Crux Mathematicorum

VOLUME 40, NO. 2 February / Fevrier 2014

Editorial Board

Published by
Canadian Mathematical Society
209 - 1725 St. Laurent Blvd.
Ottawa, Ontario, Canada K1G 3V4
FAX: 613–733–8994
email: subscriptions@cms.math.ca

Publié par
Société mathématique du Canada
209 - 1725 boul. St. Laurent
Ottawa (Ontario) Canada K1G 3V4
Téléc : 613–733–8994
email : abonnements@smc.math.ca

IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Dear ***Crux*** readers,

Have you seen this puzzle before?

> You have 12 marbles and an old-fashioned balance scale in front of you. One of the marbles is heavier than the others. Can you figure out which one if you are allowed to use the scales only three times? What if you know that one marble is different, but don't know whether it is heavier or lighter than the other 11 marbles? Can you find the different marble by using the scales only three times?

What is so special about this problem? I personally enjoy its accessibility: to solve it, you only need to know how a balance scale works. In a sense, this is problem solving in its purest form.

Starting with this issue, I am happy to introduce to ***Crux*** some materials with a slightly different flavour. Through examples and exercises, we will introduce an area of mathematics that our readers probably have not seen before – in this issue, we shall begin with Ramsey's theory. Building anything from the ground up is always an adventure and I hope you enjoy this kind of exploration into mathematics.

I am also glad to feature an excerpt from Richard Hoshino's "The Math Olympian" (***Crux*** intends to review the book; meanwhile, we will present a few excerpts) as well as our ever-popular regular sections.

We at the CMS are working hard to eliminate the journal's backlog; however, the editorial transitions and administrative issues have caused some recent delays. Be assured that we are handling it and the production will soon be back on track.

As usual, do not hesitate to contact me directly at `crux-editors@cms.math.ca`.

Kseniya Garaschuk

THE CONTEST CORNER

No. 22

Robert Bilinski

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-contest@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er juin 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

CC106. En chaque sommet d'un tétraèdre régulier de côté 3, on découpe une pyramide de façon que la surface de la découpe soit un triangle équilatéral. Les quatre triangles équilatéraux ainsi obtenus ont tous des dimensions différentes. Quelle est la longueur totale des arêtes du solide ainsi tronqué ?

CC107. Dans un triangle ABC rectangle en B tel que $BC = 1$, on place D sur le côté AC pour que $AD = AB = \frac{1}{2}$. Quelle est la longueur de DC ?

CC108. Dans un repère orthonormé, la droite $y = 5x$ coupe la parabole $y = x^2$ au point A. La perpendiculaire à OA en O coupe la parabole en B. Quelle est l'aire du triangle OAB ?

CC109. Soit E l'ensemble des réels x pour lesquels les deux membres de l'égalité sont définis :

$$\cot 8x - \cot 27x = \frac{\sin kx}{\sin 8x \sin 27x}.$$

Si cette égalité tient pour tous les x dans E, que vaut k ?

CC110. Quel est le nombre de solutions réelles de l'équation :

$$|1 + x - |x - |1 - x||| = | - x - |x - 1||.$$

. . . .

CC106. At each summit of a regular tetrahedron of side length 3, we cut off a pyramid such that the cut-off surface makes an equilateral triangle. The four equilateral triangles thus obtained have all different dimensions. What is the total length of the edges of the solid thus truncated ? Provide a proof.

CC107. In a right triangle ABC with right angle at B and $BC = 1$, we place D on side AC such that $AD = AB = \frac{1}{2}$. What is the length of DC ?

CC108. In an orthonormal system, the line with equation $y = 5x$ crosses the parabola with equation $y = x^2$ in point A. The perpendicular to OA at O intersects the parabola at B. What is the area of triangle AOB ?

CC109. Let E be the set of reals x for which the two sides of the following equality are defined :

$$\cot 8x - \cot 27x = \frac{\sin kx}{\sin 8x \sin 27x}.$$

If this equality holds for all the elements of E, what is the value of k ?

CC110. What is the number of real solutions to the equation :

$$|1 + x - |x - |1 - x||| = | - x - |x - 1||.$$

CONTEST CORNER SOLUTIONS

CC56. From the set of consecutive integers $\{1, 2, 3, \ldots, n\}$, three integers that form a geometric sequence are deleted. The sum of the integers remaining is 6125. Determine the smallest value of n and all three-term geometric sequences that make this possible.

Originally 1996 Invitational Mathematics Challenge, Grade 11, problem 5.

We present the solution by Konstantine Zelator.

We know that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ and that a, ar, ar^2 are the terms of a geometric sequence. Thus $a + ar + ar^2 + 6125 = \frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2} > 6125$. The smallest value of n which works is 111. This means that $a(1 + r + r^2) = 91$. We have two cases.

Case 1 : r is a positive integer.

Then, both a and $r^2 + r + 1$ are natural numbers. Since 91=7×13, we get the following : $a = 1, r = 9$, or $a = 7, r = 3$, or $a = 13, r = 2$.

Case 2 : $r > 1$ and r is a fraction.

This means that $r = \frac{d}{c}$, where c and d are relatively prime positive integers with $c \geq 2$ and $d \geq 3$. Then $a(1 + \frac{d}{c} + \frac{d^2}{c^2}) = 91$ or $a(d^2 + cd + c^2) = 91c^2$.

Since ar^2 is an integer, we know $\frac{ad^2}{c^2}$ is an integer and because c does not divide d, c^2 divides a. Let $a = c^2k$ for some positive integer k. Then $k(d^2 + cd + c^2) = 91$ and since we know $c \geq 2$ and $d \geq 3$ giving us $d^2 + cd + c^2 \geq 19$. Thus $k = 1$ and $d^2 + cd + c^2 = 91$. The only solution is $d = 6$ and $c = 5$ and $a = 25$.

Therefore there are four 3-term sequences that satisfy the conditions :

$$1, 9, 81 \quad 7, 21, 63 \quad 13, 26, 52 \quad 25, 30, 36.$$

CC57. Triangle DEF is acute. Circle C_1 is drawn with DF as its diameter and circle C_2 is drawn with DE as its diameter. Points Y and Z are on DF and DE respectively so that EY and FZ are altitudes of ΔDEF. EY intersects C_1 at P, and FZ intersects C_2 at Q. EY extended intersects C_1 at R, and FZ extended intersects C_2 at S. Prove that P, Q, R, and S are concyclic points.

Originally 2002 Canadian Open Mathematics Challenge, problem B4.

Solved by M. Bataille ; S. Muralidharan ; and Z. Burnett. We present the solution by S. Muralidharan.

We will show that the points P, Q, R and S lie on a circle with centre D.

Let $\angle EDF$ be denoted by D, length $DF = y$ and length $DE = z$. Since DF is the diameter of the circle C_1 and EY is perpendicular to DF, it follows that $DP = DR$. Now, $DY = z\cos D$ and $O_1P = O_1D = \frac{y}{2}$. From the right-angled triangle PYO_1, we get :

$$PY^2 = O_1P^2 - O_1Y^2 = \frac{y^2}{4} - \left(\frac{y}{2} - z\cos D\right)^2 = yz\cos D - z^2\cos^2 D.$$

From right-angled triangle DPY, we have :

$$DP^2 = PY^2 + DY^2 = yz\cos D - z^2\cos^2 D + z^2\cos^2 D = yz\cos D.$$

Thus, we have $DP = DR = yz\cos D$.

By symmetry, if we use the above argument with the circle C_2, we get

$$DQ = DS = yz\cos D.$$

Thus P, Q, R and S lie on a circle with centre D and radius $yz\cos D$.

CC58. Find all real values of x, y and z such that

$$\begin{aligned} x - \sqrt{yz} &= 42 \\ y - \sqrt{zx} &= 6 \\ z - \sqrt{xy} &= -30\,. \end{aligned}$$

Originally problem B4 of 1997 Canadian Open Mathematics Challenge.

Solved by Š. Arslanagić ; M. Bataille ; M. Coiculescu ; J. L. Díaz-Barrero ; D. Văcaru ; E. Wang ; K. Zelator ; and T. Zvonaru. We present the solution of Titu Zvonaru.

The first and second equation imply $x, y > 0$, then from $xz > 0$, we conclude $z > 0$. Hence we make a substitution $x = a^2, y = b^2, z = c^2$. Our system becomes

$$a^2 - bc = 42, \quad b^2 - ac = 6, \quad c^2 - ab = -30. \tag{1}$$

Subtracting the second equation from the first equation, and subtracting the third from the second gives us the following two equations :

$$(a - b)(a + b + c) = 36, \tag{2}$$

$$(b - c)(a + b + c) = 36. \tag{3}$$

Hence $a - b, b - c$ and $a + b + c$ are all non-zero and $a - b = b - c$. This implies $a = 2b - c$. Substituting this into (1) yields

$$4b^2 - 5bc + c^2 = 42, \tag{4}$$

$$b^2 - 2bc + c^2 = 6. \tag{5}$$

From (5), $b - c = \pm\sqrt{6}$, and hence from (4), $c = \pm\sqrt{6}$. We then conclude the solutions to the system (1) are

$$(-3\sqrt{6}, -2\sqrt{6}, -\sqrt{6}),\ (3\sqrt{6}, 2\sqrt{6}, \sqrt{6}),$$

which each yield $x = 54, y = 24, z = 6$.

CC59. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place ?

Originally Question 3 of 2013 Stanford Math Tournament, Team test.

One incorrect solution was received.

CC60. How many integer solutions are there to

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1?$$

Originally 2010 APICS Math Competition, Question 5.

Solved by Richard Hess, whose solution we present below.

We consider the more general equation, where 2010 is replaced by an arbitrary n. Then, multiplying the equation by 2, we get

$$(0 + a_0)^2 + (a_0 + a_1)^2 + (a_1 + a_2)^2 + \cdots + (a_{n-1} + a_n)^2 + (a_n + 0)^2 = 2.$$

Define $b_0 = a_0$, $b_1 = a_0 + a_1$, $\cdots$, $b_k = a_{k-1} + a_k$, $\cdots$, $b_{n+1} = a_n$. Since our a_i are integers, so will the b_i. It follows that exactly two of the b_i will be nonzero. There are $\frac{(n+1)(n+2)}{2}$ many ways to choose $k < \ell$ so that $b_k, b_\ell \neq 0$.

Notice that $b_k^2, b_\ell^2 = 1$ so $b_k = \pm 1$. Notice that once we choose a_k then all other a_i are decided : $a_0 = a_1 = \cdots = a_{k-1} = 0$, $a_k = -a_{k+1} = \cdots = (-1)^{\ell-k-1}a_{\ell-1}$, $a_\ell = \cdots = a_n = 0$. It follows then that $a_k = \pm 1$, so there are only two possible choices for a_k. So the total number of solutions is $N = 2\frac{(n+1)(n+2)}{2} = (n+1)(n+2)$.

Since $n = 2010$, we see that $N = 2011 \cdot 2012 = 4046132$.

THE OLYMPIAD CORNER

No. 320

Nicolae Strungaru

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-olympiad@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er juin 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier d'avoir traduit les problèmes.

OC166. Soit $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Déterminer la valeur maximale de

$$\sum_{n=1}^{10}(na_n^2 - n^2a_n)\,.$$

OC167. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$, telles que

$$(x-2)f(y) + f(y+2f(x)) = f(x+yf(x))$$

pour tout $x, y \in \mathbb{R}$.

