Crux Mathematicorum

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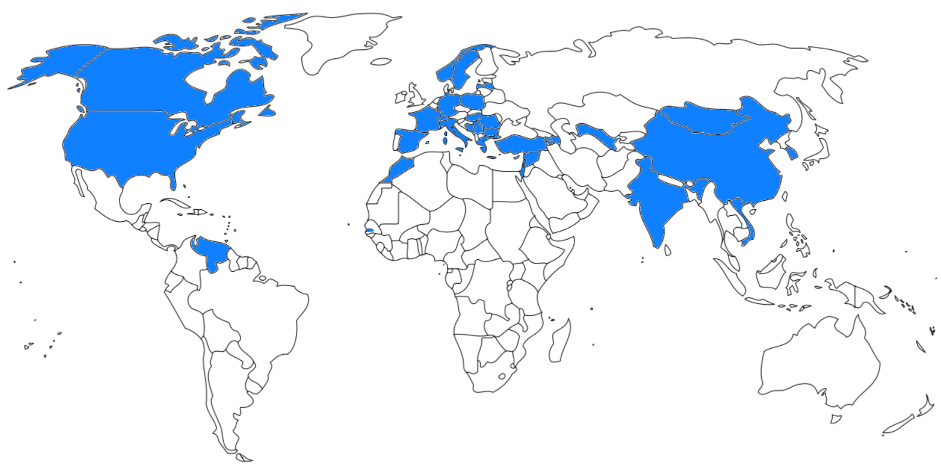
EDITORIAL

Welcome to Volume 41 of ***Crux*** !

I am thrilled to be publishing the first issue of the 2015 volume in the year of 2015. Thanks to my dedicated editors and administrative support from the Canadian Mathematical Society's office, we are getting a real leg up on our backlog.

The year 2015 at the CMS publications office was also marked by the appearance of two more books in A Taste Of Mathematics (ATOM) series including the first book in French. They are *Volume XIV: Sequences and Series* by Margo Kondratieva with Justin Rowsell and *Volume XV: Géométrie plane, avec des nombres* by Michel Bataille. These booklets serve as great enrichment materials for mathematically inclined and interested high school students, so I encourage you to check them out.

With our audience constantly expanding, here is the map of where Volume 40 solvers and proposers come from (represented by shaded regions):



We have managed to touch every continent except Antarctica and Australia. As for my own country of origin, Belarus, I am sending out the first subscription there starting with this Volume.

All in all, for Volume 41, there is still room to grow and we will start right now.

Kseniya Garaschuk

THE CONTEST CORNER

No. 31

Robert Bilinski

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **March 1, 2016**, *although late solutions will also be considered until a solution is published.*

CC134. (*Correction*). Let two tangent lines from the point $M(1,1)$ to the graph of $y = k/x$, $k < 0$ touch the graph at the points A and B. Suppose that the triangle MAB is an equilateral triangle. Find its area and the value of constant k.

CC151. Consider a non-zero integer n such that $n(n + 2013)$ is a perfect square.

a) Show that n cannot be prime.

b) Find a value of n such that $n(n + 2013)$ is a perfect square.

CC152. A square of an $n \times n$ chessboard with $n \geq 5$ is coloured in black and white in such a way that three adjacent squares in either a line, a column or a diagonal are not all the same colour. Show that for any 3×3 square inside the chessboard, two of the squares in the corners are coloured white and the two others are coloured black.

CC153. A sequence $a_0, a_1, \ldots, a_n, \ldots$ of positive integers is constructed as follows:

- if the last digit of a_n is less than or equal to 5, then this digit is deleted and a_{n+1} is the number consisting of the remaining digits; if a_{n+1} contains no digits, the process stops;
- otherwise, $a_{n+1} = 9a_n$.

Can one choose a_0 so that we can obtain an infinite sequence?

CC154. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2012}$ are written on the blackboard. Alice chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is only one number left on the board. What are the possible values of the final number?

CC155. Find all real solutions x to the equation $[x^2 - 2x] + 2[x] = [x]^2$. Here $[a]$ denotes the largest integer less than or equal to a.

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CC134. (*Correction*). Deux droites, issues du point $M(1, 1)$, sont tangentes à la courbe d'équation $y = k/x$ $(k < 0)$ aux points A et B. Sachant que le triangle MAB est équilatéral, déterminer la valeur de k et l'aire du triangle.

CC151. Considérer un entier naturel non-nul n tel que $n(n + 2013)$ soit un carré parfait.

a) Montrer que n ne peut pas être nombre premier.

b) Trouver une valeur de n tel que $n(n + 2013)$ soit un carré parfait.

CC152. Les cases d'un échiquier $n \times n$, avec $n \geq 5$, sont coloriées en noir ou en blanc de telle sorte que trois cases adjacentes sur une ligne, une colonne ou une diagonale ne soient pas de la même couleur. Montrer que pour tout carré 3×3 à l'intérieur de l'échiquier, deux de ses cases situées aux coins sont de couleur blanche et les deux autres sont de couleur noire.

CC153. Une suite d'entiers positifs de terme général a_n et de premier terme a_0 est définie pour tout entier naturel n de la façon suivante:

- si le dernier chiffre de a_n est inférieur ou égal a 5, alors ce chiffre est supprimé et les chiffres restants forment le terme a_{n+1}; si a_{n+1} ne contient pas de chiffre, le procédé s'arrête;
- autrement, $a_{n+1} = 9a_n$.

Peut-on choisir un entier naturel a_0 de sorte que la suite (a_n) soit infinie?

CC154. On a écrit les nombres $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2012}$ sur un tableau. Michèle en choisit deux, notés x et y, puis elle les efface et les remplace par le nombre $x+y+xy$. Elle continue ainsi jusqu'à ce qu'il ne reste plus qu'un seul nombre sur le tableau. Quelles sont les valeurs possibles de ce nombre?

CC155. Déterminer tous les nombres réels x tels que $[x^2 - 2x] + 2[x] = [x]^2$. Ici $[a]$ désigne le plus grand nombre entier inférieur ou égal à a.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(1), p. 4.

CC101. Find all pairs of whole numbers a and b such that their product ab is divisible by 175 and their sum $a+b$ is equal to 175.

Originally 7th secondary Mathematics Olympiad (Poland), 2nd level, question 1.

We received seven correct solutions. Below is the one by S. Muralidharan.

Since $b = 175 - a$ and $175 = 5^2 \cdot 7$, we need

$$\begin{aligned} a(175-a) \equiv 0 \pmod{175} &\iff a^2 \equiv 0 \pmod{175} \\ &\iff a \equiv 0 \pmod 5 \text{ and } a \equiv 0 \pmod 7. \end{aligned}$$

Thus $a \in \{0, 35, 70, 105, 140, 175\}$. Along with $b = 175 - a$, each of these values gives a solution.

CC102. In pentagon $ABCDE$, angles B and D are right. Prove that the perimeter of triangle ACE is at least $2BD$.

Originally 7th secondary Mathematics Olympiad (Poland), 1st level, question 5.

We received two correct submissions with similar solutions, which we present below.

Let M be the midpoint of AC and N the midpoint of EC. Since ABC and CDE are right-angled triangles, we have $MA = MB = MC$ and $NC = ND = NE$. By applying the triangle inequality twice, we obtain the inequality

$$BD \leq BM + MN + ND$$

Notice that $2BM = AC, 2MN = EA$ and $2DN = EC$, so mutiplying the inequality by 2 and substituting gives the desired inequality of

$$2BD \leq AC + CE + EC.$$

CC103. Let a and b be two rational numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{ab}$ is also rational. Prove that $\sqrt{a}$ and $\sqrt{b}$ must also be rationals.

Originally 7th secondary Mathematics Olympiad (Poland), 1st level, question 6.

There were eight solutions to this problem. We present two solutions.

Solution 1, by Šefket Arslanagić, slightly expanded by the editor.

If $a = b = 0$, then the statement holds. Otherwise it is not hard to see that a and b cannot be negative. Let r be the rational number with $r = \sqrt{a} + \sqrt{b} + \sqrt{ab}$ and note that r is positive. Then

$$\begin{aligned} &\sqrt{a} + \sqrt{ab} = r - \sqrt{b} \\ \Rightarrow\, & a + 2a\sqrt{b} + ab = r^2 - 2r\sqrt{b} + b \\ \Rightarrow\, & 2(a + r)\sqrt{b} = r^2 + b - a - ab \end{aligned}$$

Since $a + r > 0$,

$$\sqrt{b} = \frac{r^2 + b - a - ab}{2(a + r)}$$

is a rational number. Similarly it can be shown that $\sqrt{a}$ is rational.

Solution 2, a combination of the solutions by Ángel Plaza and Daniel Văcaru.

For $a = 1$, $b = 1$, or $a = b$, it is easy to see that the statement holds. Suppose otherwise. Since $\sqrt{a}+\sqrt{b}+\sqrt{ab} \in \mathbb{Q}$, its square $a+b+ab+2\sqrt{ab}+2a\sqrt{b}+2b\sqrt{a}$ is rational as well. Therefore $\sqrt{ab} + a\sqrt{b} + b\sqrt{a} \in \mathbb{Q}$. Taking the difference with $\sqrt{a} + \sqrt{b} + \sqrt{ab}$, we obtain $(a - 1)\sqrt{b} + (b - 1)\sqrt{a} \in \mathbb{Q}$. By taking the square, we can conclude that $\sqrt{ab}$ is rational and therefore

$$\sqrt{a} + \sqrt{b} \in \mathbb{Q}. \tag{1}$$

Then

$$\sqrt{a} - \sqrt{b} = \frac{a - b}{\sqrt{a} + \sqrt{b}} \in \mathbb{Q}. \tag{2}$$

Adding (1) and (2) yields $\sqrt{a} \in \mathbb{Q}$, while subtracting (2) from (1) yields $\sqrt{b} \in \mathbb{Q}$.

CC104. Compare the area of an incircle of a square to the area of its circumcircle.

Question originally proposed by the editor.

We received nine correct solutions. We present the solution of Fernando Ballesta Yagüe.

Given a square of side ℓ, the radius of the incircle is r and the radius of the circumcircle is R. As r is the perpendicular from the centre to one side, it is half a side, that is, $\frac{\ell}{2}$. As the radius of the circumcircle is the distance from the centre to one vertex, it is half a diagonal, that is, $\frac{\sqrt{2}\ell}{2}$ (since the diagonal is $\sqrt{2}\ell$). Therefore, the area of the incircle is $\pi \cdot r^2 = \pi \cdot (\frac{\ell}{2})^2 = \frac{\pi\ell^2}{4}$, and the area of the circumcircle is $\pi \cdot R^2 = \pi \cdot (\frac{\sqrt{2}\ell}{2})^2 = \frac{\pi\ell^2}{2}$.

