# *Crux Mathematicorum*

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## *Editorial Board*



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# Crux Mathematicorum

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# Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

# EDITORIAL

I routinely get asked what my favourite number is. And even though I am sure the person I'm talking to doesn't expect anything but a superficial answer, I find it hard to pick one – there are so many interesting numbers to choose from! Here are some that have made my short list today:

- 26, because it is Rubik's cube's number, meaning that every cube can be solved in at most 26 moves;
- 246, because it is currently the best proven prime gap bound, that is, we know that there are infinitely many pairs of consecutive primes of distance at most 246 apart;
- 42, because it is the "answer to The Ultimate Question of Life, the Universe, and Everything" and I've just finished reading the *The Hitchhiker's Guide to the Galaxy.*

We all have different tastes: in food, numbers and problems. At ***Crux*** , we strive to publish a variety of materials to satisfy every reader, so you always have something to pick and choose from. But our journal is build *by* our audience, not just around it. So I encourage you to send us both your solutions as well as problem proposals, articles and book reviews.

Happy reading!

Kseniya Garaschuk

# THE CONTEST CORNER

No. 23

Robert Bilinski

*The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to* `crux-contest@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

**Submissions of solutions.** *Each solution should be contained in a separate file named using the convention* LastName_FirstName_CCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

*To facilitate their consideration, solutions should be received by the editor by* **1 July 2015***, although late solutions will also be considered until a solution is published.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

**CC111**. Find all positive integers with two or more digits such that if we insert a 0 between the units and tens digits we get a multiple of the original number.

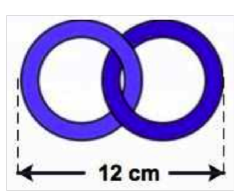
**CC112**. Jerome groups odd numbers in groups that contain successive quantities of odd number of elements such as:

$$\{1\}, \{3, 5\}, \{7, 9, 11\}, \{13, 15, 17, 19\}, \ldots$$

What is the sum of the 100th grouping?

**CC113**. If $I$ is a point inside $ABCD$ with $PA = 2$, $PB = 3$, $PC = 5$ and $PD = 6$, what is the maximum possible area of $ABCD$?

**CC114**. A chain with two links is 12 cm long. A chain with five links is 27 cm long. What is the length, in cm, of a chain with 40 links?



**CC115**. Mathias has put together 120 identical unit cubes to form a rectangular prism and painted all six sides of it. There are 24 unpainted cubes left when the prism is undone. What is the surface area of the prism?

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**CC111**. Trouver tous les entiers positifs qui ont 2 chiffres ou plus tels que si on insère un 0 entre les unités et les dizaines, on obtient un multiple du nombre original.

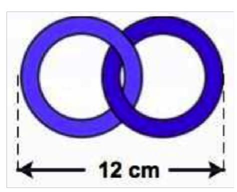
**CC112**. Jérôme groupe les nombres impairs dans des groupes contenant des quantités successives de nombres impaires comme

$$\{1\}, \{3, 5\}, \{7, 9, 11\}, \{13, 15, 17, 19\}, \ldots$$

Quelle est la somme du 100ième groupe ?

**CC113**. Si $I$ est un point intérieur à $ABCD$ tel que $PA = 2$, $PB = 3$, $PC = 5$ et $PD = 6$, quel est l'aire maximale possible pour $ABCD$ ?

**CC114**. Une chaîne avec 2 chaînons mesure 12 cm de long. Une chaîne avec 5 chaînons est 27 cm. Quelle est la longueur en cm d'une chaîne avec 40 chaînons ?




**CC115**. Mathias a mis 120 cubes unitaires ensemble pour former un prisme rectangulaire. Il en peint les 6 faces et remarque qu'il reste 24 cubes vierges quand il défait son prisme. Quelle est la surface totale du prisme ?

# CONTEST CORNER SOLUTIONS

**CC61**. We place 3 green, 4 yellow and 5 red balls in a bag. Two balls of different colours are selected at random, removed, and replaced with two balls of the third colour. Show that it is impossible for all of the remaining balls to be the same colour, no matter how many times this process is repeated.

*Originally problem 4c in 2004 Hypatia Contest.*

*Solved by R. Hess; I. Leifer; D. Lowry-Duda; H. Wang; and M. Wu. We present Han Wang's solution below.*

Consider the numbers of balls modulo 3. Then the position where all remaining balls are the same colour is congruent to $(0, 0, 0) \pmod 3$, and the starting position is congruent to $(0, 1, 2) \pmod 3$. Notice that both subtracting 1 and adding 2 balls are congruent to adding 2, when considered $\pmod 3$.

So regardless of whether we add two balls to the red, the green, or the blue coloured balls, the next position is congruent to $(2, 0, 1)$. After another move (regardless of distinguished colour), the position is congruent to $(1, 2, 0)$. After another move, the position is congruent to $(0, 1, 2)$, which is the same as the start!

So we will never reach a position congruent to $(0, 0, 0)$, and so it is impossible to get all balls to be the same colour.

**CC62**. For each real number $x$, let $[x]$ be the largest integer less than or equal to $x$. For example, $[5] = 5$, $[7.9] = 7$ and $[-2.4] = -3$. An arithmetic progression of length $k$ is a sequence $a_1, a_2, \ldots, a_k$ with the property that there exists a real number $b$ such that $a_{i+1} - a_i = b$ for each $1 \le i \le k-1$. Let $\alpha > 2$ be a given irrational number. Then $S = \{[n\alpha] : n \in \mathbb{Z}\}$ is the set of all integers equal to $[n\alpha]$ for some integer $n$. Prove that for any integer $m \ge 3$, there exist $m$ distinct numbers contained in $S$ which form an arithmetic progression of length $m$.

*Originally from 2013 Canadian Open Mathematics Challenge, problem C4a.*

*No solutions were received for this problem.*

**CC63**. A quadrilateral circumscribes a circle. Prove that the perimeter of the quadrilateral bears the same ratio to the perimeter of the circle as the area of the quadrilateral bears to the area of the circle.

*Originally 1976 Descartes Contest, problem 4.*

*Solved by S. Arslanagić; N. Evgenidis; R. Hess; E. H. Pilehrood; N. Stanciu and T. Zvonaru. We give a solution based on all submitted solutions.*

Let $ABCD$ be the quadrilateral circumscribing the circle centred at $O$; let $AB = a, BC = b, CD = c$ and $AD = d$. Let $P_c$ and $P_q$ be the respective perimeters of the circle and quadrilateral, and let $A_c$ and $A_q$ be the respective areas of the circle and quadrilateral.

We have $P_q = a + b + c + d$, where $a, b, c$, and $d$ are the side-lengths of the quadrilateral, and $P_c = 2\pi r$, where $r$ is the radius of the circle.

The area of the quadrilateral is the sum of the areas of the four triangles $ABO$, $BCO$, $CDO$, $ADO$, all of which have height $r$, since the radii are at right angles to the sides of the quadrilateral. Thus

$$\begin{aligned} A_q &= \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} + \frac{dr}{2} \\ &= \frac{1}{2}(a + b + c + d)r. \end{aligned}$$

Since the area of the circle is $A_c = \pi r^2$, if follows that

$$\frac{P_q}{P_c} = \frac{a + b + c + d}{2\pi r} = \frac{(a + b + c + d)r}{2\pi r^2} = \frac{A_q}{A_c}.$$

**CC64**. Show that a power of 2 can never be the sum of $k$ consecutive positive integers, $k > 1$.

*Originally problem 9 from 1978 Descartes Contest.*

*Solved by M. Bataille ; M. Coiculescu ; I. D. Gerganov ; R. Hess ; D. Manes ; E. H. Pilehrood ; V. Văcaru ; E. Wang ; T. Zvonaru and N. Stanciu. We present the solution by Matei Coiculescu.*

Suppose there exist positive integers $k > 1, n,$ and $m$ such that

$$2^n = m + (m + 1) + \cdots + (m + k - 1) = mk + \frac{k(k-1)}{2}$$

This gives

$$2^{n+1} = k(2m + k - 1)$$

If $k$ is even, then $(2m + k - 1)$ is odd and greater than 1, which is not possible as $2^{n+1}$ has only even factors. It is also not possible for $k$ to be odd for the same reason. This is a contradiction, hence a power of 2 can never be the sum of $k > 1$ positive integers.

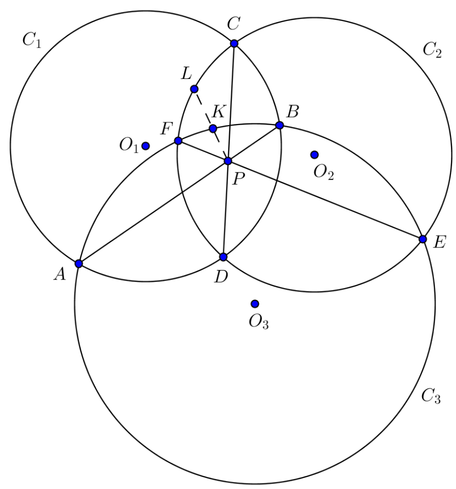
**CC65**. Suppose that three circles in the plane are located so that each pair of circles intersect in two points, thereby giving a common chord to those two circles. Prove that these three chords pass through one point.

*Originally question 8 from 2004 APICS Math Competition.*

*Solved by G. Tsapakidis ; and N. Stanciu and T. Zvonaru. We present a solution based on the one by George Tsapakidis.*

Let $C_1, C_2, C_3$ be three circles such that $C_1$ intersects $C_3$ at points $A$ and $B$ ; $C_1$ intersects $C_2$ at points $C$ and $D$ ; and, $C_2$ intersects $C_3$ at points $E$ and $F$. Let $P$ be the intersection point of the chords $AB$ and $CD$. If $E = P$, then we are done, so suppose $E \neq P$. By the Theorem of Intersecting Chords, we have

$$|AP| \cdot |PB| = |CP| \cdot |PD|. \tag{1}$$




Suppose that a line containing the segment $EP$ does not contain the chord $EF$, so that it intersects circles $C_2$ and $C_3$ at the points $L$ and $K$, respectively, where $F$, $K$ and $L$ are all distinct. Now $EK$ and $AB$ are intersecting chords of $C_3$, while $EL$ and $CD$ are intersecting chords of $C_2$. By the Theorem of Intersecting Chords, we now have

$$|EP| \cdot |PK| = |AP| \cdot |PB|, \text{ and} \tag{2}$$
$$|EP| \cdot |PL| = |CP| \cdot |PD|. \tag{3}$$

Combining Equations (1–3), we obtain $|EP| \cdot |PL| = |EP| \cdot |PK|$. We have $|EP| > 0$, since $E \neq P$, therefore $|PK| = |PL|$, and since the segments $PK$ and $PL$ lie on the same line, we must have $K = L$. This is a contradiction, so it must be the case that the chord $EF$ also passes through $P$.

*Editor's comments.* Victor Pambuccian has noted that this problem appears in Hilbert's lectures on the foundations of geometry, going back to 1896. It is true under significantly weaker assumptions than the axiom system for the standard Euclidean plane. For specifics, see V. Pambuccian's review of *David Hilbert's lectures on the foundations of geometry*, 1891–1902, in Philos. Math. (3) 21 (2013), no. 2, 255–277.

There is a subtle oversight in the wording of the original problem ; the chords need not intersect at all! No counterexample was submitted, perhaps due to the power of suggestion, so we state a very natural modification of the problem :

> Suppose that three circles in the plane are located so that each pair of circles intersect in two points, thereby giving a common chord to those two circles. Prove that if two of these chords intersect, then the third chord also intersects them at the same point.

Thus, we avoid the cases where all three chords are contained in mutually parallel secant lines, or where the secants intersect outside the three circles.

## Math Quotes

The first nonabsolute number is the number of people for whom the table is reserved. This will vary during the course of the first three telephone calls to the restaurant, and then bear no apparent relation to the number of people who actually turn up, or to the number of people who subsequently join them after the show/match/party/gig, or to the number of people who leave when they see who else has turned up.

The second nonabsolute number is the given time of arrival, which is now known to be one of the most bizarre of mathematical concepts, a recipriversexcluson, a number whose existence can only be defined as being anything other than itself. In other words, the given time of arrival is the one moment of time at which it is impossible that any member of the party will arrive. Recipriversexclusons now play a vital part in many branches of math, including statistics and accountancy and also form the basic equations used to engineer the Somebody Else's Problem field.

The third and most mysterious piece of nonabsoluteness of all lies in the relationship between the number of items on the bill, the cost of each item, the number of people at the table and what they are each prepared to pay for. (The number of people who have actually brought any money is only a subphenomenon of this field.)

*From "Life, the Universe and Everything." New York : Harmony Books, 1982 by Adams, Douglas.*

# THE OLYMPIAD CORNER

## No. 321

Nicolae Strungaru

*The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to* `crux-olympiad@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

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*The editor thanks Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, for translations of the problems.*

**OC171**. Find all 3-digit numbers for which the ratio between the number and the sum of its digits is minimal.

**OC172**. Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

**OC173**. Each integer is coloured with one of two colours, red or blue. It is known that, for every finite set $A$ of consecutive integers, the absolute value of the difference between the number of red and blue integers in the set $A$ is at most 1000. Prove that there exists a set of 2000 consecutive integers in which there are exactly 1000 red numbers and 1000 blue numbers.

**OC174**. Suppose that $a$ and $b$ are two distinct positive real numbers with the property that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for all positive integers $n$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. Prove that $a$ and $b$ are integers.

**OC175**. Suppose $O$ is the center of the circumcircle of triangle $ABC$. Let $P$ be the midpoint of the arc $\widehat{BAC}$ and $QP$ be a diameter. Let $I$ be the incentre of the triangle $ABC$ and let $D$ be the intersection of $PI$ and $BC$. The circumcircle of $\triangle AID$ and the extension of $PA$ meet at $F$. Let $E$ be a point on the line segment $PD$ such that $DE = DQ$. Let $r, R$ be the radius of the inscribed circle and circumcircle of $\triangle ABC$, respectively.

If $\angle AEF = \angle APE$, prove that

$$\sin^2(\angle BAC) = \frac{2r}{R}.$$

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**OC171**. Détermine tous les nombres à 3 décimales tels que le ratio entre le nombre et la somme de ses chiffres est minimal.

**OC172**. Déterminer tous les polynômes $P(x)$ à coefficients réels tels que

$$(x+1)P(x-1) - (x-1)P(x)$$

est un polynôme constant.

**OC173**. On colore les entiers, chacun ayant l'une des couleurs rouge ou bleu. Or on sait que, pour tout ensemble fini $A$ consistant d'entiers consécutifs, la différence en valeur absolue entre les nombres d'entiers rouges et bleus dans $A$ est au plus 1000. Démontrer qu'il existe un ensemble de 2000 entiers consécutifs qui contient exactement 1000 entiers rouges et 1000 entiers bleus.

**OC174**. Supposons que $a$ et $b$ sont des nombres réels positifs distincts tels que $\lfloor na \rfloor$ divise $\lfloor nb \rfloor$ pour tout entier positif $n$, oú $\lfloor x \rfloor$ désigne le plus grand entier inférieur ou égal à $x$. Démontrer que $a$ et $b$ sont entiers.

**OC175**. Supposons que $O$ est le centre du cercle circonscrit du triangle $ABC$ ; soit $P$ le point milieu de l'arc $\widehat{BAC}$ et soit $QP$ un diamètre. Soit $I$ le centre du cercle inscrit du triangle $ABC$ et soit $D$ l'intersection de $PI$ et $BC$. Le cercle circonscrit du triangle $AID$ et le prolongement de $PA$ se rencontrent au point $F$. Soit $E$ un point sur le segment $PD$ tel que $DE = DQ$. Soient $r$ et $R$ les rayons des cercles inscrit et circonscrit du triangle $ABC$ respectivement.

Si $\angle AEF = \angle APE$, démontrer que

$$\sin^2(\angle BAC) = \frac{2r}{R}.$$

# OLYMPIAD SOLUTIONS

**OC111**. Let $x, y$ and $z$ be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

*Originally question 1 from the 2012 Canadian Mathematical Olympiad.*

*Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; R. Hess; D. Manes; P. Perfetti; and T. Zvonaru. We give the solution used by most of the solvers.*

$$\begin{aligned} x^2 + xy^2 + xyz^2 - 4xyz + 4 &= x^2 - 4x + 4 + 4x - 4xy + xy^2 + 2xy + xyz^2 - 4xyz \\ &= (x-2)^2 + x(y-2)^2 + xy(z-2)^2 \geq 0. \end{aligned}$$

*Editor's comment.* Zvonaru remarked that we only used $x, y \geq 0$, but $z$ could be any real number, which is obvious from the solution.

**OC112**. Find all pairs of natural numbers $(a, b)$ that are not relatively prime ($\gcd(a, b) \neq 1$) such that

$$\gcd(a, b) + 9\,\mathrm{lcm}[a, b] + 9(a + b) = 7ab\,.$$

*Originally question 4 from the Albanian team selection test, 2012.*

*Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; and K. Zelator. We give the solution similar to the solutions of Šefket Arslanagić and Konstantine Zelator (done independently).*

We claim that the only solutions are $(4, 38), (38, 4)$ and $(4, 4)$.

We will use the well known equality that $\gcd(a, b) \cdot \mathrm{lcm}[a, b] = a \cdot b$. The equation then becomes

$$\gcd(a, b) + 9\frac{ab}{\gcd(a, b)} + 9(a + b) = 7ab\,.$$

Let $d := \gcd(a, b)$ and write $a = dx, b = dy$, with $\gcd(x, y) = 1$. Then

$$\begin{aligned} d + \frac{9d^2xy}{d} + 9d(x + y) = 7d^2xy &\iff 1 + 9xy + 9(x + y) = 7dxy \\ &\iff d = \frac{1 + 9xy + 9(x + y)}{7xy}. \end{aligned}$$

Let us now observe that if $d \geq 5$ we have

$$\frac{1 + 9xy + 9(x + y)}{7xy} \geq 5 \iff 26xy \leq 1 + 9x + 9y.$$

But this is not possible, since $9x \leq 9xy$ and $9y \leq 9xy$ which would imply $8xy \leq 1$.

This shows that $d \leq 4$. Moreover, as $9xy > 7xy$ we have $d > 1$. This shows that $d \in \{2, 3, 4\}$.

If $d = 2$ we get,

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 2 \iff 5xy = 1 + 9x + 9y.$$

Therefore,

$$y = \frac{9x + 1}{5x - 9} = 2 - \frac{x - 19}{5x - 9}\,.$$

It follows that $5x - 9$ divides $x - 19$ and hence it also divides

$$(5x - 9) - 5(x - 19) = 86\,.$$

This shows that $5x - 9 \in \{\pm 1, \pm 2, \pm 43, \pm 86\}$. As $x$ is an integer, the only possibilities are

$$5x - 9 = 1 \implies x = 2$$

and

$$5x - 9 = 86 \implies x = 19\,.$$

This leads to $(a, b) = (4, 38)$ and $(a, b) = (38, 4)$.

If $d = 3$, we get

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 3 \iff 12xy - 9x - 9y = 1.$$

But this is not possible, as the left hand side is divisible by 3, and the right hand side is not. Therefore, there is no solution in this case.

If $d = 4$, we get

$$\frac{1 + 9xy + 9(x + y)}{7xy} = 4 \iff 19xy - 9x - 9y = 1 \iff y = \frac{9x + 1}{19x - 9}.$$

Therefore, as $x, y \geq 1$ we have $19x - 9 \geq 0$ and

$$9x + 1 \geq 19x - 9 \Rightarrow x \leq 1\,.$$

This shows that $x = 1$, and then $y = 1$. In this case we get $(a, b) = (4, 4)$.

**OC113**. Prove that among any $n$ vertices of a regular $(2n - 1)$-gon we can find 3 which form an isosceles triangle.

*Originally question 4 from Day 1 of China Western Mathematical Olympiad 2012.*

*Solved by O. Geupel; and D. Văcaru. We give the solution of Oliver Geupel.*

Our proof is by contradiction. Suppose that $\mathcal{V}$ is a set of $n$ vertices of a regular $(2n-1)-$gon $P_1P_2\ldots P_{2n-1}$ that does not allow to form an isosceles triangle with vertices in $\mathcal{V}$. We consider the cases $n=3$ and $n\geq 4$ in succession.

First consider the case $n=3$.

There is no loss of generality in assuming that $P_1\in\mathcal{V}$. Then, either $P_2$ or $P_5$ is not a member of $\mathcal{V}$, because otherwise we would have an isosceles triangle. Also, either $P_3$ or $P_4$ is not a member of $\mathcal{V}$. On the other hand, two points out of $P_2$, $P_3$, $P_4$, and $P_5$ must be members of $\mathcal{V}$.

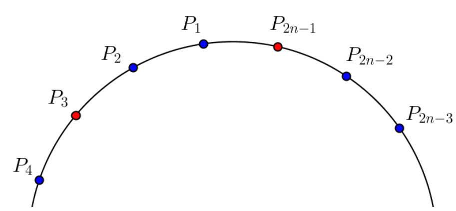
Hence, each of the pairs $\{P_2,P_5\}$ and $\{P_3,P_4\}$ contains exactly one member of $\mathcal{V}$. Without loss of generality let $P_2\in\mathcal{V}$. But both $P_1P_2P_3$ and $P_1P_2P_4$ are isosceles triangles, a contradiction, which completes the case $n=3$.

It remains to consider the case $n\geq 4$.

Again suppose that $P_1\in\mathcal{V}$.

Then, each of the $n-1$ pairs $\{P_k,P_{2n+1-k}\}$ where $2\leq k\leq n$, contains exactly one member of $\mathcal{V}$. Specifically, one out of the points $P_2$ and $P_{2n-1}$ is in $\mathcal{V}$; say $P_2\in\mathcal{V}$ and $P_{2n-1}\notin\mathcal{V}$.

Then $P_3\notin\mathcal{V}$, because $P_1P_2P_3$ is an isosceles triangle.




Thus, $P_{2n-2}\in\mathcal{V}$. Since $P_2\in\mathcal{V}$, one out of $P_4$ and $P_{2n-1}$ must be in $\mathcal{V}$.

But we saw that $P_{2n-1}\notin\mathcal{V}$. Therefore, $P_4$ is a member of $\mathcal{V}$. Since $P_{2n-2}\in\mathcal{V}$, one out of $P_{2n-3}$ and $P_{2n-1}$ must be in $\mathcal{V}$.

But we saw that $P_{2n-1}\notin\mathcal{V}$. Therefore, $P_{2n-3}$ is a member of $\mathcal{V}$. Now we have obtained the isosceles triangle $P_1P_4P_{2n-3}$ in $\mathcal{V}$, a contradiction which completes the proof.

**OC114**. Let $ABC$ be a scalene triangle. Its incircle touches $BC,AC,AB$ at $D,E,F$ respectively. Let $L,M,N$ be the symmetric points of $D,E,F$ with respect to $EF$, $FD$, respectively $DE$. The line $AL$ intersects $BC$ at $P$, the line $BM$ intersects $CA$ at $Q$, and the line $CN$ intersects $AB$ at $R$. Prove that $P,Q,R$ are collinear.

*Originally question 3 from China Team Selection Test 1, 2012.*

*No solutions were received for this problem.*

**OC115**. Find the smallest positive integer $n$ for which there exists a positive integer $k$ such that the last 2012 decimal digits of $n^k$ are all 1's.

*Originally question 4 from Brazil National Olympiad 2012.*

*Solved by Richard Hess. We provide his solution modified by the editor.*

We claim that the smallest such integer is $n = 71$.

First let us observe that for $n^k$ to end in 1, the last digit of $n$ can only be $1, 3, 7$ or 9. We claim the the last digit must be 1.

Indeed, if the last digit is 3 or 7, as the order of those elements modulo 10 is four, $k$ must be a multiple of 4. Then $n^k - 1$ is divisible by $n^4 - 1$. As $n$ is odd, then $n^2 - 1$ is divisible by 4 and therefore so is $n^4 - 1$. Moreover, by Fermat Little Theorem, $n^4 - 1$ is divisible by 5.

This shows that 20 divides $n^4 - 1$, and hence it also divides $n^k - 1$. But this implies that the second last digit of $n^k$ is even, which is a contradiction.

Same way, if the last digit is $n$ is 9, it follows that $k$ is even. Then $n^k - 1$ is divisible by $n^2 - 1 = (n-1)(n+1)$. As the last digit of $n$ is 9 we have $n + 1$ is divisible by 10 and $n - 1$ is even. This shows again that $n^2 - 1$, and therefore $n^k - 1$ is divisible by 20. Exactly as above, this is not possible.

This shows that the last digit of $n$ is one. We can then write

$$n = 10m + 1\,.$$

As $10m|n-1|n^k-1$, we get again $m|\frac{n^k-1}{10} = 111..1$ and hence $m$ is odd.

As $n^k \equiv 3 \pmod 4$ it also follows that $k$ must be odd. Thus, since $n^2 \equiv 1 \pmod 8$, we have

$$7 \equiv n^k \equiv n \pmod 8\,.$$

This shows that $n \equiv 7 \pmod 8$, which shows that $n \neq 11, 51$. To complete the proof, we need to show that $n \neq 31$ and that 71 works.

Assume by contradiction that $n = 31$.

As

$$31^k \equiv 111 \equiv 31^7 \pmod{1000}\,,$$

we get that $k - 7$ is divisible by the order of 31 modulo 1000, which is 50. Therefore $k = 50l + 7$, and then

$$31^k - 111 = 31^k - 31^7 + 31^7 = 31^7(31^{k-7} - 1) + 31^7 - 111.$$

Now, $31^{k-7} - 1$ is divisible by $31^{50} - 1$, and

$$31^{50} \equiv 1 \pmod{125} \quad \text{and} \quad 31^{50} \equiv 1 \pmod{16}\,.$$

Therefore $31^{50}-1$ is divisible by 2000. A short computation, shows that $31^7-111$ is also divisible by 2000. Hence,

$$1000 \equiv 31^k - 111 \equiv 0 \pmod{2000} ,$$

which is a contradiction. This proves that $n = 31$ cannot work.

We must now show that $n = 71$ works.

Let us start by observing that

$$71^{13} \equiv 31111 \left(\bmod\ 10^5\right) \quad \text{and} \quad 71^{250} \equiv 30001 \left(\bmod\ 10^5\right).$$

We claim that

$$71^{10^a} \equiv 3 \cdot 10^{a+3} + 1 \left(\bmod\ 10^{a+4}\right) .$$

We prove this by induction. For $a = 1$, it is obvious. Next we show that it being true for $a$ implies it is true for $a+1$. We have :

$$71^{25\times 10^a} = 3 \times 10^{a+3} + 1 + b(10^{a+4}) .$$

Then

$$\begin{aligned} 71^{25\times 10^{a+1}} &\equiv \left(3\times 10^{a+3}+1+b(10^{a+4})\right)^{10} \left(\bmod\ 10^{a+5}\right) \\ &\equiv 1 + 10\cdot\left(3\times 10^{a+3}+b(10^{a+4})\right) \\ &\qquad\qquad + \binom{10}{2}\cdot\left(3\times 10^{a+3}+b(10^{a+4})\right)^2 + ... \left(\bmod\ 10^{a+5}\right) \\ &\equiv 1 + 3\times 10^{a+4} \left(\bmod\ 10^{a+5}\right) \end{aligned}$$

This completes the induction.

Finally, we can build recursively a sequence $r_t$ such that

$$71^{r_t} \equiv \underbrace{111..1}_{\text{t times}} \left(\bmod\ 10^t\right) .$$

Indeed, we can pick $r_5 = 13$ and then, since

$$71^{r_t} \equiv \underbrace{111..1}_{\text{t times}} \left(\bmod\ 10^t\right),$$

we get that

$$71^{r_t} - \underbrace{111..1}_{\text{t times}} = 10^t \cdot x.$$

Since

$$71^{10^{t-3}} \equiv 3 \cdot 10^t + 1 \left(\bmod\ 10^{t+1}\right) ,$$

it follows from the binomial theorem that

$$71^{u\cdot 10^{t-3}} \equiv 3u \cdot 10^t + 1 \pmod{10^{t+1}} .$$

and hence

$$71^{r_t+u\cdot 10^{t-3}} \equiv (3u+x)\cdot 10^t + \underbrace{111..1}_{\text{t times}} \pmod{10^{t+1}} .$$

Therefore, if $3u + x \equiv 1 \pmod{10}$ and we define

$$r_{t+1} = r_t + u \cdot 10^{t-3} ,$$

we get

$$71^{r_{t+1}} \equiv \underbrace{111..1}_{\text{t+1 times}} \pmod{10^{t+1}} .$$

# BOOK REVIEWS

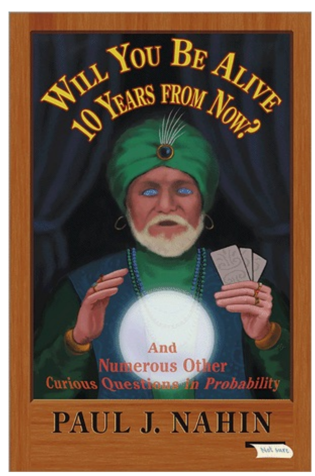
Robert Bilinski

*Will You Be Alive 10 Years from Now ? And Numerous Other Curious Questions in Probability* by Paul J. Nahin
ISBN 978-0-691-15680-4, hardcover, 220+xxvi pages
Published by Princeton University Press, 2014, $27.95 (US)

*Reviewed by* ***Ed Barbeau****, University of Toronto, Toronto, ON*

Many of us are intrigued and challenged by probability questions. During the half millennium they have been around, they have often been the source of controversy among some of the greatest mathematical thinkers. Even today, it is difficult to find agreement on some probabilistic conundrums.

This is evident in the problems presented by Paul Nahin who zealously and zestfully invites us to savour their mysteries. He taught probability theory and its applications to electrical engineers at the University of New Hampshire and the University of Virginia for about three decades, and during this time honed his skill at exposing the mathematics in a clear and comely way. This is his third book on the topic, all published by Princeton. *Digital Dice : Computational Solutions to Practical Probability Problems* appeared in 2008, while *Dueling Idiots and Other Probability Puzzlers* came out in 2012. While he writes with a light touch, this book is not for the casual reader. Background in algebra and combinatorics is required along with experience in following mathematical arguments. It is a useful source of examples for anyone teaching an elementary probability course at the tertiary level. While it advertises itself as "a collection of not so well-known mathematical mind-benders", many readers will probably encounter something familiar.




Even in the preface, Nahin engages the reader with problems. Here is one : *100 passengers each with a seat assignment board a plane. However, the first to board takes a seat at random. The other more docile passengers board in turn, taking their assigned seats if available, but taking a vacant seat at random if occupied. What is the probability that the last passenger to board gets the seat assigned ?* Elsewhere, Marilyn Vos Savant, a columnist in the American *Parade* magazine gets chided for getting the wrong end of the stick on another probability question.

The preface is followed by an introduction with nine "classic problems from the past" such as gambler's ruin. Some of the classics are not old and not quite probability questions. The last of these deals with three separate situations that illustrate Simpson's paradox, a radio-direction finding problem and the "spaghetti" problem. Before we get to the essays proper, there is another river for the reader to cross in the form of twelve challenge problems "for you to think about as you read".

There are twenty-five essays, each focussing on a particular question. The problem is posed, sometimes with an anecdote or description of its origin followed by a theoretical analysis and a description and implementation of a computer simulation. While the author used MATLAB, he claims to have written the computer codes "in such a low-level way that you should have little trouble converting any of them to your favourite language". This simulation is an important part of the book. Because it is not always easy to formulate a probability problem mathematically, a Monte Carlo experiment provides a useful check ; in particular, it forces you to clarify exactly what is being randomized and come to grips with the underlying probability distribution. Two appendices provide solutions to the challenge problems and a technical note on MATLAB's random number generator.

Most of the problems are treated within a half dozen pages. For example, one was suggested by a colleague who assigned 20 students at random into five tutorial groups and discovered that he had put a pair of twins in the same group ; what is the chance of this happening ? The author describes his own analysis leading to the answer 3/19, and then gives a more elegant solution provided by a reviewer. Ten million simulations provided the result 0.1579092, very close to the theoretical value. Other problems deal with medical tests, gambling, the dying out of a chain letter, random checks to find undocumented immigrants, and sporting results. The elegant derivation of the probability that a candidate winning an election actually led all the way through the counting of the ballots is one that also appears in Sherman Stein's *How the Other Half Thinks : Adventures in Mathematical Reasoning.* The final chapter treats the prisoner's dilemma and Newcomb's paradox.

One of the problems has to do with the number of typographical errors that remain in a manuscript after it has been proofread on two separate occasions. For this book, the probability that the proofreaders have rooted out all slips is less than 1. On page 104, after the first equals sign in the displayed equation for $E(\mathbf{T})$, there is $1-$ that should not be there. On pages 128 and 129, the diagrams need a straightforward correction. I was not completely comfortable with the mathematics on page 170 in Chapter 23 treating the question that gave the book its title. Letting $p(x)$ be the probability that a person alive *now* (not necessarily at birth) will be alive $x$ years from now and $\phi(x)$ be the life expectancy *then* (after $x$ years), Nahin puts the overall life expectancy at $p(x)\phi(x)$. Then he says that "but we know that the life expectancy at age $x$ is $\phi(x)$, and so $p(x)\phi(x) = \phi(x)$". I was confused as to exactly what $\phi(x)$ was. He wriggles out of the conclusion that $p(x) = 1$ by recasting the situation as an integral equation, but the resulting mathematics still needs a critical eye.

However, the rest of the mathematical analysis in the book is surefooted, and I found it both enjoyable and enlightening. I am happy to recommend it.

By the way, the answer to the airplane seating problem is 1/2.

# FOCUS ON...

No. 11

Michel Bataille

## The Partial Sums of some Divergent Series

### Introduction

Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \omega > 1$$

(possibly $\omega = \infty$). As the ratio test shows, the series $\sum a_n$ is divergent, the sequence of its partial sums $A_n = \sum_{k=1}^{n} a_k$ satisfying $\lim_{n\to\infty} A_n = \infty$. The sequence $(a_n)$ also satisfies $\lim_{n\to\infty} a_n = \infty$. The proof is straightforward : pick a real number $\rho$ in the interval $(1, \omega)$ and fix a positive integer $N$ such that $\frac{a_{n+1}}{a_n} > \rho$ for all integers $n \geq N$. Then for $n \geq N$ we have

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \cdot \frac{a_{N+2}}{a_{N+1}} \cdot \cdots \cdot \frac{a_n}{a_{n-1}} \geq \rho^{n-N}$$

so that $a_n \geq \rho^n \cdot \frac{a_N}{\rho^N}$ and $\lim_{n\to\infty} a_n = \infty$ follows (since $\lim_{n\to\infty} \rho^n = \infty$).

Thus, the sequences $(A_n)$ and $(a_n)$ have the same (infinite) limit as $n \to \infty$. Investigating further, we will establish a few results about the behaviour of the sequences $(A_n)$ and $\left(\frac{A_{n+1}}{A_n}\right)$ in comparison with the sequences $(a_n)$ and $\left(\frac{a_{n+1}}{a_n}\right)$ and offer some applications.

### Three results

Keeping the above notations, we prove the following :

(i) if $\omega$ is a real number, then $\quad \lim_{n\to\infty} \frac{A_n}{a_n} = \frac{\omega}{\omega - 1} \quad$ and $\quad \lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \omega.$

(ii) if $\omega = \infty$, then $\quad \lim_{n\to\infty} \frac{A_n}{a_n} = 1 \quad$ and $\quad \lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \infty.$

Thus, the sequences $\left(\frac{a_{n+1}}{a_n}\right)$ and $\left(\frac{A_{n+1}}{A_n}\right)$ have the same limit. More can be said in that direction as the following striking result shows :

(iii) if $\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n} - n\alpha\right) = \beta$ where $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$, then $\lim_{n\to\infty} \left(\frac{A_{n+1}}{A_n} - n\alpha\right) = \beta$ as well.

The main tool in the proofs below is the powerful Stolz-Cesàro theorem (shortened to SCT) which appears more and more often in solutions (see for example problems **3470** [2009 : 396,399 ; 2010 : 412], **3764** [2012 : 285,286 ; 2013 : 330]). Various forms of this theorem can be found, but I particularly like the following version : let $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ be two sequences of real numbers with $v_n > 0$ for all $n$. Let $U_n = \sum_{k=1}^{n} u_k$ and $V_n = \sum_{k=1}^{n} v_k$. If the series $\sum v_n$ is divergent and $\lim_{n\to\infty} \frac{u_n}{v_n} = \omega$ (finite or not), then $\lim_{n\to\infty} \frac{U_n}{V_n} = \omega$. (For a proof of SCT, see for example [1].)

*Proofs.* (i) Let $a_0 = 0$ and $b_n = a_n - a_{n-1}$ $(n \geq 1)$. Note that $B_n = \sum_{k=1}^{n} b_k = a_n$. In addition, we have

$$\frac{b_n}{a_n} = 1 - \frac{a_{n-1}}{a_n} \qquad (1)$$

for all positive integers $n$, hence

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 1 - \frac{1}{\omega} = \frac{\omega - 1}{\omega}.$$

From SCT, we also have $\lim_{n\to\infty} \frac{B_n}{A_n} = \lim_{n\to\infty} \frac{a_n}{A_n} = \frac{\omega-1}{\omega}$ and the result follows.

The fact that $\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \omega$ directly follows from $\frac{A_{n+1}}{A_n} = \frac{A_{n+1}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n} \cdot \frac{a_n}{A_n}$.

(ii) The proof is similar, but this time we have $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$ (from (1) since $\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = 0$) and so $\lim_{n\to\infty} \frac{a_n}{A_n} = 1$. We deduce (as above) $\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \infty$.

(iii) Note that we are now considering a particular case of (ii). Let $c_0 = 0$ and $c_n = A_{n+1} - n\alpha A_n$ for $n \geq 1$. For $n \geq 2$, a short calculation gives $c_n - c_{n-1} = a_{n+1} - n\alpha a_n - \alpha A_{n-1}$ so that

$$\frac{c_n - c_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} - n\alpha - \alpha \cdot \frac{A_n - a_n}{a_n} = \left(\frac{a_{n+1}}{a_n} - n\alpha\right) - \alpha\left(\frac{A_n}{a_n} - 1\right).$$

It follows that $\lim_{n\to\infty} \frac{c_n - c_{n-1}}{a_n} = \beta - \alpha \cdot 0 = \beta$. From SCT, we obtain

$$\beta = \lim_{n\to\infty} \frac{c_n}{A_n} = \lim_{n\to\infty} \frac{A_{n+1} - n\alpha A_n}{A_n} = \lim_{n\to\infty} \left(\frac{A_{n+1}}{A_n} - n\alpha\right),$$

as desired.

## Applications

A very simple application of (i) is obtained by taking $a_n = \frac{x^n}{n}$ where $x$ is a real number such that $x > 1$. In that case $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$, hence $\lim_{n\to\infty} \frac{A_n}{a_n} = \frac{x}{x-1}$. Unexpectedly, we can resort to this result to give a solution to **2683** [2001 : 462 ; 2002 : 539 ) completely different from Li Zhou's featured one. This problem is asking for

$$\lim_{n\to\infty} \left( \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n+1}{2k+1} \binom{n+1}{2k+1} \right).$$

Let $S_n = \sum\limits_{k=0}^{\lfloor n/2\rfloor} \frac{1}{2k+1}\binom{n+1}{2k+1}$. Using the binomial theorem, it is easy to see that

$$2S_n = \int_0^1 \frac{(1+x)^{n+1} - (1-x)^{n+1}}{x}\,dx = \int_0^2 (1 + u + \cdots + u^n)\,du.$$

For the second equality, note that

$$\int_0^1 \frac{(1+x)^{n+1} - 1}{x}\,dx = \int_1^2 \frac{u^{n+1} - 1}{u-1}\,du \quad \text{and}$$
$$\int_0^1 \frac{(1-x)^{n+1} - 1}{x}\,dx = -\int_0^1 \frac{u^{n+1} - 1}{u-1}\,du.$$

Thus, $2S_n = \sum\limits_{k=1}^{n+1} \frac{2^k}{k}$ and from (i),

$$\lim_{n\to\infty} \frac{2S_n}{\frac{2^{n+1}}{n+1}} = \frac{2}{2-1} = 2.$$

It is quickly deduced that the desired limit is 2.

Our second example is from a more recent problem : Let $p$ be a positive real number. Find

$$\lim_{n\to\infty} \frac{1 + \sqrt{2}(2!)^p + \sqrt[3]{3}(3!)^p + \cdots + \sqrt[n]{n}(n!)^p}{(n!)^p}.$$

[This is, slightly extended, problem **3469** [2009 : 396,398 ; 2010 : 411].]

Observing that $\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} \exp\left(\frac{\ln(n)}{n}\right) = 1$, we see that

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{n+1}((n+1)!)^p}{\sqrt[n]{n}(n!)^p} = \lim_{n\to\infty} (n+1)^p = \infty.$$

From (ii), we have

$$\lim_{n\to\infty} \frac{1 + \sqrt{2}(2!)^p + \sqrt[3]{3}(3!)^p + \cdots + \sqrt[n]{n}(n!)^p}{\sqrt[n]{n}(n!)^p} = 1,$$

and it follows that the required limit is 1.

As an application of (iii), we find

$$\lim_{n\to\infty} \left( \frac{\sum_{j=1}^{n+2} j^j}{\sum_{j=1}^{n+1} j^j} - \frac{\sum_{j=1}^{n+1} j^j}{\sum_{j=1}^{n} j^j} \right).$$

[This is problem 821 of *The College Mathematics Journal* proposed in Vol. 37, March 2006.]

Let $a_n = n^n$ and $A_n = \sum\limits_{j=1}^{n} j^j = \sum_{j=1}^{n} a_j$. We calculate

$$\begin{aligned}\frac{a_{n+1}}{a_n} = (n+1)\left(1+\frac{1}{n}\right)^n &= n\left(1+\frac{1}{n}\right)\cdot\exp\left(n\ln\left(1+\frac{1}{n}\right)\right)\\ &= ne\left(1+\frac{1}{n}\right)\cdot\exp\left(n\ln\left(1+\frac{1}{n}\right)-1\right)\\ &= ne\left(1+\frac{1}{n}\right)\cdot\exp\left(-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)\\ &= ne\left(1+\frac{1}{n}\right)\cdot\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right) = ne+\frac{e}{2}+o(1)\end{aligned}$$

as $n \to \infty$. From the result above, we deduce that $\frac{A_{n+1}}{A_n} = ne + \frac{e}{2} + o(1)$ and so

$$\frac{A_{n+2}}{A_{n+1}} - \frac{A_{n+1}}{A_n} = (n+1)e + \frac{e}{2} + o(1) - (ne + \frac{e}{2} + o(1)) = e + o(1).$$

Thus, the required limit is $e$.

**Exercises**

In conclusion, we propose two easy exercises in the same vein as the applications above :

1. Find $\rho, \alpha$ and $\ell > 0$ such that

$$\lim_{n\to\infty} \rho^n n^\alpha \sum_{k=1}^{n} \frac{5^n}{n\binom{2n-1}{n}} = \ell.$$

2. Find $\alpha$ for the following sequence to be convergent

$$\left(\frac{\sum\limits_{k=1}^{n+1} k!\csc(\pi/2^k)}{\sum\limits_{k=1}^{n} k!\csc(\pi/2^k)} - n\alpha\right)_{n\geq 1}.$$

What is its limit in that case ?

# References

[1] G. Nagy, *The Stolz-Cesàro Theorem* available at `www.math.ksu.edu/~nagy/snippets/stolz-cesaro.pdf`

# Apples, Oranges, and Bananas

Warut Suksompong

**Puzzle 1** : Some apples and oranges are distributed among 99 boxes. Prove that we can choose 50 boxes so that together they contain at least half of all the apples and at least half of all the oranges.

This puzzle was given in the All-Russian Mathematical Olympiad in 2005. We invite the reader to try this puzzle before moving on.

First, if there are only apples, and we would like to choose 50 boxes so that together they contain at least half of the apples, that's easy. We just choose the 50 boxes with the greatest number of apples. Also, we can see that the bound 50 is tight. If every box contains one apple, and we are only allowed to choose 49 boxes, any effort will be fruitless – we will always get less than half of the apples no matter what we do.

But we have apples and oranges. What are we going to do? We want to choose a subset with 50 boxes that satisfy two (symmetric) properties simultaneously. If we can show that the number of subsets that satisfy one property is greater than half of the number of all subsets of size 50, we would be done – some subset satisfies both properties at once. What we are guessing is stronger than what the puzzle asks for, so we are making a gamble. Nevertheless, the statement is true. To show that, we will make a little set-theoretic detour.

Recall that given a set of size $2n+1$, the number of subsets of size $n$ is equal to the number of subsets of size $n+1$. That means we can pair up the subsets of size $n$ with the subsets of size $n+1$. What is more interesting is that we can do it in a rather special way : Each subset of size $n$ is a subset of its pair of size $n+1$.

**Lemma 1** *Let $N = \{1, 2, \ldots, n\}$ and $0 \leq k < n/2$. Let $A$ be the collection of all subsets of $N$ of size $k$, and $B$ be the collection of all subsets of $N$ of size $k+1$. Construct a bipartite graph $G$ with vertices $A \cup B$ in such a way that there is an edge joining $X \in A$ and $Y \in B$ if and only if $X \subseteq Y$. Then there exists a matching between all the elements of $A$ and (not necessarily all) the elements of $B$.*

This Lemma is given as an exercise without solution in [**3**]. The following proof is based on the hint for the exercise.

*Proof.* We define a matching $f$. For any set $X \in A$ (which has size $k$), let $f(X)$ be a subset of $N$ of size $k+1$ that contains $X$; the single element that is added to $f(X)$ is determined in the following way :

Suppose $X = \{a_1, a_2, \ldots, a_k\}$, where $a_1 < a_2 < \ldots < a_k$, and define $a_0 = 0$. Let $m$ be the largest among the indices $0 \leq i \leq k$ for which $2i - a_i$ is maximized. Then $f(X) = X \cup \{a_m + 1\}$.

First, we show that $f(X)$ is well-defined. To that end, we prove that $a_m$ cannot be the largest element, $n$. Indeed, if that were the case, then $2m - a_m = 2m - n \leq 2k - n < 0 = 2 \cdot 0 - a_0$, contradicting the definition of $m$. We also prove that $a_m + 1$

cannot already be present in $X$. Indeed, if it were, then it would have index $m+1$. But then $2(m+1) - a_{m+1} = 2m+2-(a_m+1) = 2m - a_m + 1 > 2m - a_m$, once again contradicting the definition of $m$.

Now we show that for any two distinct sets $X, Y \in A$, the sets $f(X)$ and $f(Y)$ are distinct. Equivalently, any subset $Z$ of $N$ of size $k+1$ can be an image of at most one subset of size $k$. Consider the elements of $Z = \{a'_1, a'_2, \dots, a'_{k+1}\}$. We show that the added element corresponds to the *first* index that maximizes the value $2i - a_i$. Originally, the index $m$ had the value $2m - a_m = r$ maximized. The values $2i - a'_i$ for $i = 1, 2, \dots, m$ are unchanged. The new element, $a'_{m+1}$, has the corresponding value $2(m+1) - a_{m+1} = 2(m+1) - (a_m+1) = r+1$. For $a'_{m+2}, a'_{m+3}, \dots, a'_{k+1}$, the corresponding values $2i - a'_i$ are $2(i+1) - a_i = 2i - a_i + 2$. By the definition of $m$, we have that $2i - a_i < 2m - a_m$, which is equivalent to $2i - a_i \leq 2m - a_m - 1$. Hence $2i - a_i + 2 \leq 2m - a_m + 1 = r + 1$. This means $m+1$ is the first index that maximizes the value $2i - a_i$, as desired. Hence, $Z$ is an image of at most one $k$-element set $X$ with respect to the matching $f$. ■

We are now ready to show that the number of subsets with 50 boxes that contain at least half of all the apples is greater than half of the number of all subsets with 50 boxes. First, each subset with 50 boxes can be uniquely paired with its complement, which is a subset with 49 boxes. Of the two subsets, at least one has at least half of all the apples. If at least half of the subsets with 50 boxes have at least half of the apples, we are done. Otherwise, at least half of the subsets with 49 boxes have at least half of all the apples. Using Lemma 1, the subsets with 50 boxes that are matched with these subsets with 49 boxes also have at least half of the apples, which means we are done.

But are we ? We want to show that the number of subsets with 50 boxes that contain at least half of all the apples is *greater than* half of the number of all subsets with 50 boxes. We have only shown that it is *at least* half, so there is more work to do. The only case in which it might be *exactly* half is when exactly half of the subsets with 49 boxes and exactly half of the subsets with 50 boxes contain at least half of all the apples, and they correspond to one another in the matching induced by Lemma 1. Assume that the boxes contain $a_1 \geq a_2 \geq \ldots \geq a_{99}$ apples, and assume without loss of generality that one pair in the matching is $\{a_3, a_5, \dots, a_{99}\}$ and $\{a_1, a_3, a_5, \dots, a_{99}\}$. The former set contains less than half of all the apples. Indeed, $a_3 + a_5 + \ldots + a_{99} \leq a_2 + a_4 + \ldots + a_{98} < a_1 + a_2 + a_4 + \ldots + a_{98}$. (Unless $a_1 = 0$, but that would imply $a_1 = a_2 = \ldots = a_{99} = 0$, and the situation can be handled separately.) On the other hand, the latter set contains at least half of all the apples. Indeed, $a_1 + a_3 + a_5 + \ldots + a_{99} \geq a_1 + a_3 + \ldots + a_{97} \geq a_2 + a_4 + \ldots + a_{98}$. So we are really done this time.

The good news is that we have solved the puzzle. The bad news is that the Russian Olympiad is organized for students of several class years separately, and it is sometimes the case that the jury proposes similar puzzles to different class years, adjusting the difficulty accordingly. This is in fact the case here, and the puzzle we have solved is for 8th grade students. Here is the one for 9th graders :

**Puzzle 2** : Some apples and oranges are distributed among 100 boxes. Prove that we can choose 34 boxes so that together they contain at least one-third of all the apples and at least one-third of all the oranges.

Unfortunately, it seems hard to apply the same method when the concerned sets include only around one-third of the boxes. Let us start by again sorting the boxes according to the number of apples, say, $a_1 \geq a_2 \geq \ldots \geq a_{100}$. The crucial observation is that choosing boxes $1, 4, 7, \ldots, 100$ is enough to guarantee at least one-third of all the apples. Indeed, $a_1+a_4+a_7+\ldots+a_{100} \geq a_2+a_5+\ldots+a_{98}$ and $a_1+a_4+a_7+\ldots+a_{100} \geq a_3+a_6+\ldots+a_{99}$. This hints at splitting the boxes into 34 groups : $\{1\}, \{2, 3, 4\}, \{5, 6, 7\}, \cdots, \{98, 99, 100\}$. By the observation we just made, choosing any 34 boxes, one from each group, will guarantee us at least one-third of all the apples. Consequently, we can choose the box with the greatest number of oranges from each group and guarantee ourselves at least one-third of all the oranges as well.

This solution can also be applied to our first puzzle quite easily. Moreover, it has the merit of being constructive – if we actually have to choose the boxes rather than just proving their existence, it gives us a method to do it within a reasonable amount of time.

Before we move on, it is worth asking what happens in the first puzzle when we change the number 99 to 100 (or any other even number). What is the least number for which there always exists some subset with that number of boxes that together contain at least half of the apples and at least half of the oranges ? If we choose 50 out of 99 boxes using our previous method, and then choose the 100th box, we achieve our goal using 51 boxes. Is that the best we can do ? It is, when 49 boxes contain one apple each and no oranges and the remaining 51 boxes contain one orange each and no apples.

Now comes a remarkable thing. Even if we have a third type of fruit, we can still choose 51 out of the 100 boxes so that they together contain at least half of each type of fruit. This is exactly what 11th graders were tasked in the Olympiad. Here is the formal statement :

**Puzzle 3** : Some apples, oranges, and bananas are distributed among 100 boxes. Prove that we can choose 51 boxes so that together they contain at least half of all the apples, at least half of all the oranges, and at least half of all the bananas.

This puzzle is quite difficult, and none of the students managed to solve it in the actual Olympiad. The methods we have so far don't seem to extend easily to accommodate for the third type of fruit. Focusing on just one type of fruit leaves us in trouble in satisfying the guarantee for the other two types. We would like to somehow deal with two types of fruit at once. We begin with the following cute lemma.

**Lemma 2** *Let* $(a_1, b_1), (a_2, b_2), \ldots, (a_{2n}, b_{2n})$ *be pairs of positive integers. They can be partitioned into two groups of* $n$ *pairs each so that if* $A_1$ *and* $A_2$ *denote the sum of the* $a_i$ *in the first and second group respectively, then* $|A_1 - A_2| \leq \max_i a_i$*,* *and an analogous statement holds for $b_i$.*

*Proof.* We sort the pairs according to $a_i$ in descending order, and split the pairs into $n$ groups : $\{(a_1, b_1), (a_2, b_2)\}, \{(a_3, b_3), (a_4, b_4)\}, \ldots, \{(a_{2n-1}, b_{2n-1}), (a_{2n}, b_{2n})\}$. If we choose $n$ pairs, one from each group, the difference in $a_i$ will not exceed $a_1$. Indeed, the maximum difference is $a_1 - a_2 + a_3 - a_4 + \ldots + a_{2n-1} - a_{2n} \leq a_1$.

We first choose any $n$ pairs, one from each group. Suppose that the difference in $b_i$ exceeds $\max_i b_i$. Assume without loss of generality that the sum of $b_i$ in the first group is greater than that in the second group. This means there exists two pairs, say $(a_{2i-1}, b_{2i-1})$ and $(a_{2i}, b_{2i})$, such that $b_{2i-1} > b_{2i}$, the first pair belongs to the first group, and the second pair belongs to the second group. We switch these two pairs. The difference in $b_i$ changes by no more than $2b_1$, so the absolute value strictly decreases. We continue this process as long as the difference in $b_i$ exceeds $\max_i b_i$. Since there are only finitely many partitions, the process necessarily stops, giving us the desired partition. ■

We are now ready to solve the 11th grade puzzle. In fact, the solution will be short and sweet. We first choose a box with the greatest number of apples, and from the remaining boxes we choose a box with the greatest number of oranges. Now, thanks to Lemma 2, we can partition the remaining 98 boxes into two groups of 49 boxes so that the difference between the apples does not exceed the box with the greatest number of apples, and likewise with oranges. We choose the 49 boxes with more bananas. Combined with the first two boxes, we have our desired set of 51 boxes.

Finally, what if there are more types of fruit ? If there are apples, oranges, bananas, and pears distributed among 100 boxes, what is the least number for which there always exists some subset with that number of boxes that together contain at least half of each type of fruit ? We leave this as an open puzzle for the reader.

**Note** : Credit for the puzzles from the Olympiad is due to I. Bogdanov, G. Chelnokov, and E. Kulikov. The solutions are also due to them, except for the non-constructive solution to the first puzzle, which the author came up with.

## References

[**1**] *Problem 110178* (in Russian),
`http://problems.ru/view\_problem\_details\_new.php?id=110178`

[**2**] *Problem 110198* (in Russian),
`http://problems.ru/view\_problem\_details\_new.php?id=110198`

[**3**] László Lovász and Michael D. Plummer, *Matching Theory*, Akadémiai Kiadó, 1986.

# PROBLEMS

*Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please email your submissions to* `crux-psol@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

**Submissions of solutions.** *Each solution should be contained in a separate file named using the convention* LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

**Submissions of proposals.** *Original problems are particularly sought, but other interesting problems are also accepted provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention* LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

*To facilitate their consideration, solutions should be received by the editor by* **1 July 2015***, although late solutions will also be considered until a solution is published.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.*

*An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.*

## 3921. *Proposed by Michel Bataille.*

Let $AUVW$ be a rectangle with $UV = 1$. In each of the following cases (a) $VW = 8$ and (b) $VW = 9$, is it possible to construct with ruler and compass points $B$ on ray $[AU)$ and $C$ on ray $[AW)$ such that $V$ and the orthogonal projection of $A$ onto $BC$ are symmetric about the midpoint of $BC$?

## 3922. *Proposed by Marcel Chiriţă.*

Let $M$ be a point inside a triangle $ABC$. Show that

$$\frac{(x+y+z)^9}{729xyz} \geq a^2b^2c^2,$$

where $MA = x, MB = y, MC = z$ and $BC = a, AC = b, AB = c$.

**3923**. *Proposed by George Apostolopoulos.*

Prove that in any triangle $ABC$,

$$\frac{\sin^3\frac{A}{2}}{\sin^3\frac{A}{2}+\cos^3\frac{A}{2}}+\frac{\sin^3\frac{B}{2}}{\sin^3\frac{B}{2}+\cos^3\frac{B}{2}}+\frac{\sin^3\frac{C}{2}}{\sin^3\frac{C}{2}+\cos^3\frac{C}{2}}\leq\frac{3R}{2(r+s)},$$

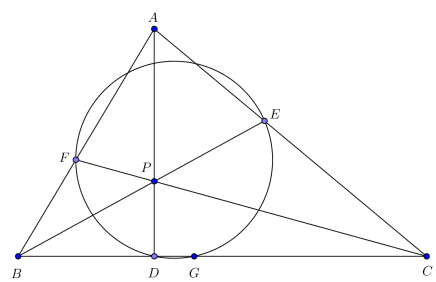
where $s, r$ and $R$ are the semiperimeter, the inradius and the circumradius, respectively, of the triangle $ABC$.

**3924**. *Proposed by Michel Bataille.*

Let $\{F_k\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for every positive integer $k$. If $m$ and $n$ are positive integers with $m$ odd and $n$ not a multiple of 3, prove that $5F_m^2 - 3$ divides $5F_{mn}^2 + 3(-1)^n$.

**3925**. *Proposed by Ilker Can Çiçek.*

Let $ABC$ be a scalene triangle. Let $D$ be the foot of the altitude from the vertex $A$. Let $P$ be the point on the segment $AD$ ($P \neq A, P \neq D$), such that for the points $E$ and $F$ defined by $BP \cap AC = E$ and $CP \cap AB = F$, the equality $BF \cdot CD = BD \cdot CE$ holds. Let $G$ be the intersection point of the circumcircle of the triangle $DEF$ and the segment $BC$ with $G$ lying between $D$ and $C$ :




Prove that $AB + AC = BC + AE$ if and only if $BF + CG = CE + BD$.

**3926**. *Proposed by George Apostolopoulos.*

Let $a, b$ and $c$ be positive real numbers such that $a+b+c = 1$. Find the minimum value of the expression

$$\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}.$$

**3927**. *Proposed by Marcel Chiriţă.*

Let $ABCO$ be a tetrahedron with the face angles at $O$ all right angles. If we denote the altitude from $O$ by $h$, the inradius by $r$, and the angles that the lines $OA, OB, OC$ make with the face $ABC$ by $x, y, z$, show that

$$r\sqrt{2} \geq h(\cos 2x + \cos 2y + \cos 2z).$$

**3928**. *Proposed by Michel Bataille.*

Let $A \in \mathcal{M}_n(\mathbb{C})$ with $\operatorname{rank}(A) \le 1$ and complex numbers $x_1, x_2, \ldots, x_n$ such that $\sum\limits_{k=1}^{n} x_k^2 = 1$. If

$$B = \left(\begin{array}{c|c} 0 & x_1 \cdots x_n \\ \hline x_1 & \\ \vdots & A \\ x_n & \end{array}\right)$$

and $I_{n+1}$ is the unit matrix of size $n+1$, prove that

$$\det(I_{n+1} + B) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

**3929**. *Proposed by Péter Ivády.*

Show that for all $0 < x < \pi/2$, the following inequality holds :

$$\left(1 + \frac{1}{\sin x}\right)\left(1 + \frac{1}{\cos x}\right) \ge 5\left[1 + x^4\left(\frac{\pi}{2} - x\right)^4\right].$$

**3930**. *Proposed by José Luis Díaz–Barrero.*

In a triangle $ABC$, let $a$, $b$ and $c$ denote the lengths of the sides $BC$, $CA$ and $AB$. Show that

$$\sqrt{\frac{a\,\sin^{1/2} B}{4a+b+c}} + \sqrt{\frac{b\,\sin^{1/2} C}{a+4b+c}} + \sqrt{\frac{c\,\sin^{1/2} A}{a+b+4c}} \le \frac{3}{2}\sqrt[8]{\frac{4}{27}}.$$

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**3921**. *Proposé par Michel Bataille.*

Soit $AUVW$ un rectangle où $UV = 1$. Dans chacun des cas suivants (a) $VW = 8$ et (b) $VW = 9$, est-il possible de construire à la règle et au compas des points $B$ sur la demi-droite $[AU)$ et $C$ sur la demi-droite $[AW)$ tels que $V$ et le projeté orthogonal de $A$ sur $BC$ soient symétriques par rapport au milieu de $BC$ ?

**3922**. *Proposé par Marcel Chiriţă.*

Soit $M$ un point à l'intérieur d'un triangle $ABC$. Soit $x = MA$, $y = MB$, $z = MC$, $a = BC$, $b = CA$ et $c = AB$. Démontrer que

$$\frac{(x+y+z)^9}{729xyz} \ge a^2b^2c^2.$$

**3923**. *Proposé par George Apostolopoulos.*

Démontrer que dans n'importe quel triangle $ABC$,

$$\frac{\sin^3 \frac{A}{2}}{\sin^3 \frac{A}{2} + \cos^3 \frac{A}{2}} + \frac{\sin^3 \frac{B}{2}}{\sin^3 \frac{B}{2} + \cos^3 \frac{B}{2}} + \frac{\sin^3 \frac{C}{2}}{\sin^3 \frac{C}{2} + \cos^3 \frac{C}{2}} \leq \frac{3R}{2(r+p)},$$

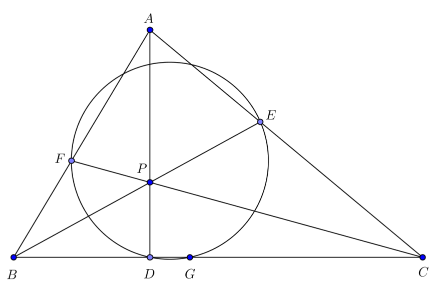
$p$ étant le demi-périmètre du triangle, $r$ le rayon du cercle inscrit dans le triangle et $R$ le rayon du cercle circonscrit au triangle.

**3924**. *Proposé par Michel Bataille.*

Soit $\{F_k\}$ la suite de Fibonacci définie par $F_0 = 0$, $F_1 = 1$ et $F_{k+1} = F_k + F_{k-1}$ pour tout entier strictement positif $k$. Étant donné deux entiers strictement positifs $m$ et $n$, $m$ étant impair et $n$ n'étant pas un multiple de 3, démontrer que $5F_m^2 - 3$ est un diviseur de $5F_{mn}^2 + 3(-1)^n$.

**3925**. *Proposé par Ilker Can Çiçek.*

Soit un triangle scalène $ABC$ et $D$ le pied de la hauteur issue de $A$. On considère un point $P$ sur le segment $AD$ ($P \neq A, P \neq D$) et on trace les céviennes $BE$ et $CF$ qui passent par $P$. Soit $G$ le point d'intersection du segment $BC$ et du cercle circonscrit au triangle $DEF$, situé entre les points $C$ et $D$.




Sachant que $BF \cdot CD = BD \cdot CE$, démontrer que $AB + AC = BC + AE$ si et seulement si $BF + CG = CE + BD$.

**3926**. *Proposé par George Apostolopoulos.*

Soit $a, b$ et $c$ des nombres réels strictement positifs tels que $a+b+c = 1$. Déterminer la valeur minimale de l'expression

$$\sqrt{a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2}} + \sqrt{b^2 + c^2 + \frac{1}{b^2} + \frac{1}{c^2}} + \sqrt{a^2 + c^2 + \frac{1}{a^2} + \frac{1}{c^2}}.$$

**3927**. *Proposé par Marcel Chiriţă.*

Soit un tétraèdre $ABCO$ dont les angles des faces au sommet $O$ sont tous droits. Soit $h$ la hauteur issue de $O$, $r$ le rayon de la sphère inscrite dans le tétraèdre et

$x$, $y$ et $z$ les mesures des angles formés par les segments respectifs $OA$, $OB$ et $OC$ et la face $ABC$. Démontrer que

$$r\sqrt{2} \geq h(\cos 2x + \cos 2y + \cos 2z).$$

**3928**. *Proposé par Michel Bataille.*

Soit $A \in \mathcal{M}_n(\mathbb{C})$ telle que $\text{rang}(A) \leq 1$ et $x_1, x_2, \ldots, x_n$ des nombres complexes tels que $\sum_{k=1}^{n} x_k^2 = 1$. Sachant que

$$B = \left(\begin{array}{c|c} 0 & x_1 \cdots x_n \\ \hline x_1 & \\ \vdots & A \\ x_n & \end{array}\right)$$

et que $I_{n+1}$ est la matrice identité d'ordre $n+1$, démontrer que

$$\det(I_{n+1} + B) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

**3929**. *Proposé par Péter Ivády.*

Démontrer que pour tout $x$, $0 < x < \pi/2$, on a

$$\left(1 + \frac{1}{\sin x}\right)\left(1 + \frac{1}{\cos x}\right) \geq 5\left[1 + x^4\left(\frac{\pi}{2} - x\right)^4\right].$$

**3930**. *Proposé par José Luis Díaz–Barrero.*

On considère un triangle $ABC$. Soit $a$, $b$ et $c$ les longueurs des côtés respectifs $BC$, $CA$ et $AB$. Démontrer que

$$\sqrt{\frac{a\,\sin^{1/2} B}{4a+b+c}} + \sqrt{\frac{b\,\sin^{1/2} C}{a+4b+c}} + \sqrt{\frac{c\,\sin^{1/2} A}{a+b+4c}} \leq \frac{3}{2}\sqrt[8]{\frac{4}{27}}.$$

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

## 3821. *Proposed by Farrukh Rakhimjanovich Ataev.*

Prove that any triangle can be divided into five triangles such that one of the triangles is equilateral, one is isosceles, one is right angled, one is acute and one is obtuse.

*Solved by P. Woo; and by T. Zvonaru and N. Stanciu; and the proposer. We present the solution by Titu Zvonaru and Neculai Stanciu, slightly modified by the editor.*

Let $ABC$ be the given triangle. We will use the following claim : "If $XYZ$ is a right nonisosceles triangle and $T$ is the midpoint of the hypotenuse $YZ$, then of the two isosceles triangles $XTZ$ and $XTY$, one is acute and the other is obtuse." The claim can be verified by considering a circle with diameter $YZ$ to see that $|TX| = |TY| = |TZ|$. Consider two cases :

(a) If $ABC$ is equilateral, let $M$ be the midpoint of $BC$, $N$ the midpoint of $AB$, $E$ the midpoint of $AM$, and $D$ the projection of $M$ onto $AC$. We have $\Delta BMN$ equilateral, $\Delta ANM$ obtuse, $\Delta MDC$ right-angled. By the claim, one of $\Delta MDE$ and $\Delta ADE$ is acute and the other is isosceles. (One can check that $\Delta MDE$ is the acute triangle, as it is in fact equilateral.)

(b) If $ABC$ is not equilateral, let $A$ be the largest angle, so $A > 60°$. Without loss of generality, suppose $B < 60°$. Let $M$ be a point on $BC$ such that $\angle BAM = 60°$. Since $B < 60°$, $AB > AM$, and we can find a point $N$ on $AB$ for which $|AN| = |AM|$. Then $\Delta AMN$ is equilateral and $\Delta BNM$ is obtuse. Let $D$ be the projection of $M$ onto $AC$, which lies on $AC$ because $C < 90°$. If at least one of the two right triangles $\Delta MDC$ and $\Delta ADM$ is nonisosceles, then by the claim, we can divide it into an acute triangle and an isosceles triangle. If both $\Delta MDC$ and $\Delta ADM$ are isosceles, then $C = 45°$, $A = 60° + 45°$, and $B = 30°$, in which case we swap the roles of $B$ and $C$ in this argument.

## 3822. *Proposed by M. N. Deshpande.*

Let $ABC$ be an isosceles triangle with $AB = AC$ and $\angle A = \alpha$. Further, let $G$ be its centroid and circle $\Gamma$ passes through $B$, $C$ and $G$. Point $D$ is on the circle, different from $G$, such that $BD = CD$ and let $\angle BDC = \delta$. Show that

(i) $\alpha + \delta \geq 120°$.

(ii) $\left(\dfrac{\cos\alpha + \cos\delta}{1 + \cos\alpha\cos\delta}\right)$ does not depend on $\alpha$.

*Solved by AN-anduud Problem Solving Group ; Ş. Arslanagić ; M. Bataille ; P. De ; O. Kouba ; S. Malikić ; M.R. Modak ; C. R. Pranesachar ; C. Sánchez-Rubio ; D. Văcaru ; P. Woo ; T. Zvonaru and N. Stanciu ; and the proposer. We present the solution by the AN-anduud Problem Solving Group.*

(i) Let $\angle GCA = \psi$. Since $BDCG$ is cyclic, the angles at $B$ and $C$ sum to $180°$, so that (in quadrilateral $BDCA$) $\alpha+\delta+180°+2\psi = 360°$. To prove $\alpha+\delta \geq 120°$, therefore, we need only to prove that

$$\psi \leq 30°.$$

Let $GT$ be the perpendicular from $G$ to $AC$. Then $GT \leq \frac{1}{3}m_b = \frac{1}{3}m_c$, whence

$$\sin\psi = \frac{GT}{GC} = \frac{GT}{\frac{2}{3}m_c} \leq \frac{\frac{1}{3}m_c}{\frac{2}{3}m_c} = \frac{1}{2}.$$

Hence $\sin\psi \leq \frac{1}{2} = \sin 30°$. Since the function $y = \sin x$ is increasing on the interval $\left(0, \frac{\pi}{2}\right)$ we get $\psi \leq 30°$, as desired.

(ii) Observe that

$$\angle GBC = \angle GCB = \frac{1}{2}\angle BDC = \frac{1}{2}\delta.$$

Thus

$$\tan\frac{\delta}{2} = \frac{\frac{1}{3}m_a}{\frac{1}{2}a} = \frac{2m_a}{3a} \qquad \text{and} \qquad \tan\frac{\alpha}{2} = \frac{\frac{a}{2}}{m_a} = \frac{a}{2m_a}.$$

Therefore

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{4m_a^2-a^2}{4m_a^2+a^2},\ \cos\delta = \frac{1-\tan^2\frac{\delta}{2}}{1+\tan^2\frac{\delta}{2}} = \frac{9a^2-4m_a^2}{9a^2+4m_a^2},$$

and, finally,

$$\frac{\cos\alpha+\cos\delta}{1+\cos\alpha\cos\delta} = \frac{64a^2m_a^2}{80a^2m_a^2} = \frac{4}{5},$$

which, indeed, does not depend on $\alpha$.

**3823**. *Proposed by Neculai Stanciu and Titu Zvonaru.*

Let $ABC$ be a triangle with height $AD$, where $E$ and $F$ are the midpoints of sides $AC$ and $AB$ respectively. For any point $P$ in the plane of the triangle $ABC$, let $Y$ and $Z$ be its symmetric from the points $E$ and $F$, respectively. If $P'$ is the midpoint of $DP$ and $M = BY \cap CZ$, then prove that the line through $M$ and $P'$ passes through a fixed point.

*Solved by M. Bataille ; O. Geupel ; O. Kouba ; M.R. Modak ; C. Sánchez-Rubio ; P. Woo ; and the proposers. We present the solution by Omran Kouba.*

There is no need to restrict $D$ to be the foot of the altitude from $A$ : We shall see that for any point $D$ in the plane of the triangle $ABC$, the barycenter of the points $A, B, C$, and $D$ lies on all the lines $MP'$.

Since $\overrightarrow{PY} = 2\overrightarrow{PE}$ and $\overrightarrow{PZ} = 2\overrightarrow{PF}$, we conclude that

$$\overrightarrow{YZ} = 2(\overrightarrow{PF} - \overrightarrow{PE}) = 2\overrightarrow{EF} = \overrightarrow{CB}.$$

Thus $BCYZ$ is a parallelogram, and $M$ is the intersection point of its diagonals, so $M$ is the midpoint of $BY$. Therefore

$$\begin{aligned} 2(\overrightarrow{DP'} + \overrightarrow{DM}) &= \overrightarrow{DP} + \overrightarrow{DY} + \overrightarrow{DB} \quad (P' \text{ and } M \text{ are the midpoints of } DP \text{ and } BY) \\ &= 2\overrightarrow{DE} + \overrightarrow{DB} \quad (E \text{ is the midpoint of } PY) \\ &= \overrightarrow{DA} + \overrightarrow{DC} + \overrightarrow{DB} \quad (E \text{ is the midpoint of } AC). \end{aligned}$$

Thus, if $X$ is the midpoint of $P'M$ then we conclude from the previous result that

$$4\overrightarrow{DX} = \overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC} + \overrightarrow{DD}.$$

This means that $X$ is the barycenter of the four points $A$, $B$, $C$, and $D$, and all the lines through $M$ and $P'$ pass through this fixed point (which is the common midpoint of all segments $MP'$).

*Editor's comment.* Bataille observed that $M$ is not well defined should $P$ be chosen on the line through $A$ that is parallel to $BC$. In that case the result continues to hold if $M$ is defined to be the common midpoint of $BY$ and $CZ$.

**3824**. *Proposed by Edward T. H. Wang and Dexter Wei.*

Let

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + + \frac{1}{\sqrt{n}}$$

where $n \in \mathbb{N}$. It is well known that $S_n \geq 2(\sqrt{n+1} - 1)$. Prove or disprove the stronger inequality that

$$S_n \geq \frac{2n}{1 + \sqrt{n}}\,.$$

*Solved by AN-anduud Problem Solving Group; Ş. Arslanagić; M. Bataille; M. Dincă; O. Kouba; K-W. Lau; S. Malikić; C.R. Pranesachar; N. Stanciu and T. Zvonaru; D. Văcaru; P. Y. Woo; and the proposers. There was one incorrect submission. We present two solutions.*

*Solution 1, by Omran Kouba.*

We will prove that the stronger inequality is valid for every positive integer $n$. Indeed, note that for positive $a$ and $b$ we have

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = 4 + \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 \geq 4,$$

or equivalently, we obtain

$$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) \geq 2\left(\frac{a-b}{a^2-b^2}\right).$$

Taking $a=\sqrt{k+1}$ and $b=\sqrt{k}$, we find that for $k \geq 1$ we have

$$\frac{1}{2}\left(\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}\right) \geq 2\left(\sqrt{k+1}-\sqrt{k}\right).$$

Adding these inequalities for $k=1,2,\ldots,n-1$ we get

$$S_n - \frac{1}{2}\left(1+\frac{1}{\sqrt{n}}\right) \geq 2(\sqrt{n}-1) = \frac{2(n-1)}{\sqrt{n}+1},$$

or equivalently

$$S_n \geq \frac{2n}{\sqrt{n}+1} + \frac{1}{2}\left(1+\frac{1}{\sqrt{n}}-\frac{4}{\sqrt{n}+1}\right) = \frac{2n}{\sqrt{n}+1} + \frac{(\sqrt{n}-1)^2}{2(n+\sqrt{n})}.$$

This proves the desired inequality, with equality if and only if $n=1$.

*Solution 2, by AN-anduud Problem Solving Group.*

Since the function $f(x)=\frac{1}{\sqrt{x}}$ is strictly decreasing and strictly convex on $x \geq 1$, inspection of a graph of the integral of $\frac{1}{\sqrt{x}}$ and a left-endpoint Riemann sum for $\frac{1}{\sqrt{x}}$ yields :

$$\begin{aligned} S_n = 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} &\geq \int_1^n \frac{dx}{\sqrt{x}} + \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)+\frac{1}{\sqrt{n}} \\ &= 2(\sqrt{n}-1)+\frac{1}{2}(1-\frac{1}{\sqrt{n}})+\frac{1}{\sqrt{n}} \\ &= \frac{4n+1-3\sqrt{n}}{2\sqrt{n}}. \end{aligned}$$

Now we prove the following inequality for any positive integer $n$,

$$\frac{4n+1-3\sqrt{n}}{2\sqrt{n}} \geq \frac{2n}{1+\sqrt{n}}.$$

We have

$$\begin{aligned} \frac{4n+1-3\sqrt{n}}{2\sqrt{n}} \geq \frac{2n}{1+\sqrt{n}} &\Leftrightarrow (4n+1-3\sqrt{n})(1+\sqrt{n}) \geq 4n\sqrt{n} \\ &\Leftrightarrow (4n+1)-3\sqrt{n}+(4n\sqrt{n}+\sqrt{n}-3n) \geq 4n\sqrt{n} \\ &\Leftrightarrow n+1-2\sqrt{n} \geq 0 \\ &\Leftrightarrow (\sqrt{n}-1)^2 \geq 0, \end{aligned}$$

and this last inequality is clearly true. Hence by combining the two inequalities, we have $S_n \geq \dfrac{2n}{1+\sqrt{n}}$ for any positive integer $n$.

*Editor's comments.* The featured solutions were the only two submitted solutions that do not rely on induction. The induction proof is reasonably straightforward, although it is possible to reduce work by applying calculus techniques (as Bataille did using second derivatives and the Mean Value Theorem), or using rough approximations to eliminate the numerous square roots that appear (as Malikić did, bounding $n(n+1)$ and $n+1$ above by squares).

**3825**. *Proposed by Brian Brzycki.*

Triangle $ABC$ is acute. Points $X$ and $Y$ trisect side $BC$, with $X$ closer to $B$. Semicircles centred at $X$ and $Y$ and tangent to $AB$ and $AC$ are drawn, respectively.

(a) Prove that the two semicircles must intersect.

(b) If the semicircles intersect at $Z$, and $\angle XZY = \theta$, prove that

$$\cos(2B) + \cos(2C) + 4\sin(B)\sin(C)\cos(\theta) = 0\,.$$

*Solved by M. Amengual Covas; G. Apostopoulos; Ş. Arslanagić; M. Bataille; J. Heuver; O. Kouba; S. Malikić; C. R. Pranesachar; D. Smith; D. Stone and J. Hawkins; T. Zvonaru and N. Stanciu; P. Y. Woo; and the proposer. We present the solution given by most of the solvers.*

(a) Without loss of generality, let $BX = XY = YC = 1$; let $x$ and $y$ be the respective radii of the semi-circles based on $BC$ with centres $X$ and $Y$. Since the triangle is acute, $B + C > 90^\circ$. Therefore,

$$x + y = \sin B + \sin C = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} > 2\sin 45^\circ \cos 45^\circ = 1 = XY$$

and the two semi-circles intersect.

(b) By the Law of Cosines applied to triangle $ZXY$,

$$1 = x^2 + y^2 - 2xy\cos\theta = \sin^2 B + \sin^2 C - 2\sin B \sin C \cos\theta.$$

Since $\cos 2B = 1 - 2\sin^2 B$ and $\cos 2C = 1 - 2\sin^2 C$, the desired result follows.

*Editor's comment.* All the solvers used the Law of Cosines for part (b). However, for (a), Heuver, Kouba and the proposer used similar triangles to obtain

$$x : XY = x : BX = h_c : BC,$$

$$y : XY = y : YC = h_b : BC,$$

where $h_c$ and $h_b$ are the respective altitudes in triangle $ABC$ from $C$ and $B$. Since these altitudes meet in the interior of the triangle, $h_c + h_b > BC$, so that $x + y > XY$.

**3826**. *Proposed by Ovidiu Furdui.*

Let $f : [0,1] \mapsto [0,\infty)$ and let $g : [0,1] \mapsto [0,\infty)$ be two continuous functions. Find the value of

$$\lim_{n\to\infty} \sqrt[n]{f\left(\frac{1}{n}\right) g\left(\frac{n}{n}\right) + f\left(\frac{2}{n}\right) g\left(\frac{n-1}{n}\right) + \cdots + f\left(\frac{n}{n}\right) g\left(\frac{1}{n}\right)}.$$

*Solved by O. Kouba; P. Perfetti; and the proposer. There was one incorrect submission. We present the solution by Omran Kouba, modified slightly by the editor.*

Let $\ell = \int_0^1 f(x)g(1-x)dx$. We will prove that if $\ell > 0$ then the considered limit is equal to 1, while the limit *might not exist* if $\ell = 0$. Indeed, let

$$R_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) g\left(1 - \frac{k}{n}\right) \quad \text{and} \quad T_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) g\left(1 - \frac{k-1}{n}\right).$$

By the Triangle Inequality, we have

$$|R_n - T_n| \leq n \left\|f\right\|_\infty \omega\left(g, \frac{1}{n}\right),$$

where $\left\|f\right\|_\infty = \sup_{[0,1]} |f|$ and $\omega(g,\epsilon)$ is a modulus of uniform continuity, given by

$$\omega(g,\epsilon) = \sup\{|g(u) - g(v)| : (u,v) \in [0,1]^2, |u - v| \leq \epsilon\}.$$

($\omega(g,\epsilon)$ is finite because a continuous function on a closed and bounded interval is uniformly continuous.) Thus, since $\lim\limits_{\epsilon\to 0} \omega(g,\epsilon) = 0$, we have

$$\lim_{n\to\infty} \frac{|R_n - T_n|}{n} = 0.$$

On the other hand, observe that $\dfrac{1}{n}R_n$ is a Riemann sum for the function $x \mapsto f(x)g(1-x)$, and consequently

$$\lim_{n\to\infty} \frac{R_n}{n} = \int_0^1 f(x)g(1-x)dx = \ell.$$

From these two equations, we conclude that

$$\lim_{n\to\infty} \frac{T_n}{n} = \ell.$$

Now, if $\ell > 0$, then there is $n_0$ such that for $n \geq n_0$ we have $an \leq T_n \leq bn$, with $a = \ell/2$ and $b = 3\ell/2$. Thus

$$\sqrt[n]{an} \leq \sqrt[n]{T_n} \leq \sqrt[n]{bn}, \qquad \text{for } n \geq n_0.$$

But $\lim_{n\to\infty} \sqrt[n]{cn} = 1$ for every $c > 0$, so by taking the limit as $n$ tends to $\infty$, we conclude that $\lim_{n\to\infty} \sqrt[n]{T_n} = 1$, in the case $\ell > 0$.

Next, we consider the case $\ell = 0$, which is equivalent to $f(x)g(1-x) = 0$ for every $x \in [0,1]$, because the integrand is continuous and non-negative. We will give an example where the proposed limit does not exist. Indeed, let

$$f(x) = g(x) = \max\left(0, x - \frac{1}{2}\right).$$

With this choice, note that $g(1-x) = \max(0, \frac{1}{2} - x)$, so that almost all of the terms in $T_n$ are zero. Treating even and odd $n$ separately, we have $T_{2n} = 0$ and

$$T_{2n+1} = f^2\left(\frac{n+1}{2n+1}\right) = \left(\frac{(2n+2)-(2n+1)}{2(2n+1)}\right)^2 = \frac{1}{4(2n+1)^2},$$

and therefore $\lim_{n\to\infty} \sqrt[2n]{T_{2n}} = 0$, and $\lim_{n\to\infty} \sqrt[2n+1]{T_{2n+1}} = 1$, so, the proposed limit does not exist in this case.

*Editor's comments.* This solution exhibits a reasonably important problem solving technique. If a quantity *almost* looks like it converges to something nice, but it doesn't immediately do so, perhaps it's a good idea to approximate it with something that converges in a more obvious fashion, and work from there. For example, here we are dealing with $T_n$, but $\frac{1}{n}T_n$ isn't exactly a Riemann sum for that integral. However, $\frac{1}{n}R_n$ is, and the two sums approximate each other.

## 3827. *Proposed by Jung In Lee.*

For integer $k$, let $f(k)$ be the largest prime factor of $k$. The sequences $\{a_n\}$, $\{b_n\}$ are defined by $a_0 = b_0 = pq$, $a_{n+1} = a_n + pf(a_n)$, $b_{n+1} = b_n + qf(b_n)$ for $n \geq 1$, for given positive integers $p$ and $q$. Prove that there are infinitely many pairs of integers $(c, d)$ that satisfy

$$\frac{a_c}{p} = \frac{b_d}{q}.$$

*No solutions to this problem were received. The problem remains open.*

## 3828. *Proposed by George Apostolopoulos.*

Let $ABC$ be an acute angled triangle with $\angle B = 2\angle C$ and altitude $AD$. Drop perpendiculars $DK$ and $DL$ from $D$ to the sides $AB$ and $AC$ respectively.

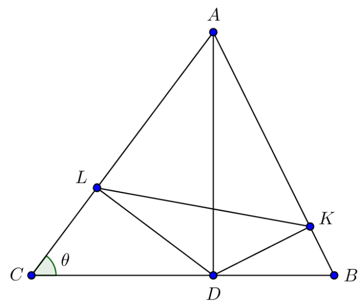
(a) Prove that $\sin A > \dfrac{2\sin^2 C}{1+\cos C}$.

(b) If $\frac{AD}{KL} = \sqrt{5} - 1$, find the angles of the triangle $ABC$.

*Solved by A. Alt; Ş. Arslanagić; R. Barbara; M. Bataille; P. De; O. Geupel; O. Kouba; V. Konečný (2 solutions); K.-W. Lau; S. Malikić; M. R. Modak; C.R.*

*Pranesachar; D. Smith; D. Stone and J. Hawkins; G. Tsapakidis; H. Wang; T. Zvonaru; and the proposer. We present the solution by Kee-Wai Lau expanded by the editor.*

Let $\angle C = \theta$, so that $\angle B = 2\theta$ and $\angle A = \pi - 3\theta$.




a) Since $ABC$ is an acute triangle, we have that

$$2\theta < \frac{\pi}{2} \quad \text{and} \quad \pi - 3\theta < \frac{\pi}{2},$$

which imply

$$\frac{\pi}{6} < \theta < \frac{\pi}{4},$$

so $\sin\theta < \dfrac{1}{\sqrt{2}}$. Hence

$$4\sin^2\theta < 2 \quad \text{or} \quad 3 - 4\sin^2\theta > 1,$$

from which we have :

$$\begin{aligned}\sin A &= \sin 3\theta = 3\sin\theta - 4\sin^3\theta = (3 - 4\sin^2\theta)\sin\theta > \sin\theta \\ &= \frac{(\sqrt{2}+1)\sin\theta}{\sqrt{2}+1} > \frac{2\sin\theta}{\sqrt{2}+1} = \frac{2\left(\frac{1}{\sqrt{2}}\right)\sin\theta}{1+\frac{1}{\sqrt{2}}} > \frac{2\sin^2\theta}{1+\cos\theta} = \frac{2\sin^2 C}{1+\cos C},\end{aligned}$$

since $\sin\theta < \dfrac{1}{\sqrt{2}} < \cos\theta$.

b) Since $\triangle ACD \sim \triangle ADL$ and $\triangle ABD \sim \triangle ADK$, we have $\angle ADL = \angle ACD = \theta$ and $\angle ADK = \angle ABD = 2\theta$. Since $\angle AKD = \angle ALD = \dfrac{\pi}{2}$, $AKDL$ is a cyclic quadrilateral. Hence,

$$\angle DKL = \angle DAL = \frac{\pi}{2} - \angle ADL = \frac{\pi}{2} - \theta.$$

Applying the sine law to $\triangle DKL$, we have

$$AD = \frac{DL}{\cos\theta} = \frac{DL}{\cos\left(\frac{\pi}{2} - \angle DKL\right)} = \frac{DL}{\sin\angle DKL} = \frac{KL}{\sin\angle KDL} = \frac{KL}{\sin\angle A} = \frac{KL}{\sin 3\theta}.$$

So

$$\sin 3\theta = \frac{KL}{AD} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}.$$

It is well-known that $\sin 54^\circ = \dfrac{\sqrt{5}+1}{4}$ (*Editor's comment : this can be derived by elementary procedure and several solvers provided a detailed proof for this.*) Since

$$\angle A < \frac{\pi}{2} \quad \text{and} \quad \sin\angle A = \sin(\pi - 3\theta) = \sin 3\theta = \frac{\sqrt{5}+1}{4},$$

we conclude that $\angle A = 54^\circ = \dfrac{3\pi}{10}$, from which it follows that

$$\angle C = \theta = \frac{1}{3}(\pi - \angle A) = \frac{1}{3}\left(\pi - \frac{3\pi}{10}\right) = \frac{7\pi}{30} = 42^\circ$$

and $\angle B = 2\theta = \dfrac{7\pi}{15} = 84^\circ$.

**3829**. *Proposed by Michel Bataille.*

Let $a$, $b$, $c$ be positive real numbers and $\Delta = a^2+b^2+c^2-(ab+bc+ca)$. Improve the well known inequality $\Delta \geq 0$ by proving that

$$\Delta \geq \left(\frac{a(a-b)^2(a-c)^2 + b(b-c)^2(b-a)^2 + c(c-a)^2(c-b)^2}{a+b+c}\right)^{\frac{1}{2}}.$$

*Solved by A. Alt ; AN-anduud Problem Solving Group ; Ş. Arslanagić ; R. Barbara ; D. Bailey, E. Campbell and C. Diminnie ; M. Dincă ; N. Evgenidis ; O. Kouba ; D. Koukakis ; K. -W. Lau ; S. Malikić ; P. McCartney ; C.R. Pranesachar ; D. Smith ; T. Zvonaru and N. Stanciu ; and the proposer. There was one flawed solution. The more efficient approaches are summarized below.*

*Preliminaries.* We establish notation and basic facts. The summation sign will refer to cyclic sums :

$$\sum f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b).$$

$$\begin{aligned}\Delta &= a^2+b^2+c^2-ab-bc-ca\\ &= (a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b)\\ &= (a-b)(a-c)+(b-c)^2 = (b-c)(b-a)+(c-a)^2 = (c-a)(c-b)+(a-b)^2\\ &= \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \geq 0.\end{aligned}$$

Then we have :

$$\begin{aligned} A &= a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \\ &= \sum a^3 - \sum(a^2b + ab^2) + 3abc. \end{aligned}$$

$$\begin{aligned} B &= a(a-b)^2(a-c)^2 + b(b-a)^2(b-c)^2 + c(c-a)^2(c-b)^2 \\ &= \sum a^5 + \sum(a^3b^2 + a^2b^3) + 4\sum a^3bc - 3\sum ab^2c^2 - 2\sum(a^4b + ab^4) \\ &= \Delta A. \end{aligned}$$

Finally,

$$\begin{aligned} \Gamma = (a+b+c)\Delta^2 - B = \Delta[\Delta(a+b+c) - A] &= \Delta\left[\sum(a^2b + ab^2) - 6abc\right] \\ &= \Delta[(a+b)(b+c)(c+a) - 8abc]. \end{aligned}$$

The problem requires it to be shown that $\Gamma \geq 0$. Equality will occur if and only if $a = b = c$.

*Solution 1, by Ş. Arslanagić; Kee-Wai Lau; Salem Malikić; and Phil McCartney (all independently).*

$$\Gamma = \Delta(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - 6abc) \geq 0,$$

by the arithmetic-geometric means inequality.

*Edtor's comment.* E. Nikolaos used the fact that $\Gamma = \Delta[(a+b)(b+c)(c+a) - 8abc]$ and noted that $a + b \geq 2\sqrt{ab},\ b + c \geq 2\sqrt{bc},\ c + a \geq 2\sqrt{ca}$.

*Solution 2, by Titu Zvonaru and Neculai Stanciu.*

$$\begin{aligned} \Gamma &= a[\Delta - (a^2 - ab - ac + bc)][\Delta + (a^2 - ab - ac + bc)] \\ &\qquad + b[\Delta - (b^2 - bc - ba + ca)][\Delta + (b^2 - bc - ba + ca)] \\ &\qquad + c[\Delta - (c^2 - ca - ca + ab][\Delta + (c^2 - ca - cb + ab)] \\ &= (ab^4 + a^4b + bc^4 + b^4c + ca^4 + c^4a) + 6(ab^2c^2 + a^2bc^2 + a^2b^2c) \\ &\qquad - 8(a^3bc + ab^3c + abc^3) \\ &= a(b-c)^4 + b(c-a)^4 + c(a-b)^4 \geq 0. \end{aligned}$$

*Solution 3, by the AN-anduud Problem Solving Group; and Dimitrios Koukakis (independently).*

Observe that

$$(uv + vw + wu)^2 = u^2v^2 + v^2w^2 + w^2u^2$$

when $u + v + w = 0$. Therefore, setting $(u, v, w) = (a - b, b - c, c - a)$, we obtain

$$\begin{aligned} (a+b+c)\Delta^2 &= (a+b+c)[(a-b)^2(a-c)^2 + (b-c)^2(b-a)^2 + (c-a)^2(c-b)^2] \\ &\geq a(a-b)^2(a-c)^2 + b(b-c)^2(b-a)^2 + c(c-a)^2(c-b)^2 = B. \end{aligned}$$

*Solution 4, by Omran Kouba, modified by the editor.*

Without loss of generality, assume that $a \geq b \geq c$. Then

$$\Delta \geq (a-b)(a-c) \geq |b-a|(b-c);$$
$$\Delta \geq (c-a)(c-b) = (a-c)(b-c).$$

Then $\Delta^2$ is not less than each of $(a-b)^2(a-c)^2$, $(b-a)^2(c-a)^2$ and $(c-a)^2(c-b)^2$. Therefore $\Delta^2$ is not less than the weighted average $B/(a+b+c)$ of these terms.

*Solution 5, by Dionne Bailey, Elsie Campbell and Charles Diminnie.*

Since $2\Delta = (a-b)^2 + (b-c)^2 + (c-a)^2$, then

$$4\Delta^2 = 4(a-b)^2(c-a)^2 + [(a-b)^2-(c-a)^2]^2 + (b-c)^4 + 2(a-b)^2(b-c)^2 + 2(b-c)^2(c-a)^2,$$

so that $\Delta^2 \geq (a-b)^2(a-c)^2$.

Similarly, $\Delta^2 \geq (b-c)^2(b-a)^2$ and $\Delta^2 \geq (c-a)^2(c-b)^2$. Hence the right side of the inequality does not exceed $(a+b+c)^{-1/2}(a\Delta^2 + b\Delta^2 + c\Delta^2)^{1/2} = \Delta$.

*Solution 6, by Arkady Alt.*

Note that

$$\begin{aligned} B &= \sum a(a-b)(a-c)[\Delta - (b-c)^2] \\ &= \Delta \sum a(a-b)(a-c) + (a-b)(b-c)(c-a) \sum a(b-c) \\ &= \Delta \sum a[\Delta - (b-c)^2] + 0 = \Delta^2(a+b+c) - \Delta \sum a(b-c)^2. \end{aligned}$$

Hence

$$\Gamma = \Delta \sum a(b-c)^2 \geq 0.$$

*Solution 7, by C.R. Pranesachar.*

$$\begin{aligned} \Gamma &= (a+b+c)\Delta^2 - B = \Delta[(a+b+c)B - A] \\ &= (b+c)(a-b)^2(a-c)^2 + (a+c)(b-a)^2(b-c)^2 + (a+b)(c-a)^2(c-b)^2 \geq 0. \end{aligned}$$

**3830**. *Proposed by Tigran Hakobyan.*

Let $a > 0$. Define the sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers by

$$a_1 = a, a_{n+1} = a_n + \{a_n\}, n \geq 1$$

where $\{x\}$ is the fractional part of $x$. Find all $a > 0$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined above is bounded.

*Solved by A. Alt; R. Barbara; O. Kouba; K. Lewis; P. Perfetti; D. Stone and J. Hawkins; D. Văcaru; and the proposer. Two incorrect solutions were received. We present a composite of solutions by the listed solvers.*

We prove first that the sequence is bounded if and only if it is eventually an integer.

Suppose first that the sequence $\{a_n\}_{n=0}^{\infty}$ is bounded. Since the sequence is monotone increasing, it is convergent and thus Cauchy. Thus, there is a positive integer $N$ such that for all $n \geq N$, $|a_n - a_N| < \frac{1}{2}$ and $a_n < a_N + 1$. The first of these inequalities can be written in the equivalent form $\{a_n\} < \frac{1}{2}$ for all $n \geq N$. We have $a_{N+1} = \lfloor a_N \rfloor + 2\{a_N\}$, $a_{N+2} = \lfloor a_N \rfloor + 4\{a_N\}$, and

$$\lfloor a_N \rfloor + 2^M \cdot \{a_N\} = a_{N+M} < a_N + 1 < \lfloor a_N \rfloor + 2$$

for all positive integers $M$. Hence,

$$2^M \cdot \{a_N\} < 2$$

for all $M$, so $\{a_N\} = 0$. Thus $a_N$ is an integer.

Conversely, suppose $a_N$ is an integer, for some positive integer $N$. Then $a_n = a_N$ for all $n \geq N$, so that the sequence is bounded above by $a_N$.

Now, we show that the sequence is eventually an integer if and only if it starts with a dyadic rational.

1. Suppose $a_d$ is an integer for some nonnegative integer $d$. If $d = 0$, then $a$ is an integer, and thus a dyadic rational. If $d > 0$, then $a = a_0 = k + \frac{m}{2^d}$, where $k$ and $m$ are nonnegative integers.

2. Suppose that $a$ is a dyadic rational. Write $a = k + \frac{m}{2^d}$, where $k$, $m$, and $d$ are nonnegative integers, with $d \geq 1$ and $m = 0$ or odd. Then there are odd integers $m_1, ..., m_d$ such that $\{a_1\} = \frac{m_1}{2^{d-1}}$, and for $n < d$, $\{a_n\} = \frac{m_n}{2^{d-n}}$, so that $\{a_{d-1}\} = \frac{1}{2}$ and $\{a_d\} = 0$. Hence $a_d$ is an integer.

# Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

## Proposers

George Apostolopoulos, Messolonghi, Greece : 3923, 3926

Michel Bataille, Rouen, France : 3921, 3924, 3928

Marcel Chiriţă, Bucharest, Romainia : 3922, 3927

Ilker Can Çiçek, Istanbul High School, Istanbul, Turkey : 3925

José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain : 3930

Péter Ivády, Budapest, Hungary : 3929

## Solvers - individuals

Arkady Alt, San Jose, CA, USA : OC111, OC112, 3828, **3829**, 3830

Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : 3825

George Apostolopoulos, Messolonghi, Greece : 3825, 3828

Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : CC63, OC111, **OC112**, 3822, 3824, 3825, 3828, 3829

Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan : 3821

Roy Barbara, Lebanese University, Fanar, Lebanon : 3828, 3829, 3830

Michel Bataille, Rouen, France : CC64, OC111, OC112, 3822, 3823, 3824, 3825, 3828, 3829

Brian Brzycki, Troy High School, Whittier, CA, USA : 3825

Matei Coiculescu, East Lyme High School, East Lyme, CT, USA : **CC64**

Chip Curtis, Missouri Southern State University, Joplin, MO, USA : OC111, OC112

Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India : 3822, 3828

Marian Dincă, Bucharest, Romania : 3824, 3829

M. N. Deshpande, Magpur, India : 3822

Nikolaos Evgenidis, Gymnasium of Agia, Thessalia, Greece : CC63, 3829

Ovidiu Furdui, Campia Turzii, Cluj, Romania : 3826

Ivan Dinkov Gerganov, student, Bulgaria, Kardzhali town : CC64

Oliver Geupel, Brühl, NRW, Germany : **OC113**, 3823, 3828

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