OC168. Soit $ABCD$ un carré. Déterminer tous les points P dans le plan, différents de A, B, C, D, tels que

$$\angle APB + \angle CPD = 180^\circ\,.$$

OC169. Déterminer tous les entiers positifs $n \geq 2$ tels que, pour tous les entiers $0 \leq i, j \leq n$, les nombres $i + j$ et $\binom{n}{i} + \binom{n}{j}$ ont la même parité.

OC170. Soit ABC un triangle. Les bissectrices des angles $\angle CAB$ et $\angle ABC$ intersectent les segments BC et AC à D et E respectivement. Démontrer que

$$DE \leq (3 - 2\sqrt{2})(AB + BC + CA)\,.$$

. . . .

OC166. Let $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Find the maximum value of

$$\sum_{n=1}^{10} (na_n^2 - n^2 a_n)\,.$$

OC167. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x - 2)f(y) + f(y + 2f(x)) = f(x + yf(x))$$

for all $x, y \in \mathbb{R}$.

OC168. Let $ABCD$ be a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB + \angle CPD = 180^\circ\,.$$

OC169. Find all positive integers $n \geq 2$ such that for all integers $0 \leq i, j \leq n$ the numbers $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have the same parity.

OC170. Let ABC be a triangle. The internal bisectors of angles $\angle CAB$ and $\angle ABC$ intersect segments BC, respectively AC at D, respectively E. Prove that

$$DE \leq (3 - 2\sqrt{2})(AB + BC + CA)\,.$$

OLYMPIAD SOLUTIONS

OC106. Find all the positive integers n for which all the n digit integers containing $n-1$ ones and 1 seven are prime.

Originally question 3 from Macedonia National Olympiad 2011.

Solved by R. Hess; D. E. Manes; and D. Văcaru. We give two solutions.

Solution 1, by David E. Manes.

If $n = 1$ then the number is 7 which is prime.

If $n = 2$ then the only two possibilities are 71 or 17, which are both primes.

We claim that there is no $n \geq 3$ which works. To see this, we look at the remainder of n when divided by 6.

If $n = 6k$ or $n = 6k + 3$, the sum of the digits of all these numbers is $n + 6$ which is divisible by 3. Therefore, none of the numbers is prime.

If $n = 6k+1$ with $k \geq 1$, then since $111111 = 7*11*13$, it follows that the number

$$7\underbrace{111\ldots1}_{6k}$$

is divisible by 7, and strictly larger than 7. Therefore, it is not prime.

If $n = 6k+2$, then, as $n > 2$ it follows that $k \geq 1$. Then $n = 6k'+8$ where $k' \geq 0$. Then, as 11171111 is a multiple of 7, it follows that

$$11171111\underbrace{111\ldots1}_{6k'}$$

is divisible by 7, and therefore is not prime.

If $n = 6k + 4$, then since $7111 = 13 * 547$, it follows that

$$7111\underbrace{111\ldots1}_{6k}$$

is divisible by 13, hence not prime.

Finally, if $n = 6k + 5$, then since 11711 is a multiple of 7, it follows that

$$11711\underbrace{111\ldots1}_{6k'}$$

is divisible by 7, and therefore is not prime.

Solution 2, by Richard I. Hess.

It is easy to see that $n = 1, n = 2$ work.

Case $n = 3$ doesn't work because all numbers are divisible by 9.

Case $n = 4$ doesn't work because $1711 = 29 * 59$.

Case $n = 5$ doesn't work because $11711 = 7 * 1763$.

We now prove that no $n \geq 6$ works. First, let us observe that for each n every number has the form

$$\underbrace{111\ldots1}_{n} + 6 * 10^k\,,$$

for some $1 \leq k \leq n$. Next, note that

$$7|\underbrace{111\ldots1}_{n} \quad \text{if and only if} \quad 7|\underbrace{999\ldots999}_{n} \quad \text{if and only if} \quad 7|10^n - 1.$$

This is equivalent to $10^n \equiv 1 \pmod 7 \Leftrightarrow 6 = \text{ord}_7(10)|n$. We split now the problem in two cases.

Case 1: $6|n$. Here all the numbers are divisible by 3, and hence none is prime.

Case 2: $6 \nmid n$. In this case, we saw above that $7 \nmid \underbrace{111\ldots1}_{n}$. Therefore, $\underbrace{111\ldots1}_{n}$ is invertible modulo 7. As 10 is a primitive root modulo 7, there exists a $1 \leq k \leq 6 \leq n$ such that

$$\underbrace{111\ldots1}_{n} \equiv 10^k \pmod 7\,,$$

Therefore, for this k, 7 divides $\underbrace{111\ldots1}_{n} + 6 * 10^k$ and hence this number is not prime.

It follows that $n = 1$ or $n = 2$.

OC107. ABC is a triangle of perimeter 4. Point X is marked on the ray AB and point Y is marked on the ray AC such that $AX = AY = 1$. BC intersects XY at M. Prove that one of the triangles ABM or ACM has perimeter 2.

Originally question 4 from Russia National Olympiad 2012, Grade 10 Day 1.

Solved by Michel Bataille whose solution we present below.

Let $BC = a, CA = b, AB = c$. From the hypotheses, $a = 4 - b - c$ and

$$\overrightarrow{AX} = \frac{1}{c}\overrightarrow{AB}, \quad \overrightarrow{AY} = \frac{1}{b}\overrightarrow{AC}\,.$$

In real coordinates relative to (A, B, C), we have

$$X = (c - 1 : 1 : 0), \quad Y = (b - 1 : 0 : 1), \quad XY : x + y(1 - c) + z(1 - b) = 0$$

so that $M = (0 : 1 - b : c - 1)$. Since M is interior to segment BC, $1 - b$ and $c - 1$ must have the same sign.

Without loss of generality, we suppose that $c > 1$ and $b < 1$ in what follows. Then

$$(c-b)\overrightarrow{BM} = (c-1)\overrightarrow{BC}$$

and

$$(c-b)\overrightarrow{CM} = (1-b)\overrightarrow{CB},$$

so that

$$(c-b)BM = (c-1)a = (c-1)(4-b-c), \tag{6}$$
$$(c-b)CM = (1-b)a = (1-b)(4-b-c)\,. \tag{7}$$

Also,

$$(c-b)\overrightarrow{AM} = (1-b)\overrightarrow{AB} + (c-1)\overrightarrow{AC}$$

so that

$$(c-b)^2AM^2 = (1-b)^2c^2 + (c-1)^2b^2 + (1-b)(c-1)(2\overrightarrow{AB}\cdot\overrightarrow{AC})$$

with

$$2\overrightarrow{AB}\cdot\overrightarrow{AC} = b^2 + c^2 - a^2 = b^2 + c^2 - (4-b-c)^2 = 8b + 8c - 2bc - 16\,.$$

A short calculation gives $(c-b)^2AM^2 = (3b+3c-2bc-4)^2$, hence

$$(c-b)AM = |3b+3c-2bc-4|\,. \tag{8}$$

Now, if $3b+3c-2bc-4 \geq 0$, then using (6), (7) and (8) , we obtain

$$(c-b)(AM+AC+MC) = 3b+3c-2bc-4+b(c-b)+(1-b)(4-b-c) = 2(c-b)\,,$$

and the perimeter of AMC is 2.

If $3b+3c-2bc-4 < 0$, then similarly,

$$(c-b)(AM+AB+MB) = -3b-3c+2bc+4+c(c-b)+(c-1)(4-b-c) = 2(c-b)\,,$$

and the perimeter of AMB is 2. The result follows.

OC108. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$2f(x) = f(x+y) + f(x+2y)\;\forall x \in \mathbb{R}, y \in [0,\infty)\,.$$

Originally question 1 from Romania Team Selection Test 2011, Day 1.

Solved by M. Bataille; D. Văcaru; and T. Zvonaru and N. Stanciu. We give the common solution of Titu Zvonaru and Neculai Stanciu.

If we replace y by $2y$ in the given relation we get

$$2f(x) = f(x+2y) + f(x+4y)\;\forall x \in \mathbb{R}, y \in [0,\infty)\,.$$

Therefore, for all $x \in \mathbb{R}, y \in [0, \infty)$ we have

$$f(x+y) = 2f(x) - f(x+2y) = f(x+4y)\,.$$

Replacing x by $x - y$ we get

$$f(x) = f(x+3y)\,\forall x \in \mathbb{R}, y \in [0,\infty)\,.$$

Then, if $a < b$ are any two real numbers, by setting $x = a$ and $y = \frac{b-1}{3}$ we get

$$f(a) = f(b)\,.$$

This proves that f is a constant function. Conversely, it is easy to check that all constant functions satisfy the given condition.

OC109. Let $a_1, a_2, \ldots, a_n, \ldots$ be a permutation of the set of positive integers. Prove that there exist infinitely many positive integers i so that $\gcd(a_i, a_{i+1}) \leq \frac{3}{4}i$.

Originally question 2 from China Team Selection 2011, test 3 Day 2.

No solution was received to this problem.

OC110. Let G be a graph, not containing K_4 as a subgraph. If the number of vertices is $3k$, with k integer, what is the maximum number of triangles in G?

Originally question 3 from Mongolia National Olympiad 2011, Team Selection Test Day 2.

Solved by Oliver Geupel. We present his solution below.

We prove that the maximum number of triangles is k^3.

First, we give an example of a graph G with k^3 triangles. The set of $3k$ vertices of G is split into three subsets of cardinality k each. Every vertex of G has an edge to every vertex in the two other subsets, but has no edge to any vertex in its own subset. [*Editor's Comment: G is called the complete tri-partite graph and is usually denoted by* $K_{k,k,k}$.] Clearly, G contains k^3 triangles but does not contain K_4 as a subgraph.

It remains to show that k^3 is also an upper bound.

Let G be a graph with $3k$ vertices, not containing K_4 as a subgraph. We claim that the number of triangles in G does not exceed k^3. Our proof is by induction on the number k.

For $k = 0$ the claim is obviously true.

Now let us assume that $k \geq 1$ and that the claim holds true for the number $k - 1$.

If G does not contain a triangle then there is nothing to prove. Otherwise consider any triangle in G. The set of $3k$ vertices of G is split into the subset V containing the three vertices of the triangle and the subset W of the $3(k-1)$ remaining vertices. The set T of triangles in G is split into four subsets

- the set T_1 of triangles with three vertices in V,
- the set T_2 of triangles with three vertices in W,
- the set T_3 of triangles with one vertex in V and two vertices in W and
- the set T_4 of triangles with two vertices in V and one vertex in W.

Clearly, $|T_1| = 1$.

The node set W induces a subgraph H of G which satisfies the induction hypothesis. Therefore, H contains at most $(k-1)^3$ triangles by induction, which implies $|T_2| \le (k-1)^3$.

Consider a triangle in T_3. If an edge of H would be combined with two distinct vertices in V to form two triangles, then these four vertices would constitute a 4-clique, which is impossible by hypothesis. Hence, every edge of H can occur in at most one triangle in T_3. Turan's theorem states that a K_{r+1}-free graph with n vertices has at most $\dfrac{(r-1)n^2}{2r}$ edges. Thus, H has not more than $3(k-1)^2$ edges, so that $|T_3| \le 3(k-1)^2$.

