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THE CONTEST CORNER

No. 15

Shawn Godin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_CCProblemNumber *(example Doe_Jane_CC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-contest@cms.math.ca`. *Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 September 2014**, *although solutions received after this date will also be considered until the time when a solution is published.*

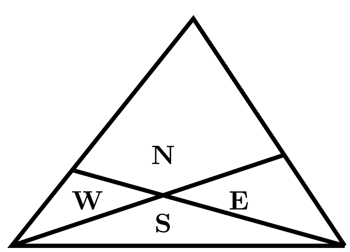
Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the Solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC71. A bag is filled with red and blue balls. Before drawing a ball, there is a $\frac{1}{4}$ chance of drawing a blue ball. After drawing out a ball, there is now a $\frac{1}{5}$ chance of drawing a blue ball. How many red balls are in the bag?

CC72. From the set of natural numbers $1, 2, 3, \ldots, n$, four consecutive even numbers are removed. The remaining numbers have an average value of $51\frac{9}{16}$. Determine all sets of four consecutive even numbers whose removal creates this situation.

CC73. A farmer owns a triangular field, as shown. He reckons 5 sheep can graze in the west field, 10 sheep can graze in the south field, and 8 can graze in the east field. (All sheep eat the same amount of grass.) How many sheep can graze in the north field?



CC74. Let $1000 \leq n = ABCD_{10} \leq 9999$ be a positive integer whose digits $ABCD$ satisfy the divisibility condition:

$$1111 \mid (ABCD + AB \times CD).$$

Determine the smallest possible value of n.

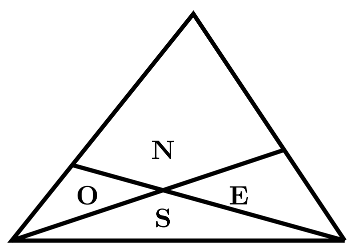
CC75. Let P be a point inside the triangle ABC such that $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ and $\angle ABC = 40°$. Determine $\angle BPC$.

. . . .

CC71. Un sac contient des boules rouges et des boules bleues. Si on pige au hasard une boule du sac, la probabilité de choisir une boule bleue est de $\frac{1}{4}$. Après avoir pigé une boule, la probabilité de choisir une boule bleue est maintenant de $\frac{1}{5}$. Combien y a-t-il de boules rouges dans le sac ?

CC72. On enlève quatre entiers pairs consécutifs de l'ensemble contenant les entiers positifs $1, 2, 3, \ldots, n$. Les nombres qui restent ont une moyenne de $51\frac{9}{16}$. Déterminer tous les ensembles de quatre entiers pairs consécutifs que l'on aurait pu enlever.

CC73. Un fermier possède un champ de forme triangulaire, comme dans la figure ci-dessous. Il calcule que 5 brebis peuvent brouter dans le champ ouest, 10 brebis peuvent brouter dans le champ sud et 8 brebis peuvent brouter dans le champ est. (Toutes les brebis mangent la même quantité d'herbe.) Combien de brebis peuvent brouter dans le champ nord ?



CC74. Soit n $(1000 \leq n = ABCD_{10} \leq 9999)$ un entier positif dont les chiffres $ABCD$ vérifient la condition de divisibilité suivante :

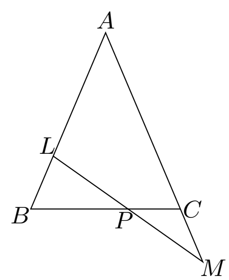
$$1111 \mid (ABCD + AB \times CD).$$

Déterminer la plus petite valeur possible de n.

CC75. Soit P un point à l'intérieur du triangle ABC de manière que $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ et $\angle ABC = 40°$. Déterminer la mesure de l'angle BPC.

CONTEST CORNER SOLUTIONS

CC21. In the diagram ΔABC is isosceles with $AB = AC$. Prove that if $LP = PM$, then $LB = CM$.
(*Originally question # 10 from the 2008 Manitoba Mathematical Competition.*)



Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany; Richard I. Hess, Rancho Palos Verdes, CA, USA; David Jonathan, Palembang, Indonesia; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Jacques Vernin, Marseille, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We present the solution of Heuver.

Let $LX \parallel AM$ meet BC in X. Then triangles LPX and MPC are congruent. Furthermore triangle BLX is isosceles with $BL = LX$. It follows that $BL = LX = CM$ as required.

CC22. Points $A_1, A_2, \ldots, A_{2k}$ are equally spaced around the circumference of a circle and $k \geq 2$. Three of these points are selected at random and a triangle is formed using these points as its vertices. Determine the probability that the triangle is acute.
(*Originally question # 10 b) from the 2006 Euclid Competition.*)

One incorrect solution was received.

CC23. The three-term geometric progression $(2, 10, 50)$ is such that

$$(2 + 10 + 50) \times (2 - 10 + 50) = 2^2 + 10^2 + 50^2.$$

(a) Generalize this (with proof) to other three-term geometric progressions.

(b) Generalize this (with proof) to geometric progressions of length n.

(*Originally question #5 from the 2000 APICS Competition.*)

Solved by Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; and Titu Zvonaru, Cománeşti, Romania. Partial solutions by Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Greg Cook, Angelo State University, San Angelo, TX, USA; Jacques Vernin, Marseille, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Zvonaru modified by the editor.

a) Suppose a geometric progression has first term a and common ratio r, so that its terms are a, ar and ar^2. We shall assume the geometric progression is non-trivial, so $r \neq \pm 1$. We want to show

$$(a + ar + ar^2)(a + a(-r) + a(-r)^2) = a^2 + (ar)^2 + (ar^2)^2.$$

Evaluating the left side, we have

$$\begin{aligned}(a + ar + ar^2)(a + a(-r) + a(-r)^2) &= a\left(\frac{r^3 - 1}{r - 1}\right) \cdot a\left(\frac{(-r)^3 - 1}{-r - 1}\right) \\ &= a^2 \frac{(r^3 - 1)(r^3 + 1)}{(r - 1)(r + 1)} \\ &= a^2 \frac{r^6 - 1}{r^2 - 1} \\ &= a^2(1 + r^2 + r^4) \\ &= a^2 + (ar)^2 + (ar^2)^2,\end{aligned}$$

as desired.

b) To generalize part a), we establish a similar identity but split the cases when n is even or odd. First consider when n is odd. We will establish that for any a, r with $r \neq \pm 1$,

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{n-1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{n-1}) \\ &= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^{n-1})^2.\end{aligned}$$

Similar to part a), we have

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{n-1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{n-1}) \\ &= a^2 \frac{(r^n - 1)((-r)^n - 1)}{(r - 1)(-r - 1)} \\ &= a^2 \frac{r^{2n} - 1}{r^2 - 1} \\ &= a^2(1 + r^2 + r^4 + \cdots + r^{2n-2}) \\ &= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^{n-1})^2,\end{aligned}$$

as desired.

Now consider when n is even, say $n = 2k + 2$ where $k \geq 0$. Then the terms of the geometric progression are $a, ar, ar^2, \cdots, ar^{2k+1}$. We will prove that

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{2k+1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{2k+1})\\&= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^k)^2 - (ar^{k+1})^2 - \cdots - (ar^{2k+1})^2.\end{aligned}$$

Indeed we have

$$\begin{aligned}&a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^k)^2 - (ar^{k+1})^2 - \cdots - (ar^{2k+1})^2\\&= (a^2 + (ar)^2 + \cdots + (ar^k)^2)(1 - ar^{2k+2})\\&= \frac{a^2r^{2k+2} - 1}{r^2 - 1}(1 - ar^{2k+2})\\&= -\frac{(a^2r^{2k+2} - 1)^2}{r^2 - 1}\end{aligned}$$

whereas

$$\begin{aligned}&(a + ar + ar^2 + \cdots + ar^{2k+1})(a + a(-r) + a(-r)^2 + \cdots + a(-r)^{2k+1})\\&= \frac{ar^{2k+2} - 1}{r - 1} \cdot \frac{a(-r)^{2k+2} - 1}{-r - 1}\\&= -\frac{(a^2r^{2k+2} - 1)^2}{r^2 - 1}.\end{aligned}$$

exactly as desired.

CC24. Given the equation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24.$$

(a) Prove that the equation has no integer solutions.
(b) Does this equation have rational solutions? If yes, give an example. If no, prove it.
(*Originally question #2 from the 2009 Memorial University of Newfoundland Undergraduate Mathematics Competition.*)

Solved by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany; Billy Jin, Waterloo Collegiate Institute and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comǎneşti, Romania. We present Curtis' solution.

We can rearrange the equation,

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24 \tag{1}$$

into the factored form,

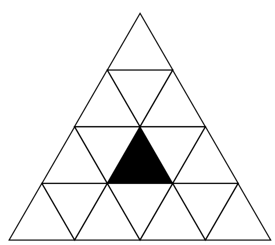
$$-(x + y + z)(y + z - x)(z + x - y)(x + y - z) = 24 \tag{2}$$

(a) If x, y and z are integers, then all four factors have the same parity since, for example, $x+y+z$ and $y+z-x$ differ by the even number $2x$. But the product of four odd numbers is odd, so our four factors must be even. There do not exist four even numbers whose product is -24, as any such product must be divisible by 16, which -24 is not. Hence the equation (1) has no integer solutions.

(b) One the other hand, $(-\frac{5}{2}, -\frac{1}{2}, -1)$ is a solution to (2) and hence (1) in rational numbers since

$$\begin{aligned}\left[\left(-\frac{5}{2}\right)+\left(-\frac{1}{2}\right)+(-1)\right]&\left[\left(-\frac{1}{2}\right)+(-1)-\left(-\frac{5}{2}\right)\right]\\ &\times\left[(-1)+\left(-\frac{5}{2}\right)-\left(-\frac{1}{2}\right)\right]\left[\left(-\frac{5}{2}\right)+\left(-\frac{1}{2}\right)-(-1)\right]\\ &=(-4)(1)(-3)(-2)=24\end{aligned}$$

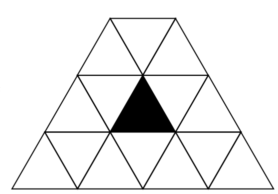
CC25. Alphonse and Beryl are playing a game, starting with the geometric shape shown. Alphonse begins the game by cutting the original shape into two pieces along one of the lines. He then passes the piece containing the black triangle to Beryl, and discards the other piece. Beryl repeats these steps with the piece she receives; that is to say, she cuts along the length of a line, passes the piece containing the black triangle back to Alphonse, and discards the other piece. This process continues, with the winner being the player who, at the beginning of his or her turn, receives only the black triangle. Is there a strategy that Alphonse can use to be guaranteed that he will win?
(*Originally question B3 b) from the 2000 Canadian Open Mathematics Challenge.*)

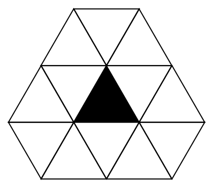


Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

We present his solution below (with a little correction from the editor).

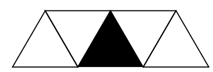
Alphonse's strategy is as follows. On his first move, he should remove one of the three corners, leaving the shape on the right. He then passes things off to Beryl, who has one of two possible actions.

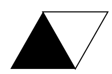




If Beryl snips another small corner, Alphonse responds by snipping off the remaining corner to give the shape on the left. Then Beryl must snip either a row of 3 triangles or a row of 5 triangles. Alphonse will then make an appropriate parallel cut and hand the shape below to Beryl.

