Crux Mathematicorum

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Editorial Board

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EDITORIAL

As I was reviewing my last editorial, I thought of how mathematics is taught across the globe. Surely, math is math everywhere, but students' and teachers' approaches to it vary culture to culture. I have experienced this first-hand: I moved to Canada when I was 18 having gone through the school system and even one year of university in Belarus (yes, I was born in the USSR). For the sake of strengthening my English, I decided to start university from scratch in Canada. And now that I work in mathematical education, I often compare my two freshman years. Belorussian first-year calculus was Canadian third-year analysis: rigorous, technical, precise. We could find limits only using ϵ and δ, we dealt with functions purely analytically and we memorized a lot of proofs. In Canada, my experience was the complete opposite: all we did was direct computations, graph functions and generalize patterns. The former was theoretical, the latter — practical. Which one is better?

Clearly, you need both. My theoretical experience prepared me for doing math thoroughly, with great attention to detail and with forethought; my practical experience built up my intuition for math and taught me to always look for and make connections between various representations of the same mathematical object. But habits are persistent and we tend to stick to what we are more comfortable with. So my Canadian students resist the theoretical side of math calling it "dry" and "irrelevant", while my Belorussian types often object to graphing and describing math in words since it is "watered-down" and "not mathy". You can't please everyone.

At ***Crux***, it is also clear that people (both amongst our subscribers and within the Editorial Board) have different preferred methods and tend to be faithful to their favourite approaches. Whatever your taste, you will find something to your liking on the pages of our journal. And if not, then clearly we are missing your submissions! So send them along to `crux-psol@cms.math.ca` for problem proposals and to `crux-articles@cms.math.ca` for articles.

Kseniya Garaschuk

THE CONTEST CORNER

No. 32

Robert Dawson

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 avril 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

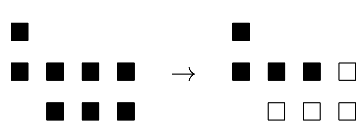
CC156. Décrire et réaliser un croquis précis de la région qui représente l'ensemble

$$\{(x, y, z) : \ |x| + |y| \leqslant 1, \ |y| + |z| \leqslant 1, \ |z| + |x| \leqslant 1\}.$$

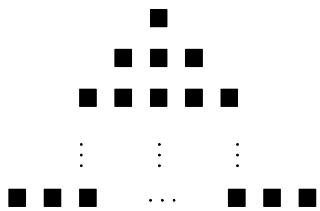
CC157. Étant donné une matrice 5×5 dont chacun des nombres est un 0 ou un 1, démontrer qu'il doit exister une sous-matrice 2×2 (c'est-à-dire l'intersection de la réunion de deux rangées avec la réunion de deux colonnes) dont tous les nombres sont soit 0, soit 1.

CC158. On considère un point mobile A sur la partie positive de l'axe des abscisses, un point mobile B sur la partie positive de l'axe des ordonnées et l'origine O de manière que le triangle ABO ait toujours une aire de 4. Déterminer l'équation d'une courbe, définie dans le premier quadrant, qui est tangente à chacun des segments AB.

CC159. La disposition de huit tuiles carrées, ci-dessous à gauche, peut être divisée en deux groupes congruents de quatre tuiles, comme sur la droite. (On remarque qu'un groupe est le reflet de l'autre dans un miroir, ce qui est permis.)



Déterminer une façon de diviser la disposition suivante de 100 tuiles en deux groupes congruents de 50 tuiles, ou démontrer qu'il est impossible de le réaliser.



(19 tuiles dans la 10^e^ rangée)

CC160. Déterminer tous les triplets (f, g, h) de fonctions continues à valeurs réelles définies sur $\mathbb{R}$ telles pour tout nombre réel x,

$$f(g(x)) = g(h(x)) = h(f(x)) = x.$$

. . . .

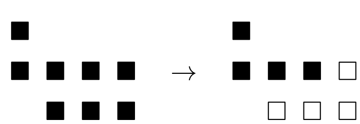
CC156. Describe and accurately sketch the region

$$\{(x, y, z) : \ |x| + |y| \leqslant 1, \ |y| + |z| \leqslant 1, \ |z| + |x| \leqslant 1\}.$$

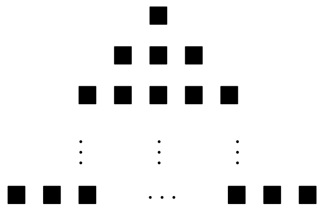
CC157. Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 submatrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

CC158. Suppose movable points A, B lie on the positive x-axis and y-axis, respectively, in such a way that $\triangle ABO$, where O is the origin, always has area 4. Find an equation for a curve in the first quadrant which is tangent to each of the line segments AB.

CC159. The following pattern of eight square tiles can be divided into two congruent sets of four tiles as shown. (Note that one set is the mirror image of the other — this is legal.)



Find a way to divide the following pattern of 100 tiles into two congruent sets of fifty tiles, or show it cannot be done.



(19 tiles in the 10th row)

CC160. Find all triples of continuous functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f(g(x)) = g(h(x)) = h(f(x)) = x\ .$$

Three really magic squares

Having received his yearly salary in silver coins, the royal Mathematician arranged the coins into vertical stacks and placed them on a 3×3 square so that the numbers representing the amount of coins in each stack formed a magic square, that is a square such that the sum of the numbers along every row, column and diagonal of the square is the same. Some stacks came out being quite tall, but none were higher than 300 coins tall.

The King liked the arrangement but lamented over the fact that all the numbers came out being composite. "If your majesty gives me 9 more coins, I will add one to each stack; the magic square property will be preserved but all the new numbers will be prime", replied the Mathematician. The King nearly agreed, but was interrupted by the Joker, who took away one coin from each stack and the new numbers all became prime (and the square, of course, remained a magic square).

What was the original magic square composed by the Mathematician?

From Kvant, 1981 (9), p.31.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section apparaissent initialement dans 2014: 40(2), p. 51–52.

CC106. At each summit of a regular tetrahedron of side length 3, we cut off a pyramid such that the cut-off surface makes an equilateral triangle. The four equilateral triangles thus obtained have all different dimensions. What is the total length of the edges of the solid thus truncated? Provide a proof.

Originally problem 29 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2008.

We received two incomplete submissions to this problem, neither of which adequately proved that the four tetrahedral corners that are removed from the original tetrahedron must be regular. We present an editor's solution.

Let A, B, C, and D be the vertices of the given tetrahedron, and let P, Q, and R be the points on DA, DB, and DC, respectively, such that PQR is the equilateral triangle formed by cutting off the pyramid containing the vertex D. Suppose that the polyhedron has been labeled so that $DP \geq DQ \geq DR$. We will first prove that these segments must, in fact, be equal. Compare triangles DPR and DQR. We have $RP = RQ$ and the angles at D are both 60°. Because we assume that RD is no larger than either DP or DQ, the angles at P and Q must be acute. From the sine law (applied to both triangles) we have

$$\sin\angle DQR = DR\frac{\sin 60^\circ}{RQ} = DR\frac{\sin 60^\circ}{RP} = \sin\angle DPR,$$

from which we conclude that the two triangles are congruent, whence $DQ = DP$. Focusing now on the 60° angle PDQ, we note that the length of the segment PQ increases monotonically as the lengths $DP = DQ$ increase, so there will be exactly one position of P and Q for which $PQ = QR$ $(= RP)$, namely where the lengths DP, DQ, DR are all equal. Because the angles at D are all 60°, the three faces at D are equilateral triangles, and the tetrahedron $DPQR$ is therefore regular.

Returning to the problem, because all edges of a regular tetrahedron have the same length, $DP + DQ + DR = PQ + QR + RP$ and we conclude that the truncation at the vertex D does not change the sum of the edge lengths. Of course, the same can be said about the truncation at the other vertices, so the total length of the edges of the truncated tetrahedron must equal 18.

CC107. In a right triangle ABC with right angle at B and $BC = 1$, we place D on side AC such that $AD = AB = \frac{1}{2}$. What is the length of DC?

Originally problem 6 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2002.

There were eight solution submitted for this question, all essentially the same.

By the Pythagorean Theorem we have

$$AC = \sqrt{BC^2 + AB^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

and

$$DC = AC - AD = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}.$$

CC108. In an orthonormal system, the line with equation $y = 5x$ crosses the parabola with equation $y = x^2$ in point A. The perpendicular to OA at O intersects the parabola at B. What is the area of triangle AOB?

Originally problem 20 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.

We received six correct solutions, and one incorrect solution. We present the solution of Titu Zvonaru.

It is easy to deduce that $A(5, 25)$. The slope of OB is $-1/5$. Solving the system $y = -\frac{1}{5}x$, $y = x^2$ we obtain $B(-\frac{1}{5}, \frac{1}{25})$.

It follows that $OA = \sqrt{5^2 + 25^2} = 5\sqrt{26}$, $OB = \sqrt{\frac{1}{5^2} + \frac{1}{25^2}} = \frac{\sqrt{26}}{25}$. Hence the area of the triangle is $AOB = \frac{OA \cdot OB}{2} = \frac{26}{10} = \frac{13}{5}$.

CC109. Let E be the set of reals x for which the two sides of the following equality are defined:

$$\cot 8x - \cot 27x = \frac{\sin kx}{\sin 8x \sin 27x}.$$

If this equality holds for all the elements of E, what is the value of k?

Originally problem 21 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.

We received seven submitted solutions to this problem, one of which was incorrect and five were incomplete. We present the only correct solution by Paolo Perfetti modified by the editor.

Note first that $E = \{x \in \mathbb{R} | x \neq \frac{m\pi}{8}$ and $x \neq \frac{m\pi}{27}$ for any $m \in \mathbb{Z}$. For $x \in E$, the given equality is equivalent to

$$\sin 8x \cdot \sin 27x(\cot 8x - \cot 27x) = \sin kx. \quad (1)$$

We shall prove that the only value of k for which (1) holds for all $x \in E$ is $k = 19$.

Since

$$\begin{aligned}\sin 8x \cdot \sin 27x(\cot 8x - \cot 27x) &= \sin 27x \cos 8x - \cos 27x \sin 8x \\ &= \sin(27x - 8x) = \sin 19x,\end{aligned}$$

$k = 19$ satisfies (1).

Next, suppose (1) holds for all $x \in E$ and some $k \in \mathbb{Z}$ with $k \neq 19$.

If $k = -19$, then from (1) we have $2\sin 19x = 0$ for all $x \in E$, which is false (for example, if $x = \frac{\pi}{38}$, then $x \in E$, but $\sin 19x = \sin\frac{\pi}{2} = 1 \neq 0$). Hence $k \neq -19$.

From (1), we also have

$$2\sin\left(\frac{19-k}{2}x\right)\cos\left(\frac{19+k}{2}x\right) = 0. \qquad (2)$$

Since $\sin\left(\frac{19-k}{2}x\right) = 0$ if and only if $\frac{19-k}{2}x = m\pi$ or $x = \frac{2m\pi}{19-k}$ and $\cos\left(\frac{19+k}{2}x\right) = 0$ if and only if $\frac{19+k}{2}x = (m+\frac{1}{2})\pi$ or $x = \frac{(2m+1)\pi}{19+k}$ for some $m \in \mathbb{Z}$, there must be some $x \in E$ that does not satisfy (2). (To be more precise, the set of all x such that $x = \frac{2m\pi}{19-k}$ or $x = \frac{(2m+1)\pi}{19+k}$ for some $m \in \mathbb{Z}$ is countable while E is clearly uncountable.) This is a contradiction and our proof is complete.

CC110. What is the number of real solutions to the equation:

$$|1 + x - |x - |1 - x||| = |-x - |x - 1||.$$

Originally problem 26 from Demi-finale du Concours Maxi de Mathématiques de Belgique 2009.

We have received four correct solutions and one incorrect submission. We present the solution by Henry Ricardo.

We compute the left-hand side (LHS) and the right-hand side (RHS) on three intervals that cover the real number line.