Consider a triangle in T_4. If a vertex in W would occur in two distinct triangles in T_4 then this vertex, being combined with the three vertices in V, would constitute a 4-clique, which is impossible by hypothesis. Hence, every vertex in W can occur in at most one triangle in T_4, so that $|T_4| \le 3(k-1)$.

Putting everything together, we obtain

$$|T| = |T_1| + |T_2| + |T_3| + |T_4| \le 1 + (k-1)^3 + 3(k-1)^2 + 3(k-1) = k^3,$$

which completes the induction.

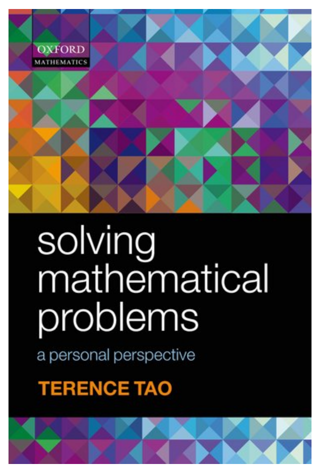
BOOK REVIEWS

Robert Bilinski

Solving Mathematical Problems: a personal perspective by Terence Tao
ISBN: 978-0-19-920560-8
Oxford University Press, 2006, $35 (US), 103pages

Reviewed by ***Robert Bilinski****, Collège Montmorency*

Terence Tao is now a world renowned Fields Prize winning mathematician, but his exploits started much earlier as he participated for the first time in the IMO in 1986 at the tender age of 10. He won a bronze medal that year. He represented Australia again the next two years and won his next two IMO medals, first a silver medal then a gold. He is, to this date, the youngest medal winner ever for his bronze medal and the youngest gold medal winner. After that, he started university and retired from his IMO career at the age of 13. The first edition of this book was written by him at the age of 15 and was reworked in 2006.



Curiously, it has never been reviewed in ***Crux***. I discovered this book by chance as I received the helm of the book review column and found it a fitting start for my hopefully long lasting return to the ***Crux*** editorial Board. I previously was Skoliad editor and am returning as Contest Corner Editor, both of which focus on problems oriented towards younger problem solvers. This book embodies the spirit of ***Crux*** and the inclusion of all problem solvers, the sharing of mathematical gems and of the beauty of their solutions. This book differs from other contest books, textbook, vulgarization of mathematics or any other general interest books I have read. It could have been taken out of a "Problem of the Month" or "Problem solver's toolkit" sections of ***Crux***. The aim of the book is not to show solutions to many problems; it is to highlight the thought process behind problem solving: rewriting, trial and error, backtracking, simplifying, generalizing and "sideways thinking" (changing math fields, using geometry in algebra or algebra in geometry...).

Through a limited list of typical contest problems handpicked by Tao, we are taken on a tour of problem solving. The chapters are organized around the techniques used in the problem solving: The first chapter deals with general techniques, the next chapters go through number theory, algebra and analysis, Euclidean geometry, then analytic geometry to end up with problems necessitating a multidisciplinary approach. But it is not a long list of problems per chapter that is exposed. Tao analyzes each problem in a real-time fashion that involves dead-ends and brainstorming bouts, and slowly builds up to a full resolution of each problem.

To give an example, the problem in the first chapter is centred on solving a triangle whose sides are in an arithmetic progression. Does one then go for algebra, geometry or number theory? Through an 8 step process specific to the situation, Tao guides us to a solution that ultimately borrows a bit from each. There are also a few problems in each section whose solutions are not provided, most of which were composed by the author, that are left as exercises for the readers to utilize the reasoning applied in that section.This book is not for you if you are looking for a compendium of 300 olympiad level problems, or for a pared down optimal solution guide or a book that gives out recipes for mathematical Olympic gold. This book is akin to a master class of problem solving, but given by a 15 year old virtuoso, full of maturity and skill, but also fun and simplicity.

As a problem solver, I enjoyed thoroughly reading Tao's perspective on problem solving and his analysis of each problem. Especially since it was written when he was only 15. I encourage our younger readers to pick up this book to get an insight into problem solving. I also encourage our more seasoned readers to be tempted: despite the fact that we each have our own styles, in problems we like and in the way we solve them, it is worthwhile to gain insight into the workings of one of the great mathematical minds of our age. All in all, the tone and style of the book make it eminently readable and down to earth.

Good reading!

Ramsey's Theory Through Examples, Exercises, and Problems: Part I

Veselin Jungić

1 Introduction

Ramsey theory is a contemporary mathematical field that is part of combinatorics. There are applications of Ramsey theory in number theory, geometry, topology, set theory, logic, ergodic theory, information theory, and theoretical computer science. In the words of Imre Leader [3],

> The fundamental kind of question Ramsey theory asks is: can one always find order in chaos? If so, how much? Just how large a slice of chaos do we need to be sure to find a particular amount of order in it?

The starting point in studying Ramsey theory is the 'pigeonhole principle':

Theorem 1 *Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$, then at least one pigeonhole contains at least two pigeons. More generally, there is at least one pigeonhole containing at least $\lceil n/k \rceil$ pigeons.*

Exercise 1 *Use the pigeonhole principle to prove that, for any natural number n, if $a_1, a_2, \ldots, a_{n+1}$ are distinct natural numbers between 1 and $2n$, then there exist i, j, $i \neq j$, such that a_i divides a_j.*

Exercise 1 is known as one of Erdős's favourite questions to ask of an ε. [1]

In Ramsey theory, it is often possible to state difficult problems in a way that any numerically literate person can understand them. As an example, here is a long standing open problem [2]:

Problem 1 *If the set of natural numbers is partitioned in a finite number of cells, must there exist x, y (with x and y not both equal to 2) such that $x + y$ and xy belong to the same cell?*

Ramsey theory is named after British mathematician, economist, and philosopher Frank Ramsey. He was born in 1903 in Cambridge, England, into a family of a Cambridge mathematics professor. The oldest of four siblings, Ramsey married when he was 22 years old and had two daughters. Ramsey died in 1930 at the age of 27. His youngest sister, Margaret Paul [4], suggested that the probable cause of his death was a liver illness brought on by the Hepatitis B virus that Ramsey contracted while swimming in the River Cam. Ramsey was a lifelong literature and music enthusiast and he enjoyed hiking during his vacations. Ramsey significantly

[1] In mathematics, the Greek letter ε is often used to denote a small positive real number. Paul Erdős, a famous Hungarian mathematician and the father of Ramsey theory, used to call young people "epsilons".

contributed to the fields of mathematics, economics, and philosophy while only in his twenties.

2 Ramsey's Theorem

As an introduction to Ramsey's theorem, we look at the following exercises. Consider the global population at the present and imagine that you can form all possible groups of, for example, 10 people. Next, partition this newly formed set of groups of 10 people in, for example, 100 mutually disjoint cells following any criterion you prefer.

Exercise 2 *Use the website called Worldometers* [2] *to find the estimate of the size of the current global population. Call this estimate* m.

Exercise 3 *Use your estimate* m *from Exercise 2 to find the number of different groups of 10 people. Call this number* g_{10} *and write it in scientific notation.*

Exercise 4 *Use your estimate* m *from Exercise 2 to find the number of different groups of 10 people that any given person would belong to. Write your answer in scientific notation.*

Exercise 5 *Use the number* g_{10} *from Exercise 3 to find the number of ways in which you can partition the set of groups of 10 people in 100 mutually disjoint cells. Approximate your answer with a power of* 10.

The enormous size of the number obtained in Exercise 5 illustrates what Leader meant when he asked "can one always find order in chaos?" Is there a pattern that is unavoidable regardless in which way we partition the set of groups of 10 people in 100 mutually disjoint cells? For example,

Question 1 *Can you be sure that for any fixed partition there would be 1000 people so that all groups of 10 that contain only individuals from those chosen 1000 people belong to the same partition cell?*

The answer to Question 1 is, "Yes, if there were enough people on Earth," since according to *Ramsey's Theorem*:

Theorem 2 *Given any* r, n, *and* μ *we can find an* m_0 *such that, if* $m \geq m_0$ *and the* r*-combinations of any* Γ_m *are divided in any manner into* μ *mutually exclusive classes* C_i $(i = 1, 2, \ldots, \mu)$, *then* Γ_m *must contain a sub-class* Δ_n *such that all the* r*-combinations of members of* Δ_n *belong to the same* C_i.

In our example $r = 10$, $n = 1000$, $\mu = 100$, and m represents the size of the global population Γ_m at a certain moment in time. The phrase "r-combinations" in the theorem matches our phrase "the set of groups of 10 people". Moreover, the phrase "a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i" refers to our "1000 people so that all groups of 10 that contain

[2] http://www.worldometers.info/world-population/

only individuals from those chosen 1000 people belong to the same partition cell". For these values of r, n, and μ, the value of the number m_0 is unknown, but almost certainly the world population will never reach the required m_0.

The above theorem by Frank Ramsey appears in *On a Problem of Formal Logic* in the Proceedings of the London Mathematical Society in 1930 [5]. Ron Graham and Bruce Rothschild, pioneers in Ramsey theory, described Ramsey's theorem in the following way [1]:

> The theorem is a profound generalization of the 'pigeonhole principle' or 'Dirichlet box principle'. As is the case with many beautiful ideas in mathematics, Ramsey's theorem extends just the right aspect of an elementary observation and derives consequences which are extremely natural although far from obvious.

Exercise 6 *For which values of r and n does Ramsey's theorem become the 'pigeonhole principle'?*

3 Ramsey's Theorem: Friends and Strangers

Consider the following so-called 'dinner party problem':

Problem 2 *How many people must be at dinner to ensure that there are either three mutual acquaintances or there are three mutual strangers?*

Exercise 7 *For which values of r, n, and μ does Ramsey's theorem become the 'party problem'?*

To solve Problem 2, we use so-called *edge 2-colourings of complete graphs* on 5 and 6 vertices, K_5 and K_6. The complete graph K_m on m vertices is represented by a drawing in which we first draw m points in the plane so that no three points are collinear and then we draw a line segment between each pair of those m points. The points are called *vertices* and the line segments are called *edges* (Figure 1).

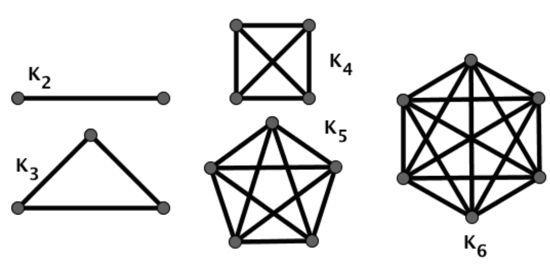


Figure 1: Complete graphs K_2, K_3, K_4, K_5 and K_6

Exercise 8 *Use two colours, say red and blue, to colour the edges of K_6. Each edge can be coloured by only one colour. (In more mathematical terms, you are asked to perform an edge 2-colouring of K_6.)*

Note that in Exercise 8 you have two colours available, but you may wish to use only one colour.

Question 2 *How many different edge 2-colourings of K_6 are there?*

Question 3 *Can you find a monochromatic triangle in your colouring; i.e., can you find three edges coloured by the same colour that form a triangle? (Note that any triangle represents K_3.)*

Problem 3 *Does any edge 2-colouring of K_6 yield a monochromatic K_3?*

Exercise 9 *Explain in a short paragraph why Problem 2 and Problem 3 are equivalent.*

The answer to Problem 3 is, "Yes, it does." To see this, follow these two steps.