As $\frac{\pi\ell^2}{2} = 2\frac{\pi\ell^2}{4}$, the area of the circumcircle is twice the area of the incircle.

CC105. Knowing that $3.3025 < \log_{10} 2007 < 3.3026$, determine the left-most digit of the decimal expansion of 2007^{1000}.

Originally from 2007 AMQ Cegep contest.

We received five correct solutions. We present the solution of S. Muralidharan.

Given that

$$3.3025 < \log_{10} 2007 < 3.3026,$$

we have

$$3302.5 < 1000 \log_{10} 2007 = \log_{10} 2007^{1000} < 3302.6.$$

Hence

$$10^{3302}10^{0.5} < 2007^{1000} < 10^{3302}10^{0.6}$$

or

$$10^{3302}10^{\frac{3}{6}} < 2007^{1000} < 10^{3302}10^{\frac{3}{5}}.$$

Now, $10^{\frac{3}{6}} = 1000^{\frac{1}{6}}$ and since $3^6 < 1000 < 4^6$, it follows that

$$3 < 10^{\frac{3}{6}} < 4.$$

Also, $10^{\frac{3}{5}} = 1000^{\frac{1}{5}}$ and since $3^5 < 1000 < 4^5$, it follows that

$$3 < 10^{\frac{3}{5}} < 4.$$

Hence

$$3 \times 10^{3302} < 10^{\frac{3}{6}} \times 10^{3302} < 2007^{1000} < 10^{\frac{3}{5}} \times 10^{3302} < 4 \times 10^{3302}.$$

Therefore, the left most decimal digit of 2007^{1000} is 3.

THE OLYMPIAD CORNER

No. 329

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **March 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, for translations of the problems.

OC211. Find the maximum value of

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$$

where a, b, c are real numbers in the interval $[-2, 2]$.

OC212. Let $ABCDE$ be a pentagon inscribed in a circle (O). Let $BE \cap AD = T$. Suppose the line parallel to CD which passes through T cuts AB and CE at X and Y, respectively. If ω be the circumcircle of triangle AXY, prove that ω is tangent to (O).

OC213. Suppose $p > 3$ is a prime number and

$$S = \sum_{2 \le i < j < k \le p-1} ijk.$$

Prove that $S + 1$ is divisible by p.

OC214. Let ABC be an acute-angled triangle with $AC \neq BC$, let O be the circumcentre and F the foot of the altitude through C. Furthermore, let X and Y be the feet of the perpendiculars dropped from A and B respectively to (the extension of) CO. The line FO intersects the circumcircle of FXY a second time at P. Prove that $OP < OF$.

OC215. Let $n > 1$ be an integer. The first n primes are $p_1 = 2, p_2 = 3, \ldots, p_n$. Set $A = p_1^{p_1} p_2^{p_2} \ldots p_n^{p_n}$. Find all positive integers x, such that $\frac{A}{x}$ is even, and $\frac{A}{x}$ has exactly x divisors.

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OC211. Déterminer la valeur maximale de

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$$

où a, b, c sont des nombres réels dans l'intervalle $[-2, 2]$.

OC212. Soit $ABCDE$ un pentagone inscrit dans un cercle (O). Soit $BE \cap AD = T$. Supposons que la ligne parallèle à CD et passant par T intersecte AB et CE aux points X et Y. Si ω est le cercle circonscrit du triangle AXY, démontrer que ω est tangent à (O).

OC213. Supposons $p > 3$ un nombre premier et soit

$$S = \sum_{2 \leq i < j < k \leq p-1} ijk.$$

Démontrer que $S + 1$ est divisible par p.

OC214. Soit ABC un triangle aigu tel que $AC \neq BC$, soit O le centre du cercle circonscrit et soit F le pied de l'altitude passant par C. De plus, soient X et Y les pieds des perpendiculaires de A et B (respectivement) vers CO et son prolongement. La ligne FO intersecte le cercle circonscrit de FXY en un deuxième point P. Démontrer que $OP < OF$.

OC215. Soit $n > 1$ entier et soient les n premiers nombres premiers $p_1 = 2, p_2 = 3, \ldots, p_n$. Posons $A = p_1^{p_1} p_2^{p_2} ... p_n^{p_n}$. Déterminer tous les entiers positifs x tels que $\frac{A}{x}$ est entier et $\frac{A}{x}$ possède exactement x diviseurs.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(1), p. 9–10.

OC151. Let ABC be a triangle. The tangent at A to the circumcircle intersects the line BC at P. Let Q, R be the symmetrical of P with respect to the lines AB respectively AC. Prove that $BC \perp QR$.

Originally question 1 from the Japan Mathematical Olympiad.

We received five solutions. We give the solution of Michel Bataille.

We shall denote by $\angle(m, n)$ the directed angle from line m to line n (measured modulo π).

We have $\angle(PQ, PR) = \angle(AB, AC)$ (since $PQ \perp AB$ and $PR \perp AC$) and because A is the circumcentre of ΔPQR (note that $AQ = AP = AR$), we also have $\angle(PQ, PR) = \angle(\ell, AR) = \angle(AQ, \ell)$ where ℓ is the perpendicular bisector of QR. It follows that

$$\angle(AB, AC) = \angle(AQ, \ell) \tag{1}$$

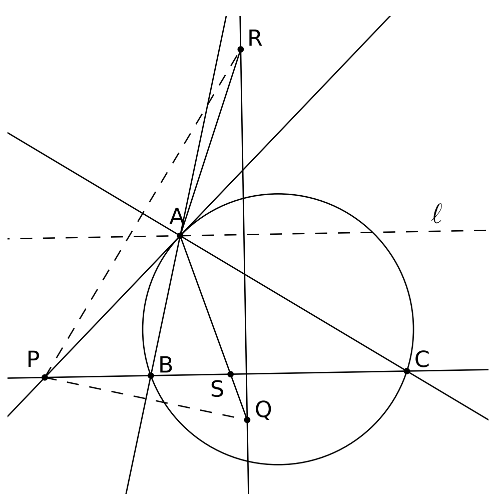
Because AP is tangent at P to the circumcircle of ΔABC, we have $\angle(AP, AB) = \angle(CA, CB)$. Therefore,

$$\angle(AB, AQ) = \angle(CA, CB)$$

and

$$\angle(AQ, BC) = \angle(AQ, AB) + \angle(BA, BC) = \angle(CB, CA) + \angle(BA, BC) = \angle(AB, AC).$$

Thus, AQ is not parallel to BC and if AQ intersects BC at S, we have $\angle(SA, SB) = \angle(AQ, BC) = \angle(AB, AC)$, hence $\angle(SA, SB) = \angle(SA, \ell)$ (by (1)). As a result, $\angle(\ell, SB) = 0$ and ℓ is parallel to BC. Since ℓ is perpendicular to QR, we conclude that BC is perpendicular to QR.



OC152. Find all non-constant polynomials $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients whose roots are exactly the numbers $a_0, a_1, \ldots, a_{n-1}$ with the same multiplicity.

Originally question 3 from day 2 of the France TST 2012.

We received three solutions all of which assumed that the coefficients (except for possibly the first) needed to be distinct. The editor believes that the question, while possibly ambiguous, understands that repetition of coefficients is allowed and that the multiplicity condition means that each root is not repeated unless multiple coefficients are the same. As a result, the editor will give a full solution allowing repetition.

Suppose that we have such a polynomial. Write

$$P(x) = \prod_{i=0}^{n-1}(x - a_i).$$

Then, via Vieta's formulas, we have that $a_0 a_1 ... a_{n-1} = (-1)^n a_0$. If $a_0 = a_1 = a_2 = ... = a_{k-1} = 0$ for $1 \leq k < n$, then divide out by the largest power of k and look at the remaining polynomial. This will introduce a factor of x^k. Thus, without loss of generality, we suppose that $a_0 \neq 0$. From here, clearly $n > 1$ since for $x + a_0$, we have root $-a_0$ and a_0 and $-a_0$ are distinct when $a_0 \neq 0$. So suppose $n \geq 2$. Then comparing constant terms again and cancelling the a_0 term gives $a_1 a_2 ... a_{n-1} = (-1)^n$. Hence each root is either $a_0 \neq 0$ or ± 1 (we will include the factors of x^k at the end). Rewrite $P(x)$ as

$$P(x) = (x-1)^\ell (x+1)^m (x - a_0).$$

for integers ℓ, m. Using a binomial expansion, we see that

$$\begin{aligned} P(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= x^{\ell+m+1} + (m - \ell - a_0)x^{\ell+m} + \cdots \\ &\quad + (a_0(\ell(-1)^{\ell-1} + m(-1)^\ell) + (-1)^\ell)x + a_0(-1)^{\ell+1}. \end{aligned}$$

Comparing coefficients of the constant term yields that ℓ is odd. Hence, we see that 1 is a root of the polynomial and thus

$$0 = P(1) = 1 + \sum_{i=0}^{n-1} a_i.$$

Let S be the summation above. Squaring both sides and expanding yields

$$0 = (1+S)^2 = 1 + 2S + S^2 = 2(S+1) - 1 + S^2 = -1 + S^2.$$

Further, since $a_i^2 = 1$ when $0 < i < n$, we have via Vieta's formulas,

$$S^2 = \left(\sum_{i=0}^{n-1} a_i\right)^2 = \sum_{i=0}^{n-1} a_i^2 + 2\sum_{0 \leq i < j < n} a_i a_j = (n-1) + a_0^2 + 2a_{n-2}.$$

Combining shows that
$$n = 2 - a_0^2 - 2a_{n-2}.$$
Suppose now that $n = 2$, then $a_{n-2} = a_{2-2} = a_0$. Substituting above shows that $0 = a_0(a_0 + 2)$. Hence $a_0 = -2$. By Vieta again in this case, we see that $a_0 + a_1 = -a_1$ and so $a_1 = 1$. This gives the polynomial
$$P(x) = x^2 + x - 2.$$
Now, if $n \geq 3$, then $a_{n-2} \neq a_0$. Thus, $a_{n-2} \in \{\pm 1\}$. Hence recalling that $n = 2 - a_0^2 - 2a_{n-2}$, we see that $a_{n-2} = -1$ and $a_0 = \pm 1$ (recall $a_0 \neq 0$ and $n > 0$). This gives $n = 3$. Thus, $a_1 = -1$ and solving (from Vieta once more) $a_0 + a_1 + a_2 = -a_2$ and $a_0 a_1 a_2 = -a_0$, we see that $a_2 = 1$ and $a_0 = -1$. This gives
$$P(x) = x^3 - x^2 - x + 1.$$
Therefore, combining with the zeroes we excluded, we get the following possible solutions (for suitable n):

1. $P(x) = x^n$
2. $P(x) = x^{n-2}(x^2 + x - 2)$
3. $P(x) = x^{n-3}(x^3 - x^2 - x + 1)$

completing the proof.

