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Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

The future is here. I notice it most at the beginning of each academic year as students have to do more and more things using various available technologies. First, the course registration moved online, next came online assignments, now discussion forums and even online tests. Some courses use clickers, some use learninganalytics. WebWork, WebAssign, Launchpad — these are just some of the technologies my students had to learn in the first week of classes. And this is on top of having to navigate the university registration system (which is far from intuitive or crash-proof).

As mathematicians, we also notice the changes brought into our lives by computers and the Internet. Getting access to needed resources used to be non-trivial: journals were only available in hard copies in the libraries and if your university library didn't have the subscription to a particular journal, you'd have to request an interlibrary loan and wait for (gasp!) weeks to get your hands on the paper you want to read. We are now on the other end of the spectrum. You type something into Google and the search for the needle in the haystack begins.

With this issue, Volume 40 of ***Crux*** comes to a close. In spite of (or because of?) the abundance of mathematics on the Internet, ***Crux*** is as popular as ever. As Editor-in-Chief, I know how different the journal operates today as opposed to 10 years ago: our entire production is done online, all of our issues are available electronically on the CMS website, we receive most of our submissions via email, we have Facebook presence, etc. Yet, like most of our subscribers, it is still the little purple book I look for in my mailbox.

Some things are simply timeless.

Kseniya Garaschuk

THE CONTEST CORNER

No. 30

Kseniya Garaschuk

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

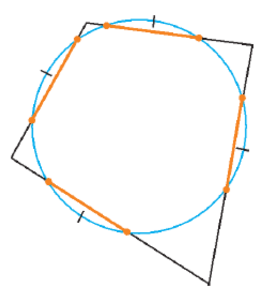
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

CC146. Déterminer le nombre de solutions entières (x, y) à l'équation

$$xy = x + y + 999,999,999.$$

CC147. Un cercle intersecte chaque côté d'un quadrilatère de façon à ce que les côtés du quadrilatère découpent des arcs de même longueur.



Démontrer que ce quadrilatère possède un cercle inscrit.

CC148. À l'aide de cubes de taille $1 \times 1 \times 1$, Alexandria forme une brique rectangulaire de taille $6 \times 10 \times 15$. Combien de petits cubes se trouvent sur la diagonale principale de la grosse brique ?

CC149. Déterminer toutes les valeurs positives x, y, z telles que pour tout triangle avec côtés de longueurs a, b, c il existe un triangle avec côtés de longueurs ax, by, cz.

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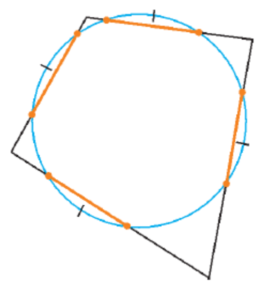
CC150. Saleh écrit tous les nombres de 1 à 2015, à l'aide de plumes rouge et bleu. Le plus gros nombre bleu est égal au nombre de nombres bleus ; le plus petit nombre rouge est égal à la moitié du nombre de nombres rouges. Combien de nombres rouges Saleh a-t-il écrits ?

. . . .

CC146. Determine the number of integer solutions (x, y) to the equation

$$xy = x + y + 999,999,999.$$

CC147. A circle intersects every side of a quadrilateral in such a way that the sides of the quadrilateral cut away equal length arcs.



Show that you can inscribe a circle into this quadrilateral.

CC148. Using cubes of size $1 \times 1 \times 1$, Amanda puts together a rectangular brick of size $6 \times 10 \times 15$. How many little cubes does the main diagonal of the big brick cross ?

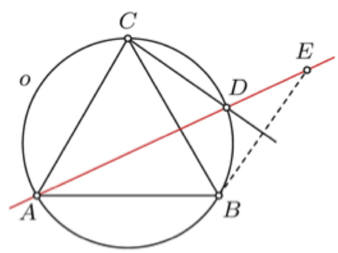
CC149. Find all positive numbers x, y, z such that for any triangle with side lengths a, b, c there exists a triangle with sides ax, by, cz.

CC150. Shane writes down all numbers from 1 to 2015 in red and blue pen. The largest blue number is equal to the number of blue numbers ; the smallest red number is equal to half the number of red numbers. How many red numbers did Shane write down ?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section apparaissent dans 2013 : 39(10), p. 435–436.

CC96. An equilateral triangle ABC is inscribed in a circle o. Point D is on arc BC of o. Point E is the symmetric of B with respect to line CD. Prove that A, D and E are collinear.



Originally 10th secondary mathematics olympiad (Poland), First level, question 2.

We received two correct solutions, and one incorrect solution. We present the solution of John Heuver, slightly modified by the editor.

Quadrilateral $ABDC$ is cyclic, which implies that $\measuredangle BDC = 180 - \measuredangle CAB = 120^\circ$. By symmetry, $\measuredangle BDC = \measuredangle EDC = 120^\circ$, and hence $\measuredangle BDE = 120^\circ$. Further, $\measuredangle ACB = \measuredangle ADB = 60^\circ$, since the two angles are subtended by the same arc. Thus $\measuredangle ADB + \measuredangle BDE = 60^\circ + 120^\circ = 180^\circ$; that is, A, D and E are collinear.

CC97. Find the smallest value of the expression $a+b^3$ where a and b are positive numbers whose product is 1.

Originally 10th secondary mathematics olympiad (Poland), First level, question 3.

We received nine correct submissions and one incorrect submission. We present the solution by Henry Ricardo.

Using the fact that $ab = 1$, the Arithmetic-Geometric Mean inequality yields

$$a + b^3 \;=\; a + \frac{1}{a^3} \;=\; \frac{a}{3} + \frac{a}{3} + \frac{a}{3} + \frac{1}{a^3} \;\geq\; 4\sqrt[4]{\frac{a^3}{27a^3}} \;=\; \frac{4}{\sqrt[4]{27}} \;\approx\; 1.754765351.$$

This minimum value is attained when $\dfrac{a}{3} = \dfrac{1}{a^3}$: that is, when $a = \sqrt[4]{3}$.

CC98. Are there real numbers x and y such that $\sqrt{x^2+1} + \sqrt{y^2+1} = x + y\,$?

Originally 7th secondary mathematics olympiad (Poland), First level, question 1.

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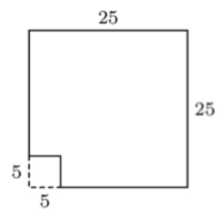
We received seven correct submissions. We present the solution by Henry Ricardo.

The answer is no.

We have $\sqrt{x^2+1}+\sqrt{y^2+1} > \sqrt{x^2}+\sqrt{y^2} = |x|+|y| \geq x+y$.

CC99. We cut off a square of side 5 from the corner of a square of side 25. Can we cut the remaining part into 100 rectangles of dimension either 1×6 or 2×3 ?



Originally 10th secondary mathematics olympiad (Poland), First level, question 6.

We present the solution by Titu Zvonaru.

It is not possible. Consider the original square of side length 25 as a board with 625 cells of side length 1. Colour the cells with three colours red, white, and blue such that the bottom left corner is coloured red, and such that the colours red, white, and blue alternate in that order along each row (from left to right) and each column (from bottom to top). If we cut out the bottom left square of side length 5, the remaining board contains 201 red cells, 199 white cells, and 200 blue cells. As any 1×6 or 2×3 rectangle covers exactly 2 red, 2 white, and 2 blue cells, it is not possible to cut this board into 100 such rectangles.

CC100. In a 6-team tournament, each team played with each other team exactly once. A team gets 3 points for a victory, 1 point for a draw and 0 for a defeat. After the tournament, the sum of the scores by all the teams is 41. Prove that there exists a group of 4 teams where each team tied at least once.

Originally 7th secondary mathematics olympiad (Poland), Second level, question 2.

We received three correct submissions. We present the solution by Titu Zvonaru.

Let x be the number of games that end in a draw. Since 3 total points are awarded for a win and 2 total points for a tie, we obtain

$$2x+3(15-x)=41.$$

Solving yields $x=4$, so there were 4 games which ended in a draw. Amongst 3 teams a total of 3 games are played, so there must have been at least 4 teams that were involved in ties, so there is a group of 4 teams where each team tied at least once.

THE OLYMPIAD CORNER

No. 328

Nicolae Strungaru and Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC206. Deux cercles K_1 et K_2, de rayons différents, intersectent aux points A et B. Soient C et D deux points sur K_1 et K_2 respectivement, tels que A est le mipoint du segment CD. Le prolongement de DB rencontre K_1 à un second point E et le prolongement de CB rencontre K_2 à un second point F. Soient l_1 et l_2 les bissectrices perpendiculaires de CD et EF respectivement.

1. Démontrer que l_1 et l_2 ont un point commun unique, dénoté P.
2. Démontrer que les longueurs CA, AP et PE sont les longueurs d'un triangle rectangle.

OC207. Déterminer toutes les fonctions injectives $f : \mathbb{Z} \to \mathbb{Z}$ telles que

$$|f(x) - f(y)| \leq |x - y|$$

pour tout $x, y \in \mathbb{Z}$.

OC208. Déterminer toutes les valeurs x non entières telles que

$$x + \frac{13}{x} = \lfloor x \rfloor + \frac{13}{\lfloor x \rfloor}$$

où $\lfloor x \rfloor$ dénote le plus grand entier n plus petit ou égal à x.

OC209. La séquence $a_1, a_2, \ldots, a_n$ consiste des nombres $1, 2, \ldots, n$ dans un certain ordre. Pour quels entiers positifs n est-il possible que les $n + 1$ nombres $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$ ont tous des restes différents lorsqu'ils sont divisés par $n + 1$?

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OC210. Déterminer tous les entiers positifs a tels que pour tout entier $n \geq 5$, on a $2^n - n^2 \mid a^n - n^a$.

. . . .

OC206. Two circles K_1 and K_2 of different radii intersect at two points A and B. Let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.

1. Show that l_1 and l_2 have a unique common point (denoted by P).
2. Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

OC207. Find all injective functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy:

$$|f(x) - f(y)| \leq |x - y|$$

for any $x, y \in \mathbb{Z}$.

OC208. Find all non-integers x such that

$$x + \frac{13}{x} = \lfloor x \rfloor + \frac{13}{\lfloor x \rfloor}$$

where $\lfloor x \rfloor$ means the greatest integer n less than or equal to x.

OC209. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n + 1$ numbers $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_n$ all have different remainders when divided by $n + 1$?

OC210. Find all positive integers a such that for any positive integer $n \geq 5$ we have $2^n - n^2 \mid a^n - n^a$.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section initialement apparaissent dans 2013: 39(10), p. 440–441.

OC146. Let ABC be an isosceles triangle with $AC = BC$. Take points D on side AC and E on side BC. Let F be the intersection of bisectors of angles DEB and ADE. If F lies on side AB, prove that F is the midpoint of AB.

Originally from Moldova Junior Balkan Team Selection Test Day 2, Problem 3.

We received seven correct submissions. We present the solution by Adnan Ali.

Let $d(X, PQ)$ denote the perpendicular distance of point X from line PQ. Then we observe that since F lies on the angle bisector of $\angle ADE$,

$$d(F, DE) = d(F, DA) = d(F, AC).$$

And since F lies on the angle bisector of $\angle DEB$, we also have

$$d(F, DE) = d(F, EB) = d(F, BC).$$

This shows that F is equidistant from AC and BC and so it must lie on the angle bisector of $\angle ACB$. Therefore FC is the bisector of the angle ACB. As $AC = BC$, it follows that FC is also the midline of the triangle, and hence F is the midpoint of AB.

OC147. Suppose a_1 is a natural number and $\{a_n\}_n$, is defined by the rule:

$$a_{n+1} = a_n + 2d(n)\,,$$

where $d(n)$ denotes the number of different divisors of n (including 1 and n). Does there exist an a_1 such that two consecutive members of the sequence are squares of natural numbers?

Originally from the Bulgaria Mathematical Olympiad Day 1, Problem 2.

We present the solution by Oliver Geupel. There were no other submitted solutions.

We show that the answer is No.

Suppose by contradiction that, for some properly chosen initial value $a_1 \geq 1$, the members a_n and a_{n+1} are perfect squares. Let $a_n = m^2$.

Since $a_{n+1} - a_n = 2d(n)$ and $(m+1)^2 - m^2 = 2m + 1$ is odd, we cannot have $a_{n+1} = (m+1)^2$.

Therefore, $a_{n+1} \geq (m+2)^2$, and hence

$$4m + 4 = (m+2)^2 - m^2 \leq a_{n+1} - a_n = 2d(n) \leq 4\sqrt{n}.$$

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This implies
$$(m+1)^2 \le n\,.$$

Since, $a_1 \ge 1$, $d(1) = 1$, and $d(k) \ge 2$ for $k \ge 2$, by induction we have $a_n \ge 4n-5$. Therefore
$$m^2 = a_n \ge 4n-5 \ge 4(m+1)^2 - 5,$$
and hence $3m^3+8m \le 1$, which is a contradiction. This shows that our assumption is wrong, and therefore there is no value of a_1 for which two consecutive terms of the sequence are perfect squares.

OC148. Complex numbers x_i, y_i satisfy $|x_i| = |y_i| = 1$ for all $1 \le i \le n$. Let $x = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, $y = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$ and $z_i = xy_i + yx_i - x_iy_i$. Prove that
$$\sum_{i=1}^{n} |z_i| \le n\,.$$

Originally from the China Team Selection Test 2012, Problem 1.

We received two correct submissions. We present the solution by Michel Bataille.

For each i such that $1 \le i \le n$, we have $z_i = xy - (x_i - x)(y_i - y)$, hence
$$|z_i| \le |x||y| + |x_i - x||y_i - y| \le \frac{|x|^2 + |y|^2}{2} + \frac{|x_i - x|^2 + |y_i - y|^2}{2}\,.$$

It follows that
$$\sum_{i=1}^{n} |z_i| \le \frac{1}{2}\left(n|x|^2 + \sum_{i=1}^{n} |x_i - x|^2 + n|y|^2 + \sum_{i=1}^{n} |y_i - y|^2\right)\,. \qquad (1)$$

Now, from
$$\begin{aligned}|x_i|^2 = |(x_i - x) + x|^2 &= ((x_i - x) + x)\cdot\left(\overline{(x_i - x)} + \overline{x}\right)\\ &= |x_i - x|^2 + (x_i - x)\overline{x} + \overline{(x_i - x)}x + |x|^2,\end{aligned}$$
we deduce
$$\sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} |x_i - x|^2 + \overline{x}\sum_{i=1}^{n}(x_i - x) + x\overline{\sum_{i=1}^{n}(x_i - x)} + n|x|^2 = n|x|^2 + \sum_{i=1}^{n} |x_i - x|^2\,,$$
with the last equality following from
$$\sum_{i=1}^{n}(x_i - x) = \Big(\sum_{i=1}^{n} x_i\Big) - nx = 0\,.$$

A similar result holds for $\sum_{i=1}^{n} |y_i|^2$ and returning to (1), we obtain

$$\sum_{i=1}^{n} |z_i| \le \frac{1}{2}\left(\sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |y_i|^2\right).$$

The required result follows now as $|x_i| = |y_i| = 1$.

OC149. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$, such that for all integers x, y we have

$$f\left(x + y + f(y)\right) = f(x) + ny\,,$$

where n is a fixed integer.

Originally from China TST 2012 Day 1, Problem 3.

We received two correct submissions. We present the solution by Oliver Geupel.

It is straightforward to verify that the two functions

$$f(x) = mx \qquad \text{and} \qquad f(x) = -(m+1)x$$

are solutions of the given functional equation if the number n can be written in the form

$$n = m(m+1)$$

with an integer m. We prove that there are no further solutions.

We show that if f is any solution of the given functional equation, then $f(x) = lx$ for some integer l so that $l(l+1) = n$. Let f be a solution to the equation.

For $k \in \mathbb{Z}$, let $P(k)$ denote the assertion that the identity

$$f(k(x + f(x))) = f(0) + knx$$

holds for every integer x. We prove that $P(k)$ holds for every $k \in \mathbb{Z}$. First, we show $P(k)$ for $k \ge 0$ by induction on k.

The base case $k = 0$ is obvious.

Assuming $P(k)$, we obtain

$$\begin{aligned} f((k+1)(x + f(x))) &= f(k(x + f(x)) + x + f(x)) = f(k(x + f(x))) + nx \\ &= f(0) + knx + nx = f(0) + (k+1)nx\,, \end{aligned}$$

that is, $P(k+1)$, which completes the induction.

It remains to show $P(-k)$ for $k \ge 0$ by induction on k. Assume $P(-k)$. Then,

$$\begin{aligned} f(0) - knx &= f(-k(x + f(x))) = f(-(k+1)(x + f(x)) + x + f(x)) \\ &= f(-(k+1)(x + f(x))) + nx\,; \end{aligned}$$

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whence $f(-(k+1)(x+f(x))) = f(0)-(k+1)nx$, which is $P(-(k+1))$, thus completing the induction. Thus $P(k)$ holds for every integer k.

As a consequence, we successively obtain

$$f(0)+(x+f(x))ny = f((x+f(x))(y+f(y))) = f(0)+(y+f(y))nx$$

and

$$xf(y) = yf(x)$$

for all integers x and y. Setting $y = 1$ gives

$$f(x) = xf(1) = lx\,.$$

where $l = f(1)$. Plugging this into the original functional equation, we obtain

$$f(1)(x+y+yf(1)) = f(x+y+f(y)) = f(x)+ny = xf(1)+ny.$$

Therefore

$$n = f(1)(f(1)+1) = l(l+1).$$

This completes the proof that there are no further solutions.

OC150. Let ABC be an isosceles triangle with $AB = AC$ and let D be the leg of perpendicular from A. P is an interior point of triangle ADC such that $\angle APB > 90^\circ$ and $\angle PBD + \angle PAD = \angle PCB$. Let Q be the intersection of CP and AD, and let R be the intersection of BP and AD. Let T be a point on $[AB]$ and let S be a point on $[AP]$ not belonging to $[AP]$ such that $\angle TRB = \angle DQC$ and $\angle PSR = 2\angle PAR$. Prove that $RS = RT$.

Originally from Turkey National Olympiad Second Round 2012, Day 1, Problem 2.

We received two correct submissions. We present the solution by Oliver Geupel.

Fixing the ambiguity in the problem statement, suppose that S is a point on *the half-line from point A through point P*, not belonging to the segment $[AP]$.

Let

$$\alpha = \angle QBA\,,\ \beta = \angle RBQ \ \text{ and } \ \gamma = \angle DBR\,.$$

The following relations are found by routine angle chasing:

$$\begin{aligned}
&\angle PAC = 90^\circ - \alpha - 2\beta - \gamma\,, && \angle PBA = \alpha + \gamma\,,\\
&\angle BAP = 90^\circ - \alpha - \gamma\,, && \angle CBP = \gamma\,,\\
&\angle PCB = \beta + \gamma\,, && \angle ACP = \alpha
\end{aligned}$$

By the trigonometric version of Ceva's theorem, for the triangle ABC and the point P it holds

$$\begin{aligned}&\frac{\cos(\alpha+2\beta+\gamma)}{\cos(\alpha+\gamma)}\cdot\frac{\sin(\alpha+\beta)}{\sin\gamma}\cdot\frac{\sin(\beta+\gamma)}{\sin\alpha}\\ &\qquad=\frac{\sin\angle PAC}{\sin\angle BAP}\cdot\frac{\sin\angle PBA}{\sin\angle CBP}\cdot\frac{\sin\angle PCB}{\sin\angle ACP}=1\,.\end{aligned}\tag{1}$$

and for the triangle ATS and the point R we have

$$\frac{\sin\beta}{\cos(\alpha+\beta+\gamma)}\cdot\frac{\cos(\gamma-\alpha)}{\sin\angle STR}\cdot\frac{\sin\angle RST}{\sin 2\beta}=1\,.\tag{2}$$

From (1) we deduce

$$\begin{aligned}0&=\cos(\alpha+2\beta+\gamma)\sin(\alpha+\beta)\sin(\beta+\gamma)-\cos(\alpha+\gamma)\sin\gamma\sin\alpha\\ &=\frac{1}{4}\left(\cos(2\alpha+2\beta)+\cos(2\beta+2\gamma)+\cos(2\gamma+2\alpha)-\cos 2\alpha-\cos 2\gamma\right.\\ &\qquad\qquad\left.-\cos(2\alpha+4\beta+2\gamma)\right)\\ &=\frac{1}{2}\big(\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\big)\big(\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)\big)\\ &=\sin\beta\sin(\alpha+\beta+\gamma)\big(\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)\big),\end{aligned}$$

whence

$$\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)=0.$$

Therefore,

$$\begin{aligned}&\sin\beta\cos(\gamma-\alpha)-\cos(\alpha+\beta+\gamma)\sin 2\beta\\ &=-\sin\beta\big(2\cos(\alpha+\beta+\gamma)\cos\beta-\cos(\alpha-\gamma)\big)\\ &=-\sin\beta\big(\cos(\alpha+2\beta+\gamma)-\cos(\alpha-\gamma)+\cos(\alpha+\gamma)\big)\\ &=0\,.\end{aligned}$$

By (2), we conclude that $\angle RST=\angle STR$, that is, the triangle RST is isosceles with $RS=RT$. The proof is complete.

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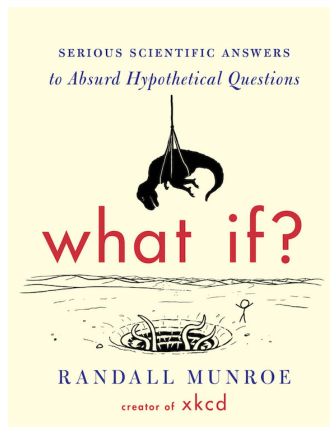
BOOK REVIEWS

Robert Bilinski

What if? Serious Scientific Answers to Absurd Hypothetical Questions by Randall Munroe
ISBN 978-0-544-45686-0, 303 pages
Houghton Mifflin Harcourt

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

Randall Munroe is a physicist who worked at NASA, then started the famous XKCD, a mathematics and physics based comic strip. As a widely read author, he has received a tremendous amount of fan mail containing the strangest questions; he then tried answering these questions in a scientifically sound way, respecting the known laws of physics, chemistry, biology and mathematics. Over the years, he maintained a blog with the answers to these questions. He compiled some of them into this book, updating older articles with comments and remarks from other specialists reading his blog and added a lot of new material. The result is an unconventional and original book for the curious, if not of the answers, then of the wide ranging selection of odd and unusual questions about our world.



To grasp this book, one has to get a feel for the type of questions addressed. Here are a few of them, randomly selected by flipping through the book:

> "I read about some researchers who were trying to produce sperm from bone marrow stem cells. If a woman were to have sperm cells made from her own stem cells and impregnated herself, what would be her relationship to her daughter?
>
> "If you suddenly began rising steadily at 1 foot per second, how exactly would you die? Would you freeze or suffocate first? Or something else?"
>
> "What if a rainstorm dropped all of its water in a single giant raindrop?"

As we can see, curiosity is, by its own nature, wide ranging and without limits.

One of the immense pleasures of XKCD is its special kind of weird humour that requires the reader to have mathematical maturity to tickle their funny bone. This book is naturally full of it, which is further amplified by the unexpected hypothetical questions whose craziness seems to have no bounds. The humour transpires through the use of proofs reductio ad absurdum to "real life", or the search of an exact mathematical proof to a messed up little problem. In sum, the book is rife with sentences like the following:

> "Because the moles form a literal fur coat, when frozen, they would insulate the interior of the planet and slow the loss of heat to space."

> "At those speeds, you don't really have to worry about the heating from the air – a quick back-of-the-envelope calculation suggests that if your body were doing that much work, your core temperature would reach fatal levels in a matter of seconds."

I will leave you with a riddle: What questions would have these sentences as part of their answers? I hope your imagination is in full gear, as the real questions will surpass your wildest imaginings.

As a bonus, Mr. Munroe also has added a long list of questions he doesn't even attempt to answer because of their weirdness:

> Question: "What if you strapped C4 to a boomerang? Could this be an effective weapon, or would it be as stupid as it sounds?"

> Answer: "Aerodynamics aside, I'm curious what tactical advantage you're expecting to gain by having the high explosive fly back at you if it misses the target."

Through the answers, we also notice some disturbing facts, namely that the government also has to answer questions such as "Why don't we try to destroy tropical cyclones by nuking them?" The government actually issued a response to this, to which Mr. Munroe replies "It makes me happy that an arm of the US government has, in some official capacity, issued an opinion on the subject of firing nuclear missiles at hurricanes."

A very appealing characteristic of the answers proposed by the author is that he builds them all from scratch and the ground up, so we embark on a very hands-on and real life application tour of mathematics. After all, math is as good at proving things as it is at disproving them, something that is often overlooked in our current iteration of schooling. Another very important premise is that quick calculations can very powerfully disprove stupid ideas, even when said calculations are just rough estimations. There is definitely more than meets the eye in this book.

The virtue of works by Mr. Munroe is that they are still mathematically sound, but sometimes applied in a flippant way corresponding to the absurdity of the reasoning itself. I can't remember a book that has made me laugh so much. Clearly, one buys a book like this for its entertainment value. We can only lament that humorous mathematically inclined entertainment is so rare, and console ourselves that this one in particular is so good.

Good reading.

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Chebyshev polynomials and recursive relations (II)

N. Vasiliev and A. Zelevinskiy

Continued from ***Crux***, Volume 40 (8). In the first part of this article, we defined Chebyshev polynomials of the first and second type and exhibited some of their properties. Here we continue this discussion. Recall the following numbered equations from part I of this article:

$$P_{n+1}(x) = x \cdot P_n(x) - P_{n-1}(x), \quad \text{where} \quad P_0(x) = 1,\ P_1(x) = x; \tag{1}$$

$$P_n(2\cos\phi) = \frac{\sin((n+1)\phi)}{\sin\phi}; \tag{2}$$

for polynomials $Q_i(x)$ with $Q_0(x) = 2, Q_1(x) = x$ that satisfy (1) we have:

$$2\cos n\phi = Q_n(2\cos\phi); \tag{2'}$$

$$P_n(x) = \frac{(x+\sqrt{x^2-4})^{n+1} - (x-\sqrt{x^2-4})^{n+1}}{2^{n+1}\sqrt{x^2-4}}. \tag{3}$$

3 Polynomial roots and various products

Many interesting problems that involve symmetrical combinations of n numbers (or letters) can be easily solved if these numbers are viewed as roots of some polynomials of degree n. Given n numbers $\gamma_k = 2\cos\frac{k\pi}{n+1}$, for $k = 1, 2, \ldots, n$, the corresponding polynomial is the aforementioned $P_n(x)$: indeed, if you substitute $\phi = \frac{\pi}{n+1}, \frac{2\pi}{n+1}, \ldots, \frac{n\pi}{n+1}$ into (2), we see that $\gamma_n = 2\cos\frac{k\pi}{n+1}$ are the roots of $P_n(x)$. Here we need the following well-known result: if γ is a root of a polynomial $F(x)$, then $F(x)$ is divisible by $x - \gamma$. Therefore, the polynomial $P(x)$ is divisible by all the polynomials $x - \gamma_k$ and hence is divisible by their product. Since $P_n(x)$ is of degree n and has leading coefficient equal to 1, it is equal to the product $\prod_{1\le k\le n}(x - \gamma_k)$. So we have:

$$P_n(x) = \prod_{1\le k\le n}\left(x - 2\cos\frac{k\pi}{n+1}\right). \tag{4}$$

Exercise 5.

a) Prove the identity $Q_n(x) = \prod_{1\le k\le n}\left(x - 2\cos\frac{(2k-1)\pi}{2n}\right)$ (4′)

b) Check (4) and (4′) for $n = 2, 3, 4, 5$.

Next, we will derive one curious identity, which follows from combining (2) and (4). Let us calculate $P_{2m}(0)$ for $m > 0, m \in \mathbb{Z}$ in two different ways and equate the results. On the one hand, from (2) we have:

$$P_{2m}(0) = P_{2m}\left(2\cos\frac{\pi}{2}\right) = \left(\sin\frac{(2m+1)\pi}{2}\right) / \sin\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + m\pi\right) = (-1)^m.$$

On the other hand, from (4) we get:

$$P_{2m}(0) = \prod_{1\le k\le 2m}\left(-2\cos\frac{k\pi}{2m+1}\right).$$

For $m+1 \le k \le 2m$, replace each $\cos\frac{k\pi}{2m+1}$ by $-\cos\left(\pi - \frac{k\pi}{2m+1}\right)$ to get

$$P_{2m}(0) = (-1)^m\left[2^m \prod_{1\le k\le m}\cos\frac{k\pi}{2m+1}\right]^2.$$

The expression in square brackets is positive because it involves only cosines of acute angles; it is therefore equal to 1 (which is easily seen after equating the two expressions for $P_{2m}(0)$), that is

$$\prod_{1\le k\le m}\cos\frac{k\pi}{2m+1} = \frac{1}{2^m}. \tag{5}$$

In words, (5) can be described as follows: for $m > 0$, the geometric mean of cosines of acute angles that are multiples of $\frac{\pi}{2m+1}$ is equal to $\frac{1}{2}$.

Exercise 6.

a) Find $P_n(1), P_n(-1), Q_n(1), Q_n(-1)$.

Prove the following identities similar to (5):

b) $\prod_{1\le k\le m}\sin\dfrac{k\pi}{2m} = \dfrac{\sqrt{m}}{2^{m-1}},\ \ m \ge 1;$

c) $\prod_{1\le k\le m}\tan\dfrac{k\pi}{2m+1} = \sqrt{2m+1},\ \ m \ge 1;$

d) $\prod_{1\le k\le m}\cos\dfrac{(2k-1)\pi}{4m} = \dfrac{\sqrt{2}}{2^m},\ \ m \ge 1.$

Exercise 7. Determine for which values of m and n

a) polynomial P_n is divisible by polynomial P_m;

b) polynomial Q_n is divisible by polynomial Q_m.

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4 Generating functions, series and coefficients

In this section, we will explore a method that is useful in many areas of mathematics — analysis, combinatorics, probability theory — the method of *generating functions*. This method allows us to use separate elements of a sequence to receive information about the whole sequence.

For a sequence $a_0, a_1, a_2, \ldots$, its generating function is given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots = \sum_{n \geq 0} a_n z^n.$$

These expressions are also known as *power series*. You can add, subtract and multiply power series like any other polynomials, you can divide one power series by another (if the divisor's constant term is not equal to 0), you can differentiate and integrate the whole series term by term; in fact, you can use all of these operations to get new series. It is often possible to use a recurrence relation definition of the function to find a simple formula for its generating function and vice versa — use the generating function to find a formula or a relation for elements of the sequence.

For a finite sequence $a_0, a_1, \ldots a_n$, its generating function is polynomial $f(z) = a_0 + a_1 z + \ldots + a^n z^n$. For example, the polynomial $f_n(z) = (1+z)^n$ is a generating function for the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, which appear in the nth row of Pascal's triangle (Table 2 in Part I of this article):

$$\sum_{0 \leq k \leq n} \binom{n}{k} z^k = (1+z)^n. \tag{6}$$

Differentiating (6) enough times and then setting $z = 0$, we find that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot \cdots \cdot k}.$$

Consider the obvious identity $(1+z)(1+z)^n = (1+z)^{n+1}$ written as

$$(1+z)\left(\sum_{0 \leq k \leq n} \binom{n}{k} z^k\right) = \sum_{0 \leq k \leq n+1} \binom{n+1}{k} z^k.$$

Opening the brackets and considering the coefficients of z^m on both sides of the above expression, we get an important identity

$$\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}.$$

Amongst all infinite series, geometric series have a particularly nice closed form of the power series. Let $b_0 = b$ and $b_n = q b_{n-1}$. Then replace each b_n by $q b_{n-1}$ in the sum

$$f(z) = \sum_{n \geq 0} b_n z^n = b + \sum_{n \geq 1} b_n z^n$$

to get

$$f(z) = b + qz \sum_{n\geq 1} b_{n-1} z^{n-1} = b + qzf(z),$$

from which we have $f(z)(1-qz) = b$, and hence

$$f(z) = \sum_{n\geq 0} b_n z^n = \frac{b}{1-qz}. \tag{7}$$

This is, of course, the well-known formula for the sum of a geometric series for $|qz| < 1$. But using this method, we can also get the generating function for our sequence of polynomials $P_n(x)$.

Let $\Phi(z) = \sum_{n\geq 0} P_n(x) z^n = 1 + xz + \sum_{n\geq 2} P_n(x) z^n$. Here x is a parameter and below, for ease of notation, we will use P_i instead of $P_i(x)$. By (1), replace each P_n (for $n \geq 2$) by $xP_{n-1} - P_{n-2}$. Then

$$\begin{aligned}\Phi(z) &= 1 + xz + \sum_{n\geq 2} xP_{n-1} z^n - \sum_{n\geq 2} P_{n-2} z^n \\ &= 1 + xz + xz \sum_{n\geq 2} P_{n-1} z^{n-1} - z^2 \sum_{n\geq 2} P_{n-2} z^{n-2} \\ &= 1 + xz + xz(\Phi(z) - 1) - z^2 \Phi(z).\end{aligned}$$

Therefore, $\Phi(z) \cdot (z^2 - xz + 1) = 1$ so

$$\Phi(z) = \frac{1}{z^2 - xz + 1}. \tag{8}$$

This simple formula contains the entire sequence of polynomials P_n that we have been studying! We can obtain separate P_n, that hide within it, in two different ways.

Method 1. For $|x| > 2$, the quadratic equation $z^2 - xz + 1 = 0$ has two roots:

$$u = \frac{x + \sqrt{x^2-4}}{2}, \quad v = \frac{x - \sqrt{x^2-4}}{2}. \tag{9}$$

Since $z^2 - xz + 1 = (z-u)(z-v)$ and $uv = 1$, we get:

$$\Phi(z) = \frac{1}{(z-u)(z-v)} = \left(\frac{u}{1-zu} - \frac{v}{1-zv}\right)\frac{1}{u-v} = \sum_{n\geq 0} \frac{u^{n+1} - v^{n+1}}{u-v} z^n,$$

that is $P_n(x) = \dfrac{u^{n+1} - v^{n+1}}{u-v}$, which is formula (3).

Method 2. Let us use (8) to find separate coefficients for each polynomial $P_n(x)$. Here is how:

$$\begin{aligned}\Phi(z) &= \frac{1}{1-(xz-z^2)} = \sum_{k\geq 0} (xz - z^2)^k \\ &= \sum_{k\geq 0} \left(\sum_{0\leq i\leq k} (-1)^i \binom{k}{i} x^{k-i} z^{k+i} \right) = \sum_{n\geq 0} z^n \left(\sum_i (-1)^i \binom{n-i}{i} x^{n-2i} \right),\end{aligned}$$

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where we used (7), (6) and then separated the coefficient of z^n, which is the needed $P_n(x)$. So we have:

$$P_n(x) = \sum_i (-1)^i \binom{n-i}{i} x^{n-2i}. \tag{10}$$

For example, $P_6(x) = \binom{6}{0}x^6 - \binom{5}{1}x^4 + \binom{4}{2}x^2 - \binom{3}{3} = x^6 - 5x^4 + 6x^2 - 1.$

Of course, one can prove the existing formulas (3) and (10) without generating functions; but the most exciting thing is how they appear, very easily, from the simple formula (8).

Note: we could perform all the operations with the power series above because for small values of z all the above series converge (for instance, (7) converges for $|z| < 1/|q|$).

Exercise 8. Consider the Fibonacci sequence given by

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = u_n + u_{n-1}.$$

a) Prove that its generating function is equal to $\dfrac{z}{1 - z - z^2}$.

b) Deduce that $u_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$.

c) Prove the identity $u_n = \sum_i \binom{n-i-1}{i}$.

Exercise 9.

a) Find the generating function for the sequence of polynomials $Q_n(x)$ define at the beginning of this article. Use it to show that for $|x| > 2$ we have

$$Q_n(x) = u^n - v^n,$$

where u and v are given by (9). Note: this is exercise 3c.

b) Show that formulas (3) and (3′) for $|x| < 2$ turn into formulas (2) and (2′). (Hint: $u = \cos\phi + i\sin\phi, v = \cos\phi - i\sin\phi$ if $x = 2\cos\phi$.)

This article originally appeared in Kvant, 1982 (1). It has been translated and adapted with permission.

Variations on a theme: The sum of equal powers of natural numbers (II)

Arkady Alt

This is the second of a series of notes, organized around the problem of finding a closed form for :

$$S_p(n) := \sum_{k=1}^{n} k^p = 1^p + 2^p + ... + n^p \text{ where } p, n \in \mathbb{N}.$$

In the first article, which appeared in issue 8 of this volume, we showed that $S_p(n)$ is always a polynomial in n of degree $p+1$, and we went over various ways to find that polynomial. As a review, the reader is encouraged to use one or more of these techniques to verify that

$$\begin{aligned} S_2(n) &= \frac{n(n+1)(2n+1)}{6} = S_1(n) \cdot \frac{2n+1}{3}, \\ S_3(n) &= \frac{n^2(n+1)^2}{4} = S_1^2(n), \\ S_4(n) &= \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30} = S_2(n) \cdot \frac{6S_1(n)-1}{5}, \\ S_5(n) &= \frac{n^2(n+1)^2}{12}\left(2n^2+2n-1\right) = S_1^2(n) \cdot \frac{4S_1(n)-1}{3}. \end{aligned}$$

The frequency with which the factor $S_1(n)$ appears is striking. In what follows, we will investigate the reasons for this.

The structure of $S_p(n)$

Because $S_1(n)$ appears so often, we will abbreviate it as S. We thus have (see the article by V. S. Abramovich in issue 6 of this volume):

$$\begin{aligned} S_2(n) &= S \cdot \frac{2n+1}{3} \\ S_3(n) &= S^2 \\ S_4(n) &= S_2(n) \cdot \frac{6S-1}{5} = \frac{S(6S-1)}{5} \cdot \frac{2n+1}{3} \\ S_5(n) &= S^2 \cdot \frac{4S-1}{3} = S_3 \cdot \frac{4S-1}{3} \end{aligned}$$

We observe that $S_4(n) = P(S)S_2(n)$ and $S_5(n) = Q(S)S_3(n)$, where P and Q are first-degree polynomials with rational coefficients. Does this pattern continue? We conjecture that $S_p(n) = S^{\delta(p)} Q_p(S) \cdot M_p(n)$, where

$$M_p(n) = \begin{cases} 1 & \text{if } p \text{ is odd,} \\ \frac{2n+1}{3} & \text{if } p \text{ is even,} \end{cases} \qquad \delta(p) = \begin{cases} 2 & \text{if } p \text{ is odd,} \\ 1 & \text{if } p \text{ is even,} \end{cases}$$

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and $Q_p(S)$ is a polynomial of degree $\left[\frac{p+1}{2}\right]-\delta(p)$ with rational coefficients. Because we are looking at patterns that skip a value of p, it will be convenient to find recurrences that do the same thing – using $S_1, S_3, \ldots, S_{2n-1}$ to find S_{2n+1} and $S_2, S_4, \ldots, S_{2n-2}$ to find S_{2n}.

Recurrence relations for S_p for odd p and for even p

Exercise 1 *Let $p \geq 3$ be odd. Expand $(t+1)^{p+2} + (t-1)^{p+2}$ and show that this is equal to*

$$2\sum_{i=0}^{\frac{p+1}{2}} \binom{p+2}{2i} t^{p+2-2i} .$$

Use this to show that

$$S_p(n) = \frac{(n+1)^{p+2} - n^{p+2} - 1 - (p+2)\left(n^2+n\right) - 2\sum_{i=2}^{\frac{p-1}{2}} \binom{p+2}{2i} S_{p+2-2i}(n)}{(p+2)(p+1)} . \tag{1}$$

Exercise 2 *Let $p \geq 2$ be even. Expand $(t+1)^{p+1} - (t-1)^{p+1}$ and show that this is equal to*

$$2\sum_{i=0}^{p/2} \binom{p+1}{2i+1} t^{p-2i} .$$

Use this to show that

$$S_p(n) = \frac{(n+1)^{p+1} + n^{p+1} - 1 - 2\sum_{i=1}^{p/2} \binom{p+1}{2i+1} S_{p-2i}(n)}{2(p+1)} . \tag{2}$$

Exercise 3 *Using the above recursions or otherwise, prove that*

$$S_6(n) = \frac{n(n+1)(2n+1)\left(3n^4+6n^3-3n+1\right)}{42} = \frac{S_2(n)\left(6S^2-6S+1\right)}{7}$$

and further that

$$S_6(n) = \frac{S(6S^2-6S+1)}{7} \cdot \frac{2n+1}{3}$$

and

$$S_7(n) = S^2 \cdot \frac{6S^2-4S+1}{3}.$$

So we have sufficient grounds to formulate two hypotheses (these are problems 4 and 5 in the Abramovich article):

a) For any odd $p \geq 3$, the quotient $\frac{S_p(n)}{S_3} = \frac{S_p(n)}{S^2}$ is a polynomial in S with rational coefficients;

b) For any even $p \geq 2$, the quotient $\frac{S_p(n)}{S_2(n)} = \frac{3S_p(n)}{S\cdot(2n+1)}$ is a polynomial in S with rational coefficients. (Or, using the above notation, prove that the quotient $\frac{S_p(n)}{S^{\delta(p)}M_p(n)}$ is a polynomial in S with rational coefficients.)

We will prove both hypotheses in an upcoming article, but for now we note one interesting property of $S_p(n)$ when p is odd. For any odd p, and any natural number n, the natural number $S_p(n)$ is divisible by the natural number $S_1(n)$. Indeed, upon reordering the terms we see that

$$S_p(n) = \sum_{k=1}^{n} k^p = \sum_{k=0}^{n-1} (n-k)^p . \tag{3}$$

Note that $a^p + b^p$ is divisible by $a+b$ if p is odd since $a^p + b^p = a^p - (-b)^p$ and $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$. Then $k^p + (n-k)^p$ is divisible by $k + n - k = n$ for any $k = 1, 2, \ldots, n-1$ and therefore

$$\begin{aligned} 2S_p(n) = \sum_{k=1}^{n} k^p + \sum_{k=0}^{n-1} (n-k)^p &= \sum_{k=1}^{n-1} (k^p + (n-k)^p) + 2n^p \\ &= \sum_{k=1}^{n-1} \left(k^p + \sum_{i=0}^{p} \binom{p}{i} n^i (-k)^{p-i} \right) + 2n^p . \end{aligned}$$

As p is odd, we have

$$2S_p(n) = \sum_{k=1}^{n-1} \sum_{i=1}^{p} \left(\binom{p}{i} n^i (-k)^{p-i} \right) + 2n^p$$

and every term of the right-hand side is divisible by n.

Exercise 4 *By reindexing the right-hand side of (3), show that $(n+1)|2S_p(n)$. Conclude that the natural number $S_1(n)$ divides the natural number $S_p(n)$.*

Remark 1 *It is important to understand the subtlety of this statement that we just proved! Despite the fact that the coefficients of the polynomials $S_p(n)$ and $S_1(n)$ are rational numbers, their values and the value of the quotient $\frac{S_p(n)}{S_1(n)}$ are integers for any integer n. But that cannot be said about the quotient $\frac{S_p(n)}{S_1^2(n)}$ (find examples for which this quotient is not an integer). In fact, this quotient for odd $p \geq 3$ is also a polynomial with rational coefficients: this was confirmed above for $p = 3, 5, 7$ and will be proven for all p later.*

To prove hypothesis a) above, we need the following result.

Lemma 1 *For any odd $n \geq 5$, there are polynomials $K_n(t)$ with integer coefficients such that*

$$(x+1)^n - x^n - 1 - n(x^2 + x) = (x^2 + x)^2 K_n(x^2 + x).$$

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Proof sketch. Let $t = x^2 + x$ and let $L_n = (x+1)^n - x^n$, $n \in \mathbb{N} \cup \{0\}$. Then $K_n = \frac{(x+1)^n - x^n - 1 - nt}{t}$ and $L_n = t^2 K_n + nt + 1$. We will prove that for any odd $n \geq 5$, K_n is a polynomial in t with integer coefficients.

Note that L_n can be defined by the recurrence

$$L_n = L_{n+2} - (2x+1)L_{n+1} + tL_n$$

for $n \in \mathbb{N} \cup \{0\}$, with $L_0 = 0$ and $L_1 = 1$ (check this!). Then substituting $L_n = t^2 K_n + nt + 1$ into this recurrence yields the recurrence

$$K_{n+4} = (2t+1)K_{n+2} - t^2 K_n - (n-2)t + 3$$

for odd $n \geq 5$ with $K_5 = 5, K_7 = 7(t+2)$. Therefore, K_n is a polynomial in t with integer coefficients. □

Exercise 5 *Using the lemma above and recurrence (1) for $S_p(n)$ for odd p, prove hypothesis a), that is that for any odd $p \geq 3$ there is a polynomial $Q_p(x)$ with rational coefficients such that $S_p(n) = S^2 \cdot Q_p(S)$; find a recursion for $Q_p(S)$.*

Remark 2 *Polynomials $S^2 \cdot Q_p(S)$, which equal to $S_p(n)$ for odd $p \geq 3$ are called* Faulhaber's polynomials *in honour of the German mathematician Johann Faulhaber (1580–1635) who first discovered this representation of $S_p(n)$ and computed the first seventeen of the polynomials. Recurrence (3) for calculating $Q_p(S)$ can be practically considered as a recurrence for Faulhaber's polynomials.*

Exercise 6 *Using recurrence (2) for $S_p(n)$ for even p, prove hypothesis b).*

References

Abramovich, V. S., *Sums of equal powers of natural numbers*, ***Crux* 40 (6)**, p. 248–252.

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PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2016***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

3991. *Proposé par Michel Bataille.*

Soit ABC un triangle tel que $BC = a, CA = b, AB = c, \angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$ et soient $m_a = AA', m_b = BB', m_c = CC'$ où A', B', C' sont les mi points de BC, CA, AB. Démontrer que

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}.$$

3992. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soient α, a, b, c des nombres réels positifs tels que $a+b+c+3 = 6abc$. Déterminer la valeur maximale de

$$\frac{1}{a^\alpha + b^\alpha + 1} + \frac{1}{b^\alpha + c^\alpha + 1} + \frac{1}{c^\alpha + a^\alpha + 1}.$$

3993. *Proposé par Dragoljub Milošević.*

Soient h_a, h_b et h_c les altitudes d'un triangle et soit r le rayon de son cercle inscrit. Démontrer que

$$\frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} \geq \frac{3}{5}.$$

3994. *Proposé par George Apostolopoulos.*

Soient a, b, c des nombres réels positifs tels que $a + b + c = 1$. Démontrer que

$$a^4 + b^4 + c^4 \geq abc.$$

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3995. *Proposé par Michel Bataille.*

Pour x et y positifs, posons $\mathcal{M}_0(x,y) = \sqrt{xy}$ et $\mathcal{M}_\alpha(x,y) = \left(\frac{x^\alpha+y^\alpha}{2}\right)^{\frac{1}{\alpha}}$ où α est un nombre réel non nul. Étant donné un triangle équilatéral ABC, déterminer pour quelles valeurs de α la propriété suivante tient: $\mathcal{M}_\alpha(PB, PC) \leq PA$ pour tout point P sur la ligne BC distinct de B et C.

3996. *Proposé par Marcel Chiriţă.*

Soit $a \in (1, \infty)$ et $b, c \in \mathbb{R}$. Déterminer toute fonction différentiable $f : [1, \infty) \to \mathbb{R}$ telle que

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c$$

pour tout $\lambda \in (0, \infty)$ et $x \in [1, \infty)$.

3997. *Proposé par Mihaela Berindeanu.*

Soient a, b, c des nombres positifs dont le produit est 8. Démontrer que

$$\frac{a^4+b^4}{c^3} + \frac{a^4+c^4}{b^3} + \frac{b^4+c^4}{a^3} \geq 64\left(\frac{1}{a^5} + \frac{1}{b^5} + \frac{1}{c^5}\right) + 6.$$

3998. *Proposé par George Apostolopoulos.*

Soient a_i, $i = 1, 2, \ldots, n$ des nombres réels positifs tels que $\sum_{i=1}^n a_i = n$. Démontrer que

$$\sum_{i=1}^n \left(\frac{a_i^3+1}{a_i^2+1}\right)^4 \geq n.$$

3999. *Proposé par Leonard Giugiuc et Diana Trailescu.*

Soient a, b, c des nombres réels tels que $a \geq 1 \geq b \geq c > -3$ et

$$ab + bc + ca = 3.$$

Démontrer que $a + b + c \geq 3$.

4000. *Proposé par Marcel Chiriţă.*

Soient $x_1, x_2, \ldots x_n$ tels que $x_1 > x_2 > \ldots > x_n > 0$ puis $x_1 x_2 \ldots x_n = 1$ où $n \geq 3$. Démontrer que

$$\frac{x_1^2+x_2^2}{x_1-x_2} \cdot \frac{x_2^2+x_3^2}{x_2-x_3} \cdots \frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} > 2^{3/n}.$$

. . . .

3991. *Proposed by Michel Bataille.*

Let ABC be a triangle with $BC = a, CA = b, AB = c, \angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$ and let $m_a = AA', m_b = BB', m_c = CC'$ where A', B', C' are the midpoints of BC, CA, AB. Prove that

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}.$$

3992. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let α, a, b, c be positive real numbers such that $a + b + c + 3 = 6abc$. Find the maximum value of the expression

$$\frac{1}{a^\alpha + b^\alpha + 1} + \frac{1}{b^\alpha + c^\alpha + 1} + \frac{1}{c^\alpha + a^\alpha + 1}.$$

3993. *Proposed by Dragoljub Milošević.*

Let h_a, h_b and h_c be the altitudes and r the inradius of a triangle. Prove that

$$\frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} \geq \frac{3}{5}.$$

3994. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$a^4 + b^4 + c^4 \geq abc.$$

3995. *Proposed by Michel Bataille.*

For positive x and y, let $\mathcal{M}_0(x, y) = \sqrt{xy}$ and $\mathcal{M}_\alpha(x, y) = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}}$ if α is a nonzero real number. Given an equilateral triangle ABC, determine for which values of α the following property holds: $\mathcal{M}_\alpha(PB, PC) \leq PA$ for every point P distinct from B and C on the line BC.

3996. *Proposed by Marcel Chiriţă.*

Let $a \in (1, \infty)$ and $b, c \in \mathbb{R}$. Find all differentiable functions $f : [1, \infty) \to \mathbb{R}$ such that

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c$$

for all $\lambda \in (0, \infty)$ and $x \in [1, \infty)$.

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3997. *Proposed by Mihaela Berindeanu.*

Let a, b, c be positive numbers with product 8. Prove that

$$\frac{a^4+b^4}{c^3}+\frac{a^4+c^4}{b^3}+\frac{b^4+c^4}{a^3} \geq 64\left(\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}\right)+6.$$

3998. *Proposed by George Apostolopoulos.*

Let a_i, $i = 1, 2, \ldots, n$ be positive real numbers such that $\sum_{i=1}^{n} a_i = n$. Prove that

$$\sum_{i=1}^{n}\left(\frac{a_i^3+1}{a_i^2+1}\right)^4 \geq n.$$

3999. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Consider real numbers a, b, c such that $a \geq 1 \geq b \geq c > -3$ and

$$ab + bc + ca = 3.$$

Prove that $a + b + c \geq 3$.

4000. *Proposed by Marcel Chiriţă.*

Let $x_1, x_2, \ldots x_n$ with $x_1 > x_2 > \ldots > x_n > 0$, $x_1 x_2 \ldots x_n = 1$ and $n \geq 3$. Show that

$$\frac{x_1^2+x_2^2}{x_1-x_2}\cdot\frac{x_2^2+x_3^2}{x_2-x_3}\cdots\frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n}\cdot\frac{x_1^2+x_n^2}{x_1-x_n} > 2^{3/n}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2013: 39(10), p. 456–460.

3891. *Proposed by Michel Bataille.*

Let ABC be a triangle that is neither equilateral nor right-angled and let O be its circumcentre. Let the altitudes from A, B and C meet the circumcircle again at A', B' and C', respectively. If U, V and W denote the circumcentres of triangles OBC, OCA and OAB, respectively, prove that the lines UA', VB' and WC' are concurrent and identify their common point.

We received seven solutions, all correct. We present four of these solutions.

Solution 1, by the proposer.

Let $\mathbf{I}$ be the inversion in the circumcircle Γ of triangle ABC and $\mathbf{J}$ be the inversion in the circumcircle Δ of triangle OBC. Let H be the orthocentre of triangle ABC. The points H and A' are reflections of each other in the axis BC; any circle through these two points is orthogonal to BC. The inversion $\mathbf{I}$ fixes A' and carries the line BC to Δ. Let $P = \mathbf{I}(H)$. Then, because $\mathbf{I}$ preserves orthogonality, any circle through A' and P is orthogonal to Δ, so that $\mathbf{J}(A') = P$. Therefore $A'P$ passes through the centre U of Δ. [See R.A. Johnson, *Advanced Euclidean Geometry*, Houghton-Mifflin, 1929, reprinted by Dover, 1960, paragraph 80, p. 55.] Thus UA', and similarly VB' and WC', all pass through the point $P = \mathbf{I}(H)$.

Solution 2, by Oliver Geupel (slightly modified by the editor).

We prove that the three lines are concurrent at the inverse point P of the orthocentre H with respect to inversion in the circumcircle of the triangle ABC.

Consider the problem in the complex plane where the circumcircle is the unit circle and the geometric inverse of z is $1/\bar{z}$. Let a, a', b, c, h, p, u denote the affixes of the respective points A, A', B, C, H, P, U. Noting that two vectors z and w are perpendicular if and only if $z/\bar{z} = -w/\bar{w}$, we deduce from $AA' \perp BC$ that

$$-aa' = \frac{a-a'}{(1/a)-(1/a')} = \frac{a-a'}{\bar{a}-\bar{a'}} \quad \text{and} \quad -\frac{b-c}{\bar{b}-\bar{c}} = -\frac{b-c}{(1/b)-(1/c)} = bc,$$

whence

$$a' = -\frac{bc}{a} \qquad \text{and} \qquad \bar{a'} = -\frac{a}{bc}.$$

The condition $u\bar{u} = (u-b)(\bar{u}-\bar{b})$ leads to $\bar{b}u + b\bar{u} = 1$. Similarly $\bar{c}u + c\bar{u} = 1$. From this, we find that

$$u = \frac{bc}{b+c} \qquad \text{and} \qquad \bar{u} = \frac{1}{b+c}.$$

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Since $h = a+b+c$ and $h\bar{p} = 1$,

$$\bar{p} = \frac{1}{a+b+c} \qquad \text{and} \qquad p = \frac{abc}{ab+bc+ca}.$$

A straightforward computation yields

$$\begin{aligned}(a'-u)(\bar{a'}-\bar{p}) &= \left(\frac{bc}{a}+\frac{bc}{b+c}\right)\left(\frac{a}{bc}+\frac{1}{a+b+c}\right) = \frac{a^2+ab+bc+ca}{a(b+c)} \\ &= \left(\frac{a}{bc}+\frac{1}{b+c}\right)\left(\frac{bc}{a}+\frac{abc}{ab+bc+ca}\right) = (\bar{a'}-\bar{u})(a'-p).\end{aligned}$$

Hence the complex number $(a'-u)/(a'-p)$, being equal to its conjugate, is real. Thus the line $A'U$ passes through the point P. Similarly, the lines $B'V$ and $C'W$ pass through P. This completes the proof.

Solution 3, by Titu Zvonaru.

Conventionally, let a, b, c, R and H be the sides, circumradius and orthocentre of triangle ABC. Suppose D on BC is the foot of the altitude from A. It is known that $HD = DA' = 2R\cos B\cos C$. Since BC subtends the angle $2A$ at O and the angle $180^\circ - 2A$ at any point on the arc of the circumcircle of OBC opposite O, it follows that

$$OU = BU = \frac{a}{2\sin 2A} = \frac{2R\sin A}{4\sin A\cos A} = \frac{R}{2\cos A}.$$

Let UA' and OH intersect at P. Because $OU \| HA'$,

$$\frac{PO}{PH} = \frac{OU}{HA'} = \frac{R}{2\cos A}\cdot\frac{1}{4R\cos B\cos C} = \frac{1}{8\cos A\cos B\cos C}.$$

Therefore, UA', VB' and WC' are concurrent at P, which lies on the Euler line OH and satisfies

$$\frac{PO}{PH} = \frac{1}{8\cos A\cos B\cos C}.$$

(This argument can be adapted if one angle is obtuse.)

Solution 4, by Prithwijit De.

We assume that the triangle is acute and not equilateral. Let $X = UW \cap OB$ and $Y = UV \cap OC$. Since UW right bisects OB, $\angle UXO = 90^\circ$. Similarly, $\angle UYO = 90^\circ$, so that the quadrilateral $UXOY$ is concyclic. Therefore,

$$\angle WUV = \angle XUY = 180^\circ - \angle XOY = 180^\circ - \angle BOC = 180^\circ - 2A = \angle C'A'B'.$$

To see the last equality, note that

$$\angle AA'C' = \angle ACC' = 90^\circ - \angle BAC = \angle ABB' = \angle AA'B$$

(so that the line AA' bisects $\angle B'A'C'$). Similarly, $\angle VWU = \angle B'C'A'$ and $\angle UVW = \angle A'B'C'$. Thus triangles $A'B'C'$ and UVW are directly similar, and the lines UA', VB' and WC' concur at the centre P of homothety.

Since the line AA' bisects $\angle B'A'C'$, and, similarly, BB' and CC' bisects respectively $\angle A'B'C'$ and $\angle A'C'B'$, the incentre of triangle $A'B'C'$ is the orthocentre H of triangle ABC.

Since O is the midpoint of the arc BC of the circumcircle with centre U of triangle OBC, $\angle OUB = \angle OUC$, and OU bisects $\angle WUV$. Similarly OV bisects $\angle UVW$ and OW bisects $\angle UWV$. Therefore the incentre of triangle UVW is O. The line joining the incentres of triangles $A'B'C'$ and UVW passes through the centre of homothety, so that O, H and P are collinear.

From solution 2, note that $A'B'/UV = A'H/UO = 8\cos A\cos B\cos C$. Also

$$\begin{aligned} OH^2 &= (a/2 - c\cos B)^2 + (2R\cos B\cos C - R\cos A)^2 \\ &= R^2[(\sin A - 2\sin C\cos B)^2 + (2\cos B\cos C - \cos A)^2] \\ &= R^2[1 - 4\sin A\sin C\cos B + 4\cos^2 B - 4\cos A\cos B\cos C] \\ &= R^2[1 - 4\cos B(\cos A\cos C + \sin A\sin C + \cos(A+C)] \\ &= R^2[1 - 8\cos A\cos B\cos C]. \end{aligned}$$

Therefore,

$$1 - \left(\frac{OH}{R}\right)^2 = 8\cos A\cos B\cos C = \frac{A'B'}{UV} = \frac{PH}{PO}.$$

This simplifies to $OH \cdot OP = R^2$, from which we see that P is the inversion of H in the circumcircle of triangle ABC.

3892. *Proposed by George Apostolopoulos.*

Let a, b and c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that

$$\frac{R}{r} \geq \frac{2}{3}(\cos A + \cos B + \cos C) + \frac{a^3 + b^3 + c^3}{3abc}.$$

We received 11 correct solutions and one incorrect solution. We present a composite of similar solutions by Šefket Arslanagić, Michel Bataille, Scott Brown, John Heuver, Kee-Wai Lau, Dragoljub Milošević, and Titu Zvonaru.

Using the identities

$$\cos A + \cos B + \cos C = \frac{R+r}{R},$$

$$a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)$$

and

$$abc = 4Rrs,$$

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[for example in D. S. Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989], the claimed inequality is equivalent to

$$\frac{R}{r} \geq \frac{2}{3}\left(\frac{R+r}{R}\right) + \frac{2s(s^2-3r^2-6Rr)}{12Rrs}.$$

This can be rewritten as $6R^2+2Rr-r^2 \geq s^2$. Gerretsen's Inequality asserts that

$$4R^2+4Rr+3r^2 \geq s^2,$$

so it suffices to show that

$$6R^2+2Rr-r^2 \geq 4R^2+4Rr+3r^2.$$

But this is equivalent to

$$0 \leq 2R^2-2Rr-4r^2 = 2(R-2r)(R+r),$$

which holds by Euler's inequality $R \geq 2r$.

3893. *Proposed by Ovidiu Furdui.*

Let $n \geq 1$ be an integer and let the decimal part of a real number a be defined by $\{a\} = a - \lfloor a \rfloor$. Evaluate

$$\int_0^{\frac{\pi}{2}} \sin 2x \left\{\ln^{2n-1} \tan x\right\} \mathrm{d}x.$$

We received six correct solutions. We present the solution of the AN-anduud Problem Solving Group.

With the substitution $u = \tan x$, the integral becomes

$$\begin{aligned} I &= \int_0^\infty \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du \\ &= \int_0^1 \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du + \int_1^\infty \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du. \end{aligned}$$

If in this first integral we substitute $v = \frac{1}{u}$, we obtain

$$\begin{aligned} I &= \int_1^\infty \frac{2v}{(1+v^2)^2}\{-\log^{2n-1} v\}\, dv + \int_1^\infty \frac{2u}{(1+u^2)^2}\{\log^{2n-1} u\}\, du \\ &= \int_1^\infty \frac{2x}{(1+x^2)^2}\left(\{-\log^{2n-1} x\} + \{\log^{2n-1} x\}\right) dx \\ &= \int_1^\infty \frac{2x}{(1+x^2)^2}\, dx = \frac{1}{2}, \end{aligned}$$

using the equation $\{y\} + \{-y\} = 1$, which holds for all $y \in \mathbb{R} \setminus \mathbb{Z}$.

3894. *Proposed by Paul Bracken.*

a) Prove that for $n \in \mathbb{N}$,

$$1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

b) For the equation in part a), evaluate the limit of the right-hand side as $n \to \infty$ and compute the sum in closed form.

We received 12 correct submissions. We present two solutions for part a, and three solutions for part b, each utilized by multiple solvers.

For ease of solution-wide notation, let S_n be the left-hand-side of the identity, and let T_n be the right-hand-side.

Part a, Solution 1. It is easily checked that $\frac{1}{(3k)^3-3k} = \frac{1}{2}\left(\frac{1}{3k-1} + \frac{1}{3k+1} - \frac{2}{3k}\right)$. It follows that

$$\begin{aligned} 1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} &= \sum_{k=0}^{n} \frac{1}{3k+1} + \sum_{k=1}^{n} \frac{1}{3k-1} - 2\sum_{k=1}^{n} \frac{1}{3k} \\ &= \left(\sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{3k}\right) - 2\sum_{k=1}^{n} \frac{1}{3k} \\ &= \sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{3n+1} \frac{1}{k} \end{aligned}$$

that is,

$$1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

Part a, Solution 2. It is simple to check that for $n = 1$, both sides of the proposed equality evaluate to $\frac{13}{12}$. After assuming that the identity is true for n, in order to prove that it is also true for $n+1$ it is enough to prove that $S_{n+1} - S_n = T_{n+1} - T_n$, for then we may add $S_n = T_n$ to both sides to obtain the result. So we must prove

$$\frac{2}{(3(n+1))^3 - 3(n+1)} = \sum_{k=1}^{2n+3} \frac{1}{k+n+1} - \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

But the right-hand side of the last equation is

$$\frac{1}{3n+4} + \frac{1}{3n+3} + \frac{1}{3n+2} - \frac{1}{n+1} = \frac{2}{(3(n+1))^3 - 3(n+1)}.$$

Part b, Solution 1. From the above calculation in part a, the sum $T_n = \sum_{k=1}^{2n+1} \frac{1}{k+n}$ satisfies

$$T_n = \sum_{k=1}^{3n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = H_{3n+1} - H_n$$

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where as usual H_m denotes $\sum_{k=1}^{m} \frac{1}{k}$. It is well-known that for all positive integer m, we have $H_m = \ln(m) + \gamma + \varepsilon(m)$ where γ is the Euler-Mascheroni constant and $\lim_{m\to\infty} \varepsilon(m) = 0$. As a result, we may write

$$T_n = \ln(3n+1) + \gamma + \varepsilon(3n+1) - \ln(n) - \gamma - \varepsilon(n) = \ln\left(\frac{3n+1}{n}\right) + \varepsilon(3n+1) - \varepsilon(n)$$

and so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\left(3 + \frac{1}{n}\right) = \ln(3).$$

Part b, Solution 2. The right-hand side of the identity is bounded:

$$\int_{n+1}^{3n+2} \frac{1}{x}\,dx \le \sum_{k=n+1}^{3n+1} \frac{1}{k} \le \int_{n}^{3n+1} \frac{1}{x}\,dx.$$

Evaluating both sides of the inequality gives

$$\ln\left(\frac{3n+2}{n+1}\right) \le \sum_{k=n+1}^{3n+1} \frac{1}{k} \le \ln\left(\frac{3n+1}{n}\right),$$

and taking the limit as n tends to infinity yields

$$1 + 2\sum_{k=1}^{\infty} \frac{1}{(3k)^3 - 3k} = \lim_{n\to\infty} T_n = \ln(3).$$

Part b, Solution 3. Observe that we have:

$$\begin{aligned} T_n = \sum_{k=1}^{2n+1} \frac{1}{k+n} &= \sum_{k=1} n \frac{1}{k+n} + \sum_{k=n+1}^{2n} \frac{1}{k+n} + \frac{1}{3n+1} \\ &= \sum_{k=1} n \frac{1}{1+\frac{k}{n}} \frac{1}{n} + \sum_{k=1} n \frac{1}{2+\frac{k}{n}} \frac{1}{n} + \frac{1}{3n+1}. \end{aligned}$$

The two sums in the last line are Riemann sums for integrals of $\frac{1}{1+x}$ and $\frac{1}{2+x}$ over $[0,1]$, respectively, with constant step size equal to $\frac{1}{n}$, and right-hand endpoints. Taking n to infinity gives:

$$\begin{aligned} \lim_{n\to\infty} T_n &= \int_0^1 \frac{1}{1+x}\,dx + \int_0^1 \frac{1}{2+x}\,dx + \lim_{n\to\infty} \frac{1}{3n+1} \\ &= \ln(2) - \ln(1) + \ln(3) - \ln(2) + 0 = \ln(3). \end{aligned}$$

Editor's Comments. All solvers basically used some combination of these solutions. The induction proof in Part 1, Solution 2 is a difference equation technique. The Riemann sum argument is clever, but needs careful treating; that is, T_n itself is *not* a Riemann sum, but it *is* a Riemann sum plus a term that goes to 0. Finally, the comment about "a closed form" was, I believe, intended to mean "compute the value of the limit of this expression", not to find a closed form expression for the finite sum T_n.

3895. *Proposed by Neculai Stanciu and Titu Zvonaru.*

In the acute triangle ABC with $AB \neq AC$, let A' be the foot of the altitude from A, and let the bisector of the angle at A meet BC at D and the circumcircle at M. Finally, for a point T on the segment AD, let P and N be its projections on AA' and BC, respectively. Prove that if M, N, and P are collinear, then T is the incentre of the triangle.

Seven solutions were submitted, all correct. We give two solutions.

Solution 1, by Michel Bataille.

We first establish that T is the incentre of triangle ABC if and only if $MB = MT$. For, T is the incentre $\Leftrightarrow$ it lies on the bisector of angle $B \Leftrightarrow \angle CBT = \angle ABT$

$$\Leftrightarrow \angle TBM - \angle CBM = \angle BTM - \angle BAM = \angle BTM - \angle CBM$$
$$\Leftrightarrow \angle TBM = \angle BTM \Leftrightarrow MB = MT.$$

Suppose that M, N, P are collinear and that U is the midpoint of BC. Wolog, let $b = AC > AB = c$ so that $DC > DB$ (since $DC : DB = b : c$). Since $BU/BM = \cos(\angle MBC) = \cos(\angle MAC) = \cos(A/2)$, it follows that $MB = a/(2\cos A/2)$, where $a = BC$. Let M' be the point diametrically opposite to M on the circumcircle of triangle ABC. Since $\angle MAM' = 90^\circ$ and

$$\angle AMM' = 90^\circ - \angle AM'M = 90^\circ - \angle ACM = 90^\circ - \left(C + \frac{A}{2}\right) = \frac{B-C}{2},$$

we have that

$$MA = MM' \cos(\angle AMM') = 2R\cos\frac{B-C}{2}.$$

Also

$$\frac{a}{2}\tan\frac{A}{2} = BU\tan\frac{A}{2} = MU = MD\cos\frac{B-C}{2},$$

whereupon

$$MD = \frac{a\tan A/2}{2\cos((B-C)/2).}$$

Since $PT \| ND$ and $AP \| NT$, the homothety with centre M that takes $N \longrightarrow P$ also takes $D \longrightarrow T$ and $T \longrightarrow A$. Therefore $MA : MT = MT : MD$, and so

$$MT^2 = MA \cdot MD = 2R\cos\frac{B-C}{2} \cdot \frac{a\tan A/2}{2\cos((B-C)/2)}$$
$$= aR\tan\frac{A}{2} = \frac{a^2}{2\sin A}\cdot\tan\frac{A}{2} = \frac{a^2}{4\cos^2(A/2)} = MB^2.$$

Thus $MT = MB$ and the desired result follows.

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Solution 2, by Madhav R. Modak.

Let $AB < AC$, so that the points on BC are in the order $BA'NDC$. Let $AT = x$ and $TD = y$. Since M, N and P are collinear, Menelaus' theorem for triangle $AA'D$ and transversal PNM yields

$$\frac{A'N}{ND} \cdot \frac{DM}{MA} \cdot \frac{AP}{PA'} = -1. \tag{1}$$

Since $AA' \| TN$ and $PT \| A'D$, we have that $AP/PA' = A'N/ND = x/y$, so (1) gives numerically,

$$\frac{DM}{MA} = \frac{y^2}{x^2}. \tag{2}$$

Since $\angle DBM = \angle CAM = \frac{1}{2}A$ and $\angle BMD = \angle BCA = C$, the Sine Law applied to triangle BDM give $BD/\sin C = DM/\sin\frac{1}{2}A$. Since $\angle ABM = B + \frac{1}{2}A$ and $\angle BMA = C$, the Sine Law applied to triangle ABM gives $MA/\sin(B + \frac{1}{2}A) = AB/\sin C$. These give

$$\frac{DM}{MA} = \frac{BD}{AB} \cdot \frac{\sin\frac{1}{2}A}{\sin(B + \frac{1}{2}A)}. \tag{3}$$

In triangle ABD, $\angle ABD = 180^\circ - (B + \frac{1}{2}A)$ and $\angle BAD = \frac{1}{2}A$, so that, by the Sine Law, $BD/\sin\frac{1}{2}A = AB/\sin(B + \frac{1}{2}A)$. With (2) and (3), this yields that

$$\frac{y^2}{x^2} = \frac{DM}{MA} = \left(\frac{BD}{AB}\right)^2 \qquad \text{or} \qquad \frac{BD}{AB} = \frac{y}{x}.$$

Thus, in triangle ABD, BT bisects angle B and T lies on the bisectors of both angle A and B. Therefore T is the incentre of triangle ABC.

3896. *Proposed by Dao Thanh Oai and Nguyen Minh Ha.*

Let $[WXYZ]$ represent the signed area of the quadrilateral $WXYZ$ (where W, X, Y, Z can be any four points in the plane), namely half the signed area of the parallelogram formed by the vectors $\overrightarrow{WY}$ and $\overrightarrow{XZ}$:

$$[WXYZ] = \frac{1}{2}|\overrightarrow{WY}||\overrightarrow{XZ}|\sin(\overrightarrow{WY}, \overrightarrow{XZ}).$$

If $A_1A_2\ldots A_{2n}$ and $B_1B_2\ldots B_{2n}$ are two similarly oriented regular $2n$-gons in the plane, prove that $[A_iA_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]$ is constant for any i, $1 \le i \le 2n$, where the indices are reduced modulo $2n$.

We received four correct submissions. We present the solution independently submitted by Michel Bataille and by C.R. Pranesachar.

We first show that if the points W, X, Y, Z are represented by the complex numbers w, x, y, z, respectively, then

$$[WXYZ] = \frac{1}{2}\,\mathrm{Im}\left((z - x)(\overline{y} - \overline{w})\right). \tag{1}$$

Proof. Let $\alpha = \arg(z-x)$ and $\beta = \arg(y-w)$. Then, $\sin(\overrightarrow{WY}, \overrightarrow{XZ}) = \sin(\alpha - \beta)$, whence

$$\begin{aligned}[WXYZ] &= \frac{1}{2}\,|y-w|\cdot|z-x|\sin(\alpha-\beta)\\ &= \frac{1}{2}\,|y-w|\cdot|z-x|\mathrm{Im}(e^{i(\alpha-\beta)}) = \frac{1}{2}\,\mathrm{Im}(|z-x|e^{i\alpha}\cdot|y-w|e^{-i\beta});\end{aligned}$$

that is, $[WXYZ] = \frac{1}{2}\,\mathrm{Im}\left((z-x)(\overline{y-w})\right)$ as claimed.

Turning to the problem, we will suppose without loss of generality that the affixes of vertices $A_1, A_2, \ldots, A_{2n}$ of the first regular $2n$-gon are

$$a_1 = \omega, a_2 = \omega^2, \ldots, a_{2n} = \omega^{2n} = 1,$$

where $\omega = e^{\pi i/n}$. Because $B_1B_2\ldots B_{2n}$ is directly similar to $A_1A_2\ldots A_{2n}$, the affixes of its vertices are

$$b_1 = a\omega + b, \qquad b_2 = a\omega^2 + b, \ldots, \qquad b_{2n} = a + b$$

for some complex numbers a, b with $a \neq 0$. Then, using (1), we obtain

$$\begin{aligned}2[A_iA_{i+1}B_{i+1}B_i] &= \mathrm{Im}((b_i - a_{i+1})(\overline{b_{i+1}} - \overline{a_i})\\ &= \mathrm{Im}(a\omega^i + b - \omega^{i+1})(\overline{a}\omega^{-(i+1)} + \overline{b} - \omega^{-i}))\\ &= \mathrm{Im}(c + a\overline{b}\omega^i - \overline{b}\omega^{i+1} + \overline{a}b\omega^{-(i+1)} - b\omega^{-i}),\end{aligned}$$

where $c = \frac{|a|^2}{\omega} + |b|^2 - (a + \overline{a}) + \omega$.

Similarly,

$$\begin{aligned}2\quad &[A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]\\ &= \mathrm{Im}(c + a\overline{b}\omega^{n+i} - \overline{b}\omega^{n+i+1} + \overline{a}b\omega^{-(n+i+1)} - b\omega^{-(n+i)})\\ &= \mathrm{Im}(c - a\overline{b}\omega^i + \overline{b}\omega^{i+1} - \overline{a}b\omega^{-(i+1)} + b\omega^{-i}),\end{aligned}$$

where the second equality follows from $\omega^n = e^{i\pi} = -1$. Thus,

$$[A_iA_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}] = \frac{1}{2}\mathrm{Im}(2c) = (1 - |a|^2)\sin\frac{\pi}{n}.$$

This quantity is independent of i ($1 \leq i \leq 2n$), and the proof is complete.

Editor's Comments. The solutions of Bataille and Pranesachar were nearly identical except for notation — Pranesachar used Cartesian coordinates instead of Bataille's complex numbers. The use of complex numbers resulted in a more compact presentation.

3897. *Proposed by Yakub Aliyev.*

Let the cevians BB_1 and CC_1 of a triangle ABC intersect at the point O. Prove that if a line is drawn through O meeting line segment BC_1 at X and line segment B_1C at Y, then

$$\frac{|BX|}{|XC_1|} > \frac{|B_1Y|}{|YC|}.$$

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We received ten correct submissions. We present the solution by Roy Barbara.

Let ℓ denote the line through C that is parallel to AB. Denote by B_2 and Z the points where ℓ meets BB_1 and XY, respectively. Finally, the line through B_1 that is parallel to YZ meets ℓ at U. Because all points of ℓ except C lie outside the given triangle, B_1 must lie between O and B_2, so that U lies between Z and B_2, which implies that

$$B_2Z > UZ. \tag{1}$$

Similar triangles BXO and B_2ZO provide $\frac{BX}{B_2Z} = \frac{XO}{ZO}$; similar triangles C_1XO and CZO provide $\frac{XC_1}{ZC} = \frac{XO}{ZO}$. From this pair of equalities we get

$$\frac{BX}{XC_1} = \frac{B_2Z}{ZC}. \tag{2}$$

Since YZ is parallel to the base UB_1 of ΔUB_1C, we have

$$\frac{B_1Y}{YC} = \frac{UZ}{ZC}. \tag{3}$$

Using (2), (1), (3) in this order we obtain

$$\frac{BX}{XC_1} = \frac{B_2Z}{ZC} > \frac{UZ}{ZC} = \frac{B_1Y}{YC}.$$

Editors comments. Only Bataille observed explicitly that it is necessary to assume that the line segments mentioned in the statement of the problem exclude their endpoints. Moreover, everybody assumed tacitly that the points B_1 and C_1 were chosen to lie on the sides of ΔABC. Should they lie on the sides extended away from the vertex A beyond B and C, then the featured solution could be easily modified to prove that the required inequality would be reversed.

3898. *Proposed by Dragoljub Milošević.*

On the extension of the side AB of the regular pentagon $ABCDE$, let the points F and G be placed in the order F, A, B, G so that $AG = BF = AC$. Compare the area of triangle FGD to the area of pentagon $ABCDE$.

There were 14 solutions, all of them correct. Seven solvers used variants of the strategy of Solution 1. All but one of the remaining solvers, with more or fewer complications, provided a computational solution along the lines of Solution 3. However, Dag Jonsson had a different slant, which we reproduce in Solution 2.

Solution 1, by various solvers.

Since any side of a regular pentagon is parallel to the diagonal not containing either of its endpoints, $BF \| EC$. Since $BF = EC$, $BFEC$ is a parallelogram and $FE \| BC \| DA$. Therefore $[ADF] = [ADE]$. Also, $[BDG] = [BDC]$. Hence

$$[ABCDE] = [ABD] + [ADE] + [BDC] = [ABD] + [ADF] + [BDG] = [FGD].$$

Solution 2, by Dag Jonsson modified by the editor.

Let s and d be the side and diagonal lengths of the pentagon and let AC and BD intersect at H. From the similarity of the triangles ADB and BAH, we deduce that $s/d = (d-s)/s$, whence $s^2 - d^2 + ds = 0$. The pentagon $ABCDE$ is the union of five $72^\circ - 54^\circ - 54^\circ$ isosceles triangles whose bases are the sides of the pentagon of length s and whose common apex is the circumcentre of the pentagon. Each such triangle has area ks^2 for some contant k.

Since $AG = AD$, it follows that

$$\angle GFD = \angle FGD = \angle AGD = \angle ADG = \frac{1}{2}(180^\circ - \angle DAG) = 54^\circ,$$

so that triangle FGD is a $72^\circ - 54^\circ - 54^\circ$ isosceles triangle with base $2d - s$ similar to each component triangle of the pentagon. Hence its area is $k(2d-s)^2$ and so

$$[ABCDE] - [FGD] = 5ks^2 - k(2d-s)^2 = 4k(s^2 - d^2 + ds) = 0.$$

Solution 3, by various solvers.

If $t = \cos 36^\circ$, then $-t = \cos 144^\circ = 2(2t^2-1)^2 - 1 = 8t^4 - 8t^2 + 1$. Since $0 = 8t^4 - 8t^2 + t + 1 = (t+1)(2t-1)(4t^2 - 2t - 1)$ and $t \neq -1, \frac{1}{2}$, we have that $\cos 36^\circ = \frac{1}{4}(1+\sqrt{5})$ from which $\cot 36^\circ = \frac{1}{5}\sqrt{25 + 10\sqrt{5}}$. Let 1 be the side length of the pentagon. Then the diagonal length is $2\cos 36^\circ = \frac{1}{2}(1+\sqrt{5})$ and the length of FG is $4\cos 36^\circ - 1 = \sqrt{5}$.

The distance from the circumcentre to each side of the pentagon is $\frac{1}{2}\tan 54^\circ$, so

$$[ABCDE] = \frac{5}{4}\tan 54^\circ = \frac{5}{4}\cot 36^\circ = \frac{\sqrt{25 + 10\sqrt{5}}}{4}.$$

The distance from each vertex of the pentagon to its opposite edge is $\frac{1}{2}\tan 72^\circ$, so

$$\begin{aligned} [FGD] = \frac{\sqrt{5}}{4}\tan 72^\circ &= \frac{\sqrt{5}}{2}\left(\frac{\cot 36^\circ}{\cot^2 36^\circ - 1}\right) \\ &= \frac{5\sqrt{5}}{2}\left(\frac{\sqrt{25+10\sqrt{5}}}{10\sqrt{5}}\right) \\ &= \frac{\sqrt{25+10\sqrt{5}}}{4} = [ABCDE]. \end{aligned}$$

3899. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a + b + c = 3$. Prove that

$$\left(\frac{a^3+1}{a^2+1}\right)^2 + \left(\frac{b^3+1}{b^2+1}\right)^2 + \left(\frac{c^3+1}{c^2+1}\right)^2 \geq ab + bc + ca.$$

When does equality hold?

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We received 20 correct submissions. There were a variety of different solutions to this problem; we will feature two solutions.

Solution 1, by Salem Malikić.

Editor's comment. This approach, utilized by most of the solvers, consists of first finding a lower bound for the square expression on the left hand side and then showing that the right hand side is a lower bound for the resulting function.

By the Cauchy-Schwarz inequality we have that for all positive reals x:

$$(x^3+1)(x+1) \geq (x^{\frac{3}{2}} \cdot x^{\frac{1}{2}} + 1)^2 = (x^2+1)^2.$$

Using this inequality we obtain

$$\left(\frac{x^3+1}{x^2+1}\right)^2 = \frac{(x^3+1)(x+1)(x^2-x+1)}{(x^2+1)^2} \geq \frac{(x^2+1)^2(x^2-x+1)}{(x^2+1)^2} = x^2-x+1.$$

Then

$$\begin{aligned} &\left(\frac{a^3+1}{a^2+1}\right)^2 + \left(\frac{b^3+1}{b^2+1}\right)^2 + \left(\frac{c^3+1}{c^2+1}\right)^2 \\ &\qquad \geq (a^2+b^2+c^2) - (a+b+c) + 3 \\ &\qquad = a^2+b^2+c^2 \\ &\qquad = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + (ab+bc+ca) \\ &\qquad \geq ab+bc+ca. \end{aligned}$$

From the last inequality, it is obvious that equality is only achieved by $a = b = c = 1$.

Solution 2, by Oliver Geupel.

Editor's comment. This is a more involved but also more general solution. The problem is solved with the method of Lagrange multipliers which was recently presented in this journal, see [2013 : 24].

Let us start with the following preliminary observation: The function

$$h : [0,3] \to \mathbb{R} : x \mapsto \frac{2x(x^3+1)(x^3+3x-2)}{(x^2+1)^3} + x - 3$$

is convex on $[0,1)$ and increasing on $[1,3]$ – as can be seen fairly easily from the first and second derivatives – and satisfies $h(0) < h(1) < h(3)$, implying $h(x) < h(1)$ for all $x \in [0,1]$. Hence, for real numbers $a, b \in [0,3]$ with the property $a+b=3$, the condition $h(a) = h(b)$ can only be satisfied if $a = b = 3/2$. Furthermore, for real numbers $a, b, c \in [0,3]$ with the property $a+b+c=3$, the condition $h(a) = h(b) = h(c)$ can only be satisfied if $a = b = c = 1$.

Now let us turn to the proposed problem. Denote by m the minimum value of the continuous function

$$f(x,y,z) = \left(\frac{x^3+1}{x^2+1}\right)^2 + \left(\frac{y^3+1}{y^2+1}\right)^2 + \left(\frac{z^3+1}{z^2+1}\right)^2 - (xy+yz+zx)$$

on the compact region

$$K = \{(x,y,z) \in \mathbb{R}^3 \ : \ x,y,z \geq 0 \text{ and } x+y+z=3\}.$$

We prove that $m=0$ and that $f(a,b,c)=0$ is equivalent to $a=b=c=1$.

Assume that $f(a,b,c)=m$. We have $m \leq f(1,1,1)=0$.

First we show that (a,b,c) is an interior point of K. Suppose (a,b,c) is on the boundary of K, say $c=0$. Since $f(3,0,0)=f(0,3,0)>0$, the point (a,b,c) cannot be a vertex of the triangular region K. Hence, (a,b) is an interior point of the region

$$\{(x,y) \in \mathbb{R}^2 \ : \ x,y \geq 0 \text{ and } x+y=3\}.$$

Then, there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$h(a) = \partial_1 f(a,b,0) = \lambda = \partial_2 f(a,b,0) = h(b).$$

By our preliminary observation, we obtain $a=b=3/2$, which is impossible because $f(3/2,3/2,0)>0$. We have proved that (a,b,c) is an interior point of K.

Now, there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$h(a) = \partial_1 f(a,b,c) = \lambda, \quad h(b) = \partial_2 f(a,b,c) = \lambda, \quad h(c) = \partial_3 f(a,b,c) = \lambda.$$

By the preliminary observation, we obtain $a=b=c=1$.

As a consequence, the proposed inequality follows. The equality holds if and only if $a=b=c=1$.

3900. *Proposed by Abdilkadir Altintaş and Halit Çelik.*

In a triangle ABC, $AB=AC$, $m(BAC)=20°$, D is the point on AC such that $m(DBC)=25°$ and E is the point on AB such that $m(BCE)=65°$. Find the measure of the angle CED.

We received 14 correct solutions, and 12 incorrect or incomplete solutions, most of which used calculators and/or rounded values. We present two solutions.

Solution 1, by Dag Jonsson, slightly modified by the editor.

Note that $\measuredangle ABC = \measuredangle ACB = 80°$ ($\triangle ABC$ is isosceles), which allows us to calculate $\measuredangle DCE = 15°$, $\measuredangle DBE = 55°$, $\measuredangle BEC = 35°$ and $\measuredangle BDC = 75°$.

We draw the normal EF to the side AC. Let $\alpha = \measuredangle EDF$, then

$$\frac{AF}{FD} = \frac{EF}{FD} \div \frac{EF}{AF} = \frac{\tan\alpha}{\tan 20°}. \tag{1}$$

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The Sine Law applied to $\triangle ABD$ gives

$$\frac{AD}{AB} = \frac{\sin 55^\circ}{\sin(180^\circ - 75^\circ)} = \frac{\sin 55^\circ}{\sin 75^\circ}, \tag{2}$$

and similarly from $\triangle AEC$ we find that

$$\frac{AE}{AC} = \frac{\sin 15^\circ}{\sin 35^\circ}. \tag{3}$$

Since $AB = AC$, we get from (2) and (3) that

$$\frac{AE}{AD} = \frac{\sin 15^\circ \cdot \sin 75^\circ}{\sin 35^\circ \cdot \sin 55^\circ}$$

By construction, $\triangle AFE$ is a right triangle, and so $AF = AE \cdot \sin 70^\circ$. Hence,

$$\begin{aligned}\frac{AF}{AD} = \frac{AE \cdot \sin 70^\circ}{AD} &= \frac{\sin 15^\circ \cdot \sin 75^\circ \cdot \sin 70^\circ}{\sin 35^\circ \cdot \sin 55^\circ} \\ &= \frac{\sin 15^\circ \cdot \cos 15^\circ \cdot \sin 70^\circ}{\sin 35^\circ \cdot \cos 35^\circ} \\ &= \frac{\frac{1}{2}\sin 30^\circ \cdot \sin 70^\circ}{\frac{1}{2}\sin 70^\circ} = \frac{1}{2}.\end{aligned}$$

But this means that $AF = FD$, and by (1) that $\tan\alpha = \tan 20^\circ$, i.e. $\alpha = 20^\circ$. Thus, $\measuredangle CED = \alpha - \measuredangle ECD = 5^\circ$.

Solution 2, by C.R. Pranesachar, modified by the editor.

We shall show that $\measuredangle CED = 5^\circ$. Denote by O the intersection of BD and CE. Since $\measuredangle BOC = 180^\circ - (\measuredangle OBC + \measuredangle OCB) = 180^\circ - (25^\circ + 65^\circ) = 90^\circ$, BD and CE intersect at right angles. We also calculate that $\measuredangle ABC = \measuredangle ACB = 80^\circ$, and hence $\measuredangle EBD = 55^\circ$, $\measuredangle ECD = 15^\circ$.

Denote by a the length of BC. From the right angle triangles around O, we get $BO = a\cos 25^\circ$; $EO = BO\tan 55^\circ = a\cos 25^\circ \tan 55^\circ$; $CO = a\sin 25^\circ$; $DO = CO\tan 15^\circ = a\sin 25^\circ \tan 15^\circ$.
Let $\theta = \measuredangle CED$. From $\triangle DOE$ we have

$$\tan\theta = \frac{DO}{OE} = \frac{a\sin 25^\circ \tan 15^\circ}{a\cos 25^\circ \tan 55^\circ} = \frac{\tan 25^\circ \tan 15^\circ}{\tan 55^\circ}.$$

For readability, denote $\tan 5^\circ$ by t. Then, using the difference of angles formula for tangent and simplifying, we get

$$\tan\theta = \tan 15^\circ \cdot \frac{\tan(30^\circ - 5^\circ)}{\tan(60^\circ - 5^\circ)} = \tan 15^\circ \cdot \frac{1 - \sqrt{3}t}{\sqrt{3} + t} \cdot \frac{1 + \sqrt{3}t}{\sqrt{3} - t} = \tan 15^\circ \cdot t \cdot \frac{1 - 3t^2}{3t - t^3}.$$

By the triple angle formula for tangent, $\frac{1-3t^2}{3t-t^3} = \frac{1}{\tan(3\cdot 5^\circ)}$, so it follows from the above calculation that $\tan\theta = t = \tan 5^\circ$. But $\theta \in [0, 180^\circ]$, so it follows that $\theta = 5^\circ$, as claimed.

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Solvers - collaborations

YEAR-END FINALE

With this issue, I finish my first whole volume as Editor-in-Chief. It has been an eventful year for me at ***Crux*** (not to mention outside of it as I have successfully defended my PhD and moved cities). I have seen several changes to the Editorial Board and I have enjoyed acquainting myself with the ***Crux*** audience. Together with the CMS head office, we have moved the entire production of ***Crux*** online and I have CMS office staff to thank for their help and support. I cannot thank all my editors enough for their patience in many different ways: from getting used to the new online systems to their careful proofreading of my drafts. I thank you for your open-mindedness and advice and, most of all, your willingness to always contribute an opinion and share your expertise.

Editorial Board changes are bittersweet: I dislike saying goodbye to editors I have gotten to know and enjoyed working with, but I also look forward to working with the new people. In this volume, we are saying goodbye to Nicolae Strungaru, Olympiad Corner Editor, and welcome Carmen Bruni in that position. I also welcome Dennis Epple and Magda Georgescu in the positions of Problems Editors. Joseph Horan, Amanda Malloch, Robin Koytcheff, Alejandro Erickson and Mallory Flynn – my Guest Editors – have been invaluable to me during this fast-paced production year.

Crux readers have been an inspiration to me this entire year. Your interest in and contributions to the publication make my job a meaningful affair. I am always glad to receive your letters, hear your opinions and I'm looking forward to spending more time with you!

Finally, I would like to thank my friends and family who have listened to me talk about ***Crux*** a lot and listened close enough to provide advice and help out where needed.

We have already started working on issue 1 and I am excited to have a 2015 issue come out in 2015! Here is to another great year to come!

Kseniya Garaschuk

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