A.

Case 1 : $n = 0$. In this case, no other column can contain more than one zero, since otherwise that column and column A would share a 2×2 submatrix of zeroes. Then any two columns other than A must each have at least 4 ones, so they share at least three rows of ones. Thus, there are 2×2 submatrices of ones using any pair of columns besides A.

Case 2 : $n = 1$. If column A contains only one one, then it has zeroes in 4 rows. In those four rows, no other column will contain a pair of zeroes, so each of the other columns must have at least 3 ones in those four rows. Therefore any two columns (excluding A) will have ones in two of the same rows, so again we have a 2×2 submatrix of ones.

Case 3 : $n = 2$. Each of the other columns must contain at least two ones in the rows in which A has zeroes. If any column has three ones in these rows, then it will share a 2×2 submatrix of ones with any of the others. If not, each of these four column has two ones among the three rows. But there are only $\binom{3}{2} = 3$ ways to do this, so two columns must have the same arrangement. Then these two contain a 2×2 submatrix of ones.

Case 4 : $n \geq 3$. In this case, each column has at least three ones. So, given any two columns, they either share two rows with ones, or between them have at least one one in each row. The first arrangement gives us the 2×2 submatrix of ones that we are looking for. In the second arrangement, any third column must share two rows of ones with one of the original two columns, again giving us a 2×2 submatrix of ones.

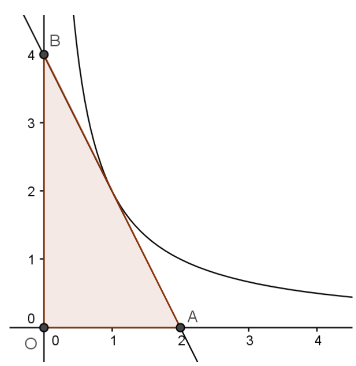
Therefore, in every case, if we don't have a 2×2 submatrix of zeroes, we must have a 2×2 submatrix of ones.

CC158. Suppose movable points A, B lie on the positive x-axis and y-axis, respectively, in such a way that $\triangle ABO$, where O is the origin, always has area 4. Find an equation for a curve in the first quadrant which is tangent to each of the line segments AB.

Originally problem 6 of the 2014 Science Atlantic Math Contest.

We received three solutions, two of which were completely correct. We present the solution by Andrea Fanchini.

We consider a generic point A that lies on the positive x-axis, at $(t, 0)$. If the triangle ABO has area 4 then the point B must be at $(0, 8/t)$. The family of lines AB thus satisfy the equation $y = -\frac{8}{t^2}(x - t)$, or $f(x, y, t) = t^2y + 8x - 8t = 0$.



The envelope of this family of lines is defined as the set of points for which

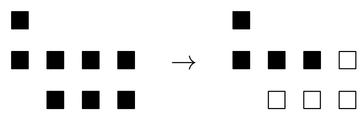
$$\begin{cases} f(x, y, t) = 0, \\ \dfrac{\partial f(x, y, t)}{\partial t} = 0, \end{cases}$$

so we have

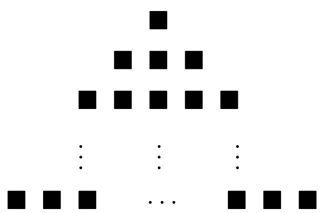
$$\begin{cases} t^2y + 8x - 8t = 0, \\ 2yt - 8 = 0. \end{cases}$$

Solving, we obtain the equation of the envelope that is the hyperbola $xy = 2$.

CC159. The following pattern of eight square tiles can be divided into two congruent sets of four tiles as shown. (Note that one set is the mirror image of the other — this is legal.)



Find a way to divide the following pattern of 100 tiles into two congruent sets of fifty tiles, or show it cannot be done.



(19 tiles in the 10th row)

Originally problem 2 of the 2015 Science Atlantic Math Contest.

We received no submissions to this problem. We present a solution sketch by the editor.

First, prove the following two facts :

- The bottom two corners are the only two tiles with one coordinate equal and the other differing by 18.
- The only pairs in which both coordinates differ by 9 consists of the top corner and one of the bottom corners.

Now, suppose for a contradiction that a partition exists. By the pigeonhole principle, one set contains two of the three corners. By the two facts above, the other set cannot contain two tiles equivalently situated.

CC160. Find all triples of continuous functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f(g(x)) = g(h(x)) = h(f(x)) = x\ .$$

Originally problem 3 of the 2015 Science Atlantic Math Contest.

We received one correct solution. We present the solution by Konstantine Zelator.

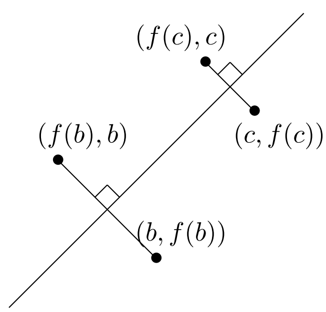
First, we observe that $f(g(x)) = x$ implies that $g(x) = f^{-1}(x)$. Furthermore, $g(h(x)) = x$ implies that $h(x) = g^{-1}(x)$. Combining these facts, $h(x) = (f^{-1})^{-1}(x) = f(x)$. So then $x = h(f(x)) = f(f(x))$. So f (and by similar reasoning, g and h) is its own inverse. Furthermore, $f = g = h$.

We claim that $y = f(x)$ intersects $y = x$ at least once, and if it intersects more than once then $f(x) = x$. If it intersects $y = x$ exactly once then it is a line of the form $f(x) = -x + k$ for some $k \in \mathbb{R}$.

Suppose that $f(x) \neq x$. Then we can find a so that $(a, f(a))$ is not on the line $y = x$. But since f is its own inverse it follows that $f(f(a)) = a$ and so $(f(a), a)$ is on $y = f(x)$. But these points are on opposite sides of $y = x$, so by the continuity of $f(x)$ it must intersect with $y = x$ at least once.

Now, suppose that $y = f(x)$ intersects $y = x$ more than once, say at points (a, a) and (b, b) with $a < b$. Since it is invertible, f must be one-to-one, and hence increasing or decreasing. In this case, f is clearly increasing. Suppose that there is some $(c, f(c))$ not on $y = x$. Then $c < f(c)$, but as we've stated before, $(f(c), c)$ is also on the curve, contradicting its increasing property. Thus no c can exist and every point must lie on $y = x$. So $f(x) = x$.

If, on the other hand, $y = f(x)$ and $y = x$ intersect only at one point, say (a, a) then by considering another point $(b, f(b))$ with $b \neq f(b)$ we can conclude that f is decreasing since $(f(b), b)$ is also on $y = f(x)$. Consider a third point, $(c, f(c))$ with $c \neq f(c)$. If $(b, f(b))$ and $(c, f(c))$ don't fall on a line perpendicular to $y = x$ then we can see, by considering the four points $(b, f(b))$, $(c, f(c))$, $(f(b), b)$, and $(f(c), c)$ that f will not be decreasing, a contradiction.



So all the points on $y = f(x)$ must lie on a line perpendicular to x. Therefore, $f(x) = -x + 2a$.

In summation, $f(x) = g(x) = h(x) = x$ or $-x + k$ for some $k \in \mathbb{R}$.

THE OLYMPIAD CORNER

No. 340

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC266. Soit D un point sur le côté BC d'un triangle acutangle ABC. Soit O_1 et O_2 les centres respectifs des cercles circonscrits aux triangles ABD et ACD. Démontrer que la droite qui joint le centre du cercle circonscrit au triangle ABC et l'orthocentre du triangle O_1O_2D est parallèle à BC.

OC267. On a empilé des disques rouges et des disques bleus de même grandeur de manière à former une pile de forme triangulaire. Le niveau supérieur de la pile compte un disque et chaque niveau compte un disque de plus que le niveau immédiatement au-dessus. Chaque disque qui n'est pas au niveau le plus bas touche à deux disques au-dessous de lui et ce disque est bleu si les deux disques sont de la même couleur. Autrement, il est rouge.

Supposons que le niveau le plus bas compte 2048 disques dont 2014 sont rouges. Quelle est la couleur du disque au niveau supérieur ?

OC268. Soit $\mathbb{Z}_{\geq 0}$ l'ensemble des entiers supérieurs ou égaux à 0. Déterminer toutes les fonctions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ qui vérifient la relation

$$f(f(f(n))) = f(n+1) + 1$$

pour tout $n \in \mathbb{Z}_{\geq 0}$.

OC269. Soit x, y, z les nombres réels qui satisfont à

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 8 \quad \text{et} \quad x^3 + y^3 + z^3 = 1.$$

Déterminer la valeur minimale de $x^4 + y^4 + z^4$.

OC270. Étant donné un entier pair strictement positif n, on place chacun des nombres $1, 2, \ldots, n^2$ sur une des cases d'un damier $n \times n$. Soit S_1 la somme des nombres placés sur les cases noires et S_2 la somme des nombres placés sur les cases blanches. Déterminer tous les n pour lesquels il est possible d'obtenir $\dfrac{S_1}{S_2} = \dfrac{39}{64}$.

. . . .

OC266. In an acute triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

OC267. Blue and red circular disks of identical size are packed together to form a triangle. The top level has one disk and each level has 1 more disk than the level above it. Each disk not at the bottom level touches two disks below it and its colour is blue if these two disks are of the same colour. Otherwise its colour is red.

Suppose the bottom level has 2048 disks of which 2014 are red. What is the colour of the disk at the top ?

OC268. Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

OC269. Let x, y, z be the real numbers that satisfy the following :

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 8, x^3 + y^3 + z^3 = 1.$$

Find the minimum value of $x^4 + y^4 + z^4$.

OC270. For even positive integer n we put all numbers $1, 2, ..., n^2$ into the squares of an $n \times n$ chessboard (each number appears once and only once). Let S_1 be the sum of the numbers put in the black squares and S_2 be the sum of the numbers put in the white squares. Find all n such that we can achieve $\dfrac{S_1}{S_2} = \dfrac{39}{64}$.

OLYMPIAD SOLUTIONS

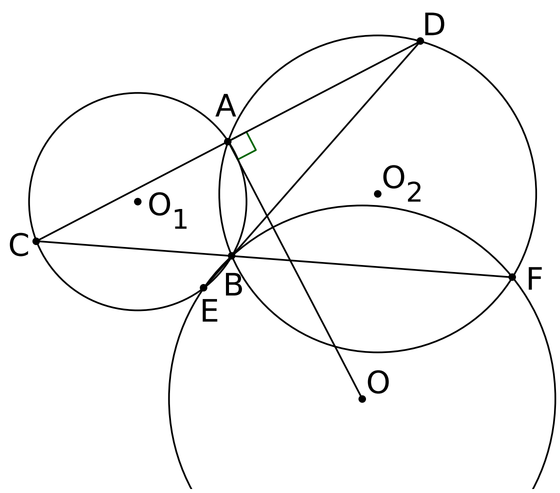
Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(10), p. 417–419.

OC206. Two circles K_1 and K_2 of different radii intersect at two points A and B, and let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let ℓ_1 and ℓ_2 be the perpendicular bisectors of CD and EF, respectively.

1. Show that ℓ_1 and ℓ_2 have a unique common point (denoted by P).
2. Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

Originally problem 1 of the 2013 China National Olympiad.

We received two correct submissions. We present the solution by Oliver Geupel.



Let K be the circumcircle of $\triangle BEF$. Let O, O_1, O_2, r, r_1, and r_2 be the centres and radii of K, K_1, and K_2, respectively. For a point X and a circle Γ, let $\mathcal{P}(X,\Gamma)$ denote the power of X with respect to Γ. We have $O \in \ell_2$ and

$$\mathcal{P}(C,K) = \mathcal{P}(C,K_2) = CA \cdot CD = DA \cdot DC = \mathcal{P}(D,K_1) = \mathcal{P}(D,K).$$

We deduce $CO = DO$; whence $O \in \ell_1$. Thus $O \in \ell_1 \cap \ell_2$.

To complete part 1, it is enough to show that $\ell_1 \neq \ell_2$. We prove it by contradiction. Suppose to the contrary that $\ell_1 = \ell_2$. Then $CD \parallel EF$, so that $\triangle BCD$ and $\triangle BFE$ are homothetic. Hence the points A, B, and the midpoint G of EF lie on a common line l. But A, G and O lie on $\ell_1 = \ell_2$. It follows $l = \ell_1$, so that $B \in \ell_1$. As a consequence CD is parallel to O_1O_2. If the distance of the lines is d, we obtain

$$r_1^2 = d^2 + \frac{AC^2}{4} = d^2 + \frac{AB^2}{4} = r_2^2,$$

which contradicts the hypothesis $r_1 \neq r_2$. Part 1 is complete.

Observe that

$$CO^2 - r^2 = \mathcal{P}(C, K) = \mathcal{P}(C, K_2) = 2CA^2.$$

In the right triangle ACO we have $AO^2 + CA^2 = CO^2 = 2CA^2 + r^2$. Consequently, $AO^2 = CA^2 + r^2 = CA^2 + OE^2$. By the converse of the Pythagorean Theorem, CA, AO, and OE are the side lengths of a right triangle. This completes part 2.

OC207. Find all injective functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy :

$$|f(x) - f(y)| \leq |x - y|$$

for any $x, y \in \mathbb{Z}$.

Originally problem X-3 of the 2013 Romanian National Olympiad.

We received four correct submissions. We present the solution by Michel Bataille.

We show that the solutions are the functions $x \mapsto x + a$ and $x \mapsto -x + a$ where a is an arbitrary integer.

Such a function is clearly a solution. Conversely, let f be any solution and let $g : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $g(x) = f(x) - f(0)$. Then g is injective, satisfies $|g(x) - g(y)| \leq |x - y|$ for any $x, y \in \mathbb{Z}$ and in addition, $g(0) = 0$. Thus, we may as well suppose that $f(0) = 0$ from the beginning and show that $f(x) = x$ for all $x \in \mathbb{Z}$ or $f(x) = -x$ for all $x \in \mathbb{Z}$.

Let f be a solution such that $f(0) = 0$. Then, $|f(x)| \leq |x|$ for any integer x and in particular $|f(1)| \leq 1$. In addition, since f is injective, we have $f(1) \neq f(0)$, that is, $f(1) \neq 0$. It follows that $f(1) = 1$ or $f(1) = -1$.

First, we suppose that $f(1) = 1$. Assume that for some positive integer n, we have $f(k) = k$ for each element k of $\{0, 1, \ldots, n\}$. Then, from $|f(n+1) - f(n)| \leq |(n+1) - n| = 1$ and $f(n+1) \neq f(n)$, we deduce that $f(n+1) - f(n) = 1$ or -1. However, $f(n+1) - f(n) = -1$ implies $f(n+1) = n - 1 = f(n-1)$, contradicting f injective. Thus, $f(n+1) = n+1$ and so $f(k) = k$ for each element k of $\{0, 1, \ldots, n+1\}$. By induction, we have proved that for any positive integer n, we have $f(k) = k$ for each element k of $\{0, 1, \ldots, n\}$ and in particular, $f(n) = n$.

The function $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(x) = -f(-x)$ is injective, satisfies $|h(x) - h(y)| \leq |x - y|$ for any $x, y \in \mathbb{Z}$ and $h(0) = 0$. Thus, $h(n) = n$ for any positive integer n, which means that $f(-n) = -n$ for any positive integer n. Gathering the results, we see that $f(x) = x$ for any integer x.

In the case when $f(1) = -1$, from what has been already obtained, the function $-f$ satisfies $(-f)(x) = x$ for any integer x, hence $f(x) = -x$ for any integer x.

OC208. Find all non-integers x such that $x+\frac{13}{x}=\lfloor x\rfloor+\frac{13}{\lfloor x\rfloor}$ where $\lfloor x\rfloor$ means the greatest integer n less than or equal to x.

Originally problem 5 of the 2013 China Northern Mathematical Olympiad.

We received seven correct submissions. We present the solution by Digby Smith.

Let $x = m + a$ with $m = \lfloor x\rfloor$ and $a \in \mathbb{R}$ between 0 and 1. Note that if $0 < x < 1$, then $m = 0$ and there is no solution (the right hand side above is undefined). Thus, suppose that $m \neq 0$. Substituting into the above equation yields

$$m+a+\frac{13}{m+a}=m+\frac{13}{m}$$

Simplifying yields

$$m(m+a)=13$$

We proceed in cases. When $m \geq 4$, we see that $m(m+a) \geq 16$ which is a contradiction. When $m \in \{1, 2, 3\}$ then $m(m+a) < m(m+1) \leq 12$, also a contradiction. Now, if $m \leq -5$ then $m(m+a) > 20$ and once again there is no solution. For $m \in \{-1, -2, -3\}$, we see that $m(m+a) < m^2 \leq 9$, also a contradiction. Thus, this leaves only the case $m = -4$. Substituting this into the equation gives

$$(-4)(-4+a)=13 \qquad \Rightarrow \qquad a=3/4$$

Hence, $x = m + a = -4 + 3/4 = -13/4$ and this is the only solution.

OC209. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$ all have different remainders when divided by $n+1$?

Originally problem 2 of the 2013 Canadian Mathematical Olympiad.

We present the solution by The Missouri State University Problem Solving Group. There were no other submissions.

Since for any arrangement we have $\displaystyle\sum_{i=1}^{n} a_i = \frac{n(n+1)}{2}$, if n is even, then this sum leaves a remainder of 0 when divided by $n+1$ meaning that this case is impossible in this case. Thus, suppose that n is odd. Consider the arrangement given by

$$1, n-1, 3, n-3, 5, n-5, \cdots, n.$$

This arrangement satisfies the given criteria. Indeed, observe that the sequence modulo $n+1$ is equivalent to

$$1, -2, 3, -4, 5, -6, \cdots, n$$

and so the sequence $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$ modulo $n+1$ is equivalent to

$$0, 1, -1, 2, -2, 3, -3, \cdots, (n+1)/2.$$

OC210.

Find all positive integers a such that for any positive integer $n \geq 5$ we have $2^n - n^2 \mid a^n - n^a$.

Originally problem 8 of the 2013 China Western Mathematical Olympiad.

We received no submissions to this problem.

Math Quotes

In the mathematics I can report no deficience, except that it be that men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect it maketh a quick eye and a body ready to put itself into all postures; so in the mathematics, that use which is collateral and intervenient is no less worthy than that which is principal and intended.

Roger Bacon in John Fauvel and Jeremy Gray (eds.) "A History of Mathematics : A Reader", Sheridan House, 1987.

Limits before epsilons and deltas

Margo Kondratieva

Introduction

It has been observed that when somebody, in a casual chat mentions that their job is related to teaching or research in mathematics, this statement often kills the conversation. One of the reasons is that mathematics tends to be very formal, in order to be rigorous and precise in its conclusions. While this is a norm for a mathematician, other people often cannot grasp the essence behind formal derivations and quickly lose interest with the subject. Most regretfully, this happens to students attending mathematics classes. For example, many students of calculus who have seen the formal definition of the limit in terms of epsilon and delta can start the sentence, "For any positive epsilon, there exists a delta such that ...". Unfortunately, not so many can complete the definition in a meaningful way. If people do not understand mathematics, they get bored, frustrated, and are reluctant to discuss it.

However, there are some mathematical statements that produce quite an opposite effect. Try for example, asking your friend whether the following is true or false :

$$0.999\ldots = 1. \tag{1}$$

Here the left-hand side represents an infinite decimal fraction while the right-hand side is just a natural number 1. Or you can enquire whether the equality of the following infinite nested square root and the number holds :

$$\sqrt{4\sqrt{4\sqrt{4\ldots}}} = 4. \tag{2}$$

Before reading further you are advised to respond to these questions yourself and then compare with our discussion below. When answering the second question, it is also useful to think about the following generalization : for which values of a, b and n does the following equality hold ?

$$\sqrt[n]{a\sqrt[n]{a\sqrt[n]{a\ldots}}} = b \tag{3}$$

It is not obvious that the left- and right-hand sides of the equation (1) or (2) are equal, and you might sense it from people's responses based on their intuition. The popular belief that both statements are false is contradictory to the result obtained in calculus. Well, at least people are willing to form and defend their opinion about a mathematical equation !

From a psychological point of view, the expression on the left-hand side is infinitely long, so it never stops. People tend to consider it as an infinite process of writing either 9's or $\sqrt{4}$'s. This is obviously quite different from writing a single digit on the right-hand side. However, the infinite process can be encapsulated to form an object. If we ask ourselves about the value of the infinitely long expression, we must see beyond the infinite process towards its completion. Since the time of Aristotle, philosophers distinguish the potential and actual infinity. This distinction becomes relevant to our examples. The process of writing either 9's or $\sqrt{4}$'s is potentially infinite as we can keep writing but will never have it completed. At the same time grasping the final result of this process as a totality corresponds to Aristotle's notion of the actual infinity, as it refers to the expression perceived as truly containing infinitely many 9's or $\sqrt{4}$'s.

One may ask, what all these psychological and philosophical considerations have to do with mathematical analysis? It appears that they are extremely important for our discussion about limits. But as promised in the title, we will try to do it less formally, by means that avoid the epsilon and delta technique.

The Heine approach

Our first example is a good starting point for talking about infinite sequences. If we do not want to rely on our intuition with infinitely long decimals and trust only decimals that have finite representation, we may look at the numbers 0.9, 0.99, 0.999, etc. What we have here is a sequence of numbers $S(n)$ each of which has exactly n 9's, where $n = 1, 2, 3, \ldots$. We want to know the value of the expression with infinitely many 9's. In mathematical language, we need to find the limit of the sequence $S(n)$ when n approaches infinity.

We will discuss a method in the spirit of Heine. But before we do, we need to explain a couple of mathematical notions. First, an interval on a number line that includes all points x between numbers A and B is *open*, if it does not include the endpoints A and B. It is denoted by (A, B). Thus, x belongs to an open interval (A, B) means $A < x < B$. For example, all numbers that satisfy the inequalities $1 < x < 2$ form the open interval $(1, 2)$. Second, we say that *almost all* terms of a sequence $S(n)$, $n \geq 1$ belong to an open interval (A, B) if all but a finite number of terms of the sequence satisfy the inequality $A < a_n < B$.

German mathematician Heinrich Eduard Heine, who lived in the 19th century, proposed the following definition of the limit. His approach is equivalent to the one found in standard calculus textbooks, but it avoids epsilon-delta language.

Definition. A number S is called the *limit* of a sequence $S(n)$ as n approaches infinity if **every** open interval containing S also contains almost all terms of the sequence.

We can use this definition to show that $S = 1$. Observe that every open interval that contains 1 has the form $(1 - a, 1 + b)$ for some $a > 0$ and $b > 0$. Now, take $a > 0$ and choose a natural r such that $10^{-r} < a$. Then, $S(r) = 1 - 10^{-r} > 1 - a$. From the above relation, we can also see that as n increases, the value of $S(n)$ grows as well, particularly, $1 - a < S(r) < S(r + 1) < S(r + 2) < \cdots$. In addition, note that $S(n) < 1$ for all $n \geq 1$. Thus, every open interval $(1 - a, 1 + b)$ contains all terms $S(n)$ for $n \geq r$. That is, every open interval of the form $(1 - a, 1 + b)$ contains both 1 and almost all terms of the sequence, so the limit of the sequence $S(n)$ as n approaches infinity, is 1.

As an exercise, we can also show that both cases $S < 1$ and $S > 1$ are not possible according to the definition.

Indeed, suppose there exists a limit $S < 1$. Let $S < 1 - 10^{-r}$ for some natural number r. In this case the interval of the form $(0, 1 - 10^{-r})$ contains S but it contains only a finite number of terms of the sequence, namely, only terms $S(n)$, where $n < r$. This contradicts the requirement that "every open interval containing S also contains almost all terms of the sequence."

Now suppose $S > 1$. Let $S > 1 + 10^{-r}$ for some natural number r. Then the interval $(1 + 10^{-r}, 2)$ contains S but none of the terms of the sequence. This again contradicts the requirement that "every open interval containing S also contains almost all terms of the sequence."

In a sense, this approach formalizes the following logic : if you believe that $0.999\ldots$ is less than 1, "you must tell how much the deficit is, and when you do, we can show you a large enough chunk of (finitely) many 9's in this decimal which are closer to 1 than the deficit" ([3], p. 62).

Finally, note that our result is consistent with the following algebraic derivation. Suppose that $0.999\ldots$ is a real number and denote it by $S = 0.999\ldots$. Then $S - 0.9 = 0.099\cdots = S/10$. Equivalently, $0.9S = 0.9$. Solving for S, we obtain (1).

Equation (2) can be treated similarly. The sequence of which we need to find the limit is

$$Q(1) = \sqrt{4}, \quad Q(2) = \sqrt{4\sqrt{4}}, \quad Q(3) = \sqrt{4\sqrt{4\sqrt{4}}}, \ldots$$

Again, we regard the infinite nested square root as an object rather than a process and denote it by Q. If we look carefully, we observe that $\sqrt{4Q} = Q$. Squaring both sides we get $4Q = Q^2$. Since $Q(1) > 0$ and $Q(1) < Q(2) < Q(3) < \cdots$, we need $Q > 0$. Thus $Q = 4$. Observe that $Q(1) = 4^{1/2}$, $Q(2) = 4^{1/2+1/4}$, $Q(3) = 4^{1/2+1/4+1/8}$, and the term that contains exactly n nested roots can be written as $Q(n) = 4^{M(n)}$, where $M(n) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n$, $n \geq 1$.

Note that $M(n)$ is a finite geometric series. It is well known how to compute its sum. However, even if you do not remember the general formula, here is a way to do so. Multiply $M(n)$ by $1/2$ and observe that if you subtract the product from $M(n)$, most of the terms cancel, and you obtain $M(n) - \frac{1}{2}M(n) = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}$. Thus, $M(n) = 1 - 2^{-n}$. By using the definition proposed by Heine, we can justify that the limit of the sequence $M(n)$ as n approaches infinity is 1. Then we can conclude that the value of the infinite nested root is $Q = 4^1 = 4$.

More ways to show that the infinite sum of powers of $1/2$ is 1 are given in [2]. Some of these methods are easily adaptable to other geometric series in order to illustrate that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad \text{for } -1 < x < 1. \tag{4}$$

Various viewpoints

So far we have considered several methods which allowed us to establish the equality of an infinite expression and a real number. From a mathematical standpoint, we calculated either the limits of appropriate sequences or the sums of appropriate series. One philosophical idea was to treat the infinite expression as an object rather that an infinite process and manipulate this object mathematically. As an exercise you can show that (3) holds for $a = b^{n-1}$, $b \geq 0$ and integer $n \geq 2$, for example, if $a = 4$, $n = 3$ and $b = 2$, we have $\sqrt[3]{4\sqrt[3]{4\sqrt[3]{4\ldots}}} = 2$.

One remark is in order. In fact, infinite expressions could be assigned different meanings (and values) in different theoretical frameworks and such assignments should be validated in order to avoid possible inconsistencies. In this article we rely on Heine's definition when, for example, value 1 is assigned to the infinite expression $0.999\ldots$. However, seemingly reasonable mathematical derivations can bring us to the conclusion that an integer $111\ldots$ is equal to $-1/9$. Indeed, denote $T = 111\ldots$ and observe that $10T+1 = T$, then solve for T. The same result could be *derived* from (4) by letting $x = 10$, that is, applying formula (4) beyond the interval $|x| < 1$ in which it was previously considered. Another famous assignment is Ramanujan's celebrated summation : $1 + 2 + 3 + 4 + \cdots = -1/12$. Note that even if formally established, to be validated such assignments require a different framework as we can see that they do not meet the criteria of Heine's definition of the limit. Moreover, they are inconsistent with a view that an unrestrictedly growing sequence of numbers cannot represent a finite rational number. However, a discussion of frameworks required for validation of the above assignments (e.g. p-adic analysis or complex analysis) is beyond the scope of this paper.

Telescoping

Telescoping is another useful technique for investigating infinite series. The main idea is to change the appearance of the sum and look for similar terms to cancel each other. Consider the following example. Find the numerical value of P, where

$$P = \frac{1}{10\cdot 11} + \frac{1}{11\cdot 12} + \frac{1}{12\cdot 13} + \cdots .$$

Here then the sequence of which we need to find the limit is $P(1) = \frac{1}{10\cdot 11}$, $P(2) = \frac{1}{10\cdot 11} + \frac{1}{11\cdot 12}$, $P(3) = \frac{1}{10\cdot 11} + \frac{1}{11\cdot 12} + \frac{1}{12\cdot 13}$, $\ldots$ Now, observing that

$$\frac{1}{10\cdot 11} = \frac{1}{10} - \frac{1}{11}, \quad \frac{1}{11\cdot 12} = \frac{1}{11} - \frac{1}{12}, \quad \frac{1}{12\cdot 13} = \frac{1}{12} - \frac{1}{13}, \quad \ldots,$$

we get $P(n) = \frac{1}{10} - \frac{1}{10+n}$ for $n \geq 1$. Using Heine's definition we can argue that the limit of this sequence is $P = 1/10$. This is because the positive numbers $(1/11, 1/12, 1/13, \ldots)$ which are subtracted from $1/10$ get smaller and smaller as you add more summands.

One can contrast this derivation with the following example. Consider the infinite sum

$$\ln\frac{10}{11} + \ln\frac{11}{12} + \ln\frac{12}{13} + \cdots .$$

The sum $R(n)$, which contains exactly the first n terms, can be written as

$$R(n) = \ln 10 - \ln 11 + \ln 11 - \ln 12 + \cdots + \ln(9+n) - \ln(10+n) = \ln 10 - \ln(10+n),$$

$n \geq 1$. Note that, as in the previous example, cancellations also take place. However, the infinite sum is not equal to $\ln 10$ (and in fact does not have any finite value) because the positive numbers $(\ln 11, \ln 12, \ldots)$ which are subtracted from $\ln 10$ get larger and larger as you add more summands, and so the sequence $R(n)$ deviates from, rather than approaches $\ln 10$, as n approaches infinity.

Conclusion

As the above examples show, knowing simple techniques from algebra and geometry may be sufficient to obtain numerical answers. At the same time, the study of series and sequences constitutes an important branch in analysis. There are mathematical subtleties that require special attention as well as new ideas that are needed for evaluating some limits. For example, note that the formula for summation of a geometric series (4) can be viewed as a representation of function $f(x) = 1/(1-x)$ by a polynomial with infinitely many terms, called the power series. This representation is valid for $|x| < 1$. More examples of power series representing a function on the interval $|x| < 1$ can be obtained by either taking the (first, second, etc.) derivative or integrating both sides of (4) term by term. Similarly, other functions can be represented by power series, e.g. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$ for all x, and (by taking the derivative of both sides) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$ for all x. Some interesting relations can be derived from these equations if they are evaluated for particular values of x. We leave it for the reader to show that

$$\frac{2\pi^2}{3!} - \frac{4\pi^4}{5!} + \frac{6\pi^6}{7!} - \frac{8\pi^8}{9!} + \cdots = 1.$$

Exercises

1. Let $a \geq 2$ be an integer. Show that $a^{1/a} \cdot (a^2)^{1/a^2} \cdot (a^3)^{1/a^3} \cdots = a^{a/(a-1)^2}$.

2. Show that

$$\frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \cdots = \frac{9}{16}.$$

3. Let F_n, $n \geq 1$ be the sequence of Fibonacci numbers, that is, the first two terms are $F_1 = F_2 = 1$, and every following term is the sum of the two previous terms in the sequence, $F_3 = 1+1 = 2, F_4 = 1+2 = 3, F_5 = 2+3 = 5, F_6 = 3+5 = 8, \ldots$. Evaluate $\dfrac{F_3}{F_2 \cdot F_4} + \dfrac{F_4}{F_3 \cdot F_5} + \dfrac{F_5}{F_4 \cdot F_6} + \cdots$

4. Is the following equality true?

$$2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \ldots}}}} = 3$$

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Approaching the Extremum

L. Kurlyandchik

The concept of the derivative, which is introduced in high school, provides an easy way to find the maximum and minimum values of a given one-variable function $y = f(x)$. However, some problems require finding maximum and minimum values of a multivariate function. The methods involving a derivative in this case are much more complicated and are not studied in high school; therefore, one needs to develop elementary approaches to such problems. This article describes one such method devised in 1885 by German mathematician, Rudolf Sturm.

Let us begin by studying two problems.

Problem 1. Which convex n-gon inscribed in a unit circle has the largest area?

Problem 2. Find n numbers such that their sum is equal to 1 and the sum of their squares is as small as possible.

In Figure 1a), we have a triangle ABC inscribed in a circle with $|AB| > |BC|$. What will happen to the area of this triangle if, without changing the position of points A and C, we move the point B along the arc AC so that the lengths of sides AB and BC get closer to each other (that is, the difference $|AB| - |BC|$ gets smaller)? Let B' be the point on the arc AC on the same side of AC as B such that $|AB'| = |CB|$. To bring the lengths of AB and BC closer together, we can replace B by any point on the arc BB'. Then the altitude BH of ABC increases and hence the area of ABC increases.

This observation shows that a non-regular n-gon cannot be a solution to Problem 1. Indeed, if AB and BC are non-equal neighbouring sides of such an n-gon (see Figure 1b)), then, by replacing the point B by any point on the arc BB', we increase the area of the triangle ABC and hence the area of the n-gon.

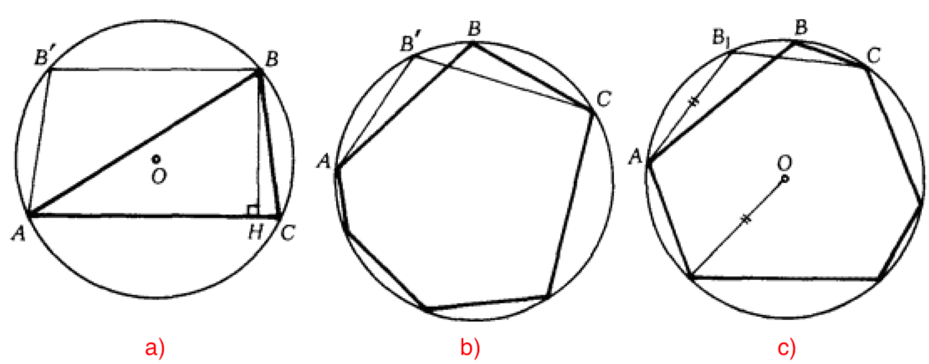


Figure 1: Illustration for Problem 1.

Moving onto Problem 2, let us see what happens to the sum of the squares of two numbers if we bring these numbers closer together without changing their sum.

Let a and b be the two given numbers with $a < b$ and let $0 < \epsilon < b - a$. Then

$$(a+\epsilon)^2 + (b-\epsilon)^2 = a^2 + b^2 - 2\epsilon(b-a-\epsilon) < a^2 + b^2.$$

We see that the sum of their squares got smaller. This observation shows that a set of n numbers with the sum of 1 which contains two unequal numbers cannot be a solution to Problem 2. Indeed, by bringing two numbers closer together, while keeping their sum constant, we reduce the sum of their squares.

One might think that we have solved Problems 1 and 2 : the solution to Problem 1 is a regular n-gon inscribed in a unit circle and the solution to Problem 2 is a set of numbers $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$. However, all we have proved is that no non-regular n-gon can be a solution to Problem 1 and that no set of n numbers not equal to $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$ can be a solution to Problem 2. But there is one more possibility : no n-gon is a solution to Problem 1 and no set of n numbers is a solution to Problem 2. Proving existence of a mathematical object with given properties is often no small feat. But solving any given problem is usually easier if you know a potential answer, which is the case with Problems 1 and 2.

Solution to Problem 1. Let us prove that the area of any n-gon inscribed in a given circle is no greater than the area of the regular n-gon inscribed in the same circle. This would mean that the regular n-gon is indeed the solution to Problem 1, whereas before we have only proved that a non-regular n-gon is not the solution to Problem 1.

Note that if we replace point B of the inscribed n-gon with B' (see Figure 1b)), we interchange the larger and smaller sides without changing the area. Repeating this operation, we can interchange the sides of a given non-regular n-gon to obtain an n-gon of the same area whose smallest and largest sides share a vertex.

Let P be the inscribed non-regular n-gon. The smallest side of this n-gon subtends an arc with the central angle less than $\frac{360^\circ}{n}$, while its largest side subtends an arc with the central angle greater than $\frac{360^\circ}{n}$. Switch the sides of the n-gon so that the smallest and the largest sides are next to each other. In the hexagon of Figure 1c), these are the sides AB and BC. On the arc AC, measure out the arc AB_1, whose central angle equals $\frac{360^\circ}{n}$. Replacing B with B_1, we will get an n-gon P_1 whose area is larger than the area of P. Moreover, the n-gon P_1 has at least one side subtending an arc with central angle equal to $\frac{360^\circ}{n}$ and, hence, has one side equal to the side of the regular n-gon. If the n-gon P_1 is not regular, we repeat the process to get an n-gon P_2 of larger area with at least one more side equal to the side of a regular n-gon. After a finite number of steps, we will arrive at the regular n-gon whose area is greater than the area of P. □

Solution to Problem 2. Let the sum of numbers $a_1, a_2, \ldots, a_n$ equal 1. If not all of these numbers are equal, then the smallest one is less than $\frac{1}{n}$ and the largest one is greater than $\frac{1}{n}$. Suppose without loss of generality that $a_1 < \frac{1}{n}$ and $a_2 > \frac{1}{n}$. Replacing a_1 with $\frac{1}{n}$ and a_2 with $a_1 + a_2 - \frac{1}{n}$, we bring a_1 and a_2 closer together thereby increasing the sum of their squares without changing the total sum. As such, we get a new set of numbers that sum to 1 while the sum of their squares is less than that of the original set. Moreover, this new set has at least one number equal to $\frac{1}{n}$. If not all numbers in the new set equal $\frac{1}{n}$, we repeat the process. After a finite number of repetitions, we will arrive at the set $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$ with the sum of squares less than the sum of squares of the original set. Therefore, set $\left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}$ is the solution to Problem 2. □

If we take some constant S, then the above argument shows that among all sets of numbers $a_1, a_2, \ldots, a_n$ that sum to S, the set with the smallest sum of squares is $\left\{\frac{S}{n}, \frac{S}{n}, \ldots, \frac{S}{n}\right\}$. Therefore, given an arbitrary set of numbers $a_1, a_2, \ldots, a_n$, the following inequality holds :

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq n\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2,$$

and therefore

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2.$$

Hence, the arithmetic mean of the squares of a set of numbers is no less than the square of the arithmetic mean of these numbers themselves. Further, equality holds only if $a_1 = a_2 = \ldots = a_n$ (left as an exercise to the reader).

Of course you notice that even though Problems 1 and 2 seemed unrelated, they were solved using essentially the same method. Both problems asked to find an extreme value of a multivariate function. The solution was constructed by repeatedly massaging only two variables at a time. In Problem 1, we examined how the area of an n-gon changed when working with two neighbouring sides ; in Problem 2, we examined how the sum of the squares of n numbers changed when working with two of these numbers. This way of finding the extremum of a multivariate function is called *Sturm's method* and will be used throughout the remainder article.

Let us start with three Lemmas.

First, consider the product of two numbers a and b ($b > a > 0$) when these numbers get closer together while their sum stays constant. Let $0 < \epsilon < b - a$. Then

$$(a + \epsilon)(b - \epsilon) = ab + \epsilon(b - a - \epsilon) > ab.$$

Therefore, we have

Lemma 1. The product of two numbers with a constant sum increases as their difference decreases.

It is slightly more complicated to figure out what happens in a similar situation with the sum of powers of two numbers $a^k + b^k$ for a natural number $k \geq 2$. Let $\epsilon > 0$ and let us compare $a^k + b^k$ and $(a + \epsilon)^k + (b - \epsilon)^k$. An easy way to do so is to study the function $f(\epsilon) = (a + \epsilon)^k + (b - \epsilon)^k$ using its derivative. Since $f'(\epsilon) = k((a + \epsilon)^{k-1} - (b - \epsilon)^{k-1})$, we have that $f'(\epsilon)$ is negative for $a + \epsilon < b - \epsilon$; that is, for $\epsilon < \frac{b-a}{2}$. Therefore, the function f is decreasing on the interval $[0, \frac{b-a}{2})$. Furthermore, $f(0) = a^k + b^k$ and hence for $0 < \epsilon < \frac{b-a}{2}$ we have

$$a^k + b^k > (a + \epsilon)^k + (b - \epsilon)^k.$$

This inequality holds also for $\epsilon \in (0, b-a)$ since the graph of the function f is symmetrical about the line $\epsilon = \frac{b-a}{2}$. So we have

Lemma 2. The sum of the k-th powers of two numbers ($k \geq 2$) with a constant sum increases as their difference decreases.

Now let us investigate the case where the numbers a and b get closer together while their product is kept constant. What happens to the sum $a+b$ in this case? Suppose $a < b$. Compare $a+b$ and $\lambda a + \frac{b}{\lambda}$ for some $\lambda > 1$ (note that the product of $a+b$ and $\lambda a + \frac{b}{\lambda}$ and is equal to ab) :

$$(a+b) - \left(\lambda a + \frac{b}{\lambda}\right) = (\lambda - 1)\left(\frac{b}{\lambda} - a\right).$$

Therefore, for $\lambda \in (1, \frac{b}{a})$, we have $a + b > \lambda a + \frac{b}{\lambda}$. This gives us

Lemma 3. The sum of two positive numbers with a constant product decreases as their difference decreases.

Lemma 1 provides an easy proof of the inequality which compares the arithmetic mean A with the geometric mean G : if $a_1, a_2, \ldots, a_n$ are positive numbers that are not all equal, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Indeed, if the numbers are not all equal, then the smallest is less than A and the largest one is greater than A. Suppose that $a_1 < A$ and $a_2 > A$. By replacing a_1 with A and a_2 with $a_1 + a_2 - A$, we bring them closer together while keeping their sum unchanged. At the same time, the arithmetic mean A stays the same, while the geometric mean G get larger (see Lemma 1). If the numbers of the new set are still not all equal to the arithmetic mean, we repeat the process. Since every time we increase the geometric mean and keep the arithmetic mean constant, in the final set of numbers the arithmetic mean equals the final geometric mean, which exceeds the original geometric mean, as claimed.

Exercise 1. Using the proof technique of Lemma 1, find 25 numbers $n_1, n_2, \ldots, n_{25}$ that sum to 1981 so that the product $n_1! n_2! \ldots n_{25}!$ is minimized.

Exercise 2. Using Lemma 2, prove that for positive numbers $a_1, a_2, \ldots, a_n$ and a natural number k, we have

$$\frac{a_1^k + a_2^k + \cdots + a_n^k}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^k.$$

Exercise 3. Using Lemma 3, prove *Huygens' inequality* :

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \geq (1 + \sqrt[n]{x_1 x_2 \cdots x_n})^n.$$

In all the above problems, a function's extremum was achieved when all the variables were equal. This is why while looking for these extrema, we were using the method of bringing the variable values closer together. In the following exercise, try using a similar approach, but instead of bringing the variables closer together, consider what happens to the function when the difference between variables increases.

Exercise 4. Let $x_1, x_2, \ldots, x_n \in [a, b]$, where $0 < a < b$. Prove that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \leq \frac{(a+b)^2}{4ab}n^2.$$

Now try your hand at these further exercises.

Exercise 5. Which n-gon inscribed in a unit semi-circle (so that one of its sides coincides with the diameter) has the maximum area?

Exercise 6. Prove that the equilateral triangle has the smallest perimeter among all triangles with a given area.

Exercise 7. Investigate what happens to the product $(1 + \frac{1}{a})(1 + \frac{1}{b})$ when the positive numbers a and b get closer together while their sum is kept constant. Show that if the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1, then

$$\left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \cdots \left(1 + \frac{1}{x_n}\right) \geq (n+1)^n.$$

Exercise 8. Investigate what happens to the sum $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$ when the positive numbers a and b get closer together while their sum is kept constant. Show that if the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1, then

$$\left(x_1 + \frac{1}{x_1}\right)^2 + \left(x_2 + \frac{1}{x_2}\right)^2 + \cdots + \left(x_n + \frac{1}{x_n}\right)^2 \geq \frac{(n^2+1)^2}{n}.$$

Exercise 9. Investigate what happens to the fraction $\frac{(1-a)(1-b)}{ab}$ when the positive numbers a and b, $a + b = 1$, get closer together. Show that if the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1, then

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{x_1 x_2 \cdots x_n} \geq (n-1)^n.$$

Exercise 10. Investigate what happens to the sum $\frac{1}{1+a} + \frac{1}{1+b}$ when the product of positive numbers a and b stays constant. Then prove the following inequalities.

a) If the numbers $x_1, x_2, \ldots, x_n$ are all greater than 1, then

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \geq \frac{n}{1 + \sqrt[n]{x_1 x_2 \cdots x_n}}.$$

b) If the positive numbers $x_1, x_2, \ldots, x_n$ are all less than 1, then

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \le \frac{n}{1+\sqrt[n]{x_1 x_2 \cdots x_n}}.$$

Exercise 11. Let $\alpha, \beta, \gamma, \delta \in (0, \frac{\pi}{2})$ and suppose that $\alpha + \beta + \gamma + \delta = \pi$. Show that $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta \ge 4$.

Exercise 12. Let $\alpha, \beta, \gamma, \delta$ be positive numbers such that $\alpha + \beta + \gamma + \delta = \pi$. Show that $\sin\alpha \sin\beta \sin\gamma \sin\delta \le \frac{1}{4}$.

Exercise 13.

a) Let a, b, c, d be positive numbers. Prove that

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \ge \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}.$$

b) Let $x_1, x_2, \ldots, x_n$ be positive numbers and let k be a natural number with $k \ge 2$. Prove that

$$\sqrt[k]{\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}} \ge \sqrt[n-1]{\frac{x_1 x_2 \cdots x_n \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}{n}}.$$

Exercise 14. Given two positive numbers a and b, $a < b$, find the n numbers $x_1, x_2, \ldots, x_n \in (a, b)$ so that the fraction

$$\frac{(a+x_1)(x_1+x_2)(x_2+x_3)\cdots(x_{n-1}+x_n)(x_n+b)}{x_1 x_2 \cdots x_n}$$

is minimized.

Exercise 15. The sum of some set of nonnegative numbers is equal to 3 and the sum of their squares is bigger than 1. Prove that you can pick three of these numbers such that their sum exceeds 1.

This article appeared in Russian in Kvant, 1981(1), p. 21–25. It has been translated and adapted with permission.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, d'avoir traduit les problèmes.

4085. *Proposé par José Luis Díaz-Barrero. Correction.*

Soit ABC un triangle acutangle. Démontrer que

$$\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}<\frac{3\sqrt{2}}{2}.$$

4111. *Proposé par Mihaela Berindeanu.*

Soit O le centre du cercle circonscrit au triangle ABC et R son rayon. P est un point quelconque sur le côté BC du triangle. Déterminer la valeur de R, sachant que le produit $PA\cdot PB\cdot PC$ a une valeur maximale de 2016.

4112. *Proposé par Ardak Mirzakhmedov et Leonard Giugiuc.*

Soit ABC un triangle acutangle. Démontrer que

$$\sqrt{96\cos^2 A+25}+\sqrt{96\cos^2 B+25}+\sqrt{96\cos^2 C+25}\geq 21.$$

4113. *Proposé par Dragoljub Milošević.*

Soit m_a, m_b et m_c les longueurs des médianes d'un triangle, w_a, w_b et w_c les longueurs des bissectrices, r le rayon du cercle inscrit et R le rayon du cercle circonscrit au triangle. Démontrer que

$$\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\leq\frac{1}{2}+\frac{r}{R}+\frac{R}{r}.$$

4114. *Proposé par Michel Bataille.*

Soit BC un segment de droite dans le plan. Déterminer le lieu géométrique des points A pour lesquels le centre de gravité G du triangle ABC vérifie $\angle GAB = \angle GBC$ et $\angle GAC = \angle GCB$.

4115. *Proposé par Daniel Sitaru.*

Démontrer que

$$n^{\ln 2} \leq \sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}$$

pour tout entier n ($n \geq 2$).

4116. *Proposé par George Apostolopoulos.*

Soit a, b et c des réels strictement positifs tels que $a + b + c = 3$. Déterminer la valeur minimale de l'expression

$$a^2(a^2 - a + 1) + b^2(b^2 - b + 1) + c^2(c^2 - c + 1).$$

4117. *Proposé par Martin Lukarevski.*

La suite (x_n) est définie de façon récursive par $x_0 = 0, x_1 = 1$ et

$$x_{n+1} = x_n\sqrt{x_{n-1}^2 + 1} + x_{n-1}\sqrt{x_n^2 + 1}, \quad n \geq 1.$$

Déterminer une expression pour x_n.

4118. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soit $a \in (0, \frac{\pi}{2}]$ et $b \in [\frac{\pi}{2}, \pi)$ tels que $a + b = \pi$. Calculer $\int_a^b \frac{x}{\sin x} dx$.

4119. *Proposé par Ovidiu Furdui.*

Soit $m, n, p \in \mathbb{N}$ ($m \neq n$) et soit A et B des matrices 2×2 dont les éléments sont complexes et telles que $mAB - nBA = pI_2$. Démontrer que

$$(AB - BA)^2 = O_2.$$

4120. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Déterminer la valeur minimale de la fonction $f : [1, 2] \mapsto \mathbb{R}$ définie par

$$f(x) = \sqrt{\frac{8 - 3x}{x}} + 2\sqrt{4x + 1} - \sqrt{4x^2 - 8x + 49}.$$

. . . .

4085. *Proposed by José Luis Díaz-Barrero. Correction.*

Let ABC be an acute triangle. Prove that

$$\sqrt[4]{\sin(\cos A) \cdot \cos B} + \sqrt[4]{\sin(\cos B) \cdot \cos C} + \sqrt[4]{\sin(\cos C) \cdot \cos A} < \frac{3\sqrt{2}}{2}.$$

4111. *Proposed by Mihaela Berindeanu.*

The circumscribed circle of $\triangle ABC$ has circumcenter O and circumradius R. Let P be a point on the side BC. Calculate R given that the maximum value of the product $PA \cdot PB \cdot PC$ is 2016.

4112. *Proposed by Ardak Mirzakhmedov and Leonard Giugiuc.*

Let ABC be an acute triangle. Prove that

$$\sqrt{96\cos^2 A + 25} + \sqrt{96\cos^2 B + 25} + \sqrt{96\cos^2 C + 25} \geq 21.$$

4113. *Proposed by Dragoljub Milošević.*

Let m_a, m_b and m_c be the lengths of medians, w_a, w_b and w_c be the lengths of the angle bisectors, r and R be the the inradius and the circumradius, respectively, of a triangle. Prove that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \leq \frac{1}{2} + \frac{r}{R} + \frac{R}{r}.$$

4114. *Proposed by Michel Bataille.*

In the plane, let BC be a given line segment. Find the locus of A such that the centroid G of the triangle ABC satisfies $\angle GAB = \angle GBC$ and $\angle GAC = \angle GCB$.

4115. *Proposed by Daniel Sitaru.*

Prove that for all natural numbers $n \geq 2$, we have

$$n^{\ln 2} \leq \sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}.$$

4116. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers such that $a + b + c = 3$. Find the minimum value of the expression

$$a^2(a^2 - a + 1) + b^2(b^2 - b + 1) + c^2(c^2 - c + 1).$$

4117. *Proposed by Martin Lukarevski.*

The sequence (x_n) is given recursively by $x_0 = 0, x_1 = 1$,

$$x_{n+1} = x_n\sqrt{x_{n-1}^2 + 1} + x_{n-1}\sqrt{x_n^2 + 1}, \quad n \geq 1.$$

Find x_n.

4118. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $a \in (0, \frac{\pi}{2}], b \in [\frac{\pi}{2}, \pi)$ with $a + b = \pi$. Calculate $\int_a^b \dfrac{x}{\sin x} dx$.

4119. *Proposed by Ovidiu Furdui.*

Let $m, n, p \in \mathbb{N}$, $m \neq n$, and let A and B be 2×2 matrices with complex entries for which $mAB - nBA = pI_2$. Prove that

$$(AB - BA)^2 = O_2.$$

4120. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Find the minimum value of the function $f : [1, 2] \mapsto \mathbb{R}$, where

$$f(x) = \sqrt{\frac{8 - 3x}{x}} + 2\sqrt{4x + 1} - \sqrt{4x^2 - 8x + 49}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

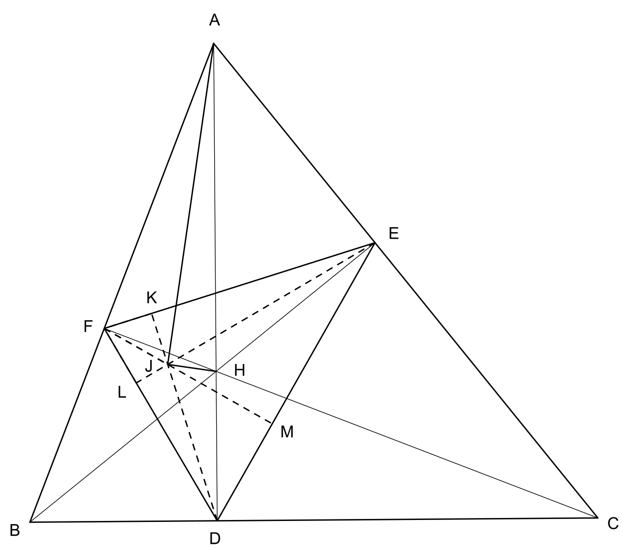
Statements of the problems in this section originally appear in 2015: 41(2), p. 71–74.

4011. *Proposed by Abdilkadir Altinas.*

In non-equilateral triangle ABC, let H be the orthocentre of ABC and J be the orthocentre of the orthic triangle DEF of ABC (that is the triangle formed by the feet of the altitudes of ABC). If $\angle BAC = 60°$, show that $AJ \perp HJ$.

We received nine solutions. Eight of the solutions used angle-chasing in cyclic quadrilaterals, and one solution used barycentric coordinates. The former type of solutions were simpler, but they all missed the fact that there are subtleties if the orthocenter is not interior to the triangle.

We present the solution by Ricardo Barroso Campos slightly modified by the editor.



Since $\angle HFA = \angle HEA = 90°$, quadrilateral $AFHE$ is cyclic. Hence

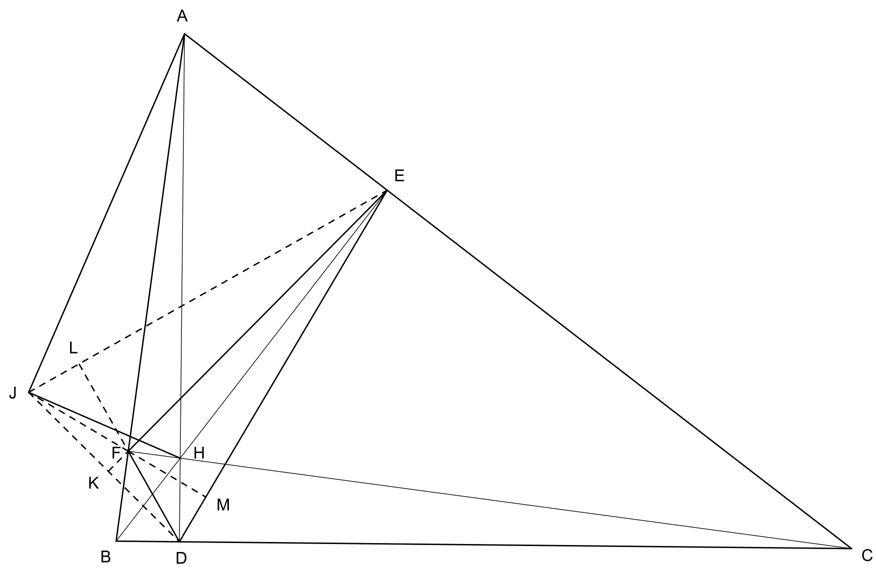
$$\angle EHF = 180° - \angle CAB = 120°.$$

Denote by Γ the circumcircle of $AFHE$, and note that AH is the diameter of this circle (since $\angle AFH = 90°$).

In the cyclic quadrilateral $BDHF$, $\angle BDF = \angle BHF = 180° - \angle EHF = 60°$. Similarly, from the cyclic quadrilateral $CDHE$, we get $\angle CDE = 60°$. Hence $\angle FDE = 180° - (\angle BDF + \angle CDE) = 60°$.

Denote the feet of the altitudes from D, E, and F by K, L and M respectively, as in the diagram. $DLJM$ is cyclic, thus $\angle LJM = 180° - \angle FDE = 120°$. Hence $\angle FJE = \angle LJM = 120°$, which implies that the quadrilateral $AFJE$ is cyclic (since $\angle FJE + \angle FAE = 180°$). It follows that J is on the circle Γ. Hence, since AH is the diameter of Γ, we get $\angle AJH = 90°$, so $AJ \perp HJ$.

Editor's Comments. The provided solution fails if one of the orthocentres is not interior to its triangle. The following diagram shows the case where the point J is not interior to DEF (for one, $\angle FJE = 60°$, not $120°$). Note however that it is not difficult to adjust the solution for these cases.



4012. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 3$. Consider real numbers a_k, $1 \leq k \leq n$ such that

$$a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 1 \geq a_n \geq 0 \quad \text{and} \quad \sum_{k=1}^{n} a_k = n.$$

Prove that

$$\frac{(n-2)(n+1)}{2} \leq \sum_{1 \leq i < j \leq n} a_i a_j \leq \frac{n(n-1)}{2}.$$

We received eight submissions of which seven were correct and complete. We present the solution by Ivan Chan Kai Chin.

Since we have

$$2 \cdot \sum_{1 \leq i \leq j \leq n} a_i a_j = (\sum_{k=1}^{n} a_k)^2 - \sum_{k=1}^{n} a_k^2 = n^2 - \sum_{k=1}^{n} a_k^2,$$

it suffices to prove that

$$n \leq \sum_{k=1}^{n} a_k^2 \leq n+2.$$

The left inequality holds by Cauchy-Schwarz, since

$$\sum_{k=1}^{n} a_k^2 \geq \frac{1}{n} \cdot (\sum_{k=1}^{n} a_k)^2 = n,$$

with equality when $a_1 = a_2 = \cdots = a_n = 1$.

For the other inequality, set $b_k = a_k - 1$ for all $1 \leq k \leq n$. Then $b_k \geq 0$ for all $1 \leq k \leq n-1$, $-1 \leq b_n \leq 0$, and $\sum_{k=1}^{n} b_i = 0$. We have

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} (1+b_k)^2 = \sum_{k=1}^{n-1} (1+b_k)^2 + (1-(b_1+b_2+\cdots+b_{n-1}))^2. \qquad (1)$$

Define the quantity $S = b_1 + b_2 + \cdots + b_{n-1} \leq 1$, and

$$f(b_1, b_2, \cdots, b_{n-1}) = \sum_{k=1}^{n-1} (1+b_k)^2.$$

For any $1 \leq i \leq j \leq n-1$,

$$f(\cdots b_i, \cdots, b_j, \cdots) \leq f(\cdots b_i + b_j, \cdots, 0, \cdots),$$

since

$$(1+b_i)^2 + (1+b_j)^2 \leq (1+b_i+b_j)^2 + 1 \iff 2b_ib_j \geq 0$$

holds for all b_i, b_j, $1 \leq i \leq j \leq n-1$. Thus we have

$$f(b_1, b_2, \cdots, b_{n-1}) \leq f(b_1 + b_2 + \cdots b_{n-1}, 0, \cdots, 0) = f(S, 0, \cdots, 0)$$

and (1) becomes

$$\begin{aligned} f(S, 0, \cdots, 0) + (1-S)^2 &= (1+S)^2 + (n-2) + (1-S)^2 \\ &= n + 2S^2 \\ &\leq n+2 \end{aligned}$$

Equality holds when $S = 1$, $b_2 = b_3 = \cdots = b_{n-1} = 0$, $b_1 = 1$ and $b_n = -1$, which corresponds to $a_1 = 2, a_2 = a_3 = \cdots = a_{n-1} = 1, a_n = 0$.

4013. *Proposed by Mehmet Şahin.*

Let a, b, c be the sides of triangle ABC, D be the foot of the altitude from A and E be the midpoint of BC. Define $\theta = \angle DAE$ and suppose that $\angle ACB = 2\theta$. Prove that the sides of the triangle satisfy

$$(a-b)^2 = 2c^2 - b^2.$$

We received 16 submissions. Among them one simply stated that the claim was incorrect and provided a counterexample, while 15 proved the claim under the additional assumption that $b > c$; moreover, 5 proved that for the claim to be correct, the assumption that $b > c$ is both necessary and sufficient, and 3 of those submissions went on to provide a complete description of triangles that satisfy the given hypotheses.

We present the solution by Joel Schlosberg, supplemented by ideas from C. R. Pranesachar.

We shall prove that if a triangle satisfies $\angle ACB = 2\angle DAE$ and, moreover, $b > c$, then $(a-b)^2 = 2c^2 - b^2$; if $b < c$ then $a = 2b$ (and the claimed equation fails to hold). Note that if $b = c$ then $A = D = E$ is the midpoint of the segment BC, and the triangle is degenerate.

Scale $\triangle ABC$ so that $b = 1$. By right-angle trigonometry, $AD = \sin 2\theta = 2\sin\theta\cos\theta$, so that

$$AD^2 = 4\sin^2\theta(1-\sin^2\theta).$$

Use signed lengths for segments on BC, with BC positive. Then

$$DE = pAD\tan\theta = 2p\sin^2\theta,$$

where $p = 1$ if B and D are on one side of E, and C is on the other (which happens if and only if $b > c$); otherwise, when E is between B and D (and, equivalently, $b < c$) then we set $p = -1$. Furthermore, we have

$$DC = \cos 2\theta = 1 - 2\sin^2\theta$$

$$EC = DC - DE = 1 - (2+2p)\sin^2\theta$$

$$a = BC = 2EC = 2 - (4+4p)\sin^2\theta$$

$$BD = BC - DC = 1 - (2+4p)\sin^2\theta.$$

By the Pythagorean theorem (using $p^2 = 1$),

$$c^2 = AD^2 + BD^2 = 1 - 8p\sin^2\theta + (16+16p)\sin^4\theta, \tag{1}$$

while (using $b = 1$)

$$\frac{(a-b)^2+b^2}{2} = \frac{1}{2}a^2 - ab + b^2 = 1 - (4+4p)\sin^2\theta + (16+16p)\sin^4\theta. \tag{2}$$

Comparing equations (1) and (2), we see that $(a-b)^2 = 2c^2 - b^2$ iff $p = 1$. On the other hand, setting $p = -1$ and $b = 1$ in equation (2) we get $a = 2$ and deduce that $a = 2b$.

Editor's Comments. This problem should be compared with problem 4008 whose solution appeared in the previous issue. It dealt with triangles for which $\angle ACB = 2\angle DAE$ and, in addition, $\angle ABC = 3\angle DAE$. One finds that this can happen if and only if $\angle A = 90°, \angle B = 54°$, and $\angle C = 36°$; of course this implies that $b > c$ and, consequently, that $(a-b)^2 = 2c^2 - b^2$.

4014. *Proposed by Mihaela Berinedanu.*

Let n be a natural number and let x, y and z be positive real numbers such that $x + y + z + nxyz = n + 3$. Prove that

$$(1 + \frac{y}{x} + nyz)(1 + \frac{z}{y} + nzx)(1 + \frac{x}{z} + nxy) \geq (n+2)^3$$

and determine when equality holds.

We received six correct solutions. We present the solution by Dionne Bailey, Elsie Campbell and Charles Diminnie (joint).

The arithmetic-geometric means inequality yields that

$$n+3 = x+y+z+nxyz \geq (n+3)\left[x \cdot y \cdot z \cdot (xyz)^n\right]^{1/n+3} = (n+3)\left[xyz\right]^{(n+1)/(n+3)},$$

so that $xyz \leq 1$.

The inequality is equivalent to

$$(x + y + nxyz)(y + z + nxyz)(z + x + nxyz) \geq (n+2)^3(xyz)$$

or

$$(n + 3 - z)(n + 3 - x)(n + 3 - y) \geq (n + 2)^3 xyz.$$

Using the arithmetic-geometric means inequality and the fact that $xyz \leq (xyz)^{2/3}$, we obtain that

$$\begin{aligned}
&(n + 3 - x)(n + 3 - y)(n + 3 - z) \\
&\qquad = (n + 3)^3 - (n + 3)^2(x + y + z) + (n + 3)(xy + yz + zx) - xyz \\
&\qquad \geq (n + 3)^3 - (n + 3)^2[(n + 3) - nxyz] + 3(n + 3)(xyz)^{2/3} - xyz \\
&\qquad \geq n(n + 3)^2 xyz + 3(n + 3)xyz - xyz \\
&\qquad = (n^3 + 6n^2 + 9n + 3n + 9 - 1)xyz = (n + 2)^3 xyz,
\end{aligned}$$

as desired, with equality if and only if $x = y = z = 1$.

4015. *Proposed by Michel Bataille.*

Find all real numbers a such that

$$a \cos x + (1 - a) \cos \frac{x}{3} > \frac{\sin x}{x}$$

for every nonzero x of the interval $(-\frac{3\pi}{2}, \frac{3\pi}{2})$.

There were four submitted solutions for this problem, all of which were correct. We present the solution by Joel Schlosberg.

We will prove that the inequality

$$a\cos(x) + (1-a)\cos\left(\frac{x}{3}\right) > \frac{\sin(x)}{x} \tag{1}$$

holds for $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$ if and only if $a \in (-\infty, 1/4]$.

Substituting $y = x/3$ and dividing by y^2, the above inequality is equivalent to

$$a\left(\frac{\cos(y) - \cos(3y)}{y^2}\right) < \frac{3y\cos(y) - \sin(3y)}{3y^3}, \tag{2}$$

for $|y| \in (0, \pi/2)$. By repeated applications of l'Hospital's Rule,

$$\lim_{y\to 0} \frac{\cos(y) - \cos(3y)}{y^2} = \lim_{y\to 0} \frac{-\sin(y) + 3\sin(3y)}{2y} = \lim_{y\to 0} \frac{-\cos(y) + 9\cos(3y)}{2} = 4,$$

and similarly,

$$\begin{aligned}\lim_{y\to 0} \frac{3y\cos(y) - \sin(3y)}{3y^3} &= \lim_{y\to 0} \frac{3\cos(y) - 3y\sin(y) - 3\cos(3y)}{9y^2} \\ &= \lim_{y\to 0} \frac{-6\sin(y) - 3y\cos(y) + 9\sin(3y)}{18y} \\ &= \lim_{y\to 0} \frac{-9\cos(y) + 3y\sin(y) + 27\cos(3y)}{18} = 1.\end{aligned}$$

Suppose that (1) holds for $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$. Then the rewritten inequality (2) holds for non-zero y in a neighbourhood of 0. Taking $y \to 0$ on both sides yields $4a \le 1$, and so $a \in (-\infty, 1/4]$.

Since both sides of (2) are even functions of y, it is sufficient to prove (2) for $y \in (0, \pi/2)$. We let

$$f(x) = 9y\cos(y) - 4\sin(3y) + 3y\cos(3y).$$

For $y \in (0, \pi/2)$, it is well-known that $\tan(y) > y$, so

$$\begin{aligned}f'(y) &= 9(\cos(y) - y\sin(y) - \cos(3y) - y\sin(3y)) \\ &= 9(4(1-\cos^2(y))\cos(y) - 4y(1-\sin^2(y))\sin(y)) \\ &= 36(\sin^2(y)\cos(y) - y\cos^2(y)\sin(y)) \\ &= 36\cos^2(y)\sin(y)(\tan(y) - y) > 0\end{aligned}$$

(via triple-angle formulas). Therefore, for $y \in (0, \pi/2)$, $f(y) > f(0) = 0$, which is equivalent to (2) when $a = 1/4$. Hence (1) holds for $a = 1/4$.

Suppose now that $a < 1/4$. For $y \in (0, \pi/2)$, we have

$$\cos(y) - \cos(3y) = 4(1-\cos^2(y))\cos(y) = 4\sin^2(y)\cos(y) > 0,$$

so that

$$a\left(\frac{\cos(y)-\cos(3y)}{y^2}\right) < \frac{1}{4}\left(\frac{\cos(y)-\cos(3y)}{y^2}\right) < \frac{3y\cos(y)-\sin(3y)}{3y^3},$$

and we are done.

Editor's Comments. All four solution methods involved similar elements: trigonometric identities used to rewrite the inequality, a limit (either by power series or by L'Hospital's Rule), and some calculus. Deiermann noted that if we set the right-hand side of the original inequality equal to 1 for $x = 0$, then we may allow equality at $x = 0$. Deiermann also suggested a generalization, propped up by Mathematica: if $n \geq 3$, then we have

$$a\cos(x) + (1-a)\cos\left(\frac{x}{n}\right) > \frac{\sin(x)}{x}.$$

for all non-zero $x \in (-\frac{3\pi}{2}, \frac{3\pi}{2}) \setminus \{0\}$ if and only if $a \leq \frac{n^2-3}{3(n^2-1)}$. A quick sketch of the argument by the editor seems to indicate that it is true, but the conclusion of the proof is still out of reach.

4016. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers. Find the maximal value of the expression

$$\frac{x+2y}{2x+3y+z} + \frac{y+2z}{2y+3z+x} + \frac{z+2x}{2z+3x+y}.$$

We received 21 submissions, all of which were correct. We present two solutions.

Solution 1, by Arkady Alt.

Let $S(x,y,z)$ denote the given expression. Then by using Cauchy-Schwarz Inequality we have

$$\begin{aligned} 3 - S(x,y,z) &= \sum_{cyc}\left(1 - \frac{x+2y}{2x+3y+z}\right) \\ &= \sum_{cyc}\frac{x+y+z}{2x+3y+z} \\ &= \frac{6(x+y+z)}{6}\sum_{cyc}\frac{1}{2x+3y+z} \\ &= \frac{1}{6}\sum_{cyc}(2x+3y+z)\cdot\sum_{cyc}\frac{1}{2x+3y+z} \\ &\geq \frac{1}{6}\cdot 9 = \frac{3}{2}. \end{aligned}$$

Hence, $S(x,y,z) \leq \dfrac{3}{2}$ and $S(x,x,x) = \dfrac{3}{2}$.

Solution 2, by Šefket Arslanagić.

Since the given inequality is homogeneous, we may assume that $x+y+z=1$. By the AM-HM Inequality, we have

$$\begin{aligned} S(x,y,z) &= 3-\left(\frac{1}{1+x+2y}+\frac{1}{1+y+2z}+\frac{1}{1+z+2x}\right) \\ &\le 3-\frac{9}{(1+x+2y)+(1+y+2z)+(1+z+2x)} \\ &= 3-\frac{9}{6}=\frac{3}{2}. \end{aligned}$$

Hence, the maximum value of $S(x,y,z)$ is $\frac{3}{2}$ attained when $x=y=z$.

Editor's Comments. Kee-Wai Lau made an interesting and not-so-easy-to-see observation that

$$S(x,y,z)-\frac{3}{2}=-\frac{\sum(3x+y+2z)(x+y-2z)^2}{6\prod(2x+3y+z)}\le 0.$$

4017. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that one of the numbers $PU\cdot BC, PV\cdot CA, PW\cdot AB$ is the sum of the other two.

From the 6 correct submissions we received, we present a composite of the similar solutions by Šefket Arslanagić, Ricard Peiró i Estruch, and Joel Schlosberg.

Since $PV\perp CA$ and $PW\perp AB$, $\angle VPW$ is either equal to or supplementary to $\angle BAC$, so

$$\sin\angle VPW=\sin\angle BAC=\sin A;$$

similarly,

$$\sin\angle WPU=\sin B \quad \text{and} \quad \sin\angle UPV=\sin C.$$

Moreover, because $PUVW$ is cyclic we have

$$\sin\angle VPW=\sin\angle VUW,\ \sin\angle WPU=\sin\angle WVU,\ \sin\angle UPV=\sin\angle UWV.$$

Finally, the Law of Sines applied to ΔUVW implies

$$\frac{VW}{WU}=\frac{\sin\angle VUW}{\sin\angle WVU} \quad \text{and} \quad \frac{UV}{WU}=\frac{\sin\angle UWV}{\sin\angle WVU},$$

while applied to ΔABC implies

$$\frac{\sin A}{\sin B}=\frac{BC}{CA} \quad \text{and} \quad \frac{\sin C}{\sin B}=\frac{AB}{CA}.$$

Let us suppose that the diagram has been labeled so that the quadrilateral $PUVW$ is cyclic in that order, whence PV is the diagonal, and Ptolemy's theorem says that $PV \cdot WU = PU \cdot VW + PW \cdot UV$. Putting it all together, we get

$$\begin{aligned} PV &= PU \cdot \frac{VW}{WU} + PW \cdot \frac{UV}{WU} \\ &= PU \cdot \frac{\sin\angle VUW}{\sin\angle WVU} + PW \cdot \frac{\sin\angle UWV}{\sin\angle WVU} \\ &= PU \cdot \frac{\sin\angle VPW}{\sin\angle WPU} + PW \cdot \frac{\sin\angle UPV}{\sin\angle WPU} \\ &= PU \cdot \frac{\sin A}{\sin B} + PW \cdot \frac{\sin C}{\sin B} \\ &= PU \cdot \frac{BC}{CA} + PW \cdot \frac{AB}{CA}. \end{aligned}$$

Thus, we conclude that $PV \cdot CA = PU \cdot BC + PW \cdot AB$; in other words, the product involving the diagonal of the quadrilateral equals the sum of the products involving the sides.

Additionally, the proposer observed (and proved) that the area of ΔUVW is independent of the choice of P on γ.

4018. *Proposed by Ovidiu Furdui.*

Let

$$I_n = \int_0^1 \cdots \int_0^1 \ln(x_1 x_2 \cdots x_n) \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n,$$

where $n \geq 1$ is an integer. Prove that this integral converges and find its value.

We received three solutions, all of which were correct and complete. We present the solution by the proposer.

The integral equals

$$n\left(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)\right),$$

where ζ denotes the Riemann zeta function.

We have, based on symmetry reasons, that for all $i, j = 1, 2, \ldots, n$

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \ln x_i \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n \\ &\quad = \int_0^1 \cdots \int_0^1 \ln x_j \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n, \end{aligned}$$

and this implies that

$$\begin{aligned}
I_n &= n\int_0^1 \cdots \int_0^1 \ln x_1 \ln(1 - x_1 x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n \\
&= n\int_0^1 \cdots \int_0^1 -\ln x_1 \sum_{k=1}^{\infty} \frac{(x_1 \cdots x_n)^k}{k} dx_1 dx_2 \cdots dx_n \\
&\overset{(*)}{=} n\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 \cdots \int_0^1 -\ln x_1 (x_1 \cdots x_n)^k dx_1 dx_2 \cdots dx_n \\
&= n\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 (-x_1^k \ln x_1) \mathrm{d}x_1 \int_0^1 x_2^k \mathrm{d}x_2 \cdots \int_0^1 x_n^k \mathrm{d}x_n \\
&= n\sum_{k=1}^{\infty} \frac{1}{k(k+1)^{n+1}}.
\end{aligned}$$

We used at step (*) Tonelli's Theorem for nonnegative functions, which allows us to interchange the integration sign and the summation sign.

Let $S_{n+1} = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)^{n+1}}$. Since

$$\frac{1}{k(k+1)^{n+1}} = \frac{1}{k(k+1)^n} - \frac{1}{(k+1)^{n+1}},$$

we have, by summation, that $S_{n+1} = S_n - (\zeta(n+1) - 1)$. This implies, since $S_1 = 1$, that

$$S_{n+1} = S_1 - (\zeta(2) + \zeta(3) + \cdots + \zeta(n+1) - n) = n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1).$$

Hence

$$I_n = n\left(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)\right),$$

and the problem is solved.

4019. *Proposed by George Apostolopoulos.*

A triangle with side lengths a, b, c has perimeter 3. Prove that

$$a^3 + b^3 + c^3 + a^4 + b^4 + c^4 \geq 2(a^2b^2 + b^2c^2 + c^2a^2).$$

We received 21 correct solutions. We present the solution by AN-anduud Problem Solving Group.

The claimed inequality is equivalent to

$$(a^3 + b^3 + c^3)(a + b + c) + 3(a^4 + b^4 + c^4) \geq 6(a^2b^2 + b^2c^2 + c^2a^2)$$

or

$$[(a^3b+b^3a)+(a^3c+c^3a)+(b^3c+c^3b)]+[2(a^4+b^4)+2(b^4+c^4)+2(c^4+a^4)]$$
$$\geq 6(a^2b^2+b^2c^2+c^2a^2).$$

By the AM-GM Inequality we have

$$a^3b+b^3a \geq 2a^2b^2,\ a^3c+c^3a \geq 2a^2c^2,\ b^3c+c^3b \geq 2b^2c^2$$

and

$$a^4+b^4 \geq 2a^2b^2,\ b^4+c^4 \geq 2b^2c^2,\ c^4+a^4 \geq 2a^2c^2.$$

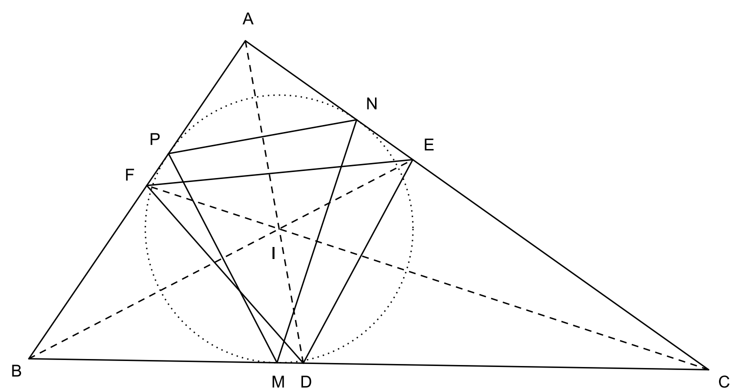
Adding the above inequalities, we obtain the desired inequality. Equality holds if and only if $a=b=c=1$.

4020. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a triangle and let the internal bisectors from A, B and C intersect the sides BC, CA and AB in D, E and F, respectively. The incircle of ΔABC touches the sides BC, CA and AB in M, N, and P, respectively. Prove that $[MNP] \leq [DEF]$, where $[\cdot]$ denotes the area of the specified triangle.

We received eleven submissions, of which nine were correct and complete. We present the solution by Šefket Arslanagić, slightly modified by the editor.

Denote by α, β and γ the angles BAC, ABC and respectively ACB of the triangle, and let r be the radius of the incircle.



From the quadrilateral $PIMB$ note that $\angle PIM = 180^\circ - \angle ABC = 180^\circ - \beta$, whence

$$[PIM] = \frac{PI \cdot MI}{2}\sin(\angle PIM) = \frac{r^2}{2}\sin(180^\circ - \beta) = \frac{r^2}{2}\sin\beta.$$

Similarly, we calculate $[MIN]$ and $[NIP]$, and we get

$$\begin{aligned}[MNP] &= [PIM]+[MIN]+[NIP] \\ &= \frac{r^2}{2}\cdot(\sin\beta+\sin\gamma+\sin\alpha).\end{aligned} \tag{1}$$

On the other hand, we have $\angle FID = \angle AIC = 180^\circ - \frac{\alpha}{2} - \frac{\gamma}{2}$, and so

$$[FID] = \frac{ID \cdot IF}{2} \sin(\angle FID) = \frac{ID \cdot IF}{2} \sin \frac{\alpha+\gamma}{2}.$$

Similarly, calculate the area of $\triangle EIF$ and $\triangle DIE$. We have

$$\begin{aligned} [DEF] &= [DIE] + [EIF] + [FID] \\ &= \frac{ID \cdot IE}{2} \sin \frac{\alpha+\beta}{2} + \frac{IE \cdot EF}{2} \sin \frac{\beta+\gamma}{2} + \frac{ID \cdot IF}{2} \sin \frac{\alpha+\gamma}{2}. \end{aligned} \tag{2}$$

The triangles $\triangle PIF$, $\triangle MID$ and $\triangle NIE$ are all right-angled triangles, from which it follows that $IF \geq r$, $ID \geq r$ and $IE \geq r$. Hence, from the formula for $[DEF]$ above we get

$$[DEF] \geq \frac{r^2}{2} \left(\sin \frac{\alpha+\beta}{2} + \sin \frac{\beta+\gamma}{2} + \sin \frac{\alpha+\gamma}{2} \right).$$

Comparing this with the formula for $[MNP]$ in (1), in order to show that $[MNP] \leq [DEF]$ it is sufficient to show that

$$\sin\alpha + \sin\beta + \sin\gamma \leq \sin \frac{\alpha+\beta}{2} + \sin \frac{\beta+\gamma}{2} + \sin \frac{\alpha+\gamma}{2}. \tag{3}$$

However, using the sum to product trigonometric formula, we have

$$\sin\alpha + \sin\beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \leq 2 \sin \frac{\alpha+\beta}{2},$$

where the inequality follows from the fact that $\sin \frac{\alpha+\beta}{2} \geq 0$ and $\cos \frac{\alpha-\beta}{2} \leq 1$. Similarly we have $\sin\beta + \sin\gamma \leq 2 \sin \frac{\beta+\gamma}{2}$ and $\sin\gamma + \sin\alpha \leq 2 \sin \frac{\alpha+\gamma}{2}$, which we can add to get the inequality in (3), the final step we need in order to conclude that $[MNP] \leq [DEF]$.

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