At this point, Beryl has two options, snip off a single triangle, or snip off two triangles. Whatever she does, Alphonse performs the opposite cut (cutting off two triangles if she snips one, and snipping off one if she snips off two triangles). This leaves Beryl with the final figure, which she must cut and then hand only the black piece to Alphonse.





Alternatively, she could snip off more than a single triangle. Alphonse's response is to make a cut parallel to Beryl's to produce the 5-triangle winning position shown above, and proceed as discussed.

[*Ed.: The submitted solution was correct up until the row of five triangles, where their solution indicated a mirroring strategy would work. This would work for a simple misinterpretation of the rules, where the winner is the one who hands the black triangle away at the end of their turn. We give Mr. Hess the benefit of the doubt and present his otherwise fine solution.*]

THE OLYMPIAD CORNER

No. 313

Nicolae Strungaru

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file* and *a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-olympiad@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on* each *solution, and each solution should start on a separate page.*

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The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC131. Find all $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$g(f(x) - y) = f(g(y)) + x,$$

for all $x, y \in \mathbb{R}$.

OC132. Find all primes p and q such that

$$(p+q)^p = (q-p)^{(2q-1)}.$$

OC133. Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. Find the maximum of

$$F = \sum_{1 \leqslant i < j \leqslant n} \min\left\{f(x_i), f(x_j)\right\},$$

where $x_1, x_2, \ldots, x_n \geqslant 0$ are real numbers satisfying $x_1 + x_2 + \ldots + x_n = 1$.

OC134. Let ABC be an acute-angled triangle with $AB \neq AC$. Let Γ be the circumcircle of ABC, H the orthocentre of ABC and O the centre of Γ. Let M be the midpoint of BC. The line AM meets Γ again at N and the circle with diameter AM crosses Γ again at P. Prove that the lines AP, BC and OH are concurrent if and only if $AH = HN$.

OC135. Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < \cdots < a_n$ such that $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$ where ϕ denotes the Euler ϕ function.

. . . .

OC131. Déterminer toutes $f, g : \mathbb{R} \to \mathbb{R}$ telles que

$$g(f(x) - y) = f(g(y)) + x$$

pour tous $x, y \in \mathbb{R}$.

OC132. Déterminer tous les nombres premiers p et q tels que

$$(p+q)^p = (q-p)^{(2q-1)}.$$

OC133. Soit $f(x) = (x+a)(x+b)$ où $a, b > 0$. Déterminer le maximum de

$$F = \sum_{1 \leqslant i < j \leqslant n} \min \{f(x_i), f(x_j)\},$$

où $x_1, x_2, \ldots, x_n \geqslant 0$ sont des nombres réels satisfaisant $x_1 + x_2 + \ldots + x_n = 1$

OC134. Soit ABC un triangle à angles aigus tel que $AB \neq AC$. Soit Γ le cercle circonscrit de ABC, H l'orthocentre de ABC, O le centre de Γ, et M le mipoint de BC. La ligne AM rencontre Γ de nouveau à N. Le cercle avec diamètre AM croise Γ de nouveau à P. Démontrer que les lignes AP, BC et OH sont concourantes si et seulement si $AH = HN$.

OC135. Démontrer que pour tout $n \in \mathbb{N}$ il existe des nombres naturels $a_1 < a_2 < \cdots < a_n$ an tels que $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$ où ϕ dénote la fonction ϕ d'Euler.

OLYMPIAD SOLUTIONS

OC71. Define a_n a sequence of positive integers by $a_1 = 1$ and a_{n+1} being the smallest integer so that

$$\mathrm{lcm}(a_1, .., a_{n+1}) > \mathrm{lcm}(a_1, .., a_n)\,.$$

Find the set $\{a_n | n \in \mathbb{Z}\}$.
(*Originally question 4 from the 2011 Austrian Mathematical Olympiad.*)

One incorrect solution was received to this problem.

As this problem is similar to Problem 2 from the fourth test of Romania IMO Selection Test 1995, which the editor wrote, we give a modified version of the Editor's solution to that problem.

Let $b_1 = 1, b_2 = 2, b_3 = 3, \ldots$ be the sequence of positive integers which are divisible by a most one prime, that is $b_1 = 1$ and $b_2, \ldots, b_n, \ldots$ are exactly the positive integers which are powers of primes.

Claim 1: Let n, k be so that $b_n \le k < b_{n+1}$. Then

$$\mathrm{lcm}\{1, 2, \ldots, k\} = \mathrm{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

Proof of Claim 1. As $\{b_1, b_2, \ldots, b_n\} \subset \{1, 2, \ldots, k\}$ we have

$$\mathrm{lcm}\{1, 2, \ldots, k\} \ge \mathrm{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

To complete the proof, we show that $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$ is a common multiple of $\mathrm{lcm}\{1, 2, \ldots, k\}$.

Let $1 \le m \le k$, and let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}\,,$$

be the prime factorization of m. Then for each $1 \le i \le j$ there exists some k_i such that

$$p_i^{\alpha_i} = b_{k_i}\,.$$

As

$$1 \le p_i^{\alpha_i} \le m \le k < b_{n+1}\,,$$

we get $k_i \le n$. Therefore $p_i^{\alpha_i} = b_{k_i}$ divides $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$.

As $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $\ldots$, $p_j^{\alpha_j}$ are pairwise relatively prime, and each one divides $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$, it follows that their product m also divides $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$.

Therefore $\mathrm{lcm}\{b_1, b_2, \ldots, b_n\}$ is divisible by all $1 \le m \le k$, which completes the proof of Claim 1.

Claim 2: For all $n \ge 1$ we have

$$\mathrm{lcm}\{b_1, b_2, \ldots, b_n\} < \mathrm{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

Proof of Claim 2. As $\{b_1, b_2, \ldots, b_n\} \subset \{b_1, b_2, \ldots, b_n, b_{n+1}\}$ we have

$$\operatorname{lcm}\{b_1, b_2, \ldots, b_n\} \leq \operatorname{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

To prove our claim, we only need to show that

$$\operatorname{lcm}\{b_1, b_2, \ldots, b_n\} \neq \operatorname{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

Let $b_{n+1} = p^k$ for some prime p and some $k \geq 1$. Then, $p^k \mid \operatorname{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}$.

If $k = 1$, then b_{n+1} is prime and none of $b_1, b_2, \ldots, b_n$ can be divisible by p, as $b_n < b_{n+1}$. Therefore

$$p \nmid \operatorname{lcm}\{b_1, b_2, \ldots, b_n\}\,;\, p \mid \operatorname{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,,$$

which shows that they cannot be equal.

If $k \geq 2$, then

$$p^k \mid \operatorname{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\}\,.$$

We claim that

$$p^k \nmid \operatorname{lcm}\{b_1, b_2, \ldots, b_n\}\,.$$

Indeed, each of $b_1, b_2, \ldots, b_n$ is either relatively prime to p, or of the form p^i where $1 \leq i \leq k-1$. It follows immediately from here that the power of p in $\operatorname{lcm}\{b_1, b_2, \ldots, b_n\}$ is exactly $k-1$.

Claim 3: For all $n \geq 1$ we have

$$a_n = b_n\,.$$

Proof of Claim 3. We prove this result by strong induction on n.

The initial step is obvious, as $b_1 = 1 = a_1$.

We now prove the inductive step. We know that $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$ and we need to show that $a_{n+1} = b_{n+1}$.

Let $b_n \leq k < b_{n+1}$. then by *Claim 1* we have

$$\operatorname{lcm}\{a_1, a_2, \ldots, a_n\} = \operatorname{lcm}\{b_1, b_2, \ldots, b_n\} = \operatorname{lcm}\{1, 2, \ldots, k\} \geq \operatorname{lcm}\{a_1, a_2, \ldots, a_n, k\}\,.$$

Therefore

$$\operatorname{lcm}\{a_1, a_2, \ldots, a_n, k\} \leq \operatorname{lcm}\{a_1, a_2, \ldots, a_n\}\,,$$

for all $k < b_{n+1}$.

Moreover, by *Claim 2*, we have

$$\operatorname{lcm}\{a_1, a_2, \ldots, a_n, b_{n+1}\} = \operatorname{lcm}\{b_1, b_2, \ldots, b_n, b_{n+1}\} > \operatorname{lcm}\{b_1, b_2, \ldots, b_n\} = \operatorname{lcm}\{a_1, a_2, \ldots, a_n\}\,.$$

Hence, the definition of a_{n+1} implies

$$a_{n+1} = b_{n+1}\,,$$

which proves the inductive step.

OC72. Prove that there are infinitely many positive integers so that the arithmetic and geometric mean of their divisors are integers.
(*Originally question 4 from the 2011 Kazahstan Math Olympiad, Grade 10.*)

Solved by Daniel Văcaru, Piteşti, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Zelator.

We show that if p is a prime of the form $3k+1$, then p^2 has the desired properties. As there are infinitely many primes of the form $3k+1$, this solves the problem.

Indeed, the divisors of p^2 are $1, p$ respectively p^2. Their arithmetic mean is

$$\frac{1+p+p^2}{3}\,.$$

As

$$1 \equiv p \equiv p^2 \pmod 3\ ,$$

we have $3 \mid p^2+p+1$, and hence

$$\frac{1+p+p^2}{3} \in \mathbb{Z}$$

as claimed.

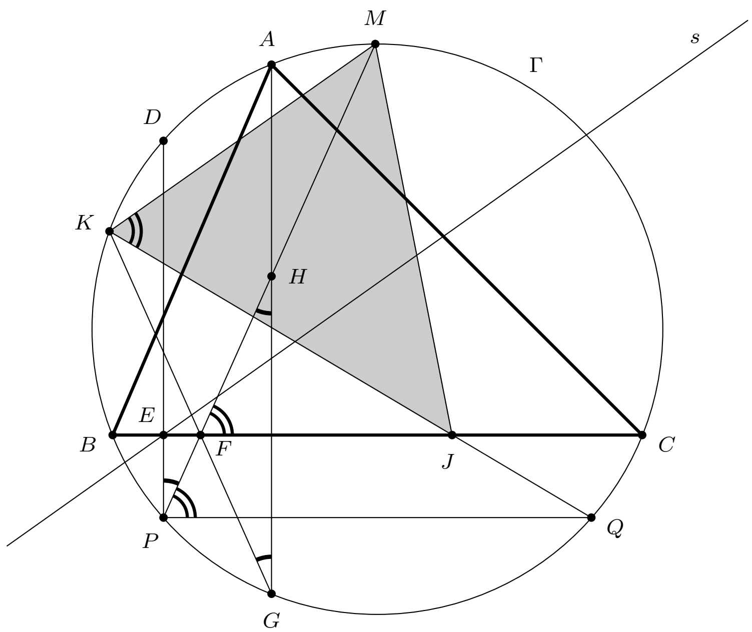
As geometric mean of $1, p, p^2$ is $p \in \mathbb{Z}$, we therefore see that p^2 has the desired property, as claimed.

OC73. Find all non-decreasing sequences $a_1, a_2, a_3, \ldots$ of natural numbers such that for each $i, j \in \mathbb{N}$, $i+j$ and $a_i + a_j$ have the same number of divisors. (a non-decreasing sequence is a sequence such that for all $i < j$, we have $a_i \leq a_j$.)
(*Originally question 3 from the 2011 Iran National Olympiad, Day 2.*)

No solution to this problem was received.

OC74. Let H be the orthocenter of an acute triangle ABC with circumcircle Γ. Let P be a point on the arc BC (not containing A) of Γ, and let M be a point on the arc CA (not containing B) of Γ such that H lies on the segment PM. Let K be another point on Γ such that KM is parallel to the Simson line of P with respect to triangle ABC. Let Q be another point on Γ such that $PQ \parallel BC$. Segments BC and KQ intersect at a point J. Prove that $\triangle KJM$ is an isosceles triangle.
(*Originally question 1 from the 2011 China team selection test, Day 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.



Let the perpendicular from P onto BC intersect Γ and BC at points D respectively E, where $D \neq P$. Let s be the Simson line of P. Let F be the intersection of BC and MP, and let the line AH meet Γ at point $G \neq A$. By a well-known property of the Simson line (see [1] § 288), we have $AD \parallel s$. Hence, $AD \parallel KM$ and $\widehat{AK} = \widehat{MD}$. As the quadrilateral $ABGC$ is cyclic and because the sides of $\angle BAC$ are perpendicular to the sides of $\angle BHC$, we have $\angle CGB = 180^\circ - \angle BAC = \angle BHC$. Thus, BC is the perpendicular bisector of GH. In other words, the reflection of the orthocentre in the side of a triangle is a point of the circumcircle; this is, in fact, Theorem 178 of [1].

It follows that $\angle AGK = \angle MPD = \angle FPE = \angle FHG = \angle AGF$. Thus, F, G, and K are collinear. Whence, $\angle JKM = \angle QKM = \angle QPM = \angle JFM$, that is, the points F, J, M, and K are concyclic. Therefore,

$$\begin{aligned}\angle KMJ &= 180^\circ - \angle JKM - \angle MJK = 180^\circ - \angle JFM - \angle MFK \\ &= 180^\circ - \angle JFK = \angle KFB = \angle GFJ = \angle JFH = \angle JFM = \angle JKM.\end{aligned}$$

Consequently, $JK = JM$, and $\triangle KJM$ is isosceles.

References

[1] N. Altshiller-Court, *College Geometry*, 2nd ed., New York, 1952

OC75. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$ be two polynomials with integral coefficients so that $a_n - b_n$ is a prime, $a_nb_0 - a_0b_n \neq 0$, and $a_{n-1} = b_{n-1}$. Suppose that there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.
(*Originally question 3 from the 2011 India Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, Franceand Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille, modified by the editor.

Let $a_n - b_n = p$.

If $n = 1$ then $P(x) - Q(x) = px$ and hence $r = 0 \in \mathbb{Z}$.

Now assume $n \geq 2$. Since $a_{n-1} = b_{n-1}$, we have

$$P(x) - Q(x) = px^n - c_{n-2}x^{n-2} - \cdots - c_0\,,$$

with $c_j = b_j - a_j$, $0 \leq j \leq n-2$. As $P(r) = Q(r) = 0$ we have

$$pr^n = c_{n-2}r^{n-2} + \cdots + c_1r + c_0\,. \tag{1}$$

If $r = 0$ we are done. Otherwise, we have $r = \dfrac{l}{m}$ for some co-prime integers l, m with $l \neq 0$ and $m \geq 1$. Multiplying 1 by m^{n-2} we get

$$\frac{pl^n}{m^2} = c_{n-2}l^{n-2} + c_{n-3}l^{n-3}m + \cdots + c_1lm^{n-3} + c_0m^{n-2}\,.$$

Therefore $\dfrac{pl^n}{m^2} \in \mathbb{Z}$, which implies $m^2 \mid pl^n$. As m, n are co-prime, m^2 and l^n are also co-prime, and hence m^2 must divide p. Since p is prime, it follows that $m^2 = 1$, and hence $m = 1$. This completes the proof.

Note, the condition that $a_nb_0 - a_0b_n \neq 0$ is not needed.

BOOK REVIEWS

John McLoughlin

Excursions in Classical Analysis: Pathways to Advanced Problem Solving and Undergraduate Research by Hongwei Chen
The Mathematical Association of America (Classroom Resource Materials), 2010
ISBN: 978-0-88385-768-7 (print), 978-0-88385-935-3 (electronic), Hardcover/e-book, 315 + xiv pages, US$59.95 (print), US$30.00 (electronic)

Reviewed by ***Robert Dawson***, *Saint Mary's University, Halifax, NS*

There is a common perception among some mathematicians who do not read ***Crux*** (and maybe even a few who do) that contest-style problem solving, while all very well as an alternative to crosswords or a way to keep the undergraduates off the streets, hasn't got a great deal to do with real mathematics. (I suspect that this opinion varies with subject area; I have used my old "Putnam skills" often in my research on discrete geometry, but rarely while working on category theory.)

This book goes a long way to closing the gap. It explores a large number of fascinating topics in classical analysis (rather loosely interpreted, as some sections might better be classified as algebra or even combinatorics). These are illustrated with problems from the Putnam, from the *Monthly*, and other sources; and these problems, in turn, motivate much of the theory. A small selection from the topics covered is listed here: inequalities (including a powerful and little–known elementary proof technique); trig identities via complex numbers; advanced trig identities; telescoping sums (including Gosper's algorithm); powers of Fibonacci numbers; series involving binomial coefficients; and parametric differentiation and integration.

Who are the readers for this book? It is probably inaccessible to most "recreational math" amateurs without some university training in mathematics. A very few high school students — those who are training for the IMO, and have a very advanced background — might benefit from it; but I suspect that much of the most relevant material is already known to their trainers, and passed on when appropriate.

Strong undergraduates training for the Putnam may be able to use this book independently. However, it should be noted that most of what is in here is not necessary to solve Putnam problems, which can usually be solved with comparatively simple tools. The book's real strength is in exhibiting common themes and "power tools" at a rather high level. Probably the natural readership for the book is among advanced undergraduate students, graduate students, and career mathematicians wanting to enhance their skills. These readers will find a rich and rewarding set of techniques, amply illustrated with challenging problems.

The book has solutions to all problems except those taken from the *Monthly* or the Putnam, for which the reader is referred to the usual sources.

FOCUS ON ...

No. 7

Michel Bataille

Decomposition into Partial Fractions

Introduction

The decomposition of a rational fraction into partial fractions is primarily taught because of its applications to integration. The rather heavy calculations involved often lead to some resentment against this technique. The few examples selected here only require a small amount of calculations and might, if need be, reconcile one to this useful algebraic tool.

Using the polynomial part

Let F be any field and $A(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$, $D(x)$ are coprime elements of $F[x]$. The polynomial part of $A(x)$ is the quotient in the division of $N(x)$ by $D(x)$: if $N(x) = D(x)Q(x) + R(x)$ with $\text{degree}(R(x)) < \text{degree}(D(x))$, this polynomial part, which we will denote $pp(A(x))$, is just $Q(x)$. Then we can write $A(x) = pp(A(x)) + A_1(x)$ where $A_1(x) = \dfrac{R(x)}{D(x)}$. We have achieved the very first step of the decomposition of the fraction $A(x)$; the rest follows from the general theorems which apply to fractions with a numerator of degree less than the denominator (to review these theorems, see for example [1]). Of course, $pp(A(x)) = 0$ if $\text{degree}(N(x)) < \text{degree}(D(x))$, and $pp(A(x)) = 1$ if $N(x)$, $D(x)$ are both monic polynomials of the same positive degree. Another useful example is $A(x) = \dfrac{x^m}{x-w}$ where m is a positive integer and w a nonzero element of F: writing $x^m = x^m - w^m + w^m$, we readily see that

$$pp(A(x)) = x^{m-1} + wx^{m-2} + \cdots + w^{m-2}x + w^{m-1}.$$

An important, not often explicated property, is the linearity of the function pp:

$$pp(\alpha A(x) + \beta B(x)) = \alpha pp(A(x)) + \beta pp(B(x))$$

whenever α, β are in F and $A(x), B(x)$ are rational fractions on F. As an application, consider n distinct elements $w_1, w_2, \ldots, w_n$ of F and the associated sums

$$S_n(m) = \sum_{i=1}^{n} \frac{w_i^m}{\displaystyle\prod_{k=1, k\neq i}^{n} (w_i - w_k)}$$

where m is a nonnegative integer. We revisit the calculation of these sums that have been introduced in several Crux problems (compare with solution *III* of problem **2487** [2000 : 512] or the solution to **KLAMKIN–10** [2006 : 323]). Let $D(x) = \prod_{i=1}^{n}(x - w_i)$ so that $\prod_{k=1,k\neq i}^{n}(w_i - w_k)$ is just $D'(w_i)$ and

$$\frac{1}{D(x)} = \sum_{i=1}^{n} \frac{1}{D'(w_i)} \cdot \frac{1}{x - w_i} \tag{1}$$

($D'(x)$ is the derivative of $D(x)$). Now, multiply each side of (1) by x^n and equate the polynomial parts to obtain

$$\begin{aligned} 1 &= \sum_{i=1}^{n} \frac{x^{n-1} + w_i x^{n-2} + \cdots + w_i^{n-2}x + w_i^{n-1}}{D'(w_i)} \\ &= x^{n-1}S_n(0) + x^{n-2}S_n(1) + \cdots + xS_n(n-2) + S_n(n-1) \end{aligned}$$

and, in one sweep,

$$\sum_{i=1}^{n} \frac{w_i^m}{\prod\limits_{k=1,k\neq i}^{n}(w_i - w_k)} = 0, \text{ for } m = 0, 1, \ldots, n-2,$$

$$\text{and} \quad \sum_{i=1}^{n} \frac{w_i^{n-1}}{\prod\limits_{k=1,k\neq i}^{n}(w_i - w_k)} = 1. \tag{2}$$

What if $m > n - 1$? Well, the method still applies but becomes more and more involved as m increases. Indeed, as above, multiply (1) by x^m. If we set $m = n + s$ and take (2) into account, the polynomial part of the right-hand side is

$$x^s + S_n(n)x^{s-1} + \cdots + S_n(n+s-2)x + S_n(n+s-1).$$

It remains to compare this to the polynomial part of the left-hand side $\frac{x^{n+s}}{D(x)}$ (directly obtained by division). For example, it is easily checked that

$$S_n(n) = w_1 + w_2 + \cdots + w_n, \quad S_n(n+1) = (w_1 + w_2 + \cdots + w_n)^2 - \sum_{i<j} w_i w_j.$$

[The readers familiar with Faa di Bruno's formula will derive a general expression of $S_n(n + s - 1)$ as a polynomial in the elementary symmetric functions of $w_1, w_2, \ldots, w_n$.]

The decomposition of $\dfrac{1}{x^n-1}$

Here we take $F = \mathbb{C}$, the field of complex numbers. The decomposition of $\dfrac{1}{x^n-1}$ is easy to obtain and deserves to be well-known. Let $\omega = \exp(-2\pi i/n)$. Here,

$$D(x) = x^n - 1 = \prod_{k=0}^{n-1}(x - \omega^{-k}) \qquad \text{and} \qquad D'(\omega^{-k}) = n\omega^{-k(n-1)} = n\omega^k.$$

It follows that

$$\frac{1}{x^n-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\omega^k x - 1}. \tag{3}$$

To see this decomposition at work, consider the following trigonometric identity: Let α, β be real numbers with $\alpha, \beta \notin \pi\mathbb{Z}$ and n a positive integer. Then

$$\sum_{k=0}^{n-1}\left(\cot\left(\frac{\alpha - k\pi}{n}\right) - \cot\left(\frac{\beta - k\pi}{n}\right)\right) = n(\cot\alpha - \cot\beta).$$

Proof. From $\cot(\alpha) = i + \dfrac{2i}{e^{2i\alpha}-1}$ (easily checked), we deduce

$$n(\cot\alpha - \cot\beta) = 2ni\left(\frac{1}{e^{2i\alpha}-1} - \frac{1}{e^{2i\beta}-1}\right).$$

Now, from (3), we have

$$\frac{1}{e^{2i\alpha}-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\omega^k e^{2\alpha i/n}-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{e^{2i(\alpha-k\pi)/n}-1}$$

hence

$$\begin{aligned} n(\cot\alpha - \cot\beta) &= \sum_{k=0}^{n-1}\left(\frac{2i}{e^{2i(\alpha-k\pi)/n}-1} - \frac{2i}{e^{2i(\beta-k\pi)/n}-1}\right) \\ &= \sum_{k=0}^{n-1}\left(\cot\left(\frac{\alpha-k\pi}{n}\right) - \cot\left(\frac{\beta-k\pi}{n}\right)\right). \end{aligned}$$

An unexpected application

Consider the following problem: Let $a_1, a_2, \ldots, a_n$ be n distinct positive real numbers, where $n \geq 2$. For $k = 1, 2, \ldots, n$, let $p_k = \prod_{j\neq k}(a_j - a_k)$. Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{p_k\sqrt{a_k}} > 0$.

Rather surprisingly, a simple and elegant solution follows from the decomposition of a rational fraction. We introduce

$$A(x) = \frac{1}{(x^2+a_1)(x^2+a_2)\cdots(x^2+a_n)},$$

whose decomposition in $\mathbb{R}(x)$ is of the form

$$A(x) = \sum_{k=1}^{n} \frac{\alpha_k x + \beta_k}{x^2 + a_k} \tag{4}$$

for some real numbers α_k, β_k, where $k = 1, 2, \ldots, n$. Multiplying both sides of (4) by $x^2 + a_k$ and taking $x = i\sqrt{a_k}$ in the deduced equality immediately show that $\alpha_k = 0$, $\beta_k = \frac{1}{p_k}$, hence

$$A(x) = \sum_{k=1}^{n} \frac{1}{p_k} \cdot \frac{1}{x^2 + a_k}. \tag{5}$$

We will use (5) to compute an integral (at last!). Since $\displaystyle\int_0^\infty \frac{dx}{x^2+a_k} = \frac{\pi}{2\sqrt{a_k}}$, we see that

$$\int_0^\infty A(x)\,dx = \frac{\pi}{2}\sum_{k=1}^{n} \frac{1}{p_k\sqrt{a_k}}.$$

The desired inequality follows since this integral is positive (recall that $A(x) > 0$ for $x \in [0, \infty)$).

Exercises

Here are some examples that are close to those treated above.

(a) Consider the sums $S_n(m)$ again and suppose $w_i \neq 0$ $(i = 1, 2, \ldots, n)$. Calculate $S_n(-1)$ and $S_n(-2)$ (hint: for the latter, first compute the derivative of $\frac{1}{D(x)}$).

(b) Using the decomposition of $\frac{1}{x^n - 1}$, rework problem **2657** [2001 : 336 ; 2002 : 401]: Prove that

$$\sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k}\right) = \frac{2k}{1 + (-1)^{k+1}\sqrt{2}\sin\theta}.$$

(c) Problem **3140** ([2006 : 238,240; 2007 : 243]) required a proof of the inequality $\displaystyle\prod_{k=1}^{n} a_k^{\frac{1}{p_k}} < 1$ (with the notations of the last paragraph). Find an alternative to Walther Janous's featured proof.

References

[1] Algebra, J. W. Archbold, Pitman, 1964, pp. 134-151.

PROBLEM OF THE MONTH

No. 6

Stéphane Baune

This column is dedicated to the memory of former ***CRUX with MAYHEM*** *Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

A Test of Time

As I began working on this problem, I glanced at my clock. As I finished solving the problem, I glanced again at the clock. To my surprise, the hour and minute hands of the clock had switched places. How long did it take me to solve this problem?

Although solvable with basic algebra, it is always interesting to tackle problems from various angles. In doing so, this has led me to consider the roots of the generalized linear polynomial

$$ax + b\lfloor cx + d\rfloor + e, \tag{1}$$

where $\lfloor\cdot\rfloor$ is the floor function.

In searching the literature, no general formula for the roots of (1) was to be found. Moreover, the Internet revealed but a trail of inquiries relating to (1) but no substantial answers. I was therefore left to work out my own formula, presented below. Based on this general formula, a solution to the puzzle is then given.

General formula

Let $abc \neq 0$. Replacing $cx + d$ by X, dividing by b, rearranging, setting $A = \frac{a}{bc}$ and $E = \frac{ec-ad}{bc}$, leads to solving for X in the equivalent equation

$$AX + \lfloor X\rfloor + E = 0. \tag{2}$$

Let X be a root. Therefore there exists $m \in \mathbb{Z}$ such that $X \in [m, m+1)$. Equation (2) becomes $AX + m + E = 0$ which yields $X = -\frac{E+m}{A}$, meaning that there is at most one root per range $[m, m+1)$. Next, the requirement $m \leq X < m+1$ becomes $m \leq -\frac{E+m}{A} < m+1$ implying that $m \in \left(-\frac{E+A}{1+A}, -\frac{E}{1+A}\right]$. Each m in this range generates a root.

Note 1. If X is an integer root, then (2) becomes $X = -\frac{E}{1+A}$. Since this is a unique number ($A \neq -1$), there can be at most one integer root; other roots will be nonintegers.

Note 2. If $A=-1$, then $X=E+m$ for all $m\in\mathbb{Z}$ provided $m\leq -\frac{E+m}{A}<m+1$, which becomes $0\leq E<1$. If this is not satisfied while $A=-1$ then there are no solutions.

Expressing these results with the original parameters gives:

General formula: *For $a\neq 0$, the roots of $ax+b\lfloor cx+d\rfloor+e$ are $x_m=-\frac{e+bm}{a}$, such that*

- *if $a+bc\neq 0$ then $m\in\left(\frac{ad-ce-a}{a+bc},\frac{ad-ce}{a+bc}\right]$*
- *if $a+bc=0$ then $m\in\mathbb{Z}$ provided that $0\leq d+\frac{e}{b}<1$, without which there are no solutions.*

Cases with $a=0$ lead to trivial solutions : if $bc\neq 0$ then for all $x_0\in\left[-\frac{e}{b},1-\frac{e}{b}\right)$, $x=\frac{x_0-d}{c}$ is a solution provided $\frac{e}{b}\in\mathbb{Z}$. If $bc=0$ then then there are no variables in the equation to solve for.

Solution to the puzzle

Let the clock be gauged from 0 to 1. Note that the time indicated by a clock can be characterized solely by the position of the hour hand. Let s_1 be the initial time and s_2 the final time. In other words, s_1 is the initial position of the hour hand and s_2 the initial position of the minute hand. Similarly, s_2 is the final position of the hour hand and s_1 the final position of the minute hand. Recall that we are working with values between 0 and 1 and not 0 to 12.

When the hour hand points to s_1, the minute hand points to $s_2=12s_1-\lfloor 12s_1\rfloor$. Similarly, when the hour hand points to s_2 the minute hand points to $s_1=12s_2-\lfloor 12s_2\rfloor$. Combining these last two equalities and simplifying gives $143s_1-\lfloor 144s_1\rfloor=0$. From the general formula above, the roots of this equation are $s_1=\frac{n}{143}$ for $n=0,1,\ldots,142$. There are thus 143 distinct beginning times such that a perfect switch occurs (s_1 belonging to the first turn, "midnight to noon"). For instance, with $n=1$ we get $s_1=\frac{1}{143}$ and $s_2=\frac{12}{143}$. Converting to HH:MM:SS gives beginning time $s_1=$ 0:05:02 and end time $s_2=$ 1:00:25. This is a 0:55:23 time interval (times rounded to the nearest second).

But we are not interested in when these occur but the time interval they generate, as per requested by the puzzle. The time interval is $\Delta s=s_2-s_1=(12s_1-\lfloor 12s_1\rfloor)-s_1$ which, after simplification equals $\frac{n}{13}-\left\lfloor\frac{n}{13}+\frac{n}{143}\right\rfloor$, $n=0,1,\ldots,142$. Enumerating, or doing a little algebra, we find that this has a 13 cycle giving the 13 answers $\Delta s=\frac{(0,1,2,3,4,5,6,7,8,9,10,11,-1)}{13}$. For instance, $\Delta s=\frac{1}{13}$ is the 0:55:23 time interval and $\Delta s=\frac{11}{13}$ is a 10:09:14 time interval. The $\Delta s=\frac{-1}{13}$ needs to be addressed. It means that $s_2<s_1$ or equivalently that the minute hand is before the hour hand during "midnight to noon". Therefore to switch, the hour hand must pass over "noon" and is the only answer that has this property. In other words, this particular switch can not be done in the same half–day. Alternatively, we may take s_2 as the start time and s_1 as the end time, giving $\Delta s=\frac{12}{13}=$ 11:04:37 switch time. This is the switched $\frac{1}{13}=$ 0:55:23; the $\Delta s=\frac{1}{13}$ is simply

the "missing" $\frac{12}{13}$. This switch starts at 1:00:25, the hour hand crossing over the "12" and ends at 0:05:02, a time interval of 11:04:37.

There are thus a class of 13 answers to the riddle: $\Delta s = \frac{m}{13}$, $m = 0, 1, \ldots, 12$. Converting gives the 13 answers: $\Delta t = \frac{12m}{13}$ hours, $m = 0, 1, \ldots, 12$. Other solutions are found with $m > 12$ and represent adding multiples of 12 h to the previous answers. The table below displays the first thirteen answers.

Note. Only the first thirteen time intervals have been explicated. These are not the times at which these intervals occur. During a day, each time interval can occur several times. For instance, the time interval of $\frac{1}{13}$ = 0:55:23 can begin at s_1 =0:05:02 AM and end at s_2 = 1:00:25 AM, or begin at 0:05:02 PM and end at 1:00:25 PM, or begin at 1:10:29 AM and end at 2:05:52 AM, and there are many more. All these values may be found by applying $s_1 = \frac{n}{143}$ to two turns (a whole day).

Time to switch

m	**hours**	**hh:mm:ss**
0	0	0
1	0.9231	0:55:23
2	1.8462	1:50:46
3	2.7692	2:46:09
4	3.6923	3:41:32
5	4.6154	4:36:55
6	5.5385	5:32:18
7	6.4615	6:27:43
8	7.3846	7:23:05
9	8.3077	8:18:28
10	9.2308	9:13:51
11	10.1538	10:09:14
12	11.0769	11:04:37

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Electronic submissions are preferable, with each solution contained in a separate file. Solution files should be named using the convention LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions to the editor at* `crux-editors@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Original problems are particularly sought, but other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 September 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3841. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Let ABC be a triangle with $a = BC$, $b = CA$, $c = AB$, $\angle A \le \angle B \le \angle C$ and $a^2 + b^2 = 2Rc$, where R is the circumradius of ABC. Determine the measure of $\angle C$.

3842. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

Let $d(n)$ be the number of positive divisors of n. For given positive integers a and b, there exist infinitely many positive integers m such that $d(a^m) \ge d(b^m)$; there also exist infinitely many positive integers n such that $d(a^n) \le d(b^n)$. Prove that $d(a^k) = d(b^k)$ for any positive integer k.

3843. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b be distinct real numbers such that

$$a^4 + b^4 - 3(a^2 + b^2) + 8 \leq 2(a + b)(2 - ab).$$

Find the value of the expression

$$A = (ab)^n + (ab + 1)^n + (ab + 2)^n,$$

where n is a positive integer.

3844. *Proposed by Michel Bataille, Rouen, France.*

Find the intersection of the surface with equation

$$(x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2 = (x + y)(y + z)(z + x)$$

with the plane $x + y + z = 2$.

3845. *Proposed by Dao Thanh Oai, Kien Xuong, Thai Binh, Viet Nam.*

Let the six points $A_1, A_2, \ldots, A_6$ lie in that order on a circle, and the six points $B_1, B_2, \ldots, B_6$ lie in that order on another circle. If the quadruples $A_i, A_{i+1}, B_{i+1}, B_i$ lie on circles with centres C_i for $i = 1, 2, \ldots, 5$, then prove that A_6, A_1, B_1, B_6 must also lie on a circle. Furthermore, if C_6 is the centre of the new circle, then prove that lines C_1C_4, C_2C_5, and C_3C_6 are concurrent.

3846. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r be a positive real number. Prove that the inequality

$$\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} + \frac{1}{1 + c + c^2} \geq \frac{3}{1 + r + r^2}$$

holds for any positive a, b, c such that $abc = r^3$ if and only if $r \geq 1$.

3847. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

Prove that there are no distinct positive integers a, b, c and nonnegative integer k that satisfy the conditions

$$a^{b+k} \mid b^{a+k}, \quad b^{c+k} \mid c^{b+k}, \quad c^{a+k} \mid a^{c+k}.$$

3848. *Proposed by Rudolf Fritsch, University of Munich, Munich, Germany.*

We define an altitude of the plane $(2n + 1)$-gon $A_0A_1 \ldots A_{2n}$ to be the line through vertex A_i perpendicular to the *opposite side* $A_{i-n}A_{i+n}$ (where indices are reduced modulo $2n+1$). Prove that if $2n$ of the altitudes are concurrent, then the remaining altitude passes through the point of concurrence.

3849. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

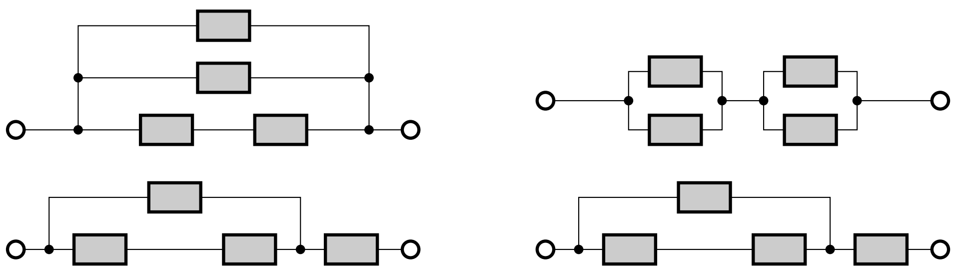
Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients having all its zeros in the disc $\mathcal{C} = \{z \in \mathbb{C} : |z| \leq \sqrt{6}\}$. Show that

$$|A(3z)| \geq \left(\frac{2}{3}\right)^{n/2} |A(2z)|$$

for any complex number z with $|z| = 1$.

3850. *Proposed by Lee Sallows, Nijmegen, The Netherlands; Brian Trial, Ferndale, MI, USA; and Stan Wagon, Macalester College, St. Paul, MN, USA.*

Each of the four networks shown uses the same four distinct integer-valued resistors a, b, c, d, and the total resistances of the networks themselves are again a, b, c, d. Find values of a, b, c, d that work.



[*Note: The proposers solved the problem using a computer. We will accept computer aided solutions, but would be interested to see if the problem can be solved by hand.*]

3805. *Correction. Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle et I le centre de son cercle inscrit. Soit A' sur le rayon IA au-delà de A de sorte que $A'A = BC$. Soit B' et C' définis de manière analogue, de sorte que $B'B = CA$ et $C'C = AB$. Montrer que

$$\frac{[A'B'C']}{[ABC]} \geq (1 + \sqrt{3})^2,$$

où $[\cdot]$ désigne la surface.

3841. *Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Soit ABC un triangle avec $a = BC$, $b = CA$, $c = AB$, $\angle A \leq \angle B \leq \angle C$ et $a^2 + b^2 = 2Rc$, où R est le rayon du cercle circonscrit de ABC. Trouver la mesure de $\angle C$.

3842. *Proposé par Jung In Lee, École Secondaire Scientifique de Séoul, Séoul, République de Corée.*

Soit $d(n)$ le nombre de diviseurs positifs de n. Pour deux entiers positifs donnés a et b, il existe une infinité d'entiers positifs m tels que $d(a^m) \geq d(b^m)$; et il existe aussi une infinité d'entiers positifs n tels que $d(a^n) \leq d(b^n)$. Montrer que $d(a^k) = d(b^k)$ pour tout entier positif k.

3843. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b deux nombres réels distincts tels que

$$a^4 + b^4 - 3(a^2 + b^2) + 8 \leq 2(a + b)(2 - ab).$$

Trouver la valeur de l'expression

$$A = (ab)^n + (ab + 1)^n + (ab + 2)^n,$$

où n est un entier positif.

3844. *Proposé par Michel Bataille, Rouen, France.*

Trouver l'intersection de la surface d'équation

$$(x^2 + y^2)^2 + (y^2 + z^2)^2 + (z^2 + x^2)^2 = (x + y)(y + z)(z + x)$$

avec le plan $x + y + z = 2$.

3845. *Proposé par Dao Thanh Oai, Kien Xuong, Thai Binh, Viet Nam.*

Soit $A_1, A_2, \ldots, A_6$ six points situés dans cet ordre sur un cercle et six autres points $B_1, B_2, \ldots, B_6$ situés dans cet ordre sur un autre cercle. Si les quadruplets $A_i, A_{i+1}, B_{i+1}, B_i$ sont situés sur des cercles de centres C_i pour $i = 1, 2, \ldots, 5$, montrer qu'alors A_6, A_1, B_1, B_6 doivent aussi être situés sur un cercle. De plus, si C_6 est le centre du nouveau cercle, montrer qu'alors les droites C_1C_4, C_2C_5 et C_3C_6 sont concourantes.

3846. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit r un nombre réel positif. Montrer que l'inégalité

$$\frac{1}{1 + a + a^2} + \frac{1}{1 + b + b^2} + \frac{1}{1 + c + c^2} \geq \frac{3}{1 + r + r^2}$$

est satisfaite pour tout triplet positif a, b, c tel que $abc = r^3$ si et seulement si $r \geq 1$.

3847. *Proposé par Jung In Lee, École Secondaire Scientifique de Séoul, Séoul, République de Corée.*

Montrer qu'il n'existe pas trois entiers positifs distincts a, b, c et un entier non négatif k satisfaisant les conditions

$$a^{b+k} \mid b^{a+k}, \quad b^{c+k} \mid c^{b+k}, \quad c^{a+k} \mid a^{c+k}.$$

3848. *Proposé par Rudolf Fritsch, Université de Munich, Munich, Allemagne.*

On définit une hauteur dans un (2n+1)-gone plan $A_0A_1\ldots A_{2n}$ comme la droite passant par le sommet A_i et perpendiculaire au *côté opposé* $A_{i-n}A_{i+n}$ (où les indices sont réduits modulo $2n+1$). Montrer que si $2n$ des hauteurs sont concourantes, alors la hauteur restante passe par leur point d'intersection.

3849. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

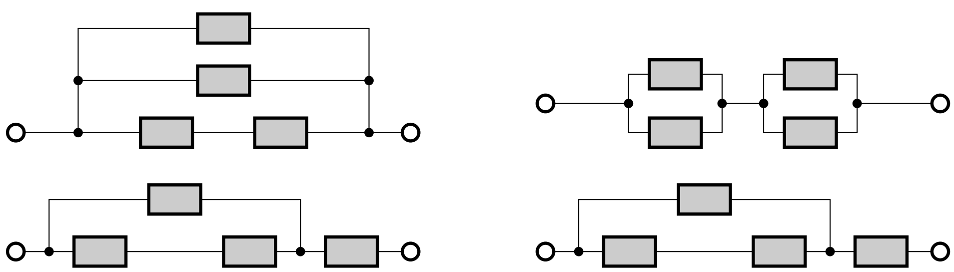
Soit $A(z)=\sum_{k=0}^{n} a_k z^k$ un polynôme de degré n à coefficients complexes ayant tous ses zéros dans le disque $\mathcal{C}=\{z\in\mathbb{C}:|z|\le\sqrt{6}\}$. Montrer que

$$|A(3z)|\ge\left(\frac{2}{3}\right)^{n/2}|A(2z)|$$

pour tout nombre complexe z avec $|z|=1$.

3850. *Proposé par Lee Sallows, Nijmegen, The Netherlands ; Brian Trial, Ferndale, MI, USA ; et Stan Wagon, Macalester College, St. Paul, MN, É-U.*

Chacun des quatre réseaux ci-dessous utilise les quatre mêmes résistances à valeurs entières a, b, c, d, et les valeurs totales des résistances des réseaux eux-mêmes sont aussi a, b, c, d. Trouver les valeurs de a, b, c, d adéquates.



[*Note : Les proposeurs ont résolu le problème en utilisant un ordinateur. Nous allons accepter des solutions assistées par ordinateur, mais nous aimerions savoir si le problème peut se résoudre indépendamment.*]

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3587★. [2010 : 461,463; 2011 : 480] *Proposed by Ignotus, Colegio Manablanca, Facatativá, Colombia.*

Define the *prime graph* of a set of positive integers as the graph obtained by letting the numbers be the vertices, two of which are joined by an edge if and only if their sum is prime.

(a) Prove that given any tree T on n vertices, there is a set of positive integers whose prime graph is isomorphic to T.

(b) For each positive integer n, determine $t(n)$, the smallest number such that for any tree T on n vertices, there is a set of n positive integers each not greater than $t(n)$ whose prime graph is isomorphic to T.

Solutions to part a) by Oliver Geupel, Brühl, NRW, Germany.

We proceed by mathematical induction on n. It is easy to verify the result holds when $n \in \{1, 2\}$.

Assume the result holds for $n = k \geq 2$, and consider a tree T with $k+1$ vertices. The tree T has a vertex v of degree 1. By the induction hypothesis, the tree $T' = T - \{v\}$ is isomorphic to the prime graph of a set $A = \{a, a_1, a_2, \ldots, a_{k-1}\}$ of positive integers. We label the vertices of T', each with its corresponding number. Let a be the label of the vertex adjacent to v in T. We must find an appropriate label for v, that is a positive integer $b \notin A$ such that $a + b$ is prime, while $a_i + b$ is composite for each $i \in \{i, \ldots, k-1\}$.

Let p_i be distinct primes such that $\gcd(a - a_1, p_i) = 1$ for $i \in \{1, \ldots, k-1\}$. By the Chinese Remainder Theorem, the simultaneous congruences

$$x \equiv a - a_i \pmod{p_i} \qquad (i \in \{1, \ldots, k-1\}$$

are equivalent to the single congruence

$$x \equiv c \pmod{p_1 p_2 \cdots p_{k-1}}\,.$$

By Dirichlet's Theorem, infinitely many primes p satisfy this congruence. Choose such a prime $q > 2a + \max\{p_1, p_2, \ldots p_{k-1}\} + \max\{a_1, a_2, \ldots a_{k-1}\}$ and let $b = p - a$. Since b is greater than each element of A, b is not a member of A. The number $a + b = p$ is prime. Also, $a_i + b = p - (a - a_i) \equiv 0 \pmod{p_1}$ and $a_i + b > p_i$, so the numbers $a_i + b$ are composite for $i \in \{1, \ldots, k-1\}$, completing the induction.

Part a) was also solved by MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA. They established, with the aid of a computer, that $t(1) = 1$, $t(2) = 2$, $t(3) = 3$, $t(4) = 5$, $t(5) = 9$, $t(6) = 13$, *and* $t(7) = 16$.

3741. [2012 : 194, 196] *Proposed by Péter Ivády, Budapest, Hungary.*

Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{ax}{a+x^2} < \sin x < \frac{bx}{b+x^2},$$

hold for all $0 < x < \frac{\pi}{2}$.

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Kee-Wai Lau, Hong Kong, China.

We show that $a = \dfrac{\pi^2}{2(\pi-2)}$ and $b = 6$.

By simple computations, it is easy to show that the given inequalities are equivalent to

$$a < \frac{x^2 \sin x}{x - \sin x} < b. \tag{1}$$

To find the largest value of a and the smallest value of b for which (1) holds for $0 < x < \dfrac{\pi}{2}$, we let

$$f(x) = \frac{x^2 \sin x}{x - \sin x}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Then

$$\begin{aligned} f'(x) &= \frac{1}{(x-\sin x)^2}\Big((x-\sin x)(2x\sin x + x^2\cos x) - (x^2 \sin x)(1-\cos x)\Big) \\ &= \frac{1}{(x-\sin x)^2}(x^2\sin x + x^3\cos x - 2x\sin^2 x) = \frac{xg(x)}{(x-\sin x)^2} \end{aligned} \tag{2}$$

where $g(x) = x\sin x + x^2\cos x - 2\sin^2 x$.

Since

$$0 < x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$$

and

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

we have

$$\begin{aligned} g(x) &< x\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) - 2\left(x - \frac{x^3}{6}\right)^2 \\ &= \left(2x^2 - \frac{2}{3}x^4 + \frac{1}{20}x^6\right) - \left(2x^2 - \frac{2}{3}x^4 + \frac{1}{18}x^6\right) = -\frac{1}{180}x^6 < 0. \end{aligned} \tag{3}$$

From (2) and (3) we have $f'(x) < 0$ so $f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ which implies

$$\lim_{x \to \frac{\pi}{2}^-} f(x) < f(x) < \lim_{x \to 0^+} f(x). \tag{4}$$

Since

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \frac{\left(\frac{\pi}{2}\right)^2}{\frac{\pi}{2}-1} = \frac{\pi^2}{2(\pi-2)}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x\sin x}{1-\frac{\sin x}{x}} = \lim_{x\to 0^+} \frac{x\left(x-\frac{x^3}{3!}+\cdots\right)}{1-\left(1-\frac{x^2}{3!}+\cdots\right)}$$
$$= \lim_{x\to 0^+} \frac{x^2+o(x^3)}{\frac{x^2}{6}+o(x^3)} = 6,$$

we have from (4) that $\dfrac{\pi^2}{2(\pi-2)} < f(x) < 6$ which completes the proof.

Also solved by JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; and the proposer. There was also an incorrect solution.

3742. [2012 : 194, 196] *Proposed by Michel Bataille, Rouen, France.*

In a scalene triangle ABC, let K, L, M be the feet of the altitudes from A, B, C, and P, Q, R be the midpoints of BC, CA, AB, respectively. Let LM and QR intersect at X, MK and RP at Y, KL and PQ at Z. Show that AX, BY, CZ are parallel.

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain.

Since $BMLC$ is a cyclic quadrilateral, $\angle MLA = \angle ABC = \angle ARQ$. Therefore $LQRM$ is cyclic and $XQ \cdot XR = XL \cdot XM$.

Let Γ be the circumcircle of ARQ. Since ABC maps to ARQ by a dilatation with factor $\frac{1}{2}$, its centre is the midpoint S of the segment AO, where O is the circumcentre of triangle ABC. Let AX intersect this circle at $U \neq A$. Then

$$XQ \cdot XR = XU \cdot XA.$$

Let Δ be the circle whose diameter is AH. Since AH subtends right angles at L and M, this circle is the circumcircle of ALM with diameter AH. Accordingly, its centre is the midpoint T of AH. Let AX intersect this circle at $V \neq A$. Then

$$XL \cdot XM = XV \cdot XA.$$

Since $XU \cdot XA = XQ \cdot XR = XL \cdot XM = XV \cdot XA$, $U = V$. Thus, AU is a chord of both Γ and Δ, so its right bisector contains both the centres S and T. But ST is the image of OH under a dilatation with centre A, so $ST \| OH$. It follows that $AX \perp OH$. Similarly it can be shown that both BY and CZ are perpendicular to OH, from which the desired result follows.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

We add the hypothesis that ABC is not a right triangle. Consider position vectors with the origin at point A. Then

$$\mathbf{K} = \frac{(\tan B)\mathbf{B} + (\tan C)\mathbf{C}}{\tan B + \tan C}, \quad \mathbf{M} = \frac{(\tan B)\mathbf{B}}{\tan A + \tan B}, \quad \mathbf{P} = \frac{\mathbf{B} + \mathbf{C}}{2}, \quad \mathbf{R} = \frac{\mathbf{B}}{2}.$$

The point Y is the intersection of the lines RP and MK. Hence, there are real numbers s and t for which

$$\mathbf{Y} = s\mathbf{P} + (1-s)\mathbf{R} = \frac{1}{2}\mathbf{B} + \frac{s}{2}\mathbf{C},$$

$$\mathbf{Y} = t\mathbf{K} + (1-t)\mathbf{M} = \left(\frac{t\tan B}{\tan B + \tan C} + \frac{(1-t)\tan B}{\tan A + \tan B}\right)\mathbf{B} + \frac{(t\tan C)\mathbf{C}}{\tan B + \tan C}.$$

Comparing the coefficients yields that

$$t = \frac{(\tan A - \tan B)(\tan B + \tan C)}{2\tan B(\tan A - \tan C)}, \quad s = \frac{\tan C}{\tan B} \cdot \frac{\tan A - \tan B}{\tan A - \tan C}.$$

Therefore,

$$\mathbf{Y} - \mathbf{B} = -\frac{1}{2}\mathbf{B} + \frac{1}{2} \cdot \frac{\tan C}{\tan B} \cdot \frac{\tan A - \tan B}{\tan A - \tan C}\mathbf{C}.$$

Similarly, we find that

$$\mathbf{Z} - \mathbf{C} = -\frac{1}{2}\mathbf{C} + \frac{1}{2} \cdot \frac{\tan B}{\tan C} \cdot \frac{\tan A - \tan C}{\tan A - \tan B}\mathbf{B}.$$

Consequently,

$$\mathbf{Z} - \mathbf{C} = -\frac{\tan B}{\tan C} \cdot \frac{\tan A - \tan C}{\tan A - \tan B}(\mathbf{Y} - \mathbf{B}).$$

Thus, BY and CZ are parallel. Analogously, we obtain that CZ and AX are parallel. This completes the proof.

III. Solution by the proposer.

We show that AX is perpendicular to the Euler line through the circumcentre O and orthocentre H. A similar argument applies to BY and CZ, so that AX, BY and CZ are parallel.

Let I denote the inversion with centre A that interchanges L and Q. Since $AM \cdot AR = AQ \cdot AL$, the inversion interchanges M and R. But L, Q, M, R all lie on the nine-point circle $\mathbf{N}$, so it follows that $I(\mathbf{N}) = \mathbf{N}$.

The circle with diameter AH contains the points L and M, and is carried by the inversion I to a line through $Q = I(L)$ and $R = I(M)$. The point $J = I(H)$ is therefore on the intersection of AH and QR.

Since $AH \perp BC$ and $BC \| QR$, it follows that $AJ \perp QR$ and the circle $\mathbf{\Gamma}$ with diameter AX passes through J. The points A and X are conjugate with respect to $\mathbf{N}$, so that $\mathbf{\Gamma}$ and $\mathbf{N}$ are orthogonal. Thus $I(\mathbf{\Gamma})$ is a diameter of $I(\mathbf{N}) = \mathbf{N}$ that contains $I(J) = H$. Since the centre of $\mathbf{N}$ lies on OH, $I(\mathbf{\Gamma}) = OH$. Since AX is orthogonal to $\mathbf{\Gamma}$ and is carried to itself by I, $AX \perp OH$ and the desired result follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; EDMUND SWYLAN, Riga, Latvia; and PETER Y. WOO, Biola University, La Mirada, CA, USA. The first and third solutions may need slight adaptation depending on the configuration.

3743. [2012 : 194, 196] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Two equal circles are tangent to the parabola $y = x^2$ at the same point. One of the circles is also tangent to the x–axis, while the other is tangent to the y– axis. Find the radius of the circles. *This problem was inspired by problem* **3732** *[2012 : 149, 151].*

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{P}$ be the given parabola and let $T(u, u^2)$ be a point of $\mathcal{P}$. The equal circles with centres A and B, radius $r > 0$, and touching $\mathcal{P}$ at T are suitable if and only if the following three conditions hold:

(1) $A(2u - r, r),\ B(r, 2u^2 - r)$

(T is the midpoint of AB, and the circles are tangent to the x-axis and y-axis, respectively.),

(2) $(r - u) + 2u(u^2 - r) = 0$

(AB is perpendicular to the tangent to $\mathcal{P}$ at T; that is, AB has slope $-\frac{1}{2u}$),

(3) $(r - u)^2 + (u^2 - r)^2 = r^2 \quad (AT = r)$.

We will assume that $u \geq 0$, even $u > 0$ (because of (3)), keeping in mind that for any suitable pair of circles, a second suitable pair is obtained by reflection in the y-axis.

From (3), for some real θ,

$$r - u = r\cos\theta \quad \text{and} \quad r - u^2 = r\sin\theta.$$

Note that $\cos\theta \neq \pm 1$ and $\sin\theta \neq 0$ (otherwise $u^2 = r$ and $r - u = \pm r$, contradicting (2)) and that $\cot\theta = 2u$ (from (2)).

Since $u = r(1 - \cos\theta),\ u^2 = r(1 - \sin\theta)$, we have $\cot\theta = \dfrac{2u^2}{u} = \dfrac{2(1 - \sin\theta)}{(1 - \cos\theta)}$; that is,

$$2\sin\theta = 2 + \cos\theta - 3\cos^2\theta.$$

Squaring both sides, we readily see that $(\cos\theta - 1)(9\cos^2\theta + 3\cos\theta - 4) = 0$, whence there are two possibilities for $\cos\theta$, namely

$$-\frac{\sqrt{17}+1}{6}, \quad \text{and} \quad \frac{\sqrt{17}-1}{6}.$$

Conversely, if $\cos\theta$ takes one of these values and $\sin\theta\cos\theta > 0$, then setting $u = \dfrac{\cot\theta}{2}$ and $r = \dfrac{\cos\theta}{2\sin\theta(1 - \cos\theta)}$, we can easily reverse the calculations and obtain (2) and (3). This yields

- if $\cos\theta = -\frac{1+\sqrt{17}}{6}$:

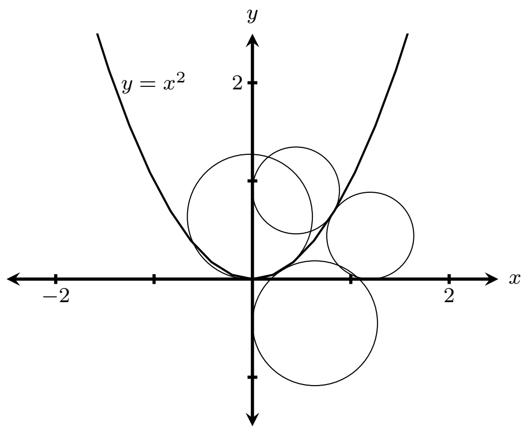
$$\sin\theta = \frac{1-\sqrt{17}}{6}, \quad u = \frac{9+\sqrt{17}}{16} \approx .8202, \quad r = \frac{69-3\sqrt{17}}{128} \approx .4424;$$

- if $\cos\theta = \frac{\sqrt{17}-1}{6}$:

$$\sin\theta = \frac{\sqrt{17}+1}{6}, \quad u = \frac{9-\sqrt{17}}{16} \approx .3048, \quad r = \frac{69+3\sqrt{17}}{128} \approx .6357.$$

In conclusion, the radius of suitable circles is

$$\text{either} \qquad \frac{69-3\sqrt{17}}{128} \qquad \text{or} \qquad \frac{69+3\sqrt{17}}{128}.$$



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incorrect submissions.

Only Bataille, Stallion, and Stoënescu solved the problem as it was stated. Everybody else did the problem that the proposer seems to have intended, namely to find the common radius when the outer circle is tangent to the x-axis and the inner circle to the y-axis, thereby rejecting the second solution as extraneous. This second solution reminds us that a circle which is tangent to a parabola always *intersects it in two further points, although those two points might be imaginary or coincident. The accompanying figure shows the two solutions that are tangent to the parabola at points in the first quadrant; note that the circle of the second solution that is tangent to the x-axis touches it slightly to the left of the origin (where $x \approx -.0261$). The four circles that are tangent to the parabola in the second quadrant are mirror images in the y-axis of the four that are shown.*

Sands observed that the radical $\sqrt{17}$ mysteriously pops up also in problem 3732 (concerning the contact point of the unit circle that is tangent to $y = x^2$) that inspired his problem. Might $\sqrt{17}$ one day replace the golden section as Nature's favorite irrational?

3744. [2012 : 194, 196] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers with sum 4. Prove that

$$\frac{a^8+b^8}{(a^2+b^2)^2}+\frac{b^8+c^8}{(b^2+c^2)^2}+\frac{c^8+a^8}{(c^2+a^2)^2}+abc \geq a^3+b^3+c^3.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since $a+b+c=4$, the given inequality is equivalent to

$$\frac{4(a^8+b^8)}{(a^2+b^2)^2}+\frac{4(b^8+c^8)}{(b^2+c^2)^2}+\frac{4(c^8+a^8)}{(c^2+a^2)^2}+abc(a+b+c) \geq (a+b+c)(a^3+b^3+c^3). \quad (1)$$

Using the trivial inequality $2(x^2+y^2) \geq (x+y)^2$ twice, we have for $x^2+y^2 \neq 0$,

$$\frac{4(x^8+y^8)}{(x^2+y^2)^2} \geq \frac{2(x^4+y^4)^2}{(x^2+y^2)^2} \geq \frac{2(x^4+y^4)^2}{2(x^4+y^4)} = x^4+y^4.$$

Therefore,

$$\frac{4(a^8+b^8)}{(a^2+b^2)^2}+\frac{4(b^8+c^8)}{(b^2+c^2)^2}+\frac{4(c^8+a^8)}{(c^2+a^2)^2} \geq 2(a^4+b^4+c^4). \quad (2)$$

Furthermore, we have, by Schur's Inequality

$$\begin{aligned} &2(a^4+b^4+c^4)+abc(a+b+c)-(a+b+c)(a^3+b^3+c^3) \\ &\quad = a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b) \geq 0 \end{aligned}$$

so

$$2(a^4+b^4+c^4)+abc(a+b+c) \geq (a+b+c)(a^3+b^3+c^3). \quad (3)$$

Combining (2) and (3) we obtain (1) and the proof is complete.

*Also solved by *AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; *ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; *MICHEL BATAILLE, Rouen, France; *CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; *THANOS MAGKOS, Thessaloniki, Greece; *SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; *PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; *TITU ZVONARU, Comăneşti, Romania; and *the proposer. (A * indicates that the solution given by the solver also employed Schur's Inequality and is similar to the one featured above.)*

Cao, Geupel and Zvonaru pointed out that equality holds if and only if $a=b=c=4/3$. Magkos remarked that the given inequality actually holds for all real a, b, c as long as no two of them are simultaneously equal to zero. As usual, Wagon's solution is based on an argument using Mathematica to show that there are no counterexamples to the negation of the given inequality. He also claimed, without proof, that the inequality holds true when the sum $a+b+c=4$ is replaced by 5, 6, 7, 100, or 401/100, and that in general, the result might even be true for all a, b, and c with $a+b+c \geq 4$.

3745. [2012 : 195, 196] *Proposed by Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

In the square $ABCD$ the semicircle with diameter AD intersects the quarter circle with centre C and radius CD in the point P. Show that $PB = \sqrt{2}AP$.

I. Solution by Mihaï-Ioan Stoënescu, Bischwiller, France.

Sans réduire la généralité du problème, soit 2 la longueur du côté du carré $ABCD$. On considère le repère d'origine D et d'axes DC et DA. Ainsi on a $D(0,0), A(0,2), B(2,2), C(2,0), E(0,1)$ et finalement $P(a,b)$. En utilisant la formule pour la distance entre deux points, on a

$$a^2 + (b-1)^2 = 1, \tag{1}$$

car $PE^2 = 1$, et

$$(a-2)^2 + b^2 = 4, \tag{2}$$

car $PC^2 = 4$. Alors

$$\frac{PB^2}{PA^2} = \frac{(a-2)^2 + (b-2)^2}{a^2 + (b-2)^2} = \frac{4 - b^2 + (b-2)^2}{1 - (b-1)^2 + (b-2)^2},$$

en utilisant (1) et (2). En développant, on tire que $\dfrac{PB^2}{PA^2} = \dfrac{8-4b}{4-2b} = 2$. Ainsi $\dfrac{PB}{PA} = \sqrt{2}$, C.Q.F.D.

II. Composite of similar solutions by Dimitrios Koukakis, Kilkis, Greece, and by the proposer.

We know that for any point P in the plane of a rectangle $ABCD$

$$PA^2 + PC^2 = PB^2 + PD^2.$$

[It is easier to prove the equality than to find a reference: Simply apply the formula for the length of a median in terms of the sides to triangles PAC and PBD, while noting that these triangles share the median from P and have equal bases $AC = BD$.] Because triangle APD is inscribed in a semicircle, it has a right angle at P so that

$$PA^2 + PD^2 = AD^2.$$

Moreover, we are given $PC = DC = AD$. Plugging $PC^2 = AD^2 = PA^2 + PD^2$ into the first equation, we get

$$PA^2 + (PA^2 + PD^2) = PB^2 + PD^2,$$

which reduces $2PA^2 = PB^2$, or $PB = \sqrt{2}AP$, as desired.

III. Composite of similar solutions by Michel Bataille, Rouen, France; Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy; Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain; and Itachi Uchiha, Hong Kong, China.

Because the angle between a tangent to a circle and a chord through the point of contact equals the angle in the alternate segment, three applications of the alternate segment theorem tells us that $\angle PAB = \angle PDA = \angle PBD$ and $\angle PBA = \angle PDB$. Hence $\Delta APB \sim \Delta BPD$ (since corresponding angles are equal), so that

$$\frac{PB}{PA} = \frac{BD}{AB} = \sqrt{2},$$

and the result follows.

IV. Composite of similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by the AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

If the square has been labeled counterclockwise, a clockwise rotation through 90° about A takes D to B and P to a point P' outside the square. Then $AP = AP'$ and $\angle PAP' = 90^\circ$. Since $\Delta AP'B$ is the image of the right triangle ΔAPD (inscribed in the semicircle), $AP'B = 90^\circ$. Thus,

$$\angle PP'B = 90^\circ - 45^\circ = 45^\circ. \tag{3}$$

Because the angle between a tangent to a circle and a chord through the point of contact equals half the angle at the center that is subtended by the chord, $\angle PBA = \frac{1}{2}\angle PCB$ and $\angle ABP' = \angle ADP = \frac{1}{2}\angle DCP$. Therefore,

$$\angle PBP' = \angle PBA + \angle ABP' = \frac{1}{2}(\angle DCP + \angle PCB) = \frac{1}{2}\angle DCB = 45^\circ. \tag{4}$$

From equations (3) and (4) $\Delta PP'B$ is isosceles with $PB = PP'$. By the Pythagorean Theorem applied to $\Delta AP'P$,

$$PB^2 = PP'^2 = AP^2 + AP'^2 = 2AP^2;$$

consequently, $PB = \sqrt{2}AP$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GREG COOK, Student, Angelo State University, San Angelo, TX; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain (3 further solutions); CHASEN GRADY SMITH, Georgia Southern University, Statesboro, GA, USA; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comăneşti, Romania.

The majority of the submissions used coordinates much like the first featured solution.

3746. [2012 : 195, 196] *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let $Q(n)$ denote the sum of the digits of the positive integer n. Prove that there are infinitely many positive integers n such that

$$Q(n) + Q(n^2) + Q(n^3) = [Q(n)]^2.$$

This is an extension of problem **3506** *[2010 : 45, 47; 2011 : 57, 58].*

Composite of submitted solutions.

Let $n = a \cdot 10^k + b$. Then $n^2 = a^2 \cdot 10^{2k} + 2ab \cdot 10^k + b^2$ and $n^3 = a^3 \cdot 10^{3k} + 3a^2b \cdot 10^{2k} + 3ab^2 \cdot 10^k + b^3$. If k is at least as great as the number of digits in $3a^2b$, $3ab^2$ and b^3, then $Q(n) = Q(a)+Q(b)$, $Q(n^2) = Q(a^2)+Q(2ab)+Q(b^2)$ and $Q(n^3) = Q(a^3) + Q(3a^2b) + Q(3ab^2) + Q(b^3)$.

When $(a, b) = (2, 7), (7, 2), (1, 17), (17, 1), (3, 15), (15, 3)$, we find that $Q(n) = 9$, $Q(n^2) = 27$ and $Q(n^3) = 45$, when k is not less than 3, 3, 4, 4, 5, 5, respectively. Thus, we can obtain infinitely many examples.

In addition, there are individual values of n not comprised in the foregoing lists. The smallest such examples are 207, 414, 702, 1062, 1134, 1161, 1206, 1215, 1233, 1323, 1332, 1341, 1431, 1503, 2007, 2016, 2034, 2070, 2124. Other values are 3204 and 5301.

Multiplying any suitable value of n by a power of 10 provides other examples.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; DIGBY SMITH, Mount Royal University, Calgary, AB; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

The family of solutions provided by either of $(a, b) = (2, 7), (7, 2)$ was given by nine of the solvers. The validity of 207 or 702 was picked up by six solvers. The family provided by either of $(a, b) = (1, 17), (17, 1)$ was discovered by Bailey, Campbell and Diminnie, as well as by Barbara and Wagon. Barbara also identified the cases $(a, b) = (3, 15), (15, 3)$. Hess and Stone and Hawkins gave a list of the smallest values of n. The example 3204 is due to Curtis and 5301 to Hess and Manes.

Four solvers found their infinite family by determining one value of n and then taking its product with powers of 10, providing a solution that is technically correct but perhaps not in the spirit of the problem.

3747. [2012 : 195, 196] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be real numbers with $a + b + c = 0$ and $c \geq 1$. Prove that

$$a^4 + b^4 + c^4 - 3abc \geq \frac{3}{8}.$$

Composite of similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; George Apostolopoulos, Messolonghi, Greece; and Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since $(x+y)^2 \le 2(x^2+y^2)$ for all x and y we have

$$(a+b)^4 \le 4(a^2+b^2)^2 \le 8(a^4+b^4)$$

which together with $ab \le \frac{1}{4}(a+b)^2$ then yield

$$\begin{aligned} a^4+b^4+c^4-3abc &\ge \frac{1}{8}(a+b)^4+c^4-\frac{3}{4}(a+b)^2c \\ &= \frac{1}{8}c^4+c^4-\frac{3}{4}c^3 = \frac{3}{4}c^3\left(\frac{3}{2}c-1\right) \ge \frac{3}{4}\left(\frac{3}{2}-1\right) = \frac{3}{8}. \end{aligned}$$

Clearly, equality holds if and only if $a=b$ and $c=1$, that is, $a=b=-\frac{1}{2}$ and $c=1$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 more solutions); DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC (3 solutions); DIMITRIOS KOUKAKIS, Kilkis, Greece; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, Thessaloniki, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; OCTAVIAN STROE, Piteşti, Romania; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer (2 solutions).

3748⋆. [2012 : 195, 197] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given three mutually external circles in general position, there will exist six distinct lines that are common internal tangents to pairs of the circles. Prove that if three of those common tangents, one to each pair of the circles, are concurrent, then the other three common tangents are also concurrent.

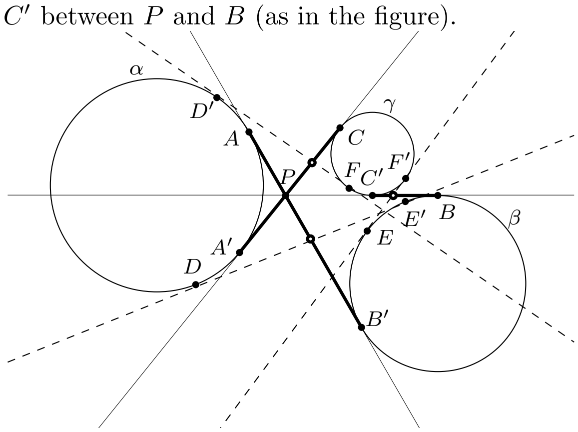
Comment by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

Here is a promising approach to the problem. We will see that when given three external circles situated so that three of the six common internal tangent lines pass through a point, then the length of one of the common tangents is the sum of the other two. It seems plausible that the converse also holds, hence

> **Conjecture**. *Given three mutually external circles in general position, there will be two internal common tangents (equal in length) between each pair of the circles, resulting in six distinct straight lines containing these six common tangents. These lines can be split into two subsets of three concurrent lines if and only if one of the lengths is the sum of the other two.*

We assume that we are given external circles α, β, and γ, and that three of their six distinct common internal tangent lines pass through a point P. By carefully labeling the figure, we can reduce the number of special cases to two. Circle α will be assigned the contact points A, A', D, D', while β's are B, B', E, E', and γ's are C, C', F, F'. They are labeled so that $AB' = DE'$ are the longest of the common internal tangents. Because there are only two tangent lines through P to any single circle, it is clear that for three of the common tangents to pass through P, each would necessarily belong to a different pair of circles. Thus we label our three lines through P to be AB', BC', and CA'. We wish to label the points where the tangents touch the circles so that $AB' > BC', CA'$ (and, consequently, $DE' > EF', FD'$). There will be two cases to consider:

- *P lies in the interior of the segment AB'.* The lines AB' and CA' divide the plane into four regions; because the lines are internal tangents, the circles β and γ must lie in the region vertically opposite the region enclosing α. The line $C'B$ will necessarily intersect α, and our decision to make $AB' > CA'$ will place C' between P and B (as in the figure).



- *B' lies between A and P.* In this case P is exterior to the three common tangents lying along the tangent lines through it. Because $PA = PA', PB = PB', PC = PC'$, our assumption that AB' be the longest internal common tangent forces B' to lie between A and P, B between C' and P, and C between A' and P (as in the figure).

With our notation in place we are ready to prove the claim that $AB' = BC' + CA'$ (and, consequently, $DE' = EF' + FD'$).

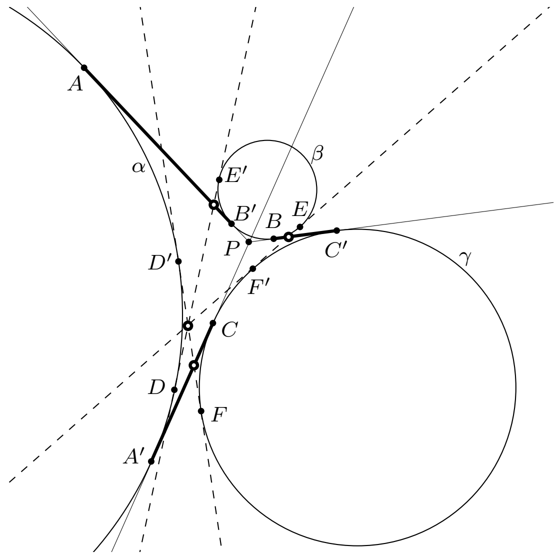
Case 1: P lies between A and B'.

$$\begin{aligned} AB' &= AP + PB' \\ &= A'P + PB \\ &= (A'C - PC) + (PC' + C'B) \\ &= A'C + C'B \end{aligned}$$

Case 2: B' lies between A and P.

$$\begin{aligned} AB' &= AP - PB' \\ &= A'P - PB \\ &= (A'C + PC) - (PC' - C'B) \\ &= A'C + C'B \end{aligned}$$

To see where difficulties arise with the converse, suppose that the lines



AB', BC', CA' all pass through P, so that $AB' = BC' + CA'$. Observe that because $CA' = FD'$, we also have $AB' = BC' + FD'$ even though the lines AB', BC', FD' are not concurrent. Somehow one must discover how to make use of the hypothesis that the lines AB', BC', CA' are concurrent (which implies that $AB' = BC' + CA'$ and, therefore, that $DE' = EF' + FD'$). The conjecture appears even harder to prove; for the conjecture one would be given only $AB' = BC' + CA'$ with some assignment of labels, and would have to describe how to correctly select a subset of three concurrent lines from the given set of six common internal tangent lines.

No solutions have been received; the problem remains open.

3749. [2012 : 195, 197] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let D and E be arbitrary points on the sides BC and AC of a triangle ABC. Prove that

$$\sqrt{[ADE]} + \sqrt{[BDE]} \le \sqrt{[ABC]},$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Arkady Alt, San Jose, CA, USA.

Let $t = \dfrac{BD}{BC}$, $s = \dfrac{CE}{CA}$, and $F = [ABC]$. Then $t, s \in [0,1]$,

$$[BDE] = Fst,$$

and

$$[ADE] = F(1-s)(1-t).$$

Hence the claimed inequality is equivalent to the inequality

$$\sqrt{(1-s)(1-t)} + \sqrt{st} \le 1,$$

which follows immediately from the Cauchy-Schwarz inequality applied to the pairs $\left\langle \sqrt{1-s}, \sqrt{s} \right\rangle$ and $\left\langle \sqrt{1-t}, \sqrt{t} \right\rangle$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DIMITRIOS KOUKAKIS, Kilkis, Greece; KEE-WAI LAU, Hong Kong, China; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

3750. [2012 : 195, 197] *Proposed by Michel Bataille, Rouen, France.*

Let $T_k = 1 + 2 + \cdots + k$ be the k^{th} triangular number. Find all positive integers m, n such that $T_m = 2T_n$.

Composite of submitted solutions.

The equation is equivalent to $m(m+1) = 2n(n+1)$, which in turn becomes $x^2 - 2y^2 = -1$ when $x = 2m+1$ and $y = 2n+1$. The positive solutions $(x, y) = (x_k, y_k)$ of the Pellian equation $x^2 - 2y^2 = -1$ are all odd and given by

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})(3+2\sqrt{2})^k$$

for $k = 0, 1, 2, \ldots$.

There are various ways of describing these solutions:

1. $(x_0, y_0) = (1, 1)$, and $(x_{k+1}, y_{k+1}) = (3x_k + 4y_k, 2x_k + 3y_k)$;
2. $(x_0, y_0) = (1, 1)$, $(x_1, y_1) = (7, 5)$, and
$$(x_{k+2}, y_{k+2}) = 6(x_{k+1}, y_{k+1}) - (x_k, y_k);$$
3.
$$x_k = \frac{1}{2}\left[(1+\sqrt{2})^{2k+1} + (1-\sqrt{2})^{2k+1}\right],$$
$$y_k = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2k+1} - (1-\sqrt{2})^{2k+1}\right];$$

4.

$$x_k = \sum_{i=0}^{k} \binom{2k+1}{2i} 2^i,$$
$$y_k = \sum_{i=0}^{k} \binom{2k+1}{2i+1} 2^i.$$

A complete set of solution pairs (m, n) can be derived from the pairs (x_k, y_k) when $k \geq 1$; the smallest pairs are

$$(3, 2), (20, 14), (119, 84), (696, 492), (4059, 2870), (23660, 16730), (137903, 97512).$$

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA and GEORGE MELKI, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NERMIN HODZIĆ, Dobosnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Curtis rewrote the equation as $m^2 + (m+1)^2 = (2n+1)^2$, from which we see that the solution amounts to finding Pythagorean triplets whose smallest entries differ by 1. Stone and Hawkins pointed out that the pairs (m, n) satisfy the recurrence relation (with $(m_0, n_0) = (0, 0)$)

$$\begin{pmatrix} m_{k+1} \\ n_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m_k \\ n_k \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

They also wonder whether, given a triangular array of T_m dots, there is a transparent way to see that the bottom $m - n$ rows contain T_n dots.

Both Arslanagić and the pair Hodzić-Malakić pointed out that the problem appears in College Math. J. *25:3 (May, 1994), 241-243, as well as in the chapter that treats Pell's equation in the book* Mathematical Olympiad Challenges *by Titu Andreescu and Razvan Gelca. The theory of Pell's equation appealed to in the solution can be found in many elementary number theory textbooks.*

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