OC153. Find all non-decreasing functions from real numbers to itself such that for all real numbers x, y we have
$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)\,.$$
Originally question 3 from day 1 of the Turkish National Olympiad Second Round 2012.

We received two correct submissions. We give the solution of Michel Bataille.

The identity function $x \mapsto x$ is obviously a solution. We show that there are no other solutions. To this end, we consider an arbitrary solution f and denote by $E(x, y)$ the equality $f(f(x^2) + y + f(y)) = x^2 + 2f(y)$.

First, we show that $f(0) = 0$. Let $a = f(0)$. From $E(0, 0)$, we have $f(2a) = 2a$ and $E(0, 2a)$ then yields $f(5a) = 4a$. It follows that $a \leq 0$ since otherwise we would have $f(3a) = 4a$ (from $E(\sqrt{2a}, 0)$) and then $E(\sqrt{3a}, 0)$ leads to $f(5a) = 5a$, in contradiction with $f(5a) = 4a$. Now, from $E(\sqrt{-2a}, 0)$, we obtain $f(a+f(-2a)) = 0$. But, from $E(0, y)$ we deduce that $f(y) = 0$ implies $f(a + y) = 0$ and iterating, $f(2a + y) = 0$ and $f(3a + y) = 0$. Thus, we have $f(4a + f(-2a)) = 0$. However, we also have $f(4a + f(-2a)) = 2a$ (by $E(\sqrt{-2a}, 2a)$) and so $a = 0$.

Let us show that $f(z) = z$ whenever $z > 0$. Since $f(0) = 0$, we have $f(f(x^2)) = x^2$ for all x (by $E(x, 0)$) and so $f(x^2) = x^2$ because $f(x^2) < x^2$ implies $f(f(x^2)) \leq$

$f(x^2)$, that is, $x^2 \leq f(x^2)$, a contradiction. Similarly $f(x^2) > x^2$ leads to a contradiction.

Consider now $y \in (-\infty, 0)$. The relation $E(x,y)$ now writes as $f(x^2+y+f(y)) = x^2 + 2f(y)$ and in particular $E(\sqrt{-y}, y)$ gives $f(f(y)) = -y + 2f(y)$. Note that because f is increasing, $f(y) \leq 0$ and $2f(y)-y = f(f(y)) \leq 0$, hence $3f(y)-y \leq 0$.

We distinguish the cases $y - f(y) \geq 0$ and $y - f(y) < 0$. In the former case, $f(y-f(y)) = y - f(y)$ since $y - f(y) \geq 0$ and $f(y-f(y)) = 0$ by $E(\sqrt{-2f(y)}, y)$, hence $f(y) = y$. On the other hand, if $y - f(y) < 0$, since $3f(y) - y \leq 0$, we can apply $E(\sqrt{y - 3f(y)}, f(y))$ and we obtain $0 = f(y) - y$. In both cases, $f(y) = y$.

We may conclude that $f(x) = x$ for all x, negative or not, and we are done.

OC154. For $n \in \mathbb{Z}^+$ we denote

$$x_n := \binom{2n}{n}.$$

Prove there exist infinitely many finite sets A, B of positive integers, such that $A \cap B = \emptyset$, and

$$\frac{\prod_{i \in A} x_i}{\prod_{j \in B} x_j} = 2012\,.$$

Originally question 3 from day 1 of the China TST 2012.

We received one correct submission by Oliver Geupel, which we present below.

For every positive integer n, we have

$$\frac{x_{n+1}}{x_n} = \frac{2(2n+1)}{n+1}.$$

and

$$\frac{x_{n+2}}{x_n} = \frac{x_{n+2}}{x_{n+1}} \cdot \frac{x_{n+1}}{x_n} = \frac{2(2n+3)}{n+2} \cdot \frac{2(2n+1)}{n+1} = \frac{4(2n+1)(2n+3)}{(n+1)(n+2)}.$$

Hence,

$$\begin{aligned} \frac{x_{n+1}}{x_n} \cdot \frac{x_{2n}}{x_{2n+2}} &= \frac{2(2n+1)}{n+1} \cdot \frac{(2n+1)(2n+2)}{4(4n+1)(4n+3)} = \frac{(2n+1)^2}{(4n+1)(4n+3)} \\ &= \frac{8n^2+8n+2}{2(16n^2+16n+3)} = \frac{x_{8n^2+8n+1}}{x_{8n^2+8n+2}}. \end{aligned}$$

Thus,

$$\frac{x_{n+1}}{x_n} \cdot \frac{x_{2n}}{x_{2n+2}} \cdot \frac{x_{8n^2+8n+2}}{x_{8n^2+8n+1}} = 1.$$

Moreover,

$$x_1 \cdot x_5 \cdot \frac{x_{252}}{x_{251}} = 2 \cdot 252 \cdot \frac{2 \cdot 503}{252} = 2012.$$

Therefore, we may put

$$A = \{1, 5, 252, n+1, 2n, 8n^2 + 8n + 2\}, \quad B = \{251, n, 2n+2, 8n^2 + 8n + 1\}$$

for any $n > 252$.

OC155. There are 42 students taking part in the Team Selection Test. It is known that every student knows exactly 20 other students. Show that we can divide the students into 2 groups or 21 groups such that the number of students in each group is equal and every two students in the same group know each other.

Originally question 3 from Vietnam Team Selection Test 2012.

No submissions were received.

From *Mathematical Cartoons* by Charles Ashbacher.

FOCUS ON...

No. 15

Michel Bataille

A Formula of Euler

Introduction

In this number, we consider the sums $S(n,m)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^m$ where n is a positive integer and m a nonnegative integer. Euler found a very simple closed form of $S(n,m)$ as long as $m\le n$:

$$S(n,m)=0 \text{ if } m=0,1,\dots,n-1 \text{ and } S(n,n)=n!.$$

This result is often called Euler's formula. To appreciate its power, we give a quick solution to problem 11212 posed in the *American Mathematical Monthly* in March 2006:

> Show that for an arbitrary positive integer n, $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=0$.

Just notice that

$$\begin{aligned}\binom{2n-2r}{n-1}&=\frac{1}{(n-1)!}(2n-2r)(2n-1-2r)\cdots(n+2-2r)\\&=a_{n-1}r^{n-1}+a_{n-2}r^{n-2}+\cdots+a_1r+a_0,\end{aligned}$$

where the coefficients $a_0,a_1,\dots,a_{n-1}$ do not depend on r. Thus,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=\sum_{j=0}^{n-1}a_j\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^j=(-1)^n\sum_{j=0}^{n-1}a_jS(n,j)$$

and the result immediately follows since each $S(n,j)$ vanishes. To use the formula in its full extent, we can also calculate $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n}$; with the help of $S(n,n)=n!$, the result 2^n is readily obtained. This illustrates the polynomial version of Euler's formula: if $P(x)$ is a polynomial whose coefficients are free of k, then

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(k)=0$$

if the degree of P is less than n and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(k)=a_nn!$$

if $P(x)=a_nx^n+\cdots$ is of degree n.

We will give three elementary proofs of Euler's formula, favoring approaches that bring out connections with algebra, analysis and combinatorics. Quite a ubiquitous formula! The reader will find other proofs and links with more advanced tools (difference operator Δ, Stirling numbers, *etc.*) in the survey article [1].

First approach: polynomials and linear algebra

We introduce the polynomials

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = x(x-1),\ \ldots,\ P_n(x) = x(x-1)\cdots(x-n+1)$$

and recall that $(P_0, P_1, \ldots, P_n)$ is a basis of the linear space formed by all polynomials of degree less than or equal to n. Now, consider $Q(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k$. By successive differentiations, we readily obtain

$$Q^{(m)}(x) = \sum_{k=m}^{n}(-1)^k\binom{n}{k}P_m(k)x^{k-m}$$

for $0 \le m \le n$. On the other hand, since $Q(x) = (1-x)^n$, we also have

$$Q^{(m)}(x) = (-1)^m P_m(n)(1-x)^{n-m}.$$

Comparing the two results and taking $x = 1$ yields

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}P_m(k) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}P_m(k) = Q^{(m)}(1) = 0$$

if $m < n$ and $(-1)^n P_n(n) = (-1)^n n!$ if $m = n$. Because any polynomial $P(x)$ with degree$(P) = d \le n$ is a linear combination of $P_0, P_1, \ldots, P_d$, Euler's formula is derived at once.

A recourse to the polynomial version of Euler's formula can be found in solutions II and III of **3670** [2012 : 301,302]. Another example is the following identity, extracted from a problem of the St. Petersburg Contest:

> Show that if n is an integer such that $n \ge 2$ and x, y are complex numbers with $x \ne 0$, then
>
> $$\sum_{k=0}^{n-1}\binom{n-1}{k}(x+k)^{k-1}(y+n-k)^{n-k-1} = \frac{(x+y+n)^{n-1}}{x}.$$

To prove this identity, we first transform the left-hand side L into a double sum:

$$\begin{aligned}
L &= \sum_{k=0}^{n-1} \binom{n-1}{k} (x+k)^{k-1} (x+y+n-(x+k))^{n-k-1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (x+k)^{k-1} \sum_{j=0}^{n-k-1} (-1)^{n-k-1-j} \binom{n-k-1}{j} (x+y+n)^j (x+k)^{n-k-1-j} \\
&= \sum_{j=0}^{n-1} (x+y+n)^j \sum_{k=0}^{n-j-1} \binom{n-1}{k} \binom{n-k-1}{j} (-1)^{n-k-1-j} (x+k)^{n-2-j} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} (x+y+n)^j \sum_{k=0}^{n-j-1} \binom{n-1-j}{k} (-1)^{n-k-1-j} (x+k)^{n-2-j} \qquad (1)
\end{aligned}$$

where we have used the equality $\binom{n-1}{k}\binom{n-k-1}{j} = \binom{n-1}{j}\binom{n-1-j}{k}$.

From Euler's formula, the inner sum in (1) is 0 except when $j = n-1$. Thus,

$$L = \binom{n-1}{n-1} (x+y+n)^{n-1} \binom{0}{0} (-1)^0 (x+0)^{-1} = \frac{1}{x}(x+y+n)^{n-1}.$$

Second approach: using a Maclaurin expansion

Consider the function f defined by

$$f(x) = (e^x - 1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{kx}.$$

Clearly, $S(n,m) = f^{(m)}(0)$ where $f^{(m)}$ denotes the mth derivative of f. Thus, the values of $S(n,m)$ $(m = 0, 1, \ldots, n)$ can be obtained from the corresponding coefficients $\frac{f^{(m)}(0)}{m!}$ of the Maclaurin expansion of f. Since $f(x) = x^n(1+\frac{x}{2}+\cdots)^n$, Euler's formula follows.

Note that this proof makes it possible to easily obtain the value of $S(n, n+s)$ if the positive integer s is small; for example, $S(n, n+1) = \frac{n(n+1)!}{2}$ since $(1+\frac{x}{2}+\cdots)^n = 1 + \frac{nx}{2} + \cdots$.

