Crux Mathematicorum

VOLUME 41, NO. 7 September / Septembre 2015

IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

We get a lot of problem proposals (keep them coming!) and, inevitably, some start with a statement isomorphic to "using standard notation", which always makes me smile. Mathematics is a language in itself and different notations are its dialects: some I read past without even noticing and some I have to look up to understand.

Even something as basic as numbers and algorithms get lost in translation: on North American blackboards, 7 is suddenly stripped of its horizontal bar and 9 loses its resemblance to a lowercase "g", while long division looks nothing like it did in my grade school. The quirks of the English language do not make it any easier: the redundancy in "straight lines" makes you question whether there are any curvy ones and "combination locks" should really be called "permutation locks" because the order of the numbers does matter.

Moving on to something less culture-dependent, we often see different representations of the same thing: $\arctan x$ versus $\tan^{-1} x$, C_n^k versus ${}_nC_k$ versus $C(n,k)$ versus $\binom{n}{k}$. Ambiguous, but widely accepted, notations are a whole other story; in this realm, we have $\sin^2 x$, f^{-1}, $f'(x)$, $f^{(5)}(x)$, etc. Some words are much over-used (regular, homogeneous, normal, trivial) and so are some symbols. Consider a simple $\pm$ as it can mean both a whole interval of values (in statistics, we use $\mu \pm \sigma$) or just one of the two possibilities (in quadratic formula, $x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$). Simpler yet, the vertical bar symbol: it means "such as" in set-builder notation, "divides" in integer arithmetic, "given" in probability, "or" in logic, "evaluate at" or "restrict to" in function notation, ...

As with language dialects, our notation preferences show our heritage and background, both mathematical and cultural. At ***Crux***, we value the diversity and distinctiveness of our readers and their contributions. So submit your problem proposals and your solutions to `crux-psol@cms.math.ca` and we will gladly decipher your "standard notations".

Kseniya Garaschuk

THE CONTEST CORNER

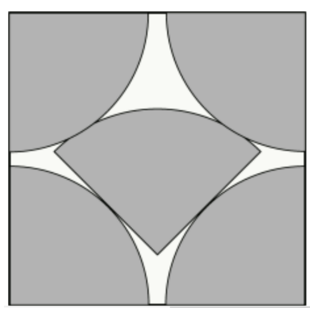
No. 37

Robert Bilinski

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

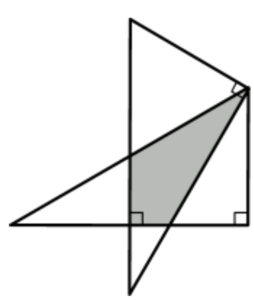
To facilitate their consideration, solutions should be received by the editor by **September 1, 2016**, *although late solutions will also be considered until a solution is published.*

CC181. A delivery boy decides to optimize the transportation of his pizzas. In his system, each box contains entirely and without any overlap 5 identical quarter pizzas (drawn in grey).



The box is square, the drawing is symmetrical with respect to the center line and all contacts seen are mathematically perfect. The radius of a pizza is 16 cm. What is the minimal area of the bottom of the box rounded off to the nearest integer?

CC182. We rearrange two halves of an equilateral triangle (cut along one of its altitudes) in the following way:



The area of the equilateral triangle was 600 cm^2. What is the area of the overlap region common to the 2 triangles?

CC183. We call a number *productive* if all the products of consecutive digits of the number can be found in its written form. 2013 and 1261 are examples of such numbers. Taking the first one as an example, we get the following consecutive products $2 \times 0 = 0, 0 \times 1 = 0$ and $1 \times 3 = 3$ which can all be found in the written form of 2013. For the second number, the products are $1 \times 2 = 2, 2 \times 6 = 12$ and $6 \times 1 = 6$ which can all be read in 1261. What is the smallest productive number which can be written using all the digits from 0 to 9?

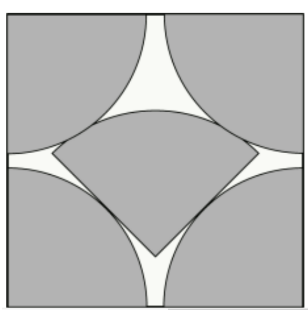
CC184. We are looking for two positive integers such that the difference of their squares is a cube and the difference of their cubes is a square. What is the value of the greatest of the two given that it is smaller than 20?

CC185. Each asterisk in the following multiplication can only be replaced by a digit in the set $\{2, 3, 5, 7\}$. Complete the multiplication.

```
      * * *
  ×     * *
  ---------
    * * * *
  * * * *
  ---------
  * * * * *
```

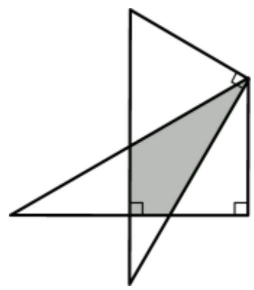
. . . .

CC181. Un livreur décide d'optimiser le transport des pizzas. Dans son système, chaque boîte contient entièrement et sans recouvrement 5 quarts de pizza identiques (en gris sur le dessin).



La boîte est carrée, la figure est symétrique par rapport à la droite médiane et tous les contacts entre les morceaux sont mathématiquement parfaits. Le rayon d'un quart de pizza est 16 cm. Quelle est l'aire du fond de la boîte au minimum arrondi à l'entier le plus près?

CC182. On place deux moitiés d'un triangle équilatéral (découpé le long d'une hauteur) comme dans la figure suivante.



Le triangle équilatéral avait une aire de 600 cm^2. Quelle est l'aire de la partie commune où les deux triangles chevauchent?

CC183. On appelle un nombre *productif* si tous les produits de deux chiffres consécutifs de ce nombre se lisent dans l'écriture de celui-ci. 2013 et 1261 en sont des exemples. Pour le premier, les produits sont $2\times 0 = 0, 0\times 1 = 0$ et $1\times 3 = 3$ sont tous dans l'écriture de 2013. Pour le second, les produits sont $1 \times 2 = 2, 2\times 6 = 12$ et $6\times 1 = 6$ on les retrouve tous dans 1261. Quel est le plus petit nombre productif qui utilise les 10 chiffres de 0 à 9?

CC184. On cherche 2 entiers positifs tels que la différence de leurs carrés est un cube et la différence de leurs cubes est un carré. Quelle est la valeur du plus grand d'entre eux sachant qu'il est inférieur à 20?

CC185. Chaque astérisque de la multiplication suivante ne peut être remplacé que par un chiffre parmi $\{2, 3, 5, 7\}$. Complétez la multiplication.

$$
\begin{array}{rrrrr}
 & & * & * & * \\
 & \times & & * & * \\
\hline
 & * & * & * & * \\
* & * & * & * & \\
\hline
* & * & * & * & *
\end{array}
$$

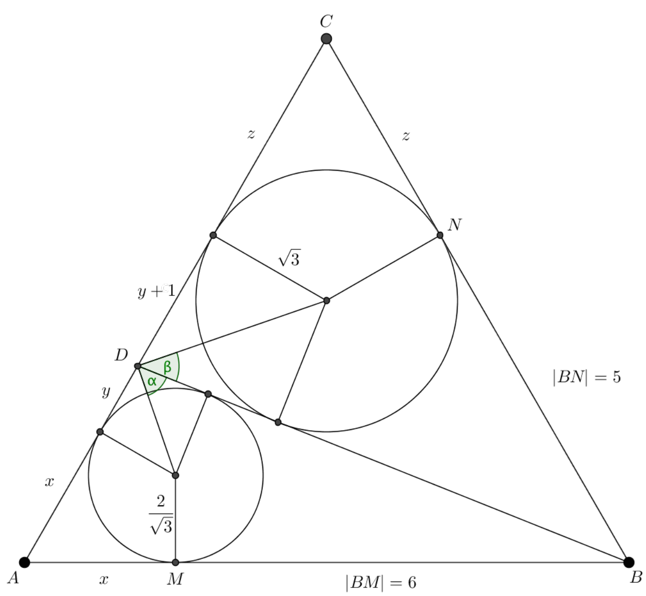
CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(7), p. 276–277 unless noted otherwise.

CC131. Let D be the point on the side AC of triangle ABC. Circle of radius $2/\sqrt{3}$ is inscribed in triangle ABD and touches AB at a point M; circle of radius $\sqrt{3}$ is inscribed in triangle BCD and touches BC at a point N. Given that $|BM| = 6$ and $|BN| = 5$, find the lengths of sides of triangle ABC.

Originally question 4 from the 1981 entrance exam to Moscow Physics Institute.

We received five correct solutions. We present the solution by Andrea Fanchini.



With reference to the figure we have $2\alpha + 2\beta = \pi$, so $\alpha + \beta = \frac{\pi}{2}$. Then

$$\arctan \frac{\frac{2}{\sqrt{3}}}{y} + \arctan \frac{\sqrt{3}}{y+1} = \frac{\pi}{2}$$

from which we obtain

$$\arctan \frac{\frac{2}{\sqrt{3}}}{y} = \arctan \frac{y+1}{\sqrt{3}} \quad \iff \quad y^2 + y - 2 = 0 \quad \iff \quad y = 1.$$

Joining the three vertices of any triangle to the centre of the inscribed circle, the triangle is divided into three triangles. If the sidelengths of the larger triangle are

a, b, and c, and the radius of the inscribed circle is r, the area of the triangle is the sum of the three smaller triangles: $\triangle = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc$, or $\triangle = r\frac{1}{2}(a+b+c)$. Using Heron's formula, with $s = \frac{1}{2}(a+b+c)$ the semiperimeter of the triangle, the area of the triangle is $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, so

$$rs = \sqrt{s(s-a)(s-b)(s-c)} \qquad (1)$$

In triangle ABD, we have $a = 7$, $b = x + 6$, $c = x + 1$, $s = x + 7$, $r = \frac{2}{\sqrt{3}}$. Substituting into (1), we have

$$\frac{2}{\sqrt{3}}(x+7) = \sqrt{6x(x+7)}$$

and hence $x = 2$. Similarly, in triangle BCD we have $a = 7$, $b = z + 2$, $c = z + 5$, $s = z + 7$, $r = \sqrt{3}$. Substituting into equation 1, we get

$$\sqrt{3}(z+7) = \sqrt{10z(z+7)},$$

so $z = 3$.

So the triangle ABC is equilateral with sides $AB = BC = CA = 8$.

CC132. You are told the following four statements about natural numbers n and k:

(a) $n + 1$ is divisible by k,

(b) $n = 2k + 5$,

(c) $n + k$ is divisible by 3,

(d) $n + 7k$ is a prime number.

Three of these statements are true and one is not. Find all possible pairs (n, k).

Originally question 2 from the 1980 entrance exam to Moscow Physics Institute correspondence math school.

There were five correct solutions for this problem. We present the solution by Digby Smith.

Suppose statements (c) and (d) are both true. Then there exists a natural number m such that $n+k = 3m$. It follows that $n+7k = (n+k)+6k = 3m+6k = 3(m+2k)$. This number cannot be prime, so one of (c) or (d) must be false.

Now suppose that (b) and (c) are both true. Then again there exists an m such that $n + k = 3m$. Given that $n = 2k + 5$ we must have $3m = n + k = 3k + 5$. However 5 is not divisible by 3, so either (b) or (c) must be false.

Since there is only one false statement, it must be that (c) is false.

Give that (a) and (b) are both true, k divides $n + 1 = 2k + 6$, so k divides 6. This gives 1, 2, 3, and 6 as the possible values for k. The corresponding n values are 7, 9, 11, and 17. Since $n + 7k$ is a prime number, this leaves only $(9, 2)$ and $(17, 6)$ as the possible pairs (n, k).

CC133. Ten numbers, not necessarily unique, are written in a row. Then, under every number, we write how many numbers in this row are smaller than it. Can the second row be

a) 9 0 0 2 5 3 6 3 6 6?

b) 5 6 1 1 4 8 5 8 0 1?

Originally question 5 from the 1982 entrance exam to All-union correspondence math school.

We received two correct solutions. We present the solution given by Digby Smith.

a) The number 9002536366 will suffice.

b) No. Suppose there exists such a number $ABCDEFGHIJ$. Since the second row is 5611485801, we need the following condition to hold

$$F = H > B > A = G > E > C = D = J > I.$$

However, this implies that there are 7 numbers less than B and not the stated 6. Thus there is a contradiction.