Case 1. Suppose that $0 \leq x \leq 1$. Then

$$\text{RHS} = |-x - (1-x)| = |-x - 1 + x| = 1.$$

When $x \in [-, \frac{1}{2}]$,

$$|1 + x - |x - (1-x)|| = |1 + x - (1 - 2x)| = 3x$$

and when $x \in (\frac{1}{2}, 1]$,

$$|1 + x - |2x - 1|| = |1 + x - (2x - 1)| = |2 - x| = 2 - x$$

so that

$$\text{LHS} = \begin{cases} 3x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2 - x & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Thus LHS $=$ RHS when either $3x = 1$ or $2 - x = 1$, which implies $x = \frac{1}{3}$ and $x = 1$ for $x \in [0, 1]$.

Case 2. If $x > 1$, we have

$$\text{LHS} = |1 + x - |x - (x - 1)|| = |1 + x - 1| = x$$

and

$$\text{RHS} = | - x - (x - 1)| = | - 2x + 1| = 2x - 1.$$

But $x > 1$ implies that $(2x - 1) - x = x - 1 > 0$, so RHS $>$ LHS and there are no solutions to the equation in the interval $(1, \infty)$.

Case 3. Finally, for $x \in (-\infty, 0)$, RHS $= 1$ and LHS $= |3x| = -3x$, so LHS $=$ RHS if and only if $-3x = 1$, which implies $x = -\frac{1}{3}$.

Thus the only solutions of the given equation are $x = -\dfrac{1}{3},\ 1,\ \dfrac{1}{3}$.

Math Quotes

The solution of problems is one of the lowest forms of mathematical research. Yet, its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem.

Benjamin Franklin Finkel.

THE OLYMPIAD CORNER

No. 330

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 avril 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC216. Soit $p = n^2 + 1$ un nombre premier. Déterminer toutes les solutions entières à l'équation suivante :

$$x^2 - (n^2+1)y^2 = n^2.$$

OC217. Soit G le centroïde du triangle rectangle ABC où $\angle BCA = 90^\circ$. Soit P le point sur le rayon AG tel que $\angle CPA = \angle CAB$, et soit Q le point sur le rayon BG tel que $\angle CQB = \angle ABC$. Démontrer que les cercles circonscrits de AQG et BPG se rencontrent à un point sur le côté AB.

OC218. Déterminer toute fonction $f : \mathbb{N} \to \mathbb{N}$ satisfaisant

$$f(mn) = \operatorname{lcm}(m,n) \cdot \gcd(f(m), f(n))$$

pour tous les entiers positifs m et n.

OC219. Pour m et n des entiers positifs donnés, démontrer qu'il existe un entier c tel que les nombres cm et cn ont le même nombre d'occurences de chaque chiffre non nul, lorsqu'ils sont exprimés en base 10.

OC220. Soit $A_1A_2...A_8$ un octagone convexe où tous les côtés sont de même longueur et où les côtés opposés sont parallèles. Pour chaque $i = 1, ..., 8$, posons B_i le point d'intersection des segments A_iA_{i+4} et $A_{i-1}A_{i+1}$, où $A_{j+8} = A_j$ et $B_{j+8} = B_j$ pour tout j. Fournir un nombre i, parmi 1, 2, 3, et 4, satisfaisant

$$\frac{A_iA_{i+4}}{B_iB_{i+4}} \leqslant \frac{3}{2}.$$

. . . .

OC216. Let $p = n^2 + 1$ be a given prime number. Find the set of integer solutions to the following equation :

$$x^2 - (n^2 + 1)y^2 = n^2.$$

OC217. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

OC218. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(mn) = \operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n))$$

for all positive integers m, n.

OC219. Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

OC220. Let $A_1A_2...A_8$ be a convex octagon such that all of its sides are equal and its opposite sides are parallel. For each $i = 1, ..., 8$, define B_i as the intersection between segments A_iA_{i+4} and $A_{i-1}A_{i+1}$, where $A_{j+8} = A_j$ and $B_{j+8} = B_j$ for all j. Show that some number i, amongst 1, 2, 3, and 4 satisfies

$$\frac{A_iA_{i+4}}{B_iB_{i+4}} \leqslant \frac{3}{2}.$$

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section apparaissent initialement dans 2014 : 40(2), p. 56–57.

OC156. Let $ABCD$ be a tetrahedron. Prove that vertex D, center of insphere and centroid of $ABCD$ are collinear if and only if the areas of triangles ABD, BCD, CAD are equal.

Originally question 2 from day 1 of the Poland Math Olympiad.

We received four correct solutions to this problem. We present the solution by Michel Bataille.

Let I and r be the center and the radius of the insphere. Let $\mathcal{V}(\cdot)$ and $\mathcal{A}(\cdot)$ denote volume and area, respectively.

Since $\mathcal{V}(IBCD) = \frac{1}{3} \cdot r \cdot \mathcal{A}(BCD)$ (and similarly for triangles CDA, DAB and ABC), we can use

$$(\mathcal{A}(BCD) : \mathcal{A}(CDA) : \mathcal{A}(DAB) : \mathcal{A}(ABC))$$

instead of

$$\mathcal{V}(IBCD) : \mathcal{V}(ICDA) : \mathcal{V}(IDAB) : \mathcal{V}(IABC))$$

for the barycentric coordinates of I relative to (A, B, C, D). It follows that

$$\sigma\mathbf{I} = (\mathcal{A}(BCD))\mathbf{A} + (\mathcal{A}(CDA))\mathbf{B} + (\mathcal{A}(DAB))\mathbf{C} + (\mathcal{A}(ABC))\mathbf{D}$$

where σ is the sum of the areas of the faces of $ABCD$ and the bold face letters represent the coordinates of the points. In particular, we have

$$\sigma\overrightarrow{DI} = (\mathcal{A}(BCD))\overrightarrow{DA} + (\mathcal{A}(CDA))\overrightarrow{DB} + (\mathcal{A}(DAB))\overrightarrow{DC}. \qquad (1)$$

Let G be the centroid of $ABCD$. Then,

$$4\mathbf{G} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D},$$

from which we deduce

$$4\overrightarrow{DG} = \overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC}. \qquad (2)$$

Now, D, I, G are collinear if and only if

$$\sigma\overrightarrow{DI} = \lambda(4\overrightarrow{DG}) \qquad (3)$$

for some real number λ. Since $\overrightarrow{DA}, \overrightarrow{DB}, \overrightarrow{DC}$ are not coplanar, (1) and (2) show that (3) occurs if and only if

$$\mathcal{A}(BCD) = \mathcal{A}(CDA) = \mathcal{A}(DAB).$$

OC157. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y, \forall x, y \in \mathbb{R}\,.$$

Originally question 4 from Kyrgyzstan National Olympiad.

We received three correct solutions and one incorrect submission. We present the solution by Henry Ricardo.

Letting $x = 0$ in the given equation, we have

$$f(f(0)^2 + f(y)) = y,\ \forall y \in \mathbb{R} \qquad (1)$$

so f is *onto*. Thus there exists $a \in \mathbb{R}$ such that $f(a) = 0$. Taking $x = a$ in the original equation gives us

$$f(f(y)) = y, \quad \forall y \in \mathbb{R}, \qquad (2)$$

which shows that f is one to one since $f(x) = f(y)$ implies $f(f(x)) = f(f(y))$, or by (2), $x = y$.

Now replace x by $f(x)$ in the original equation to obtain

$$f(x^2 + f(y)) \;=\; f(x)f(f(x)) + y \;=\; xf(x) + y \;=\; f(f(x)^2 + f(y))$$

Usnig the fact that f is one to one, we have

$$x^2 + f(y) \;=\; f(x)^2 + f(y).$$

Therefore $x^2 = f(x)^2$ and $f(x) = \pm x$.

To eliminate the possibility that $f(x) = x$ for some x and $f(y) = -y$ for some $y \neq x$, suppose that $xy \neq 0$ and $f(x) = x, f(y) = -y$. The original equation gives us $f(x^2 - y) = x^2 + y$, but we know that $f(x) = \pm x$ and so $\pm(x^2 - y) = x^2 + y$ implies either $x = 0$ or $y = 0$. Now it is clear that $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = -x$ for all $x \in \mathbb{R}$ satisfy the original equation and are the only such functions.

OC158. Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

Originally question 4 from day 1 of the Romania TST.

We received no solutions to this problem.

OC159. Let p be an odd prime number. Prove that there exists a natural number x such that x and $4x$ are both primitive roots modulo p.

Originally question 3 from the 2012 Iran National Math Olympiad Third Round.

We received one correct solution and one incorrect submission. We present the solution by Oliver Geupel.

The existence of a primitive root modulo p is a well-known fact. Suppose that a is a primitive root modulo p. Then there is a positive integer r such that $2 \equiv a^r \pmod{p}$ and therefore $4 \equiv a^{2r} \pmod{p}$. Let $p_1, p_2, \ldots, p_\ell$ denote the distinct prime divisors of $p-1$. For $1 \leqslant k \leqslant \ell$, let s_k be an integer such that $s_k \not\equiv 0 \pmod{p_k}$ and $s_k \not\equiv -2r \pmod{p_k}$. Find via the Chinese Remainder Theorem, a natural number m such that

$$m \equiv s_k \pmod{p_k}, \qquad 1 \leqslant k \leqslant \ell.$$

Then, neither m nor $m+2r$ is divisible by any p_k. Hence, each of the numbers m and $m+2r$ is coprime with $p-1$.

We obtain

$$p-1 \nmid m,\ 2m,\ 3m,\ \ldots,\ (p-2)m,$$

$$p-1 \nmid m+2r,\ 2(m+2r),\ 3(m+2r),\ \ldots,\ (p-2)(m+2r).$$

Thus,

$$a^m,\ a^{2m},\ a^{3m},\ \ldots,\ a^{(p-2)m} \not\equiv 1 \pmod{p},$$

$$a^{m+2r},\ a^{2(m+2r)},\ a^{3(m+2r)},\ \ldots,\ a^{(p-2)(m+2r)} \not\equiv 1 \pmod{p},$$

since a is a primitive root modulo the prime p. We have obtained that a^m and $a^{m+2r} \equiv 4a^m \pmod{p}$ are primitive roots modulo p. Therefore, $x = a^m$ has the required property.

OC160. The incircle of triangle ABC, is tangent to sides BC, CA and AB at D, E respectively F. Let T and S be the reflection of F with respect to B respectively the reflection of E with respect to C. Prove that the incenter of triangle AST is inside or on the incircle of triangle ABC.

Originally question 3 from day 2 of the Iran National Math Olympiad Second Round.

No solutions were received.

Extending a tetrahedron

I. Sharygin

One of the most beautiful tools that can be used when solving geometrical problems consists of replacing the geometric figure in question with another one, a more convenient one in some sense. For example, if the problem involves a triangle with a median, often it is helpful to use this triangle to construct a parallelogram therefore extending the median to turn it into the parallelogram's diagonal. In this article, we will consider several problems involving a triangular pyramid, the so-called tetrahedron, that can be solved by extending the tetrahedron to another solid, often a parallelepiped.

The first way to extend the tetrahedron is presented in Figure 1. Here, AA_1BD is the given tetrahedron. The plane of each of the faces DCC_1D_1, CBB_1C_1 and $A_1B_1C_1D_1$ of the parallelepiped passes through one vertex of the tetrahedron and is parallel to the edge of it opposite of that vertex. This way, one of the corners of the tetrahedron becomes one of the corners of the parallelepiped.

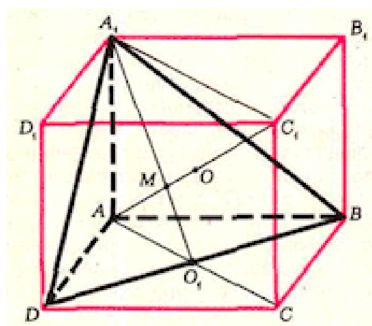


FIGURE 1: Extending the tetrahedron, method 1.