A direct application of this remark is provided by the following solution to problem 824 of the *College Mathematics Journal*, proposed in March 2006:

> Prove that the value of the sum
>
> $$\sum_{j=1}^{n} \sum_{k=1}^{n} (-1)^{j+k} \frac{1}{j!k!} \binom{n-1}{j-1} \binom{n-1}{k-1} (j+k)!$$
>
> is independent of n.

The given double sum is $V_n = \sum\limits_{j=1}^{n} \frac{(-1)^j}{j!}\binom{n-1}{j-1}U_j$ where

$$U_j = \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\frac{(j+k)!}{k!} = \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}(k+2)(k+3)\cdots(k+j+1).$$

Since $(k+2)(k+3)\cdots(k+j+1)$ is a polynomial in k with degree j, we obtain $U_j = 0$ if $j < n-1$, $U_{n-1} = (-1)^n(n-1)!$ and

$$\begin{aligned} U_n &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}(k^n + (2+3+\cdots+(n+1))k^{n-1}) \\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}k^n + \frac{n(n+3)}{2}\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}k^{n-1} \\ &= (-1)^n\frac{(n-1)n!}{2} + (-1)^n\frac{n(n+3)}{2}(n-1)! = (-1)^n(n+1)n!. \end{aligned}$$

As a result, for all positive integers n,

$$V_n = \frac{(-1)^{n-1}}{(n-1)!}\cdot(n-1)U_{n-1} + \frac{(-1)^n}{n!}U_n = -(n-1) + (n+1) = 2.$$

Third approach: combinatorics

If m, n are any positive integers and $[m] = \{1, 2, \ldots, m\}$ and $[n] = \{1, 2, \ldots, n\}$, we denote by $\sigma(m, n)$ the number of surjections from $[m]$ onto $[n]$. If its range is properly restricted, a mapping f from $[m]$ to $[n]$ can be seen as a surjection from $[m]$ onto a nonempty subset of $[n]$. Thus, we obtain all the mappings from $[m]$ to $[n]$ by choosing a subset A of $[n]$ with cardinality $k \neq 0$ and a surjection from $[m]$ onto A in all possible ways. Since the total number of mappings from $[m]$ to $[n]$ is n^m, we are led to the equality

$$n^m = \sum_{k=1}^{n}\binom{n}{k}\sigma(m,k) \tag{2}$$

Now, if (a_n) and (b_n) are two sequences such that $a_n = \sum\limits_{k=0}^{n}\binom{n}{k}b_k$ for all positive integers n, then we have $b_n = \sum\limits_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_k$ for all positive integers n (a well-known inversion formula). Taking $a_0 = b_0 = 0$ and $a_n = n^m$, $b_n = \sigma(m, n)$ for $n \geq 1$ (and fixed m), equality (2) implies that $\sigma(m, n) = \sum\limits_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^m$. Euler's formula is then deduced from the fact that there are no surjections from $[m]$ onto $[n]$ if $n > m$ and that there are $n!$ such surjections (bijections actually) if $n = m$. [We have assumed that $m \geq 1$; but if $m = 0$, then $S(n, 0) = 0$ follows from $0 = (1 - 1)^n$ and the binomial theorem.]

Here is a related problem:

If m, n are positive integers, evaluate $S = \sum_{k=1}^{n} k\binom{n}{k}\sigma(m, k)$.

We propose the following solution. Changing the order of summation, we obtain

$$S = \sum_{k=0}^{n} k\binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^m = \sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}k\binom{n}{k}\binom{k}{j}j^m.$$

Since $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{n-k}$, we see that $S = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}j^m A_j$ where

$$\begin{aligned} A_j = \sum_{k=j}^{n}(-1)^{n-k}k\binom{n-j}{n-k} &= \sum_{\ell=0}^{n-j}(-1)^{\ell}(n-\ell)\binom{n-j}{\ell} \\ &= n \cdot \sum_{\ell=0}^{n-j}(-1)^{\ell}\binom{n-j}{\ell} - \sum_{\ell=0}^{n-j}(-1)^{\ell} \cdot \ell \cdot \binom{n-j}{\ell}. \end{aligned}$$

Now, $\sum_{\ell=0}^{n-j}(-1)^{\ell}\binom{n-j}{\ell} = 0$ except for $j = n$ when the value is 1 and $\sum_{\ell=0}^{n-j}(-1)^{\ell}\ell\binom{n-j}{\ell} = 0$ except for $j = n - 1$ when the value is -1. It follows that

$$S = (n^m \cdot n + (-1) \cdot n \cdot (n-1)^m) = n(n^m - (n-1)^m).$$

We conclude with two exercises.

Exercises

1. Show that for each integer $n \geq 2$

$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{(-1)^{k-1}}{k}\left(1 - \frac{k}{n}\right)^n = \sum_{k=1}^{n-1}\frac{1}{k+1}.$$

2. For nonnegative integer n, evaluate in closed form

$$\sum_{k=0}^{n}\frac{(-1)^k}{(k!)^2} \cdot \frac{(n+k+2)!}{(n-k)!}.$$

Reference

[1] H.W. Gould, Euler's Formula for nth Differences of Powers, *American Mathematical Monthly*, Vol. 85, June-July 1978, pp. 450-467.

Double Counting

Victoria Krakovna

1 Introduction

In many combinatorics problems, it is useful to count a quantity in two ways. Let's start with a simple example.

Example 1 (Iran 2010 #2) There are n points in the plane such that no three of them are collinear. Prove that the number of triangles, whose vertices are chosen from these n points and whose area is 1, is not greater than $\frac{2}{3}(n^2 - n)$.

Solution. Let the number of such triangles be k. For each edge between two points in the set we count the number of triangles it is part of. Let the total number over all edges be T. On the one hand, for any edge AB, there are at most 4 points such that the triangles they form with A and B have the same area. This is because those points have to be the same distance from line AB, and no three of them are collinear. Thus, $T \leq 4\binom{n}{2}$. On the other hand, each triangle has 3 edges, so $T \geq 3k$. Thus,

$$k \leq \frac{T}{3} \leq \frac{4}{3}\binom{n}{2} = \frac{2}{3}(n^2 - n).$$

It's a good idea to consider double counting if the problem involves a pairing like students and committees, or an array of numbers; it's also often useful in graph theory problems.

2 Some tips for setting up the double counting

1. *Look for a natural counting.* (In the above example, the most apparent things we can sum over are points, edges and triangles. In this case edges are better than points because it is easier to bound the number of triangles that an edge is involved in. Thus, we sum over edges and triangles.)
2. *Consider counting ordered pairs or triples of things.* (For instance, in Example 1 we counted pairs of the form (edge, triangle).)
3. *If there is a desired unknown quantity in the problem, try to find two ways to count some other quantity, where one count involves the unknown and the other does not.* (This is done almost trivially in Example 1, where the expression that involves the unknown k is $T = 3k$.)

To illustrate, here are some more examples.

Example 2 (IMO 1987 #1) Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!$$

Solution. The first idea that might occur here would be to find $p_n(k)$, then multiply it by k, sum it up ... probably resulting in a big expression. However, if we look at the required result, we see that it suggests a natural counting — the left hand side is the total number of fixed points over all permutations. Another way to obtain that is to consider that each element of $\{1, 2, 3, \ldots, n\}$ is a fixed point in $(n-1)!$ permutations, so the total is $n(n-1)! = n!$. (Note that we are counting pairs of the form (point, permutation) such that the point is a fixed point of the permutation.)

Example 3 (China Hong Kong MO 2007) In a school there are 2007 girls and 2007 boys. Each student joins no more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club. Show that there is a club with at least 11 boys and 11 girls.

Solution. Since we are only given information about club membership of students of different gender, this suggests that we should consider triples of the form (boy, girl, club), where the boy and girl both attend the club. Let the total number of such triples be T.

For each pair (boy, girl), we know that they attend at least one club together, so since there are 2007^2 such pairs,

$$T \geq 2007^2 \cdot 1.$$

Assume that there is no club with at least 11 boys and 11 girls. Let X be the number of triples involving clubs with at most 10 boys, and Y be the number of triples involving clubs with at most 10 girls. Since any student is in at most 100 clubs, the number of (girl, club) pairs is at most $2007 \cdot 100$, so $X \leq 2007 \cdot 100 \cdot 10$. Similarly, $Y \leq 2007 \cdot 100 \cdot 10$. Then,

$$2007^2 \leq T \leq X + Y \leq 2 \cdot 2007 \cdot 1000 = 2007 \cdot 2000$$

which is a contradiction.

Example 4 (IMO SL 2003) Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ be real numbers. Let $A = \{a_{ij}\}$ (with $1 \leq i, j \leq n$) be an $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} 1 & \textit{if} \quad x_i + y_j \geq 0; \\ 0 & \textit{if} \quad x_i + y_j < 0. \end{cases}$$

Let B be an $n \times n$ matrix with entries 0 or 1 such that the sum of the elements of each row and each column of B equals to the corresponding sum for the matrix A. Show that $A = B$.

Solution. Unlike in the previous problems, here it is not at all obvious what quantity we should count in two ways. We want it to involve the x_i and y_i, as well as the a_{ij} and b_{ij}. It makes sense to consider something that is symmetric in both of the above pairs of variables, and equals to zero if $A = B$. Let

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i + y_j)(a_{ij} - b_{ij}).$$

On the one hand,

$$\begin{aligned} S &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i (a_{ij} - b_{ij}) + \sum_{i=1}^{n} \sum_{j=1}^{n} y_j (a_{ij} - b_{ij}) \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} (a_{ij} - b_{ij}) + \sum_{j=1}^{n} y_j \sum_{i=1}^{n} (a_{ij} - b_{ij}) = 0, \end{aligned}$$

since the column and row sums in A and B are the same.

On the other hand, using the definition of a_{ij} we have the following:

- when $x_i + y_j \geq 0$, $a_{ij} - b_{ij} = 1 - b_{ij} \geq 0$;
- when $x_i + y_j < 0$, $a_{ij} - b_{ij} = -b_{ij} \leq 0$;

so $(x_i + y_j)(a_{ij} - b_{ij}) \geq 0 \ \forall i, j$. Since $S = 0$, it follows that $(x_i + y_j)(a_{ij} - b_{ij}) = 0$ $\forall i, j$. From this it is easy to derive that $a_{ij} = b_{ij} \ \forall i, j$.

3 Cool proofs using double counting

The following is a proof of Cayley's formula, which counts the number of trees on n distinct vertices. There are several other proofs of this fact using bijection, linear algebra, and recursion, but the double counting proof is considered the most beautiful of them all. (The four proofs are given in Aigner and Ziegler's book "Proofs from THE BOOK".)

Theorem 1 ***(Cayley's Formula)*** *The number of different unrooted trees that can be formed from a set of n* distinct *vertices is $T_n = n^{n-2}$.*

Proof. We count the number S_n of sequences of $n - 1$ directed edges that form a tree on the n distinct vertices.

Firstly, such a sequence can be obtained by taking a tree on the n vertices, choosing one of its nodes as the root, and taking some permutation of its edges. Since a particular sequence can only be obtained from one unrooted tree, the number of sequences is

$$S_n = T_n \cdot n(n-1)! = T_n n!.$$

Secondly, we can start with the empty graph on n vertices, and add in $n - 1$ directed edges one by one. After k edges have been added, the graph consists of

$n-k$ rooted trees (an isolated vertex is considered a tree). A new edge can go from any vertex to a root of any of the trees (except the tree this vertex belongs to). This is necessary and sufficient to preserve the tree structure. The number of choices for the new edge is thus $n(n-k-1)$, and thus the number of choices for the whole sequence is

$$S_n = \prod_{k=1}^{n-1} n(n-k-1) = n^{n-1}(n-1)! = n^{n-2}n!$$

The desired conclusion follows.

A more unexpected use of double counting is the following proof of Fermat's Little Theorem.

Theorem 2 ***(Fermat's Little Theorem)*** *If a is an integer and p is a prime, then*

$$a^p \equiv a \pmod{p}.$$

Proof. Consider the set of strings of length p using an alphabet with a different symbols. Note that these strings can be separated into equivalence classes, where two strings are equivalent if they are rotations of each other. Here is an example of such an equivalence class (called a"necklace") for $p = 5$:

$$\{BBCCC, BCCCB, CCCBB, CCBBC, CBBCC\}.$$

Let's call a string with at least two distinct symbols in it *non-trivial*. All the rotations of a non-trivial string are distinct — since p is prime, a string cannot consist of several identical substrings of size greater than 1. Thus, all the equivalence classes have size p, except for those formed from a trivial string, which have size 1.

Then, there are two ways to count the number of non-trivial strings. Since there are a^p strings in total, a of which are trivial, we have $a^p - a$ non-trivial strings. Also, the number of non-trivial strings is p times the number of equivalence classes formed by non-trivial strings. Therefore, p divides $a^p - a$.

4 Problems

1. Prove the following identity:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

2. (Iran 2010 #6) A school has n students, and each student can take any number of classes. Every class has at least two students in it. We know that if two different classes have at least two common students, then the number

of students in these two classes is different. Prove that the number of classes is not greater that $(n-1)^2$.

3. (IMO SL 2004) There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:
   (a) Each pair of students is in exactly one club.
   (b) For each student and each society, the student is in exactly one club of the society.
   (c) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

   Find all possible values of k.

4. (IMO 1998 #2) In a competition there are m contestants and n judges, where $n \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges their ratings coincide for at most k contestants. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}$$

5. (MOP practice test 2007) In an $n \times n$ array, each of the numbers in $\{1, 2, \ldots, n\}$ appears exactly n times. Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct numbers.

6. (USAMO 1995 #5) In a group of n people, some pairs of people are friends and the other pairs are enemies. There are k friendly pairs in total, and it is given that no three people are all friends with each other. Prove that there exists a person whose set of enemies has at most $k(1-\frac{4k}{n^2})$ friendly pairs in it.

7. Consider an undirected graph with n vertices that has no cycles of length 4. Show that the number of edges is at most $\frac{n}{4}(1+\sqrt{4n-3})$.

8. (IMO 1989 #3) Let n and k be positive integers, and let S be a set of n points in the plane such that no three points of S are collinear, and for any point P of S there are at least k points of S equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

5 References

1. Mathematical Excalibur (Volume 13, Number 4), "Double Counting".
`http://www.math.ust.hk/excalibur/v13_n4.pdf`

2. M. Aigner, G. Ziegler, “Proofs from THE BOOK”.
3. Wikipedia page on Double Counting.
   http://en.wikipedia.org/wiki/Double_counting_(proof_technique)
4. Yufei Zhao, “Double Counting using Incidence Matrices”.
   http://web.mit.edu/yufeiz/www/doublecounting_mop.pdf
5. MathLinks forum posts.
   http://www.artofproblemsolving.com/Forum/index.php

6 Hints

1. Count the number of ways to choose a committee with a chairperson out of n people.
2. Count triples of the form (number of students in class, student, student), where the two students both attend a class with that number of students.
3. Count triples of the form (student, club, society) by focusing on clubs (since all the information you are given is about clubs).
4. Count triples of the form (contestant, judge, judge) where the two judges have the same rating for the contestant.
5. Assume each row and each column has less than $\sqrt{n}$ distinct numbers in it. For each row or column, consider the number of distinct numbers in it.
6. Consider the people as an undirected graph, with edges between friends. It is easier here to count the number of pairs (person, edge) where the edge is *not* between two enemies of that person.
7. Count triples of the form (u, v, w) where u, v, w are vertices such that (u, v) and (v, w) are edges.
8. Count triples of the form (u, v, w) where u, v, w are points such that u and w are equidistant from v. (There is another solution that does not involve the “no 3 are collinear” condition ;)).

This article was originally used as a handout to accompany a lecture given by the author at the Canadian Mathematical Society Summer IMO Training Camp in June 2010 at Wilfrid Laurier University in Waterloo.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **March 1, 2016***, although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

4001. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Let $a, b, c, d \in \mathbb{R}$ with $d > 2$ such that

$$(2d+1)\cdot\frac{a}{6}+\frac{b}{2}+\frac{c}{d+1}=0.$$

Prove that there exists $t \in (0, d)$ such that $at^2 + bt + c = 0$.

4002. *Proposed by Henry Aniobi.*

Let f be a convex function on an interval I. Let $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ be numbers such that $x_i + y_j$ is always in I for all $1 \leq i, j \leq n$. Let $z_1, z_2, \ldots, z_n$ be an arbitrary permutation of $y_1, y_2, \ldots, y_n$. Show that

$$\begin{aligned} f(x_1+y_1)+\ldots+f(x_n+y_n) &\geq f(x_1+z_1)+\ldots+f(x_n+z_n) \\ &\geq f(x_1+y_n)+f(x_2+y_{n-1})+\ldots+f(x_n+y_1); \end{aligned}$$

4003. *Proposed by Martin Lukarevski.*

Show that for any triangle ABC, the following inequality holds

$$\begin{aligned} \sin A\sin B\sin C\left(\frac{1}{\sin A+\sin B}+\frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}\right) \\ \leq \frac{3}{4}(\cos A+\cos B+\cos C). \end{aligned}$$

4004. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = 2$. Prove that

$$\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)} \geq 1.$$

4005. *Proposed by Michel Bataille.*

Let a, b, c be the sides of a triangle with area F. Suppose that some positive real numbers x, y, z satisfy the equations

$$x + y + z = 4 \quad \text{and}$$

$$2xb^2c^2 + 2yc^2a^2 + 2za^2b^2 - \left(\frac{4 - yz}{x}\, a^4 + \frac{4 - zx}{y}\, b^4 + \frac{4 - xy}{z}\, c^4 \right) = 16F^2.$$

Show that the triangle is acute and find x, y, z.

4006. *Proposed by Dragolijub Milošević.*

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{2}{xy + yz + zx} - \frac{1}{x + y + z} \leq \frac{1}{3}.$$

4007. *Proposed by Mihaela Berindeanu.*

Show that for any numbers $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 12$, we have

$$(a^3 + 4a + 8)(b^3 + 4b + 8)(c^3 + 4c + 8) \leq 24^3.$$

4008. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with $\angle ACB = 2\alpha$, $\angle ABC = 3\alpha$, AD is an altitude and AE is a median such that $\angle DAE = \alpha$. If $|BC| = a, |CA| = b, |AB| = c$, prove that

$$\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1}.$$

4009. *Proposed by George Apostolopoulos.*

Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{R}{2r^2},$$

where r and R are inradius and circumradius of ABC, respectively.

4010. *Proposed by Ovidiu Furdui.*

Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)^{2n} f(x)\mathrm{d}x.$$

. . . .

4001. *Proposé par Cristinel Mortici et Leonard Giugiuc.*

Soit a, b, c et d des réels, avec $d > 2$, tels que

$$(2d+1)\cdot\frac{a}{6}+\frac{b}{2}+\frac{c}{d+1}=0.$$

Démontrer qu'il existe un nombre t, $t \in (0, d)$, tel que $at^2 + bt + c = 0$.

4002. *Proposé par Henry Aniobi.*

Soit f une fonction convexe sur un intervalle I. Soit $x_1, x_2, \ldots, x_n$ et $y_1, y_2, \ldots, y_n$ des nombres tels que $x_1 \le x_2 \le \ldots \le x_n$, $y_1 \le y_2 \le \ldots \le y_n$ et $x_i + y_j$ est toujours sur I pour tous i et j avec $1 \le i, j \le n$. Soit $z_1, z_2, \ldots, z_n$ une permutation quelconque de $y_1, y_2, \ldots, y_n$. Démontrer que

$$\begin{aligned} f(x_1+y_1)+\cdots+f(x_n+y_n) &\ge f(x_1+z_1)+\cdots+f(x_n+z_n)\\ &\ge f(x_1+y_n)+f(x_2+y_{n-1})+\cdots+f(x_n+y_1) \end{aligned}$$

et que les inégalités sont renversées lorsque f est concave.

4003. *Proposé par Martin Lukarevski.*

Démontrer que pour n'importe quel triangle ABC, on a toujours

$$\begin{aligned} \sin A\sin B\sin C&\left(\frac{1}{\sin A+\sin B}+\frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}\right)\\ &\le\frac{3}{4}(\cos A+\cos B+\cos C). \end{aligned}$$

4004. *Proposé par George Apostolopoulos.*

Soit x, y, z des réels strictement positifs tels que $x + y + z = 2$. Démontrer que

$$\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)}\ge 1.$$

4005. *Proposé par Michel Bataille.*

Soit a, b, c les longueurs des côtés d'un triangle et F l'aire du triangle. Soit x, y, z des réels strictement positifs qui vérifient les équations

$$x+y+z=4 \quad \text{et}$$

$$2xb^2c^2+2yc^2a^2+2za^2b^2-\left(\frac{4-yz}{x}\,a^4+\frac{4-zx}{y}\,b^4+\frac{4-xy}{z}\,c^4\right)=16F^2.$$

Démontrer que le triangle est acutangle et déterminer x, y, z.

4006. *Proposé par Dragolijub Milošević.*

Soit x, y, z des réels positifs tels que $xyz = 1$. Démontrer que

$$\frac{2}{xy+yz+zx} - \frac{1}{x+y+z} \leq \frac{1}{3}.$$

4007. *Proposé par Mihaela Berindeanu.*

Soit a, b, c des réels strictement positifs tels que $a^2 + b^2 + c^2 = 12$. Démontrer que

$$(a^3 + 4a + 8)(b^3 + 4b + 8)(c^3 + 4c + 8) \leq 24^3.$$

4008. *Proposé par Mehmet Şahin.*

Soit ABC un triangle avec $\angle ACB = 2\alpha$ et $\angle ABC = 3\alpha$. La hauteur AD et la médiane AE sont telles que $\angle DAE = \alpha$. Sachant que $|BC| = a, |CA| = b$ et $|AB| = c$, démontrer que

$$\frac{a}{b} = 1 + \sqrt{2\left(\frac{c}{b}\right)^2 - 1}.$$

4009. *Proposé par George Apostolopoulos.*

Soit m_a, m_b, m_c les longueurs des médianes d'un triangle ABC. Démontrer que

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geq \frac{R}{2r^2},$$

r étant le rayon du cercle inscrit dans le triangle et R étant le rayon du cercle circonscrit au triangle.

4010. *Proposé par Ovidiu Furdui.*

Soit f une fonction à valeurs réelles définie et continue sur l'intervalle $[0, \frac{\pi}{2}]$. Calculer

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{2}} \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)^{2n} f(x) \mathrm{d}x.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(1), p. 28–31.

3901. *Proposed by D. M. Bătinetu-Giurgiu and Neculai Stanciu.*

Let $A, B \in M_n(\mathbb{R})$ with $\det A = \det B \neq 0$. If $a, b \in \mathbb{R} \setminus \{0\}$, prove that

$$\det\left(aA + bB^{-1}\right) = \det\left(aB + bA^{-1}\right).$$

We received 15 correct solutions, as well as one incorrect and one incomplete solution. We present two solutions.

Solution 1, by Dhananjay Mehendale; most of the received solutions were variations on this theme.

Since $\det A = \det B \neq 0$, we have

$$\det(A^{-1}) = (\det A)^{-1} = (\det B)^{-1} = \det(B^{-1}).$$

We use the fact that if $X, Y \in M_n(\mathbb{R})$, then $\det(XY) = \det X \cdot \det Y$. Let I be the identity matrix. Then the identity $\det(aAB + bI) = \det(aAB + bI)$ gives

$$\begin{aligned}\det(A^{-1}) \cdot \det(aAB + bI) &= \det(aAB + bI) \cdot \det(B^{-1}) \Leftrightarrow \\ \det\left(A^{-1}(aAB + bI)\right) &= \det\left((aAB + bI)B^{-1}\right) \Leftrightarrow \\ \det(aB + bA^{-1}) &= \det(aA + bB^{-1}),\end{aligned}$$

as was to be proved.

Solution 2, by Michel Bataille, slightly modified by the editor.

If I_n denotes the unit matrix of size n, we have

$$aA + bB^{-1} = aA\left(I_n + \frac{b}{a}A^{-1}B^{-1}\right) \quad \text{and} \quad aB + bA^{-1} = aB\left(I_n + \frac{b}{a}B^{-1}A^{-1}\right).$$

It follows that

$$\det(aA + bB^{-1}) = a^n \det(A) \cdot \det\left(I_n + \frac{1}{a}A^{-1} \cdot bB^{-1}\right)$$

and

$$\det(aB + bA^{-1}) = a^n \det(B) \cdot \det\left(I_n + bB^{-1} \cdot \frac{1}{a}A^{-1}\right).$$

The desired result now follows from $\det A = \det B$ and the general property : if $C, D \in M_n(\mathbb{R})$, then $\det(I_n + CD) = \det(I_n + DC)$. To prove this latter equality, let

$$E = \left(\begin{array}{c|c} I_n & -C \\ \hline D & I_n \end{array}\right) \qquad \text{and} \qquad F = \left(\begin{array}{c|c} I_n & O_n \\ \hline -D & I_n \end{array}\right),$$

where E, F are block-partitioned and O_n is the zero matrix of size n. Then,

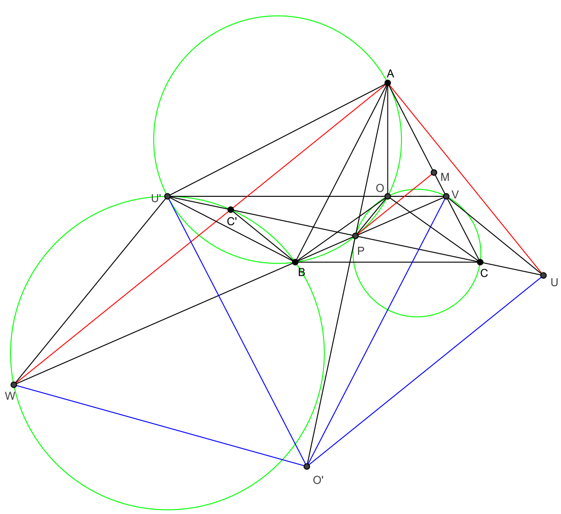
$$EF = \left(\begin{array}{c|c} I_n + CD & -C \\ \hline O_n & I_n \end{array}\right) \qquad \text{and} \qquad FE = \left(\begin{array}{c|c} I_n & -C \\ \hline O_n & I_n + DC \end{array}\right).$$

From properties of determinants, $\det(EF) = \det(FE)$. On the other hand, from the block matrix decomposition above we have $\det(EF) = \det(I_n + CD)$ and $\det(FE) = \det(I_n + DC)$, whence $\det(I_n + CD) = \det(I_n + DC)$ as claimed.

3902. *Proposed by Michel Bataille.*

Let ABC be a triangle with $AB = AC$ and $\angle BAC \neq 90^\circ$ and let O be its circumcentre. Let M be the midpoint of AC and let P on the circumcircle of $\triangle AOB$ be such that $MP = MA$ and $P \neq A$. The lines l and m pass through A and are perpendicular and parallel to PM, respectively. Suppose that the lines l and PC intersect at U and that the line PB intersect AC at V and m at W. Prove that U, V and W are not collinear and that l is tangent to the circumcircle of $\triangle UVW$.

We received four correct submissions. We present a somewhat expanded version of the solution by Glenier L. Bello-Burguet.



Let us first see why U, V, W must be distinct. Note that $P \neq A$ (given) and $P \neq C$ (because $P = C$ would place O on the circumcircle of ΔABC). Thus APC is a proper triangle, and because M is its circumcenter, it follows that $\angle CPA = 90^\circ$. So W (on a line through A that is parallel to MP) cannot coincide with U (on the perpendicular through A to MP) or V (on AC). On the other hand, should $U = V$ then the lines BP and CP would coincide, which would imply that $P \in BC$; furthermore, the lines AC and AU would coincide and we would have $PM \perp AC$. As a consequence, PMA, PMC, and PCA would all be isosceles right triangles, in which case P would be the midpoint of BC, whence $\angle BAC = 90^\circ$. This case has been excluded from the problem.

Our next goal is to prove that $OV || BC$. We shall use directed angles, where $\angle XYZ$ denotes the angle through which the line YX must be rotated in the positive direction about Y to coincide with YZ. (Otherwise some of the angles involved in the proof would have to be replaced by their supplements, depending on whether the angle at A is obtuse or acute.) Using, in turn, that $OA = OC$, AO bisects $\angle BAC$, and the quadrilateral $BPOA$ is cyclic, we obtain

$$\angle VCO = \angle OAV = \angle BAO = \angle BPO = \angle VPO.$$

Hence $OPCV$ is cyclic. By using $MA = MP$, $BPOA$ is cyclic, $OA = OB$, AO bisects $\angle BAC$ and, again, $BPOA$ is cyclic, we also have

$$\begin{aligned}\angle MPO &= \angle MPA - \angle OPA = \angle PAM - \angle OPA = \angle PAM - \angle OBA \\ &= \angle PAM - \angle BAO = \angle PAM - \angle OAM = \angle PAO = \angle PBO.\end{aligned}$$

Thus (because $\angle VPO$ is an exterior angle of ΔBPO)

$$\angle BOP = \angle VPO - \angle PBO = \angle VPO - \angle MPO = \angle VPM, \tag{1}$$

whence (using the circle $OPCV$ and $MP = MC$)

$$\begin{aligned}\angle BOV &= \angle BOP + \angle POV = \angle VPM + \angle POV = \angle VPM + \angle PCV \\ &= \angle VPM + \angle MPC = \angle VPC = \angle BPC.\end{aligned}$$

Hence,

$$\begin{aligned}\angle VOA &= \angle VOB + \angle BOA = \angle VOB + \angle BPA \\ &= \angle CPB + \angle BPA = \angle CPA.\end{aligned}$$

Because $\angle CPA = 90^\circ$, also $\angle VOA = 90^\circ$; that is, $AO \perp OV$ and (because in the isosceles triangle ABC we have $AO \perp BC$), we conclude that $OV \parallel BC$, as desired.

Let U' be the symmetric point of U with respect to P and let C' be the intersection of the lines UU' and $m = AW$. We will show that U' is the second intersection point of VO with the circumcircle of ΔAOB. Since $AC' \parallel PM$ and M is the midpoint of AC, we must have $C'P = PC$ and, therefore, $\angle U'AC' = \angle CAU$. Hence (because $\ell = AU$ is perpendicular to $m = AW$),

$$\angle U'AC = \angle C'AU = 90^\circ.$$

Since $\triangle ABC$ is isosceles and O is its circumcenter, we know that AC is tangent at A to the circumcircle of $\triangle ABO$ (because that angle between the chord AO and the line AC equals $\angle BAO$, which equals the inscribed $\angle OBA$ that is subtended by AO). Therefore, the center of circle passing through the points B, O, P and A lies on the line AU'. Note that $\angle APU' = \angle UPA = 90°$, so AU' must be a diameter of that circle. It follows that $\angle AOU' = 90°$ and, furthermore, since $AO \perp OV$, we conclude that U', O and V are collinear. Thus we see that U' lies, as claimed, on both the line VO and the circle $BPOA$. Moreover, we now have

$$U'V \parallel BC. \tag{2}$$

Let O' be the point where AP intersects the line perpendicular to AU at U. Our ultimate goal is to show that the circle of interest, namely UVW, has center O' and radius $O'U$. By symmetry with respect to P, we have $\angle O'U'A = \angle AUO' = 90°$, so (recalling that $U'AC = 90°$)

$$O'U' \parallel CA. \tag{3}$$

Using (2) and (3) we get

$$\frac{PB}{PV} = \frac{PC}{PU'} = \frac{PA}{PO'},$$

and therefore

$$VO' \parallel AB. \tag{4}$$

Now by (2), (3) and (4) we obtain that $\triangle U'O'V \sim \triangle CAB$. Since $AB = AC$ we get that $O'V = O'U'$. Furthermore, by symmetry about $O'P$ we have $O'U' = O'U$; in other words, O' is the center of the circle UVU'. It remains to show that W lies on this circle.

Using that $AW \parallel PM$ and (1) we have

$$\angle BWC' = \angle VPM = \angle BOP = \angle BU'P = \angle BU'C',$$

and, therefore, $WBC'U'$ is cyclic. On the other hand, using (2) together with the midpoint property of P, we get

$$\frac{PC'}{PU} = \frac{PC}{PU'} = \frac{PB}{PV},$$

so $UV \parallel BC'$. Hence

$$\angle U'UV = \angle PC'B = \angle U'C'B = \angle U'WB = \angle U'WV.$$

This implies that $UVU'W$ is cyclic, as claimed. Recalling that we defined $AU \perp O'U$, we conclude, finally, that AU is tangent to the circumcircle of ΔUVW. As a further consequence, we have proved that U, V and W are not collinear (since they are distinct points on a circle), and our proof is complete.

3903. *Proposed by George Apostolopoulos.*

Consider a triangle ABC with an inscribed circle with centre I and radius r. Let C_A, C_B and C_C be circles internal to ABC, tangent to its sides and tangent to the inscribed circle with the corresponding radii r_A, r_B and r_C. Show that $r_A + r_B + r_C \geq r$.

We received 18 correct submissions. We present the solution by Michel Bataille, which is similar to many other solutions received.

First, we remark that C_A is the image of C under the homothety with centre A and factor $\frac{r_A}{r}$, hence the centre I_A of C_A satisfies $\overrightarrow{AI_A} = \frac{r_A}{r}\overrightarrow{AI}$. Since we also have $II_A = r + r_A$, it follows that $\left(1 - \frac{r_A}{r}\right) AI = r + r_A$ and so

$$r_A = r \cdot \frac{AI - r}{AI + r} = r \cdot \frac{1 - \frac{r}{AI}}{1 + \frac{r}{AI}} = r \cdot \frac{1 - \sin\frac{A}{2}}{1 + \sin\frac{A}{2}}.$$

Similar results hold for r_B and r_C.
Now, let $f(x) = \frac{1-\sin x}{1+\sin x}$ $(x \in (0, \frac{\pi}{2}))$. An easy calculation gives $f''(x) = 2(1 + \sin x)^{-3}(\sin x + \cos^2 x + 1)$. Thus, $f''(x) > 0$ for all $x \in (0, \frac{\pi}{2})$ and f is convex on $(0, \frac{\pi}{2})$. From Jensen's inequality, we obtain

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \geq 3f\left(\frac{A+B+C}{6}\right) = 3 \cdot \frac{1 - \sin\frac{\pi}{6}}{1 + \sin\frac{\pi}{6}} = 1$$

and we can conclude

$$r_A + r_B + r_C = r\left(f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right)\right) \geq r.$$

Editor's comments. Bataille points out that this problem appeared in the Third Round of the Iranian Mathematical Olympiad 2002 and was previously solved in ***Crux*** [2006 : 373-374 ; 2007 : 350].

3904. *Proposed by Abdilkadir Altintaş.*

Let ABC be an equilateral triangle and let D, E and F be the points on the sides AB, BC and AC, respectively, such that $AD = 2$, $AF = 1$ and $FC = 3$. If the triangle DEF has minimum possible perimeter, what is the length of AE?

There were 20 correct solutions, with two from one solver. Thirteen exploited the reflection principle. Some used similar triangles to identify the position of E for minimum perimeter while others used vectors or analytic geometry. Five used the law of cosines to determine the side lengths of the triangle and minimized a function by differentiating. The featured solution follows the approach of the majority.

Since DF is fixed, the perimeter of the triangle is minimized when $DE + EF$ is minimized. Let G be the reflected image of F in the axis BC. Since $DE + EF = DE + EG$, by the reflection principle, the perimeter is minimized when D, E, G are collinear. In this situation, let x be the length of BE. Since $\angle DBE = \angle FCE = \angle ECG = 60°$ and $\angle DEB = \angle GEC$, triangles BDE and CGE are similar. Therefore

$$\frac{x}{2} = \frac{BE}{BD} = \frac{CE}{CG} = \frac{4-x}{3},$$

whence $x = 8/5$. By the Law of Cosines applied to triangle ABE,

$$AE^2 = 16 + x^2 - 8x\cos 60^\circ = \frac{304}{25},$$

from which $AE = 4\sqrt{19}/5$.

The minimum perimeter turns out to be $\sqrt{3} + \sqrt{21}$.

3905. *Proposed by Jonathan Love.*

A sequence $\{a_n : n \geq 2\}$ is called *prime-picking* if, for each n, a_n is a prime divisor of n. A sequence $\{a_n : n \geq 2\}$ is called *spread-out* if, for each positive integer k, there is an index N such that, for $n \geq N$, the k consecutive entries $a_n, a_{n+1}, \cdots, a_{n+k-1}$ are all distinct. For example, the sequence

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \cdots\}$$

is spread-out. Does there exist a prime-picking spread-out sequence ?

There were 2 submitted solutions for this problem, both incorrect. We present the proposer's solution, with minor clarifications.

The answer is yes. For each positive integer k and index n with $k! < n < (k+1)!$, define

$$b_n = \frac{n}{\gcd(k!, n)}.$$

Now, for any $n > k!$, suppose that p is a prime divisor of b_n. There is a non-negative exponent a for which $p^a < k \leq p^{a+1}$, so that p^{a+1} must divide n.

Then, suppose that $n > k!$, $i > 0$, and that p divides both b_n and b_{n+i}. We see that p^{a+1}, dividing both n and $n+i$, must divide i, so that $i \geq p^{a+1} > k$. It follows that the numbers $b_n, b_{n+1}, \ldots, b_{n+k-1}$ are pairwise coprime. Therefore, if we let a_n be any prime divisor of b_n, we obtain a prime-picking spread-out sequence, by the above arguments.

Editor's comments. The solutions to this problem illustrate a classic conundrum. One can pick the values a_n simply, and get one property essentially for free, and then work much harder to get the other property. Alternatively, one might work harder or more cleverly to choose the values a_n, and then work much less to obtain both properties.

The proposer's solution made a more complicated choice of a_n, but the work to obtain the requisite properties was minimal. The two submitted solutions made simpler choices of a_n, but more work was required to prove that the sequences were spread-out (since they were chosen to be trivially prime-picking), and both incorrect solutions contained errors in this work. That said, both choices of a_n were correct : letting a_n be either the largest prime divisor of n, or the prime divisor of n for which the corresponding prime power factor of n is largest, yields a prime-picking spread-out sequence.

3906⋆. *Proposed by Titu Zvonaru and Neculai Stanciu.*

If $x_1, x_2, \ldots, x_n$ are positive real numbers, then prove or disprove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

for all positive integers n.

We received one correct solution and one incorrect submission. We present the solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, modified slightly by the editor.

The proposed inequality is false in general.

Let $f(n)$ and $g(n)$ denote the expression on the left-hand side and the right-hand side of the given inequality, respectively. Then for $n = 15$ and $x_k = 2^{-k}$, $k = 1, 2, \ldots, n$, computations with the aid of a computer yield

$$f(15) = \left(\sum_{k=1}^{14} \frac{1}{2^{k-1}}\right) + \frac{1}{2^{29}} \approx 1.999938967,$$

$$g(15) = \sqrt{15 \sum_{k=1}^{15} \frac{1}{2^{2k}}} \approx 2.236067976,$$

showing that $f(15) < g(15)$.

Editor's comments. In general for the choice of x_k given above, note that

$$f(n) = \left(\sum_{k=1}^{n-1} \frac{1}{2^{k-1}}\right) + \frac{1}{2^{2n-1}} = \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} + \frac{1}{2^{2n-1}} = 2 - \frac{1}{2^{n-2}} + \frac{1}{2^{2n-1}} < 2$$

since $n - 2 < 2n - 1$. On the other hand, since

$$g(n) = \sqrt{n \sum_{k=1}^{n} \frac{1}{2^{2k}}} = \sqrt{\frac{n}{4} \sum_{k=1}^{n} \frac{1}{2^{2k-2}}} = \sqrt{\frac{n}{4} \cdot \frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)} = \sqrt{\frac{n}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)},$$

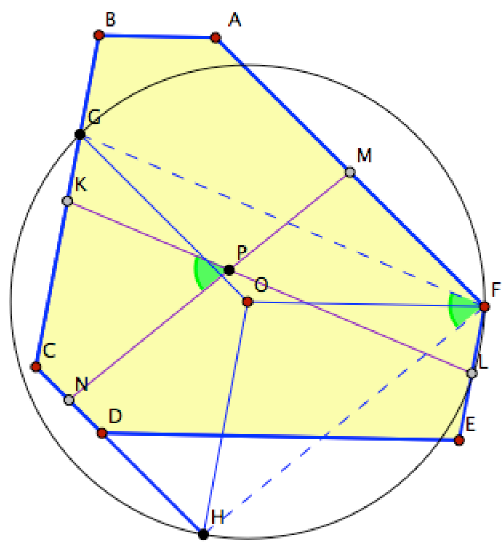
it is clear that $g(n)$ is an increasing function of n. Since we have already shown that $g(15) > 2$, it follows that $f(n) < g(n)$ for all $n \geq 15$. In fact, checking by computer reveals that the smallest n for which $f(n) < g(n)$ is $n = 12$.

3907. *Proposed by Enes Kocabey.*

Let $ABCDEF$ be a convex hexagon such that $AB + DE = BC + EF = FA + CD$ and $AB \parallel DE, BC \parallel EF, CD \parallel AF$. Let the midpoints of the sides AF, CD, BC and EF be M, N, K and L, respectively, and let $MN \cap KL = \{P\}$. Show that $\angle BCD = 2\angle KPN$.

We received four submissions, three of which were correct and one incomplete. We present the solution by Oliver Geupel.

Remark. The problem is well known. It appeared on a test for the selection of the Taiwanese team for the IMO 2014. The problem can be found with solution at `http://www.artofproblemsolving.com/community/c6h598542p3551871.`
The following solution is essentially the same as the internet solution.



Let $2c$ be the common sum of the lengths of opposite sides. We define G and H to be the points where the halflines CB and CD meet the circle with centre C and radius c. In terms of the position vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$ corresponding to the points $A, B, C, \ldots,$ we have

$$\vec{G} = \frac{1}{2}\left(\vec{B} + \vec{C} - \vec{E} + \vec{F}\right), \quad \text{and} \quad \vec{H} = \frac{1}{2}\left(-\vec{A} + \vec{C} + \vec{D} + \vec{F}\right).$$

Let O be the fourth vertex of the rhombus $CHOG$. That is,

$$\vec{O} = \frac{1}{2}\left(-\vec{A} + \vec{B} + \vec{D} - \vec{E} + 2\vec{F}\right).$$

These three assertions are easily verified :

$$2\overrightarrow{GO} = 2\left(\vec{O} - \vec{G}\right) = \vec{F} - \vec{A} + \vec{D} - \vec{C} = \overrightarrow{AF} + \overrightarrow{CD} = 2\left(\vec{H} - \vec{C}\right) = 2\overrightarrow{CH},$$

and, similarly,

$$2\overrightarrow{HO} = 2\overrightarrow{CG} = \overrightarrow{CB} + \overrightarrow{EF}.$$

Moreover, we also have

$$2\overrightarrow{FO} = 2\left(\vec{O} - \vec{F}\right) = \vec{B} - \vec{A} + \vec{D} - \vec{E} = \overrightarrow{AB} + \overrightarrow{ED}.$$

As a consequence, F lies along with G and H on the circle with centre O and radius c.

Furthermore,

$$\overrightarrow{FH} = \frac{1}{2}\left(-\overrightarrow{A} + \overrightarrow{C} + \overrightarrow{D} - \overrightarrow{F}\right) = \overrightarrow{MN}, \quad \text{and} \quad \overrightarrow{FG} = \overrightarrow{LK};$$

thus (because P lies on both MN and KL)

$$FH \parallel PN, \qquad \text{and} \qquad FG \parallel PK.$$

Putting everything together, we conclude

$$\angle BCD = \angle GCH = \angle GOH = 2\angle GFH = 2\angle KPN.$$

This completes the proof.

Editor's comments. A simple way to draw the required hexagon comes as a byproduct of the solution submitted by the proposer. First draw an equilateral hexagon $A'B'CD'E'F$ with opposite sides parallel; after having drawn the first three equal sides, the entire hexagon is completely determined because of its central symmetry. Now one can adjust the figure by choosing A and B on the lines $A'F$ and $B'C$, respectively, so that $AB||A'B'$. Then by placing D on CD' and E on FE' so that DE is parallel to $D'E'$ and its distance to $D'E'$ equals the distance between AB and $A'B'$ (which are also parallel to it). It is easy to verify that the resulting figure has opposite sides parallel and the sum of their lengths constant.

3908. *Proposed by George Apostolopoulos.*

Prove that

$$\frac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$$

for each integer $n \geq 3$.

We received 21 correct submissions, with two from one solver. Below we present four different solutions.

Solution 1, by Peter Y. Woo.

Recall Bernoulli's inequality, $(1+x)^t > 1 + tx$ when $x > -1$, $x \neq 0$ and $t \geq 1$. For $m > 1$, we have

$$\begin{aligned}\left(\frac{m+1}{m}\right)^{m+1}\left(\frac{m-1}{m}\right)^{m-1} &= \left(1+\frac{1}{m}\right)^2\left(1-\frac{1}{m^2}\right)^{m-1}\\ &> \left(1+\frac{1}{m}\right)^2\left(1-\frac{m-1}{m^2}\right)\\ &= \left(\frac{m^3+1}{m^3}\right)\left(\frac{m+1}{m}\right) > 1.\end{aligned}$$

Hence $(m+1)^{m+1} > m^{2m}(m-1)^{-(m-1)}$. Setting $m = n-1$ yields the desired result.

Solution 2, by M. Benito, Ó. Ciaurri, E. Fernández and L. Roncal.

We strengthen the inequality to

$$\frac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n \left(\frac{n^2-1}{n^2}\right)^{n-1}.$$

From Bernoulli's inequality $(1+x)^r < 1+rx$ for $x > -1$, $x \neq 0$ and $0 < r < 1$, we obtain that

$$\left(\frac{n-2}{n}\right)^{\frac{1}{n-1}} = \left(1-\frac{2}{n}\right)^{\frac{1}{n-1}} < 1 - \frac{2}{n(n-1)} = \frac{(n+1)(n-2)}{n(n-1)}.$$

The result follows.

Solution 3, by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly) ; Phil McCartney ; and Digby Smith, independently.

The function $f(x) = x \ln x$ (whose second derivative $1/x$ is positive for $x > 0$) is strictly convex on $(0, \infty)$. Hence, for $x > 2$,

$$2(x-1)\ln(x-1) = 2f(x-1) < f(x-2) + f(x) = (x-2)\ln(x-2) + x \ln x,$$

from which

$$(x-1)^{2x-2} < (x-2)^{x-2} x^x$$

as desired.

Solution 4, by Haohao Wang and Jerzy Wojdylo (jointly) ; and Angel Plaza, independently.

The function $(1+1/x)^x$ is increasing for $x > 0$. (The derivative of its logarithm is equal to $\int_x^{x+1} (t^{-1} - (x+1)^{-1}) dt$.) For $n > 2$, we have that

$$\frac{\left(1+\frac{1}{n-2}\right)^{n-2}}{\left(1+\frac{1}{n-1}\right)^{n-1}} < 1 < \frac{n}{n-1},$$

which yields the desired result.

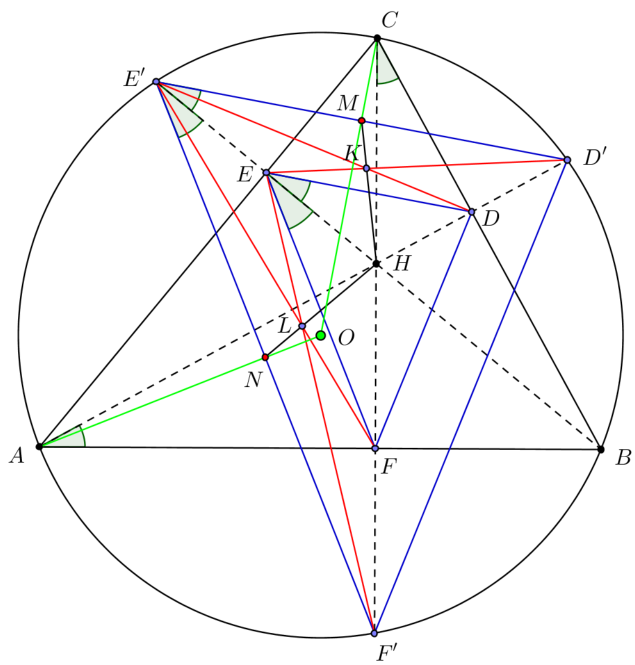
3909. *Modified proposal of Victor Oxman, Moshe Stupel and Avi Sigler.*

Given an acute-angled triangle together with its circumcircle and orthocentre, construct, with straightedge alone, its circumcentre.

Editor's Comment. The Poncelet-Steiner Theorem (1833) states that whatever can be constructed by straightedge and compass together can be constructed by straightedge alone, given a circle and its centre ; but Steiner showed that given only the circle and a straightedge, the centre cannot be found. (This shows that the orthocentre must be given in the present problem ; it cannot be constructed with the straightedge and circumcircle!) Details can be found in texts such as A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, (Blaisdell 1961), or on the internet by googling the Poncelet-Steiner Theorem.

We received five correct submissions from which we present two.

Solution 1 by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal.



Let H be the orthocentre of the given triangle ABC. Successively, we draw with the straightedge (as in the figure) the altitude AH, meeting the side BC at point D and the circumcircle, call it Γ, at point D'; the altitude BH, meeting the side AC at E and Γ at E'; the altitude CH, meeting the side AB at F and Γ at F'.

Draw the triangles DEF (the orthic triangle of ΔABC) and $D'E'F'$. These triangles are homothetic with respect to their common incentre H. Specifically, quadrilaterals $DEE'D'$ and $EFF'E'$ are trapezoids whose nonparallel sides meet at H. Draw the diagonals of each of these trapezoids : the lines DE' and $D'E$ meeting at the point K, and the lines EF' and $E'F$ meeting at the point L. Draw the line HK that meets $D'E'$ at M, and line HL that meets $E'F'$ at N. We know M is the midpoint of the segment $D'E'$, and N is the midpoint of the segment $E'F'$.

On the other hand, it is easily seen (the line $D'H$ bisects $\angle F'D'E'$, etc.) that the vertex C is the midpoint of one arc $D'E'$ of circumcircle $D'E'F'$ and, analogously, that the vertex A is the midpoint of arc $E'F'$. Then, the lines CM and AN are, respectively, the perpendicular bisectors of the segments $D'E'$ and $E'F'$. Consequently, the intersection point O of CM and AN is the circumcentre of $\Delta D'E'F'$ and the required circumcentre of ΔABC.

Solution 2 by Michel Bataille, Rouen, France.

The following construction is valid for all triangles ABC that are not right-angled. Without loss of generality, we suppose that the largest angle of the triangle is $\angle BAC$, and we denote the orthocentre by H and the circumcircle by Γ. Let the line AH intersect BC at D and Γ again at D'. Then D is between B and C and is the midpoint of HD'. Given a line segment with its midpoint, we can construct the parallel ℓ_C to HD' through C and the parallel ℓ_B to HD' through B. Let ℓ_C and ℓ_B intersect again Γ at C' and B', respectively. Note that $C' \neq C$ and $B' \neq B$ since BC is not a diameter of Γ. Since $\angle B'BC = \angle C'CB = 90^\circ$, BC' and $B'C$ are diameters of Γ. Their point of intersection is the desired centre O of Γ.

Editor's comments. Most of the submitted solutions were based on the theorem that says,

> Given a line segment XY and a line parallel to it, we can locate (with straightedge alone) the midpoint of XY; conversely, given a line segment XY with its midpoint M and a point P not on the line XY, we can draw (with straightedge alone) the line parallel to XY through P.

The proof rests upon the fact (as seen in the first solution above) that the line through the point of intersection of the diagonals of a trapezoid (trapezium in British English) and the point of intersection of its nonparallel sides bisects its parallel sides. Details can be found on pages 51-52 of A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, Pergamon Press, 1961.

3910. *Proposed by Paul Yiu.*

Two triangles ABC and $A'B'C'$ are homothetic. Show that if B' and C' are on the perpendicular bisectors of CA and AB respectively, then A' is on the perpendicular bisector of BC, and the homothetic center is a point on the Euler line of ABC.

We received four correct submissions. We present the solution of M. Bello, Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncall.

Let O and H be the circumcentre and orthocentre, respectively, of ΔABC. The lines $C'O$ and $B'O$ are altitudes of $A'B'C'$ (since, for example, $C'O$ is perpendicular to AB, which is parallel to its corresponding homothetic line $A'B'$). Hence, the point O is the orthocentre of $A'B'C'$. Thus, the line $A'O$ is the third altitude of $A'B'C'$ and, consequently, is the unique line through O that is perpendicular to $B'C'$ and (as $B'C'$ is parallel to BC), to BC. Consequently, $A'O$ is the perpendicular bisector of BC, as desired.

For the second claim, recall that the circumcentre O of ΔABC is the orthocenter of $\Delta A'B'C'$, and must therefore be the image of H under the dilatation that takes the first triangle to the second. In other words, the homothetic center must lie on the line OH. But OH is the Euler line of ΔABC (if it exists), and we are done.

AUTHORS' INDEX

Solvers and proposers appearing in this issue
(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations