Crux Mathematicorum

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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Dear readers,

Welcome to issue 8 of Volume 39. For those of you receiving hard copies of ***Crux***, this is your first ***Crux*** bundle after a long summer break and I hope it is a welcomed addition to your avocation of problem solving.

I would like to note the change of address for submission of problem proposals and numbered problem solutions by e-mail; from now on, please use the new email `crux-psol@cms.math.ca`. If you would like to contact me (Editor-in-Chief) directly, you can do so by writing to `crux-editors@cms.math.ca`. In particular, if your affiliation or location has changed, please let me know, so it can appear correctly in the following issues.

I would also like to take this opportunity to encourage everyone to submit problem proposals as they present me with a clear picture of what you, the readers, find interesting, which in turn allows me to reflect on the materials to be published in ***Crux***.

On a slightly different note, in this issue, we have a reason to celebrate! I would like to congratulate Shawn Godin on receiving the 2014 CMS Graham Wright Award for Distinguished Service (please see the official announcement on the next page). His contributions to the Canadian mathematical community as a whole and ***Crux*** in particular have been invaluable.

Keep in touch with ***Crux*** by submitting problems, solutions, articles and by liking us on Facebook at `https://www.facebook.com/CruxMathematicorum`.

Kseniya Garaschuk

AWARD ANNOUNCEMENT

Shawn Godin
(Cairine Wilson Secondary School, Ottawa)

2014 Graham Wright Award for Distinguished Service

The Canadian Mathematical Society is pleased to announce **Shawn Godin** (Cairine Wilson Secondary School, Ottawa, Ontario) as the recipient of the 2014 Graham Wright Award for Distinguished Service. The award recognizes individuals who have made sustained and significant contributions to the Canadian mathematical community and, in particular, to the Canadian Mathematical Society.

CANADIAN MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DU CANADA

Prix Graham Wright pour service méritoire 2014

La Société mathématique du Canada est heureuse d'annoncer qu'elle a décerné son prix Graham-Wright pour service méritoire 2014 à **Shawn Godin** (Cairine Wilson Secondary School, Ottawa, Ontario). Le prix récompense les personnes qui contribuent de façon importante et soutenue à la communauté mathématique canadienne et, notamment, à la Société mathmatique du Canada.

THE CONTEST CORNER

No. 18

Shawn Godin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-contest@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC86. Un hexagone H est inscrit dans un cercle. L'hexagone a trois côtés de longueur 1 et trois côtés de longueur 3. Chaque côté de longueur 1 est situé entre deux côtés de longueur 3 et chaque côté de longueur 3 est situé entre deux côtés de longueur 1. Déterminer l'aire de H.

CC87. L'index d'abondance $I(n)$ d'un entier strictement positif n est le nombre $I(n) = \frac{\sigma(n)}{n}$, $\sigma(n)$ étant égal à la somme de tous les diviseurs positifs de n, y compris 1 et n. Par exemple,

$$I(12) = \frac{1+2+3+4+6+12}{12} = \frac{7}{3}.$$

Déterminer le plus petit entier impair positif n pour lequel $I(n) > 2$. Justifier les étapes de son raisonnement.

CC88. Un chat est situé au point C, qui est à 60 mètres directement à l'ouest d'une souris située au point S. La souris tente de s'échapper à une vitesse de 7 m/s dans une direction fixe. Le chat, qui est expert en géométrie, court à une vitesse de 13 m/s le long d'une droite qui lui permettra d'intercepter la souris le plus vite possible. Supposons que la souris est interceptée après avoir parcouru une distance de d_1 mètres dans une direction particulière. Si la souris avait été interceptée après avoir parcouru une distance de d_2 mètres dans la direction opposée, démontrer que $d_1 + d_2 \geq 14\sqrt{30}$.

CC89. Soit $f : \mathbb{Z} \to \mathbb{Z}^+$ une fonction. On définit $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ comme suit : $h(x, y) = \mathrm{PGCD}(f(x), f(y))$. Sachant que $h(x, y)$ est un polynôme à deux variables, x et y, démontrer qu'il doit être constant.

CC90. Étant donné une valeur de k, $k > 0$, $n \geq 2k > 0$, on considère le carré R dans le plan formé de tous les points (x, y), $0 \leq x, y \leq n$. On colorie chaque point de R en gris si $\frac{xy}{k} \leq x + y$, et en bleu autrement. Déterminer l'aire de la région en gris en fonction de n et de k.

. . . .

CC86. A hexagon, H, is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Each side of length 1 is between two sides of length 3 and, similarly, each side of length 3 is between two sides of length 1. Find the area of H.

CC87. The abundancy index $I(n)$ of a positive integer n is $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all of the positive divisors of n, including 1 and n itself. For example,

$$I(12) = \frac{1+2+3+4+6+12}{12} = \frac{7}{3}.$$

Determine, with justification, the smallest odd positive integer n so that $I(n) > 2$.

CC88. A cat is located at C, 60 metres directly west of a mouse located at M. The mouse is trying to escape by running at 7 m/s in a fixed direction. The cat, an expert in geometry, runs at 13 m/s in a suitable straight line path that will intercept the mouse as quickly as possible. Suppose that the mouse is intercepted after running a distance of d_1 metres in a particular direction. If the mouse had been intercepted after it had run a distance of d_2 metres in the opposite direction, show that $d_1 + d_2 \geq 14\sqrt{30}$.

CC89. Let $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function, and define $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ by $h(x, y) = \gcd(f(x), f(y))$. If $h(x, y)$ is a two-variable polynomial in x and y, prove that it must be constant.

CC90. For a given $k > 0$, $n \geq 2k > 0$, consider the square R in the plane consisting of all points (x, y) with $0 \leq x, y \leq n$. Color each point in R gray if $\frac{xy}{k} \leq x + y$, and blue otherwise. Find the area of the gray region in terms of n and k.

CONTEST CORNER SOLUTIONS

CC36. For each positive integer n, define $f(n)$ to be the smallest positive integer s for which $1 + 2 + 3 + \cdots + (s - 1) + s$ is divisible by n. For example, $f(5) = 4$, because $1 + 2 + 3 + 4$ is divisible by 5 and none of 1, $1 + 2$, or $1 + 2 + 3$ is divisible by 5. Determine, with proof, the smallest positive integer k for which the equation $f(c) = f(c + k)$ has an odd positive integer solution for c.

Originally Question B4 c) on 2009 Canadian Open Mathematics Challenge.

One incorrect solution was received.

CC37. $ABCD$ is a cyclic quadrilateral, with side $AD = d$, where d is the diameter of the circle. $AB = a, BC = a$ and $CD = b$. If a, b and d are integers $a \neq b$,

a) prove that d cannot be a prime number.

b) determine the *minimum* value of d.

Originally Question 10 on 1999 Euclid contest.

Solved by Šefket Arslanagić and we present his solution.

Join A to C and since $\angle ACD$ is in a semicircle, we have $\angle ACD = 90°$. Let $\angle ABC = \alpha$, so that $\angle CDA = 180° - \alpha$ because $ABCD$ is a cyclic quadrilateral. From $\triangle ABC$, $AC^2 = a^2 + a^2 - 2a^2 \cos\alpha$. Similarly, from $\triangle ACD$, $AC^2 = d^2 - b^2$ and $\cos(180° - \alpha) = \frac{b}{d}$, i.e. $\cos\alpha = \frac{-b}{d}$. By substitution,

$$d^2 - b^2 = 2a^2 - 2a^2 \left(\frac{-b}{d}\right).$$

Simplifying yields $2a^2 = d(d - b),\ d \neq b$.

a) To prove d cannot be prime, we consider a contradiction and suppose d is prime. We have two cases from which we deduce d is composite: $d = 2$ or $d \geq 3$.

If $d = 2$ then $2a^2 = 2(2 - b)$ and hence $b + a^2 = 2$. Since a, b are positive integers, this implies $a = b = 1$ which is not possible since $a \neq b$.

Suppose $d \geq 3$. Since $d > 2$, $d|a^2$. Since d is prime, this implies $d|a$, which is impossible because d is the diameter of the circle and is larger than a.

b) We know $d \geq 2$ and can not be prime by part a), so let us check cases.

If $d = 4$, we have $a^2 = 2(4 - b)$. If $b = 1$ or 3, then $a^2 = 6$ or 2 which implies a is not an integer. If $b = 2$ then $a = 2$ but $a \neq b$ so this is impossible. Hence $d = 4$ is impossible.

If $d = 6$, we have $a^2 = 3(6 - b)$, so $b \in \{1, 2, 3, 4, 5\}$. If $b \in \{1, 2, 4, 5\}$ then a is not an integer. If $b = 3$ then $a = 3$, but as before, $a \neq b$. Thus $d = 6$ is also impossible.

If $d = 8$ then $a^2 = 4(8 - b)$. If we set $b = 7$, we have $a = 2$ and this is an acceptable solution, so the minimum possible value of d is 8.

CC38. Each vertex of a regular 11-gon is coloured black or gold. All possible triangles are formed using these vertices. Prove that there are either two congruent triangles with three black vertices or two congruent triangles with three gold vertices.

Originally Question 5 on 2011 Sun Life Financial Repêchage Competition.

Solved by G. Geupel; and T. Zvornaru and N. Stanciu. We present the solution of Gesine Geupel.

By the Pigeonhole Principle, we can find 6 vertices of the same colour. There are $\binom{6}{2} = 15$ lines joining pairs of these points. A regular 11-gon has 11 axes of symmetry, and each of these lines is parallel to one axis of symmetry, so by the Pigeonhole Principle some two lines are parallel. We get two congruent triangles by considering the two vertices of one line with each vertex on the other line.

CC39. Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \leq x, y < n$, he assigns an integer between 1 and n, inclusive, for some positive integer n. These integers satisfy the property that in any row where $y = k$, the $n - 1$ values are distinct and are never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability proportional to the integer value he assigned to it. Compute the expected value of $x + y$ for the chosen point (x, y).

Originally Question 9 on 2013 Stanford Math Tournament, Team Problems.

No solutions were received.

CC40. Define $P(1) = P(2) = 1$ and $P(n) = P(P(n - 1)) + P(n - P(n - 1))$ for $n \geq 3$. Prove that $P(2n) \leq 2P(n)$ for all positive integers n.

Originally Question 6 on 2007 University of Waterloo Big E Contest.

One incorrect solution was received.

THE OLYMPIAD CORNER

No. 316

Nicolae Strungaru

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-olympiad@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

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La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC146. Soit ABC un triangle isocèle tel que $AB = BC$. Plaçons les points D sur le côté AC et E sur le côté BC. Soit F le point dintersection des bissectrices des angles DRB et ADE. Si F se situe sur le côté AB, démontrer quil est le mi-point de AB.

OC147. Soit a_1 un nombre naturel. La suite $\{a_n\}_n$ est définie selon la règle :

$$a_{n+1} = a_n + 2d(n)\,,$$

où $d(n)$ dénote le nombre de différents diviseurs de n (incluant 1 et n). Existe-t-il a_1, tel que deux nombres consécutifs de la suite soient des carrés de nombres naturels ?

OC148. Des nombres complexes x_i, y_i satisfont $|x_i| = |y_i| = 1$ pour tout

$1 \leq i \leq n$. Soit $x = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$; $y = \frac{1}{n} \sum\limits_{i=1}^{n} y_i$ et $z_i = xy_i + yx_i - x_i y_i$. Démontrer que

$$\sum_{i=1}^{n} |z_i| \leq n.$$

OC149. Soit n un entier fixe. Déterminer toutes les fonctions $f \colon \mathbb{Z} \to \mathbb{Z}$, telles que pour tous les entiers x, y on ait

$$f\left(x + y + f(y)\right) = f(x) + ny.$$

OC150. Soit ABC un triangle isocèle tel que $AB = AD$ et soit D le pied de la perpendiculaire de A vers BC. Soit P un point intérieur au triangle ADC tel que $\angle APB > 90°$ et $\angle PBD + \angle PAD = \angle PCB$. Les segments CP et AD intersectent au point Q, et les segments BP et AD intersectent en R. Soit T un point sur $[AB]$ et soit S un point sur AP, n'appartenant pas à $[AP]$, tel que $\angle TRB = \angle DQC$ et $\angle PSR = 2\angle PAR$. Démontrer que $RS = RT$.

. . . .

OC146. Let ABC be an isosceles triangle with $AC = BC$. Take points D on side AC and E on side BC. Let F be the intersection of bisectors of angles DEB and ADE. If F lies on side AB, prove that F is the midpoint of AB.

OC147. Let a_1 be a natural number. The sequence $\{a_n\}_n$, is defined by the rule:

$$a_{n+1} = a_n + 2d(n)\,,$$

where $d(n)$ denotes the number of the different divisors of n (including 1 and n). Does there exist an a_1 such that two consecutive members of the sequence are squares of natural numbers?

OC148. Complex numbers x_i, y_i satisfy $|x_i| = |y_i| = 1$ for all $1 \leq i \leq n$. Let $x = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$; $y = \frac{1}{n} \sum\limits_{i=1}^{n} y_i$ and $z_i = xy_i + yx_i - x_i y_i$. Prove that

$$\sum_{i=1}^{n} |z_i| \leq n.$$

OC149. For a given positive integer n, find all functions $f \colon \mathbb{Z} \to \mathbb{Z}$, such that for all integers x, y we have

$$f\left(x + y + f(y)\right) = f(x) + ny.$$

OC150. Let ABC be a isosceles triangle with $AB = AC$ and let D be the foot of the perpendicular from A to BC. Let P be an interior point of triangle ADC such that $\angle APB > 90^\circ$ and $\angle PBD + \angle PAD = \angle PCB$. Segments CP and AD intersect at Q, and segments BP and AD intersect at R. Let T be a point on $[AB]$ and let S be a point on $[AP]$, not belonging to $[AP]$, such that $\angle TRB = \angle DQC$ and $\angle PSR = 2\angle PAR$. Prove that $RS = RT$.

OLYMPIAD SOLUTIONS

OC86. There were finitely many persons at a party among whom some were friends. Among any 4 of them there were either 3 who were all friends among each other or 3 who weren't friends with each other. Prove that you can separate all the people at the party in two groups in such a way that in the first group everyone is friends with each other and that all the people in the second group are not friends to anyone else in second group. (Friendship is a mutual relation).
Originally question 2 from the 2011 Croatia team selection test, day 1.

We present the solution by Oliver Geupel.

Consider the graph $G = (V, E)$ where the set V of vertices is the set of people at the party and where the set E of edges consists of exactly the pairs $\{u, v\}$ of people u and v who were friends. Since G is finite, it has a complete subgraph $K = (V_K, E_K)$ with a maximum number of vertices $|V_K|$. If $|V_K| = 1$, then G has no edges and we are done. We assume $|V_K| > 1$. It is enough to show that the subgraph $H = (V \setminus V_K,\ E_H)$ of G has no edge. The proof is by contradiction. So suppose that $\{u, v\} \in E_H$. By the maximum property of the subgraph K, there is a vertex $w \in V_K$ such that $\{u, w\} \notin E$.

Consider any $x \in V_K \setminus \{w\}$. If $\{x, v\} \notin E$, then the subgraph of G with the node set $\{u, v, w, x\}$ contains neither a triangle nor a missing triangle, which is impossible by hypothesis on G. Hence, $\{v, x\} \in E$.

We have obtained that, for each $x \in V_K \setminus \{w\}$, it holds $\{v, x\} \in E$. Hence, $\{v, w\} \notin E$ by the maximum property of the subgraph K.

Consider any $x \in V_K \setminus \{w\}$. We have $\{u, v\},\ \{v, x\},\ \{w, x\} \in E$ and $\{u, w\},\ \{v, w\} \notin E$. If $\{u, x\} \notin E$, then the subgraph of G with the node set $\{u, v, w, x\}$ contains no triangle and no missing triangle, which is impossible by hypothesis on G. Thus, $\{u, x\} \in E$. Then, the subgraph of G with the node set $(V_K \setminus \{w\}) \cup \{u, v\}$ is a complete graph. But this contradicts the maximum property of the subgraph K. The proof is complete.

OC87. Call a natural number n faithful if there exist natural numbers $a < b < c$ such that $a \mid b$, $b \mid c$ and $n = a + b + c$.

(i) Show that all but a finite number of natural numbers are faithful.

(ii) Find the sum of all natural numbers which are not faithful.

Originally question 2 from the 2011 India National Olympiad.

Solved by C. Curtis; L. Giugiuc; B. Jin and E. T. H. Wang; D. E. Manes; N. Midttun; and K. Zelator. We give the solution of Billy Jin and Edward T. H. Wang.

We show all natural numbers are faithful except for 1, 2, 3, 4, 5, 6, 8, 12, and 24, the sum of which is 65. We consider several cases.

When $n \geq 7$ is odd, then $n = 1 + 2 + (n - 3)$. Since $n - 3$ is even and at least 4, n is faithful with $a = 1, b = 2, c = n - 3$.

When n is even, we consider the cases $n \equiv 2, 4, 0 \pmod 6$ separately.

When $n = 6k + 2$ for $k \geq 3$, then $n = 2 + 6 + 6(k - 1)$ and n is faithful.

When $n = 6k + 4$ for some $k \geq 1$, then $n = 1 + 3 + 3(2k)$ and n is faithful.

When $n = 6k$, we consider cases based on whether k is a power of 2 or not.

When k is not a power of 2 we can write $n = 6 \cdot (2q + 1) \cdot 2^s$, for some $q \geq 1, s \geq 0$. We can write $n = (6q + 3)2^{s+1} = 2^{s+1} + 2^{s+2} + 3q2^{s+2}$, so n is faithful.

When k is a power of 2 we can write $n = 6 \cdot 2^s$. When $s \geq 3$ we have $n = 3 \cdot 2^{s-3} + 9 \cdot 2^{s-3} + 36 \cdot 2^{s-3}$, so n is faithful.

This leaves $n = 1, 3, 5, 2, 8, 4, 6, 12, 24$ as numbers which have not been shown to be faithful. It is straightforward to verify computationally that these numbers are not faithful.

OC88. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(f(x) - f(y)) = f(f(x)) - 2x^2 f(y) + f(y^2),$$

for all $x, y \in \mathbb{R}$.

Originally question 4 from the 2011 Japan National Olympiad.

No solution to this problem was received.

OC89. Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. Let F and G be the midpoints of AB and CD, and let ℓ be the line through G parallel to AB. Let H, K be the feet of the perpendiculars from E onto the lines ℓ and CD. Prove that $EF \perp HK$.

Originally question 2 from the 2011 Romania team selection test, day 1.

Solved by Š. Arslanagić; M. Bataille; O. Geupel; M. Stoënescu; and T. Zvonaru. We give the solution of Titu Zvonaru.

We prove a generalization: replace the condition that F and G are midpoints by

$$F \text{ on } AB \text{ and } G \text{ on } CD \text{ satisfy } \frac{AF}{FB} = \frac{DG}{GC}.$$

In fact, we shall see that if directed distances are used then F can be any point of the line AB and G the corresponding point on DC.

Let $T = EH \cap AB$; then $ET \perp AB$ (since ℓ is parallel to AB and perpendicular to EH). Because $ABCD$ is cyclic, the triangles DEC and AEB are oppositely similar. Because F and G are corresponding points in the similar triangles DEC and AEB as are K and T, we have $\angle EGK = \angle EFT$ or, using directed angles,

$$\angle KGE = \angle EFT. \tag{1}$$

We now set $S = EF \cap HK$ and want to prove that these lines are perpendicular at S. Because of the right angles at H and K, the points E, G, H, K lie on the circle whose diameter is EG, whence

$$\angle KHE = \angle KGE$$

(as directed angles). But $\angle SHE = \angle KHE$ (because $S \in KH$) and $\angle KGE = \angle EFT$ (from (1)), so it follows that $\angle SHE = \angle EFT$. Also, the vertically opposite angles at E are equal so that the triangles SHE and TFE are similar. But $\angle FTE = 90^\circ$, hence $\angle HSE = 90^\circ$; that is, $EF \perp HK$.

OC90. Let n be a positive integer. If one root of the quadratic equation $x^2 - ax + 2n = 0$ is equal to

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}},$$

prove that $2\sqrt{2n} \leq a \leq 3\sqrt{n}$.
Originally question 6 from the 2011 Kazakhstan National Olympiad, Grade 9.

Solved by G. Apostolopoulos; Š. Arslanagić; M. Bataille; C. Curtis; M. Dincă; O. Geupel; L. Giugiuc; B. Jin and E. T. H. Wang; N. Midttun; P. Perfetti; V. Pambuccian; G. Scărlătescu; D. Văcaru; and T. Zvonaru. We give a solution similar to those provided by most of the solvers.

Let

$$s := \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}.$$

Since

$$s^2 - as + 2n = 0$$

we have

$$a = s + \frac{2n}{s}.$$

Therefore, by the AM-GM inequality we get $a \geq 2\sqrt{2n}$.

To prove the other inequality we show first that $\sqrt{n} \le s \le 2\sqrt{n}$. The left hand side inequality is immediate:

$$s = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \sqrt{n}\,.$$

We next prove the right hand side inequality by induction. When $n = 1$, it states that $1 \le 2$. Next, assume the statement is true for some $n = k \ge 1$, so

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} \le 2\sqrt{k}.$$

Thus

$$\begin{aligned} \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &\le 2\sqrt{k} + \frac{1}{\sqrt{k+1}} = 2\sqrt{k} + \frac{2}{2\sqrt{k+1}} \\ &\le 2\sqrt{k} + \frac{2}{\sqrt{k} + \sqrt{k+1}} \\ &= 2\sqrt{k} + 2\left(\sqrt{k+1} - \sqrt{k}\right) = 2\sqrt{k+1} \end{aligned}$$

which completes the induction.

Now, since $\sqrt{n} \le s \le 2\sqrt{n}$, we get

$$a = s + \frac{2n}{s} \le 2\sqrt{n} + \frac{n}{\sqrt{n}} = 3\sqrt{n}\,,$$

which completes the proof.

BOOK REVIEWS

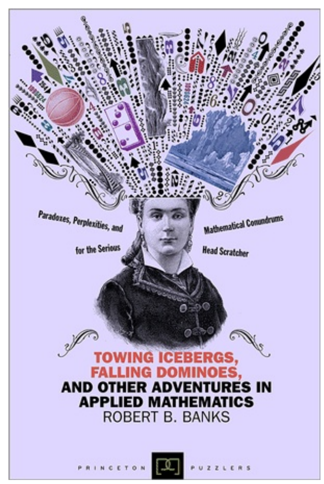
John McLoughlin

Towing Icebergs, Falling Dominoes and Other Adventures in Applied Mathematics
ISBN: 978-0-691-15818-1, paperbound 328 + xi pages, US$16.95 (2013).

Slicing Pizzas, Racing Turtles and Further Adventures in Applied Mathematics
ISBN: 978-0-691-15499-2, paperbound 286 + xi pages, US$18.95 (2012).

Both books are by Robert B. Banks, published by Princeton University Press, Princeton, New Jersey.

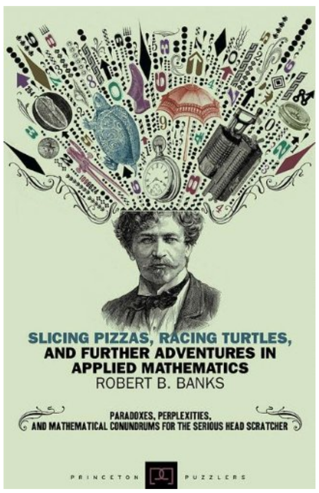
Reviewed by ***Ben Newling****, Department of Physics, University of New Brunswick, Fredericton, NB.*



I very much enjoyed reading Professor Banks' books, but at the end of the "Adventures" wasn't exactly certain with whom I should share the enjoyment.

These two titles have been reissued (chronologically backwards and interestingly differently priced) as part of the Princeton Puzzlers series, the strapline of which is "paradoxes, perplexities and mathematical conundrums for the serious head scratcher". This description is misleading at first blush and I expected some kind of puzzle book with solutions or, at least, maddening hints. What the two volumes represent instead is Professor Banks' personal collection of interesting worked examples in applied mathematics. Banks was a professor of engineering who had carefully assembled a very individual hoard of interesting problems. For the most part, each chapter or pair of chapters is a self-contained treatment of a single topic, such as the feasibility of towing the titular icebergs to the US from Antarctica as a supply of fresh water. Professor Banks gives plenty of background information about icebergs and carries out calculations on every aspect of the project, including the optimal route, the journey time, the power expended by the imagined fleet of towing vessels and the best ways to melt an iceberg in California.

My personal favourite chapter concerned the propagation of shock waves in traffic jams, but other topics include discussions of meteors, musings on the trajectories of golf balls and models of population dynamics with 23 or so disparate topics in each volume. Each chapter has a very specific title ("Big Things Falling From the Sky", "Hooks and Slices, Holes in One", "How Many People Have Ever Lived?", to name a few), but each manages to touch upon a range of situations in which similar math might apply and to incorporate some fascinating diversions, to boot. I was pleased to learn the population of Pitcairn and to be enthusiastically introduced to the Scottish mathematician, Peter Tait (1831-1901), for example. Canadian trivia included NRC research into wind turbines, the Petitcodiac river bore and even a Stephen Leacock reference. Throughout, Professor Banks' understated enthusiasm is evident. He clearly enjoyed collecting the calculations and delighted in sharing them. There is occasional very dry humour and enough breadth that any reader can find some topic that has personal relevance, whether for instance, football, jump ropes, gross domestic product, or staying dry in the rain.



So these are not puzzle books, but neither are they math education books. Professor Banks does not explain any of the mathematics or attempt to teach any mathematical techniques; he expects his readers to be familiar with exponential functions, hyperbolic sines, differential and integral calculus, for example. The mathematics is neither explained nor discussed for its own sake in the first volume, although there is one chapter about irrational numbers and another about number series in Slicing Pizzas. Banks does not show every step in every calculation by any manner of means, but instead highlights the key results in each application. I do not think that the reader will "understand more about mathematics", as Professor Banks hoped in the preface, at least not solely by reading these books. Active readers, however, will be implicitly challenged to investigate things for themselves and it is entertaining to work through Banks' calculations, question his assumptions and follow up with the wealth of references to others' studies of similar situations. The reader will certainly be impressed and entertained by the range of application.

As a teacher, I found the comprehensive bibliography an exciting resource and I was also intrigued to discover whole chapters that might be interesting to use in class or as starting points for further research. Professor Banks also suggests further calculations and projects centred around the theme of each chapter, which I always find a valuable resource when seeking inspiration.

I should mention some niggles. These books are certainly reprints rather than new editions. There are mistakes in the mathematics and in the text, which could easily have been corrected in a new edition. I might even take issue with some of the physics, such as the use of terminology surrounding momentum in the chapter about domino toppling. Some of the examples are naturally dated. Socioeconomic measures are "predicted" for the year 2000, for example and discussion of natural disasters are missing more recent examples. However, updating, comparing or extrapolating suggest natural projects for the interested reader. Most socioeconomic figures are from the US or compared to the US, but that again suggests the possibility of parallel calculations for other nations. The self-contained structure of each chapter leads to some repetition; the drag coefficient is introduced three times in almost identical fashion in the first volume. All these are minor annoyances, however. I did find myself swept along by the enthusiasm of the writing style and the eclecticism of the collection and I think, in the end, the two volumes might be enjoyably dipped into by any math, science or engineering educator and might be recommended in turn to interested students. The books could be read for pleasure or used as a teaching resource. Despite the title of the series, these are not books for someone seeking mathematical puzzles, but they are worth picking up, nevertheless, for someone interested in the application of mathematics.

Notice to Crux Readers

On behalf of CMS, I would like to take this opportunity to thank our Crux subscribers for your ongoing patience, understanding and loyalty to Crux as we continue to address the production delays. The new Editor-in-Chief, Kseniya Garaschuk and the Crux Editorial Board are currently working on the remaining 2013 issues (September to December 2013), assembling 2014 volume materials, and developing new content. If you have not yet renewed your subscription for 2014 or 2015, please visit our website at
`https://cms.math.ca/Publications/Journals/subscriptions.e`

The CMS apologizes for the ongoing production delays and can assure subscribers that everyone is working as best they can to resolve this challenging situation.

Johan Rudnick, Managing Editor

Avis aux lecteurs Crux

Au nom de la SMC, j'aimerais remercier nos abonnés du Crux de leur grande patience, de leur compréhension et de leur fidélité à la revue suite au retard de production que nous connaissons. La nouvelle rédactrice en chef, Kseniya Garaschuk, et le conseil de rédaction de la revue s'affairent en ce moment aux derniers numéros de 2013 (de septembre à décembre), à rassembler le matériel pour le volume 2014 et à créer du nouveau contenu. Si vous n'avez pas encore renouvelé votre abonnement pour 2014 ou 2015, vous pouvez le faire à l'adresse
`https://cms.math.ca/Publications/Journals/subscriptions.f`

La SMC s'excuse de ce retard et assure ses abonnés que chacun fait de son mieux pour résoudre cette situation difficile.

Johan Rudnick, Rédacteur-gérant

UNSOLVED CRUX PROBLEMS

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Chris Fisher published a list of unsolved problems from ***Crux*** *[2010 : 545, 547]. Below is a sample of two of these unsolved problems.*

339$\star$. [1978: 102, 292] *Proposed by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, New York.*

Is $\binom{37}{2}$ the only binomial coefficient $\binom{n}{r}$ whose decimal representation consists of a single digit repeated $k \geq 3$ times?

2049$\star$. [1995: 158; 1996: 183–184] *Proposed by Jan Ciach, Świętokrzyski, Poland.*

Let a tetrahedron $ABCD$ with centroid G be inscribed in a sphere of radius R. The lines AG, BG, CG, DG meet the sphere again at A_1, B_1, C_1, D_1, respectively. The edges of the tetrahedron are denoted a, b, c, d, e, f. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1} + \frac{1}{GB_1} + \frac{1}{GC_1} + \frac{1}{GD_1} \leq \frac{4\sqrt{6}}{9}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right).$$

Equality holds if $ABCD$ is regular.

This inequality, if true, would be a three-dimensional version of problem 5 of the 1991 Vietnamese Olympiad; see ***Crux*** [1994: 41].

PROBLEM SOLVER'S TOOLKIT

No. 7

J. Chris Fisher

The Problem Solver's Toolkit contains short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.

Harmonic Sets Part 4: More from the Harmonic Mean File

For this fourth and final instalment of the series, we continue our inspection of my file of results related to the harmonic mean. We come to an item that entered the file in 1973 [2]:

> *The side of a regular heptagon is half the harmonic mean of the smaller and larger diagonals.*

The left portion of Figure 1 shows the regular heptagon $ABCDEFG$. To verify that $AB = \frac{BD \cdot DA}{BD+DA}$ note that in ΔABD,

$$d = 2R\sin D, \qquad a = 2R\sin 2D,$$
$$b = 2R\sin 4D = 2R\sin 3D = 2R(3\sin D - 4\sin^3 D).$$

A deeper explanation can be found in [6], where the author shows that for $d = 1$, $ab = a + b$ is one of several identities relating the diagonals of the regular heptagon to a side, much as the golden section is related to the regular pentagon. On the right side of Figure 1 is one of the standard Euclidean constructions for the fourth proportional d of the lengths $a + b, a, b$; that is, $(a+b) : a \ = \ b : d$.

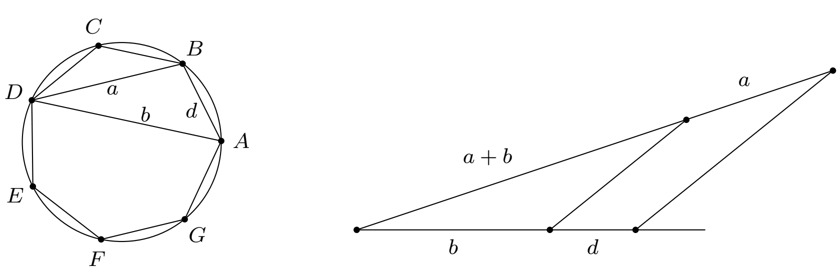


Figure 1: Two more instances of $d = \dfrac{ab}{a+b}$.

Howard Eves was another long-time collector of mathematical memorabilia. In [4] he described several familiar means while featuring ten instances of the harmonic mean that he had collected. The first two diagrams in Figure 2 come from that article; the third is Problem 1537 from *Crux* [1991: 182-183]. The six claims found in Figures 2 and 3 are to be considered exercises in the techniques discussed in

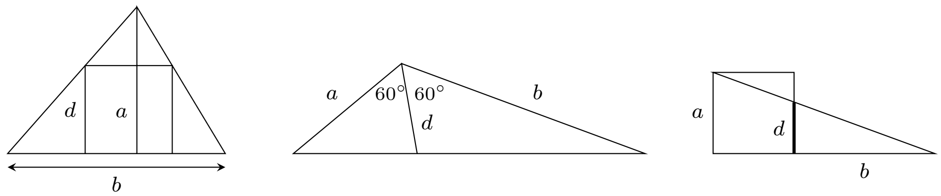


Figure 2: Three more ways to construct d to be half the harmonic mean of a and b, two from [4] and the third from [1].

the previous three parts of this series; each claim should be verified in at least two ways. For a greater challenge, the reader should explore how each diagram is related to the others by projection or perspectivity.

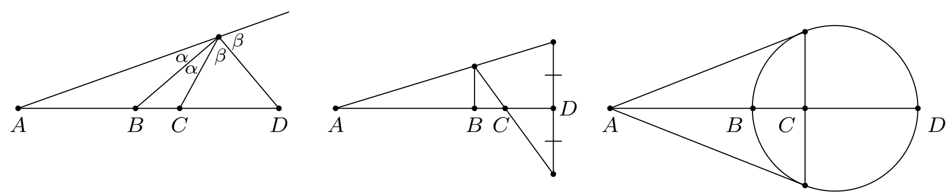


Figure 3: Three examples of harmonic sets: AC is the harmonic mean of AB and AD.

Figure 3 is also from Eves [4]; in each of the three diagrams, AC is the harmonic mean of AB and AD. As explained in part 2, this means that C is the harmonic conjugate of A with respect to B and D. Most of Eves's diagrams can also be found in his book [3, Chapter 1.6, problem 10, pages 36-37], some of which appear with further explanations.

The final item [5], reproduced in Figure 4, is a proof without words that compares several important means that are related to a circle whose radius is $\frac{a+b}{2}$. Note how cleverly the author, Sidney Kung, has labeled the points. Although no words are used, the reader is invited to use similar triangles and the Pythagorean theorem to verify the claims. The different means of a and b share several properties: Unless $a = b$ they lie strictly between a and b, they are homogeneous in the sense that $M(ka, kb) = kM(a, b)$, and they are symmetric in that $M(a, b) = M(b, a)$. It is thought that the name *contraharmonic* for CH comes from the property that it exceeds the arithmetic mean by the same amount that the arithmetic mean exceeds the harmonic mean; that is, as seen in Figure 4,

$$CH - AM = AH = AM - HM.$$

We also have that

$$AM = \frac{a+b}{2}, \quad GM = \sqrt{ab}, \quad HM = \frac{2ab}{a+b}, \quad CH = CM - HM = \frac{a^2+b^2}{a+b},$$

$$RC = \sqrt{CA \cdot CH} = \sqrt{\frac{a+b}{2}}\sqrt{\frac{a^2+b^2}{a+b}} = \sqrt{\frac{a^2+b^2}{2}}.$$

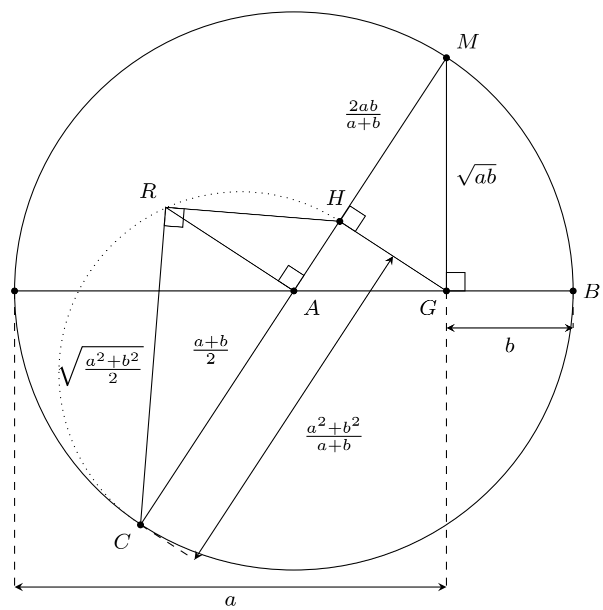


Figure 4: Harmonic Mean HM < Geometric Mean GM < Arithmetic Mean AM< Root Mean Square RC < Contraharmonic Mean CH.

The figure shows clearly that for $0 < b < a$ we have

$$b < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < a.$$

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A Quadrangle's Centroid of Vertices and van Aubel's Square Theorem

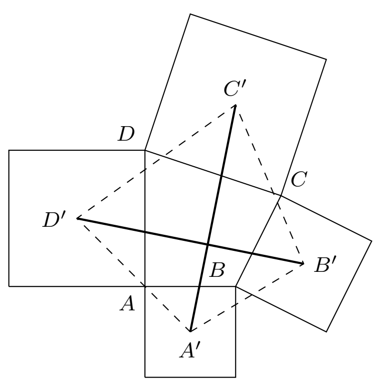
Rudolf Fritsch and Günter Pickert

This note is an English adaption of the third and last part of a more comprehensive paper written in German [4]. The authors thank Chris Fisher for his helpful comments.

We close our considerations on the various centroids of quadrangles ([5], [6]) by a discussion of van Aubel's Square Theorem and its relationship to the vertex centroid of a quadrangle. The original statement of van Aubel can be found on the internet [1] and concerns non-crossed polygons with an arbitrary number of vertices. We present it restricted to quadrangles, but in a slightly more general form.

Theorem. *On the sides of a plane quadrangle $ABCD$, let the erected squares with centers A', B', C', D' be such that the oriented angles $\angle A'AB$, $\angle B'BC$ $\angle C'CD$, $\angle D'DA$ are congruent. Then the diagonals of the quadrangle $A'B'C'D'$ have equal length and are perpendicular to each other.*

There are two possibilities for erecting the squares. The named angles are then all either $45°$ or all $-45°$.



There are many proofs of this theorem in literature and internet (for instance, [7] or [8]). The proofs that consider working in the Argand plane — with complex numbers — are especially clear. We present such a proof since it lays the foundation for what follows.

Given a plane quadrangle $ABCD$, the coordinates (complex numbers) of the vertices are denoted by a, b, c, d. Then

$$a' = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)i$$

is the coordinate of the center A' of the square erected on the side $[AB]$ such that $\angle A'AB = 45°$. For the centers of the squares erected on the further sides such that the corresponding oriented angles are congruent to the angle $\angle A'AB$ we obtain:

$$\begin{aligned}
b' &= \frac{1}{2}(b+c) + \frac{1}{2}(b-c)i,\\
c' &= \frac{1}{2}(c+d) + \frac{1}{2}(c-d)i,\\
d' &= \frac{1}{2}(d+a) + \frac{1}{2}(d-a)i.
\end{aligned}$$

If we choose the other possibility of erecting the squares, we replace i by $-i$:

$$\begin{aligned}
a' &= \frac{1}{2}(a+b) - \frac{1}{2}(a-b)i,\\
b' &= \frac{1}{2}(b+c) - \frac{1}{2}(b-c)i,\\
c' &= \frac{1}{2}(c+d) - \frac{1}{2}(c-d)i,\\
d' &= \frac{1}{2}(d+a) - \frac{1}{2}(d-a)i.
\end{aligned}$$

We consider only the first case and compute

$$\begin{aligned}
a' - c' &= \frac{1}{2}(a+b-c-d+(a-b-c+d)i),\\
b' - d' &= \frac{1}{2}(b+c-d-a+(b-c-d+a)i).
\end{aligned}$$

Evidently

$$(a'-c')i = b'-d',$$

which proves the claim.

The proof shows that we can relax most of the restrictions on the four vertices. Although the order of the four points A, B, C, D is important, there is no need for them to be distinct, and three of them are allowed to be collinear. Note, however, should $A = C$ and $B = D$ we would have $A' = C', B' = D'$; the diagonals would then have length 0 and it would be meaningless to ask for their perpendicularity. If exactly two consecutive points coincide, we have the triangle case of van Aubel's original paper. The reader is encouraged to draw different degenerate cases in order to confirm this generality. This proof also yields an interesting addendum.

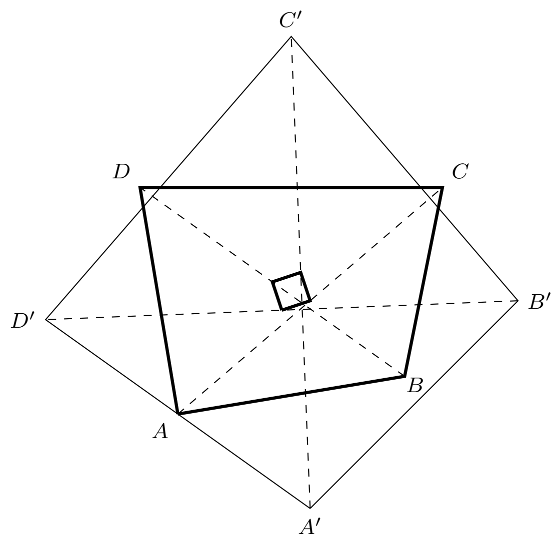
Addendum. With the notation of the preceding theorem, we have:

a) the quadrangles $ABCD$ and $A'B'C'D'$ have the same vertex centroid;

b) the midpoints of the diagonals of the quadrangles $ABCD$ and $A'B'C'D'$ form a square with the common vertex centroid as center;

c) this square collapses to a point if and only if the quadrangle $A'B'C'D'$ is a square;

d) case c) occurs if and only if the quadrangle $ABCD$ is a parallelogram.



To see this we take — as in our previous notes — the vertex centroid $S_{\mathcal{E}}$ as origin, having 0 as coordinate.

1. We have
$$a + b + c + d = 0. \tag{1}$$
which, from the above formulas, yields
$$a' + c' = (a + c)i, \;\; b' + d' = (b + d)i,$$
and, by addition,
$$a' + b' + c' + d' = 0.$$
Thus the point $S_{\mathcal{E}}$ is also the vertex centroid of the quadrangle $A'B'C'D'$.

2. The midpoints E, F, E', F' of the diagonals $[AC]$, $[CD]$, $[A'C']$, $[C'D']$ have the coordinates
$$e = \frac{1}{2}(a + c), \;\; f = \frac{1}{2}(b + d), \;\; e' = \frac{1}{2}(a' + c'), \;\; f' = \frac{1}{2}(b' + d').$$
Since $e + f = 0 = e' + f'$, the quadrangle $EE'FF'$ is centrally symmetric with the origin as center. From the equations under a), we obtain $e' = ei$, i.e. the semidiagonals of the quadrangle $EE'FF'$ have equal lengths and are perpendicular to each other; that means the quadrangle $EE'FF'$ is a square with the origin as center, which is the vertex centroid $S_{\mathcal{E}}$ of the quadrangle $ABCD$.

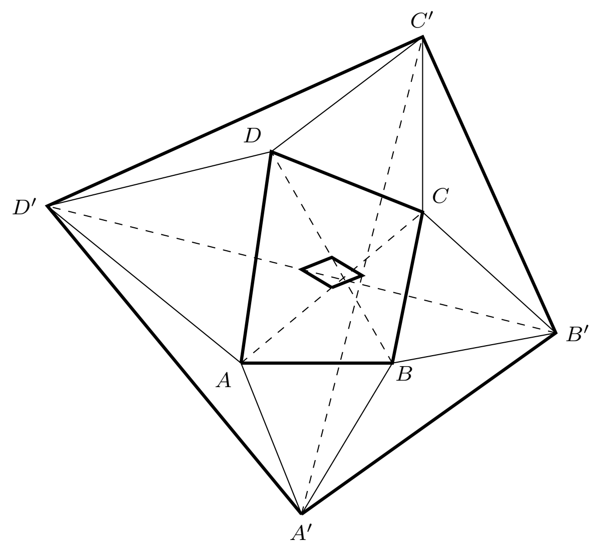
3. The Square Theorem implies further the quadrangle $A'B'C'D'$ is a square if and only if $E' = F'$ holds. In this case the quadrangle $EE'FF'$ collapses to a point.

4. Then also $E = F$ holds, i.e. the quadrangle $ABCD$ is a parallelogram.

Statement d) is is essentially the special case $n = 4$ of the Theorem of Napoleon-Barlotti [9]. The direction "$ABCD$ parallelogram $\Rightarrow$ $A'B'C'D'$ square" also appears in the literature as Thébault's Theorem [10].

Statement b) can be slightly generalized. To this end, we look at the Square Theorem from another point of view, considering the erection of isosceles right triangles instead of squares.

Theorem. *Directly similar triangles $A'AB$, $B'BC$, $C'CD$, and $D'DA$ are erected on the sides of a quadrangle $ABCD$ as in the figure below. Then:*



a) The quadrangles $ABCD$ and $A'B'C'D'$ share the vertex centroid.
b) The midpoints of the diagonals of the quadrangles $ABCD$ and $A'B'C'D'$ form a parallelogram whose center is the common vertex centroid.
c) If the point A' is different from the midpoint of the side $[AB]$, then the quadrangle $ABCD$ is a parallelogram if and only if the quadrangle $A'B'C'D'$ is a parallelogram.

*Proof.** As in the Addendum, we assume 0 as coordinate of $S_{\mathcal{E}} = 0$ so that (1) holds. Without loss of generality assume $a \neq b \neq a'$ and take

$$\rho = \frac{a - a'}{b - a'}$$

with $\rho \neq 1$ according to our assumptions. Then the required similarity of the erected triangles implies

$$\frac{b - b'}{c - b'} = \frac{c - c'}{d - c'} = \frac{d - d'}{a - d'} = \rho.$$

*This proof is considerably simpler than our original proof; we thank Chris Fisher for providing it. It is suggested by Bachmann's n-gon theory [2], in simplified form presented in [3].

Solving these equations for a', b', c', d' we obtain

$$a' = \frac{\rho b - a}{\rho - 1}, \ \ b' = \frac{\rho c - b}{\rho - 1}, \ \ c' = \frac{\rho d - c}{\rho - 1}, \ \ d' = \frac{\rho a - d}{\rho - 1}. \tag{2}$$

Summing up and looking at (1) yields $a' + b' + c' + d' = 0$ proving a).

For part (b), recall that the vertex centroid of a quadrangle is the midpoint of the segment connecting the midpoints of its diagonals. Thus, for any two quadrangles that share their vertex centroid, the segment connecting the midpoints of the diagonals of either quadrangle bisects the corresponding segment of the other, whence those four midpoints must be the vertices of a parallelogram. This proves b) not just for the pair $ABCD$ and $A'B'C'D'$, but for any two quadrangles that have the same vertex centroid.

For part (c), we take

$$\lambda = \frac{1}{1 - \rho}, \ \ \mu = \frac{\rho}{\rho - 1}$$

and – by means of the equations (2) – obtain

$$a' + c' = \lambda(a + c) + \mu(b + d), \ \ b' + d' = \lambda(b + d) + \mu(a + c).$$

We know that the quadrangle $A'B'C'D'$ is a parallelogram if and only if the equation

$$a' + c' = b' + d',$$

holds; that is,

$$\lambda(a + c - b - d) = \mu(a + c - b - d). \tag{3}$$

In view of the hypothesis in c) saying that

$$a' \neq \frac{1}{2}(a + b),$$

we have $\rho \neq -1$ and thus $\lambda \neq \mu$. Hence (3) is equivalent to

$$a + c = b + d,$$

which means that the quadrangle $ABCD$ is a parallelogram. So we have c) finishing the proof of the theorem.

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PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-psol@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX* *et* *un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec* *chaque* *solution, de même que l'établissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page.*

Comment soumettre un problème. *Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2015***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal d'avoir traduit les problèmes.

3871. *Proposé par C. Mortici.*

Trouver les nombres réels positifs qui sont solutions de

$$(x^2y-1)\ln x+(xy^2-1)\ln y=0\,.$$

3872. *Proposé par F. R. Ataev.*

Soit x, y et z les distances des sommets d'un triangle au centre de son cercle inscrit et soit r le rayon de celui-ci. Montrer que l'aire du triangle est donnée par

$$A = \frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r}\,.$$

3873. *Proposé par N. Hodžić et S. Malikić.*

Soit a, b et c trois nombres réels non négatifs tels que $a+b+c=3$. Montrer que l'inégalité suivante est correcte

$$\frac{a}{8b^3+8c^3+11bc} + \frac{b}{8c^3+8a^3+11ca} + \frac{c}{8a^3+8b^3+11ab} \geq \frac{1}{9}\,.$$

3874. *Proposé par P. Ligouras.*

Soit ABC un triangle dont les côtés satisfont $AB+AC=2BC$. Soit respectivement I et O les centres des cercles inscrit et circonscrit de ABC, et soit M l'intersection de AI avec le cercle circonscrit de ABC. Soit N un point dans le plan du triangle ABC et P le point sur la droite AI tel que $IC^2+IN^2=2IP^2$, que $\angle NIP = \angle CIP$ et que $INPC$ soit un quadrilatère cyclique. Soit Q l'intersection de AP et CN, R celle de OI et CN. Montrer que $IQ = IR$.

3875. *Proposé par D. M. Bătineţu-Giurgiu et N. Stanciu.*

Soit $(A, +, \cdot)$ un anneau avec unité, $1 \neq 0$, tel que si $xy=1$ alors $yx=1$. Si $a, b \in A$ et s'il existe un $u \in A$, où u est inversible, avec $ua=au$, $ub=bu$ tel que $uab+a+b=0$, montrer alors que $ab=ba$.

3876. *Proposé par M. Bataille.*

Soit ABC un triangle, H le pied de la hauteur issue de A et γ_b, γ_c les cercles de diamètre respectifs AB et AC. Une droite ℓ passant par H coupe respectivement γ_b, γ_c en H, et en B', C'. Si BB' et CC' se coupent en H', montrer que

$$(H'C)(HC') = k(H'B)(HB')$$

pour une certaine constante k indépendante du choix de la droite ℓ.

3877. *Proposé par M. Dincă.*

Soit $x_1, x_2, \ldots, x_m$ des nombres réels tels que $\prod_{k=1}^{m} x_k = 1$. Montrer que

$$\sum_{k=1}^{m} \frac{x_k^2}{x_k^2 - 2x_k \cos\frac{2\pi}{m} + 1} \geq 1.$$

3878. *Proposé par D. T. Oai.*

Soit H l'orthocentre du triangle ABC et soit D un point quelconque du plan, différent de A, B, C, H. Montrer que si A', B' et C' sont les points où les perpendiculaires issues de H sur DA, DB et DC coupent respectivement les droites BC, CA et AB, alors elles sont colinéaires. Inversément, si une droite (ne passant pas par un sommet ou un orthocentre) coupe BC en A', CA en B' et AB en C', montrer qu'alors les droites passant par A, B, C qui sont respectivement perpendiculaires à HA', HB', HC', sont concourantes.

3879. *Proposé par O. Furdui.*

Calculer

$$\sum_{n=2}^{\infty}\left(n\ln\left(1-\frac{1}{n}\right)+1+\frac{1}{2n}\right).$$

3880. *Proposé par I. Bluskov.*

Soit $B=[b_{ij}]$ une matrice $n\times k$ dont les éléments sont dans l'ensemble des résidus modulo v, tels que les k éléments dans chaque ligne de B sont deux à deux distincts. Construire la matrice $n\times[k(k-1)]$ des différences $D=[d_{ip}]$ où $d_{ip}=b_{ij}-b_{is}$ (mod v), $j\neq s$, $1\leq j,s\leq k$ et $1\leq i\leq n$. Soit O_q, $q=1,2,\ldots,v-1$ le nombre d'occurences du résidu q dans la matrice D. Montrer que la somme

$$\sum_{q=1}^{v-1}O_q$$

ne dépend pas de v.

. . . .

3871. *Proposed by C. Mortici.*

Solve in positive real numbers :

$$(x^2y-1)\ln x+(xy^2-1)\ln y=0.$$

3872. *Proposed by F. R. Ataev.*

Let x, y, z be the distances from the vertices of a triangle to its incentre and let r be the inradius of the triangle. Show that the area of the triangle is given by

$$A=\frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r}.$$

3873. *Proposed by N. Hodžić and S. Malikić.*

Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that the following inequality holds

$$\frac{a}{8b^3+8c^3+11bc}+\frac{b}{8c^3+8a^3+11ca}+\frac{c}{8a^3+8b^3+11ab}\geq\frac{1}{9}\,.$$

3874. *Proposed by P. Ligouras.*

In triangle ABC, the sides satisfy $AB+AC=2BC$. Let I and O be the incentre and circumcentre of triangle ABC respectively, and let M be the intersection of AI with the circumcircle of triangle ABC. Let N be a point in the plane of triangle ABC and let P be the point on the line AI such that $IC^2+IN^2=2IP^2$, $\angle NIP=\angle CIP$ and $INPC$ is a cyclic quadrilateral. Let Q be the intersection of AP and CN, and let R be the intersection of OI and CN. Prove that $IQ=IR$.

3875. *Proposed by D. M. Bătineţu-Giurgiu and N. Stanciu.*

Let $(A,\,+,\,\cdot)$ be a ring with unity, $1\neq 0$, such that if $xy=1$ then $yx=1$. If $a,b\in A$, and there is a $u\in A$, where u is invertible, with $ua=au$, $ub=bu$, such that $uab+a+b=0$, then prove that $ab=ba$.

3876. *Proposed by M. Bataille.*

Let ABC be a triangle, H the foot of the altitude from A and γ_b, γ_c the circles with diameters AB, AC, respectively. A line ℓ passing through H intersects γ_b, γ_c at H and B', C', respectively. If BB' and CC' intersect at H', show that

$$(H'C)(HC')=k(H'B)(HB')$$

for some constant k independent of the chosen line ℓ.

3877. *Proposed by M. Dincă.*

Let $x_1,x_2,\ldots,x_m$ be real numbers such that $\displaystyle\prod_{k=1}^{m}x_k=1$. Prove that

$$\sum_{k=1}^{m}\frac{x_k^2}{x_k^2-2x_k\cos\frac{2\pi}{m}+1}\geq 1.$$

3878. *Proposed by D. T. Oai.*

Let H be the orthocentre of triangle ABC and let D be any point in the plane different from A, B, C, H. Prove that if A', B', and C' are the points where the perpendiculars from H to DA, DB, and DC meet the lines BC, CA and AB, respectively, then they are collinear. Conversely, if a line (not through a vertex or orthocentre) intersects BC in A', CA in B' and AB in C', then prove that the lines through A, B, C that are perpendicular to HA', HB', HC', respectively, are concurrent.

3879. *Proposed by O. Furdui.*

Calculate
$$\sum_{n=2}^{\infty}\left(n\ln\left(1-\frac{1}{n}\right)+1+\frac{1}{2n}\right).$$

3880. *Proposed by I. Bluskov.*

Let $B=[b_{ij}]$ be an $n\times k$ matrix with entries in the set of residues modulo v, such that the k entries in each row of B are pairwise distinct. Form the $n\times[k(k-1)]$ matrix of differences $D=[d_{ip}]$ where $d_{ip}=b_{ij}-b_{is} \pmod{v}$, $j\neq s$, $1\leq j,s\leq k$ and $1\leq i\leq n$. Let O_q, $q=1,2,\ldots,v-1$ be the number of occurrences of the residue q in the matrix D. Show that the sum
$$\sum_{q=1}^{v-1}O_q$$
does not depend on v.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3771. [2012 : 334, 336] *Proposed by B. Sands.*

(a) Find infinitely many pairs (a, b) of positive rational numbers so that $\sqrt{a} - \sqrt{b}$ is a root of $x^2 + ax - b$.

(b) Find two positive integers a, b so that $\sqrt{a} - \sqrt{b}$ is a root of $x^2 + ax - b$.

Solved by Š. Arslanagić; A. Alt; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; B. D. Beasley; M. Amengual Covas; C. Curtis; R. Hess; D. Koukakis; K. E. Lewis; S. Malikić; Missouri State University Problem Solving Group; M.R.Modak; C. Sánchez-Rubio; Skidmore College Problem Group; D. Smith; T. Smith; D. R. Stone and J. Hawkins; D. Văcaru; T. Zvonaru; and the proposer. There was one incorrect solution. We present a solution adapted from that of Chip Curtis.

(a) $\sqrt{a} - \sqrt{b}$ is a root of $x^2 + ax - b$ if and only if $a - 2\sqrt{ab} + a\sqrt{a} - a\sqrt{b} = 0$, or

$$(\sqrt{a} + 2)\sqrt{b} = \sqrt{a}(\sqrt{a} + 1).$$

Thus, we obtain the possibilities

$$(a, b) = \left(c^2, \frac{c^2(c+1)^2}{(c+2)^2}\right),$$

where c is any positive rational.

Since

$$\frac{c(c+1)}{c+2} = c - 1 + \frac{2}{c+2},$$

we see that a and b cannot both be squares of integers.

However, squaring the equation relating a and b leads to

$$2\sqrt{a}(2b - a) = a^2 + a - ab - 4b.$$

If a is not the square of a rational, then

$$a = 2b = 4b + ab - a^2 = 2b(2 - b),$$

whereupon $(a, b) = (2, 1)$. This is the only case in which (a, b) is an integer pair and in which (a, b) are not both squares.

(b) $\sqrt{2} - 1$ is a root of $x^2 + 2x - 1$.

Editor's comment. Sánchez-Rubio noted that, if in the solution, we allow c to take a negative value and the square roots to be negative (in effect, asking that $\sqrt{b}-\sqrt{a}$ be a root), then we can get integer solutions with $c = 0, -1, -3, -4$. This leads to $(a, b) = (0, 0), (1, 0), (9, 36), (16, 36)$. Thus $-1 = 0 - 1$ is a root of $x^2 + x$; $3 = 6 - 3$ is a root of $x^2 + 9x - 36 = (x + 12)(x - 3)$ and $2 = 6 - 4$ is a root of $x^2 + 16x - 36 = (x + 18)(x - 2)$. On the other hand, Zvonaru set $b = k^2a$ and obtained the equation $(1 - k)\sqrt{a} = (2k - 1)$. This leads to the family

$$(a, b) = \left(\left(\frac{2k-1}{1-k}\right)^2, \left(\frac{k(2k-1)}{1-k}\right)^2\right),$$

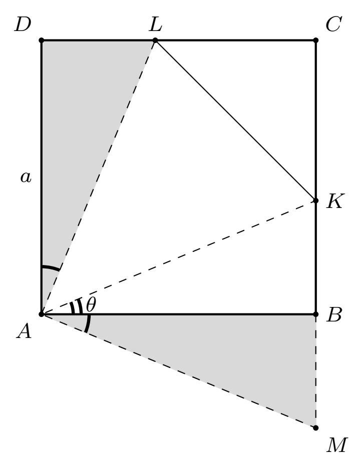
where k is a rational for which $\frac{1}{2} < k < 1$. This agrees with the family given in the solution.

3772. [2012 : 334, 336] *Proposed by G. Apostolopoulos.*

Given a square $ABCD$ with side length a. Points K and L are on BC and CD, respectively, such that the perimeter of ΔKCL is $2a$. Determine the measures of the angles of ΔAKL which minimize its area.

Solved by A. Alt ; Š. Arslanagić (2 solutions) ; D. Bailey, E. Campbell, and C. Diminnie ; M. Bataille ; C. Curtis ; J. Hawkins and D. R. Stone ; R. Hess ; O. Kouba ; D. Koukakis ; K. E. Lewis ; S. Malikić ; M.R.Modak ; M. Parmenter ; C. Sánchez-Rubio ; D. Smith ; I. Uchiha ; T. Zvonaru ; and the proposer. We present the solution by Omran Kouba.

Since the perimeter of $\triangle KCL$ is equal to $BC + CD$, we conclude that $KL = BK + DL$. Let M be a point on the line BC so that $BM = DL$, and B is between K and M as in the figure :



Since $AB = AD$ and $BM = DL$, the two right triangles $\triangle ABM$ and $\triangle ADL$ are congruent. In particular, $\angle BAM = \angle DAL$, thus $\angle LAM = 90°$. On the other hand, we have $AM = AL$ and $KM = KB + DL = KL$, so, AK is the perpendicular bisector of a line segment LM. Thus AK is the angle bisector of $\angle LAM$, and $\angle LAK = 45°$.

Now, let $\theta = \angle KAB$, then $AK = a/\cos(\theta)$ and $AL = a/\cos(45° - \theta)$, so that

$$\begin{aligned}\text{Area}(AKL) &= \frac{1}{2}AK \cdot AL \cdot \sin(45°) = \frac{a^2}{\sqrt{2}} \cdot \frac{1}{2\cos(45° - \theta)\cos\theta} \\ &= \frac{a^2}{1 + \sqrt{2}\cos(45° - 2\theta)}\end{aligned}$$

Thus, the area of $\triangle AKL$ is minimum, if and only if $\cos(45° - 2\theta) = 1$, that is $\theta = 22.5°$, in this case $AK = AL$ and consequently $\angle LKA = \angle KLA = 67.5°$. This determines the measures of the angles of $\triangle AKL$ with minimum area, and the corresponding area is $(\sqrt{2} - 1)a^2$.

3773. [2012 : 334, 336] *Proposed by M. Bataille.*

Let R and r be the circumradius and the inradius of a triangle with sides a, b, c. Under which condition on the angles of the triangle does the inequality

$$a + b + c \leq 2\sqrt{3}(R + r)$$

hold ?

Solved by A. Alt ; S. Brown ; C. Curtis ; M. Dincă ; J. G. Heuver ; V. Konečný ; O. Kouba ; K. Lau ; S. Malikić ; T. Zvonaru ; and the proposer. We present the solution by Titu Zvonaru.

By the sine law (namely, $\sum_{\text{cyclic}} a = 2R \sum_{\text{cyclic}} \sin A$) and the formula

$$1 + \frac{r}{R} = \sum_{\text{cyclic}} \cos A,$$

the given inequality is equivalent to

$$\sum_{\text{cyclic}} \sin A \ \leq \ \sqrt{3} \sum_{\text{cyclic}} \cos A,$$

and this is exactly the inequality (1) from D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, page 256. Assuming that $A \leq B \leq C$, we have :

$$\begin{aligned}&\sqrt{3} \sum_{\text{cyclic}} \cos A \leq \sum_{\text{cyclic}} \sin A \quad \text{if } B \geq \frac{\pi}{3} \qquad \text{(Triangle of Bager's Type I)}, \\ &\sqrt{3} \sum_{\text{cyclic}} \cos A \geq \sum_{\text{cyclic}} \sin A \quad \text{if } B \leq \frac{\pi}{3} \qquad \text{(Triangle of Bager's Type II)}.\end{aligned}$$

Equality holds exactly when $B = \dfrac{\pi}{3}$. Here is a proof. Using the formulas

$$\sum_{\text{cyclic}} \cos^2 A = 1 - 2\cos A \cos B \cos C$$

and

$$\sum_{\text{cyclic}} (\cos A \cos B - \sin A \sin B) = \sum_{\text{cyclic}} \cos(A+B) = -\sum_{\text{cyclic}} \cos A,$$

we get :

$$\begin{aligned}
&3\left(\sum_{\text{cyclic}} \cos A\right)^2 - \left(\sum_{\text{cyclic}} \sin A\right)^2\\
&= 3\sum_{\text{cyclic}} \cos^2 A + 6\sum_{\text{cyclic}} \cos A\cos B - \sum_{\text{cyclic}} \sin^2 A - 2\sum_{\text{cyclic}} \sin A \sin B\\
&= 3\sum_{\text{cyclic}} \cos^2 A - \sum_{\text{cyclic}} (1-\cos^2 A) + 6\sum_{\text{cyclic}} \cos A\cos B - 2\sum_{\text{cyclic}} \sin A\sin B\\
&= 4\sum_{\text{cyclic}} \cos^2 A - 3 + 4\sum_{\text{cyclic}} \cos A\cos B + 2\left(\sum_{\text{cyclic}} \cos A\cos B - \sum_{\text{cyclic}} \sin A \sin B\right)\\
&= 4(1 - 2\cos A\cos B\cos C) - 3 + 4\sum_{\text{cyclic}} \cos A\cos B - 2\sum_{\text{cyclic}} \cos A\\
&= (1-2\cos A)(1-2\cos B)(1-2\cos C).
\end{aligned}$$

Since $A \le \frac{\pi}{3}$ and $C \ge \frac{\pi}{3}$, it follows that $\cos A \ge \frac{1}{2}$ and $\cos C \le \frac{1}{2}$, hence

$$\sqrt{3}\sum_{\text{cyclic}} \cos A - \sum_{\text{cyclic}} \sin A$$

has the same sign as $2\cos B - 1$, and we are done.

3774. [2012 : 334, 336] *Proposed by P. H. O. Pantoja.*

Let a, b, c be positive real numbers. Prove that

$$\frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} + \frac{a(b^2+c^2)}{\sqrt{b^3+c^3}} + \frac{b(c^2+a^2)}{\sqrt{c^3+a^3}} \le \frac{3}{4}(a^2+b^2+c^2+a+b+c)\,.$$

Solved by A. Alt; G. Apostolopoulos; C. Curtis; O. Geupel; O. Kouba; D. Koukakis; S. Malikić; P. Perfetti; D. Smith; T. Zvonaru; and the proposer. We present the solution by Arkady Alt.

We will prove the following stronger inequality :

$$\sum_{\text{cyclic}} \frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} \leq \frac{\sqrt{2}}{2}(a^2+b^2+c^2+a+b+c). \tag{1}$$

Note first that since $(a+b)(a^3+b^3)-(a^2+b^2)^2 = ab(a-b)^2 \geq 0$, we have

$$\frac{a^2+b^2}{\sqrt{a^3+b^3}} \leq \sqrt{a+b}.$$

Hence

$$\sum_{\text{cyclic}} \frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} \leq \sum_{\text{cyclic}} c\sqrt{a+b}. \tag{2}$$

By the AM-GM Inequality, we have

$$c^2+\frac{a+b}{2} \geq 2\sqrt{c^2\left(\frac{a+b}{2}\right)} = \sqrt{2}c\sqrt{a+b},$$

so

$$a^2+b^2+c^2+a+b+c = \sum_{\text{cyclic}}\left(c^2+\frac{a+b}{2}\right) \geq \sqrt{2}\sum_{\text{cyclic}} c\sqrt{a+b} \tag{3}$$

or, equivalently,

$$\sum_{\text{cyclic}} c\sqrt{a+b} \leq \frac{1}{\sqrt{2}}(a^2+b^2+c^2+a+b+c). \tag{4}$$

From (2) and (4), our claim (1) follows immediately. It is easy to see that equality holds if and only if $a=b=c=1$.

Editor's comment. The stronger inequality featured above was also obtained by both Malikić and Perfetti. In addition, the following stronger inequality was obtained by Geupel :

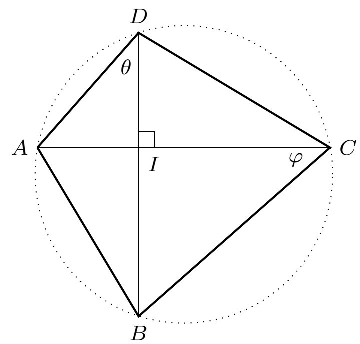
$$\sum_{\text{cyclic}} \frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} \leq \frac{2}{3}(a^2+b^2+c^2)+\frac{3}{4}(a+b+c).$$

3775. [2012 : 334, 336] *Proposed by M. Chirita.*

Let $ABCD$ be a quadrilateral with $AC \perp BD$. Show that $ABCD$ is cyclic if and only if $BC \cdot AD = IA \cdot IB + IC \cdot ID$, where I is the point of intersection of the diagonals.

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić (2 solutions); M. Bataille; O. Kouba; D. Koukakis; S. Malikić; M.R.Modak; C. Sánchez-Rubio; Skidmore College Problem Group; D. Smith; I. Uchiha; D. Văcaru; P. Y. Woo; T. Zvonaru; and the proposer. We present 2 solutions.

Solution 1 by Omran Kouba.



Let $\theta = \angle ADB$ and $\varphi = \angle ACB$. Then

$$\begin{aligned}\frac{IA \cdot IB + IC \cdot ID}{BC \cdot AD} &= \frac{ID}{AD} \cdot \frac{IC}{BC} + \frac{IA}{AD} \cdot \frac{IB}{BC}\\ &= \cos\theta\cos\varphi + \sin\theta\sin\varphi\\ &= \cos(\theta - \varphi).\end{aligned}$$

Thus, the condition $BC \cdot AD = IA \cdot IB + IC \cdot ID$, is equivalent to $\cos(\theta - \varphi) = 1$ or $\theta = \varphi$, and this, in turn, is equivalent to the fact that $ABCD$ is cyclic.

Solution 2 by Itachi Uchiha.

Since $AC \perp BD$, we have $BC^2 = IB^2 + IC^2$, $AD^2 = IA^2 + ID^2$, and also $\angle IAB = \angle IDC$ if and only if $\triangle IAB \sim \triangle IDC$. Therefore

$$\begin{aligned}ABCD \text{ is cyclic } &\Leftrightarrow \triangle IAB \sim \triangle IDC\\ &\Leftrightarrow IB \cdot ID = IC \cdot IA\\ &\Leftrightarrow (IB \cdot ID - IC \cdot IA)^2 = 0\\ &\Leftrightarrow IB^2 \cdot ID^2 + IC^2 \cdot IA^2 = 2IA \cdot IB \cdot IC \cdot ID\\ &\Leftrightarrow IB^2 \cdot ID^2 + IC^2 \cdot IA^2 = (IA \cdot IB + IC \cdot ID)^2\\ &\qquad\qquad - (IA^2 \cdot IB^2 + IC^2 \cdot ID^2)\\ &\Leftrightarrow (IB^2 + IC^2)(IA^2 + ID^2) = (IA \cdot IB + IC \cdot ID)^2\\ &\Leftrightarrow BC \cdot AD = IA \cdot IB + IC \cdot ID,\end{aligned}$$

which completes the proof.

3776. [2012 : 335, 336] *Proposed by W. Jiang.*

In ΔABC prove that

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \geq \frac{1}{2}\left(\sec\left(\frac{A}{2}\right) + \sec\left(\frac{B}{2}\right) + \sec\left(\frac{C}{2}\right)\right).$$

Solved by A. Alt ; Š. Arslanagić ; D. Bailey, E. Campbell, and C. Diminnie ; M. Bataille ; S. H. Brown ; M. Dincă; O. Geupel ; O. Kouba ; K. Lau ; S. Malikić ; C. M. Quang ; D. Smith ; P. Y. Woo ; T. Zvonaru ; and the proposer. We present 5 different solutions.

Foreword. In the following solutions, for convenience let $D = A/2$, $E = B/2$ and $F = C/2$. Then $D+E+F = 90^\circ$ and $\tan D\tan E+\tan E\tan F+\tan F\tan D = 1$.

Solution 1 by Titu Zvonaru.

Since, for example,

$$\begin{aligned}\tan D + \tan E &= \frac{\sin D\cos E + \sin E\cos D}{\cos D\cos E}\\ &= \frac{\sin(D+E)}{\cos D\cos E} = \frac{\cos F}{\cos D\cos E},\end{aligned}$$

the desired inequality can be rewritten as

$$\frac{\cos D}{\cos E\cos F} + \frac{\cos E}{\cos F\cos D} + \frac{\cos F}{\cos D\cos E} \geq \frac{1}{\cos D} + \frac{1}{\cos E} + \frac{1}{\cos F}.$$

Multiplying by $\cos D\cos E\cos F$, we see that this is equivalent to

$$\cos^2 D + \cos^2 E + \cos^2 F \geq \cos E\cos F + \cos F\cos D + \cos D\cos E.$$

However, this holds since

$$(\cos D - \cos E)^2 + (\cos E - \cos F)^2 + (\cos F - \cos D)^2 \geq 0.$$

Solution 2 by Dionne Bailey, Elsie Campbell and Charles Diminnie.

For $0 < x < \frac{\pi}{2}$, let

$$f(x) = \tan x - \frac{1}{2}\sec x - \left(x - \frac{\pi}{6}\right).$$

Since

$$f'(x) = \sec^2 x - \frac{1}{2}\sec x\tan x - 1 = \frac{\sin x(2\sin x - 1)}{2\cos^2 x},$$

it follows that $f(x)$ decreases on $(0, \pi/6)$ and increases on $(\pi/6, \pi/2)$. Therefore $f(x) \geq f(\pi/6)$, so that

$$\tan x - \frac{1}{2}\sec x \geq x - \frac{\pi}{6}$$

for $0 < x < \pi/2$ with equality if and only if $x = \pi/6$. Since D, E and F all lie between 0 and $\pi/2$,

$$\begin{aligned}\tan D + \tan E + \tan F - \frac{1}{2}(\sec D + \sec E + \sec F)&\\ \geq \left(D - \frac{\pi}{6}\right) + \left(E - \frac{\pi}{6}\right) + \left(F - \frac{\pi}{6}\right)&\\ = D + E + F - \frac{\pi}{2} = 0.&\end{aligned}$$

Solution 3 by Cao Minh Quang.

Let $x = \tan D$, $y = \tan E$ and $z = \tan F$. Taking note of the arithmetic-geometric means inequality and the fact that $xy + yz + zx = 1$, we find that

$$\begin{aligned}
&\frac{1}{2}(\sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{x^2+1}) \\
&= \frac{1}{2}(\sqrt{(x+y)(x+z)} + \sqrt{(y+z)(y+x)} + \sqrt{(z+x)(z+y)}) \\
&\le \frac{1}{4}[(2x+y+z) + (2y+z+x) + (2z+x+y)] = x+y+z.
\end{aligned}$$

This is the desired inequality.

Solution 4 by Digby Smith.

Since $\tan^2 x$ is convex on $(0, \pi/2)$, by Jensen's equality, we have that

$$\tan^2 D + \tan^2 E + \tan^2 F \ge 3\tan^2 \frac{D+E+F}{3} = 3\tan^2 \frac{\pi}{3} = 1.$$

Now,

$$\begin{aligned}
&[2(\tan D + \tan E + \tan F)]^2 \\
&= 4(\tan^2 D + \tan^2 E + \tan^2 F) + 8(\tan D \tan E + \tan E \tan F + \tan F \tan D) \\
&\ge 1 + 3(\tan^2 D + \tan^2 E + \tan^2 F) + 8 = 3(3 + \tan^2 D + \tan^2 E + \tan^2 F) \\
&= 3(\sec^2 D + \sec^2 E + \sec^2 F) \ge (\sec D + \sec E + \sec F)^2.
\end{aligned}$$

The final step exploits Cauchy's inequality. Thus

$$2(\tan D + \tan E + \tan F) \ge (\sec D + \sec E + \sec F)$$

and the desired inequality is established.

Solution 5 by Wei-Dong Jiang, the proposer.

We have :

$$\begin{aligned}
1 &= \tan D \tan E + \tan E \tan F + \tan F \tan D \\
&= \tan E \tan F + \tan D(\tan E + \tan F) \\
&\le \frac{1}{4}(\tan E + \tan F)^2 + \tan D(\tan E + \tan F) \\
&= \frac{1}{4}(\tan E + \tan F + 2\tan D)^2 - \tan^2 D.
\end{aligned}$$

Thus

$$\tan E + \tan F + 2\tan D \ge 2\sqrt{1 + \tan^2 D} = 2\sec D.$$

Adding this to its two cyclic analogues yields the desired inequality.

Editor's comment. As is often the case with a trigonometry problem, there were several different approaches exhibiting a variety of efficiency and recourse to other results. Kouba produced a solution similar to the second one above and Woo analyzed the graph of $y = 2\tan x - \sec x$ to show that it lay above the line $y = -\pi/3 + 2x$. Some solvers used the representation of the tangents and cosines of the half angles of a triangle in terms of the sides, semi-perimeter, inradius and area. Brown noted that the given inequality is equivalent to $s^2 \geq 12Rr + 3r^2$, while Lau reduced it to $3s^2 \leq (4R + r)^2$. Dincă proved this generalization : Let $A_1A_2 \dots A_n$ be a convex $n-$gon. Then

$$\sum_{k=1}^{n} \tan\frac{A_k}{2} \geq \cos\frac{\pi}{n} \sum_{k=1}^{n} \sec\frac{A_k}{2}.$$

3777. [2012 : 335, 336] *Proposed by G. Apostolopoulos.*

Let x, y, and z be positive real numbers such that $xyz = 1$ and $\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} = 3$. Determine all possible values of $x^4 + y^4 + z^4$.

Solved by A. Alt; Š.Arslanagic; D. Bailey, E. Campbell and C. Diminnie; M. Bataille; C. Curtis; R. Hess; O. Kouba; D. Koukakis; S. Malikić (2 solutions); P. Perfetti; A. Plaza; C. M. Quang; D. Smith; D. R. Stone and J. Hawkins; I. Uchiha; D. Văcaru; T. Zvonaru; and the proposer. There was also an incorrect solution. We give a solution that is a composite of virtually all solutions received.

By the AM-GM Inequality, we have

$$3 = \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} \geq 3\sqrt[3]{\frac{1}{x^4} \cdot \frac{1}{y^4} \cdot \frac{1}{z^4}} = 3.$$

Thus, we must have the equality above, which implies that $\frac{1}{x^4} = \frac{1}{y^4} = \frac{1}{z^4}$ or $x = y = z$. Since we know that $xyz = 1$, it follows that $x = y = z = 1$ and so $x^4 + y^4 + z^4 = 3$.

3778. [2012 : 335, 337] *Proposed by M. Bataille.*

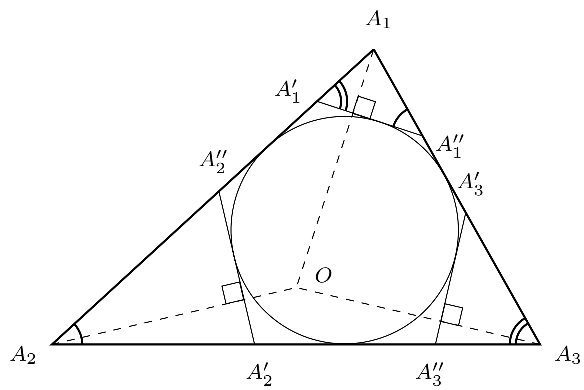
Let $\Delta A_1A_2A_3$ be a triangle with circumcentre O, incircle γ, incentre I, and inradius r. For $i = 1, 2, 3$, let A_i' on side A_iA_{i+1} and A_i'' on side A_iA_{i+2} be such that $A_i'A_i'' \perp OA_i$ and γ is the A_i–excircle of $\Delta A_iA_i'A_i''$ where $A_4 = A_1$, $A_5 = A_2$. Prove that

(a) $A_1'A_1'' \cdot A_2'A_2'' \cdot A_3'A_3'' = \dfrac{4a_1a_2a_3}{(a_1 + a_2 + a_3)^2} \cdot r^2$

(b) $A_1'A_1'' + A_2'A_2'' + A_3'A_3'' = \dfrac{a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3} \cdot IK^2 + \dfrac{3a_1a_2a_3}{a_1^2 + a_2^2 + a_3^2}$

where a_1, a_2, a_3 are the side lengths of $\Delta A_1A_2A_3$ and K is its symmedian point.

Solved by G. Apostolopoulos ; O. Geupel ; O. Kouba ; T. Zvonaru ; and the proposer. We presents the solution by Omran Kouba.



Note that, for $i = 1, 2, 3$, we have

$$\begin{aligned}\angle A_iA_i'A_i'' &= 90^\circ - \angle OA_iA_{i+1}\\ &= 90^\circ - \frac{1}{2}(180^\circ - \angle A_iOA_{i+1})\\ &= \frac{1}{2}\angle A_iOA_{i+1} = \angle A_iA_{i+2}A_{i+1}\end{aligned}$$

This proves that $\triangle A_iA_i''A_i'$ and $\triangle A_iA_{i+1}A_{i+2}$ are similar. Let h_i be the similarity that shrinks $\triangle A_1A_2A_3$ to $\triangle A_iA_i''A_i'$. If γ_i, of radius r_i, is the A_i-excircle of $\triangle A_1A_2A_3$, then $h_i(\gamma_i) = \gamma$ and $h_i(A_{i+1}A_{i+2}) = A_i''A_i'$. Thus

$$\frac{A_i'A_i''}{A_{i+1}A_{i+2}} = \frac{r}{r_i}$$

But if $s = \dfrac{a_1 + a_2 + a_3}{2}$ is the semi-perimeter of $\triangle A_1A_2A_3$, then $r_i(s - a_i) = \text{Area}(A_1A_2A_3) = rs$, so

$$A_i'A_i'' = a_i\left(\frac{s - a_i}{s}\right), \qquad \text{for } \ i = 1, 2, 3. \tag{1}$$

To prove (a) we note, using Heron's formula, that

$$\begin{aligned}A_1'A_1'' \cdot A_2'A_2'' \cdot A_3'A_3'' &= \frac{a_1a_2a_3}{s^3}(s - a_1)(s - a_2)(s - a_3)\\ &= \frac{a_1a_2a_3}{s^2} \cdot \frac{s(s - a_1)(s - a_2)(s - a_3)}{s^2}\\ &= \frac{a_1a_2a_3}{s^2} \cdot \frac{(sr)^2}{s^2} \quad = \quad \frac{a_1a_2a_3}{s^2} \cdot r^2,\end{aligned}$$

which is equivalent to (a).

To prove (b) we note first that (1) implies that

$$A_1'A_1'' + A_2'A_2'' + A_3'A_3'' = 2s - \frac{a_1^2 + a_2^2 + a_3^2}{s}. \tag{2}$$

Now, we need to express IK^2 in terms of the side lengths of $\triangle A_1A_2A_3$. Since I is the barycenter of the weighted points $(A_i; a_i)$, $i = 1, 2, 3$, and K is the barycenter of the weighted points $(A_i; a_i^2)$, $i = 1, 2, 3$, we conclude that

$$\overrightarrow{IK} = \sum_{i=1}^{3} \left(\frac{a_i^2}{q} - \frac{a_i}{2s} \right) \overrightarrow{OA_i},$$

where $q = a_1^2 + a_2^2 + a_3^2$. But, if the circumradius of $\triangle A_1A_2A_3$ is denoted by R, then $OA_i = R$ and

$$\overrightarrow{OA_i} \cdot \overrightarrow{OA}_{i+1} = R^2 \cos(2A_{i+2}) = R^2 - \frac{1}{2}(2R \sin A_{i+2})^2 = R^2 - \frac{a_{i+2}^2}{2}.$$

Thus,

$$IK^2 = R^2 \left(\sum_{i=1}^{3} \left(\frac{a_i^2}{q} - \frac{a_i}{2s} \right) \right)^2 - \sum_{i=1}^{3} a_{i+2}^2 \left(\frac{a_i^2}{q} - \frac{a_i}{2s} \right) \left(\frac{a_{i+1}^2}{q} - \frac{a_{i+1}}{2s} \right).$$

Because $\displaystyle\sum_{i=1}^{3} \left(\frac{a_i^2}{q} - \frac{a_i}{2s} \right) = 0$, the previous equation reduces to

$$\begin{aligned} IK^2 &= -a_1a_2a_3 \sum_{i=1}^{3} a_{i+2} \left(\frac{a_i}{q} - \frac{1}{2s} \right) \left(\frac{a_{i+1}}{q} - \frac{1}{2s} \right) \\ &= -a_1a_2a_3 \sum_{i=1}^{3} a_{i+2} \left(\frac{a_ia_{i+1}}{q^2} + \frac{a_{i+2}}{2sq} - \frac{1}{q} + \frac{1}{4s^2} \right) \\ &= -a_1a_2a_3 \left(\frac{3a_1a_2a_3}{q^2} + \frac{1}{2s} - \frac{2s}{q} + \frac{2s}{4s^2} \right) \\ &= a_1a_2a_3 \left(\frac{2s}{q} - \frac{3a_1a_2a_3}{q^2} - \frac{1}{s} \right). \end{aligned}$$

It follows that

$$\frac{q}{a_1a_2a_3} \cdot IK^2 + \frac{3a_1a_2a_3}{q} = 2s - \frac{q}{s},$$

and (b) follows from (2).

Editor's comment. Zvonaru reported that at the *MathWorld* site (under the heading *symmedian point*) there is an expression for IK^2 that reduces to the equation in the featured solution above.

3779. [2012 : 335, 337] *Proposed by D. Milošević.*

Let ΔABC have semi-perimeter s and let x, y, z be the distances from the centroid to the sides BC, CA, AB, respectively. Prove or disprove that

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le \frac{s}{\sqrt{3}}\,.$$

Solved by A. Alt ; Š. Arslanagić ; M. Bataille ; C. Curtis ; M. Dincă ; O. Geupel ; J. Heuver ; V. Konečný ; O. Kouba ; S. Malikić ; C. R. Pranesachar ; C. Sánchez-Rubio ; E. Suppa ; I. Uchiha ; T. Zvonaru. One incorrect solution was received. We present 2 solutions.

Foreword. In the following solutions, let a, b, c be the sides, and K, R, r the area, circumradius, and inradius of the triangle. Let h_a, h_b, and h_c be the altitudes of the triangle.

Solution 1 by Itachi Uchiha.

Since $\frac{1}{3}K = \frac{1}{2}ax$, we have $x = \frac{2K}{3a}$. Similarly, $y = \frac{2K}{3b}$ and $z = \frac{2K}{3c}$. By the RMS-AM inequality (or Power Mean inequality), the formula $K = rs = \frac{abc}{4R}$, and Euler's inequality $R > 2r$, we have

$$\begin{aligned}
\sqrt{xy} + \sqrt{yz} + \sqrt{zx} &= \frac{2K}{3}\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}\right) \\
&\le \frac{2K}{3}\sqrt{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}} = \frac{2K}{3}\sqrt{\frac{a+b+c}{abc}} \\
&= \frac{2rs}{3}\sqrt{\frac{2s}{4Rrs}} \\
&= \frac{s}{\sqrt{3}}\sqrt{\frac{2r}{R}} \le \frac{s}{\sqrt{3}},
\end{aligned}$$

with equality if and only if the triangle is equilateral.

Solution 2 is a composite of similar solutions by Sĕfket Arslanagić, Salem Malikić and Titu Zvonaru.

It is readily verified that $x = \frac{1}{3}h_a$, $y = \frac{1}{3}h_b$, $z = \frac{1}{3}h_c$. By the AM-GM inequality,

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le x + y + z = \frac{1}{3}\left(h_a + h_b + h_c\right),$$

so it suffices to prove that

$$h_a + h_b + h_c \le s\sqrt{3}.$$

This last inequality is item 6.1 on page 60 of O. Bottema et al, *Geometric Inequalities*, Wolters Noordhoff, 1969.

3780. [2012 : 335, 337] *Proposed by O. Furdui.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function and let

$$x_n = f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right).$$

Calculate $\lim\limits_{n\to\infty}(x_{n+1} - x_n)$.

Solved by A. Alt ; M. Bataille ; O. Kouba ; M. R. Modak ; P. Perfetti ; and the proposer. There were five flawed solutions, three of which applied an invalid converse of the Stolz-Cesaro theorem. We present 2 solutions.

Solution 1 by Omran Kouba.

The required limit is equal to $\int_0^1 f(x)dx$.

We first note that, if $g : [0, 1] \longrightarrow \mathbf{R}$ is a continuously differentiable function, then, using integration by parts, we have that

$$\begin{aligned}\int_0^1 \left(x - \frac{1}{2}\right) g'(x)dx &= \left[\left(x - \frac{1}{2}\right) g(x)\right]_0^1 - \int_0^1 g(x)dx \\ &= \frac{g(1) + g(0)}{2} - \int_0^1 g(x)dx.\end{aligned}$$

Apply this to the function $g(x) = f((k + x)/n)$ for $k = 0, 1, 2, \ldots, n - 1$ and add the resulting equations to obtain

$$x_n + \frac{f(0) + f(1)}{2} - n\int_0^1 f(x)dx = \int_0^1 \left(x - \frac{1}{2}\right) H_n(x)dx,$$

where

$$H_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f'\left(\frac{k + x}{n}\right).$$

Observe that, for each $x \in [0, 1]$, $H_n(x)$ is a Riemann sum for the integral $\int_0^1 f'(t)dt$ and $|H_n(x)| \le \sup_{[0.1]} |f'|$.

From the foregoing equation and its analogue for $n + 1$, we obtain that

$$x_{n+1} - x_n - \int_0^1 f(x)dx = \int_0^1 (H_{n+1}(x) - H_n(x))\left(x - \frac{1}{2}\right) dx.$$

As n tends to infinity, the integrand on the right side tends pointwise and boundedly to 0, so by the Lebesgue Dominated Convergent Theorem, we conclude that $\lim_{n\to\infty}(x_{n+1} - x_n) = \int_0^1 f(x)dx$.

Solution 2 by Paolo Perfetti.

There exists $\xi_k \in (k(n+1)^{-1}, kn^{-1})$ for which

$$\begin{aligned} x_{n+1} - x_n &= \sum_{k=1}^{n} \left(f\left(\frac{k}{n+1}\right) - f\left(\frac{k}{n}\right) \right) + f(1) \\ &= \sum_{k=1}^{n} f'(\xi_k) \left(\frac{-k}{n(n+1)}\right) + f(1) \\ &= -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} \left(f'(\xi_k) - f'\left(\frac{k}{n}\right) \right) - \frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} f'\left(\frac{k}{n}\right) + f(1) \end{aligned}$$

where $k(n+1)^{-1} < \xi_k < kn^{-1}$. Note that f' is uniformly continuous on $[0,1]$ and that

$$\left|\xi_k - \frac{k}{n}\right| \le \frac{k}{n(n+1)} < \frac{1}{n}.$$

Therefore, for each $\epsilon > 0$, when n is sufficiently large

$$\left| f'(\xi_k) - f'\left(\frac{k}{n}\right) \right| < \epsilon$$

for $1 \le k \le n$. Thus

$$\left| -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} \left(f'(\xi_k) - f'\left(\frac{k}{n}\right) \right) \right| < \epsilon \left(\frac{1}{n(n+1)}\right) \left(\frac{n(n+1)}{2}\right) = \frac{\epsilon}{2}.$$

Moreover

$$\lim_{n\to\infty} -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} f'\left(\frac{k}{n}\right) = -\int_0^1 x f'(x) dx.$$

Therefore, integrating by parts, we find that

$$\lim_{n\to\infty} (x_{n+1} - x_n) = f(1) - \int_0^1 x f'(x) dx = \int_0^1 f(x) dx.$$

Editor's comment. Malikić and Ricardo noted that the proposer poses and solves this problem in his book *Limits, Series, and Fractional Part Integrals* published by Springer in 2013. It is problem 1.32 on page 6 under Miscellaneous Limits ; the solution appears on page 52.

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

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Marcel Chirita, Bucharest, Romainia : 3775
Jan Ciach, Ostrowiec, Świętokrzyski, Poland : 2049
Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, New York : 339
Marian Dincă, Bucharest, Romania : 3877
Ovidiu Furdui, Campia Turzii, Cluj, Romania : 3879, 3780
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Dragoljub Milošević, Gornji Milanovac, Serbia : 3779
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Bill Sands, University of Calgary, Calgary, AB : 3771

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