Problem 1 *Suppose the tetrahedron AA_1BD is a right-angle triangular pyramid, that is, the edges AA_1, AB and AD are mutually perpendicular.*

a) Prove that vertex A of the tetrahedron, the point M of intersection of the medians of the face A_1BD and the centre of the circumsphere are collinear. (Note : this problem appeared on the entrance exam to the Mechanical Mathematical Department of the Moscow State University.)

b) Find the radius of the sphere circumscribed around this pyramid.

We extend tetrahedron AA_1BD to a parallelepiped (a right angle one) as in Figure 1. Then the sphere circumscribed around the tetrahedron is also circumscribed around the parallelepiped. The radius of this sphere is then equal to half the length of the diagonal of the parallelepiped, that is it is equal to $\frac{1}{2}\sqrt{AA_1^2 + AB^2 + AD^2}$, which answers part b) of the problem.

To prove part a), consider rectangle AA_1C_1C. Centre O of the sphere lies on the diagonal AC_1. Median A_1O_1 of the triangle A_1BD intersects AC_1 at a point M. Since triangles A_1C_1M and O_1AM are similar, we get :

$$\frac{A_1M}{O_1M} = \frac{A_1C_1}{O_1A} = 2,$$

meaning that M is the point of intersection of the medians of the triangle A_1BD and we are done.

Another common way to extend a tetrahedron to a parallelepiped is as follows : for every edge of the tetrahedron, construct a plane containing this edge and parallel to the opposite edge (Figure 2, left). This way, the edges of the tetrahedron become the diagonals of the faces of the parallelepiped. A little practical hint : it is easier to sketch this construction if you start with the parallelepiped first.

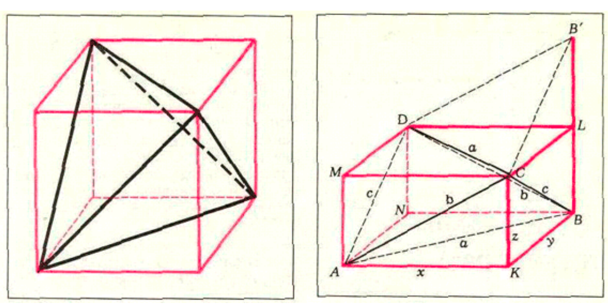


Figure 2: Extending the tetrahedron, method 2.

Problem 2 *Find the radius of the sphere that touches all the edges of a regular tetrahedron with edge length a.*

As one can easily see from Figure 2, a parallelepiped constructed in this way around a regular tetrahedron is a cube with edge length $a/\sqrt{2}$. The sphere touching the edges of the tetrahedron is the sphere inscribed in the cube. The answer is therefore $\frac{a}{2\sqrt{2}}$.

So the first method of extending a tetrahedron is useful when you are given all angles at one of the vertices (especially if those angles are all right angles) ; the second method is helpful in problems involving the opposite edges of the tetrahedron.

Problem 3 *Two opposite edges of the tetrahedron have length a, two other opposite edges have length b and the two remaining opposite edges have length c. Find the distance between the centre of the tetrahedron's insphere and the centre of the sphere that touches one face of the tetrahedron and the extension of all the others.*

Consider such a tetrahedron $ABCD$ and the parallelepiped $AMCKNDLB$ constructed as in Figure 2 on the right. Since each edge of the tetrahedron equals its opposite edge, all the faces of the parallelepiped are rectangles and hence the whole solid is rectangular.

The centre of the sphere inscribed in the tetrahedron $ABCD$ coincides with the intersection point of all the parallelepiped's diagonals (prove this fact). Without loss of generality, assume that the external sphere touches the face DCB and the extension of the other faces. Then the centre of this sphere lies at the vertex L of the parallelepiped. To see this, consider the pyramid $B'LCD$ equal to the pyramid $BLCD$, where $B'L = LB$: the points A, D, B' and C lie in one plane. Therefore, L is equidistant from the face DCB and the extension of ACD. Similarly, one can show that L is equidistant from the planes of DCB and ACB as well as DCB and ADB.

Therefore, the distance in question is equal to half the length of the diagonal of the parallelepiped. Let x, y, z denote the lengths of the edges of the parallelepiped. By Pythagoras' Theorem, we get a system of three equations :

$$x^2 + y^2 = a^2,$$
$$x^2 + z^2 = b^2,$$
$$y^2 + z^2 = c^2.$$

Adding them all up, we find that

$$\frac{AL}{2} = \frac{1}{2}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\sqrt{\frac{a^2 + b^2 + c^2}{2}}.$$

Problem 4 *Let A_1B_1CD be a tetrahedron ; let Π be the (unique) plane parallel to the lines A_1B_1 and CD and equidistant from them. Let S be the area of the cross section of A_1B_1CD cut by Π and suppose the distance between the opposite edges is h. Find the volume of the tetrahedron.*

Suppose $ABCDA_1B_1C_1D_1$ is the parallelepiped as in Figure 3.

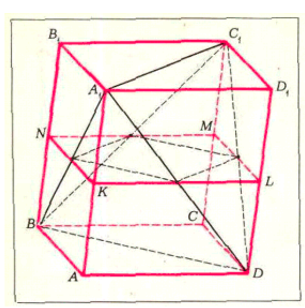


FIGURE 3: Cross section of the tetrahedron.

The volume of the tetrahedron A_1BC_1D is equal to the volume of the parallelepiped minus the volumes of the four tetrahedra, one on each face of the initial tetrahedron and each with the volume equal to one sixth of the volume of the parallelepiped (why?). Therefore, $V_{\text{tetrahedron}} = V_{\text{paralellepiped}}/3$.

Let A_1C_1 and BD be the two opposite edges and let $KLMN$ be the plane intersecting the parallelepiped in midpoints of the vertical edges AA_1, BB_1, CC_1 and DD_1. Then the vertices of the cross section defined in the problem are the midpoints of the edges of the parallelogram $KLMN$. Therefore,

$$\text{Area}(KLMN) = 2S = \text{Area}(ABCD).$$

Then

$$V_{\text{tetrahedron}} = \frac{1}{3} V_{\text{paralellepiped}} = \frac{2}{3} Sh.$$

(Using problem 4, one can easily prove Simpson's formula used to compute volumes of certain solids.)

In conclusion, let us show one example where it is more convenient to extend the tetrahedron to the triangular prism.

Problem 5 *Suppose that in a tetrahedron, areas of two faces are equal to S_1 and S_2 and the angle between them is α. Suppose further that the areas of the two other faces are equal to Q_1 and Q_2 and the angle between them is β. Show that*

$$S_1^2 + S_2^2 - 2S_1S_2 \cos\alpha = Q_1^2 + Q_2^2 - 2Q_1Q_2 \cos\beta.$$

Let us first prove that if the area of one lateral face of a prism is S and the areas of two other lateral faces are equal to S_1 and S_2 with the angle between them equal to α, then

$$S_1^2 + S_2^2 - 2S_1S_2 \cos\alpha = S^2.$$

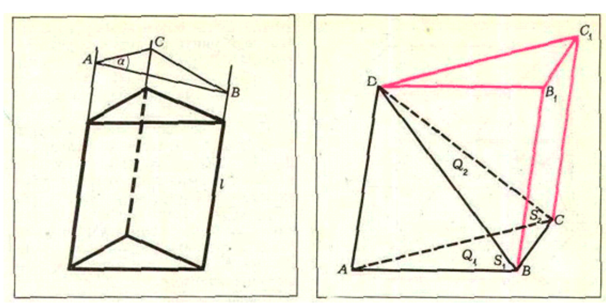


FIGURE 4: Extending the tetrahedron to a prism.

Indeed, let the plane ABC be perpendicular to the lateral faces of the prism and let $\angle BAC = \alpha$ (Figure 4, left). By Law of Cosines applied to triangle ABC, we have

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos\alpha.$$

All that is left to do now is to multiply each side by l^2, where l is the length of the lateral edge of the prism.

Now, let us come back to the main statement of Problem 5. Given the tetrahedron $ABCD$, let $S_{\triangle ABD} = S_1$, $S_{\triangle ADC} = S_2$, $S_{\triangle ABC} = Q_1$, $S_{\triangle DBC} = Q_2$, angle at the edge AD be equal to α and angle at the edge BC be equal to β. Consider the triangular prism with base ABC with AD as one of the lateral edges (Figure 4, right). Let S be the area of the parallelogram BB_1C_1C, then by the formula proved above we have :

$$4S_1^2 + 4S_2^2 - 8S_1S_2\cos\alpha = S^2.$$

Note that $S = AD \cdot BC \cdot \sin\gamma$, where γ is the angle between edges BC and AD. Similarly, considering the other triangular prism with base ACD and lateral edge BC, we get :

$$4Q_1^2 + 4Q_2^2 - 8Q_1Q_2\cos\beta = S^2.$$

Combining the above, we are done.

Exercises.

1. Prove that the sum of squares of the edge lengths of a tetrahedron is equal to four times the sum of squares of distances between the midpoints of its opposite edges.

2. Given a tetrahedron $ABCD$, show that the directions of opposite edges AD and BC are perpendicular if and only if $AB^2 + DC^2 = AC^2 + DB^2$.

3. The lengths of two opposite edges of a tetrahedron are equal to a, the other two b and the last two c. Find a) the volume of this tetrahedron ; b) the radius of the circumsphere.

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This article originally appeared in Russian in Kvant, 1976 (1), p. 60–65. It has been translated and adapted with permission.

Graphs and Edge Colouring

David A. Pike

Let's begin with a familiar scenario : suppose that we need to schedule the matches of a round-robin tournament involving n teams, so that once the tournament has concluded each team will have played against each other team exactly once. In total there would be $\binom{n}{2}$ games. Potentially they could be scheduled sequentially, but that could make for a very long and drawn out tournament. In the interest of completing the event as quickly as possible, we instead want to have several teams competing simultaneously. The question that now arises is this : given that no team can play more than one game at a time, how few time slots are needed in order to schedule the whole tournament ?

This particular question was answered long ago by modelling it with graph theory. In the 1890s Édouard Lucas [5] published a solution, for which he gave credit as follows :

> *Parmi les diverses méthodes qui nous ont été indiquées, nous exposerons, de préférence, les solutions simples et ingénieuses de M. Walecki, professeur de Mathématiques spéciales au lycée Condorcet.*

At this stage it would be good to know exactly what a graph is. Formally, a graph G consists of a set V of elements called vertices, accompanied by a set E of edges, which themselves are pairs of vertices. Any graph can easily be represented in the form of a drawing. For instance, in Figure 1 are two drawings of the graph having $V = \{a, b, c, d, e\}$ and $E = \big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{d, e\}\big\}$. Each vertex is depicted as a circular node, and each edge is illustrated by drawing a line between its two vertices.

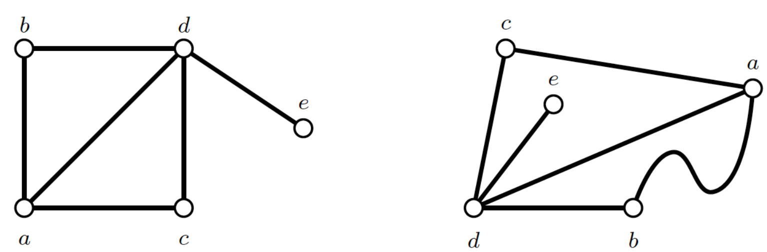

FIGURE 1: Two different drawings of a graph

Note that there is no specified location for the vertices or edges of a graph. Moreover, the edges do not need to be drawn as straight lines, but are free to have bends and curves. In practice the vertices may represent real entities (such as sports teams which do have a geographical placement) and the edges might also represent connections with physical form (such as railway lines between cities), but what is most important here is that the graph captures the existence of a relationship between pairs of vertices (such as the need for their corresponding teams to play against each other).

For our scenario of a round-robin tournament, we will want to consider a graph with n vertices such that each pair of vertices is joined by an edge. Such a graph is called a *complete graph* and is denoted by K_n. Figure 2 illustrates the graph K_7, with the vertices named 0 to 6. As can be seen in this example, we also allow edges to be drawn so that their lines intersect.