> **Step 1:** Suppose that you are given an edge 2-colouring of K_6. Fix one vertex. How many edges are coming out of this vertex? Based on which theorem can you conclude that at least three of those edges will be of the same colour?
>
> **Step 2:** Consider all possible 2-colourings of the triangle determined by the vertices adjacent by the three edges of the same colour to the originally fixed vertex.

Exercise 10 *Find an edge 2-colouring of K_5 with no monochromatic triangles.*

Exercise 11 *Based on Exercises 7, 9, and 10 and the answer to Problem 3 conclude that, in the notation of Ramsey's theorem, if $r = \mu = 2$ and $n = 3$ then $m_0 = 6$. This fact is usually written $R(3,3) = 6$ with the meaning that any 2-colouring of K_6 yields a monochromatic K_3 in the 'first' colour or a monochromatic K_3 in the 'second' colour and that there is an edge 2-colouring of K_5 with no monochromatic triangles.*

In general, for natural numbers s and t, $s, t \geq 2$, we define *the Ramsey number* $R(s,t)$ as the minimum number n for which any edge 2-colouring of K_n in red and blue contains a red K_s or a blue K_t.

Exercise 12 *Find $R(2,t)$ for any natural number $t \geq 2$. What can you tell about $R(t,2)$?*

Problem 4 *Any graph with at least 6 vertices contains a complete subgraph on 3 vertices or an independent subgraph of 3 vertices (An independent subgraph of a given graph consist of vertices of which no pair is adjacent).*

Problem 5 *True or False: Each 2-colouring of K_6 yields at least two monochromatic triangles?*

Hints and comments

Exercise 1 For each i, write a_i as $a_i = 2^{b_i} q_i$, where q_i is an odd number, and consider the sequence of odd numbers $\{q_1, \ldots, q_{n+1}\}$ in $[1, 2n]$.

Exercise 3 Note that $g_{10} = \binom{m}{10}$. Write this number in scientific notation.

Exercise 4 Note that this number is given by $\binom{m-1}{9}$.

Exercise 5 There are 100 cells and for each cell there are $G = \binom{g_{10}}{10}$ choices. Thus, the number of ways is $100^G = 10^{2G}$. Use Exercise 4 to complete this exercise.

Problem 4 Suppose that a graph G with more than 6 vertices is given. Colour all edges of the graph G red. Next, draw all missing edges and colour them blue.

Problem 5 There are 20 different triangles in K_6. Colour the edges of K_6 with red and blue and call a triangle in K_6 *2-coloured* if it is not monochromatic. Each 2-coloured triangle contains two *2-coloured* angles, i.e., two angles with sides of different colours. Conclude that the number of 2-coloured triangles is equal to one half of the number of 2-coloured angles. Count the number of the 2-coloured angles to see that the number of 2-coloured triangles is at most 18.

References

[1] Graham, R.; Rothschild, B., Ramsey Theory, Studies in Comb., G.-G. Rota ed., Math. Assoc. of Amer., 17 (1978), 80-99.

[2] Hindman, N., Leader, I., and Strauss, D., Open problems in partition regularity, Combin. Probab. Comput., 12(5-6):571-583, 2003.

[3] Leader, I., Friends and strangers, Plus magazine `http://plus.maths.org/content/friends-and-strangers` (Accessed on January 20, 2015.)

[4] Paul, M., Frank Ramsey (1903—1930) A Sister's Memoir, Smith-Gordon in cooperation with Nishimura, 2012.

[5] Ramsey, F.P., On a problem of formal logic, Proc. London Math. Soc. Ser 2, vol.30, part 4, 1930, 338-384.

Veselin Jungić
Department of Mathematics, Simon Fraser University
Burnaby, BC, Canada
`vjungic@sfu.ca`

Excerpt from The Math Olympian

Richard Hoshino

Editor's Prologue. *The Math Olympian* is a novel by Richard Hoshino, himself a former Olympian who today teaches mathematics at Quest University Canada. The story traces the growth of Bethany MacDonald from an insecure and bullied grade 5 student, who is unhappy in school, to a confident high schooler who is about to write the Canadian Mathematical Olympiad and maybe realize her dream of qualifying for the Canadian IMO team. The story begins as Bethany looks at the first problem shortly after the contest starts, at 9:00 a.m.

. . . .

The Canadian Mathematical Olympiad, Problem #1:

Determine the value of:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\cdots+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}.$$

I stare at the first problem, not sure where to start.

I circle the first term in the expression of Problem #1, the one with the ugly exponent $9^{\frac{1}{1000}}$. Am I actually supposed to calculate the 1000th root of 9? Without a calculator, I know that's not possible.

There has to be an insight somewhere. This is an Olympiad problem, and all Olympiad problems have nice solutions that require imagination and creativity rather than a calculator.

I re-read the question yet again, and confirm that I have to determine the following sum:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\cdots+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}.$$

There are 999 terms in the sum, and each term is of the form $\frac{9^x}{9^x+3}$. In the first term, x equals $\frac{1}{1000}$; in the second term, x equals $\frac{2}{1000}$; in the third term, x equals $\frac{3}{1000}$; and so on, all the way up to the last term, where x equals $\frac{999}{1000}$.

In the entire expression, there's only one doable calculation, the term right in the middle. I know I can calculate $\frac{9^{\frac{500}{1000}}}{9^{\frac{500}{1000}}+3}$, using the fact that $\frac{500}{1000}=\frac{1}{2}$.

Since raising a quantity to the exponent $\frac{1}{2}$ is the same as taking its square root, I see that:

$$\frac{9^{\frac{500}{1000}}}{9^{\frac{500}{1000}}+3}=\frac{9^{\frac{1}{2}}}{9^{\frac{1}{2}}+3}=\frac{\sqrt{9}}{\sqrt{9}+3}=\frac{3}{3+3}=\frac{3}{6}=\frac{1}{2}.$$

But other than this, I'm not sure what to do. Twirling my pen and closing my eyes, I concentrate, hoping for a spark.

One idea comes to mind: setting up a "telescoping series". My mentor, Mr. Collins, introduced me to this beautiful technique years ago at one of our Saturday afternoon sessions at Le Bistro Café. Before explaining the concept to me, Mr. Collins first gave me a simple question of adding five fractions:

$$\text{Without using a calculator, determine } \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}.$$

I solved Mr. Collins problem by finding the common denominator. In this case, the common denominator is 60, the smallest number that evenly divides into each of $2, 6, 12, 20$, and 30. So the answer is:

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}=\frac{30+10+5+3+2}{60}=\frac{50}{60}=\frac{5}{6}.$$

And then I remembered Mr. Collins' smile as he gave me another addition problem:

$$\text{Determine } \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}$$

This time, it took me almost fifteen minutes to get the answer. Most of the time was spent trying to figure out the common denominator, which I eventually determined to be 2520. But it was a tedious process of checking and re-checking all of my calculations.

After Mr. Collins congratulated me on getting the right answer, he pointed to the nine fractions on my sheet of paper and asked if there was a pattern. After staring at the numbers for a while, I saw it:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl} \mathbf{2} & = & 1\times 2 & \mathbf{6} & = & 2\times 3 & \mathbf{12} & = & 3\times 4 \\ \mathbf{20} & = & 4\times 5 & \mathbf{30} & = & 5\times 6 & \mathbf{42} & = & 6\times 7 \\ \mathbf{56} & = & 7\times 8 & \mathbf{72} & = & 8\times 9 & \mathbf{90} & = & 9\times 10 \end{array}$$

Mr. Collins suggested I write $\frac{1}{90}$ as the difference of two fractions: $\frac{1}{90}=\frac{1}{9}-\frac{1}{10}$. He then asked whether there were any other terms in this expression that could also be written as the difference of two fractions. I eventually saw that $\frac{1}{2}=\frac{1}{1}-\frac{1}{2}$ and $\frac{1}{6}=\frac{1}{2}-\frac{1}{3}$.

Once I saw the pattern, I discovered this amazing solution, called a "telescoping series": $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}$ can be re-written as:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{8}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{10}\right).$$

This is just $\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\frac{1}{6}-\frac{1}{7}+\frac{1}{7}-\frac{1}{8}+\frac{1}{8}-\frac{1}{9}+\frac{1}{9}-\frac{1}{10}$.

Since one negative fraction cancels a positive fraction with the same value, all the terms in the middle get eliminated:

$$\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{5}}-\cancel{\frac{1}{6}}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{7}}+\cancel{\frac{1}{7}}-\cancel{\frac{1}{8}}+\cancel{\frac{1}{8}}-\cancel{\frac{1}{9}}+\cancel{\frac{1}{9}}-\frac{1}{10}.$$

Like a giant telescope that collapses down to a small part at the top and a small part at the bottom, this series collapses to the difference $\frac{1}{1}-\frac{1}{10}$, which equals $\frac{9}{10}$. So the answer is $\frac{9}{10}$.

That day, Mr. Collins showed me several problems where the answer can be found using a telescoping series, where a seemingly-tedious calculation can be solved with elegance and beauty.

The key is to represent each term as a difference of the form $x-y$, where y is called the "subtrahend" and x is called the "minuend". From Mr. Collins' examples, I learned that the series telescopes every time the subtrahend of one term equals the minuend of the following term.

As I recall that lesson with Mr. Collins many years ago, I'm hopeful that I can use this technique to solve the first problem of the Canadian Math Olympiad. I look at Problem #1 again, reminding myself of what I need to determine.

I start with the general expression $\frac{9^x}{9^x+3}$ and try to write it down as the difference of two functions, so that the subtrahend of each term equals the minuend of the following term.

I try a bunch of different combinations to the difference to work out to $\frac{9^x}{9^x+3}$ such as the expression $\frac{1}{3^x}-\frac{1}{3^x+1}$ which almost works but not quite. I attempt other combinations using every algebraic method I know. All of a sudden, I realize the futility of my approach.

The denominator doesn't factor nicely, so this approach cannot work. Oh no.

9:19 a.m.

I feel the first bead of sweat on my forehead, and wonder if I'm going to get another "math contest anxiety attack". I close my eyes and take a deep breath, knowing that if I start to panic and lose focus, my chances of becoming a Math Olympian are over.

Calm down, Bethany, calm down. There's lots of time left. You can do this.

. . . .

To be continued in issue 4.

"The Math Olympian" was published by FriesenPress in January 2015. For more information, please visit `www.richardhoshino.com`.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-psol@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page.*

Comment soumettre un problème. *Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er juin 2015***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

Un astérisque ($\star$) signale un problème proposé sans solution.

La rédaction remercie Rolland Gaudet, University College of Saint Boniface, d'avoir traduit les problèmes.

3911. *Proposé par Paul Bracken.*

Soit $x_0 \in (0, 1 - 1/a]$, où $a > 1$, et soit la suite définie par $x_n = x_{n-1} - x_{n-1}^2$ pour $n \in \mathbb{N}$. Démontrer que x_n satisfait les inégalités

$$\frac{x_0}{anx_0 + 1} < x_n < \frac{x_0}{nx_0 + 1}, \quad n \in \mathbb{N}.$$

3912. *Proposé par Michel Bataille.*

Soit ABC un triangle scalène aucun angle rectangle et soit H son orthocentre. Si A_1, B_1 et C_1 sont les mi points de BC, CA et AB respectivement, démontrer que les orthocentres de HAA_1, HBB_1 et HCC_1 sont colinéaires.

3913. *Proposé par Ovidiu Furdui.*

Calculer

$$\int_0^\infty \int_0^\infty \frac{dxdy}{(e^x + e^y)^2}.$$

3914. *Proposé par George Apostolopoulos; generalisé par le comité de rédaction.*

Soit ABC un triangle avec R le rayon du cercle circonscrit, r le rayon du cercle inscrit et s le semi périmtre, tels que $s = kr$. Démontrer que $\frac{2k}{3\sqrt{3}} < \frac{R}{r} < \frac{k^2-3}{12}$.

3915. *Proposé par Marcel Chiriţă.*

Soient M et N des points sur les côtés AB et AC, respectivement, du triangle ABC, et posons $O = BN \cap CM$. Démontrer qu'il y a un nombre infini d'exemples (pas affinement équivalents) tels que les surfaces des quatre régions MBO, BCO, CNO et $AMON$ sont toutes entières.

3916. *Proposé par Nathan Soedjak.*

Soient a, b, c des nombres réels positifs. Démontrer que

$$\left(\frac{ab}{c}\right)^2 + \left(\frac{bc}{a}\right)^2 + \left(\frac{ca}{b}\right)^2 \geq 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2.$$

3917. *Proposé par Peter Y. Woo.*

À partir d'un cercle Z, son centre O et un point A sur Z, et à l'aide d'une longue règle non graduée, pouvez-vous dessiner:

i) des points B,C et D sur Z, tels que $ABCD$ ets un carreé?

ii) le carreé $AOBA'$?

iii) les points B, W'', W et W' sur Z tels que les angles AOB, AOW'', AOW et AOW' sont $90°, 60°, 45°$ and $30°$?

3918. *Proposé par George Apostolopoulos.*

Soient a, b et c des nombres réels positifs tels que $a^2 + b^2 + c^2 = 1$. Démontrer que

$$\sqrt{(ab)^{2/3} + (bc)^{2/3} + (ac)^{2/3}} < \frac{2+\sqrt{3}}{3}.$$

3919. *Proposé par Michel Bataille.*

Soit I le centre du cercle inscrit du triangle ABC. Le segment AI rencontre le cercle inscrit à M; la perpendiculaire à AM au point M intersecte BI à N. Si P est un point sur la ligne AI, démontrer que PC est perpendiculaire à AI si et seulement si PN est parallèle à BM.

3920. *Proposé par Alina Sîntămărian.*

Évaluer

$$\sum_{n=0}^{\infty} \frac{16n^2 + 20n + 7}{(4n+2)!}.$$

. . . .

3911. *Proposed by Paul Bracken.*

Let $x_0 \in (0, 1 - 1/a]$, where $a > 1$, and define the sequence $x_n = x_{n-1} - x_{n-1}^2$ for $n \in \mathbb{N}$. Prove that x_n satisfies the inequalities

$$\frac{x_0}{anx_0 + 1} < x_n < \frac{x_0}{nx_0 + 1}, \quad n \in \mathbb{N}.$$

3912. *Proposed by Michel Bataille.*

Let ABC be a scalene triangle with no right angle and H as its orthocenter. If A_1, B_1 and C_1 are the midpoints of BC, CA and AB respectively, prove that the orthocenters of HAA_1, HBB_1 and HCC_1 are collinear.

3913. *Proposed by Ovidiu Furdui.*

Calculate

$$\int_0^{\infty} \int_0^{\infty} \frac{dxdy}{(e^x + e^y)^2}.$$

3914. *Proposed by George Apostolopoulos; generalized by the Editorial Board.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s, such that $s = kr$. Prove that $\frac{2k}{3\sqrt{3}} < \frac{R}{r} < \frac{k^2 - 3}{12}$.

3915. *Proposed by Marcel Chiriţă.*

Let M and N be points on the sides AB and AC, respectively, of triangle ABC, and define $O = BN \cap CM$. Show that there are infinitely many examples (that are not affinely equivalent) for which the areas of the four regions MBO, BCO, CNO and $AMON$ are all integers.

3916. *Proposed by Nathan Soedjak.*

Let a, b, c be positive real numbers. Prove that

$$\left(\frac{ab}{c}\right)^2 + \left(\frac{bc}{a}\right)^2 + \left(\frac{ca}{b}\right)^2 \geq 3\left(\frac{ab+bc+ca}{a+b+c}\right)^2.$$

3917. *Proposed by Peter Y. Woo.*

Given a circle Z, its center O, and a point A on Z, with only a long unmarked ruler, and no compass, can you draw:

i) points B,C and D on Z so that $ABCD$ is a square?

ii) the square $AOBA'$?

iii) the points B, W'', W and W' on Z such that angles AOB, AOW'', AOW and AOW' are $90°, 60°, 45°$ and $30°$?

3918. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\sqrt{(ab)^{2/3} + (bc)^{2/3} + (ac)^{2/3}} < \frac{2+\sqrt{3}}{3}.$$

3919. *Proposed by Michel Bataille.*

Let I be the incentre of triangle ABC. The line segment AI meets the incircle at M and the perpendicular to AM at M intersects BI at N. If P is a point of the line AI, prove that PC is perpendicular to AI if and only if PN is parallel to BM.

3920. *Proposed by Alina Sîntămărian.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{16n^2+20n+7}{(4n+2)!}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3811. *Proposed by Jung In Lee.*

Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers a and b, $af(a+b) + bf(a) + b^2$ is a perfect square.

No solutions were received for this problem. The problem remains open.

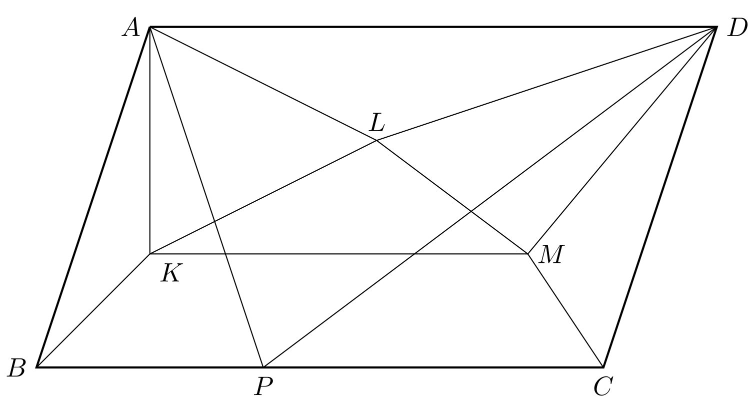
3812. *Proposed by George Apostolopoulos.*

Let $ABCD$ be a parallelogram and P be a point on side BC. Let K, L, and M be the centroids of triangles PAB, PAD and PCD, respectively. Prove that

$$[AKL] + [DLM] = [BKMC],$$

where $[\cdot]$ represents the area.

Solved by AN-anduud Problem Solving Group; M. Bataille; P. De; N. Evgenidis; O. Geupel; J. Heuver; O. Kouba; M. Modak; C. Mortici; C. Sánchez-Rubio; Skidmore College Problem Solving Group; N. Stanciu and T. Zvonaru; E. Swylan; and the proposer. We present the solution of Michel Bataille.



We use areal coordinates with reference to triangle ABC.

Let $P = tB + (1-t)C$ where $t \in [0,1]$. Observing that $D = A - B + C$, we have

$$3K = A + B + P = A + (1+t)B + (1-t)C$$
$$3L = A + P + D = 2A + (t-1)B + (2-t)C$$
$$3M = A + (t-1)B + (3-t)C$$

It follows that

$$\begin{aligned}[AKL] &= \frac{1}{9}|\delta_1|[ABC],\\ [DLM] &= \frac{1}{9}|\delta_2|[ABC],\\ [BKM] &= \frac{1}{9}|\delta_3|[ABC],\\ [BMC] &= \frac{1}{3}|\delta_4|[ABC],\end{aligned}$$

where $\delta_i,\ i=1,2,3,4$ are the following determinants, with columns from the areal coordinates of the vertices:

$$\delta_1 = \begin{vmatrix} 1 & 1 & 2\\ 0 & 1+t & t-1\\ 0 & 1-t & 2-t\end{vmatrix},\quad \delta_2 = \begin{vmatrix} 1 & 2 & 1\\ -1 & t-1 & t-1\\ 1 & 2-t & 3-t\end{vmatrix},$$

$$\delta_3 = \begin{vmatrix} 0 & 1 & 1\\ 1 & 1+t & t-1\\ 0 & 1-t & 3-t\end{vmatrix},\quad \delta_4 = \begin{vmatrix} 0 & 1 & 0\\ 1 & t-1 & 0\\ 0 & 3-t & 1\end{vmatrix}.$$

A simple calculation gives $\delta_1 = 3-t,\ \delta_2 = 2+t,\ \delta_3 = 2,\ \delta_4 = 1$ and so

$$[AKL]+[DLM] = (3-t+2+t)\frac{[ABC]}{9} = \frac{5\cdot[ABC]}{9} = [BKM]+[BMC] = [BKMC].$$

3813. *Proposed by Michel Bataille.*

Find the smallest constant C such that the inequality

$$(a^7+b^7+c^7)^6 \le C(a^6+b^6+c^6)^7$$

holds for all real numbers a, b, c such that $a+b+c=0$.

Solved by R. Barbara; R. Hess; O. Kouba; K.-W. Lau; N. Hodzić and S. Malikić; C. Mortici; P. Perfetti; S. Wagon; and the proposer. There was one incomplete solution. We present the solution of Roy Barbara, which is both efficient and provides a generalization.

The answer is

$$\frac{(2^6-1)^6}{2(1+2^5)^7} = \frac{3^5\cdot 7^6}{2\cdot 11^7} = \frac{28588707}{38974342} = 0.733526\ldots$$

We prove a more general result. Consider positive integers p and q both of which are odd and m and n both of which are even that satisfy $pm = qn$. Then the smallest constant C for which the inequality

$$(a^p+b^p+c^p)^m \le C(a^n+b^n+c^n)^q$$

holds for all real numbers a, b, c with $a + b + c = 0$ is

$$C_0 \doteq \frac{(2^p - 2)^m}{(2^n + 2)^q}.$$

If $abc = 0$ the left member is 0 and the inequality holds for all positive C. Suppose henceforth that $abc \neq 0$. Since $a + b + c = 0$, two of the variables have one sign and the other the opposite. Since the expressions on both sides of the inequality do not change if we replace each variable by its negative, we may suppose wolog that $a, b > 0$ and $c < 0$. Furthermore, since both sides of the inequality are homogeneous with degree $pm = qn$, we may suppose that $a + b = 2$. Therefore, the given inequality is equivalent to

$$(2^p - (a^p + b^p))^m \leq C(2^n + a^n + b^n)^q$$

with a, b positive and summing to 2.

Using the fact that $2^p - (a^p + b^p) \geq 0$ and the convexity of the function x^k for $k \geq 1$, we have that

$$a^p + b^p \geq 2\left(\frac{a+b}{2}\right)^p$$

and

$$a^n + b^n \geq 2\left(\frac{a+b}{2}\right)^n,$$

whence

$$\begin{aligned}(2^p - (a^p + b^p))^m &\leq (2^p - 2)^m = C_0(2^n + 2)^q \\ &\leq C_0(2^n + a^n + b^n)^q.\end{aligned}$$

Since equality occurs when $a = b = 1$, we conclude that C_0 is the minimum value of C.

Editor's comments. Because of the condition $a + b + c = 0$ and the homogeneity of the inequality, many solvers reduced the problem to maximizing a function of a single variable, for example $f(x) = (x^7 + 1 - (x+1)^7)^6(x^6 + 1 + (x+1)^6)^{-7}$. Several solvers relied on mathematical software to negotiate the technical complexities. Stan Wagon of Macalester College used `Mathematica` and Lagrange Multipliers to optimize $(a^7 + b^7 + c^7)^6(a^6 + b^6 + c^6)^{-7}$. Replacing 6 and 7 by low values of m and $m + 1$, he found that the optimal values were rational and wondered whether this was always so.

Another approach was taken by the proposer and one other submitter. The numbers a, b, c are roots of the cubic

$$(x - a)(x - b)(x - c) = x^3 - qx - r,$$

where $q = -(ab + bc + ca)$ and $r = abc$. Consider the case that $r \neq 0$. For $n \geq 1$, let $s_n = a^n + b^n + c^n$. Then $s_1 = 0$, $0 < s_2 = 2q$, $s_3 = s_1^3 + 3(qs_1 + 3r) - 6r = 3r$

and $s_n = qs_{n-2} + rs_{n-3}$ for $n \geq 4$. This leads to $s_4 = qs_2 + rs_1 = 2q^2$, $s_5 = 5qr$, $s_6 = 2q^3 + 3r^2$ and $s_7 = 7q^2r$.

The proposer expressed the inequality in terms of s_2 and s_3 and an adroit use of a weighted arithmetic-geometric means inequality led to a successful conclusion. The other submitter did not fare so well. The given inequality was rewritten as $7^6q^{12}r^6 \leq C(2q^3 + 3r^2)^7$ or

$$\frac{t^4}{(3+2t)^7} \leq \frac{C}{7^6},$$

where $t = q^3/r^2 > 0$. For $t > 0$, the left side assumes its maximum value of $16/7^7$ at $t = 2$, so that the minimum value of C for which the inequality is satisfied is apparently $16/7$.

But does t in fact range over all of the positive reals? Since $q = \frac{1}{2}(a^2 + b^2 + c^2) \geq \frac{3}{2}(a^2b^2c^2)^{1/3}$, we have that

$$\frac{q^3}{r^2} = \frac{1}{8} \cdot \frac{(a^2+b^2+c^2)^3}{a^2b^2c^2} \geq \frac{27}{8} > 2.$$

The restriction that $a + b + c = 0$ will push this lower bound even higher since the condition $a = b = c$ for equality cannot occur. We note that $(a, b, c) = (1, 1, -2)$ corresponds to $t = 27/4$ and the value $3^5/2 \cdot 11^7$ for $t^4(3 + 2t)^{-7}$.

3814. *Proposed by Marcel Chiriţă.*

Prove that for any number x in the closed interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M in the plane of the square $ABCD$ such that

$$x = \frac{AM + MC}{BM + MD}.$$

Solved by A. Alt; AN-anduud Problem Solving Group; G. Apostolopoulos; D. Bailey, E. Campbell, and C. Diminnie; R. Barbara; M. Bataille, P. De; O. Geupel; J. Hawkins and D. Stone; O. Kouba; S. Malikić; C. Mortici; C. Sánchez-Rubio; E. Swylan; D. Văcaru; and the proposer. We present 2 solutions.

Solution 1, by Prithwijit De.

Because we deal with ratios of segments, we can place the square in the Cartesian plane so that the coordinates of A, B, C and D are $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ respectively. The coordinates of any point M in the line segment BD may be assumed to be $(0, m)$, for some real number $m \in [-1, 1]$. For such a point M, $BM + MD = 2$ and $AM + MC = 2\sqrt{1 + m^2}$. Therefore

$$\frac{AM + MC}{BM + MD} = \sqrt{1 + m^2}.$$

Observe that as M moves along the line segment BD, $\sqrt{1 + m^2}$ decreases continuously from $\sqrt{2}$ to 1, then returns to $\sqrt{2}$, thereby covering the interval $[1, \sqrt{2}]$. If

M is taken on the line segment AC then we may assume that its coordinates are $(m, 0)$ for some real number $m \in [-1, 1]$. For such a point M,

$$\frac{AM + MC}{BM + MD} = \frac{1}{\sqrt{1 + m^2}}.$$

Observe that as M moves along the line segment AC, $\frac{1}{\sqrt{1+m^2}}$ increases continuously from $\frac{\sqrt{2}}{2}$ to 1 and then returns to $\frac{\sqrt{2}}{2}$, thereby covering the interval $[\frac{\sqrt{2}}{2}, 1]$. Thus for any x in the interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M on a diagonal of the square such that

$$x = \frac{AM + MC}{BM + MD}.$$

Solution 2 is a composite of similar arguments from George Apostolopoulos, Omran Kouba and Marcel Chiriţă.

Let M be any point in the plane of the square $ABCD$, and consider the function

$$f(M) = \frac{AM + MC}{BM + MD}.$$

Since its denominator does not vanish, f is a continuous function. Moreover, $f(A) = \sqrt{2}/2$ and $f(B) = \sqrt{2}$. Thus, by the intermediate value theorem we know that for any number x in the closed interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M such that $f(M) = x$, which is the desired conclusion.

But we can say more, namely, that the equation $f(M) = x$ has a solution M in the plane if and only if x belongs to $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$. Indeed, note that for nonnegative u and v we have

$$u^2 + v^2 \le (u + v)^2 \le 2(u^2 + v^2),$$

with equality on the left when $uv = 0$, and on the right when $u = v$. So, if a, b, c and d are nonnegative real numbers such that $a^2 + b^2 = c^2 + d^2$ then

$$\frac{1}{\sqrt{2}} \le \frac{a + b}{c + d} \le \sqrt{2}. \tag{1}$$

Now, with O at the centre of the square we have $\overrightarrow{OA} = -\overrightarrow{OC}$ and $\overrightarrow{OB} = -\overrightarrow{OD}$, so that for every point M,

$$\begin{aligned} AM^2 + CM^2 &= (\overrightarrow{OM} - \overrightarrow{OA})^2 + (\overrightarrow{OM} + \overrightarrow{OA})^2 = 2OM^2 + 2OA^2 \\ BM^2 + DM^2 &= (\overrightarrow{OM} - \overrightarrow{OB})^2 + (\overrightarrow{OM} + \overrightarrow{OB})^2 = 2OM^2 + 2OB^2. \end{aligned}$$

That is $AM^2 + CM^2 = BM^2 + DM^2$. (Alternatively, note that MO is the median of both triangles MAC and MBD whose sides opposite M have equal lengths.) Applying (1) to $a = MA$, $b = MB$, $c = MC$ and $d = MD$, we conclude that $f(M) \in \left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$ for every point M in the plane. Moreover, analyzing the cases of equality in (1), we see that $f(M) = \sqrt{2}$ if and only if $M \in \{B, D\}$ and $f(M) = \sqrt{2}/2$ if and only if $M \in \{A, C\}$.

3815. *Proposed by Paolo Perfetti.*

Show that $x^x \leq x^2 - x + 1$ for all $0 \leq x \leq 1$.

Solved by AN-anduud Problem Solving Group; Š. Arslanagić; M. Bataille; N. Evgenidis; M. Dincă; O. Furdui; O. Geupel; H. Wang and J. Woydylo; O. Kouba; K.W. Lau; A. Li; S. Malikić (2 solutions); P. McCartney; D. Smith; and the proposer. There were 4 solutions via numerical methods to prove claims about some inequalities, which (while correct) require approximating roots of non-algebraic equations via computer assistance. We present 4 solutions, each using more powerful theorems than the previous.

Foreword: We take $0^0 = \lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln(x)} = e^0 = 1$, so we have $1 \leq 1^2 - 1 + 1 = 1$, and so the following solutions will only prove the inequality for all $x \in (0, 1]$.

Solution 1, by Omran Kouba.

Since $1 - x + x^2 > 0$ for $x \in [0, 1]$ we may consider the function

$$f : (0, 1] \to \mathbb{R},\ f(x) = \ln(1 - x + x^2) - x \ln x.$$

We have

$$f'(x) = \frac{2x - 1}{1 - x + x^2} - 1 - \ln x$$

and

$$f''(x) = \frac{1 + 2x - 2x^2}{(1 - x + x^2)^2} - \frac{1}{x} = \frac{1 - x}{x(1 - x + x^2)^2} g(x)$$

where $g(x) = x^3 + x^2 + 2x - 1$. Now, g is increasing on $[0, 1]$ and $g(0)g(1) < 0$, so there is unique $x_0 \in (0, 1)$ such that $g(x_0) = 0$. Moreover, $g(x) < 0$ for $0 < x < x_0$, $g(x) > 0$ for $x_0 < x < 1$. This proves that f' is decreasing on $(0, x_0]$ and increasing on $[x_0, 1]$. From $f'(1) = 0$ and $\lim_{x\to 0^+} f'(x) = +\infty$ we conclude that there is a unique $x_1 \in (0, x_0)$ with $f'(x_1) = 0$ and that $f'(x)$ has the sign of $x_1 - x$ on the interval $(0, 1]$. Therefore, f is increasing on $(0, x_1]$ and decreasing on $[x_1, 1]$. But $\lim_{x\to 0^+} f(x) = 0$, and $f(1) = 0$. The next table of variations illustrates this discussion:

x	0		x_1		x_0		1
$g(x)$		$-$			0	$+$	
$f'(x)$	$+\infty$	$\searrow$	0	$\searrow$	$\smile$	$\nearrow$	0
$f'(x)$		$+$	0	$-$		$-$	
$f(x)$	0	$\nearrow$	$\frown$		$\searrow$		0

So $f(x) > 0$ for $x \in (0, 1)$, and the proposed inequality follows.

Solution 2, by Michel Bataille.

Equality holds when $x = 1$. From now on, we suppose that $x \in (0,1)$. Let $h = 1 - x$. Then $h \in (0,1)$ and the required inequality becomes

$$e^{(1-h)\ln(1-h)} \leq 1 - h + h^2$$

or, equivalently,

$$(1-h)\ln(1-h) \leq \ln(1 - h(1-h)) \qquad (1).$$

Since $h(1-h) \in (0,1)$ as well, (1) rewrites as

$$-(1-h)\sum_{n=1}^{\infty} \frac{h^n}{n} \leq -\sum_{n=1}^{\infty} \frac{h^n(1-h)^n}{n}$$

that is,

$$\sum_{n=1}^{\infty} \frac{h^n}{n}\left((1-h) - (1-h)^n\right) \geq 0.$$

But this inequality clearly holds since $h > 0$ and for all positive integer n,

$$(1-h) - (1-h)^n = (1-h)(1 - (1-h)^{n-1}) \geq 0$$

(recalling that $0 < 1 - h < 1$). The proof is complete.

Solution 3, by Salem Malikić.

Make the substitution $a = \frac{1}{x}$, so $a \geq 1$. Then the required inequality is equivalent to

$$\left(\frac{1}{a}\right)^{\frac{1}{a}} \leq \frac{1}{a^2} - \frac{1}{a} + 1,$$

which, after rearranging and raising both sides to the power of a, becomes

$$\frac{1}{a} \leq \left(1 + \frac{1-a}{a^2}\right)^a.$$

We now use Bernoulli's inequality in the form $1 + bx \leq (1+x)^b$ for $x \geq -1$ and $b \geq 1$:

$$\frac{1}{a} = 1 + a \cdot \frac{1-a}{a^2} \leq \left(1 + \frac{1-a}{a^2}\right)^a,$$

so we are done.

Solution 4, by both the AN-anduud Problem Solving Group and Nikolaos Evgenidis.

By the weighted AM-GM inequality, we have, for $x \in (0,1]$:

$$x^2 - x + 1 = x \cdot x + (1-x) \cdot 1 \geq x^x \cdot 1^{1-x} = x^x.$$

Editor's comments. A variation of Solution 2 involved the use of the binomial expansion, by H. Wang and J. Woydylo. There were variations on Solution 3, usually a different substitution before applying Bernoulli's inequality. V. Konećny commented that this is one part of the two-sided inequality

$$\frac{1}{2-x} < x^x < x^2 - x + 1,$$

for $x \in (0,1)$, which with proof via the weighted AM-GM inequality is found in a text by Jiří Herman of Masaryhova University, where Konećny studied 60 years ago.

3816. *Proposed by Mehmet Şahin.*

Let ABC be a right triangle with right angle at C, and let D be the foot of the altitude from C. Let I_1 and I_2 be the incentres of triangles CAD and CBD, respectively. Let ρ and r be the inradii of triangles I_1DI_2 and ABC, respectively. Prove that

$$\frac{\rho}{r} \le \frac{1}{2+\sqrt{2}}.$$

Solved by M. Amengual Covas; AN-anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; P. De; N. Evgenidis; O. Geupel; O. Kouba; K. W. Lau; S. Malikić; M. R. Modak; N. Stanciu and T. Zvonaru; C. Sánchez-Rubio; E. Swylan; D. Văcaru; and the proposer. We present the solution by Omran Kouba.

Let us denote BC, CA and AB by a, b and c respectively. Also, let r_1 and r_2 be the inradii of triangles CAD and CBD, respectively.

First we note that $\angle I_1DI_2 = 90^\circ$ since I_1D and I_2D are the internal and external bisectors of $\angle ADC$. From the similarity of triangles CAD and BAC we conclude that $\frac{I_1D}{IC} = \frac{CA}{AB}$; analogously, from the similarity of triangles CBD and ABC we conclude that $\frac{I_2D}{IC} = \frac{BC}{AB}$. Thus,

$$\frac{I_1D}{I_2D} = \frac{AC}{AB}.$$

This proves that the right triangles I_1DI_2 and ACB are similar and consequently

$$\frac{\rho}{r} = \frac{I_1D}{AC} = \frac{\sqrt{2}\,r_1}{AC}.$$

But from the similarity of triangles CAD and BAC we see that $\frac{r_1}{AC} = \frac{r}{AB}$. So, we have proved that

$$\frac{\rho}{r} = \sqrt{2}\cdot\frac{r}{c}.$$

On the other hand,

$$\begin{aligned}\frac{ab}{a+b+c} &= \frac{ab(a+b-c)}{(a+b)^2-c^2} = \frac{a+b-c}{2} \le \frac{\sqrt{2(a^2+b^2)}-c}{2}\\ &= \frac{(\sqrt{2}-1)c}{2} = \frac{c}{2(\sqrt{2}+1)} \qquad (2)\end{aligned}$$

and the desired inequality follows by combining (1) and (2).

3817. *Proposed by Tiagran Hakobyan.*

Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Let $p_1 < p_2 < p_3 < \ldots$ be the set of primes in the progression $\{ak + b\}_{k=0}^{\infty}$. Consider

$$\alpha = 0.p_1p_2p_3\cdots,$$

where the digits of the prime numbers $p_1, p_2, p_3, \ldots$ placed side by side form the digits of α. Prove that α is irrational.

Solved by R. Barbara; O. Guepel; and the proposer. There was one incomplete solution. We present 2 solutions.

Solution 1, by the proposer.

By Dirichlet's theorem, there are infinitely many primes in the progression

$$\{ak + b\}_{k=0}^{\infty},$$

so the decimal expansion of α cannot terminate. Suppose that α is rational. Then its decimal expansion is eventually periodic with some positive period of length u consisting of digits not all zero. Pick a prime $q = ak_0 + b$ exceeding 5 in the sequence; suppose that q has v digits. Then, for $k \ge k_0$,

$$ak + b = a(k - k_0) + q,$$

so that the digits of the primes of the form $an + q$ with $n \ge 0$ constitute the tail of the decimal expansion of α.

Since $\gcd(a \cdot 10^m, q) = 1$, the subsequence $\{a \cdot 10^m \cdot k + q\}$ with $m = 2u + v$ also has infinitely many primes, and each such prime will be responsible for a block of at least $2u$ zeros in the expansion of α. Since any succession of $2u$ digits contains one full period, we are led to a contradiction.

Solution 2, by Oliver Geupel.

Assume that α is eventually periodic with period length u. By omitting a suitable finite number of initial terms in the sequence $\{ak + b\}$, we may assume that α is actually periodic. For $i \ge 1$, let n_i be the number of primes in the sequence with i decimal digits. The the sum of the reciprocals of primes p_j with i digits satisfies

$$\sum \left\{ \frac{1}{p_j} : 1 + \sum_{t=1}^{i-1} n_t \le j \le \sum_{t=1}^{i} n_t \right\} \le \frac{n_i}{10^{i-1}},$$

since each p_j exceeds 10^{i-1}.

Suppose that $n_i > u$. Then there is an increasing succession of $u+1$ primes each with i digits that give rise to $(u+1)i = ui + i$ consecutive digits of α. Because ui is a multiple of u, the first i of these digits is equal to the last i, so that two of the primes are equal and we get a contradiction. Thus, $n_i \leq u_i$ for all i and the sum of the reciprocals of the primes in the sequence does not exceed

$$u \sum_{i=1}^{\infty} (1/10^{i-1}) = 10u/9.$$

But this contradicts the strong form of Dirichlet's theorem, so the sum diverges.

Editor's comments. Barbara observed that this result holds for any base of numeration. Geupel mentioned that he was inspired by the paper B. Sung, *When is a decimal expansion irrational?* Resonance (Indian Academy of Science) 9 (2004), 78-80.

3818. *Proposed by José Luis Díaz-Barrero.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(\sqrt{a}+\sqrt{b})^4}{a+b} + \frac{(\sqrt{b}+\sqrt{c})^4}{b+c} + \frac{(\sqrt{c}+\sqrt{a})^4}{c+a} \geq 24.$$

Solved by AN-anduud Problem Solving Group; G. Apostopoulos; Š. Arslanagić; M. Bataille; E. Campbell, D.T. Bailey and C. Diminnie; P. De; M. Dincă; N. Evgenidis; K. W. Lau; S. Malikić; P. McCartney; M. Modak; C. Mortici; P. Perfetti; D. Smith; D. Văcaru; S. Wagon; H. Wang and J. Wojdylo; and the proposer. We present 2 solutions.

Solution 1, provided by most of the solvers.

From the inequality $(x+y)^4 - 8xy(x^2+y^2) = (x-y)^4 \geq 0$, we deduce that the left side of the inequality is not less than $8(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$. An application of the arithmetic-geometric means inequality yields the desired result.

Solution 2, by Nikolaos Evgenidis.

Let $(x,y,z) = (\sqrt{a}, \sqrt{b}, \sqrt{c})$. Applying the Cauchy-Schwarz Inequality $(\sum u_i v_i)^2 \leq (\sum u_i^2)(\sum v_i^2)$ with $u_1 = (x+y)^2/\sqrt{x^2+y^2}, v_1 = \sqrt{x^2+y^2}$, we see that the left side of the inequality is not less than

$$\frac{[(x+y)^2+(y+z)^2+(z+x)^2]^2}{2(x^2+y^2+z^2)}.$$

Since $xy + yz + xx \geq 3(xyz)^{1/3} = 3$ and

$$\begin{aligned}[(x+y)^2 + (y+z)^2 + (z+x)^2]^2 &= 4(x^2 + y^2 + z^2 + xy + yz + zx)^2 \\ &\geq 4(x^2 + y^2 + z^2 + 3)^2 \\ &= 4[(x^2 + y^2 + z^2 - 3)^2 + 12(x^2 + y^2 + z^2)] \\ &\geq 48(x^2 + y^2 + z^2),\end{aligned}$$

the desired inequality follows.

3819. *Proposed by Francisco Javier Garciá Capitán.*

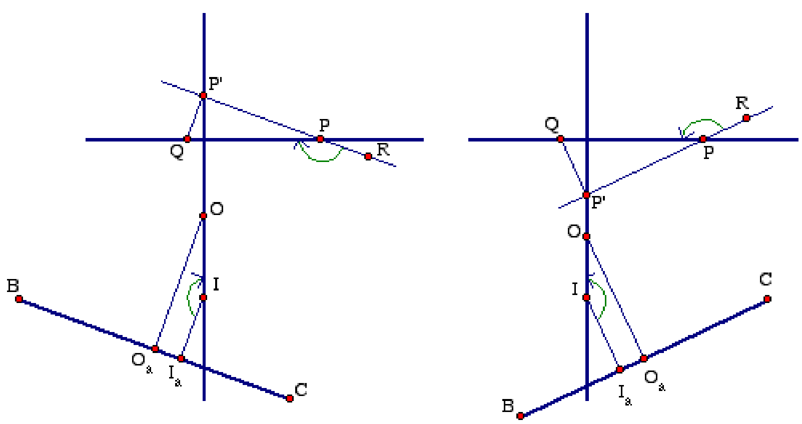
Let ABC be a triangle with circumcentre O and incentre I. Let ℓ be any line that is perpendicular to OI. Prove that for any point P on ℓ that is inside the triangle, the sum of the distances from P to the sides of ABC is constant.

Solved by M. Bataille; O. Kouba; E. Swylan; T. Zvonaru and N. Stanciu; and the proposer. We present the solution by Edmund Swylan, with details added by the editor.

With the use of signed distances, there is no need to restrict P to the inside of the given triangle. Specifically, we define the distance $d(P, YZ)$ of the point P to the side YZ of a triangle XYZ to be positive if and only if P and X lie in the same half-plane defined by the line YZ.

Here, we are given two points P and Q in the plane of an arbitrary nonequilateral triangle ABC, so that $PQ \perp OI$, and we label the points so that a rotation of 90° about P takes the vector $\overrightarrow{IO}$ into a vector that points in the same direction as $\overrightarrow{PQ}$. We are to prove that

$$\Sigma := (d(Q, BC) - d(P, BC)) + (d(Q, CA) - d(P, CA)) + (d(Q, AB) - d(P, AB)) = 0.$$



Define I_a and O_a to be the feet of the perpendiculars to BC from I and O, respectively, and $\theta_a = \angle I_a IO$ to be a signed angle (positive if labeled counterclockwise).

We introduce the notation $\overline{XY}$ for the signed distance from X to Y, where we take $\overline{IO}, \overline{AB}, \overline{BC}$, and $\overline{CA}$ to define the positive direction of the lines they determine. In this way,

$$\overline{I_aO_a} = \overline{IO}\sin\theta_a \tag{1}$$

regardless of how the lines IO and BC are related, as depicted in the figure where both $\overline{I_aO_a}$ and θ_a are negative on the left, and both are positive on the right.

Let P' be the point where the parallel to BC through P meets the perpendicular to BC through Q. We define the sign of $\overline{P'Q}$ so that $\overline{P'Q} = d(Q, BC) - d(P, BC)$. Let R be any point on the line $P'P$ for which the vectors $\overrightarrow{BC}$ and $\overrightarrow{PR}$ point in the same direction. Then because a rotation through $90°$ about P takes the vectors $\overrightarrow{II_a}$ and $\overrightarrow{IO}$ into vectors that point in the same direction as $\overrightarrow{PR}$ and $\overrightarrow{PQ}$, respectively, we have $\theta_a = \angle RPQ$. Moreover, $\angle RPQ$ has the same sign as $\overline{P'Q}$ regardless of how the lines IO and BC are related, as depicted in the figure where both $\overline{P'Q}$ and $\angle RPQ$ are negative on the left, and both are positive on the right. We conclude that

$$d(Q, BC) - d(P, BC) \;=\; \overline{P'Q} \;=\; \overline{PQ}\sin\theta_a,$$

with analogous expressions for $d(Q, CA) - d(P, CA)$ and $d(Q, AB) - d(P, AB)$. Thus, with I_b, O_b, I_c, and O_c the respective projections of I and O on the sides CA and AB, and $\theta_b := \angle I_bIO, \theta_c := \angle I_cIO$, we have reduced our sum to

$$\Sigma = \overline{PQ}(\sin\theta_a + \sin\theta_b + \sin\theta_c).$$

We now compare Σ to the expression we get by adding together the equalities analogous to (1), namely

$$\overline{I_aO_a} + \overline{I_bO_b} + \overline{I_cO_c} = \overline{IO}(\sin\theta_a + \sin\theta_b + \sin\theta_c). \tag{2}$$

Because we assume that neither $\overline{IO}$ nor $\overline{PQ}$ can be zero, Σ can vanish if and only if the sum of the three sines is zero, which can happen if and only if the sum in (2) is zero. But

$$\begin{aligned}\overline{I_aO_a} + \overline{I_bO_b} + \overline{I_cO_c} &= (\overline{BO_a} - \overline{BI_a}) + (\overline{CO_b} - \overline{CI_b}) + (\overline{AO_c} - \overline{AI_c}) \\ &= (\overline{BO_a} + \overline{CO_b} + \overline{AO_c}) - (\overline{BI_a} + \overline{CI_b} + \overline{AI_c}).\end{aligned}$$

Because each of the sums in the last line is equal to the semiperimeter of the triangle, their difference is zero, as desired.

Editor's comments. By coincidence, an article [2] appeared in the latest *Mathematics Magazine* that deals with issues related to our problem, namely Viviani's theorem and its extension to the result,

The sum of the distances from a point inside a triangle to the three sides takes every value from the smallest altitude of the triangle to the largest altitude.

(Viviani's theorem deals with the equilateral triangle, where the sum of the three distances equals the altitude.) Polster provides a simple pictorial proof, but the result is also an easy consequence of our solution to 3819. He refers to the article [1], where Abboud uses linear programming to prove that any triangle can be divided into parallel segments on which the sum is constant, but neither author observes that these parallel segments happen to be perpendicular to OI.

References:

[1] Elias Abboud, Viviani's theorem and its extensions, *College Math. J.* **41**:3 (May 2010) 203-211.

[2] Burkard Polster, Viviani á la Kawasaki: Take Two, *Math. Mag.* **87**:4 (October 2014) 280-283.

3820. *Proposed by Michel Bataille.*

Prove that

$$\frac{2x}{\sinh(2\tanh x)} < (\cosh x)^2 < \frac{2x}{\sinh(2\tanh x)} + x\sinh(2x)$$

for all nonzero real x.

Solved by R. Hess; O. Kouba; P. Perfetti; and the proposer. There was one incomplete solution, and one solution consisting solely of a Mathematica verification. We present Omran Kouba's solution.

For $t \in (0,1)$ we have

$$\frac{1-t^2}{2t}\ln\left(\frac{1+t}{1-t}\right) = \frac{(1-t^2)}{t}\tanh^{-1}(t) = (1-t^2)\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1} = 1-2\sum_{n=1}^{\infty}\frac{t^{2n}}{4n^2-1} < 1$$

and

$$\frac{\sinh(2t)}{2t} = 1 + \sum_{n=1}^{\infty}\frac{2^{2n}t^{2n}}{(2n+1)!} > 1.$$

Combining the above, we conclude that for $t \in (0,1)$ we have

$$\frac{\sinh(2t)}{2t} > (1-t^2)\frac{\tanh^{-1}(t)}{t}$$

or equivalently

$$\frac{1}{1-t^2} > \frac{2\tanh^{-1}(t)}{\sinh(2t)}.$$

Applying this with $t = \tanh x$, and noting that both sides of the obtained inequality are even functions, we obtain the first inequality.

In a similar way, for $t \in (0,1)$ we have

$$\begin{aligned}\big(1-t^2+t\sinh(2t)\big)&\frac{\tanh^{-1}(t)}{t}-\frac{\sinh(2t)}{2t}\\ &=\left(1+t^2+\sum_{n=2}^{\infty}\frac{2^{2n-1}t^{2n}}{(2n-1)!}\right)\left(\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}\right)-\sum_{n=0}^{\infty}\frac{2^{2n}t^{2n}}{(2n+1)!}\\ &=\frac{2}{3}t^2+\sum_{n=2}^{\infty}a_nt^{2n}\end{aligned}$$

with

$$a_n=\frac{4n}{4n^2-1}+\frac{(2n^2+n-1)2^{2n}}{(2n+1)!}+\sum_{k=1}^{n-2}\frac{2^{2n-2k-1}}{(2k+1)(2n-2k-1)!}>0$$

This proves that for $t \in (0,1)$ we have $\big(1-t^2+t\sinh(2t)\big)\dfrac{\tanh^{-1}(t)}{t}>\dfrac{\sinh(2t)}{2t}$. Multiplying both sides by the positive quantity $\dfrac{2t}{(1-t^2)\sinh(2t)}$, we obtain

$$\left(\frac{2}{\sinh(2t)}+\frac{2t}{1-t^2}\right)\tanh^{-1}(t)>\frac{1}{1-t^2}.$$

Applying this, with $t=\tanh x$, and noting that both sides of the obtained inequality are even functions, the second inequality follows.

Editor's comments. The featured solution utilises a substitution to simplify the problem, and then uses power series to verify each side of the inequality, thanks to how related the individual series are. Interestingly, the left-hand inequality is provable without passing to the useful but tedious power series. S. Malikić utilised the inequality $\frac{\sinh(t)}{t}>1$ for all $t \neq 0$, as follows:

$$\begin{aligned}\frac{2x}{\sinh(2\tanh(x))}&=\frac{2\tanh(x)}{\sinh(2\tanh(x))}\frac{x}{\tanh(x)}\\ &<\frac{x}{\tanh(x)}\\ &=\frac{x}{\sinh(x)}\cosh(x)<\cosh(x)<\cosh(x)^2.\end{aligned}$$

The proposer, M. Bataille, took an even more elementary approach, by computing that the derivative of $f(x)=\cosh^2(x)\sinh(2\tanh(x))-2x$ is positive for positive x, and that $f(0)=0$, therefore proving the left-hand inequality. The right-hand inequality has seen no such simple proof, although the proposer found a simpler inequality which implies the right-hand inequality, which yields a less troublesome series computation.

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Paul Bracken, University of Texas, Edinburg, TX, USA: 3911
George Apostolopoulos, Messolonghi, Greece: 3914, 3918
Michel Bataille, Rouen, France: 3912, 3919
Marcel Chiriţă, Bucharest, Romainia : 3915
Ovidiu Furdui, Campia Turzii, Cluj, Romania: 3913
Alina Sîntămărian, Technical University of Cluj-Napoca, Romania: 3920
Nathan Soedjak, Springfield, MO, USA: 3916
Peter Y. Woo, Biola University, La Mirada, CA, USA: 3917

Solvers - individuals

Arkady Alt, San Jose, USA: 3814
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : 3816
George Apostolopoulos, Messolonghi, Greece: 3813, **3814**, 3816, 3818
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina: CC58, 3815, 3816, 3818
Michel Bataille, Rouen, France: CC57, CC58, **OC107**, OC108, **3812**, 3813, 3814, **3815**, 3816, 3818, 3819, 3820
Roy Barbara, Lebanese University, Fanar, Lebanon: **3813**, 3814, 3817
Zachary Burnett, Auburn University Montgomery, Montgomery, AL, USA : CC57
Marcel Chiritiţă, Bucharest, Romania : **3814**
Matei Coiculescu, East Lyme High School, East Lyme, CT, USA: CC58
Prithwijit De, HBCSE, Mumbai, India : 3812, **3814**, 3816, 3818
José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain: CC58, 3818
Marian Dincă, Bucharest, Romania: 3815, 3818
Evgenidis Nikolaos, Gymnasium of Agia, Thessalia, Greece : 3812, **3815**, 3816, **3818**
Ovidiu Furdui, Campia Turzii, Cluj, Romania: 3815
Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain : 3819
Oliver Geupel, Brühl, NRW, Germany: **OC110**, 3812, 3814, 3815, 3816, **3817**
Tigran Hakobyan, Yerevan State University, Yerevan, Armenia : **3817**
Richard I. Hess, Rancho Palos Verdes, CA, USA: CC60, **OC106**, 3813, 3820
John G. Heuver, Grande Prairie, AB: 3812
Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria: 3812, 3813, **3814**, **3815**, **3816**, 3819, **3820**
Kee-Wai Lau, Hong Kong, China: 3813, 3815, 3816, 3818
Ang Li, Student of Southeast Missouri State University : 3815
Salem Malikić, student, Simon Fraser University, Burnaby, BC: 3814, **3815**, 3816, 3818
David E. Manes, SUNY at Oneonta, Oneonta, NY, USA: **OC106**
Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA: 3815, 3818
Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India: 3812, 3816, 3818
Cristinel Mortici, Valahia University of Târgovişte, Romania: 3812, 3813, 3814, 3818
S. Muralidharan, Chennai, India : **CC57**

Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy: 3813, 3815, 3818, 3820
Mehmet Şahin, Ankara, Turkey: 3816
Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain: 3812, 3814, 3816
Digby Smith, Mount Royal University, Calgary, AB: 3815, 3818
Edmund Swylan, Riga, Latvia: 3812, 3814, 3816, **3819**
Daniel Văcaru, Piteşti, Romania: CC58, OC106, OC108, 3814, 3816, 3818
Stan Wagon, Macalester College, St. Paul, MN, USA: 3813, 3818
Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON : CC58
Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA: **CC56**, CC58
Titu Zvonaru, Cománeşti, Romania: **CC58**

Solvers - collaborations

AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia: 3812, 3814, **3815**, 3816, 3818
Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA: 3814, 3818
Skidmore College Problem Group : 3812
Nermin Hodžić, Dobošnica, Bosnia and Herzegovina and Salem Malikić, Simon Fraser University, Burnaby, BC : 3813
John Hawkins and David R. Stone, Georgia Southern University, Statesboro, USA: 3762, 3814
Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, USA: 3815, 3818
Titu Zvonaru, Comăneşti, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania: **OC108**, 3812, 3816, 3819