CC134.

(Original version appeared in 40(7), p. 276–277.) Let two tangent lines from the point $M(1, 1)$ to the graph of $y = k/x$, $k < 0$ touch the graph at the points A and B. Suppose that the triangle MAB is a right-angle triangle. Find its area and the value of constant k.

(Correction appeared in 41(1), p. 4–5.) Let two tangent lines from the point $M(1, 1)$ to the graph of $y = k/x$, $k < 0$ touch the graph at the points A and B. Suppose that the triangle MAB is an equilateral triangle. Find its area and the value of constant k.

Originally question 4 from the 1982 entrance exam to Moscow Physics Institute.

We received 4 correct solutions. One addressed the original version of the problem; two addressed the corrected version; and one provided a solution to a generalized version of the problem. We present the solution to the corrected problem by Šefket Arslanagić.

Let A be the point in the second quadrant, with B as the point in the fourth quadrant. We have $y = \frac{k}{x}$ with $k < 0$, which has a derivative of $y' = -k/x^2$ with $k < 0, x \neq 0$. The tangent line t is thus

$$y - 1 = -\frac{k}{x^2}(x - 1),$$

so the x-coordinates of the tangent points to the curve are

$$\begin{aligned} \frac{k}{x} - 1 &= -\frac{k}{x^2}(x-1), \quad x^2 \neq 0 \\ kx - x^2 &= -kx + k \\ 0 &= x^2 - 2kx + k \\ x_{1,2} &= k \pm \sqrt{k^2 - k} \end{aligned}$$

Then the y-coordinates of the tangent points are

$$y_{1,2} = \frac{k}{x_{1,2}} = \frac{k}{k \pm \sqrt{k^2-k}} = k \mp \sqrt{k^2-k},$$

giving points $A(k - \sqrt{k^2-k}, k + \sqrt{k^2-k})$ and $B(k + \sqrt{k^2-k}, k - \sqrt{k^2-k})$. The side lengths follow as

$$\begin{aligned} (AB)^2 &= (2\sqrt{k^2-k})^2 + (-2\sqrt{k^2-k})^2 = 8(k^2-k), \\ (AM)^2 &= (k - \sqrt{k^2-k} - 1)^2 + (k + \sqrt{k^2-k} - 1)^2 \\ &= [(k-1) - \sqrt{k^2-k}]^2 + [(k-1) + \sqrt{k^2-k}]^2 \\ &= 2(k-1)^2 + 2(k^2-k) = 4k^2 - 6k + 2, \end{aligned}$$

and similarly $(BM)^2 = 4k^2 - 6k + 2$ too. The triangle ΔMAB is equilateral, from which it follows that $(AM)^2 = (BM)^2 = (AB)^2$:

$$\begin{aligned} 4k^2 - 6k + 2 &= 8(k^2 - k) \\ 4k^2 - 2k - 2 &= 0 \\ 2k^2 - k - 1 &= 0 \\ k = -1/2 \quad &\text{or} \quad k = 1, \end{aligned}$$

so we take $k = -1/2$ to satisfy $k < 0$. This gives $(AB)^2 = 8(\frac{1}{4} + \frac{1}{2}) = 6$, so $AB = \sqrt{6}$, and the area of the triangle MAB is $\frac{(\sqrt{6})^2}{4}\sqrt{3} = \frac{3}{2}\sqrt{3}$.

Editor's comment. The correct answer to the question as originally posed – with a right-angled triangle – is that there is no solution; the triangle must be acute. The generalized version of the problem (which addresses a triangle with no restrictions) finds that the triangle must be isosceles, with an area of $2(1-k)\sqrt{k(k-1)}$.

CC135. Consider the following two arithmetic progressions:

$$\log a, \log b, \log c \quad \text{and} \quad \log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a.$$

Can the values a, b, c be the lengths of the sides of a triangle? If so, find the interior angles of this triangle.

Originally question 2 from the 1980 entrance exam to Voronezh State University.

We received five correct solutions which are all similar. We present a composite of all these solutions.

From the given assumptions we have:

(i) $\log b^2 = 2\log b = \log a + \log c = \log ac$, so

$$b^2 = ac. \tag{1}$$

(ii) $2(\log 2b - \log 3c) = (\log a - \log 2b) + (\log 3b - \log a)$
$= \log 3c - \log 2b$ or $3\log 2b = 3\log 3c$, so

$$2b = 3c. \tag{2}$$

Solving (1) and (2) we obtain $(a, b, c) = (9r, 6r, 4r)$, where $r \in \mathbb{R}$, $r > 0$. Since $9r < 6r + 4r$, the values of a, b, c can be the lengths of the sides of a triangle. Using the law of cosines we find that

$$\cos A = \frac{36 + 16 - 81}{48} = -\frac{29}{48}, \quad \cos B = \frac{16 + 81 - 36}{72} = \frac{69}{72}$$
$$\text{and} \quad \cos C = \frac{81 + 36 - 16}{108} = \frac{101}{108}.$$

The approximate values of the angles in degrees, to 4 decimal places, are:
$A = 127.1689°$, $B = 32.0891°$, and $C = 20.7419°$.

THE OLYMPIAD CORNER

No. 335

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **September 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

OC241. Let n be a natural number. For all positive real numbers $x_1, \dots, x_{n+1}$ such that $x_1 x_2 ... x_{n+1} = 1$ prove that:

$$n^{1/x_1} + \cdots + n^{1/x_{n+1}} \geq n^{\sqrt[n]{x_1}} + \cdots + n^{\sqrt[n]{x_{n+1}}}.$$

OC242. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

OC243. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all the functions f, $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

OC244. $ABCD$ is a cyclic quadrilateral, with diagonals AC, BD perpendicular to each other. Let point F be on side BC, the parallel line EF to AC intersect AB at point E, line FG parallel to BD intersect CD at G. Let the projection of E onto CD be P, projection of F onto DA be Q, projection of G onto AB be R. Prove that QF bisects $\angle PQR$.

OC245. Find all sets of 2014 not necessarily distinct rationals such that: if we remove an arbitrary number in the set, we can divide the remaining 2013 numbers into three sets such that each set has exactly 671 elements and the product of all elements in each set are the same.

. . . .

OC241. Soit n un nombre naturel. Pour des nombres réels positifs $x_1, \ldots, x_{n+1}$ tels que $x_1 x_2 ... x_{n+1} = 1$ démontrer que:

$$n^{1/x_1} + \cdots + n^{1/x_{n+1}} \geq n^{\sqrt[n]{x_1}} + \cdots + n^{\sqrt[n]{x_{n+1}}}.$$

OC242. Soit k un entier positif. Deux personnes A et B s'amusent à un jeu sur une grille hexagonale infinie. Au départ, toutes les cellules sont vides. Les joueurs alternent avec A allant premier. À chaque tour, A peut choisir deux cellules adjacentes vides et placer un jeton dans chacune. À chaque tour, B peut choisir et enlever un seul jeton. Si à un quelconque moment il se trouve k cellules consécutives sur une même ligne, chacune avec un jeton, A gagne. Déterminer la valeur minimale de k telle que A ne peut pas gagner dans un nombre fini d'étapes, ou démontrer qu'aucun tel minimum existe.

OC243. Soit $\mathbb{Z}_{>0}$ l'ensemble des entiers positifs. Déterminer toutes les fonctions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ telles que

$$m^2 + f(n) \mid mf(m) + n$$

pour tout m et n entiers positifs.

OC244. Soit $ABCD$ un quadrilatère cyclique où les diagonales AC et BD sont perpendiculaires. Soit E sur AB et F sur BC de façon à ce que EF soit parallèle à AC; soit G sur CD de façon à ce que FG soit parallèle à BD; soit P la projection de E vers CD; soit Q la projection de F vers DA; soit R la projection de G vers AB. Démontrer que QF bissecte $\angle PQR$.

OC245. Déterminer tous les ensembles de 2014 rationnels, pas nécessairement distincts, tels que si on enlève un quelconque rationnel de l'ensemble, les 2013 rationnels restants peuvent être séparés en trois ensembles ayant chacun 671 éléments et dont le produit des membres de chacun de ces ensembles est le même.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(5), p. 194–195.

OC181. All the prime numbers are written in order $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on. Find all pairs of positive integers a and b with $a - b \geq 2$ such that $p_a - p_b$ divides $2(a - b)$.

Originally problem 1 from the 2013 Mexico National Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

The pair

$$a = 4, \quad b = 2$$

is a solution, since $p_4 = 7$ and $p_2 = 3$ and $p_4 - p_2 = 4$ is a divisor of $2(4 - 2) = 4$. We show that there are no other solutions.

Suppose (a, b) is any solution. If $b = 1$ then $p_b = 2$, and $p_a - p_b$ is an odd number that divides $a - b$. Moreover, $p_a \geq 2a - 1$ and $a > 2$. Hence

$$a - b \geq p_a - p_b \geq 2a - 3 > a - 1 = a - b,$$

a contradiction. Thus the numbers p_a and p_b are odd primes, which implies that $p_a - p_b \geq 2(a - b)$. We obtain $p_a - p_b = 2(a - b)$ so that all odd numbers between p_b and p_a are primes. Then, the numbers

$$p_b, \quad p_b + 2, \quad p_b + 4 \tag{3}$$

are primes. But one of the numbers (1) is divisible by 3. It follows $p_b = 3$. Since $a - b \geq 2$ and all odd numbers between 3 and p_a are primes, we have $p_a = 7$.

OC182. Let x and y be real numbers satisfying $x^2y^2 + 2yx^2 + 1 = 0$. If

$$S = \frac{2}{x^2} + 1 + \frac{1}{x} + y\left(y + 2 + \frac{1}{x}\right),$$

find the maximum and minimum values of S.

Originally problem 2 from the 2013 Uzbekistan National Olympiad.

We received five correct submissions and one incorrect submission. We present the solution by Arkady Alt.

Note that

$$\begin{aligned} x^2y^2 + 2yx^2 + 1 = 0 &\iff x^2y^2 + 2yx^2 + x^2 + 1 - x^2 = 0 \\ &\iff x^2(y+1)^2 + 1 - x^2 = 0 \\ &\iff (y+1)^2 + \frac{1}{x^2} = 1 \end{aligned}$$

This implies that there is a real number $t \neq \frac{(2n+1)\pi}{2}$ with $y+1 = \sin t$ and $\frac{1}{x} = \cos t$. Further,

$$\begin{aligned} S &= \frac{2}{x^2} + \left(2 + \frac{1}{x} + y\right) - (y+1) + y\left(y + 2 + \frac{1}{x}\right) \\ &= \frac{2}{x^2} + (y+1)\left(y + 1 + \frac{1}{x}\right) \\ &= \frac{2}{x^2} + (y+1)^2 + (y+1)\cdot\frac{1}{x} \\ &= 1 + \frac{1}{x^2} + \frac{1}{x}(y+1) \end{aligned}$$

Combining the above information yields

$$S = 1 + \cos^2 t + \sin t \cos t = \frac{3 + \cos 2t + \sin 2t}{2} = \frac{3 + \sqrt{2}\sin(2t + \pi/4)}{2}$$

and, therefore,

$$S_{\max} = \frac{3+\sqrt{2}}{2}, \qquad S_{\min} = \frac{3-\sqrt{2}}{2}.$$

Note that these values are actually obtained. For $t^* = \pi/8$ we have

$$(x^*, y^*) = \left(\frac{1}{\cos\pi/8}, \sin\pi/8 - 1\right),$$

and hence $S(x^*, y^*) = \dfrac{3+\sqrt{2}}{2}$. On the other hand, for $t_* = -3\pi/8$, we have

$$(x_*, y_*) = \left(\frac{1}{\cos(-3\pi/8)}, \sin(-3\pi/8) - 1\right),$$

and hence $S(x_*, y_*) = \dfrac{3-\sqrt{2}}{2}$, since

$$\cos\pi/8 = \frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \sin\pi/8 = \cos(-3\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2},$$
$$\sin(-3\pi/8) = -\cos\pi/8 = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

OC183. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(0) = 0$, $f(1) = 2013$ and

$$(x-y)(f(f(x)^2) - f(f(y)^2)) = (f(x) - f(y))(f(x)^2 - f(y)^2).$$

Originally problem 1 from day 2 of the 2013 Vietnam National Olympiad.

We received two correct submissions. We present the solution by Michel Bataille.

The function $x \mapsto 2013x$ is the only solution.

We set $k = 2013$ in what follows.

If $f(x) = kx$, both sides of the given equation equal $k^3(x-y)(x^2-y^2)$, as it is readily checked. Also, $f(0) = 0$ and $f(1) = k$ clearly hold. Thus $x \mapsto kx$ is a solution.
Conversely, let f be an arbitrary solution. Taking $y = 0$ in the equation, we see that

$$xf(f(x)^2) = f(x)^3 \tag{1}$$

for any real number x. With the help of (1), the given equation first becomes

$$xf(f(y)^2) + yf(f(x)^2) = f(x)f(y)^2 + f(y)f(x)^2$$

and then, multiplying by xy, $x^2f(y)^3 + y^2f(x)^3 = xy(f(x)f(y)^2 + f(y)f(x)^2)$ that is,

$$(xf(y) - yf(x))(xf(y)^2 - yf(x)^2) = 0. \tag{2}$$

Now, let $a, b \in \mathbb{R}$ with $a \neq 0$. With $x = a, y = ab$, (2) yields

$$(af(ab) - bf(a))(af(ab)^2 - bf(a)^2) = 0. \tag{3}$$

With $a = 1$ and $b < 0$, the second factor $f(b)^2 - bk^2$ is positive, hence $f(b) = kb$.

Now, with $b = -1$, (3) gives $(f(-a) + f(a))(f(-a)^2 + f(a)^2) = 0$ for all $a \neq 0$. If $a > 0$, the second factor $(k(-a))^2 + f(a)^2 = k^2a^2 + f(a)^2$ is positive, hence $f(a) = -f(-a) = -k(-a) = ka$. We conclude that $f(x) = kx$, x being positive or not. Also $f(0) = 0 = k \cdot 0$ so that $f(x) = kx$ for all real numbers x.

OC184. Let k, m and n be three distinct positive integers. Prove that

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n).$$

Originally problem 5 from day 2 of the 2013 Polish Mathematical Olympiad.

We received six correct submissions. We present the solution by Šefket Arslanagić.

Proceeding algebraically, we have the following equivalence:

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n)$$
$$(k^2-1)(m^2-1)(n^2-1) \leq k^2m^2n^2 - k^2mn - km^2n - kmn^2.$$

Rearranging gives

$$\begin{aligned}
&\iff 0 \leq k^2m^2 + k^2n^2 - k^2mn - k^2 - km^2n - kmn^2 + m^2n^2 - m^2 - n^2 + 1\\
&\iff 0 \leq (m^2 + n^2 - mn - 1)k^2 - mn(m+n)k + (m^2-1)(n^2-1)\\
&\iff 0 \leq (m^2 + n^2 - mn - 1)(k^2-1) - mn(m+n)k + m^2n^2 - mn.
\end{aligned}$$

Since $m^2 + n^2 - mn - 1 \geq mn$ holds if and only if $(m-n)^2 \geq 1$ (since $m, n \in \mathbb{N}$ and are distinct), we can prove the claim by showing the last inequality is true by noting that

$$\begin{aligned}(m^2 + n^2 - mn - 1)(k^2 - 1) - mn(m+n)k + m^2n^2 - mn \\ &\geq (k^2-1)mn - mn(m+n)k + m^2n^2 - mn \\ &= mn(k^2 - 1 - (m+n)k + mn - 1) \\ &= mn((m-k)(n-k) - 2) \geq 0,\end{aligned}$$

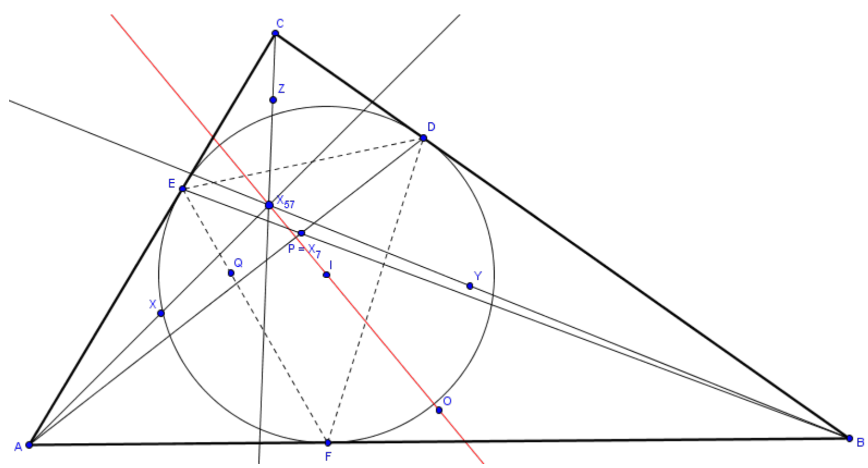
where the last inequality holds since by symmetry, we may suppose without loss of generality that $m > n > k$, and so $(m-k)(n-k) \geq 2$. Therefore, the given inequality holds.

OC185. The incircle of ΔABC touches sides BC, CA and AB at points D, E and F respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of ΔABC.

Originally problem 6 from day 2 of the 2013 Brazil National Olympiad.

We received two correct submissions. We present the solution by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations.



Let $s = \frac{a+b+c}{2}$. The points of tangency of the incircle with the sides are

$$D(0 : s-c : s-b), E(s-c : 0 : s-a) \text{ and } F(s-b : s-a : 0).$$

Then the point P is the Gergonne point (X_7 in the diagram) that has coordinates $P((s-b)(s-c) : (s-a)(s-c) : (s-a)(s-b))$. Now line EF has equation

$$\begin{vmatrix} s-c & 0 & s-a \\ s-b & s-a & 0 \\ x & y & z \end{vmatrix} = 0 \quad \Rightarrow \quad EF : (s-a)x - (s-b)y - (s-c)z = 0.$$

Then the infinite perpendicular point of the line EF is $EF_{\infty\perp}(-b-c : b : c)$.

So the equation of the line that passes through P and is perpendicular to EF is

$$\begin{vmatrix} \frac{1}{s-a} & \frac{1}{s-b} & \frac{1}{s-c} \\ -b-c & b & c \\ x & y & z \end{vmatrix} = 0$$

giving the equation

$PEF_{\infty\perp} \equiv 2(b-c)(s-a)^2x - b(s-b)(-a+b+3c)y + c(s-c)(-a+3b+c)z = 0.$

We can denote with Q the intersection point between lines $PEF_{\infty\perp}$ and EF:

$Q(2(s-b)(s-c)((s-a)(b+c)+2bc) : b(s-a)(s-c)(-a+b+3c) : c(s-a)(s-b)(-a+3b+c)).$

Now the point X is the symmetrical point of P with respect to Q and so has coordinates

$$X((s-b)(s-c)((s-a)(b+c)+bc) : b(s-a)^2(s-c) : c(s-a)^2(s-b)).$$

Cyclically, we have

$$Y(a(s-b)^2(s-c) : (s-c)(s-a)((s-b)(c+a)+ca) : c(s-b)^2(s-a)),$$

$$Z(a(s-c)^2(s-b) : b(s-c)^2(s-a) : (s-a)(s-b)((s-c)(a+b)+ab)).$$

Then, lines AX, BY and CZ have equations given by

$$\begin{aligned} AX &: -c(s-b)y + b(s-c)z = 0, \\ BY &: c(s-a)x - a(s-c)z = 0, \\ CZ &: -b(s-a)x + a(s-b)y = 0. \end{aligned}$$

So lines AX, BY and CZ are concurrent if and only if

$$\begin{vmatrix} 0 & -c(s-b) & b(s-c) \\ c(s-a) & 0 & -a(s-c) \\ -b(s-a) & a(s-b) & 0 \end{vmatrix} = 0$$

and we can see that this is true since

$$(s-a)(s-b)(s-c)\begin{vmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{vmatrix} = (s-a)(s-b)(s-c)(-abc+abc) = 0.$$

The concurrent point as intersection between the lines AX and BY is

$$AX \cap BY = (a(s-b)(s-c) : b(s-a)(s-c) : c(s-a)(s-b)).$$

This is the point X_{57} in the diagram, that is the isogonal conjugate of X_9 (Mittenpunkt) and it lies on the line IO as shown in the diagram.

FOCUS ON...

No. 18
Michel Bataille
Congruences (II)

Introduction and first examples

In this second number dedicated to congruences, we focus on congruences modulo a prime number. More specifically, we will gather some typical examples of problems related to the following elementary results: if p is a prime, then modulo p,

$$\text{for } k = 1, 2, \ldots, p-1, \quad \binom{p}{k} \equiv 0, \tag{1}$$

$$\text{for any integer } \; n, \quad n^p \equiv n, \tag{2}$$

$$(p-1)! \equiv -1. \tag{3}$$

Statement (2) is often called Fermat's Little Theorem and (3) is Wilson's Theorem. These three results should be ready-for-use in every problem-solver's outfit. We only sketch (cascading) proofs. Property (1) follows from $k\binom{p}{k} = p\binom{p-1}{k-1}$; since p and k are coprime, p must divide $\binom{p}{k}$. For (2), we may suppose $n > 0$ and use induction. The induction step follows from the binomial theorem and (1) which yield $(n+1)^p \equiv n^p + 1 \pmod{p}$. As for (3), it is obvious if $p = 2$ and if p is odd, we can use the polynomial $x^{p-1} - 1$. From (2), its roots in $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ are $1, 2, \ldots, p-1$. Their product (in $\mathbb{Z}_p$) is $(-1)^{p-1} \cdot (-1) = -1$, meaning that $(p-1)! \equiv -1 \pmod{p}$.

Various examples

Our first example is adapted from a problem set in *The Mathematical Gazette* in 2002:

> Let (L_n) be the Lucas sequence defined recursively by $L_0 = 2, L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for every positive integer n. Show that for any odd prime p we have $L_p \equiv 1 \pmod{p}$.

We use a known expression of L_n, namely, $L_n = u^n + v^n$ for all nonnegative integers n, where $u = \frac{1+\sqrt{5}}{2}$ and $v = \frac{1-\sqrt{5}}{2} = 1 - u$. Then, for any odd prime,

$$L_p = u^p + (1-u)^p = 1 + \sum_{k=1}^{p-1} (-1)^k \binom{p}{k} u^k$$

as well as

$$L_p = v^p + (1-v)^p = 1 + \sum_{k=1}^{p-1} (-1)^k \binom{p}{k} v^k.$$

Adding the two yields

$$2L_p = 2 + \sum_{k=1}^{p-1} (-1)^k \binom{p}{k} (u^k + v^k) = 2 + \sum_{k=1}^{p-1} (-1)^k \binom{p}{k} L_k.$$

Now,

$$\sum_{k=1}^{p-1} (-1)^k \binom{p}{k} L_k$$

is an integer (since all the terms of the Lucas sequence are integers) and this integer is divisible by p because of (1). It follows that $2L_p \equiv 2 \pmod{p}$ and since 2 and p are coprime, $L_p \equiv 1 \pmod{p}$.

We turn to an application of (2), extracted from a problem proposed by the University of Bilkent as problem of the month in 2008:

> Let p be a prime number, with $p \equiv 1 \pmod{6}$. Show that the integer 2^{2^p-2} is congruent to 1 modulo 127 and also modulo $2^p - 1$.

Set $p = 6k + 1$. Since from (2), $2^6 \equiv 1 \pmod{7}$, we have

$$2^p = 2^{6k+1} = (2^6)^k \cdot 2 \equiv 2 \pmod{7}$$

and so

$$2^{2^p-2} - 1 = 2^{7u} - 1 = (2^7)^u - 1$$

for some integer u. Since $2^7 = 128 \equiv 1 \pmod{127}$, we obtain

$$2^{2^p-2} - 1 \equiv 0 \pmod{127}.$$

Again from (2), we deduce $2^p \equiv 2 \pmod{p}$, hence

$$2^{2^p-2} - 1 = 2^{vp} - 1 = (2^p)^v - 1$$

for some integer v. Since $2^p \equiv 1 \pmod{2^p - 1}$, we obtain $2^{2^p-2} - 1 \equiv 0 \pmod{2^p - 1}$.

The following generalization of (2) is of interest:

> If p is a prime and r is a positive integer, then for any integer a we have
> $$a^{p^r} \equiv a^{p^{r-1}} \pmod{p^r}.$$

Noticing that (2) is just the case $r = 1$ prompts us to use induction. Let us assume that $a^{p^r} \equiv a^{p^{r-1}} \pmod{p^r}$ for some positive integer r. We want to show that $a^{p^{r+1}} \equiv a^{p^r} \pmod{p^{r+1}}$.

By assumption, $a^{p^r} = a^{p^{r-1}} + kp^r$ for some integer k, hence

$$a^{p^{r+1}} = (a^{p^{r-1}} + kp^r)^p = a^{p^r} + \sum_{j=1}^{p} \binom{p}{j} (kp^r)^j (a^{p^{r-1}})^{p-j}.$$

When $j \geq 2$, $rj \geq r+1$, hence $(kp^r)^j \equiv 0 \pmod{p^{r+1}}$ and so

$$a^{p^{r+1}} \equiv a^{p^r} + p(kp^r)(a^{p^{r-1}})^{p-1} \pmod{p^{r+1}}$$

and the desired result follows.

With the help of this generalization, we can offer a variant of solution to problem **A264** [2001 : 202; 2002 : 385]:

For any integer a and positive integer m, $a^m \equiv a^{m-\phi(m)} \pmod{m}$. $\qquad (*)$

As usual, $\phi(m)$ denotes the number of positive integers less than m and coprime to m. Note that $(*)$ implies $a^{\phi(m)} \equiv 1 \pmod{m}$ when a and m are coprime, which itself is Euler's generalization of (2)!

The result is obvious if $m = 1$ and is proved above if $m = p^r$ for some positive integer r (note that $\phi(p^r) = p^r - p^{r-1}$). Now suppose that $m = hp^r$ where h, r are integers such that $h > 1, r \geq 1$ and h coprime to p.

On the one hand, $a^m = (a^{p^r})^h \equiv (a^{p^{r-1}})^h = a^{hp^{r-1}} \pmod{p^r}$ and on the other hand,

$$\begin{aligned} a^{m-\phi(m)} = a^{hp^r - p^{r-1}(p-1)\phi(h)} &= (a^{p^r})^{h-\phi(h)} \cdot a^{p^{r-1}\phi(h)} \\ &\equiv (a^{p^{r-1}})^{h-\phi(h)} \cdot a^{p^{r-1}\phi(h)} = a^{hp^{r-1}} \pmod{p^r} \end{aligned}$$

(using the fact that $\phi(uv) = \phi(u)\phi(v)$ for coprime positive integers u, v).

Therefore $a^m \equiv a^{m-\phi(m)} \pmod{p^r}$. Since this holds for any prime divisor of m, $(*)$ follows.

Our next example is a problem that used to be set to French students sitting for a math-teacher diploma, but I do not know its exact origin.

Show that for every pair (m, n) of positive integers, the integer $mn(m^{60} - n^{60})$ is divisible by $A = 56{,}786{,}730$.

At first, mainly because of this big divisor A, the statement is rather puzzling! However, it is natural to compute the standard decomposition of A:

$$A = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 31 \times 61$$

and then to seek a possible link between 56,786,730 and 60. After a moment of reflection, it can be noticed that for any prime p dividing A, the integer $p-1$ divides 60. This is just the key to a solution. Let p be any prime divisor of A and let k be the positive integer defined by $60 = k(p-1)$. Then,

$$m^{60} - n^{60} = (m^{p-1})^k - (n^{p-1})^k = (m^{p-1} - n^{p-1}) \cdot B_p$$

for some positive integer B_p. Now, from (2),

$$mn(m^{p-1} - n^{p-1}) = n(m^p - m) - m(n^p - n)$$

is divisible by p. It follows that

$$mn(m^{60} - n^{60}) = mn(m^{p-1} - n^{p-1}) \cdot B_p$$

is also divisible by p. The desired result follows.

We conclude with an example introducing the Legendre symbol $\left(\frac{a}{p}\right)$ where p is a prime and a is coprime to p. It is defined by $\left(\frac{a}{p}\right) = 1$ if there exists an integer x such that $x^2 \equiv a \pmod{p}$ and $\left(\frac{a}{p}\right) = -1$ otherwise. With the help of property (3), we show that when p is odd, the following relation holds:

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \pmod{p}. \qquad (**)$$

Note that $(a^{\frac{p-1}{2}} - 1)(a^{\frac{p-1}{2}} + 1) = a^{p-1} - 1 \equiv 0 \pmod{p}$ so that $a^{\frac{p-1}{2}}$ is congruent to 1 or -1 modulo p. The above relation $(**)$ clarifies this point.
If $\left(\frac{a}{p}\right) = 1$, then, for some integer x coprime to p,

$$a^{\frac{p-1}{2}} \equiv (x^2)^{\frac{p-1}{2}} \equiv x^{p-1} \equiv 1 \pmod{p}$$

and $(**)$ follows.

Now suppose that $\left(\frac{a}{p}\right) = -1$. We observe that for any $k \in S = \{1, 2, \ldots, p-1\}$, there exists a unique k' in S such that $kk' \equiv a \pmod{p}$. Since $\left(\frac{a}{p}\right) = -1$, then $k \not\equiv k' \pmod{p}$ for all $k \in S$ and S is the union of $\frac{p-1}{2}$ disjoint pairs $\{k, k'\}$. Thus, the product $(p-1)!$ of the elements of S is congruent to $a^{\frac{p-1}{2}}$ and from (3), we obtain

$$-1 \equiv a^{\frac{p-1}{2}} \pmod{p},$$

that is, $(**)$.

Note the particular case: if p is an odd prime, then -1 is a square modulo p if and only if $p \equiv 1 \pmod{4}$ (see exercise 3 below).

Exercises

1. Show that if p is an odd prime number, then

$$(p+1)(p+2)\cdots(2p-1) \equiv (p-1)! \left(\mathrm{mod}\ p^2\right).$$

2. Let a, b be positive integers and p be any prime. Show that $a^p - b^p$ is either coprime to p or divisible by p^2.

3. Let m be a positive integer such that $p = 1 + 4m$ is a prime. Show that the square of $(2m)!$ is congruent to -1 modulo p.

Inequality Problems from the Chinese Mathematical Olympiad

Huawei Zhu
Guangzhou Institute of Educational Research

Prof. Zonghu Qiu, the generalissimo of Chinese mathematics competitions, had often been congratulated at international meetings on the outstanding performance of the Chinese International Mathematical Olympiad teams.

"That may be so," he would say, "but in terms of problem proposing, we are way behind the Russians and the Hungarians. I would describe a typical Chinese problem as an old sock, long and smelly. It is largely complex, just for the sake of complexity. Often, it is two or even three problems lumped into one. I hope that the new generation who will succeed us has learned valuable lessons from the masters worldwide."

I was a member of this "new generation". From 2005 to 2012, I served on the Problem Committee of the Chinese Mathematical Olympiad. My specialty being inequalities, I proposed six such problems over that period. Looking back, I see the gradual shift from complexity to elegance. I hope I succeeded to some extent, and humbly offer these problems to the readers so that they can pass judgment.

Problem 2005.

Let $a_1 = \frac{21}{16}$ and $2a_n - 3a_{n-1} = \dfrac{3}{2^{n+1}}$ for $n \geq 2$. Prove that for any integers $m \geq n \geq 2$,

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{\frac{1}{m}} \left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < \frac{m^2-1}{m-n+1}.$$

This is my initial attempt on the rarified stage of the national competition. The problem looks very imposing, and is definitely in the old style, combining several problems into one. The obvious first step is to solve the recurrence relation for a_n. The strange looking initial value suggests that a simpler sequence is lurking in the background, and it is not hard to discover what this new set of variables should be. After the determination of a_n, the real inequality problem takes over.

Problem 2006.

The real numbers $a_1,\ a_2,\ \ldots,\ a_n$ have sum 0. Prove that

$$\max_{1 \leq k \leq n} a_k^2 \leq \frac{n}{3} \sum_{i=1}^{n-1} (a_i - a_{i+1})^2.$$

Like the previous problem, the key idea is a new set of variables, but this time, the choice is more natural. For $1 \le k \le n-1$, let $d_k = a_k - a_{k+1}$. For any particular k, each of the as can be expressed in terms of a_k and the ds. This allows us to draw some conclusion about any a_k, and in particular about the largest among them.

Problem 2007.

Let a, b and c be given complex numbers. Let $m = |a+b|$ and $n = |a-b|$. Prove that if $mn \neq 0$, then

$$\max\{|ac+b|, |bc+a|\} \ge \frac{mn}{\sqrt{m^2+n^2}}.$$

This is a fairly straight-forward problem which can be solved in several different ways. One possible approach uses the idea of a linear combination.

Problem 2008.

For a positive integer n, let the real numbers $x_1 \le \cdots \le x_n$ and $y_1 \ge \cdots \ge y_n$ be such that $\sum_{i=1}^{n} ix_i \ge \sum_{i=1}^{n} iy_i$. Prove that for any real number λ,

$$\sum_{i=1}^{n} \lfloor i\lambda \rfloor x_i \ge \sum_{i=1}^{n} \lfloor i\lambda \rfloor y_i.$$

This problem is set up as an application of the method of mathematical induction. A key observation is that we must have $x_n \ge y_n$. In particular, for $n = 1$, we need $x_1 \ge y_1$ to establish the basis for the induction.

Problem 2009.

The real numbers $a_1, a_2, \ldots, a_n$, $n \ge 3$, are such that

$$\min_{1 \le i < j \le n} |a_i - a_j| = 1.$$

Determine in terms of n the minimum value of $\sum_{k=1}^{n} |a_k|^3$.

The idea behind this problem is rather simple. To minimize $\sum_{k=1}^{n} |a_k|^3$, we should bunch the as as close around 0 as possible. The condition $\min_{1 \le i < j \le n} |a_i - a_j| = 1$ means that we should take $a_k - a_{k-1} = 1$ for $2 \le k \le n$. Thus for $n = 3$, we take $a_1 = -1$, $a_2 = 0$ and $a_3 = 1$, and for $n = 4$, we take $a_1 = -\frac{3}{2}$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{2}$ and $a_4 = \frac{3}{2}$. Only technical details remain.

Problem 2011.

Let $n \geq 4$ be an integer. Let a_k and b_k, $1 \leq k \leq n$ be non-negative real numbers such that $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k > 0$. Determine the maximum value of

$$\frac{\sum_{k=1}^{n} a_k(a_k + b_k)}{\sum_{k=1}^{n} b_k(a_k + b_k)}.$$

This last problem was jointly proposed with Yunhao Fu of the next generation. Note that $\frac{\sum_{k=1}^{n} a_k(a_k + b_k)}{\sum_{k=1}^{n} b_k(a_k + b_k)}$ is unchanged if we multiply each a_k and b_k by the same non-zero constant. Hence we may assume that $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k = 1$. This provides the setting for our main idea, that of majorization among n-tuples with sum 1. At the top of the hierarchy is (1,0,0,. . . ,0) and at the bottom is $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. It seems reasonable that we should take the first one as the as and the second one as the bs. The tricky part of this problem is that there is a better choice for one of them, another n-tuple near the same end of the hierarchy.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **September 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

4061. *Proposed by Leonard Giugiuc.*

Let ABC be a non-obtuse triangle none of whose angles are less than $\frac{\pi}{4}$. Find the minimum value of $\sin A \sin B \sin C$.

4062. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let L_n denote the nth Lucas number defined by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Prove that

$$\frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} \geq \frac{2}{3} L_{n+4}^2.$$

4063. *Proposed by Marcel Chiriţă.*

Let a, b, c be real numbers greater than or equal to 3. Show that

$$\min\left(\frac{a^2b^2 + 3b^2}{b^2 + 27},\ \frac{b^2c^2 + 3c^2}{c^2 + 27},\ \frac{a^2c^2 + 3a^2}{a^2 + 27}\right) \leq \frac{abc}{9}.$$

4064. *Proposed by Michel Bataille.*

In the plane of a triangle ABC, let Γ be a circle whose centre O is not on the sidelines AB, BC, CA. Let A', B', C' be the poles of the lines BC, CA, AB with respect to Γ, respectively. Prove that

$$\frac{OA' \cdot B'C'}{OA \cdot BC} = \frac{OB' \cdot C'A'}{OB \cdot CA} = \frac{OC' \cdot A'B'}{OC \cdot AB}.$$

4065. *Proposed by Martin Lukarevski.*

Let ABC be a triangle with a, b, c as lengths of its sides and let R, r, s denote the circumradius, inradius and semiperimeter, respectively. Prove that

$$\frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} + \frac{1}{(s-c)^2} \geq \frac{2}{r}\left(\frac{1}{r} - \frac{1}{R}\right).$$

4066. *Proposed by Mihaela Berindeanu.*

Prove that for $a, b, c > 0$ and $ab + ac + bc = 2016$,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{2019^2}{2016}.$$

4067. *Proposed by Mehtaab Sawhney.*

Consider a graph G such that between any three vertices in G there are either 0 or 2 edges. Classify all such graphs G.

4068. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+2b}{2a+3b+c}+\frac{b+2c}{a+2b+3c}+\frac{c+2a}{3a+b+2c} \leq \frac{3}{2}.$$

4069. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(u_n)_{n\geq 0}$ be an arithmetic progression with a positive common difference d and with $u_1 > 0$. Let $(x_n)_{n\geq 0}$ be a sequence with $x_0 = 0, x_1 = x_2 = 1$ and

$$\sum_{k=1}^{n} u_k x_k = u_n x_{n+2} + d(x_4 - x_{n+3}) - x_2 u_1,\ \forall n \geq 0.$$

Prove that $(x_n)_{n\geq 0}$ is the Fibonacci sequence.

4070. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Compute

$$\lim_{n\to\infty}\left[\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan (n-1)}{2}+\arctan n\right)\right].$$

. . . .

4061. *Proposé par Leonard Giugiuc.*

Soit ABC un triangle non obtusangle dont la mesure de chaque angle est supérieure ou égale à $\frac{\pi}{4}$. Déterminer la valeur minimale de $\sin A \sin B \sin C$.

4062. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soit L_n le $n^{\text{ième}}$ nombre de Lucas défini par $L_0 = 2, L_1 = 1$ et $L_{n+2} = L_{n+1} + L_n$ pour tout $n \geq 0$. Démontrer que

$$\frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} \geq \frac{2}{3} L_{n+4}^2.$$

4063. *Proposé par Marcel Chiriţă.*

Soit a, b et c trois réels, chacun supérieur ou égal à 3. Démontrer que

$$\min\left(\frac{a^2b^2 + 3b^2}{b^2 + 27}, \frac{b^2c^2 + 3c^2}{c^2 + 27}, \frac{a^2c^2 + 3a^2}{a^2 + 27}\right) \leq \frac{abc}{9}.$$

4064. *Proposé par Michel Bataille.*

Dans le plan d'un triangle ABC, soit Γ un cercle dont le centre O n'est pas situé sur les droites AB, BC ou CA. Soit A', B' et C' les pôles respectifs des droites BC, CA et AB par rapport à Γ. Démontrer que

$$\frac{OA' \cdot B'C'}{OA \cdot BC} = \frac{OB' \cdot C'A'}{OB \cdot CA} = \frac{OC' \cdot A'B'}{OC \cdot AB}.$$

4065. *Proposé par Martin Lukarevski.*

Soit ABC un triangle et a, b et c les longueurs de ses côtés. Soit R le rayon du cercle circonscrit au triangle, r le rayon du cercle inscrit dans le triangle et p le demi-périmètre du triangle. Démontrer que

$$\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2} \geq \frac{2}{r}\left(\frac{1}{r} - \frac{1}{R}\right).$$

4066. *Proposé par Mihaela Berindeanu.*

Soit a, b et c des réels strictement positifs tels que $ab + ac + bc = 2016$. Démontrer que

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq \frac{2019^2}{2016}.$$

4067. *Proposé par Mehtaab Sawhney.*

On considère un graphe G de manière qu'entre n'importe quels trois sommets de G il existe 0 arc ou 2 arcs. Classifier tous les graphes G de cette sorte.

4068. *Proposé par George Apostolopoulos.*

Soit a, b et c des réels strictement positifs. Démontrer que

$$\frac{a+2b}{2a+3b+c}+\frac{b+2c}{a+2b+3c}+\frac{c+2a}{3a+b+2c}\leq\frac{3}{2}.$$

4069. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soit $(u_n)_{n\geq 0}$ une suite arithmétique dont la raison d est strictement positive et $u_1 > 0$. Soit $(x_n)_{n\geq 0}$ une suite telle que $x_0 = 0, x_1 = x_2 = 1$ et

$$\sum_{k=1}^{n} u_k x_k = u_n x_{n+2} + d(x_4 - x_{n+3}) - x_2 u_1,\ \forall n \geq 0.$$

Démontrer que $(x_n)_{n\geq 0}$ est la suite de Fibonacci.

4070. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Calculer

$$\lim_{n\to\infty}\left[\frac{1}{\ln n}\left(\frac{\arctan 1}{n}+\frac{\arctan 2}{n-1}+\cdots+\frac{\arctan(n-1)}{2}+\arctan n\right)\right].$$

Math Quotes

Games are among the most interesting creations of the human mind, and the analysis of their structure is full of adventure and surprises. Unfortunately there is never a lack of mathematicians for the job of transforming delectable ingredients into a dish that tastes like a damp blanket.

James R. Newman in J. R. Newman (ed.) "The World of Mathematics", New York : Simon and Schuster, 1956.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(7), p. 299–302.

3961. *Proposed by Michel Bataille.*

In a triangle ABC, let $\angle A \geq \angle B \geq \angle C$ and suppose that

$$\sin 4A + \sin 4B + \sin 4C = 2(\sin 2A + \sin 2B + \sin 2C).$$

Find all possible values of $\cos A$.

We received two correct solutions and one incorrect submission. We present the solution by Kee-Wai Lau, modified by the editor.

Using $A + B + C = \pi$ and trigonometric formulas, we can show that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

In detail,

$$\begin{aligned}\sin 2A + \sin 2B + \sin 2C &= \sin 2A + \sin 2B - \sin(2A + 2B)\\ &= 2\sin(A + B)\cos(A - B) - 2\sin(A + B)\cos(A + B)\\ &= 2\sin(A + B)[\cos(A - B) - \cos(A + B)]\\ &= 2\sin(A + B)(-2\sin(A)\sin(-B))\\ &= 2\sin C(2\sin A \sin B)\\ &= 4\sin A \sin B \sin C.\end{aligned}$$

Replace A, B and C in the above calculation by $2A$, $2B$ and $2C$ to get

$$\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C,$$

the only difference being in the penultimate line : since $2A+2B+2C = 2\pi$ (instead of π), we have $\sin(2A + 2B) = -\sin(2C)$, which introduces an extra minus sign.

Using these two equalities, the given relation is equivalent to

$$-4 \cos A \cos B \cos C = 1. \tag{1}$$

Since the product of cosines in (1) is negative, and $A \geq B \geq C$, we must have $A > \frac{\pi}{2} > 2C > 0$.

Using $A + B + C = \pi$, from (1) we get $4 \cos A \cos(A + C) \cos C = 1$, and so

$$4 \cos A(\cos A \cos C - \sin A \sin C) \cos C = 1.$$

Divide both sides by $\cos^2 C$ to get

$$4\cos^2 A - 4\sin A\cos A\tan C = \frac{1}{\cos^2 C}. \tag{2}$$

Let $x = \cos A$ and $t = \tan^2 C$. From $A > \frac{\pi}{2} > 2C > 0$ it follows that $-1 < x < 0$ and $0 < t < 1$. Rewrite (2) using this notation and rearrange to get

$$4x^2 - (1+t) = 4x\sqrt{1-x^2}\sqrt{t}. \tag{3}$$

Square both sides of (3) and move all the terms to one side to get

$$16(1+t)x^4 - 8(1+3t)x^2 + (t+1)^2 = 0.$$

Applying the quadratic formula, $x^2 = \frac{1+3t\pm\sqrt{3t+6t^2-t^3}}{4(1+t)}$. We check whether squaring introduced extraneous solutions. Since $-1 < x < 0$ and $0 < t < 1$, $4x^2-(1+t) < 0$ whenever (3) holds. For the solutions obtained from the quadratic formula we have $4x^2 - (1+t) = \frac{(t-t^2)\pm\sqrt{3t+6t^2-t^3}}{1+t}$ and since $t - t^2 > 0$ it is clear that, when $x^2 = \frac{1+3t+\sqrt{3t+6t^2-t^3}}{4(1+t)}$, (3) is not satisfied.

Therefore, since $x < 0$, we must have

$$x = -\frac{1}{2}\sqrt{\frac{1+3t-\sqrt{3t+6t^2-t^3}}{1+t}}. \tag{4}$$

We want to find the range of values of x for $t \in (0,1)$. Implicitly differentiate $x^2 = \frac{1+3t-\sqrt{3t+6t^2-t^3}}{4(1+t)}$ to get

$$x\frac{dx}{dt} = \frac{4\sqrt{3t+6t^2-t^3}+t^3+3t^2-9t-3}{16(1+t)^2\sqrt{3t+6t^2-t^3}}.$$

There is no $t \in (0,1)$ for which either $x = 0$ or $16(1+t)^2\sqrt{3t+6t^2-t^3} = 0$, so we conclude that the critical points of x satisfy

$$\begin{aligned}
&4\sqrt{3t+6t^2-t^3} = -t^3-3t^2+9t+3 \Leftrightarrow\\
&(3+9t-3t^2-t^3)^2 - 16(3t+6t^2-t^3) = 0 \Leftrightarrow\\
&t^6+6t^5-9t^4-44t^3-33t^2+6t+9 = 0 \Leftrightarrow\\
&(t-3)(t+1)^3(t^2+6t-3) = 0.
\end{aligned}$$

The only critical point in the range $[0,1]$ is $t = 2\sqrt{3}-3$. The corresponding value of x, obtained after a tedious but straight-forward calculation, is $\frac{1-\sqrt{3}}{2}$. From (4), we easily evaluate

$$\lim_{t\to 0^+} x = -\frac{1}{2} \quad\text{and}\quad \lim_{t\to 1^-} x = -\frac{\sqrt{2-\sqrt{2}}}{2},$$

allowing us to conclude that for $t \in (0,1)$ we have $-\frac{1}{2} < x \le \frac{1-\sqrt{3}}{2}$.

Finally, we check that for each x in this interval there is a corresponding triangle whose angles A, B and C satisfy the given relation. Suppose x_0 is such that $-\frac{1}{2} < x_0 \le \frac{1-\sqrt{3}}{2}$. Let $A = \cos^{-1}(x_0)$; since $\cos^{-1}$ is a decreasing function we have $A < \cos^{-1}(-0.5) = \frac{2\pi}{3}$. By the intermediate value theorem, since x is continuous on $[0, 2\sqrt{3}-3]$, there exists a t_0 in this interval such that $x_0 = x(t_0)$. Let $C = \tan^{-1}(\sqrt{t_0})$. Note that $\sqrt{t_0} \le \sqrt{2\sqrt{3}-3} < \sqrt{3}$, whence $C < \frac{\pi}{3}$. Let $B = \pi - A - C$; the earlier comments about the ranges for A and C imply $B > 0$.

We claim that a triangle with angles A, B and C satisfies the relation given in the problem (note : it seems likely that $B > C$ from this construction, but it is not immediately obvious, and anyway it is not needed since if A, B and C satisfy the relation but $B < C$ we can switch the labels of the vertices B and C). From the construction, $\cos C = (1 + \tan^2 C)^{-1/2} = (1 + t_0)^{-1/2}$ and $\cos A = x_0$. Moreover, x_0 and t satisfy equation (3). We calculate (using trig equalities to evaluate $\sin A$ and $\sin C$)

$$\begin{aligned}\cos B = \cos(\pi - (A + C)) &= -\cos(A + C)\\ &= -\cos A \cos C + \sin A \sin C\\ &= -x_0\sqrt{\tfrac{1}{1+t_0}} + \sqrt{1 - x_0^2}\cdot\sqrt{1 - \tfrac{1}{1+t_0}}\\ &= \sqrt{\tfrac{1}{1+t_0}}\left(-x_0 + \sqrt{1-x_0^2}\cdot\sqrt{t_0}\right).\end{aligned}$$

Since $x_0 \ne 0$ we can rearrange (3) to get $\sqrt{1-x_0^2}\cdot\sqrt{t_0} = x_0 - \frac{1+t_0}{4x_0}$; hence $\cos B = \sqrt{\frac{1}{1+t_0}}\cdot\frac{1+t_0}{-4x_0} = \frac{\cos C(1+t_0)}{-4\cos A}$. It follows that $-4\cos A\cos B\cos C = 1$, so A, B and C satisfy (1), which is equivalent to the equality given in the question.

Therefore, we conclude that the possible range of values for $\cos A$ is given by $-\frac{1}{2} < \cos A \le \frac{1-\sqrt{3}}{2}$.

3962. *Proposed by Michel Bataille.*

Let ABC be a nonequilateral triangle, Γ its circumcircle and ℓ its Euler line. Let its medians from A, B, C meet Γ again at A_1, B_1, C_1, respectively, and let $M = t_B \cap t_C$, $N = t_C \cap t_A$, $P = t_A \cap t_B$ where t_A, t_B, t_C are the tangents to Γ at A, B, C, respectively.

Prove that the lines MA_1, NB_1, PC_1 and ℓ are concurrent or parallel and that the latter occurs if and only if $\cos A \cos B \cos C = -\frac{1}{4}$.

We received three correct submissions. We present the solution by Oliver Geupel.

Consider the problem in the plane of complex numbers where Γ is the unit circle with centre O. Let $a, b, c, a_1, \dots$ denote the complex numbers representing the respective points $A, B, C, A_1, \dots$, and G with coordinate $g = (a + b + c)/3$ denote the centroid of triangle ABC. Since M is the intersection of the tangents from B and C, it must be the inverse in Γ of the midpoint $\frac{b+c}{2}$ of the chord BC, namely $m = \frac{2bc}{b+c}$ (where $x\bar{x} = 1$ is used for points of Γ). Since the midpoint of BC belongs to the chord AA_1, we have $\left(\frac{b+c}{2}-a\right)(\overline{a_1}-\bar{a})=\left(\overline{\frac{b+c}{2}}-\bar{a}\right)(a_1-a)$, so that

$$\left(\frac{b+c}{2}-a\right)\left(\frac{1}{a_1}-\frac{1}{a}\right)=\left(\frac{b+c}{2bc}-\frac{1}{a}\right)(a_1-a)\,,$$

and $a_1=\frac{bc(2a-b-c)}{a(b+c)-2bc}$.

Observing that

$$\begin{aligned}\cos A\cos B\cos C&=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2}-1=\frac{BC^2+CA^2+AB^2}{8}-1\\&=\frac{1}{8}\left((b-c)\left(\frac{1}{b}-\frac{1}{c}\right)+(c-a)\left(\frac{1}{c}-\frac{1}{a}\right)+(a-b)\left(\frac{1}{a}-\frac{1}{b}\right)\right)-1\\&=-\frac{a^2b+a^2c+ab^2+ac^2+b^2c+bc^2}{abc}-\frac{1}{4},\end{aligned}$$

the condition $\cos A\cos B\cos C=-\frac{1}{4}$ is equivalent to $\sum_{\text{sym}} a^2b=0$ (where the sum consists of six terms). We consider the cases $\cos A\cos B\cos C=-\frac{1}{4}$ and $\cos A\cos B\cos C\neq-\frac{1}{4}$ in succession.

When $\cos A\cos B\cos C=-\frac{1}{4}$, a tedious but straightforward calculation leads to

$$g(\overline{m}-\overline{a_1})-\bar{g}(m-a_1)=\frac{(a^2-bc)(b-c)^2}{3(a(b+c)-2bc)(2a-b-c)abc(b+c)}\sum_{\text{sym}}a^2b=0,$$

whence $\frac{m-a_1}{\overline{m}-\overline{a_1}}=\frac{g}{\bar{g}}$; that is, the lines MA_1 and $\ell=OG$ are parallel. Similarly, NB_1 and PC_1 are parallel to ℓ. The first case is complete.

Next we examine the case $\cos A\cos B\cos C\neq-\frac{1}{4}$. Let S be the point with coordinate $s=\frac{2abc(a+b+c)}{\sum_{\text{sym}}a^2b}$. Then,

$$s\bar{g}=\frac{2(a+b+c)(ab+bc+ca)}{3\sum_{\text{sym}}a^2b}=\bar{s}g,$$

so that the point S lies on the line $\ell=OG$. A straightforward calculation shows that

$$(m-a_1)(\overline{m}-\bar{s})\;=\;\frac{2abc(b-c)^2}{(b+c)^2\sum_{\text{sym}}a^2b}\;=\;(\overline{m}-\overline{a_1})(m-s).$$

Hence the points M, A_1, and S are collinear. Similarly, the point S lies on the lines NB_1 and PC_1. Consequently, the lines MA_1, NB_1, PC_1 and ℓ are concurrent at the point S.

Editor's comments. The other solutions found, using areal coordinates with respect to ΔABC, that the point

$$S=(a^2(b^4+c^4-a^4):b^2(c^4+a^4-b^4):c^2(a^4+b^4-c^4))$$

lies on the lines MA_1, NB_1, PC_1. Fanchini identified it as the Exeter point (X_{22} in Kimberling's *Encyclopedia of Triangle Centers*), which is known to lie on the Euler line.

3963. *Proposed by D. M. Bătineţu and Neculai Stanciu.*

Let $A \in M_n(\mathbb{R})$ such that $A^2 = 0_n \in M_n(\mathbb{R})$ and let $x, y \in \mathbb{R}$ such that $4y \geq x^2$. Prove that $\det(xA + yI_n) \geq 0$.

We received eleven correct submissions. We present the solution by Matei Coiculescu.

Let $A \in M_n(\mathbb{R})$ such that $A^2 = 0_n \in M_n(\mathbb{R})$ and let $x, y \in \mathbb{R}$ such that $4y \geq x^2$.

If $y = 0$, then $x = 0$ and the inequality is satisfied trivially. If $y \neq 0$, let

$$M = \frac{x}{2\sqrt{y}} \cdot A + \sqrt{y} \cdot I_n \in M_n(\mathbb{R}).$$

Then,

$$M^2 = (\frac{x}{2\sqrt{y}} \cdot A + \sqrt{y} \cdot I_n) \cdot (\frac{x}{2\sqrt{y}} \cdot A + \sqrt{y} \cdot I_n) = \frac{x^2}{4y} \cdot A^2 + xA + yI_n = xA + yI_n$$

based on the hypothesis, that $A^2 = 0$. Hence,

$$\det(xA + yI_n) = \det(M^2) = (\det M)^2 \geq 0.$$

We make the observation that the condition $4y \geq x^2$ is not necessary, we only need $y > 0$.

3964. *Proposed by George Apostolopoulos.*

Let P be an arbitrary point inside a triangle ABC. Let a, b and c be the distances from P to the sides BC, AC and AB, respectively. Prove that

$$\frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^4}{\sin^4 A + \sin^4 B + \sin^4 C} \leq 12R^2,$$

where R denotes the circumradius of ABC. When does the equality occur ?

We received four correct solutions. We present the solution by Oliver Geupel.

Let $x = BC$, $y = CA$, $z = AB$. By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} xyz(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 &= (\sqrt{yz}\sqrt{xa} + \sqrt{zx}\sqrt{yb} + \sqrt{xy}\sqrt{zc})^2 \\ &\leq (xy + yz + zx)(xa + yb + zc). \end{aligned} \tag{1}$$

Also note that

$$\begin{aligned} (xy + yz + zx)^2 &\leq (xy + yz + zx)^2 + \sum_{\text{cyc}} \left(\frac{3}{2}(x^2 - y^2)^2 + x^2(y - z)^2 \right) \\ &= 3(x^4 + y^4 + z^4) \end{aligned} \tag{2}$$

as well as

$$x = 2R\sin A, y = 2R\sin B, z = 2R\sin C. \qquad (3)$$

Let K denote the area of triangle ABC. Then

$$xyz = 4RK = 2R(xa + yb + zc). \qquad (4)$$

Putting (1) - (4) together, we obtain

$$\begin{aligned}\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^4}{\sin^4 A+\sin^4 B+\sin^4 C} &\le \frac{(xy+yz+zx)^2(xa+yb+zc)^2}{(xyz)^2(\sin^4 A+\sin^4 B+\sin^4 C)}\\ &\le \frac{3(x^4+y^4+z^4)}{4R^2(\sin^4 A+\sin^4 B+\sin^4 C)}\\ &= \frac{3}{4}\cdot\frac{x^4+y^4+z^4}{R^4(\sin^4 A+\sin^4 B+\sin^4 C)}\cdot R^2\\ &= 12R^2.\end{aligned}$$

This completes the proof. The equality holds in (2) if and only if $x = y = z$. Under the condition $x = y = z$, the equality in (1) then holds if and only if $a = b = c$. Hence, the equality holds in the inequality of the problem if and only if triangle ABC is equilateral and P is its midpoint.

3965. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\left(\ln\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{(-1)^n}{n}\right)x^n$$

and its value at x for each x in this interval.

We received four correct solutions and four incorrect or incomplete submissions. We present the solution by Michel Bataille.

Let $a_n = \ln\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{(-1)^n}{n}$ where n is a positive integer. Applying Taylor's formula with integral remainder to $\ln(1+x)$ between 0 and 1, we obtain

$$\begin{aligned}\ln(2) &= 1-\frac{1}{2}+\cdots+\frac{(-1)^{n+1}}{n}+(-1)^n\int_0^1\frac{(1-t)^n}{(1+t)^{n+1}}\,dt\\ &= 1-\frac{1}{2}+\cdots+\frac{(-1)^{n+1}}{n}+(-1)^n\int_0^1\frac{u^n}{1+u}\,du\end{aligned}$$

(with the help of the substitution $t = \frac{1-u}{1+u}$). It follows that

$$a_n = (-1)^{n+1}\int_0^1\frac{u^n}{1+u}\,du.$$

Now,

$$|a_n| = \int_0^1 \frac{u^n}{1+u}\,du \le \int_0^1 u^n\,du = \frac{1}{n+1}$$

for every positive integer n, so that $\lim_{n\to\infty} a_n = 0$. Morever,

$$|a_{n+1}| - |a_n| = \int_0^1 \frac{u^n(u-1)}{1+u}\,du \le 0,$$

so that the sequence $(|a_n|)_{n\ge1}$ is nonincreasing. From Leibniz's Alternating Series Test, the series $\sum_{n=1}^{\infty} a_n$ is convergent. Thus, the radius of convergence R of the power series $\sum_{n=1}^{\infty} a_n x^n$ satisfies $R \ge 1$.
On the other hand, integrating by parts, we obtain

$$\begin{aligned}\int_0^1 \frac{u^n}{1+u}\,du &= \left[\frac{u^{n+1}}{n+1}\cdot\frac{1}{1+u}\right]_0^1 + \frac{1}{n+1}\int_0^1 \frac{u^{n+1}}{(1+u)^2}\,du \\ &= \frac{1}{2(n+1)} + \frac{1}{n+1}\,J_n,\end{aligned}$$

with

$$0 \le J_n = \int_0^1 \frac{u^{n+1}}{(1+u)^2}\,du \le \frac{1}{n+2}.$$

Hence $\lim_{n\to\infty} J_n = 0$, and it follows that $\int_0^1 \frac{u^n}{1+u}\,du \sim \frac{1}{2n}$ as $n\to\infty$. From this result, we deduce that the series $\sum_{n=1}^{\infty} a_n(-1)^n$ is divergent and so $R \le 1$.

We conclude that $R = 1$ and that the interval of convergence is $(-1,1]$.

Let N be an integer such that $N > 1$ and $x \in (-1,1]$. Then,

$$\begin{aligned}\sum_{n=1}^{N} a_n x^n &= -\sum_{n=1}^{N}\int_0^1 \frac{(-ux))^n}{1+u}\,du \\ &= -\int_0^1 \frac{1}{1+u}(-ux)(1+(-ux)+(-ux)^2+\cdots+(-ux)^{N-1})\,du \\ &= \int_0^1 \frac{ux}{(1+u)(1+ux)}\,du - \int_0^1 \frac{(-1)^N(ux)^{N+1}}{(1+u)(1+ux)}\,du.\end{aligned}$$

Thus,

$$\sum_{n=1}^{N} a_n x^n = x\int_0^1 \frac{u}{(1+u)(1+ux)}\,du + (-1)^{N+1}K_N \qquad (1)$$

where $K_N = x^{N+1}\int_0^1 \frac{u^{N+1}}{(1+u)(1+ux)}\,du$.
Since

$$0 \le |K_N| \le |x|^{N+1}\int_0^1 u^{N+1}\,du = \frac{|x|^{N+1}}{N+2} \le \frac{1}{N+2},$$

we have $\lim_{N\to\infty} K_N = 0$. Thus, if $S(x) = \sum_{n=1}^{\infty} a_n x^n$, by letting $N\to\infty$, (1) yields

$$S(x) = x\int_0^1 \frac{u}{(1+u)(1+ux)}\,du. \qquad (2)$$

If $x = 1$, then

$$S(1) = \int_0^1 \frac{u}{(1+u)^2}\,du = \int_0^1 \left(\frac{1}{1+u} - \frac{1}{(1+u)^2}\right) du = \ln 2 - \frac{1}{2}$$

and if $|x| < 1$, using $\frac{u}{(1+u)(1+ux)} = \frac{1}{1-x}\left(\frac{1}{1+ux} - \frac{1}{1+u}\right)$, we obtain

$$\int_0^1 \frac{u}{(1+u)(1+ux)}\,du = \frac{\ln(1+x)}{x(1-x)} - \frac{\ln 2}{1-x}$$

and (2) gives

$$S(x) = \frac{\ln(1+x) - x\ln 2}{1-x}.$$

3966. *Proposed by Dao Hoang Viet.*

Let x and y be the legs and h the hypotenuse of a right triangle. Prove that

$$\frac{1}{2h+x+y} + \frac{1}{h+2x+y} + \frac{1}{h+x+2y} < \frac{h}{2xy}.$$

We received 14 correct solutions. We present four different solutions.

Solution 1, by Digby Smith.

Since h is greater than each of x and y, and since $h^2 = x^2 + y^2 \geq 2xy$, we have that

$$h(2h+x+y) = 2h^2 + hx + hy > 4xy + 2xy = 6xy,$$

$$h(h+2x+y) = h^2 + h(2x+y) \geq 2xy + (\sqrt{2xy})(2\sqrt{2xy}) = 2xy + 4xy = 6xy,$$

and $h(h+x+2y) \geq 6xy$. The desired inequality follows directly.

Solution 2, by Titu Zvonaru.

Since $h \geq \sqrt{2xy}$, $x + y \geq 2\sqrt{xy}$, $2x + y \geq 2\sqrt{2xy}$ and $x + 2y \geq 2\sqrt{2xy}$, we have that the left side of the desired inequality is no greater than

$$\frac{1}{2\sqrt{2xy} + 2\sqrt{xy}} + \frac{2}{\sqrt{2xy} + 2\sqrt{2xy}} = \frac{1}{\sqrt{2xy}}\left[\frac{1}{2+\sqrt{2}} + \frac{2}{3}\right]$$

$$< \frac{1}{\sqrt{2xy}} = \frac{\sqrt{2xy}}{2xy} \leq \frac{h}{2xy}.$$

Solution 3, by Michel Bataille, Oliver Geupel and Dan Jonsson (independently).

By the arithmetic-geometric means inequality, each of $2h + x + y$, $h + 2x + y$ and $h + x + 2y$ strictly exceeds $3\sqrt[3]{2hxy}$. Hence

$$\frac{1}{2h+x+y} + \frac{1}{h+2x+y} + \frac{1}{h+x+2y} < \frac{1}{\sqrt[3]{2hxy}}.$$

By Pythagoras' Theorem, $4x^2y^2 \le (x^2+y^2)^2 = h^4$, whence

$$\frac{1}{2hxy} \le \frac{h^3}{8x^3y^3}.$$

The result follows.

Solution 4, by Šefket Arslanagić, Salem Malikić and Henry Ricardo (independently).

By the arithmetic-harmonic means inequality,

$$\frac{1}{2h+x+y} = \frac{1}{(h^{-1})^{-1}+(h^{-1})^{-1}+(x^{-1})^{-1}+(y^{-1})^{-1}} < \frac{1}{16}(h^{-1}+h^{-1}+x^{-1}+y^{-1}).$$

This, along with the analogous inequalities for the other two terms, implies that the left side is less than

$$\begin{aligned}\frac{1}{4}\left(\frac{1}{h}+\frac{1}{x}+\frac{1}{y}\right) &= \frac{1}{4hxy}[xy+h(x+y)] \\ &\le \frac{1}{4hxy}\left[\frac{x^2+y^2}{2}+h\sqrt{2(x^2+y^2)}\right] = \frac{1}{4hxy}\left[\frac{h^2}{2}+h^2\sqrt{2}\right] \\ &= \frac{h}{4xy}\left[\frac{1}{2}+\sqrt{2}\right] < \frac{h}{2xy}.\end{aligned}$$

3967. *Proposed by Marcel Chiriţă.*

Determine all positive integers a, b and c that satisfy the following equation :

$$(a+b)! = 4(b+c)! + 18(a+c)!$$

We received eight correct submissions, all similar. We present the solution by Joseph DiMuro.

We shall see that there are exactly two triples that satisfy the given equation : $a = 3, b = 4, c = 2$, and $a = 11, b = 11, c = 10$. First of all, note that $(a+b)! > (b+c)!$ and $(a+b)! > (a+c)!$. Thus, we must have both $a > c$ and $b > c$.

Can we have $a > b$? If we divide the original equation through by $(a+c)!$, we get

$$\frac{(a+b)!}{(a+c)!} = \frac{4(b+c)!}{(a+c)!} + 18.$$

Since $a+b > a+c$, $\dfrac{(a+b)!}{(a+c)!}$ must be an integer. Thus, $\dfrac{4(b+c)!}{(a+c)!}$ must also be an integer. Because

$$\frac{4(b+c)!}{(a+c)!} = \frac{4}{(b+c+1)\cdots(a+c)},$$

the product in the denominator on the right must be a divisor of 4. But we are assuming here that the three integers satisfy $a > b > c > 0$, so $b+c+1 \geq 4$ and we can have just one factor in the denominator : we must have $b + c + 1 = a + c = 4$. The first equality says that $a = b + 1$, the second that $b + c = 3$; together with $b > c$ they would imply that $a = 3$, $b = 2$, and $c = 1$. But this combination does not satisfy the original equation : We cannot have $a > b$.

Similarly, we can consider the possibility that $b > a$. If we divide the original equation through by $(b + c)!$ we get

$$\frac{(a+b)!}{(b+c)!} = 4 + \frac{18(a+c)!}{(b+c)!}.$$

Since $\frac{(a+b)!}{(b+c)!}$ is an integer, $\frac{18(a+c)!}{(b+c)!}$ must also be an integer. Because

$$\frac{18(a+c)!}{(b+c)!} = \frac{18}{(a+c+1)\cdots(b+c)},$$

the product in the denominator must be a divisor of 18. But now with $b > a > c > 0$, each factor is at least 4, and should there be two or more factors in the denominator, their product would be at least 20. So in fact, there must be just one factor in the denominator : we must have $a + c + 1 = b + c$, and $b + c$ must be a divisor of 18.

We will consider the possible values for $b + c$ in turn. Note that $b = a + 1$ and $a > c$; because of that, $b \geq c + 2$. Thus, we cannot have $b + c = 3$; $b + c$ can only equal 6, 9, or 18.

If $b + c = 6$, then we have

$$\frac{(a+b)!}{6!} = 4 + \frac{18}{6} = 7,$$

so that $a + b = 7$. Consequently, in this case $a = 3$, $b = 4$, and $c = 2$, and the given equation is indeed satisfied.

If $b + c = 9$, then we would have

$$\frac{(a+b)!}{9!} = 4 + \frac{18}{9} = 6.$$

This is impossible, since the fraction on the left would then either equal 1 or be at least 10.

Similarly, we can rule out $b + c = 18$ because we would get

$$\frac{(a+b)!}{18!} = 4 + \frac{18}{18} = 5,$$

which is impossible. Thus, the only possible combination where $b > a$ is $a = 3$, $b = 4$, $c = 2$.

Finally, consider the possibility that $a = b$. Setting $b = a$ in the original equation gives us

$$(2a)! = 22(a+c)!$$
$$\frac{(2a)!}{(a+c)!} = 22$$
$$(a+c+1)\cdots(2a) = 22.$$

Each factor on the left side is at least 4, so there can be at most two such factors. But 22 cannot be written as the product of two consecutive integers. So there must be exactly one factor on the left-hand side : $a+c+1 = 2a = 22$. Thus, $a = b = 11$ and $c = 10$, which also satisfies the original equation.

3968. *Proposed by Michal Kremzer.*

Let $\{a\} = a - [a]$, where $[a]$ is the greatest integer function. Show that if a is real and $a(a - 2\{a\})$ is an integer, then a is an integer.

We received 17 correct solutions, all with the same approach. We present the solution of Kathleen E. Lewis.

Since $\{a\} = a - [a]$, we have $a - 2\{a\} = [a] - \{a\}$. Thus

$$a(a - 2\{a\}) = ([a] + \{a\})([a] - \{a\}) = [a]^2 - \{a\}^2.$$

Since $[a]^2$ is an integer, $a(a-2\{a\})$ can only be an integer if $\{a\}^2$ is also an integer. But $0 \le \{a\} < 1$ implies $0 \le \{a\}^2 < 1$. Therefore, if $a - 2\{a\}$ is an integer, then $\{a\}$ must be zero and thus a must be an integer.

3969. *Proposed by Marcel Chiriţă.*

Determine the functions $f : (\frac{8}{9}, \infty) \mapsto \mathbb{R}$ continuous at $x = 1$ such that

$$f(9x-8) - 2f(3x-2) + f(x) = 4x - 4 + \ln\frac{9x^2 - 8x}{(3x-2)^2} \qquad \text{for all } x \in \left(\frac{8}{9}, \infty\right).$$

We received four correct solutions and one incomplete submission. We present two solutions.

Solution 1, by Arkady Alt.

We have the following

$$\begin{aligned} &f(9x-8) - 2f(3x-2) + f(x) \\ &\qquad = 4x - 4 + \ln\frac{9x^2 - 8x}{(3x-2)^2} \\ &\qquad = (9x-8) - 2(3x-2) + x + \ln x + \ln(9x-8) - 2\ln(3x-2), \end{aligned}$$

so $g(9x-8) - 2g(3x-2) + g(x) = 0$, where $g(x) := f(x) - \ln x - x$. Obviously $g : \left(\frac{8}{9}, \infty\right) \longrightarrow \mathbb{R}$ is continuous at $x = 1$.

Let $y := 9x - 8$, then $3x - 2 = \frac{y+2}{3}$ and the original functional equation becomes

$$g(y) - 2g\left(\frac{y+2}{3}\right) + g\left(\frac{y+8}{9}\right) = 0. \tag{1}$$

Consider the sequence $(x_n)_{n\geq 0}$ defined recursively by $x_{n+1} = \dfrac{x_n + 2}{3}, n \geq 0$ with initial condition $x_0 := x$, where $x \in \left(\frac{8}{9}, \infty\right)$ and $x \neq 1$.Then $x_n \in \left(\frac{8}{9}, \infty\right), n \geq 0$ and by replacing y in (1) with x_n we obtain

$$g(x_n) - 2g\left(\frac{x_n+2}{3}\right) + g\left(\frac{x_n+8}{9}\right) = 0$$

i.e.

$$g(x_n) - 2g\left(x_{n+1}\right) + g\left(x_{n+2}\right) = 0, \qquad n \geq 0.$$

Since $g(x_n) - g\left(x_{n+1}\right) = g\left(x_{n+1}\right) - g\left(x_{n+2}\right)$ for any $n \geq 0$ then by induction

$$g(x_n) - g\left(x_{n+1}\right) = g(x_0) - g\left(x_1\right).$$

On the other hand, since

$$x_{n+1} = \frac{x_n+2}{3} \iff x_{n+1} - 1 = \frac{1}{3}\left(x_n - 1\right), \ n \geq 0,$$

then

$$x_n - 1 = \frac{1}{3^n}\left(x_0 - 1\right) \iff x_n = \frac{x-1}{3^n} + 1.$$

Therefore, (by continuity in $x = 1$) we have $\lim\limits_{n\to\infty} g(x_n) = g(\lim\limits_{n\to\infty} x_n) = g(1)$.

Hence,

$$g(x) - g\left(\frac{x+2}{3}\right) = g(x_0) - g\left(x_1\right) = \lim_{n\to\infty}\left(g(x_n) - g\left(x_{n+1}\right)\right) = g(1) - g(1) = 0.$$

Since $g(1) - g\left(\frac{1+2}{3}\right) = 0$, then for any $x \in \left(\frac{8}{9}, \infty\right)$ we have $g(x) - g\left(\dfrac{x+2}{3}\right) = 0$ and therefore,

$$g(x_n) - g\left(\frac{x_n+2}{3}\right) = 0 \iff g(x_n) = g\left(x_{n+1}\right), \qquad n \geq 0.$$

Since $g(x_n) = g(x_0) = g(x)$ and $\lim\limits_{n\to\infty} g(x_n) = g(1)$ we obtain $g(x) = g(1)$ for any $x \in \left(\frac{8}{9}, \infty\right)$. Therefore, $f(x) = \ln x + x + c$, where c is any real constant.

Solution 2, by Digby Smith.

Lemma. Let g be a function satisfying $g(0) = 0$ which is continuous at $t = 0$ such that for $t \in \left(-\frac{1}{9}, \infty\right)$, the following equation holds :

$$g(9t) - 2g(3t) + g(t) = 0.$$

Then $g(t)=0$ for all $t\in\left(-\frac{1}{9},\infty\right)$.

Proof. Let $s\in\left(-\frac{1}{9},\infty\right)$. For all $j\in\mathbb{N}$ with $j\geq 2$, the following holds :

$$g\left(9\cdot\frac{s}{3^j}\right)-2g\left(3\cdot\frac{s}{3^j}\right)+g\left(\frac{s}{3^j}\right)=0.$$

It then follows for $n\in\mathbb{N}$ with $n\geq 2$ that

$$\begin{aligned}\sum_{j=2}^{n}\left[g\left(9\cdot\frac{s}{3^j}\right)-2g\left(3\cdot\frac{s}{3^j}\right)+g\left(\frac{s}{3^j}\right)\right]&=0,\\ g(s)-g\left(\frac{s}{3}\right)-g\left(\frac{s}{3^{n-1}}\right)+g\left(\frac{s}{3^n}\right)&=0,\\ g(s)-g\left(\frac{s}{3}\right)=g\left(\frac{s}{3^{n-1}}\right)-g\left(\frac{s}{3^n}\right)&.\end{aligned}$$

Since g is continuous at $t=0$, it then follows that

$$g(s)-g\left(\frac{s}{3}\right)=\lim_{n\to\infty}\left(g\left(\frac{s}{3^{n-1}}\right)-g\left(\frac{s}{3^n}\right)\right)=g(0)-g(0)=0,$$

so $g(s)=g\left(\frac{s}{3}\right)$. It then follows for $m\in\mathbb{N}$ that

$$g(s)=g\left(\frac{s}{3}\right)=g\left(\frac{s}{3^2}\right)=\ldots=g\left(\frac{s}{3^m}\right).$$

Again since g is continuous at $t=0$, it follows that

$$g(s)=\lim_{m\to\infty}g\left(\frac{s}{3^m}\right)=g(0)=0.$$

That is, $g(t)=0$ for all $t\in\left(-\frac{1}{9},\infty\right)$. □

Let $f(x)=x+\ln(x)+k+g(x-1)$ with $k\in\mathbb{R}$ and $g(t)$ continuous at $t=0$ such that $g(0)=0$ satisfy the functional equation

$$f(9x-8)-2f(3x-2)+f(x)=4x-4+\ln\frac{9x^2-8x}{(3x-2)^2}.$$

Substituting, we get that f satisfies the equation if and only if

$$g(9(x-1))-2g(3(x-1))+g(x-1)=0$$

for all $x\in\left(\frac{8}{9},\infty\right)$. Applying the Lemma, it then follows that $g(x-1)=0$ for all $x\in\left(\frac{8}{9},\infty\right)$, which gives $f(x)=x+\ln(x)+k$.

Editor's Comments. Bataille noticed that $\frac{17}{18}>\frac{8}{9}$ while $9\cdot\frac{17}{18}-8=\frac{1}{2}<\frac{8}{9}$ so that $f\left(9\cdot\frac{17}{18}-8\right)$ is not defined! The intended version of the problem seems to be :

> Determine the functions $f:(0,\infty)\to\mathbb{R}$ continuous at $x=1$ such that
>
> $$f(9x-8)-2f(3x-2)+f(x)=4x-4+\ln\frac{9x^2-8x}{(3x-2)^2}\qquad\text{for all } x\in\left(\frac{8}{9},\infty\right).$$

3970. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{10a^3+9} + \frac{1}{10b^3+9} + \frac{1}{10c^3+9} \geq \frac{3}{19}.$$

We received eight correct solutions. Two submissions were incorrect and two incomplete. We present two solutions.

Solution 1, by Nermin Hodžić and Salem Malikić.

Without loss of generality, we may assume that $a + b \geq 2$. Then the following inequality holds :

$$\frac{1}{10a^3+9} + \frac{1}{10b^3+9} \geq \frac{8}{5(a+b)^3+36}.$$

To see this, note that the numerator of the difference between the left and right sides is equal to 10 times

$$\begin{aligned}&(a-b)^2[5(a+b)^4 + 5ab(a^2+b^2) + 30a^2b^2 - 27(a+b)]\\ &\qquad \geq (a-b)^2[40(a+b) - 27(a+b) + 5ab(a^2+b^2) + 30a^2b^2] \geq 0,\end{aligned}$$

with equality if and only if $a = b$.

Thus, the difference between the two sides of the required inequality is not less than

$$\frac{8}{5(c-3)^2+36} + \frac{1}{10c^3+9} - \frac{3}{19} = \frac{30c(c-1)^2(5c^3-35c^2+60c+36)}{19(5(3-c)^3+36)(10c^3+9)}.$$

Since $c \leq 1$, this quantity is nonnegative. Equality occurs when $c = 0$ or $c = 1$.

Thus, the desired inequality holds with equality exactly when

$$(a,b,c) = (1,1,1), \left(\frac{3}{2}, \frac{3}{2}, 0\right), \left(\frac{3}{2}, 0, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}, 0\right).$$

Solution 2, by Madhav Modak.

Let E be equal to the left side of the desired inequality. We may assume that $a \geq b \geq c \geq 0$. The function

$$f(x) = \frac{1}{10x^3+9}.$$

is concave on $[0, \alpha]$ and convex on $[\alpha, 3]$, where $\alpha = \sqrt[3]{9/20}$.

The tangent to the graph of f at $(1, 1/19)$ has equation $y = g(x)$ with

$$g(x) = -\frac{30}{19^2}(x-1) + \frac{1}{19} = \frac{49-30x}{19^2}.$$

Then
$$f(x) - g(x) = \frac{10(x-1)^2(30x^2+11x-8)}{19^2(10x^3+9)} \geq 0,$$
for $x \geq 1/2$. Therefore, when $c \geq 1/2$,
$$E \geq \frac{30}{19^2}[(1-a)+(1-b)+(1-c)] + \frac{3}{19} = \frac{3}{19}.$$
Equality occurs if and only if $a = b = c = 1$.

The tangent to the graph of f at $(3/2, 4/171)$ has equation $y = h(x)$ with
$$h(x) = -\frac{40}{3 \times 19^2}\left(x - \frac{3}{2}\right) + \frac{4}{9 \times 19} = \frac{256 - 120x}{9 \times 19^2}.$$
When $0 \leq x \leq 3$,
$$f(x) - h(x) = \frac{20(x-\frac{3}{2})^2(60x^2+52x+21)}{(9 \times 19^2)(10x^3+9)}.$$
Equality occurs if and only if $x = 3/2$.

Therefore,
$$E \geq \frac{512 - 120(3-c)}{9 \times 19^2} + \frac{1}{10c^3+9}.$$
When $0 \leq c \leq 1/2$, we find that
$$E - \frac{3}{19} \geq \frac{10c(120c^3 - 361c^2 + 108)}{(9 \times 19^2)(10c^3+9)} \geq 0,$$
with equality if and only if $c = 3/2$.

Thus, we obtain the desired inequality with the same conditions for equality as before.

Editor's comments. Fanchini applied Muirhead's Theorem to the numerator of the difference between the two sides to obtain the result, but as the argument uses an advanced result and the execution is tedious but straightforward, we do not present it here.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

George Apostolopoulos, Messolonghi, Greece : 4068
Michel Bataille, Rouen, France : 4064
D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania : 4062, 4069
Mihaela Berindeanu, Bucharest, Romania : 4066
Marcel Chiriţă, Bucharest, Romainia : 4063
Leonard Giugiuc, Romania : 4061
Leonard Giugiuc and Daniel Sitaru, Drobeta Turnu Severin, Romania : 4070
Martin Lukarevski, University "Goce Delcev", Macedonia : 4065
Mehtaab Sawhney, Commack High School, Commack, NY, USA : 4067

Solvers - individuals

Arkady Alt, San Jose, CA, USA : **OC182**, 3964, 3968, **3969**
George Apostolopoulos, Messolonghi, Greece : 3964
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : **CC134**, CC135, OC182, **OC184**, **3966**, 3968
Roy Barbara, Lebanese University, Fanar, Lebanon : 3963, 3968
Michel Bataille, Rouen, France : OC182, **OC183**, OC184, OC185, 3961, 3962, 3963, 3964, **3965**, **3966**, 3967, 3968, 3969
Marcel Chiriţă, Bucharest, Romania : 3966, 3967, 3969
Matei Coiculescu, East Lyme High School, East Lyme, CT, USA : CC135, **3963**
Joseph DiMuro, Biola University, La Mirada, CA, USA : **3967**, 3968
Andrea Fanchini, Cantù, Italy : **CC131**, **OC185**, 3962, 3966, 3968, 3970
Ovidiu Furdui, Technical University of Cluj-Napoca, Romania : 3965
Oliver Geupel, Brühl, NRW, Germany : **OC181**, OC182, OC183, OC184, **3962**, 3963, **3964**, **3966**, 3968
Dag Jonsson, Uppsala, Sweden : **3966**
Anastasios Kotronis, Athens, Greece : 3965
Michal Kremzer, Gliwice, Poland : 3968
Kee-Wai Lau, Hong Kong, China : **3961**, 3970
Kathleen E. Lewis, University of the Gambia, Brikama, Republic of the Gambia : **3968**
Salem Malikić, Simon Fraser University, Burnaby, BC : **3966**, 3968
Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India : **3970**
Michael Parmenter, Memorial University of Newfoundland, St. Johns, NL : 3968
Ricard Peiró i Estruch, IES "Abastos", Valencia, Spain : CC131, CC134, CC135, 3966
Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy : OC182, 3966, 3970
Angel Plaza, University of Las Palmas de Gran Canaria, Spain : 3968
Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam : 3970
Henry Ricardo, New York Math Circle, New York, USA : **3966** (2 solutions), 3970
Digby Smith, Mount Royal University, Calgary, AB : **CC132**, **CC133**, CC134, CC135, OC184, 3963, **3966**, 3967, 3968, **3969**

Solvers - collaborations