Crux Mathematicorum

VOLUME 40, NO. 5 MAY / MAI 2014

Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

Dear ***Crux*** readers,

With this issue, we will be introducing some formatting changes into ***Crux***. First and most noticeable, are the changes in author attributions. For a cleaner presentation, only the name of the featured solver(s) will now appear in the solutions, whereas all the solvers will be acknowledged in the Authors Index at the end of the issue. The instructions on how to submit solutions to each section's problems will no longer appear at the beginning of each section but rather will be included on the back cover of each printed issue and appear in a separate document online. In this issue, we will also be introducing the first of the "From the archives" materials, which I hope you will enjoy reading.

Now, just for fun, two puzzles. All the letters of the English alphabet are placed either above or below the line. Try to figure out the pattern and determine whether S, the next letter, goes above or below the line:

A							H	I				M		O				?
	B	C	D	E	F	G			J	K	L		N		P	Q	R	?

How about in this case?

A				E	F		H	I		K	L	M	N					?
	B	C	D			G			J					O	P	Q	R	?

I use these puzzles to remind my students (and myself!) that sometimes the problem solving tools lie in the most surprising places – sometimes it is a long-forgotten technique and sometimes it is barely a technique at all. Sometimes we even have to put our mathematical intuition aside and look at the problem with a completely open mind. Flip through the pages of this issue to hopefully find some challenging and unexpected results.

Kseniya Garaschuk

P.S. Need a hint for the letter puzzles? The first puzzle has a non-trivial connection to Lewis Carroll's *Through the Looking-Glass, and What Alice Found There*, so think of various symmetries. As for the second one, people who don't speak English or children who are learning to write can spot this pattern fast, so think of tracing letters.

THE CONTEST CORNER

No. 25

Kseniya Garaschuk

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **September 1, 2015**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC121. Towns A and B are situated on two straight roads intersecting at the angle of $\angle ACB = 60^\circ$. One way to get from A to B is by taking the bus which goes from A to C to B; this takes 11 minutes. Alternatively, you can walk from A directly to B, which takes an hour and 10 minutes. Finally, you can first walk from A to the road on which B is situated and then take the bus to B, but this takes longer still even if the bus comes immediately.

Find the distance from A to the intersection C if you walk at the speed of 3 km/h and the bus drives at the speed of 30 km/h.

CC122. The sequence $\{x_n\}$ is given by the following recursion formula:

$$x_1 = \frac{a}{2}, \quad x_n = \frac{a}{2} + \frac{x_{n-1}^2}{2}, \quad n \geq 2, \quad 0 < a < 1.$$

Find the limit of the sequence.

CC123. Find how many pairs of integers (x, y) satisfy the inequality

$$2^{x^2} + 2^{y^2} < 2^{1976}.$$

CC124. In a chess tournament, every participant played every other participant exactly once. In each game each participant scored 1 point for the win, 0.5 points for the tie and 0 points for the loss. At the end of the tournament, you discovered that in any group of any three participants there is one who, in the games against the other two, got 1.5 points. What is the maximum possible number of participants the tournament could have had?

CC125. Orthogonal projections of a triangle ABC onto two perpendicular planes are equilateral triangles with side length 1. If the median AD of triangle ABC has length $\sqrt{\frac{9}{8}}$, find BC.

. . . .

CC121. On considère deux villes A et B. Chacune est située sur une route droite. Les deux routes se coupent en C de manière que l'angle ACB mesure 60°. Pour se rendre de A à B, Dan peut prendre l'autobus qui va de A à C à B, ce qui prend 11 minutes. Dan peut aussi marcher directement de A à B, ce qui prend une heure et 10 minutes. Enfin, Dan peut marcher de A jusqu'à la route sur laquelle B est située, puis prendre l'autobus jusqu'à B, mais cela prend encore plus de temps, même si l'autobus arrive immédiatement.

Déterminer la distance de A à C, sachant que Dan marche à une vitesse de 3 km/h et que l'autobus avance à une vitesse de 30 km/h.

CC122. La suite $\{x_n\}$ est définie de façon récursive, a étant un nombre réel, $0 < a < 1$:

$$x_1 = \frac{a}{2}, \quad x_n = \frac{a}{2} + \frac{x_{n-1}^2}{2}, \quad n \geq 2.$$

Déterminer la limite de la suite.

CC123. Combien y a-t-il de couples (x, y) d'entiers qui vérifient l'équation $2^{x^2} + 2^{y^2} < 2^{1976}$?

CC124. Dans un tournoi d'échecs, chaque participant a rencontré chaque autre participant exactement une fois pour une partie. Après chaque partie, on a attribué 1 point au gagnant et 0 point au perdant et dans le cas d'un match nul, on a attribué 0,5 point à chaque participant. À la fin du tournoi, on constate que dans n'importe quel groupe de trois participants, il y a un participant qui a marqué un total de 1,5 point dans ses parties contre les deux autres. Quel est le nombre maximal possible de participants dans ce tournoi?

CC125. Chacune des projections orthogonales d'un triangle ABC sur deux plans mutuellement orthogonaux est un triangle équilatéral avec des côtés de longueur 1. Sachant que la médiane AD du triangle ABC a une longueur de $\sqrt{\frac{9}{8}}$, déterminer la longueur du côté BC.

CONTEST CORNER SOLUTIONS

CC71. A bag is filled with red and blue balls. Before drawing a ball, there is a $\frac{1}{4}$ chance of drawing a blue ball. After drawing out a ball, there is now a $\frac{1}{5}$ chance of drawing a blue ball. How many red balls are in the bag?

Originally problem 10 from 2012 W.J. Blundon Mathematics Contest.

We present the solution of Petros Souldis.

Let n be the total number of balls and x the number of blue balls. For picking the first ball, we have

$$\frac{1}{4} = P(\text{first ball is blue}) = \frac{x}{n}.$$

Thus, $n = 4x$. For the second ball, we consider two cases based on whether the first selected ball is blue or red.

Case 1: The first ball picked was red. Then we have:

$$\frac{1}{5} = P(\text{second ball is blue}) = \frac{x}{n-1}$$

This gives $5x = n - 1$ and substituting in $x = 4n$, we get $x = -1$, so this cannot occur.

Case 2: The first ball picked was blue. Then we have:

$$\frac{1}{5} = P(\text{second ball is blue}) = \frac{x-1}{n-1}$$

This gives $5x - 5 = n - 1$ and if we substitute in $x = 4n$ we get $x = 4$. The total number of balls will be $n = 4x = 16$ and the number of red balls will be $16 - 4 = 12$.

CC72. From the set of natural numbers $1, 2, 3, \ldots, n$, four consecutive even numbers are removed. The remaining numbers have an average value of $51\frac{9}{16}$. Determine all sets of four consecutive even numbers whose removal creates this situation.

Originally 1995 Invitational Mathematics Challenge, Grade 11, problem 4.

We present the solution by Titu Zvonaru and Neculai Stanciu.

If $n > 107$, when removing the largest 4 integers, $(n, n-1, n-2, n-3)$, the average of the remaining numbers is $(n-3)/2 > 52$.

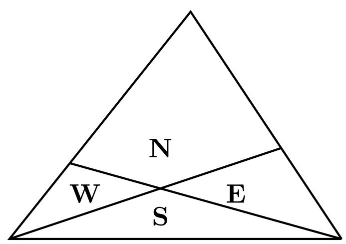
If $n < 99$, even when removing the smallest 4 integers, $(1, 2, 3, 4)$, the average is at most 51.5.

Suppose the numbers we remove are $2k, 2k+2, 2k+4, 2k+6$. Then the total is

$$\frac{n(n+1)/2 - 8k - 12}{n-4} = \frac{n^2 + n - 16k - 24}{2n-8}.$$

Since the numerator of this fraction is an integer when n and k are integers, for this to give an answer that, in reduced form, has a denominator of 16, we must have $2n - 8 \equiv 0 \pmod{16}$, or $n \equiv 4 \pmod{8}$. The only possible n of that form in the range $[99, 107]$ is $n = 100$. When $n = 100$, we get that $k = 11$, so $22, 24, 26, 28$ is the only possible set of removed numbers.

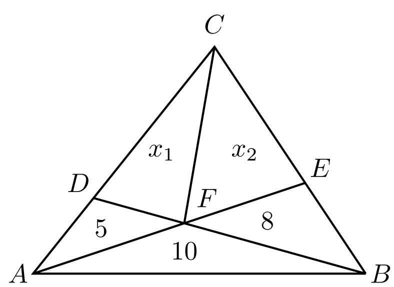
CC73. A farmer owns a triangular field, as shown. He reckons 5 sheep can graze in the west field, 10 sheep can graze in the south field, and 8 can graze in the east field. (All sheep eat the same amount of grass.) How many sheep can graze in the north field?



Originally problem 9 from 2012 W.J. Blundon Mathematics Contest.

We present the solution by Šefket Arslanagić.

We label the triangle as in the diagram:



We will now repeatedly use the result that if two triangles have the same height, the ratio of their areas is equal to the ratio of their bases:

$$\frac{x_1 + 5}{x_2} = \frac{AF}{FE} = \frac{10}{8},$$

$$\frac{x_1}{5} = \frac{CD}{AD} = \frac{x_1 + x_2 + 8}{15}$$

Solving these two equations for x_1 and x_2 yields $x_1 = 10, x_2 = 12$, so N can hold $10 + 12 = 22$ sheep.

CC74. Let $1000 \leq n = ABCD_{10} \leq 9999$ be a positive integer whose digits $ABCD$ satisfy the divisibility condition:

$$1111 \mid (ABCD + AB \times CD).$$

Determine the smallest possible value of n.

Originally 2014 Sun Life Financial Rêpechage Competition, problem 3.

Solved by Richard Hess; David Manes; and Titu Zvonaru and Neculai Stanciu. Here is the summary of all the solutions.

The answer is 1729, and $1729+17\times 29 = 2222$. Other multiples of 1111 along with their corresponding numbers are $(4444, 3142)$, $(5555, 3845)$, $(6666, 3399)$, $(8888, 8307)$, $(11110, 5890)$, $(11110, 9418)$, $(13332, 7289)$, $(13332, 9146)$.

It is straightforward to see that 1111 cannot be represented in the form $ABCD + AB \times CD$. Suppose that $A = 1$ and $0 \leq B \leq 6$. Then

$$\begin{aligned}(ABCD)_{10} &= (100)(10A + B) + (10A + B + 1)(10C + D)\\ &= (100)(10 + B) + (11 + B)(10C + D)\end{aligned}$$

cannot exceed $1600 + 1700 = 3300$ and so must be equal to 2222. Taking account of the fact that $2222 - 100(10 + B)$ must be divisible by $11 + B$ with a quotient not exceeding 99, we see that there are no possibilities for $0 \leq B \leq 6$.

Now consider $A = 1$ and $B = 7$. Since $2222 - 1700 = 522 = 18 \times 29$, we find that $ABCD = 1729$ works. Since $3333 - 1700$ is not a multiple of 18, this is the only possibility with $(A, B) = (1, 7)$.

CC75. Let P be a point inside the triangle ABC such that $\angle PAC = 10^\circ$, $\angle PCA = 20^\circ$, $\angle PAB = 30^\circ$ and $\angle ABC = 40^\circ$. Determine $\angle BPC$.

Originally 2004 MUN Undergrad Math Competition, Question 7.

We present 4 solutions.

Solution 1, by Miguel Amengual Covas.

Since $\angle CAB = \angle CBA = 40^\circ$, triangle ABC is isosceles and symmetric about its altitude from C to AB. The reflection in this altitude fixes C, switches A and B, and carries P to a point Q. In particular, $\angle CQB = \angle CPA = 150^\circ$. Since $CP = CQ$ and $\angle PCQ = \angle ACB - 2\angle ACP = 100^\circ - 40^\circ = 60^\circ$, triangle CPQ is equilateral.

Therefore $PQ = CQ$. Since also $\angle PQB = 360^\circ - \angle PQC - \angle CQB = 360^\circ - 60^\circ - 150^\circ = 150^\circ = \angle CQB$, and QB is common, triangles PQB and CQB are congruent. Therefore $BP = BC$ and so $\angle BPC = \angle BPQ + \angle CPQ = \angle BCQ + 60^\circ = 80^\circ$.

Solution 2, by Michel Bataille, and David E. Manes (independently).

Let $c = |AC| = |BC|$, $u = |CP|$, and $v = |BP|$. By the Sine Law applied to triangle APC, $u = c\sin 10^\circ/\sin 150^\circ = 2c\sin 10^\circ = 2c\cos 80^\circ$. By the Cosine Law applied to triangle BPC, $v^2 = c^2 + u^2 - 2uc\cos 80^\circ = c^2$. Therefore $v = c$, triangle BCP is isosceles and $\angle BPC = \angle BCP = 80^\circ$.

Solution 3, by Šefket Arslanagić.

Since triangle ABC is isosceles, the altitude CN from C to AB bisects angle ACB. Thus, $\angle PCN = \angle ACN - \angle PCA = 50^\circ - 20^\circ = 30^\circ$. Let the line AP meet CN at M. Since $\angle CPM = \angle PCM = 30^\circ$, $MP = CM$. Since $\angle PMC = 120^\circ$, it follows that $\angle PMN = 60^\circ$, so that $\angle PMB = \angle CMB = 120^\circ$. Triangles PMB and CMB are congruent (SAS), so that $PB = CB$ and $\angle BPC = \angle BCP = 80^\circ$.

Solution 4, by George Apostolopoulos.

Let $x = \angle PBC$. Using the Sine Law, we have that

$$\begin{aligned} 1 &= \frac{PA}{PB}\cdot\frac{PB}{PC}\cdot\frac{PC}{PA} \\ &= \frac{\sin(40^\circ - x)}{\sin 30^\circ}\cdot\frac{\sin 80^\circ}{\sin x}\cdot\frac{\sin 10^\circ}{\sin 20^\circ}, \end{aligned}$$

from which $\sin x\sin 20^\circ = 2\sin(40^\circ - x)\cos 10^\circ\sin 10^\circ$. Thus $\sin x = \sin(40^\circ - x)$ and $x = 20^\circ$. Hence

$$\angle BPC = 180^\circ - \angle PBC - \angle PCB = 180^\circ - 20^\circ - 80^\circ = 80^\circ.$$

THE OLYMPIAD CORNER

No. 323

Nicolae Strungaru and Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **September 1, 2015***, although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC181. All the prime numbers are written in order $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... Find all pairs of positive integers a and b with $a - b \geq 2$ such that $p_a - p_b$ divides $2(a - b)$.

OC182. Let x and y be real numbers satisfying $x^2y^2 + 2yx^2 + 1 = 0$. If

$$S = \frac{2}{x^2} + 1 + \frac{1}{x} + y\left(y + 2 + \frac{1}{x}\right),$$

find the maximum and minimum of S.

OC183. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(0) = 0$, $f(1) = 2013$ and

$$(x - y)(f(f(x)^2) - f(f(y)^2)) = (f(x) - f(y))(f(x)^2 - f(y)^2).$$

OC184. Let k, m and n be three distinct positive integers. Prove that

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n).$$

OC185. The incircle of ΔABC touches sides BC, CA and AB at points D, E and F respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of ΔABC.

. . . .

OC181. On écrit tous les nombre premiers en ordre : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... Déterminer tous les couples (a, b) d'entiers strictement positifs, $a - b \geq 2$, tels que $p_a - p_b$ soit un diviseur de $2(a - b)$.

OC182. Soit x et y des réels qui vérifient l'équation

$$x^2y^2 + 2yx^2 + 1 = 0.$$

Soit

$$S = \frac{2}{x^2} + 1 + \frac{1}{x} + y\left(y + 2 + \frac{1}{x}\right).$$

Déterminer les valeurs maximale et minimale de S.

OC183. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ qui vérifient $f(0) = 0$, $f(1) = 2013$ et

$$(x - y)(f(f(x)^2) - f(f(y)^2)) = (f(x) - f(y))(f(x)^2 - f(y)^2).$$

OC184. Soit k, m et n trois entiers strictement positifs distincts. Démontrer que

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n).$$

OC185. Le cercle inscrit dans le triangle ABC touche les côtés BC, CA et AB aux points respectifs D, E et F. Soit P le point d'intersection des droites AD et BE. Soit X, Y et Z les images respectives du point P par les réflexions par rapport à EF, FD et DE. Démontrer que les droites AX, BY et CZ sont concourantes en un point sur la droite IO, I étant le centre du cercle inscrit dans le triangle ABC et O étant le cercle circonscrit au triangle ABC.

OLYMPIAD SOLUTIONS

OC121. Prove that for all positive real numbers x, y, z we have

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx).$$

Originally question 2 from the 2012 Balkan Mathematical Olympiad.

We present two solutions.

Solution 1, composed of similar solution of David Manes and Paolo Perfetti.

By Cauchy-Schwarz we have $(z+x)(z+y) \geq (\sqrt{z}\sqrt{z}+\sqrt{x}\sqrt{y})^2$. Moreover, AM-GM gives $x+y \geq 2\sqrt{xy}$. Thus

$$\begin{aligned}\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} &\geq \sum_{cyc}(x+y)(z+\sqrt{xy}) \\ &= \sum_{cyc}(x+y)z + \sum_{cyc}(x+y)\sqrt{xy} \\ &\geq \sum_{cyc} xz+yz + 2\sum_{cyc} xy \\ &= 4(xy+yz+zx)\,.\end{aligned}$$

Solution 2, composed of similar solutions by Arkady Alt and Šefket Arslanagić.

Let

$$a := \sqrt{y+z},\ b := \sqrt{z+x},\ c := \sqrt{x+y}\,.$$

Then $a, b,$ are the side lengths of an acute triangle, because

$$\frac{b^2+c^2-a^2}{2} = x > 0, \quad \frac{c^2+a^2-b^2}{2} = y > 0, \quad \frac{a^2+b^2-c^2}{2} = z > 0.$$

Moreover, we have

$$\begin{aligned}4\,(xy+yz+zx) &= \sum_{cyclic}\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right) \\ &= 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4 = 16F^2,\end{aligned}$$

where F is the area of the triangle.

Let R, r, s be circumradius, inradius and semiperimeter of the triangle. Then, original inequality becomes

$$abc\,(a+b+c) \geq 16F^2 \iff 8FRs \geq 16F^2 \iff Rs \geq 2F \iff Rs \geq 2sr \iff R \geq 2r,$$

where latter inequality is the well known Euler's Inequality.

OC122. We define a sequence $f_n(x)$ of functions by

$$f_0(x) = 1, f_1(x) = x,\ (f_n(x))^2 - 1 = f_{n-1}(x)f_{n+1}(x),\ \text{for } n \geq 1.$$

Prove that for every $n, f_n(x)$ is a polynomial with integer coefficients.

Originally question 3 from the Indian national Olympiad 2012.

We give the solution of Omran Kouba.

We first prove by induction on n that for all $\theta \notin \mathbb{Q}\pi$, we have

$$f_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{1}$$

This is trivially true for $n = 0, 1$. So, let us suppose that this is true for n and $n-1$, then for $n\theta \notin \pi\mathbb{Z}$ we have

$$\begin{aligned} f_{n+1}(2\cos\theta) &= \frac{f_n^2(2\cos\theta) - 1}{f_{n-1}(2\cos\theta)} = \frac{\sin^2((n+1)\theta) - \sin^2\theta}{\sin\theta\,\sin(n\theta)} = \frac{\cos(2\theta) - \cos(2(n+1)\theta)}{2\sin\theta\,\sin(n\theta)} \\ &= \frac{2\sin((n+2)\theta)\sin(n\theta)}{2\sin\theta\,\sin(n\theta)} = \frac{\sin((n+2)\theta)}{\sin\theta}. \end{aligned}$$

Now for a given $n \geq 1$, we define the function

$$Q_n(x) = f_{n+1}(x) + f_{n-1}(x) - xf_n(x).$$

It is easy to see that all f_n are rational functions, and hence so are Q_n. Using (1), we see that $Q_n(2\cos\theta) = 0 \quad \forall\theta \notin \mathbb{Q}\pi$. Therefore, Q_n is a rational function which has infinitely many zeroes. Thus $Q_n \equiv 0$, which yields

$$f_{n+1}(x) = xf_n(x) - f_{n-1}(x).$$

As $f_0 = 1, f_1 = x$, by induction it follows immediately that $f_n \in \mathbb{Z}[X]$.

OC123. Let p be prime. Find all positive integers n for which, whenever x is an integer such that $x^n - 1$ is divisible by p, then $x^n - 1$ is also divisible by p^2.

Originally question 3 from Japan Math Olympiad 2012.

No solution was received to this problem. We give a solution by the editor.

We claim that n has the desired property if and only if $p|n$.

"$\Rightarrow$" Since $p|(1+p)^n - 1$ it follows that $p^2|(1+p)^n - 1$. Therefore

$$0 \equiv (1+p)^n - 1 \equiv np \pmod{p^2}.$$

This shows that $p|n$.

"$\Leftarrow$" This implication is an immediate consequence of the Hensel's Lemma :

> If $f(X) = X^n - 1$, then as $f(x) \equiv 0 \pmod{p}$ and $f'(x) \equiv 0 \pmod{p}$, it follows that $f(x) \equiv 0 \pmod{p^2}$.

We provide below a more elementary solution. Let $n = pk$. Then

$$x^n - 1 \equiv (x^p)^k - 1 \equiv x^k - 1 \pmod{p} .$$

Let $y := x^k$. Then we know that $y \equiv 1 \pmod{p}$, and hence

$$1 + y + y^2 + \cdots + y^{p-1} \equiv 1 + 1 + 1 + \cdots + 1 \equiv 0 \pmod{p} .$$

This shows that $p | 1 + y + y^2 + \cdots + y^{p-1}$. As p also divides $1 - y$, we get that

$$p^2 | 1 - y^p = 1 - x^n .$$

Editor's comment. For a reference, see K.H. Rosen, *Elementary Number Theory and its applications*, 6th Edition, Addison-Wesley, 2011.

OC124. Find all triples (a, b, c) of positive integers with the following property : for every prime p, if n is a quadratic residue $\pmod{p}$, then $an^2 + bn + c$ is also a quadratic residue $\pmod{p}$.

Originally question 2 from the 2012 Romanian Team Selection Test, day 5.

We present the solution by Oliver Geupel.

The triples

$$(a, b, c) = (u^2, 2uw, w^2) \tag{1}$$

with $u, w \in \mathbb{N}$ have the desired property, because $an^2 + bn + c = (un + w)^2$ is a perfect square and therefore a quadratic residue modulo every prime. We will show that there are no other solutions.

Let a, b, c be natural numbers with the desired property and consider the polynomial

$$P(x) = ax^4 + bx^2 + c .$$

Then, for every natural number n, the number $P(n)$ is a quadratic residue modulo every prime by hypothesis.

By a well-known application of Chebotarev's density theorem, an integer is a perfect square if it is a quadratic residue modulo every prime. Hence, $P(n)$ is a perfect square for every natural number n.

Let $P(x) = (F(x))^2 G(x)$ with polynomials $F, G \in \mathbb{Z}[x]$ where the polynomial

$$G(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

is square-free in $\mathbb{Z}[x]$. The degree of G is either 0 or 2 or 4. We prove by contradiction that $\deg G = 0$.

Assume that $\deg G \geq 2$. Because G is square-free, the resultant $R(G, G')$ is an integer distinct from 0. It is a well-known fact that for every integer polynomial $Q(x)$ with $\deg Q(x) > 0$ there are infinitely many primes $p_1 < p_2 < p_3 < \cdots$ and natural numbers $n_1, n_2, n_3, \ldots$ such that $p_i \mid Q(n_i)$ for $i = 1, 2, 3, \ldots$. Let p_i and n_i be such numbers in the case $Q = G$. Then,

$$G(n_i + p_i) - G(n_i) - p_i G'(n_i) = p_i^2(6a_4 n_i^2 + 4a_4 n_i p_i + 3a_3 n_i + a_3 p_i + a_2 + a_2 p_i).$$

We deduce $p_i^2 \mid G(n_i + p_i) - G(n_i) - p_i G'(n_i)$; whence $p_i \mid G(n_i + p_i)$. Because $P(n_i + p_i)$ is a perfect square, we obtain $p_i^2 \mid G(n_i + p_i)$. Also, $p_i^2 \mid G(n_i)$. Hence, $p \mid G'(n_i)$. We conclude that, for $i = 1, 2, 3, \ldots$, the prime p_i is a divisor of the integer $R(G, G')$. Consequently, $R(G, G') = 0$, a contradiction. This proves that $\deg G = 0$.

We obtain $P(x) = (F(x))^2$. Putting $F(x) = ux^2 + vx + w$, we have

$$ax^4 + bx^2 + c = (ux^2 + vx + w)^2.$$

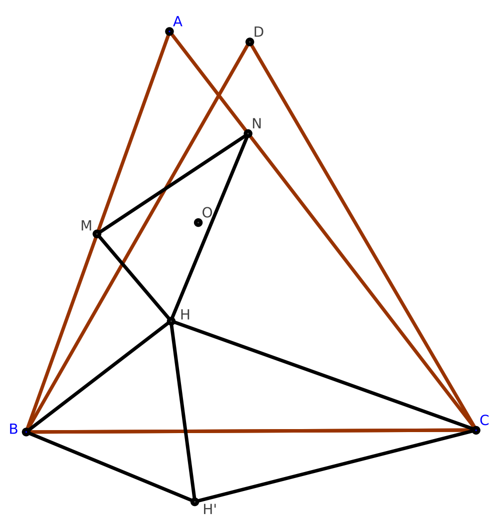
Comparing coefficients finally yields (1).

OC125. ABC is an acute angle triangle with $\angle A > 60^\circ$ and H is its orthocenter. M, N are two points on AB, AC respectively, such that $\angle HMB = \angle HNC = 60^\circ$. Let O be the circumcenter of triangle HMN. Let D be a point on the same side of BC as A such that $\triangle DBC$ is an equilateral triangle. Prove that H, O, D are collinear.

Originally question 1 from 2012 Chinese Team Selection Test, day 1.

We give the solution of Oliver Geupel.

Let ρ be the rotation about the fixed point B by an angle of 60° such that $\rho(D) = C$. Let $\rho(H) = H'$.



We have

$$\frac{HM}{HN} = \frac{HB}{HC} = \frac{HH'}{HC} \text{ and } \angle MHN = \angle BHC - 60^\circ = \angle H'HC.$$

Hence, the triangles MHN and $H'HC$ are similar. Moreover, by rotation,

$$\angle HH'C = \angle BH'C - 60^\circ = \angle BHD - 60^\circ\,.$$

Thus,

$$\begin{aligned}\angle HNM = \angle HCH' &= 180^\circ - \angle H'HC - \angle HH'C = 300^\circ - \angle BHC - \angle BHD \\ &= \angle CHD - 60^\circ.\end{aligned}$$

We obtain

$$\angle MHO = \frac{1}{2}(180^\circ - \angle HOM) = 90^\circ - \angle HNM = 150^\circ - \angle CHD = \angle MHD.$$

Consequently, the points H, O, and D are collinear.

Meetings at sea

Yes, weekly from Southampton
Great steamers, white and gold,
Go rolling down to Rio
And I'd like to roll to Rio.
Some day before I'm old!

So, weekly, say every Thursday at noon, steamers leave Southampton to roll down to Rio. This trip of total length 9800 kilometres takes a steamer exactly 14 days to complete (so it covers the distance of 700 kilometres a day) and it arrives to Rio at noon on Thursday, two weeks later. After a 4-day stop in Rio, the steamer sails back and in exactly 14 days at noon on Monday it arrives to Southampton. In 3 more days (note - again on Thursday!), the steamer rolls to Rio. And I'd like to roll to Rio. So I board a steamer in Southampton and sail over to Rio. You task is to find out :

a) How many steamers rolling back to Southampton will I see on my trip?
b) When (which days of the week) will I see those steamers?
c) How far away from Southampton will each one of them be?
d) Two steamers meet at sea. Is it true that at the same time at some other place at sea two other steamers meet? If yes, then what is the distance between these two meeting points?
e) How many steamers sail back and forth between Southampton and Rio?

From article by A. Rosenthal, Kvant, 1976 (5). Poem is by Rudyard Kipling.

BOOK REVIEWS

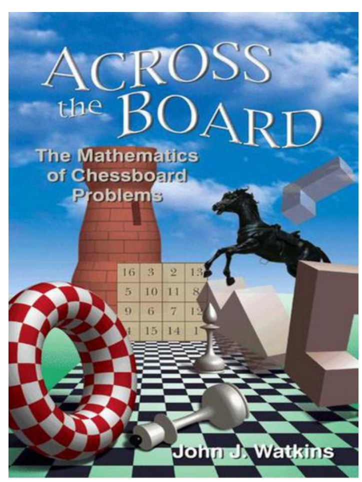
Robert Bilinski

Across the Board : the Mathematics of Chessboard Problems by John J. Watkins
ISBN 978-0-691-15498-5, 2012 reprint, 260 pages.
Published by Princeton University Press, 2004/2012.

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

John J. Watkins is a professor emeritus at Colorado College whose research spans the areas of number theory and combinatorics. During his career, he published 5 articles on chessboard problems. Occasionally, we find problems in Olympiads which are based on the chessboard, so, when I found this book, I thought it would be an ideal book for ***Crux*** readers. I had a few pre-conceptions about what the book would contain (tours, domination and independence problems), but I was surprised by how much more there actually was : the variants, the effect of the board size and shape, covering problems, links to other mathematical objects like latin squares, or games like tic-tac-toe, etc. Across thirteen chapters, the book leads the readers through the universe of the chessboard.

Across the Board is written by someone who actively published on the mathematics of chessboards. The work seems complete even if obscure configurations, like the triangular or three-dimensional toroidal with a twist chessboards, are only mentioned in passing. Hence, the results are mainly on square chessboards of varying sizes, but of differing properties. In the first eight chapters, the book's aim seems to be to introduce the main concepts of chessboard problems without being either an encyclopedia or a monograph on the subject. But after chapter eight, the book warps into a theorem, proof and application of theorem format. Then, in the final chapters, it comes back to the more "casual" format.



Now, maybe I should explain the main types of chessboard problems for those not familiar with them. The first one I mention is tours. The problem is to create a path that visits each and every square of a chessboard exactly once with a given piece. The second one I mention is domination. In this situation, the aim is to place the minimal number of the same kind of chess piece on the board so that all the squares on it are attackable by them, or you want to "cover" the board with the spheres of influence with the minimal number of one kind of piece. The third problem is independence. The purpose is to place as many pieces of one kind on the board so that each is exempt from attack from the other pieces. The last type of chessboard problem is covering problems. It has nothing to do with chess pieces, but is concerned with dominoes and polyomino coverings of chessboards.

The mathematics broached in this book is pretty far reaching. There is the topology of the spaces in which the chessboards find themselves : the author discusses, among others, normal, cylindrical, toroidal, full 3D cubic, box 3D cubic, triangular and spherical chessboards. There is the graph representation of the moves possible on said chessboards and the combinatorics counting possible moves or arrangements of pieces. Logic is also used to prove some simple results. Gray codes are explained and then used to create tours for chess pieces on the board (though the historical reference is off since the code was first invented in 1872 by a French notary called Luc-Agathon Gros, not the American engineer Frank Gray, who rediscovered it 1953 and applied it to computers). In chapter 10, a bit of number theory is used to ascertain which board sizes have a certain solution to the n queen independence problem.

The book's intended readership is clearly mathematically inclined, but the book is not written only for mathematicians. Before chapter 8, it contains only the most easily comprehensible and fairly short proofs. Then the bar is raised, the proofs get longer and they need a bit of concentration to assimilate. But, after all, you cannot make an omelet without breaking the eggs and to do these problems at this level you have to be good at math. The presentation of the results is interspersed with problems aimed at picking the readers' curiosity. The book will also satisfy the history buffs with references to the enigmas of Dudeney, Stewart and Gardner, and with references to the much older roots of chess problems, be they African, Persian, Arabic, European or Indian. I was surprised by how old cylindrical and toroidal chessboard problems were and that, a few centuries ago, there were variants of knights called "camels" (they move three straight and one to the side). Though this seems like a bit of a paradox to me since the book does not fully cite much more recent results : for example, the 1995 lower bound theorem by Weakley is presented and proved in full, but it is not dated. Personally, I would have also been interested in seeing some mathematics of real chess games, but I will have to look somewhere else for that.

All in all, a most enjoyable book that will surely offer new and original avenues for problem solvers of all kinds in need of new techniques, approaches or problems to solve. I definitely recommend this book as a reference on the subject as it seems to be the lone contender in the field. With a 41 article bibliography, it can also act as a stepping board for someone wanting to tackle the open problems in the field. Maybe give it as a gift to a non-mathematician chess player to interest them in a new aspect of this workspace. Happy readings!

FOCUS ON...

No. 12

Michel Bataille

Intersecting Circles and Spiral Similarity

Introduction

Let two circles C_1, C_2, with centres O_1, O_2 and radii r_1, r_2, intersect at points U, V. Among the spiral similarities transforming C_1 into C_2, those with centre U or V deserve a special interest. Specifically, let σ be the one with centre U. Of course, the factor of σ is $\frac{r_2}{r_1}$ and $\sigma(O_1) = O_2$, so that its angle is $\theta = \angle(\overrightarrow{UO_1}, \overrightarrow{UO_2})$, the directed angle from vector $\overrightarrow{UO_1}$ to vector $\overrightarrow{UO_2}$. But a special feature of this transformation, emphasized in this number, is the very simple way the image of any point P of C_1 is obtained : $P' = \sigma(P)$ is the second point of intersection of the circle C_2 with the line through P and V.

The proof is easy. First note that P' certainly is on C_2, hence we just have to show that P, V, P' are collinear. With the help of the usual properties of angles subtending arcs of a circle, we calculate

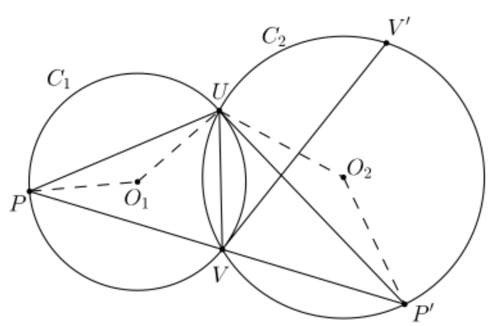
$$\begin{aligned}\angle(VP, VP') &= \angle(VP, VU) + \angle(VU, VP') \\ &= \frac{1}{2}\angle(\overrightarrow{O_1P}, \overrightarrow{O_1U}) + \frac{1}{2}\angle(\overrightarrow{O_2U}, \overrightarrow{O_2P'}).\end{aligned}$$

Since a spiral similarity preserves directed angles, we have

$$\angle(\overrightarrow{O_2U}, \overrightarrow{O_2P'}) = \angle(\overrightarrow{O_1U}, \overrightarrow{O_1P})$$

and finally $\angle(VP, VP') = 0 \pmod{\pi}$. The conclusion follows.

In the proof above, it is understood that P is different from V. But as P approaches V on C_1, the limiting position of the line VP is the tangent to C_1 at V. Therefore $V' = \sigma(V)$ is the point where this tangent meets C_2 again (see figure below).



For convenience, this result about spiral similarities will be called $(\mathcal{R})$ in what follows.

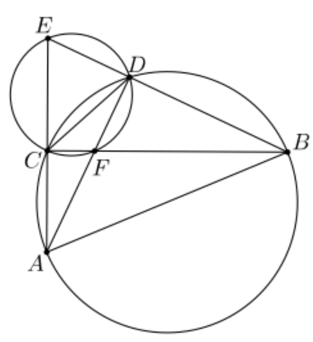
Two applications

(a) Keeping the above notations, let λ be a fixed real number and suppose that an arbitrary line ℓ through V meets again C_1 at P_1 and C_2 at P_2. What is the locus of $R_\lambda = \lambda P_1 + (1-\lambda)P_2$ as ℓ turns around V? (This is, slightly modified, Walther Janous's problem **2706** [2002 : 54; 2003 : 54]). Here is a simple solution based on the properties of spiral similarities and prompted by $(\mathcal{R})$. Let $\ell_0 = P_1^0 P_2^0$ be the position of ℓ parallel to O_1O_2 and let $R_\lambda^0 = \lambda P_1^0 + (1-\lambda)P_2^0$. Then, from $(\mathcal{R})$, $\sigma(P_1^0) = P_2^0$ and for any other position of ℓ, $\sigma(P_1) = P_2$. Thus, the spiral similarity with centre U transforming P_1^0 into P_1 also transforms P_2^0 into P_2 and, as it preserves collinearity and signed ratio, transforms R_λ^0 into R_λ. As a result, the spiral similarity σ_λ with centre U such that $\sigma_\lambda(P_1^0) = R_\lambda^0$ satisfies $\sigma_\lambda(P_1) = R_\lambda$. Since P_1 traverses C_1 as ℓ varies, the locus of R_λ is $\sigma_\lambda(C_1)$ that is, the circle with centre $\sigma_\lambda(O_1)$ passing through U (note that $R_\lambda = U$ when $P_1 = U$).

(b) As a second example where a call to $(\mathcal{R})$ is quite natural, consider the following question (extracted from [1]) :

> Let A, B, C, D be four concyclic points such that AC, BD intersect at E and AD, BC intersect at F. If C, D, E, F are concyclic, show that EF is perpendicular to AB.

Consider the spiral similarity σ with centre C transforming the circle $(CDEF)$ into the circle $(ABCD)$. From $(\mathcal{R})$, we have $\sigma(E) = B$ and $\sigma(F) = A$ and it follows that $\angle(\overrightarrow{CE}, \overrightarrow{CB}) = \angle(\overrightarrow{CF}, \overrightarrow{CA})$. Hence $\angle(CE, CB) = \angle(CB, CA)$ and, since E, C, A are collinear, the angle of σ must be a right angle. In consequence, EF is perpendicular to its image AB.



Coming across Ptolemy in Croatia

Our last example is #3 of the Croatian Mathematical Olympiad 2006 [2009 : 293] :

> The circles Γ_1 and Γ_2 intersect at the points A and B. The tangent line to Γ_2 through the point A meets Γ_1 again at C and the tangent line

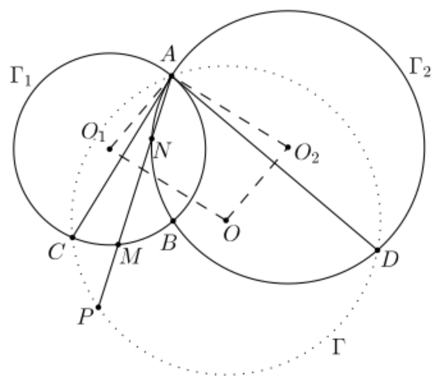
to Γ_1 through A meets Γ_2 again at D. A half-line through A, interior to the angle $\angle CAD$, meets Γ_1 at M, meets Γ_2 at N, and meets the circumcircle of ΔACD at P. Prove that $AM = NP$.

Amengual Covas's neat solution is based on similar triangles [2010 : 444]; we propose a variant using property $(\mathcal{R})$.

Let Γ be the circumcircle of ΔACD and O_1, O_2, O be the centres of $\Gamma_1, \Gamma_2, \Gamma$, respectively. Let σ_B denote the spiral similarity with centre B transforming Γ_1 into Γ_2. From $(\mathcal{R})$, we have $\sigma_B(M) = N$ and $\sigma_B(A) = D$, hence

$$\frac{ND}{AM} = \frac{r_2}{r_1} \qquad (1)$$

where r_1, r_2 are the radii of Γ_1, Γ_2, respectively.



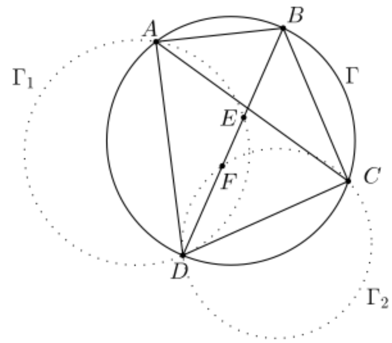
In the same way, if σ_D is the spiral similarity with centre D transforming Γ_2 into Γ, we have $\sigma_D(N) = P$ and $\sigma_D(O_2) = O$. Thus,

$$\frac{ND}{NP} = \frac{O_2D}{O_2O} = \frac{O_2D}{AO_1} = \frac{r_2}{r_1}$$

(note that AO_2OO_1 is a parallelogram, its sides being parallel). A comparison with (1) gives $AM = NP$, as desired.

And what about Ptolemy? Well, in some way, his famous theorem is hidden in this problem! Ptolemy's Theorem states that if A, B, C, D are four points in this order on a circle, then $AB \cdot CD + BC \cdot AD = AC \cdot BD$. The following proof is closely related to the above problem.

Let Γ be the circumcircle of $ABCD$ and let σ_A be the spiral similarity with centre A such that $\sigma_A(C) = D$. Let $\sigma_A(B) = E$. If Γ_1 is the circumcircle of triangle ADE, we have $\Gamma_1 = \sigma_A(\Gamma)$ and property $(\mathcal{R})$ tells us that E is on BD (between B and D as $\angle CAD < \angle BAD$). Similarly, if σ_C is the spiral similarity with centre C such that $\sigma_C(A) = D$, then $F = \sigma_C(B)$ is on $\Gamma_2 = \sigma_C(\Gamma)$ and on the line segment BD.



Now, because of ($\mathcal{R}$) again, CD (resp. AD) is tangent at D to Γ_1 (resp. Γ_2), so we recognize the configuration of the problem and derive $DE = FB$.

To conclude, since $\frac{AD}{AC} = \frac{DE}{BC}$ and $\frac{CD}{CA} = \frac{DF}{AB}$, we have

$$AB \cdot CD + BC \cdot AD = AC(DF + DE) = AC(DF + FB) = AC \cdot BD.$$

Exercise

The result ($\mathcal{R}$) should help the reader to solve the following problem, adapted from Gerry Leversha's problem **2457** [1999 : 308; 2000 : 316]). Let $ABCD$ be a quadrilateral such that AD and BC intersect at E. Suppose that $ID = IC, JA = JC, KB = KD$ and $\angle(\overrightarrow{ID}, \overrightarrow{IC}) = \angle(\overrightarrow{JA}, \overrightarrow{JC}) = \angle(\overrightarrow{KD}, \overrightarrow{KB}) = \angle(\overrightarrow{EA}, \overrightarrow{EB})$. Show that E, I, J, K are collinear.

Reference

[1] Problème E.271, *Quadrature*, No 63, janvier-mars 2007.

A Laboratory on Cubic Polynomials

Arkady Alt

Most ***Crux*** readers will be familiar with quadratic polynomials, and know how to solve the corresponding equations by completing the square or using the quadratic formula. Far fewer people know how to solve a cubic equation! In this lab, we will study polynomials of third degree $P(x) = ax^3 + bx^2 + cx + d$, corresponding to the cubic equation

$$ax^3 + bx^2 + cx + d = 0 ,$$

(We will always assume the coefficients of the polynomial to be real, though we will consider roots which are not.)

We can reduce the polynomial to monic form (in which the leading coefficient equals 1) in various ways. The simplest way is, of course, to divide the entire polynomial by a. But we can also do it by a change of variables : letting $x = (t/a)$ and multiplying by a^2, we obtain the monic equation

$$t^3 + bt^2 + act + a^2d = 0 .$$

Such a reduction is useful when we want to keep the coefficients as integers! In the particular case where $d = 1$, obviously $x = 0$ is not a root, and we can also reduce the equation to monic form by substituting $x = (1/t)$.

In what follows, we suppose that the cubic equation has – somehow or other – the form

$$x^3 + rx^2 + px + q = 0 \qquad \text{for } p, q, r \text{ real.} \tag{1}$$

Problem 1 *Prove that the equation (1) always has at least one real root.*

Hint : Show that there exists a pair of numbers m, M with $P(M)P(m) < 0$. Then use the Intermediate Value Theorem.

Problem 2 *Prove that, for any $a \in R$, $f(x) = (x - a) \cdot g(x) + f(a)$, where $g(x)$ is a quadratic polynomial.*

Hint : Which quadratic polynomial?

Let $x = a$ be a root of $f(x)$. Then $f(x) = (x - a) \cdot g(x)$, where $g(x)$ is a quadratic polynomial. If $g(a) \neq 0$ we shall say that $x = a$ is a *simple root* of the equation $f(x) = 0$; otherwise it is a *multiple root*. If $g(a) = 0$, we have two possibilities : either $g(x) = (x-a)^2$ or $g(x) = (x-a)\cdot(x-b)$ for $b \neq a$. In the first case the root a has multiplicity 3 and $f(x) = (x - a)^3$. In the second case, it has multiplicity 2 and $f(x) = (x - a)^2 \cdot (x - b)$.

Problem 3 *Prove that for any cubic equation of form (1) one of the following must hold; and find an example of each case.*

1. *$f(x) = 0$ has one simple real root and no other real roots*
2. *$f(x) = 0$ has three simple real roots*

3. *$f(x) = 0$ has one real root of multiplicity 2 and one simple real root*
4. *$f(x) = 0$ has one real root of multiplicity 3.*

What conditions on p, q, r force each case ?

If the equation $f(x) = 0$ has all of its roots real, then one of (2-4) must be the case. Only the case of one simple real root is eliminated.

Problem 4 *Prove that the three real numbers x_1, x_2, x_3 (some or all of which could be equal) are roots of equation (1) if and only if they satisfy the following three conditions :*

$$\begin{cases} x_1 + x_2 + x_3 = -r \\ x_1x_2 + x_2x_3 + x_3x_1 = p \\ x_1x_2x_3 = -q \end{cases} \tag{2}$$

This is the cubic case of Viète's theorem *and the above system of equations is called* Viète's system.

Before we go on, we note that equation 1 can be reduced by the further substitution $x = y - \frac{r}{3}$ to the form

$$y^3 + b \cdot y + c = 0 \tag{3}$$

with no quadratic term.

Problem 5 *Derive formulae for b, c in terms of p, q, r.*

Now we consider this equation in more detail.

Problem 6 *Show that if $b > 0$, then equation (3) has a unique real root, which if we hold b fixed (say $b = 1$) can be considered as a function $y(c)$ of the other coefficient. Show that if, furthermore, $c > 0$, then $y(c) < 0$.*

Problem 7 *Show that $y(c)$ as defined above is monotone decreasing, continuous, and twice differentiable on $(-\infty, \infty)$. Find the derivative dy/dc.*

Problem 8 *Let $y(c)$ be as defined above.*

a) *Find the second derivative d^2y/dc^2.*

b) *Show that $y(c)$ is concave down for $c < 0$, concave up for $c > 0$, and has a point of inflection at $c = 0$. (You can prove this part without using derivatives.)*

We now return to the question of computing the roots of equation (3). The case $b = 0$ is fairly trivial (what are the roots in this case ?), so we concentrate on the remaining cases.

Case I : $b > 0$. Make the substitution

$$y = \sqrt{\frac{b}{3}} \cdot \left(t - \frac{1}{t}\right)$$

in equation (3) and manipulate it to obtain a quadratic equation in t^3.

- Is it always soluble within the real number system ?
- Is it always equivalent to the original equation in the sense of having the same set of roots ?
- If this calculation has introduced *extraneous* roots that are not roots of the original equation, how can they be removed ?

Case II : $b < 0$. Make the substitution

$$y = \sqrt{\frac{b}{3} \cdot \left(t + \frac{1}{t}\right)}$$

in equation (3).

- In this case, it is possible that the resulting quadratic equation has no roots - when does this happen ?
- What conditions on b and c will guarantee roots ?
- The quadratic equation may have two roots - what do we do with them ?
- Are there extraneous roots ?
- Have we found all the roots of equation (1), and if not how do we find the rest ?

The next approach uses a trigonometric transformation.

Problem 9 *Consider the cubic equation*

$$4t^3 - 3t = d \tag{4}$$

where $|d| \leq 1$. *Set* $d = \cos(\alpha)$ *and use the substitution* $t = \cos(\phi)$ *to show that*

$$\left\{t_0 := \cos\left(\frac{\alpha}{3}\right),\ t_1 := \cos\left(\frac{\alpha + 2\pi}{3}\right),\ t_2 := \cos\left(\frac{\alpha + 4\pi}{3}\right)\right\}$$

is the full set of real roots of (4). How does the multiplicity of the roots depend on d *?*

Problem 10 *Suppose that* $|d| > 1$ *in equation (4). Prove that*

$$t_0 := \frac{\sqrt[3]{d + \sqrt{d^2 - 1}} - \sqrt[3]{d - \sqrt{d^2 - 1}}}{2}$$

is the unique real root.

Hint : prove that $|t_0| \geq 1$.

Problem 11 *Consider the cubic equation*

$$4t^3 + 3t = d \tag{5}$$

Prove that

$$t_0 := \frac{\sqrt[3]{\sqrt{d^2-1}+d} - \sqrt[3]{\sqrt{d^2-1}-d}}{2}$$

is its unique real root.

Problem 12 *Show that the equation $y^3 + by + c = d$ can be put into the form (4) or (5) by the substitution $y = 2\sqrt{|b|/3}t$, depending on the sign of b.*

Problem 13 *Use Problem (12) to formulate the results of problems (9-11) for the equation $y^3 + by + c = 0$. How does the value $D := (q/2)^2 + (p/3)^3$ help classify the possible cases ?*

We now consider cubic equations with real coefficients but with *complex* roots. For convenience, we will write our original equation in the form

$$x^3 - rx^2 + px - q = 0 \; . \tag{6}$$

Problem 14 *Prove that this equation always has three solutions in the set of complex numbers, counting each root with its correct multiplicity.*

We will introduce a *discriminant* for cubic equations. Readers will recall that for a reduced *quadratic* $x^2 + bx + c$ with roots x_1, x_2, the discriminant is defined to be

$$\Delta := b^2 - 4c = (x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \; .$$

When this is positive, the quadratic has two distinct real roots. When it is zero, it has a double root, and when it is negative, a complex-conjugate pair.

For a reduced *cubic* polynomial, the discriminant is defined to be

$$\Delta := (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2 \; .$$

This has similar properties.

Problem 15 *Prove the following statements :*

1. $\Delta > 0 \Leftrightarrow$ *all three roots are distinct and real ;*

2. $\Delta = 0 \Leftrightarrow$ *at least two roots are equal and all are real ;*

3. $\Delta < 0 \Leftrightarrow$ *one root is real and the others are complex conjugates*

Problem 16 *Suppose the cubic polynomial to be in the form (3). Show that $\Delta = -4a^3 - 27b^2$.*

The discriminant Δ doesn't give us complete information about the nature of the roots : when $\Delta = 0$ we don't know whether we have one double root and one single root, or one triple root. We can determine this using

$$\Delta_1 := (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \; .$$

Problem 17 *Suppose that $\Delta = 0$. Show that there is a triple root if and only if $\Delta_1 = 0$.*

Problem 18 *For a cubic polynomial in the form (6), prove that*

$$\Delta = r^2p^2 - 4r^3q - 4p^3 - 27q^2 + 18rpq\ .$$

Problem 19 *For a cubic polynomial in the form (6), prove that*

$$\Delta_1 = 2(r^2 - 3p)\ .$$

Problem 20 *For a cubic polynomial in the form (6), prove that necessary and sufficient conditions for all roots to be real and positive are*

$$r, p, q, \Delta \geq 0\ .$$

. . . .

The article is adapted with permission from an article by Arkady Alt in Delta *(Nov. 1994).*

Arkady Alt
San Jose, CA, USA

Math Quotes

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection.

From "Mathematics Is an Edifice, Not a Toolbox", by Hugo Rossi, Notices of the AMS, v. 43, no. 10, October 1996.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **September 1, 2015***, although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

3941. *Proposed by Dinu Bălceşti, Ovidiu Vâlcea, and Gabriel Romania.*

Prove that

$$\frac{x_1}{x_2+x_3+x_4+\cdots+x_n-x_1}+\frac{x_2}{x_1+x_3+x_4+\cdots+x_n-x_2}+\cdots$$
$$+\frac{x_n}{x_1+x_2+x_3+\cdots+x_{n-1}-x_n}\geq\frac{n}{n-2},$$

where $x_i \in \mathbb{R}^+$, $x_i \neq 0$, $x_i < x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n$, $i \in \{2, \ldots, n\}$, $n \in \mathbb{N}$, $n > 2$.

3942. *Proposed by Marcel Chiriţă.*

Consider a sequence $(x_n)_{n\geq 1}$ with $x_1 = 1$ and $x_{n+1} = \frac{1}{n+1}\left(x_n + \frac{1}{x_n}\right)$ for $n \geq 1$. Find $x_n\sqrt{n}$.

3943. *Proposed by Michel Bataille.*

Let a be a positive real number. Evaluate

$$\lim_{n\to\infty}\left(n\cdot\int_0^a\left(\frac{\cosh t}{\cosh a}\right)^{2n+1}dt\right).$$

3944. *Proposed by Bill Sands.*

With the elimination of the penny in Canada, purchase totals in stores are rounded off to the nearest multiple of 5 cents. For example, if you bought several items in a store with total price $9.97, you would only pay $9.95, but if your items totalled to $9.98 then you would pay $10. Suppose you go into a dollar store and want to buy 123 items worth 1 cent, 2 cents, 3 cents, . . ., $1.23. You are allowed to group the 123 items into any number of groups of any sizes, and each group would be a separate purchase. How could you group the 123 items so as to pay the smallest possible total amount ?

3945. *Proposed by J. Chris Fisher.*

Given circles (A) and (B) with centres A and B, and a circle (C) with centre C that meets (A) in points A_1 and A_2 that are not on (B), and meets (B) in points B_1 and B_2 that are not on (A), prove that the unique conic with foci A and B that is tangent to the perpendicular bisector of A_2B_2 is tangent also to the perpendicular bisector ℓ of A_1B_1.

3946. *Proposed by George Apostolopoulos.*

Prove that in any triangle ABC

a) $\dfrac{a^2}{w_bw_c} + \dfrac{b^2}{w_aw_c} + \dfrac{c^2}{w_aw_b} \geq 4,$

b) $\left(\dfrac{a}{w_bw_c}\right)^2 + \left(\dfrac{b}{w_aw_c}\right)^2 + \left(\dfrac{c}{w_aw_b}\right)^2 \geq \left(\dfrac{4}{3R}\right)^2,$

where R is the circumradius of ABC and w_a, w_b, w_c are the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.

3947. *Proposed by Michel Bataille.*

Let $A_1A_2A_3$ be a non-isosceles triangle and I its incenter. For $i = 1, 2, 3$, let D_i be the projection of I onto $A_{i+1}A_{i+2}$ and U_i, V_i be the respective projections of A_{i+1}, A_{i+2} onto the line IA_i (indices are taken modulo 3). Prove that

(a) $\dfrac{U_1D_1}{V_1D_1} \cdot \dfrac{U_2D_2}{V_2D_2} \cdot \dfrac{U_3D_3}{V_3D_3} = \dfrac{U_1D_2}{V_1D_3} \cdot \dfrac{U_2D_3}{V_2D_1} \cdot \dfrac{U_3D_1}{V_3D_2} = 1,$

(b) $\dfrac{[D_1U_1V_1]}{\sin^2 \frac{\alpha_2-\alpha_3}{2}} + \dfrac{[D_2U_2V_2]}{\sin^2 \frac{\alpha_3-\alpha_1}{2}} + \dfrac{[D_3U_3V_3]}{\sin^2 \frac{\alpha_1-\alpha_2}{2}} = [A_1A_2A_3]$, where α_i is the angle of $\Delta A_1A_2A_3$ at vertex A_i $(i = 1, 2, 3)$ and $[XYZ]$ denotes the area of ΔXYZ.

3948. *Proposed by George Apostolopoulos.*

Let $a_1, a_2, \ldots a_n$ be real numbers such that $a_1 > a_2 > \ldots > a_n$. Prove that

$$\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n} + a_1 - a_n \geq 2(n-1).$$

When does the equality hold?

3949. *Proposed by Arkady Alt.*

For any positive real a and b, find

$$\lim_{n\to\infty} \left((n+1)\left(\frac{a^{\frac{1}{n+1}} + b^{\frac{1}{n+1}}}{2}\right)^{n+1} - n\left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2}\right)^{n} \right).$$

3950. *Proposed by Cristinel Mortici.*

Let $A \subset \{1, 2, 3, ..., n\}$ be a $\left(\left[\frac{n}{3}\right] + 2\right)$-element set, not containing two consecutive numbers, where $[\cdot]$ denotes the greatest integer function. Prove that there exist elements $x < y < z$ of A such that either (x, y, z) is an arithmetic progression, or $(x, y, z - 1)$ is an arithmetic progression.

. . . .

3941. *Proposé par Dinu Bălceşti, Ovidiu Vâlcea, and Gabriel Romania.*

Démontrer que $\forall n \in \mathbb{N}$, $n > 2$

$$\frac{x_1}{x_2 + x_3 + x_4 + \cdots + x_n - x_1} + \frac{x_2}{x_1 + x_3 + x_4 + \cdots + x_n - x_2} + \dots + \frac{x_n}{x_1 + x_2 + x_3 + \cdots + x_{n-1} - x_n} \geq \frac{n}{n-2},$$

où $x_i \in \mathbb{R}^+$, $x_i \neq 0$, $x_i < x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n$, $i \in \{2, \ldots, n\}$.

3942. *Proposé par Marcel Chiriţă.*

Soit la suite $(x_n)_{n\geq 1}$, où $x_1 = 1$ et $x_{n+1} = \frac{1}{n+1}\left(x_n + \frac{1}{x_n}\right)$ lorsque $n \geq 1$. Déterminer $x_n\sqrt{n}$.

3943. *Proposé par Michel Bataille.*

Soit a un nombre réel strictement positif. Évaluer

$$\lim_{n\to\infty}\left(n \cdot \int_0^a \left(\frac{\cosh t}{\cosh a}\right)^{2n+1} dt\right).$$

3944. *Proposé par Bill Sands.*

Depuis l'élimination de la pièce de monnaie de 1 cent au Canada, la facture totale d'un achat est arrondie aux 5 cents près lorsqu'on paie comptant. Par exemple, si la facture totale d'un achat est de 9,97 \$, on paie 9,95 \$ comptant, mais si la facture totale est de 9,98 \$, on paie 10 \$ comptant. On suppose que l'on veut acheter 123 items dans un magasin à un dollar et que ces items coûtent respectivement 1 cent, 2 cents, 3 cents, . . . , 1,23 \$. Il est permis de regrouper ces items en n'importe quel nombre de groupes contenant chacun n'importe quel nombre d'items de manière que chaque groupe soit un achat distinct. Comment peut-on regrouper les 123 items de manière à payer le moins possible ?

3945. *Proposé par J. Chris Fisher.*

On considère deux cercles (A) et (B) de centres A et B, un cercle (C) de centre (C) qui coupe (A) aux points A_1 et A_2 qui ne sont pas sur (B), et qui coupe (B) aux points B_1 et B_2 qui ne sont pas sur (A). Démontrer que la seule conique ayant pour foyers A et B et qui est tangente à la médiatrice de A_2B_2 est aussi tangente à la médiatrice ℓ de A_1B_1.

3946. *Proposé par George Aposolopoulos.*

Soit un triangle ABC, a, b et c les longueurs respectives des côtés opposés aux angles A, B et C et w_a, w_b et w_c les longueurs des bissectrices respectives de ces angles à l'intérieur du triangle. Démontrer que

a) $\dfrac{a^2}{w_bw_c} + \dfrac{b^2}{w_aw_c} + \dfrac{c^2}{w_aw_b} \geq 4$ et

b) $\left(\dfrac{a}{w_bw_c}\right)^2 + \left(\dfrac{b}{w_aw_c}\right)^2 + \left(\dfrac{c}{w_aw_b}\right)^2 \geq \left(\dfrac{4}{3R}\right)^2$,

R étant le rayon du cercle circonscrit au triangle ABC.

3947. *Proposé par Michel Bataille.*

Soit $A_1A_2A_3$ un triangle non isocèle et I le centre du cercle inscrit dans le triangle. Soit D_i le projeté de I sur $A_{i+1}A_{i+2}$ et U_i, V_i le projeté de A_{i+1}, A_{i+2} sur la droite IA_i $(i = 1, 2, 3)$ (on considère les indices modulo 3). Démontrer que

(a) $\dfrac{U_1D_1}{V_1D_1} \cdot \dfrac{U_2D_2}{V_2D_2} \cdot \dfrac{U_3D_3}{V_3D_3} = \dfrac{U_1D_2}{V_1D_3} \cdot \dfrac{U_2D_3}{V_2D_1} \cdot \dfrac{U_3D_1}{V_3D_2} = 1$ et

(b) $\dfrac{[D_1U_1V_1]}{\sin^2 \frac{\alpha_2-\alpha_3}{2}} + \dfrac{[D_2U_2V_2]}{\sin^2 \frac{\alpha_3-\alpha_1}{2}} + \dfrac{[D_3U_3V_3]}{\sin^2 \frac{\alpha_1-\alpha_2}{2}} = [A_1A_2A_3]$, α_i étant l'angle du triangle $A_1A_2A_3$ au sommet A_i $(i = 1, 2, 3)$ et $[XYZ]$ étant l'aire du triangle XYZ.

3948. *Proposé par George Aposolopoulos.*

Soit $a_1, a_2, \ldots a_n$ des nombres réels tels que $a_1 > a_2 > \ldots > a_n$. Démontrer que

$$\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n} + a_1 - a_n \geq 2(n-1).$$

Quand y a-t-il égalité ?

3949. *Proposé par Arkady Alt.*

Déterminer

$$\lim_{n\to\infty} \left((n+1) \left(\frac{a^{\frac{1}{n+1}} + b^{\frac{1}{n+1}}}{2} \right)^{n+1} - n \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2} \right)^n \right),$$

a et b étant des réels strictement positifs.

3950. *Proposé par Cristinel Mortici.*

Soit A un sous-ensemble de l'ensemble $\{1, 2, 3, ..., n\}$ contenant $\left[\frac{n}{3}\right] + 2$ éléments et ne contenant pas deux entiers consécutifs où $[\cdot]$ est la fonction partie entière. Démontrer qu'il existe des éléments x, y et z de A, $x < y < z$, tels que (x, y, z) ou $(x, y, z - 1)$ soit une suite arithmétique.

How to measure heights ?

Suppose you are charged with a task to measure the height of a building given a long rope and a barometer of known dimensions. Below we describe 8 methods of solving this problem.

Method 1 (trivial). Climb to the top of the building, tie barometer to the end of the rope, slowly drop it down until it hits the ground. Measure the rope.

Method 2 (direct). Climb the side of the building (or take the inside stairs) with barometer at hand, measuring the length of the device along the wall. Count the number of barometer lengths used.

Method 3 (aerostatic). Measure the air pressure at the street level and on the roof of the building. Use the difference in the two measures to calculate the height.

Method 4 (geometrical). Wait for a sunny day, place the barometer at the street level and measure the length of its shadow. Measure the length of the building's shadow. Use similar triangles to figure out the building's height.

Method 5 (sociological). Poll all the building's residents and take the average of their estimates of the building's height. Offer the barometer as the prize for the best guess.

Method 6 (kinematic). Knowing your heart rate, use your pulse to estimate how long it takes barometer to hit the ground when dropped from the roof of the building. Use the formula $h = gt^2/2$ to calculate the building's height.

Method 7 (bureaucratic). Find the architecture company that built this structure and ask for its measurements. Barometer might no longer be available if you are doing these experiments in order.

Method 8 (pedagogical). Maybe our readers will offer another method ? Assume the rope and the barometer are intact.

Translated and adapted from the note by M. Tulchinskyi, Kvant, 1976 (12).

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge Edmund Swylan for the late correct solution to the problem 3761.

3841. *Proposed by Marcel Chiriţă.*

Let ABC be a triangle with $a = BC$, $b = CA$, $c = AB$, $\angle A \leq \angle B \leq \angle C$ and $a^2 + b^2 = 2Rc$, where R is the circumradius of ABC. Determine the measure of $\angle C$.

Many correct and three incorrect solutions were received.

Solution 1, by Cristóbal Sánchez-Rubio.

Let O be the circumcircle of triangle ABC. Let A' be the other end of the diameter through A and let B' be a point such that $AB' = \sqrt{a^2+b^2}$. Then

$$a^2 + b^2 = 2Rc \Leftrightarrow AB' = \sqrt{a^2+b^2} = \sqrt{2R \cdot c} \to c \leq AB' \leq 2R.$$

But

$$\begin{array}{rcl} C < 90^\circ & \Rightarrow & AB' > 2R > c \\ C > 90^\circ & \Rightarrow & AB' < c < 2R. \end{array}$$

Both are impossible, so $C = 90^\circ$.

Solution 2, by the proposer Marcel Chiriţă.

From $a^2 + b^2 = 2Rc$, we have $4R^2 \sin^2 A + 4R^2 \sin^2 B = 4R^2 \sin C$, so that

$$\sin^2 A + \sin^2 B = \sin C.$$

Rewriting gives successively

$$\begin{array}{rcl} \sin^2 A + \sin^2 B & = & \sin(A+B), \\ \sin^2 A + \sin^2 B & = & \sin A \cos B + \cos A \sin B, \\ \sin A(\sin A - \cos B) & = & \sin B(\cos A - \sin B). \end{array}$$

Consider the following cases, noting that $A \leq B < 90^\circ$.

- If $0 < \cos B < \sin A$, then $0 < \sin B < \cos A$, and $1 = \cos^2 B + \sin^2 B < \sin^2 A + \cos^2 A = 1$, a contradiction.
- If $0 < \sin A < \cos B$, then $0 < \cos A < \sin B$, and $1 = \sin^2 A + \cos^2 A < \cos^2 B + \sin^2 B = 1$, a contradiction.

- Otherwise, $\sin A = \cos B$ and $\cos A = \sin B$, implying that $A + B = 90^\circ$ and $C = 90^\circ$.

Solution 3, by Omran Kouba.

As in solution 2, we have $\sin^2 A + \sin^2 B = \sin C$, which can be rewritten as

$$\begin{aligned} 2\sin C &= 2 - \cos(2A) - \cos(2B) \\ &= 2 - 2\cos(A-B)\cos(A+B) \\ &= 2 + 2\cos(A-B)\cos C, \end{aligned}$$

or equivalently $\sin C = 1 + \cos(A - B)\cos C \geq 1$. Hence $C = \frac{\pi}{2}$.

3842. *Proposed by Jung In Lee.*

Let $d(n)$ be the number of positive divisors of n. For given positive integers a and b, there exist infinitely many positive integers m such that $d(a^m) \geq d(b^m)$; there also exist infinitely many positive integers n such that $d(a^n) \leq d(b^n)$. Prove that $d(a^k) = d(b^k)$ for any positive integer k.

We present the solution by Joseph DiMuro.

Let $a = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ and $b = q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s}$ be the prime factorizations of a and b. Then for any positive integer k :

$$d(a^k) = d(p_1^{ka_1} p_2^{ka_2} \cdots p_r^{ka_r}) = \prod_{i=1}^{r} (ka_i + 1), \text{ and}$$

$$d(b^k) = d(q_1^{kb_1} q_2^{kb_2} \cdots q_s^{kb_s}) = \prod_{j=1}^{s} (kb_j + 1).$$

Consider the polynomial

$$f(x) = \prod_{i=1}^{r} (a_i x + 1) - \prod_{j=1}^{s} (b_j x + 1).$$

For a positive integer x, $f(x) = d(a^x) - d(b^x)$. We were given that $f(x) \geq 0$ for infinitely many positive integer values of x, and $f(x) \leq 0$ for infinitely many positive integer values of x. Since $f(x)$ is continuous, this implies that $f(x)$ has infinitely many zeroes and is therefore the 0 polynomial. Thus, $f(x) = 0$ for all real values of x, so $d(a^k) = d(b^k)$ for any positive integer k.

3843. *Proposed by George Apostolopoulos.*

Let a, b be distinct real numbers such that

$$a^4 + b^4 - 3(a^2 + b^2) + 8 \leq 2(a + b)(2 - ab).$$

Find the value of the expression

$$A = (ab)^n + (ab+1)^n + (ab+2)^n,$$

where n is a positive integer.

We present the solution by Salem Malikić and Nermin Hodžić (done independently).

The given condition is equivalent to

$$2(a^4 + b^4 - 3(a^2+b^2) + 8 - 2(a+b)(2-ab)) \leq 0,$$

which may be rewritten as

$$(a-b)^2(a+b-1)^2 + (a^2+b^2+a+b-4)^2 \leq 0.$$

This implies that a and b must satisfy the two equations

$$a^2+b^2+a+b-4 = 0, \quad (a-b)(a+b-1) = 0.$$

Since a and b are distinct, the last equation implies that $a+b-1=0$, so that $a+b=1$, and the first equation becomes $a^2+b^2=3$. Then

$$ab = \frac{(a+b)^2 - (a^2+b^2)}{2} = -1,$$

so that

$$A = (ab)^n + (ab+1)^n + (ab+2)^n = (-1)^n + 1 = \begin{cases} 0 & n \text{ odd}, \\ 2 & n \text{ even}. \end{cases}$$

Editor's comments : There were two main solution types. The first was to rearrange inequalities and equations to solve for a and b, similar to the above; the second was to use calculus to find a and b (as they are critical points of the polynomial that is being set less than 0). Malikić and Hodžić used the first solution method, but instead of solving for a and b, they simply solved for ab, which saves some work.

3844. *Proposed by Michel Bataille.*

Find the intersection of the surface with equation

$$(x^2+y^2)^2 + (y^2+z^2)^2 + (z^2+x^2)^2 = (x+y)(y+z)(z+x)$$

with the plane $x+y+z=2$.

Among all the received solutions, one was incorrect. We present three solutions.

Solution 1, by Nermin Hodžić.

We have

$$\begin{aligned}
&(x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2 \\
&\qquad \geq^{(1)} \frac{(x^2+y^2+y^2+z^2+z^2+x^2)^2}{3} = \frac{4(x^2+y^2+z^2)^2}{3} \\
&\qquad \geq^{(2)} \frac{4(|x|+|y|+|z|)}{27} \\
&\qquad \geq^{(3)} \frac{8(|x|+|y|+|z|)^3}{27} \\
&\qquad \geq^{(4)} \frac{|x+y|+|y+z|+|z+x|)^3}{27} \\
&\qquad \geq^{(5)} |(x+y)(y+z)(z+x)| \\
&\qquad \geq (x+y)(y+z)(z+x),
\end{aligned}$$

with equality if and only if $x = y = z = \frac{2}{3}$. Here (1) follows from Cauchy's inequality applied to the vectors $< x^2+y^2, y^2+z^2, z^2+x^2 >$ and $< 1, 1, 1 >$; (2) is the AM-QM inequality; (3) follows from the hypothesis $x+y+z=2$, (4) follows from the triangle inequality, and (5) follows from the AM-GM inequality.

Solution 2, by Omran Kouba, modified slightly by the editor.

By the AM-QM inequality, $2(x^2+y^2) \geq (|x|+|y|)^2$, and by Muirhead's inequality, $a^4+b^4+c^4 \geq (a+b+c)abc$ for any nonnegative real numbers a, b, c, so that

$$\begin{aligned}
&4((x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2) \\
&\qquad \geq (|x|+|y|)^4+(|y|+|z|)^4+(|z|+|x|)^4 \\
&\qquad \geq 2(|x|+|y|+|z|)(|x|+|y|)(|y|+|z|)(|z|+|x|) \\
&\qquad \geq 2|x+y+z||x+y||y+z||z+x|,
\end{aligned}$$

with equality if and only if $x = y = z$. Hence for $x+y+z=2$, we see that

$$(x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2 \geq |(x+y)(y+z)(z+x)|,$$

with equality if and only if $x = y = z = \frac{2}{3}$. This proves that $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ is the unique point of intersection.

Solution 3, composite of similar solutions by Salem Malikić and Sĕfket Arslanagić.

At a point of intersection,

$$2[(x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2]-(x+y+z)(x+y+(y+z)(z+x)=0,$$

which is successively equivalent to

$$\begin{aligned}
4(x^4+y^4+z^4)+2(x^2y^2+y^2z^2+z^2x^2)-4(x^2yz+xy^2z+xyz^2)& \\
-(x^3y+xy^3+y^3z+yz^3+z^3x+zx^3)&=0.
\end{aligned}$$

Now,

$$\frac{(x^2-xy)^2+(x^2-xz)^2+(y^2-yx)^2+(y^2-yz)^2+(z^2-zx)^2+(z^2-zy)^2}{2}$$
$$+(x^2-yz)^2+(y^2-zx)^2+(z^2-xy)^2+(x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2$$
$$+(xy-yz)^2+(yz-zx)^2+(zx-xy)^2=0.$$

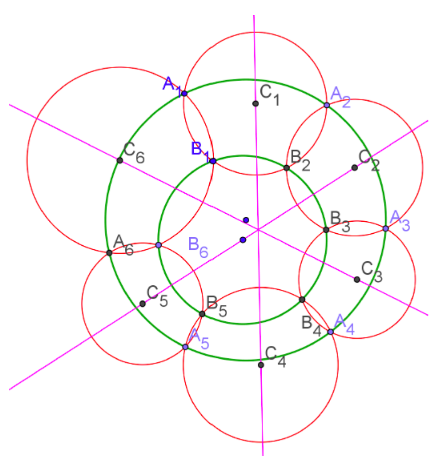
For this to hold, each term must equal 0, which in turn implies that $x=y=z=\frac{2}{3}$. Since $\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$ is readily verified to be a solution, it is the unique solution.

3845. *Proposed by Dao Thanh Oai.*

Let the six points $A_1, A_2, \ldots, A_6$ lie in that order on a circle, and the six points $B_1, B_2, \ldots, B_6$ lie in that order on another circle. If the quadruples $A_i, A_{i+1}, B_{i+1}, B_i$ lie on circles with centres C_i for $i = 1, 2, \ldots, 5$, then prove that A_6, A_1, B_1, B_6 must also lie on a circle. Furthermore, if C_6 is the centre of the new circle, then prove that lines C_1C_4, C_2C_5, and C_3C_6 are concurrent.

No solutions to this problem were received. We present a solution by a member of Editorial Board (J. Chris Fisher). We pose one part of the solution as problem 3945.

The order of the points on their respective circles is not important (except for making a nice picture) :



The first claim follows immediately from two applications of Miquel's theorem, which says that a chain of three circles will close with a fourth circle ; more precisely, in the statement of our problem restricted to the quadruples $A_i, A_{i+1}, B_{i+1}, B_i$ lying on circles with i running from 1 to 3, the theorem states that the points A_1, B_1, B_4, A_4 must lie on a circle. Now we have a second chain of three circles, namely the new circle $A_1B_1B_4A_4$ with the remaining two circles $A_4A_5B_5B_4$ and $A_5A_6B_6B_5$, whence (by Miquel's theorem) the remaining four points A_6, A_1, B_1, B_6 will also lie on a circle, as required. A similar argument will work for any chain having an even number of circles (with $2k$ points on both the A and the B circles, and i running from 1 to $2k-1$) : the points A_{2k}, A_1, B_1, B_{2k} must also lie on a circle.

For the second claim we will see that the sides of the hexagon $C_1C_2C_3C_4C_5C_6$ are tangent to a conic whose foci are the centres, call them A and B, of the circles A_i and B_i ; by Brianchon's theorem, the lines joining opposite vertices of the hexagon, namely C_1C_4, C_2C_5, and C_3C_6, must be concurrent. To this end, we wish to show that for each i, the unique conic with foci A and B that is tangent to the line C_iC_{i-1} (joining the centres of consecutive circles) coincides with the unique conic with those foci that is tangent to the line C_iC_{i+1}. Note that the conic will be an ellipse if the tangent C_iC_{i+1} misses the line segment AB ; it is a hyperbola if the tangent intersects AB between A and B. (To avoid the line passing through A or B we should insist that none of the A_i lie on the circle containing the B_i, and vice versa.) The second claim thereby reduces to a theorem that seems as if it should have been known a century ago, for which it seems to be easier to find a proof than a reference. The editor J. Chris Fisher now poses this proof as problem 3945, which appears in this issue of *Crux*.

3846. *Proposed by Arkady Alt.*

Let r be a positive real number. Prove that the inequality

$$\frac{1}{1+a+a^2}+\frac{1}{1+b+b^2}+\frac{1}{1+c+c^2}\geq\frac{3}{1+r+r^2}$$

holds for any positive a, b, c such that $abc=r^3$ if and only if $r\geq 1$.

We present the proof by the proposer, modified and expanded by the editor.

We first prove the following lemma :

Lemma. Let r be a given positive number. Then

$$\frac{1}{1+a+a^2}+\frac{1}{1+b+b^2}\geq\frac{2}{1+r+r^2} \tag{1}$$

for any positive a and b with $ab=r^2$ if and only if $r\geq r_0$, where r_0 is the unique positive root of the equation $4x^3+3x^2-3x-1=0$.

[Editor : Let $f(x)=4x^3+3x^2-3x-1$. Then $f(0)=-1<0$ and $f(1)=3>0$, so f has a real root $r_0\in(0,1)$. By Rule of signs, r_0 is the only positive root.]

Proof. Note that if (1) holds for any positive a and b with $ab=r^2$, then

$$\lim_{a\to\infty}\left(\frac{1}{1+a+a^2}+\frac{1}{1+b+b^2}\right)=1\geq\frac{2}{1+r+r^2}$$

if and only if $r^2+r-1\geq 0$, so $r\geq\dfrac{\sqrt{5}-1}{2}$.

Now, suppose $a, b > 0$ with $ab = r^2$, where $r \geq \dfrac{\sqrt{5}-1}{2}$. Let $x = a + b$, then $x \geq 2\sqrt{ab} = 2r$, and

$$\begin{aligned}
&\frac{1}{1+a+a^2} + \frac{1}{1+b+b^2} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{2+a+b+a^2+b^2}{1+a+b+ab+ab(a+b)+a^2+b^2+a^2b^2} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{2+x+x^2-2r^2}{1+x+r^2+(r^2)x+x^2-2(r^2)+r^4} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{x^2+x+2-2r^2}{x^2+r^2x+x-r^2+r^4+1} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{P(x)}{Q(x)},
\end{aligned}$$

where $Q(x) = (x^2+r^2x+x-r^2+r^4+1)(1+r+r^2)$ and by tedious computations together with synthetic division, we have :

$$\begin{aligned}
P(x) &= (x^2+x+2-2r^2)(1+r+r^2) - 2(x^2+r^2x+x-r^2+r^4+1) \\
&= (r^2+r-1)x^2 + (-r^2+r-1)x + 2(1-r^2)(r^2+r+1) - 2(1-r^2+r^4) \\
&= (r^2+r-1)x^2 - (r^2-r+1)x - 4r^4 - 2r^3 + 2r^2 + r \\
&= (x-2r)((r^2+r-1)x + 2r^3 + r^2 - r - 1).
\end{aligned}$$

Since $x \geq 2r$ and clearly $Q(x) > 0$, we have $\dfrac{P(x)}{Q(x)} \geq 0$ if and only if

$$(r^2+r-1)x + 2r^3 + r^2 - r - 1 \geq 0. \tag{2}$$

Since $x \geq 2r$ and $r^2+r-1 \geq 0$, (2) holds if and only if it holds for $x = 2r$; that is, $2r(r^2+r-1)+2r^3+r^2-r-1 \geq 0$ or $4r^3+3r^2-3r-1 \geq 0$ or $4r^2+3r-3-\frac{1}{r} \geq 0$.

The function $g(r) = 4r^2 + 3r - 3 - \frac{1}{r}$ is increasing on $(0, \infty)$ and $g(\frac{1}{2}) = -\frac{5}{2} < 0, g(\frac{3}{4}) = \frac{1}{6} > 0$, so it has only one root r_0 and $r_0 \in (\frac{1}{2}, \frac{3}{4})$. Hence, r_0 is the smallest value of r such that (2) holds for all $x \geq 2r$.

Furthermore, if we set $r_1 = \frac{\sqrt{5}-1}{2}$, then

$$4r_1^3+3r_1^2-3r_1-1 = 4r_1(r_1^2+r_1-1)-r^2+r_1-1+2(r_1-1) = 2(r_1-1) = \sqrt{5}-3 < 0,$$

so $r_1 < r_0 < \dfrac{3}{4}$.

In particular, (1) holds for all $a, b > 0$ such that $ab = r^2$ if $r \geq 1$ and this completes the proof of the lemma. ■

Using this lemma, we now prove that for all $a, b, c > 0$ with $abc = r^3$,

$$\frac{1}{1+a+a^2} + \frac{1}{1+b+b^2} + \frac{1}{1+c+c^2} \geq \frac{3}{1+r+r^2}$$

if and only if $r \geq 1$.

Necessity. Setting $c = \frac{r^3}{x^2}$ and $a = b = n$ in the given inequality where n is an arbitrary natural number, and passing to the limit, we have

$$1 = \lim_{n\to\infty}\left(\frac{2}{1+n+n^2} + \frac{1}{1+\frac{r^3}{n^2}+\frac{r^6}{n^4}}\right) \geq \frac{3}{1+r+r^2},$$

which implies $r^2 + r - 2 \geq 0$ or $(r+2)(r-1) \geq 0$, so $r \geq 1$.

Sufficiency. Let $a, b, c > 0$ with $abc = r^3, r \geq 1$. Without loss of generality, assume that $a \geq b \geq c$. Then $c^3 \geq abc = r^3$, so $c \geq r$. Set $x = \sqrt{ab}$. Then $c = \frac{r^3}{x^2}$ and $x \geq c$, so $x \geq r \geq 1$. Since $ab = x^2$, we have, by the lemma, that

$$\frac{1}{1+a+a^2} + \frac{1}{1+b+b^2} \geq \frac{2}{1+x+x^2}.$$

Hence, it suffices to prove that for any x and r with $x \geq r \geq 1$, we have

$$\frac{2}{1+x+x^2} + \frac{1}{1+\frac{r^3}{x^2}+\frac{r^6}{x^4}} \geq \frac{3}{1+r+r^2}. \tag{3}$$

Let

$$D(x) = \frac{2}{1+x+x^2} + \frac{1}{1+\frac{r^3}{x^2}+\frac{r^6}{x^4}} - \frac{3}{1+r+r^2}.$$

Then

$$D(x) = \frac{1}{1+\frac{r^3}{x^2}+\frac{r^6}{x^4}} - \frac{1}{1+r+r^2} - 2\left(\frac{1}{1+r+r^2} - \frac{1}{1+x+x^2}\right) \tag{4}$$

$$= \frac{r+r^2-\frac{r^3}{x^2}-\frac{r^6}{x^4}}{(1+\frac{r^3}{x^2}+\frac{r^6}{x^4})(1+r+r^2)} - \frac{x+x^2-r-r^2}{(1+r+r^2)(1+x+x^2)} = \frac{A(x)}{1+r+r^2}, \tag{5}$$

where

$$A(x) = \frac{x^2r\left(x^2-r^2\right)+r^2\left(x^4-r^4\right)}{x^4+x^2r^3+r^6} - \frac{2\left(1+x+r\right)\left(x-r\right)}{1+x+x^2} \tag{6}$$

$$= \frac{(x^2-r^2)(rx^2+r^2(x^2+r^2))(x^2+x+1) - 2(x+r+1)(x-r)(x^4+r^3x^2+r^6)}{(x^4+x^2r^3+r^6)(x^2+x+1)} \tag{7}$$

$$= \frac{(x-r)B(x)}{(x^4+x^2r^3+r^6)(x^2+x+1)}, \tag{8}$$

where

$$\begin{aligned}
B(x) &= (x+r)(x^2+x+1)(rx^2+r^2(x^2+r^2)) - 2(x+r+1)(x^4+r^3x^2+r^6) \\
&= (rx+r^2)(x^2+r(x^2+r^2))(x^2+x+1) - 2(x+r+1)(x^4+r^3x^2+r^6) \\
&= (r^2+r-2)x^5 + (r^3+2r^2-r-2)x^4 + (r^4-r^3+2r^2+r)x^3 \\
&\quad + (r^5-r^4-r^3+r^2)x^2 + (-2r^6+r^5+r^4)x - 2r^7 - 2r^6 + r^5 \\
&= (x-r)E(x),
\end{aligned}$$

where

$$E(x) = (r^2 + r - 2)x^4 + (2r^3 + 3r^2 - 3r - 2)x^3 + (3r^4 + 2r^3 - r^2 - r)x^2 \\ + (4r^5 + r^4 - 2r^3)x + (2r^6 + 2r^5 - r^4).$$

It then suffices to prove that $E(x) \geq 0$ for all $x \geq r$. Since

$$\begin{aligned} E(r) &= r^6 + r^5 - 2r^4 + 2r^6 + 3r^5 - 3r^4 - 2r^3 + 3r^6 + 2r^5 - r^4 - r^3 \\ &\quad + 4r^6 + r^5 - 2r^4 + 2r^6 - 2r^5 - r^4 \\ &= r^7 + 12r^6 + 8r^5 - 9r^4 - 4r^3 \\ &= r^7 + 3r^6 + 4r^5 + r^3(r-1)(9r+4)(r+1) \geq 0 \end{aligned}$$

for $r \geq 1$ and since all the coefficients of $E(x)$ are clearly nonnegative as well, we conclude that $E(x) \geq E(r) > 0$ for all $x \geq r$. Hence, $B(x) \geq 0$ and from (8) $A(x) \geq 0$ and finally from (5) $D(x) \geq 0$, which establishes (3) and completes the proof.

Editor's comment. Perfetti's solution was computer assisted and Pranesachar's solution used Maple.

3847. *Proposed by Jung In Lee.*

Prove that there are no distinct positive integers a, b, c and nonnegative integer k that satisfy the conditions

$$a^{b+k} \mid b^{a+k}, \quad b^{c+k} \mid c^{b+k}, \quad c^{a+k} \mid a^{c+k}.$$

We present the solution by Joseph DiMuro.

We prove the stronger result that there are no distinct positive integers a, b, c and nonnegative real number k that satisfy the conditions

$$a^{b+k} \leq b^{a+k}, \;\; b^{c+k} \leq c^{b+k}, \;\; c^{a+k} \leq a^{c+k}. \tag{1}$$

Suppose (1) holds. Then from $a^{b+k} \leq b^{a+k}$, we have

$$\ln(a^{b+k}) \leq \ln(b^{a+k}) \;\text{ or }\; (b+k)\ln a \leq (a+k)\ln b,$$

so

$$\frac{\ln a}{a+k} \leq \frac{\ln b}{b+k}.$$

Similarly, from the other inequalities in (1), we deduce that

$$\frac{\ln b}{b+k} \leq \frac{\ln c}{c+k} \;\text{ and }\; \frac{\ln c}{c+k} \leq \frac{\ln a}{a+k}.$$

Therefore, we have

$$\frac{\ln a}{a+k} = \frac{\ln b}{b+k} = \frac{\ln c}{c+k}.$$

Now, let $f_k(x) = \frac{\ln x}{x+k}$, $x \in (0, \infty)$. Then $f_k(x)$ is a continuous function such that

$$f_k(a) = f_k(b) = f_k(c) \qquad (2)$$

and it suffices to show that (2) cannot hold. Since

$$f_k'(x) = \frac{\frac{1}{x}(x+k) - \ln x}{(x+k)^2} = \frac{1 + \frac{k}{x} - \ln x}{(x+k)^2},$$

we have $f_k'(x) = 0$ if and only if $1 + \frac{k}{x} - \ln x = 0$. Let $g(x) = 1 + \frac{k}{x} - \ln x$. Then $g'(x) = -\frac{k}{x^2} - \frac{1}{x} < 0$, so $g(x)$ is a strictly decreasing function. Since

$$\lim_{x\to 0^+} g(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = -\infty,$$

we see that $g(x) = 0$ for exactly one value of x, so $f_k(x)$ has exactly one critical value. Hence, (2) cannot hold and our proof is complete.

3848. *Proposed by Rudolf Fritsch.*

We define an altitude of the plane $(2n+1)$-gon $A_0A_1 \dots A_{2n}$ to be the line through vertex A_i perpendicular to the *opposite side* $A_{i-n}A_{i+n}$ (where indices are reduced modulo $2n+1$). Prove that if $2n$ of the altitudes are concurrent, then the remaining altitude passes through the point of concurrence.

We present a composite of the similar solutions by Oliver Geupel and by Günter Pickert.

For an arbitrary set of $2n+1$ vectors $\vec{A}_i$, whose indices $-n \le i \le n$ are considered to be integers modulo $2n+1$ (or, more precisely, to be the elements of the factor group $\mathbb{Z}_{2n+1} = \mathbb{Z}/(2n+1)\mathbb{Z}$), one has

$$\sum_{-n}^{n} \vec{A}_i \cdot \vec{A}_{i-n} = \sum_{-n}^{n} \vec{A}_{j+n} \cdot \vec{A}_j = \sum_{-n}^{n} \vec{A}_i \cdot \vec{A}_{i+n},$$

and thus

$$\sum_{-n}^{n} \vec{A}_i \cdot (\vec{A}_{i+n} - \vec{A}_{i-n}) = 0. \qquad (1)$$

We now turn to the problem. Assume that the altitudes from $A_1,\ A_2,\ \dots,\ A_{2n}$ are concurrent in the point O and consider position vectors relative to the origin O. We are given that for $-n \le i \le n,\ i \ne 0$, the line A_iO is perpendicular to the side $A_{i+n}A_{i-n}$, which is expressed by the equation

$$\vec{A}_i \cdot \left(\vec{A}_{i+n} - \vec{A}_{i-n}\right) = 0.$$

That is, $2n$ of the summands in the sum (1) vanish, and we conclude that the remaining summand must also vanish, namely

$$\vec{A}_0 \cdot \left(\vec{A}_{-n} - \vec{A}_n\right) = 0.$$

Thus the line OA_0 is perpendicular to the side $A_{-n}A_n$, which proves that the altitude from the vertex A_0 also passes through the point O.

Editor's comments. The proposer observed that the result holds for a quite general definition of polygon : the proof makes clear that its vertices can be any ordered set of $2n+1$ points, not necessarily distinct, that satisfy the hypothesis.

Professor Pickert died on February 11, 2015, a few months before his 98th birthday. He was active until the end, as can be seen in the above solution and in the three-part article (co-authored by Rudolf Fritsch) that appeared last year in Volume 39 of *Crux*.

3849. *Proposed by José Luis Díaz-Barrero.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients having all its zeros in the disc $\mathcal{C} = \{z \in \mathbb{C} : |z| \le \sqrt{6}\}$. Show that

$$|A(3z)| \ge \left(\frac{2}{3}\right)^{n/2} |A(2z)|$$

for any complex number z with $|z| = 1$.

We present a mixture of the solutions by Omran Kouba and Alexander Mangerel.

We prove a stronger statement than the given problem, namely the above problem with $\frac{2}{3}$ replaced with $\frac{3}{2}$. To do this, we use the following claim.

Claim : For $w \in \mathbb{C}$, $|3z-w| \ge \sqrt{\frac{3}{2}}\,|2z-w|$ for any $z \in \mathbb{C}$ with $|z| = 1$ iff $|w| \le \sqrt{6}$.

To see this, square both sides of the first inequality, rewrite using conjugates, and expand to obtain

$$9 - 3(\bar{z}w + z\bar{x}) + |w|^2 \ge \frac{3}{2}(4 - 2(\bar{z}w + z\bar{x}) + |w|^2).$$

Upon simplification, this yields $|w|^2 \le 6$, and taking square roots proves the claim, as we are dealing with positive quantities whenever square roots are taken.

The proof of the main statement follows by writing $A(z) = a_n \prod_{i=1}^{n} (z - w_i)$ where w_i are the roots of A, letting $|z| = 1$, and taking absolute values, using the fact that each root of A has $|w| \le \sqrt{6}$:

$$|A(3z)| = |a_n| \prod_{i=1}^{n} |3z - w_i| \ge |a_n| \prod_{i=1}^{n} \sqrt{\frac{3}{2}}\,|2z - w_i| = \left(\frac{3}{2}\right)^{\frac{n}{2}} |A(2z)|.$$

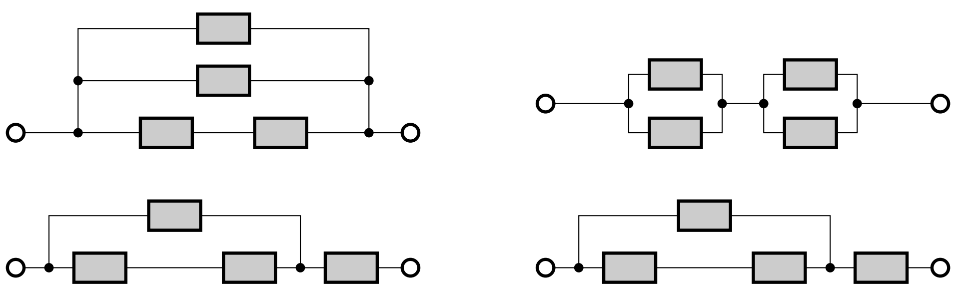
Editor's comments. Mangerel noted that he believed that the intent of the problem was to have the fraction flipped, and proved the tighter bound via slightly different calculations. The specific problem may be abstracted to the following :

> Let $a \geq b \in \mathbb{R}$, and let $A(z)$ be a polynomial with complex coefficients such that all of its roots are in the contained in the disk with radius $\sqrt{ab}$. Show that
>
> $$|A(az)| \geq \left(\frac{a}{b}\right)^{\frac{n}{2}} |A(bz)|$$
>
> for any complex z with $|z| = 1$.

The proof is the same as above when $a \neq b$, with a replacing 3 and b replacing 2, and the case where $a = b$ is obviously trivial.

3850. *Proposed by Lee Sallows and Stan Wagon.*

Each of the four networks shown uses the same four distinct integer-valued resistors a, b, c, d, and the total resistances of the networks themselves are again a, b, c, d. Find values of a, b, c, d that work.



We present a summary of the solutions by Richard Hess and the proposer.

Richard Hess found the solution $(a, b, c, d) = (k, 2k, 3k, 4k)$ which for the four resistors gives

$$\begin{aligned}
R_1 &= \frac{1}{\frac{1}{2k} + \frac{1}{4k} + \frac{1}{k+3k}} = k, \\
R_2 &= \frac{1}{\frac{1}{2k} + \frac{1}{3k}} + \frac{1}{\frac{1}{k} + \frac{1}{4k}} = 2k, \\
R_3 &= \frac{1}{\frac{1}{4k} + \frac{1}{k+3k}} + 2k = 4k, \\
R_4 &= \frac{1}{\frac{1}{3k} + \frac{1}{2k+4k}} + k = 3k.
\end{aligned}$$

The proposers used an exhaustive computer search which confirms that this is the only solution.

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Arkady Alt, San Jose, CA, USA : 3949
George Apostolopoulos, Messolonghi, Greece : 3946, 3948
Michel Bataille, Rouen, France : 3943, 3947
Marcel Chiriţă, Bucharest, Romainia : 3942
Dinu Bălceşti, Ovidiu Vâlcea, and Gabriel Romania, Bălceşti, Vălcea, Romania : 3941
J. Chris Fisher, Cleveland Heights, Ohio : 3945
Cristinel Mortici, Valahia University of Târgovişte, Romania : 3950
Bill Sands, University of Calgary, Calgary, AB : 3944

Solvers - individuals

Arkady Alt, San Jose, CA, USA : **OC121**, OC122, 3841, **3846**
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : **CC75**
George Apostolopoulos, Messolonghi, Greece : **CC75**, 3841, 3843
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : **CC73**, **CC75**, **OC121**, **3844**
Michel Bataille, Rouen, France : **CC75**, OC121, OC122, 3841, 3842, 3844, 3848, 3849
Roy Barbara, Lebanese University, Fanar, Lebanon : 3841
Marcel Chiritiţă, Bucharest, Romania : **3841**
Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India : 3841
José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain : 3849
Joseph DiMuro, Biola University, La Mirada, CA, USA : **3842**, 3843, **3847**
Marian Dincă, Bucharest, Romania : OC121
Rudolf Fritsch, Ludwig-Maxmilians-Universitat Munchen, Munich, Germany : 3848
Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany : CC71
Oliver Geupel, Brühl, NRW, Germany : OC121, OC122, OC124, OC125, 3847, **3848**
Richard I. Hess, Rancho Palos Verdes, CA, USA : CC73, CC74, CC75, **3850**
John G. Heuver, Grande Prairie, AB : OC121
Nermin Hodžić, Dobošnica, Bosnia and Herzegovina : **3843**, **3844**
Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria : **OC122**, **3841**, 3843, **3844**, 3846, **3849**
Jung In Lee, Seoul Science High School, Republic of Korea : 3842, 3847
Salem Malikić, student, Simon Fraser University, Burnaby, BC : **3843**, **3844**
David E. Manes, SUNY at Oneonta, Oneonta, NY, USA : CC74, **CC75**, **OC121**
Alexander Mangerel, University of Toronto, Toronto, ON : 3847, **3849**
Somasundaram Muralidharan, Chennai, India : CC73
Dao Thanh Oai, Kien Xuong, Thai Binh, Viet Nam : **3845**
Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy : **OC121**, 3846
Günter Pickert, Justus-Liebig-Universitat Giessen, Giessen, Germany : **3848**
C.R. Pranesachar, Indian Institute of Science, Bangalore, India : 3841, 3843, 3846, 3848
Henry Ricardo, Tappan, NY, USA : OC121
Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain : **3841**

Petros Soulidis, High School "Panagia Prousiotissa", Agrinio, Greece : **CC71**
Digby Smith, Mount Royal University, Calgary, AB : 3844
George Tsapakidis, Agrinio, Greece : 3843
Edmund Swylan, Riga, Latvia : 3761
Daniel Văcaru, Piteşti, Romania : CC71, OC121, OC122
Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA : CC72, CC75

Solvers - collaborations

Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA : 3844
John Hawkins and David R. Stone, Georgia Southern University, Statesboro, USA : 3841
Lee Sallows and Stan Wagon, Macalester College, St. Paul, MN, USA : **3850**
Titu Zvonaru and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania : **CC72**, CC73, CC74, CC75, OC121, OC122, 3841, 3843, 3844