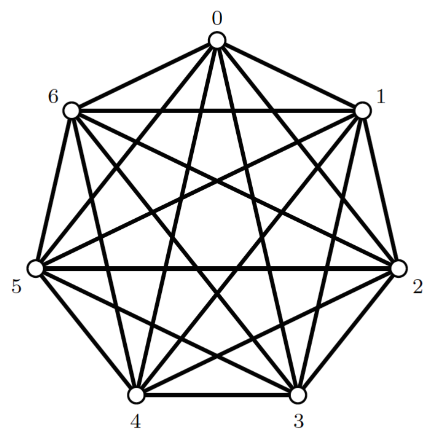

FIGURE 2: The complete graph K_7

If we have seven teams that must each play against each other during a tournament, then each edge of K_7 corresponds to an individual game that has to be played. But we still need to find a way to schedule the games, preferably into as few time slots as possible. Since each edge of K_7 represents a distinct game that must be played, then the games within a single time slot correspond to a set of edges, no two of which share a vertex. So to find a schedule, we need only find a way to partition the edges of K_7 into sets of this form.

Exercise 1 *Find a schedule for these 21 games.*

Having just found a schedule for the games that the seven teams must play, we now have to consider whether there might be a better schedule.

Question 1 *Is there a schedule that uses fewer time slots ?*

If not, then how can you be sure that yours is indeed the best ? To determine just how few time slots there are in an optimal schedule, we need to do some mathematical thinking.

Observe that each of the seven teams has to play six games, so right away we know that the number of time slots that are in any valid schedule has to be at least six. Were you able to find a schedule that only used six time slots ?

Question 2 *Can you find a schedule with six time slots ? If not, then can you prove that there is no schedule with only six time slots ?*

As it happens, there is no way to schedule the tournament with seven teams so that all of the games fit into only six time slots. To convince yourself that this is the case, note that the teams that are playing games within a single time slot are playing in pairs. So with seven teams, at most six of them can actually be playing at any given time, which in turn means that at most three games can take place at a time. In total, there are 21 games that must be played, and with at most three that can be scheduled per time slot, we need at least seven time slots for the tournament. With this argument in mind, we can now conclude that any schedule that uses only seven time slots must in fact be an optimal solution.

Let's move away from the example of $n = 7$ now and consider what might happen when n is even. Is it still the case that $n - 1$ time slots is impossible ?

Exercise 2 *Use K_4 and K_6 to find optimal schedules for $n = 4$ and $n = 6$.*

You should find that for these two small examples it is actually possible to find schedules with as few as 3 and 5 time slots, respectively. To try to see a general pattern we will now consider $n = 8$. With teams named 0 to 6 and ∞, Figure 3 illustrates how to form four pairs of teams for the first time slot of the tournament, and then how to form four pairs for the second time slot. Looking at this figure, a general approach ought to become apparent : rotate the edges clockwise for each subsequent time slot. To be a bit more technical, for each edge $\{u, v\}$ of one time slot, for the next time slot use the edge $\{u + 1, v + 1\}$ where we treat $\infty + 1$ as ∞ and $(n - 2) + 1$ as 0.

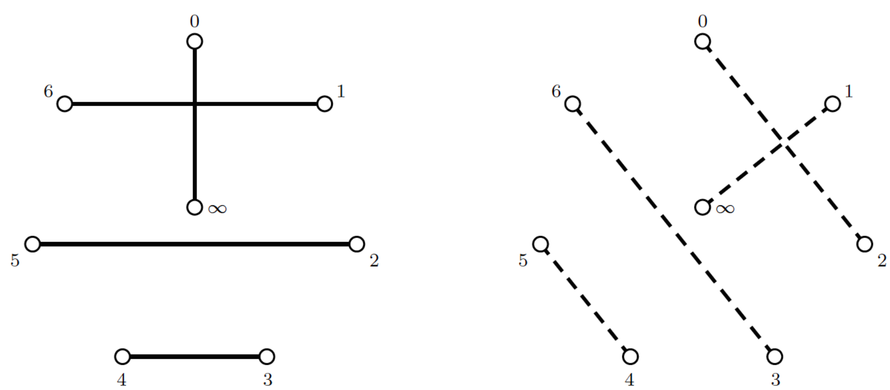

FIGURE 3: Team pairings for two different time slots in K_8

So for even n, an optimal solution is to use this technique with the edges $\{0, \infty\}$ and $\{1, n-2\}, \{2, n-3\}, \ldots, \{\frac{n-2}{2}, \frac{n}{2}\}$ for the initial time slot. The resulting schedule will have a total of $(n - 1)$ time slots for the tournament.

For odd n, however, recall that each time slot has to have a team that sits out. We can find a schedule that uses n time slots by introducing a phantom team called ∞, building an optimal schedule for $(n + 1)$ teams (note that $n + 1$ is even), and then assigning byes to teams whenever they are paired with the phantom team.

So when n is odd, we know that a schedule with n time slots can be achieved, although for $n \neq 7$ we have not yet proved that $n-1$ time slots are insufficient.

At this point hopefully you are beginning to wonder what any of this has to do with colouring, although perhaps you've already discovered that each time slot can be associated with a distinct colour. If we colour the edges of a graph so that two edges that meet at a common vertex are not allowed to share the same colour, then it is possible to use the colouring to form a schedule of pairings. Alternatively, if we do not actually have colours to work with (such as this black-and-white article), we can emulate the idea of colours with dashed lines, etc., similar to what we have done for the graph shown in Figure 4. The solid black edges $\{a, c\}$ and $\{b, d\}$ could be used to indicate two games to be played during the first day of a competition (with team e having a bye), the short-dashed edge $\{c, d\}$ provides for just one game on the second day, the dotted edges $\{a, b\}$ and $\{d, e\}$ tell us which games are to take place on the third day, and finally the long-dashed edge $\{a, d\}$ corresponds to the sole game on the fourth day of the competition. Note that in this example the competition is not a round-robin tournament (since not every pair of teams will play against each other).

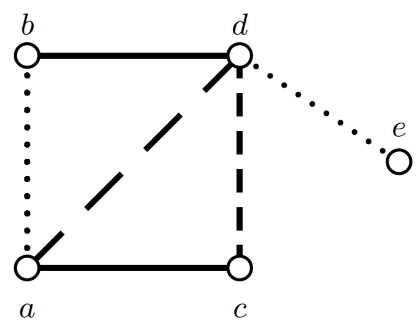

FIGURE 4: Example of an edge colouring

An edge colouring for which edges of the same colour never meet at a common vertex is called a *proper* edge colouring. An easy way to obtain a proper edge colouring is to give each edge a distinct colour, but this would result in no games taking place at the same time. As before, our goal is to determine how few time slots are needed. With our new terminology, given a graph G we want to know the smallest number of colours for which a proper edge colouring exists ; this value is called the *chromatic index* of the graph and is denoted by $\chi'(G)$. An obvious lower bound on the chromatic index is that $\chi'(G) \geqslant \Delta(G)$, where $\Delta(G)$ denotes the number of edges at any vertex with the most edges (for the graph in Figure 4, $\Delta(G)$ is 4 thanks to vertex d belonging to four edges).

For complete graphs, we have already seen that $\chi'(K_{2\ell}) = 2\ell - 1$ and $\chi'(K_{2\ell+1}) \leqslant 2\ell + 1$. So already we have examples of graphs, some of which have $\chi'(G) = \Delta(G)$ and some for which $\chi'(G)$ might be as high as $\Delta(G) + 1$. As it turns out, provided that each pair of vertices is joined by either one edge or none, then these are the only two possible values for $\chi'(G)$, as was proved by Vadim Vizing in the 1960s (this result is proved in most graph theory textbooks, such as [5]).

Vizing's Theorem *If for each pair of vertices of a graph G there is at most one edge between them, then $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$.*

Given that there are only two possible values for the chromatic index, it has become common practice to say that graphs for which $\chi'(G) = \Delta(G)$ are Class 1 and that graphs for which $\chi'(G) = \Delta(G) + 1$ are Class 2. Examples of Class 1 graphs include complete graphs with an even number of vertices, as well as all bipartite graphs ; a graph is called *bipartite* if its vertex set V can be partitioned into two subsets A and B so that every edge of the graph has one of its two vertices in A and the other in B. Bipartite graphs have numerous applications ; indeed, whole books have been written just on bipartite graphs (see [1] for one of them). Class 2 graphs include odd-length cycles (*e.g.*, the graph having $V = \{1, 2, \dots, 2\ell+1\}$ and $E = \big\{\{1,2\}, \{2,3\}, \dots, \{2\ell, 2\ell+1\}, \{1, 2\ell+1\}\big\}$.

Questions regarding how to identify which class a particular graph might be are natural to ask. As an example, in Figure 5 is the famous Petersen graph.

Exercise 3 *Determine whether the Petersen graph is Class 1 or Class 2.*

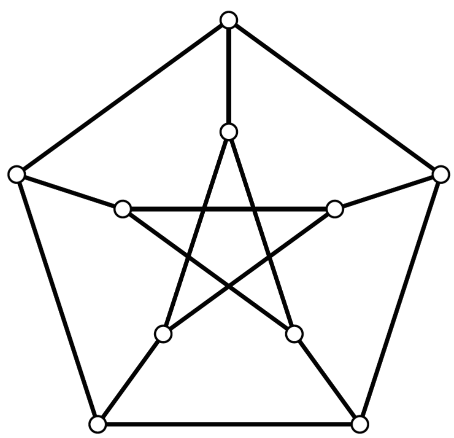

FIGURE 5: The Petersen graph

For some graphs, there is an easy way to determine their class. To give a definition, we will say that a graph G is *overfull* if $|E|$ strictly exceeds $\Delta(G)\lfloor\frac{|V|}{2}\rfloor$, where the notation $|S|$ denotes the cardinality of the set S and $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x (so for example $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$ and $\lfloor -\pi \rfloor = -4$).

Amanda Chetwynd and Anthony Hilton pioneered some of the research on overfull graphs [2]. It is easy to prove that any graph that is overfull must be Class 2. By way of contradiction, suppose that there exists an overfull graph G that happens to be Class 1. Since it is Class 1, $\chi'(G) = \Delta(G)$. Moreover, each colour can be used on at most $\lfloor \frac{|V|}{2} \rfloor$ edges, for if a colour occurred on any more edges then at least two edges of that colour would have to meet at a common vertex. With $\Delta(G)$ colours, each occurring on at most $\lfloor \frac{|V|}{2} \rfloor$ edges, it follows that $|E| \leqslant \Delta(G)\lfloor \frac{|V|}{2} \rfloor$, in violation of the graph being overfull. Thus we have obtained the desired contradiction, from which we conclude that the graph cannot be Class 1.

Having previously determined that $\chi'(K_n) \leqslant \Delta(G) + 1$ when n is odd was not itself a proof that complete graphs with an odd number of vertices are Class 2. However, by verifying that any complete graph with an odd number of vertices is overfull, we can now confirm that $K_{2\ell+1}$ is Class 2. The Petersen graph is also Class 2.

It is not too hard to deduce that if a graph is overfull then it necessarily must have an odd number of vertices. This condition is not sufficient though, for there do exist Class 1 graphs having an odd number of vertices (simply refer to Figure 4 to see an example).

The examples that we have seen so far have not been very difficult. However, determining whether a given graph is Class 1 versus Class 2 is generally not an easy problem. Indeed, Ian Holyer proved in 1981 that it is so hard that it is NP-complete [3]. Nevertheless, motivated both by scientific curiosity as well as the applications that exist for edge colourings, this continues to be an active area of research whereby people try to find faster colouring algorithms and also try to establish that certain types of graphs are Class 1 versus Class 2.

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[2] A.G. Chetwynd and A.J.W. Hilton. Star multigraphs with three vertices of maximum degree. *Math. Proc. Camb. Phil. Soc.* 100 (1986) 303–317.

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[5] D.B. West. *Introduction to Graph Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1996.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 avril 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

3960. *Proposé par George Apostolopoulos. Correction.*

Soient a, b, c des nombres réels non négatifs tels que $a + b + c = 4$. Démontrer que

$$\frac{a^2b}{3a^2+b^2+4ac}+\frac{b^2c}{3b^2+c^2+4ab}+\frac{c^2a}{3c^2+a^2+4bc}\leqslant\frac{1}{2}.$$

4011. *Proposé par Abdilkadir Altinas.*

Dans un triangle non équilatéral ABC, soient H l'orthocentre de ABC et J l'orthocentre du triangle orthique DEF de ABC (c'est-à-dire le triangle formé par les altitudes de ABC). Si $\angle BAC = 60°$, montrer que $AJ \perp HJ$.

4012. *Proposé par Leonard Giugiuc.*

Soit n un entier tel que $n \geqslant 3$. Considérer des nombres réels a_k, $1 \leqslant k \leqslant n$ tels que

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{n-1} \geqslant 1 \geqslant a_n \geqslant 0 \quad \text{et} \quad \sum_{k=1}^{n} a_k = n.$$

Démontrer que

$$\frac{(n-2)(n+1)}{2} \leqslant \sum_{1\leqslant i<j\leqslant n} a_i a_j \leqslant \frac{n(n-1)}{2}.$$

4013. *Proposé par Mehmet Şahin.*

Soient a, b et c les côtés du triangle ABC, D le pied de l'altitude émanant de A et E le mi point de BC. Poser $\theta = \angle DAE$ et supposer que $\angle ACB = 2\theta$. Démontrer que les côtés du triangle vérifient

$$(a-b)^2 = 2c^2 - b^2.$$

4014. *Proposé par Mihaela Berinedanu.*

Soit n un nombre naturel et soient x, y et z des nombres réels positifs tels que $x + y + z + nxyz = n + 3$. Démontrer que

$$(1 + \frac{y}{x} + nyz)(1 + \frac{z}{y} + nzx)(1 + \frac{x}{z} + nxy) \geqslant (n + 2)^3.$$

4015. *Proposé par Michel Bataille.*

Déterminer tous les nombres réels a tels que

$$a \cos x + (1 - a) \cos \frac{x}{3} > \frac{\sin x}{x}$$

pour tout x non nul dans l'intervalle $(-\frac{3\pi}{2}, \frac{3\pi}{2})$.

4016. *Proposé par George Apostolopoulos.*

Soient x, y et z des nombres réels positifs. Déterminer la valeur maximale de l'expression

$$\frac{x + 2y}{2x + 3y + z} + \frac{y + 2z}{2y + 3z + x} + \frac{z + 2x}{2z + 3x + y}.$$

4017. *Proposé par Michel Bataille.*

Soit P un point sur le cercle inscrit γ du triangle ABC. Les perpendiculaires vers BC, CA et AB, passant par P, rencontrent γ aux points U, V et W respectivement. Démontrer qu'un des nombres $PU \cdot BC, PV \cdot CA, PW \cdot AB$ est égal à la somme des deux autres.

4018. *Proposé par Ovidiu Furdui.*

Soit

$$I_n = \int_0^1 \cdots \int_0^1 \ln(x_1x_2 \cdots x_n) \ln(1 - x_1x_2 \cdots x_n) dx_1 dx_2 \cdots dx_n,$$

où $n \geqslant 1$ est entier. Démontrer que l'intégrale converge et déterminer sa valeur.

4019. *Proposé par George Apostolopoulos.*

Un triangle avec longueurs de côtés a, b et c possède un périmètre de longueur 3. Démontrer que

$$a^3 + b^3 + c^3 + a^4 + b^4 + c^4 \geqslant 2(a^2b^2 + b^2c^2 + c^2a^2).$$

4020. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit ABC un triangle. Supposer que les bissectrices internes de A, B et C intersectent les côtés BC, CA et AB en D, E et F respectivement. Le cercle inscrit de ΔABC touche les côtés BC, CA et AB en M, N et P respectivement. Démontrer que $[MNP] \leqslant [DEF]$, où $[\cdot]$ dénote la surface du triangle en question.

. . . .

3960. *Proposed by George Apostolopoulos. Correction.*

Let a, b, c be nonnegative real numbers such that $a + b + c = 4$. Prove that

$$\frac{a^2b}{3a^2+b^2+4ac} + \frac{b^2c}{3b^2+c^2+4ab} + \frac{c^2a}{3c^2+a^2+4bc} \leqslant \frac{1}{2}.$$

4011. *Proposed by Abdilkadir Altinas.*

In non-equilateral triangle ABC, let H be the orthocenter of ABC and J be the orthocenter of the orthic triangle DEF of ABC (that is the triangle formed by the feet of the altitudes of ABC). If $\angle BAC = 60°$, show that $AJ \perp HJ$.

4012. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geqslant 3$. Consider real numbers a_k, $1 \leqslant k \leqslant n$ such that

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{n-1} \geqslant 1 \geqslant a_n \geqslant 0 \quad \text{and} \quad \sum_{k=1}^{n} a_k = n.$$

Prove that

$$\frac{(n-2)(n+1)}{2} \leqslant \sum_{1 \leqslant i < j \leqslant n} a_i a_j \leqslant \frac{n(n-1)}{2}.$$

4013. *Proposed by Mehmet Şahin.*

Let a, b, c be the sides of triangle ABC, D be the foot of the altitude from A and E be the midpoint of BC. Define $\theta = \angle DAE$ and suppose that $\angle ACB = 2\theta$. Prove that the sides of the triangle satisfy

$$(a-b)^2 = 2c^2 - b^2.$$

4014. *Proposed by Mihaela Berinedanu.*

Let n be a natural number and let x, y and z be positive real numbers such that $x + y + z + nxyz = n + 3$. Prove that

$$(1 + \frac{y}{x} + nyz)(1 + \frac{z}{y} + nzx)(1 + \frac{x}{z} + nxy) \geqslant (n+2)^3$$

and determine when equality holds.

4015. *Proposed by Michel Bataille.*

Find all real numbers a such that

$$a\cos x + (1-a)\cos\frac{x}{3} > \frac{\sin x}{x}$$

for every nonzero x of the interval $(-\frac{3\pi}{2}, \frac{3\pi}{2})$.

4016. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers. Find the maximal value of the expression

$$\frac{x+2y}{2x+3y+z} + \frac{y+2z}{2y+3z+x} + \frac{z+2x}{2z+3x+y}.$$

4017. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that one of the numbers $PU \cdot BC, PV \cdot CA, PW \cdot AB$ is the sum of the other two.

4018. *Proposed by Ovidiu Furdui.*

Let

$$I_n = \int_0^1 \cdots \int_0^1 \ln(x_1x_2\cdots x_n)\ln(1-x_1x_2\cdots x_n)dx_1dx_2\cdots dx_n,$$

where $n \geqslant 1$ is an integer. Prove that this integral converges and find its value.

4019. *Proposed by George Apostolopoulos.*

A triangle with side lengths a, b, c has perimeter 3. Prove that

$$a^3+b^3+c^3+a^4+b^4+c^4 \geqslant 2(a^2b^2+b^2c^2+c^2a^2).$$

4020. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a triangle and let the internal bisectors from A, B and C intersect the sides BC, CA and AB in D, E and F, respectively. The incircle of ΔABC touches the sides BC, CA and AB in M, N, and P, respectively. Prove that $[MNP] \leqslant [DEF]$, where $[\cdot]$ denotes the area of the specified triangle.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014: 40(2), p. 73–76, unless otherwise specified.

An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

2832⋆. *Proposed by Walther Janous.* [2003: 176; 2004: 184]

Let n be a positive integer, and let

$$a(n) = \left|\sum_{j=0}^{3n}(-2)^j\left(\binom{6n+2-j}{j+1}+\binom{6n+1-j}{j}\right)\right|.$$

Prove that
(a) $a(n) = 3$ if and only if $n = 1$, and
(b) the sequence $\{a(n)\}_{n=1}^{\infty}$ is strictly increasing.

We give the solution to part (b) by C. R. Pranesachar.

We shall prove that the statement is false. In fact, there exist infinitely many positive integers n such that $a(n) > a(n+1)$.

Let

$$b(n) = \sum_{j\geqslant 0}(-2)^j\left(\binom{n+1-j}{j+1}+\binom{n-j}{j}\right), \quad n \geqslant 0.$$

Also let $c(n) = b(6n+1)$. Then $a(n) = |c(n)|$, as the full range of j is used. Since $\sum_{j\geqslant 0}\binom{n-j}{j}$ is the coefficient of x^n in the series

$$1 + x(1+x) + x^2(1+x)^2 + \cdots + x^n(1+x)^n + \cdots = \frac{1}{1-x-x^2}$$

we see that $\sum_{j\geqslant 0}(-2)^j\binom{n-j}{j}$ is the coefficient of x^n in the series

$$1 + x(1-2x) + x^2(1-2x)^2 + \cdots + x^n(1-2x)^n + \cdots = \frac{1}{1-x+2x^2}.$$

Now,

$$\sum_{j\geqslant 0}(-2)^j\binom{n+1-j}{j+1} = \sum_{j\geqslant 1}(-2)^{j-1}\binom{n+2-j}{j} = \frac{1}{2} - \frac{1}{2}\sum_{j\geqslant 0}(-2)^j\binom{n+2-j}{j},$$

so this sum is the coefficient of x^{n+2} in

$$\frac{1}{2(1-x)} - \frac{1}{2(1-x+2x^2)}.$$

Thus $b(n)$ is the coefficient of x^{n+2} in

$$\begin{aligned}\frac{x^2}{1-x+2x^2}+\frac{1}{2(1-x)}-\frac{1}{2(1-x+2x^2)} &= \frac{1}{2(1-x)}+\frac{2x^2-1}{2(1-x+2x^2)}\\ &= 1+\frac{1}{2(1-x)}+\frac{x-2}{2(1-x+2x^2)}.\end{aligned}$$

Hence $b(n)=\frac{1}{2}+A_0\;\alpha^n+B_0\;\beta^n$, where α and β are the roots of $\lambda^2-\lambda+2=0$, and A_0, B_0 are two fixed numbers. We have

$$\alpha=\frac{1+i\sqrt{7}}{2},\;\; \beta=\frac{1-i\sqrt{7}}{2}.$$

Using the initial values $b(0)=2$ and $b(1)=3$, we get easily that

$$A_0=\frac{3-i\sqrt{7}}{4},\;\; B_0=\frac{3+i\sqrt{7}}{4}.$$

Since $a(n)=|c(n)|$, we compute $c(n)$. In fact

$$\begin{aligned}c(n)=b(6n+1)=&\frac{1}{2}+\left(\frac{3-i\sqrt{7}}{4}\right)\alpha^{6n+1}+\left(\frac{3+i\sqrt{7}}{4}\right)\beta^{6n+1}\\ =&\frac{1}{2}+\left(\frac{5+i\sqrt{7}}{4}\right)\alpha^{6n}+\left(\frac{5-i\sqrt{7}}{4}\right)\beta^{6n}.\end{aligned}$$

Since

$$\alpha^6=\left(\frac{1+i\sqrt{7}}{2}\right)^6=\frac{9+i\sqrt{7}}{2} \quad \text{and} \quad \beta^6=\frac{9-i\sqrt{7}}{2},$$

we have

$$c(n)=\frac{1}{2}+\left(\frac{5+i\sqrt{7}}{4}\right)\left(\frac{9+i\sqrt{7}}{2}\right)^n+\left(\frac{5-i\sqrt{7}}{4}\right)\left(\frac{9-i\sqrt{7}}{2}\right)^n,\quad n\geqslant 1.$$

If we set

$$A=\frac{5+i\sqrt{7}}{4},\;\; B=\frac{5-i\sqrt{7}}{4},$$

then

$$A^2=\frac{9+5i\sqrt{7}}{8},\;\; B^2=\frac{9-5i\sqrt{7}}{8}.$$

Hence

$$b(6n+1)=\frac{1}{2}+A^2(4A^2)^n+B(4B^2)^n=\frac{1}{2}+4^n(A^{2n+1}+B^{2n+1}).$$

Now we may take $A=\sqrt{2}e^{i\theta}, B=\sqrt{2}e^{-i\theta}$, where $\theta=\arccos\left(\frac{5}{4\sqrt{2}}\right)$ is acute. So

$$c(n)=\frac{1}{2}+8^n(2\sqrt{2})\cos(2n+1)\theta.$$

Hence

$$a(n) = |c(n)| = \left|\frac{1}{2} + 8^n(2\sqrt{2})\cos(2n+1)\theta\right|, \quad n \geqslant 1.$$

Now we exploit the properties of θ. Surprisingly, θ has some nice 'solution-friendly' properties that are precisely needed. Firstly,

$$\cos 2\theta = 2\cos^2\theta = 2\cos^2\theta - 1 = 2\left(\frac{25}{32}\right) - 1 = \frac{9}{16}.$$

From this we infer that θ cannot be a rational multiple of π (the proof is left as an exercise for the reader). Therefore, the set $\{\cos(2n+1)\theta : n \in \mathbb{N}\}$ is dense in $[-1,1]$. The next property of θ that we use is as follows: since $\frac{5}{4\sqrt{2}} > \frac{3}{2}$, we have $\cos\theta > \cos 30^\circ$, and so $\theta < 30^\circ$. Hence, if $\phi \in (\theta, 30^\circ)$, an interval of positive length, we may write $\phi = \theta + \tau$ for some suitable $\tau \in (0, 30^\circ - \theta)$.

Further

$$\begin{aligned}\cos\phi - 8\cos(\phi + 2\theta) &= \cos(\theta+\tau) - 8\cos(3\theta+\tau)\\ &= \cos\tau\ (\cos\theta - 8\sin 3\theta) + \sin\tau\ (8\sin 3\theta - \sin\theta)\\ &= \sin\tau\ (8\sin 3\theta - \sin\theta) > 0.\end{aligned}$$

(In fact, one has $\sin 3\theta > \sin\theta$ since $0 < \theta < 3\theta < 90^\circ$.)

Thus $\cos\phi > 8\cos(\phi + 3\theta)$. Now the set $\{\cos(2n+1)\theta : n \in \mathbb{N}\}$ being dense in the interval $(\cos 30^\circ, \cos\theta)$, we have that for some $m \in \mathbb{N}$, we get $(2m+1)\theta = 2k\pi + \phi$, where $k \in \mathbb{N}$ and $\phi \in (\theta, 30^\circ)$. Hence

$$\cos(2m+1)\theta = \cos\phi > 8\cos(\phi + 2\theta) = 8\cos(2m+3)\theta > 0.$$

This is sufficient to infer that $c(m) > c(m+1) > 0$, and hence $a(m) > a(m+1)$. Thus $a(n) > a(n+1)$, for infinitely many n as $(2n+1)\theta$ visits $(\theta, 30^\circ)$ (mod 2π) infinitely often. Also one has $\theta = 27^\circ\ 55'\ 8''$ (approximately) and so we can as well extend the interval $(\theta, 30^\circ)$ to $(\theta, 34^\circ)$ safely, as $34^\circ + 2\theta$ is still less than 90°.

Note that we have proved the result only for positive values of $c(n)$. It may happen that $|c(n)| > |c(n+1)|$ for some negative values of $c(n)$ also. Values of n less than 100 for which this happens are given below. These values can be obtained by giving the above numerical value for θ and relevant values of n.

$$\begin{aligned}
&c(13) = 1305410163123, && c(14) = 286249224103;\\
&c(42) \approx -2.078035580 \cdot 10^{38}, && c(43) \approx -1.327909464 \cdot 10^{38};\\
&c(55) \approx -1.079614797 \cdot 10^{50}, && c(56) \approx 1.974401435 \cdot 10^{49};\\
&c(84) \approx 1.723273540 \cdot 10^{76}, && c(85) \approx 4.481080748 \cdot 10^{75};\\
&c(97) \approx 8.905486681 \cdot 10^{87}, && c(98) \approx -5.427940879 \cdot 10^{87}.
\end{aligned}$$

Although (b) is false, (a) may be still true and it is believed by this solver that (a) is in fact true.

Editor's comments. The approach above might provide a solution to part (a). Starting with the expression for $c(n)$ derived above, express everything as a rational expression in α: writing $x = \alpha^{6n+3}$, manipulate $c(n) = \pm 3$ into monic quadratic expressions and get a value of x as a complex number. A simple recurrence for the real part of powers of α, in reduced form, should then yield the desired result. The reader should work out the details to see if there is an unforeseen pitfall.

3911. *Proposed by Paul Bracken.*

Let $x_0 \in (0, 1 - 1/a]$, where $a > 1$, and define the sequence $x_n = x_{n-1} - x_{n-1}^2$ for $n \in \mathbb{N}$. Prove that x_n satisfies the inequalities

$$\frac{x_0}{anx_0 + 1} < x_n < \frac{x_0}{nx_0 + 1}, \quad n \in \mathbb{N}.$$

We have received eight correct solutions. We present the solution by Arkady Alt slightly modified by the editor.

Note first that since $x_0 \in (0, 1)$ and $x_1 - x_0 = -x_0^2 < 0$, we have $x_1 < x_0$. Furthermore, since $x_0 > x_0^2$, we have that $x_1 = x_0 - x_0^2 > 0$. Hence, $0 < x_1 < 1$. By similar argument and induction, it is easily shown that the sequence (x_n) is strictly decreasing and $x_n \in (0, 1)$ for all $n \in \mathbb{N}$.

Since

$$\frac{1}{x_k} = \frac{1}{x_{k-1} - x_{k-1}^2} = \frac{1}{x_k(1 - x_{k-1})} = \frac{1}{x_{k-1}} + \frac{1}{1 - x_{k-1}},$$

we have

$$\frac{1}{x_k} - \frac{1}{x_{k-1}} = \frac{1}{1 - x_{k-1}}$$

for all $k \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{1}{x_n} - \frac{1}{x_0} &= \sum_{k=1}^{n} \left(\frac{1}{x_k} - \frac{1}{x_{k-1}} \right) = \sum_{k=1}^{n} \frac{1}{1 - x_{k-1}} \\ &\leqslant \sum_{k=1}^{n} \frac{1}{1 - x_0} = \frac{n}{1 - x_0} < \frac{n}{1 - (1 - \frac{1}{a})} = an,\end{aligned}$$

from which we get

$$\frac{1}{x_n} < \frac{1}{x_0} + an = \frac{anx_0 + 1}{x_0},$$

so

$$\frac{x_0}{anx_0 + 1} < x_n. \tag{1}$$

Using (1), we get

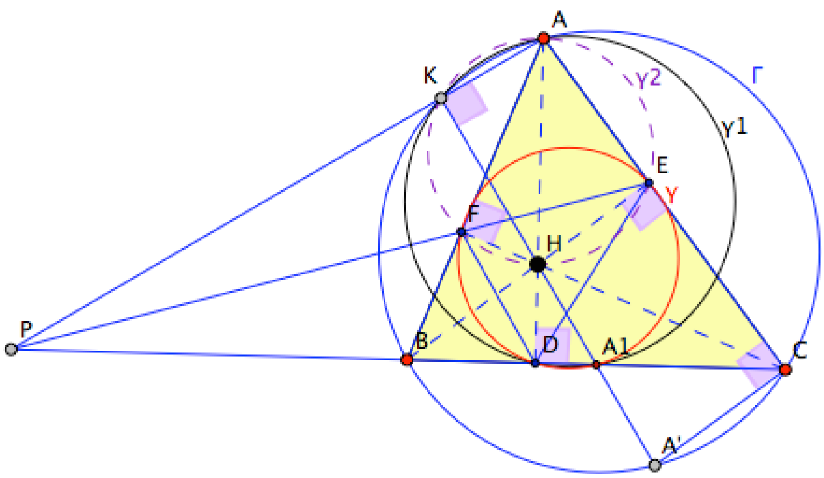
$$\begin{aligned}\frac{1}{x_n} - \frac{1}{x_0} &= \sum_{k=1}^{n} \frac{1}{1 - x_{k-1}} \geqslant \sum_{k=1}^{n} \frac{1}{1 - x_n} > \sum_{k=1}^{n} \frac{1}{1 - \frac{x_0}{anx_0 + 1}} \\ &= \frac{n}{1 - \frac{x_0}{anx_0+1}} = \frac{n(anx_0 + 1)}{anx_0 + 1 - x_0} > n,\end{aligned}$$

so
$$\frac{1}{x_n} > \frac{1}{x_0} + n = \frac{nx_0+1}{x_0}.$$
Hence,
$$x_n < \frac{x_0}{nx_0+1}. \qquad (2)$$
From (1) and (2), the proof is complete.

3912. *Proposed by Michel Bataille.*

Let ABC be a scalene triangle with no right angle and H as its orthocenter. If A_1, B_1 and C_1 are the midpoints of BC, CA and AB respectively, prove that the orthocenters of HAA_1, HBB_1 and HCC_1 are collinear.

We received seven submissions, five of which were correct and two incomplete. We present a composite of the solutions by Šefket Arslanagić and by the proposer.



Define the point A' to be the reflection of H in A_1. Then $HBA'C$ is a parallelogram (because its diagonals bisect one another), which implies that $A'C \perp AC$ (because BH is parallel to $A'C$ and perpendicular to AC). Similarly, $A'B || CH$ so that $A'B \perp AB$; consequently, AA' is a diameter of the circumcircle Γ of ΔABC.

Let K be the orthogonal projection of A onto the line A_1H (note that $K \neq A$ because $\angle BAC \neq 90°$). This point K is on the circle γ_1 with diameter AA_1 and, from the preceding remark (which implies that A', A_1, K, H are collinear), is also on Γ. It follows that AK is the radical axis of the circles Γ and γ_1.

Let γ denote the Euler (or nine-point) circle of ΔABC (which passes through the feet of the altitudes and the midpoints of the sides). Since $\angle BAC \neq 90°$, we have $\gamma \neq \gamma_1$. In addition, both γ_1 and γ pass through A_1 and the orthogonal projection D of A onto BC (which are distinct points because $AB \neq AC$). Thus, the line BC is the radical axis of γ_1 and γ. Now, the orthocentre P of ΔAHH_1, which is the point of intersection of AK and BC, is the radical center of the circles Γ, γ, γ_1. Thus, P is on the radical axis of the circles Γ and γ. The same is true of the orthocentres of triangles HBB_1 and HCC_1. The desired result follows immediately.

But we can deduce yet more: these three orthocentres lie on the *orthic axis* of ΔABC (which, consequently, must coincide with the radical axis of Γ, γ), as we now show. If E and F are the feet of the altitudes from the vertices B and C of ΔABC, then they lie on the circle whose diameter is AH, and that circle, call it γ_2, also contains K (because $AK \perp HK$). Thus the radical axis of γ_2 and γ_1 must be AK, while the radical axis of γ_2 and the Euler circle γ is EF. Putting these lines together with the radical axis BC of circles γ and γ_1, we see that the radical centre of these three circles must be the common point of AK, BC, and EF, which we know to be P (the intersection of BC and AK). By analogous arguments, the orthocentres of triangles HBB_1 and HCC_1 must likewise be the intersections of the sides DE and DF of the orthic triangle DEF with the corresponding sides AB and AC of the initial triangle. Of course, a triangle and its orthic triangle are perspective from the orthocentre, whence they must be perspective from a line, namely the orthic axis. We have just seen that the corresponding sides of the two triangles intersect in the orthocentres of HAA_1, HBB_1, HCC_1, which completes a second proof that these three points are collinear.

Editor's comments. Both incomplete submissions provided neat arguments to show that the three orthocentres satisfy one of the conditions required for the converse of Menelaus's theorem, but (as was pointed out in the editorial comments following problem 3885 [2014 : 399]) a second condition must be satisfied: zero or two of the orthocentres must lie on the sides of ΔABC (while one or three lie on the extensions of those sides).

3913. *Proposed by Ovidiu Furdui.*

Calculate

$$\int_0^\infty \int_0^\infty \frac{dxdy}{(e^x + e^y)^2}.$$

We received ten correct solutions and two incorrect submissions. We present the solution by Madhav Modak.

Denote the repeated integral by I. Then by change of variables we have

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{e^{-2y} e^{-2x}}{(e^{-x} + e^{-y})^2} dxdy \\ &= \int_0^\infty \int_{1+e^{-y}}^{e^{-y}} \frac{-e^{-2y}}{t^2} (t - e^{-y}) dtdy \end{aligned}$$

where $e^{-x}+e^{-y}=t$, and $-e^{-x}dx=dt$. Proceeding with the integral, we have

$$\begin{aligned}
I &= \int_0^\infty \int_{e^{-y}}^{1+e^{-y}} e^{-2y}\Big(\frac{1}{t}-\frac{e^{-y}}{t^2}\Big)dtdy\\
&= \int_0^\infty e^{-2y}\Big[\log t+\frac{e^{-y}}{t}\Big]_{e^{-y}}^{1+e^{-y}} dy\\
&= \int_0^\infty e^{-2y}\Big[\log\frac{1+e^{-y}}{e^{-y}}+e^{-y}\Big(\frac{1}{1+e^{-y}}-\frac{1}{e^{-y}}\Big)\Big]dy\\
&= \int_0^\infty e^{-2y}\Big[\log(1+e^{-y})+y\Big]dy+\int_0^\infty\Big(\frac{e^{-3y}}{1+e^{-y}}-e^{-2y}\Big)dy\\
&= \Big[\frac{e^{-2y}}{-2}\log(1+e^{-y})\Big]_0^\infty-\int_0^\infty\frac{e^{-2y}}{-2}\cdot\frac{-e^{-y}}{1+e^{-y}}\,dy+\int_0^\infty ye^{-2y}dy\\
&\qquad+\int_0^\infty\Big(\frac{e^{-3y}}{1+e^{-y}}-e^{-2y}\Big)dy\\
&= \frac12\log 2-\frac12\int_0^\infty\frac{e^{-3y}}{1+e^{-y}}\,dy+\Big[y\frac{e^{-2y}}{-2}\Big]_0^\infty-\int_0^\infty\frac{e^{-2y}}{-2}dy\\
&\qquad+\int_0^\infty\frac{e^{-3y}}{1+e^{-y}}dy-\int_0^\infty e^{-2y}dy\\
&= \frac12\log 2+\frac12\int_0^\infty\frac{e^{-3y}}{1+e^{-y}}\,dy-\frac12\int_0^\infty e^{-2y}dy.
\end{aligned}$$

Letting $1+e^{-y}=w$, and $-e^{-y}dy=dw$, we have

$$\begin{aligned}
&= \frac12\log 2+\frac12\int_2^1\frac{-(w-1)^2}{w}\,dw-\frac12\Big[\frac{e^{-2y}}{-2}\Big]_0^\infty\\
&= \frac12\log 2+\frac12\int_1^2(w-2+\frac1w)dw-\frac14\\
&= \frac12\log 2+\frac12\Big(\frac32-2+\log 2\Big)-\frac14,
\end{aligned}$$

so that

$$I=\log 2-\frac12.$$

3914. *Proposed by George Apostolopoulos; generalized by the Editorial Board.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s, such that $s=kr$. Prove that

$$\frac{2k}{3\sqrt3}<\frac{R}{r}<\frac{k^2-3}{12}.$$

We received 13 correct solutions. We present a hybrid of several solutions that efficiently applied formulae (implicitly and explicitly) from O. Bottema et al., Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.

As noted in most of the solutions, we assume that the triangle is not equilateral, as both of the inequalities become equations if the given triangle is equilateral.

Using $k = s/r$, the left-hand inequality is equivalent to $2s < 3R\sqrt{3}$, which is inequality 5.3 from Bottema et al.

The right-hand inequality is equivalent to

$$3r(4R + r) < s^2,$$

which is given both in 5.5 and 5.6 of Bottema et al.

3915. *Proposed by Marcel Chiriţă.*

Let M and N be points on the sides AB and AC, respectively, of triangle ABC, and define $O = BN \cap CM$. Show that there are infinitely many examples (that are not affinely equivalent) for which the areas of the four regions MBO, BCO, CNO and $AMON$ are all integers.

We received four correct solutions to this problem, each utilizing a different construction. We feature three of them.

Solution 1, based on the construction by Digby Smith.

For arbitrary $p, q \in \mathbb{N}$ with $p > q$, the numbers $p^2 - q^2$, $2pq$, and $p^2 + q^2$ (and any multiples thereof) form a Pythagorean Triple. The configuration $AMBCN$ from the problem is defined by

$$\begin{aligned} BC &= 4pq(p^2 - q^2)(p^2 + q^2), \\ BM = CN &= 4pq(p^2 - q^2)(2pq), \\ BN = CM &= 4pq(p^2 - q^2)(p^2 - q^2). \end{aligned}$$

Then ABC is isosceles and BMC and CNB are congruent and right angled. By their definitions, the areas of both BMC and CNB are equal to $BM \cdot CM/2$ and thus integers. If we define D to be the midpoint of $\overline{BC}$, then $BDO \sim BNC$ and

$$DO = \frac{NC \cdot BD}{BN} = \frac{8p^2q^2(p^2 - q^2) \cdot 2pq(p^2 - q^2)(p^2 + q^2)}{4pq(p^2 - q^2)^2} = 4p^2q^2(p^2 + q^2).$$

The area of BCO is equal to $BD \cdot DO/2$ and therefore integer. It follows that the areas of BMO and CNO are also integers. Finally $ADC \sim BNC$ and thus

$$AD = \frac{BN \cdot DC}{NC} = \frac{4pq(p^2 - q^2)^2 \cdot 2pq(p^2 - q^2)(p^2 + q^2)}{8p^2q^2(p^2 - q^2)} = (p^2 - q^2)^2(p^2 + q^2).$$

The area of ABC is equal to $BC \cdot AD/2$ and therefore integer. By subtracting the areas of BMO, BCO, and CNO, we obtain finally that the area of $AMON$ is also an integer. As p, q can be arbitrarily chosen, we obtain an infinite number of configurations that are not affinely equivalent (e.g. choose all pairs p, q with $\gcd(p, q) = 1$).

Solution 2, abridged version of the solution by the Missouri State University Problem Solving Group.

Let $A = (0,0)$, $B = (1,0)$, $C = (0,1)$, $M = (a,0)$, and $N = (0,b)$, where a and b are rational and $0 < a < b < 1$. The equations of the lines $\overline{BN}$ and $\overline{CM}$ have rational coefficients, so the coordinates of O are rational. The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\frac{1}{2}\left|\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right|.$$

Therefore, the areas of MBO, BCO, CNO, and $AMON$ are all rational. By stretching the triangle ABC, the corresponding areas can be made to be integers. Since stretching does not alter the ratios AM/MB and AN/NC, the configurations are not affinely equivalent for distinct choices of a and b.

Solution 3, by Titu Zvonaru.

Let a, b, m, n be positive integers and let ABC be a triangle with $BC = 2a$ and $h_A = b(m+1)(n+1)(m+n+1)$. Choose the points M and N on $\overline{AB}$ and $\overline{AC}$ such that

$$\frac{BM}{BA} = \frac{1}{m+1}, \quad \frac{CN}{CA} = \frac{1}{n+1}.$$

Denote by $[XY \ldots Z]$ the area of the polygon $XY \ldots Z$. Then

$$[BMC] = \frac{[ABC]}{m+1}, \quad [CNB] = \frac{[ABC]}{n+1}.$$

Suppose that $\overline{AO}$ intersects $\overline{BC}$ at A'. By Van Aubel's Theorem for Cevian triangles we obtain

$$\frac{AO}{OA'} = \frac{AM}{MB} + \frac{AN}{NC} = m + n$$

and therefore $OA' = AA'/(m+n+1)$. It follows that

$$[BOC] = \frac{[ABC]}{m+n+1}.$$

Thus the areas $[ABC]$, $[BMC]$, $[CNB]$, and $[BOC]$ are all integers and by taking differences of these areas so are $[MBO]$, $[CNO]$, and $[AMON]$.

3916. *Proposed by Nathan Soedjak.*

Let a, b, c be positive real numbers. Prove that

$$\left(\frac{ab}{c}\right)^2 + \left(\frac{bc}{a}\right)^2 + \left(\frac{ca}{b}\right)^2 \geq 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2.$$

There were 23 correct solutions, with two solutions from one solver, as well as a Maple verification. We present a sampling of the different approaches.

Solution 1, by Mohammed Aassila.

Note that $x^2+y^2+z^2 \geqslant xy+yz+zx$ and $(x+y+z)^2 \geqslant 3(xy+yz+zx)$ for real x, y, z. The left side of the inequality is not less than $b^2+c^2+a^2$. However

$$(a^2+b^2+c^2)(a+b+c)^2 \geqslant (ab+bc+ca)[3(ab+bc+ca)] = 3(ab+bc+ca)^2,$$

and the desired result follows.

Solution 2, by Michel Bataille.

By homogeneity, we may suppose that $a+b+c=1$. The inequality is then equivalent to

$$(ab)^4+(bc)^4+(ca)^4 \geqslant 3(a^2b^2c^2)(ab+bc+ca)^2.$$

Observe that

$$\begin{aligned} x^4+y^4+z^4 &= \frac{1}{4}[(x^4+x^4+y^4+z^4)+(x^4+y^4+y^4+z^4)+(x^4+y^4+z^4+z^4)] \\ &\geqslant x^2yz+xy^2z+xyz^2 = xyz(x+y+z), \end{aligned}$$

and $(x+y+z)^2 \geqslant 3(xy+yz+zx)$. Applying these inequalities leads to

$$\begin{aligned} (ab)^4+(bc)^4+(ca)^4 &= [(ab)^4+(bc)^4+(ca)^4][(a+b+c)^2] \\ &\geqslant [a^2b^2c^2(ab+bc+ca)][3(ab+bc+ca)] \\ &= 3(a^2b^2c^2)(ab+bc+ca)^2, \end{aligned}$$

as desired.

Solution 3, by Dionne Bailey, Elsie Campbell, and Charles Dimminnie; Angel Plaza; Cao Minh Quang; and Edmund Swylan, independently.

Since $x^2+y^2+z^2 \geqslant xy+yz+zx$,

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} = \frac{(ab)^2+(bc)^2+(ca)^2}{abc} \geqslant \frac{abc(a+b+c)}{abc} = a+b+c.$$

Using either the convexity of the function x^2 or the inequality of the root-mean-square and arithmetic mean, we find that

$$\begin{aligned} \left(\frac{ab}{c}\right)^2+\left(\frac{bc}{a}\right)^2+\left(\frac{ca}{b}\right)^2 &\geqslant \frac{1}{3}\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^2 \\ &\geqslant \frac{1}{3}(a+b+c)^2 = \frac{1}{3}\frac{(a+b+c)^4}{(a+b+c)^2} \\ &\geqslant \frac{[3(ab+bc+ca)]^2}{3(a+b+c)^2} = 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2. \end{aligned}$$

Solution 4, by Paolo Perfetti.

Since

$$\frac{1}{4}\left(\frac{x^2y^2}{z^2}+\frac{x^2y^2}{z^2}+\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}\right)\geqslant xy$$

by the arithmetic-geometric means inequality, we can follow the strategy of Solution 2 to obtain

$$\left(\frac{ab}{c}\right)^2+\left(\frac{bc}{a}\right)^2+\left(\frac{ca}{b}\right)^2\geqslant ab+bc+ca=\frac{3(ab+bc+ca)^2}{3(ab+bc+ca)}\geqslant\frac{3(ab+bc+ca)^2}{(a+b+c)^2}$$

as desired.

Solution 5 by Henry Ricardo.

We have

$$\begin{aligned}\left(\frac{ab}{c}\right)^2+\left(\frac{bc}{a}\right)^2+\left(\frac{ca}{b}\right)^2&=\frac{1}{2}\left[a^2\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+b^2\left(\frac{a^2}{c^2}+\frac{c^2}{a^2}\right)+c^2\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)\right]\\&\geqslant a^2+b^2+c^2\\&\geqslant\frac{(a+b+c)^2}{3}\\&\geqslant 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2.\end{aligned}$$

3917. *Proposed by Peter Y. Woo.*

Given a circle Z, its center O, and a point A on Z, with only a long unmarked ruler, and no compass, can you draw:

i) points B,C and D on Z so that $ABCD$ is a square?

ii) the square $AOBA'$?

iii) the points B, W'', W and W' on Z such that angles AOB, AOW'', AOW and AOW' are $90°, 60°, 45°$ and $30°$?

There were five correct solutions to this problem. We feature the one by the Missouri State University Problem Solving Group.

We need the following basic construction: Given three collinear points A, B, C such that $AB = BC$ and a point P not on $\overleftrightarrow{AC}$, we want to construct a line through P parallel to $\overleftrightarrow{AC}$. To do this, we choose a point Q on the ray $\overrightarrow{AP}$ such that P is between A and Q. Denote the intersection of $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CP}$ by R and denote the intersection of $\overleftrightarrow{AR}$ and $\overleftrightarrow{QC}$ by S. We claim that $\overleftrightarrow{PS}$ is the line we seek. By Ceva's theorem,

$$\frac{AB}{BC}\cdot\frac{CS}{SQ}\cdot\frac{QP}{PA}=1,$$

but since $AB = BC$ this yields $\dfrac{QP}{PA} = \dfrac{QS}{SC}$ and therefore $\overleftrightarrow{PS}$ and $\overleftrightarrow{AC}$ are parallel.

i) The intersection of $\overleftrightarrow{OA}$ and Z gives C. Choose any point X on Z other than A or C and use the basic construction above to obtain a line through X parallel to line AC. If this line only meets Z in a single point let $B = X$ and D be the intersection of line OB and Z. If the line meets Z in two distinct points, S and T, let U be the intersection of $\overleftrightarrow{AS}$ and $\overleftrightarrow{CT}$. Then B and D are the intersections of $\overleftrightarrow{UO}$ with Z. Note that by symmetry, $\overleftrightarrow{UO}$ is perpendicular to $\overleftrightarrow{AC}$, which makes $ABCD$ a square.

ii) Using the basic construction, we draw a line through A parallel to $\overleftrightarrow{BD}$ and a line through B parallel to $\overleftrightarrow{AC}$. Their intersection is the point A'.

iii) We constructed B in part i). The point where $\overline{OA'}$ meets Z gives W. Let E denote the intersection of $\overline{OA'}$ and $\overline{AB}$. Using the basic construction, if we draw a line ℓ through E parallel to $\overleftrightarrow{AC}$, the point of intersection of ℓ and Z that lies between A and B gives W' (note that ℓ bisects $\overline{OB}$, which gives $\sin(\sphericalangle AOW') = 1/2$). Similarly, a line through E parallel to $\overleftrightarrow{BD}$ gives W''.

3918. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\sqrt{(ab)^{2/3} + (bc)^{2/3} + (ac)^{2/3}} < \frac{2 + \sqrt{3}}{3}.$$

We received 22 correct solutions and one incorrect solution. We present the solution by Cristinel Mortici, slightly modified by the editor.

Recall the Power Mean Inequality: for $x, y, z > 0$ and $m \geqslant n$

$$\left(\frac{x^m + y^m + z^m}{3}\right)^{1/m} \geqslant \left(\frac{x^n + y^n + z^n}{3}\right)^{1/n}.$$

The Power Mean Inequality with $m = 1$ and $n = 2/3$ gives

$$\left(\frac{(ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3}}{3}\right)^{3/2} \leqslant \frac{ab + bc + ca}{3} \leqslant \frac{a^2 + b^2 + c^2}{3} = \frac{1}{3}.$$

It follows that

$$\left((ab)^{2/3} + (bc)^{2/3} + (ca)^{2/3}\right)^{1/2} \leqslant 3^{1/6}.$$

Finally, the Geometric Mean-Arithmetic Mean Inequality gives us

$$3^{1/6} = (1 \cdot 1 \cdot \sqrt{3})^{1/3} < \frac{1 + 1 + \sqrt{3}}{3} = \frac{2 + \sqrt{3}}{3},$$

completing the proof.

3919. *Proposed by Michel Bataille.*

Let I be the incentre of triangle ABC. The line segment AI meets the incircle at M and the perpendicular to AM at M intersects BI at N. If P is a point of the line AI, prove that PC is perpendicular to AI if and only if PN is parallel to BM.

We received seven correct solutions. We present a composite of the similar solutions by Šefket Arslanagić and by Peter Woo.

On the one hand,

$$PN||BM \iff \Delta BIM \sim \Delta NIP \iff \frac{IM}{IB} = \frac{IP}{IN}.$$

On the other hand,

$$PC \perp AI \iff \Delta CPI \text{ has a right angle at } P \iff \cos\angle PIC = \frac{IP}{IC}.$$

Let D be the foot of the perpendicular from I to BC. Then $ID = IM = r$ (the inradius), and in right triangle BDI we have $IB = \frac{r}{\sin\frac{B}{2}}$, whence

$$\frac{IM}{IB} = \frac{r}{\left(\frac{r}{\sin\frac{B}{2}}\right)} = \sin\frac{B}{2}. \tag{1}$$

In right triangle IDC, we have

$$IC = \frac{r}{\sin\frac{C}{2}}. \tag{2}$$

Because $\angle NIM$ is exterior to ΔBIA, $\angle NIM = \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}$; consequently, in right triangle NMI we have

$$\cos\angle NIM = \sin\frac{C}{2} = \frac{IM}{IN} = \frac{r}{IN}$$

and, with the help of equation (2),

$$IN = \frac{r}{\sin\frac{C}{2}} = IC. \tag{3}$$

Because $\angle PIC$ is an exterior angle of ΔAIC, $\angle PIC = \frac{A}{2} + \frac{C}{2} = 90^\circ - \frac{B}{2}$, whence

$$\cos\angle PIC = \sin\frac{B}{2}. \tag{4}$$

Putting the pieces together, we deduce

$$\begin{aligned} PN||BM &\iff \frac{IM}{IB} = \frac{IP}{IN} \\ &\iff \sin\frac{B}{2} = \frac{IM}{IB} = \frac{IP}{IC} \quad \text{(from (1) and (3))} \\ &\iff \frac{IP}{IC} = \cos\angle PIC \quad \text{(from (4))} \\ &\iff PC \perp AI. \end{aligned}$$

3920. *Proposed by Alina Sîntămărian.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!}.$$

There were 15 submitted solutions for this problem, 14 of which were correct. We present three solutions, representative of the two main solution methods utilized together with one variant.

Solution 1, by the AN-anduud Problem Solving Group.

Consider the following two power series,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \quad \text{and} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

Hence, we have

$$\sin 1 = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} = \sum_{n=0}^{\infty} \left(\frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right),$$

and

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}.$$

Using the above considerations, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!} &= \sum_{n=0}^{\infty} \frac{(4n+2)(4n+1)+2(4n+2)+1}{(4n+2)!} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \left(\frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right) \\
&= e + \sin 1.
\end{aligned}$$

Solution 2, by the group of Dionne Bailey, Elsie Campbell, and Charles Diminnie.

To begin, we note that for $n \geqslant 0$,

$$\begin{aligned}
\frac{16n^2+20n+7}{(4n+2)!} &= \frac{(4n+2)\,(4n+1)+2\,(4n+2)+1}{(4n+2)!} \\
&= \frac{1}{(4n)!} + \frac{2}{(4n+1)!} + \frac{1}{(4n+2)!},
\end{aligned}$$

and hence,

$$\sum_{n=0}^{\infty}\frac{16n^2+20n+7}{(4n+2)!}=\sum_{n=0}^{\infty}\frac{1}{(4n)!}+2\sum_{n=0}^{\infty}\frac{1}{(4n+1)!}+\sum_{n=0}^{\infty}\frac{1}{(4n+2)!}$$

(since the Ratio Test easily confirms that each of the three series on the right converges).

The remainder of this solution depends on the following known series:

$$\sin 1=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!},\quad \cos 1=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!},$$
$$\sinh 1=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!},\quad \cosh 1=\sum_{k=0}^{\infty}\frac{1}{(2k)!}.$$

Since we have

$$(-1)^k+1=\begin{cases}2 & \text{if } k \text{ is even}\\ 0 & \text{if } k \text{ is odd}\end{cases}\quad\text{and}\quad(-1)^{k+1}+1=\begin{cases}0 & \text{if } k \text{ is even}\\ 2 & \text{if } k \text{ is odd}\end{cases},$$

we obtain:

$$\sin 1+\sinh 1=\sum_{k=0}^{\infty}\frac{(-1)^k+1}{(2k+1)!}=\sum_{n=0}^{\infty}\frac{2}{[2(2n)+1]!}=2\sum_{n=0}^{\infty}\frac{1}{(4n+1)!},$$
$$\cos 1+\cosh 1=\sum_{k=0}^{\infty}\frac{(-1)^k+1}{(2k)!}=\sum_{n=0}^{\infty}\frac{2}{[2(2n)]!}=2\sum_{n=0}^{\infty}\frac{1}{(4n)!},$$
$$-\cos 1+\cosh 1=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}+1}{(2k)!}=\sum_{n=0}^{\infty}\frac{2}{[2(2n+1)]!}=2\sum_{n=0}^{\infty}\frac{1}{(4n+2)!}.$$

Therefore, we obtain,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{16n^2+20n+7}{(4n+2)!}&=\frac{\cos 1+\cosh 1}{2}+(\sin 1+\sinh 1)+\frac{-\cos 1+\cosh 1}{2}\\&=\sin 1+\sinh 1+\cosh 1\\&=\sin 1+\frac{e-e^{-1}}{2}+\frac{e+e^{-1}}{2}\\&=\sin 1+e.\end{aligned}$$

Solution 3, by Paolo Perfetti.

First, we have:

$$\frac{16n^2+20n+7}{(4n+2)!}=\frac{1}{(4n)!}+\frac{2}{(4n+1)!}+\frac{1}{(4n+2)!}.$$

Let
$$f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!},$$
so that we obtain:
$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!}, \quad f''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!}, \quad f'''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!}$$
$$f^{iv}(x) = \sum_{n=1}^{\infty} \frac{x^{4n-4}}{(4n-4)!} = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} = f(x).$$

Thus $f(x)$ satisfies $f^{iv}(x) = f(x),\ f(0) = 1,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = 0,$ whose unique solution is $f(x) = \frac{1}{2}\cosh x + \frac{1}{2}\cos x$. Evaluating, we get
$$f(1) = \frac{1}{2}\cosh 1 + \frac{1}{2}\cos 1 = \sum_{n=0}^{\infty} \frac{1}{(4n)!}.$$

Moreover, if we define
$$g(x) = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!},$$
we get $g(1) = \sum_{n=0}^{\infty} \frac{1}{(4n+1)!}$ and $g'(x) = f(x),\ g(0) = 0$. This implies
$$g(x) = \frac{1}{2}\sinh x + \frac{1}{2}\sin x, \quad g(1) = \frac{1}{2}\sinh 1 + \frac{1}{2}\sin 1.$$

Finally, defining
$$h(x) = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!},$$
we get $h(1) = \sum_{n=0}^{\infty} \frac{1}{(4n+2)!}$ and $h'(x) = g(x),\ h(0) = 0$. This implies
$$h(x) = \frac{1}{2}\cosh x - \frac{1}{2}\cos x, \quad h(1) = \frac{1}{2}\cosh 1 - \frac{1}{2}\cos 1.$$

Summing up the terms, we obtain
$$f(1) + 2g(1) + h(1) = e + \sin 1.$$

Editor's Comment. The presented solutions illustrate three techniques: rearrange the summations wisely to get a simple expression, rearrange the summations and then recall other atypical power series that make things work, and solve a couple of DEs to avoid having to work too much with power series. Wagon commented that the sum can be explicitly computed when the numerator is an arbitrary quadratic in n.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations