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Editorial Board

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IN MEMORIAM

Christopher J. Bradley, 1938 – 2013

We have just learned that Christopher J. Bradley passed away on July 11, 2013. Christopher has been a valued contributor to ***Crux Mathematicorum*** for over 20 years. The first problem he proposed for us was 1779 [1992 : 238; 1993 : 216-217]:

> *Two circles C_1 and C_2 are given with the centre A of circle C_1 lying on C_2. BC is the common chord. The chord AD of C_2 meets BC at E. From D lines DF and DG are drawn tangent to C_1 at F and G. Prove that E, F, G are collinear.*

which attracted 17 solutions.

Within the UK Mathematics Trust, he is well-known for his books as well as his work with the British IMO teams as deputy leader and for the large number of problems which he created to support the British Mathematical Olympiad and international mathematics competitions. He was to receive the UKMT gold medal for his service to maths competitions later in 2013.

The UKMT has created the "Christopher Bradley elegance prize". The prize will be awarded to the candidate or candidates who, in the opinion of the markers, has submitted the most elegant solution or solutions to a BMO2 problem or problems.

He will be sorely missed by the mathematics community.

Errata

Reader Stan Wagon points out an error in the last Problem of the Month [2013 : 27–30]. On page 27 the statement

> *the value of B that gives the sequence of maximum length is either $[\tau A]$ or $[\tau A]+1$, where $\tau = \frac{1+\sqrt{5}}{2}$, is the golden ratio,*

should have read

> *the value of B that gives the sequence of maximum length is either $[\lambda A]$ or $[\lambda A]+1$, where $\lambda = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$, is the* **reciprocal** *of the golden ratio,*

See p. 193 of Honsberger's *More Mathematical Morsels.*

THE CONTEST CORNER

No. 13

Shawn Godin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_CCProblemNumber *(example Doe_Jane_CC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* crux-contest@cms.math.ca. *Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 July 2014**, *although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the Solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC61. We place 3 green, 4 yellow and 5 red balls in a bag. Two balls of different colours are selected at random, removed, and replaced with two balls of the third colour. Show that it is impossible for all of the remaining balls to be the same colour, no matter how many times this process is repeated.

CC62. For each real number x, let $[x]$ be the largest integer less than or equal to x. For example, $[5] = 5$, $[7.9] = 7$ and $[-2.4] = -3$. An arithmetic progression of length k is a sequence $a_1, a_2, \dots, a_k$ with the property that there exists a real number b such that $a_{i+1} - a_i = b$ for each $1 \le i \le k-1$. Let $\alpha > 2$ be a given irrational number. Then $S = \{[n\alpha] : n \in \mathbb{Z}\}$, is the set of all integers that are equal to $[n\alpha]$ for some integer n. Prove that for any integer $m \ge 3$, there exist m distinct numbers contained in S which form an arithmetic progression of length m.

CC63. A quadrilateral circumscribes a circle. Prove that the perimeter of the quadrilateral bears the same ratio to the perimeter of the circle as the area of the quadrilateral bears to the area of the circle.

CC64. Show that a power of 2 can never be the sum of k consecutive positive integers, $k > 1$.

CC65. Suppose that three circles in the plane are located so that each pair of circles intersect in two points, thereby giving a common chord to those two circles. Prove that these three chords pass through one point.

. . . .

CC61. On place 3 boules vertes, 4 boules jaunes et 5 boules rouges dans un sac. On choisit au hasard deux boules de couleurs différentes qu'on enlève du sac et on place dans le sac deux boules de la troisième couleur. Démontrer qu'il est impossible d'obtenir un sac contenant des boules de la même couleur, quel que soit le nombre de fois que l'on recommence l'opération.

CC62. Pour tout nombre réel x, $[x]$ représente le plus grand entier inférieur ou égal à x. Par exemple, $[5] = 5$, $[7,9] = 7$ et $[-2,4] = -3$. Une suite arithmétique de longueur k est une suite $a_1, a_2, \ldots, a_k$ pour laquelle il existe un nombre réel b tel que $a_{i+1} - a_i = b$ pour toute valeur de i $(1 \leq i \leq k-1)$. Soit α $(\alpha > 2)$ un nombre irrationnel. Alors $S = \{[n\alpha] : n \in \mathbb{Z}\}$ est l'ensemble des entiers qui peuvent être écrits sous la forme $[n\alpha]$, n étant n'importe quel entier. Montrer que pour tout entier m $(m \geq 3)$, il existe m nombres distincts dans S qui forment une suite arithmétique de longueur m.

CC63. Un quadrilatère est circonscrit à un cercle. Démontrer que le rapport du périmètre du quadrilatère au périmètre du cercle est égal au rapport de l'aire du quadrilatère à l'aire du cercle.

CC64. Démontrer qu'une puissance de 2 ne peut être égale à la somme de k entiers consécutifs strictement positifs, $k > 1$.

CC65. On considère trois cercles dans un plan de manière que chaque deux cercles se coupent en deux points, créant ainsi une corde commune aux deux cercles. Démontrer que les prolongements des trois cordes ainsi créées passent par un même point.

CONTEST CORNER SOLUTIONS

The editor would like to acknowledge MICHEL BATAILLE, Rouen, France, whose solutions to problems CC3, CC5 and CC10 were overlooked. The editor apologizes sincerely for the oversight.

CC11. Ten boxes are arranged in a circle. Each box initially contains a positive number of golf balls. A move consists of taking all of the golf balls from one

of the boxes and placing them into the boxes that follow it in a counterclockwise direction, putting one ball into each box. Prove that if the next move always starts with the box where the last ball of the previous move was placed, then after some number of moves, we get back to the initial distribution of golf balls in the boxes.
(*Originally question # 10 from the 2009 Sun Life Financial Repêchage Competition.*)

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA. There were no other solutions.

We will show the result is true when we have $n \geq 2$ boxes arranged in a circle, each containing a positive number of balls, and a starting box for the next move. We first observe that there are a finite number of ways that the n balls can be distributed in the k boxes. This tells us that if we repeatedly apply the given move, we will eventually have some arrangement of balls and starting box occur twice. The sequence of moves to get from this arrangement to itself will then repeat.

We now must show that our starting arrangement is in this sequence of repeated moves. Given s, an arrangement of balls and starting box b, we claim that there is only one arrangement of balls and starting box we could have started with to arrive at s. To find this arrangement, we remove a ball from each box, starting with b' and proceeding clockwise, until we try to remove a ball from an empty box. Let s' be the arrangement where we place all the removed balls into this empty box b' and let it be the starting box. It is easy to see that if we perform the move from s' and b' we will get to s and b.

To see see that no other starting position would give this arrangement, we observe that there could not be more total balls used, since the box b' would thus have to contain more balls. There also cannot be fewer total balls used, since this would mean that we did not empty the starting box. Thus, the move is uniquely reversible, so our starting arrangement must be in the repeated sequence of moves.

CC12. Prove that $\sum \frac{1}{i_1 i_2 \cdots i_k} = 2001$, where the summation taken is over all non-empty subsets $\{i_1, i_2, \cdots, i_k\}$ of the set $\{1, 2, \cdots, 2001\}$.
(*Originally question # 1 from the 2001 University of Waterloo Special K Contest.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; Henry Ricardo, Tappan, NY, USA; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We present the solution of Ricardo.

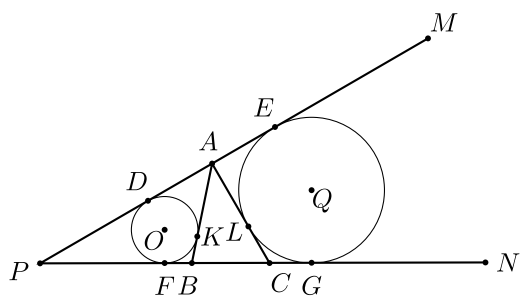
We will prove the more general result:

$$\sum \frac{1}{i_1 i_2 \cdots i_k} = n,$$

where the summation is taken over all non-empty subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$. If we expand the product $\prod\limits_{k=1}^{n} \left(1 + \frac{1}{k}\right)$ we see that we get all of the terms in our summation, and also a term of 1. Thus, we have:

$$\begin{aligned}\sum \frac{1}{i_1 i_2 \cdots i_k} &= \prod_{k=1}^{n}\left(1+\frac{1}{k}\right)-1\\ &= \prod_{k=1}^{n}\left(\frac{k+1}{k}\right)-1\\ &= \frac{n+1}{1}-1\\ &= n.\end{aligned}$$

CC13. Triangle ABC has its base on line segment PN and vertex A on line PM. Circles with centres O and Q, having radii r_1 and r_2, respectively, are tangent to both PM and PN, and to the triangle ABC externally at K and L (as shown in the diagram).



(a) Prove that the line through K and L cuts the perimeter of triangle ABC into two equal pieces.
(b) Let T be the point of contact of BC with the circle inscribed in triangle ABC. Prove that $(TC)(r_1) + (TB)(r_2)$ is equal to the area of triangle ABC.
(*Originally question B4 from the 2005 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Mihaï-Ioan Stoënescu, Bischwiller, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We present the solution of Bataille.

(a) Points D, F are symmetric in the line through P, O, Q (the internal bisector of $\angle MPN$) and so are E, G. Thus, $DE = FG$, that is, $DA + AE = FB + BC + CG$ or $AK + AL = KB + BC + CL$ and the result follows.

(b) We shall make use of the following two results whose proofs are provided at the end:

$$BF \cdot BT = rr_1, \quad CT \cdot CG = rr_2 \qquad (1)$$

where r is the inradius of ΔABC, and

$$\frac{AL \cdot CG}{PC} = \frac{AK \cdot BF}{PB}\,. \tag{2}$$

Let h be the distance from A to the line BC. Then, $2[ABC] = h \cdot TB + h \cdot TC$, $2[APC] = h \cdot PC$, $2[APB] = h \cdot PB$ where $[\cdot]$ denotes area. It follows that

$$[ABC] = TB \cdot \frac{[APC]}{PC} + TC \cdot \frac{[APB]}{PB} = r_2 TB\left(1 - \frac{AL}{PC}\right) + r_1 TC\left(1 + \frac{AD}{PB}\right)$$

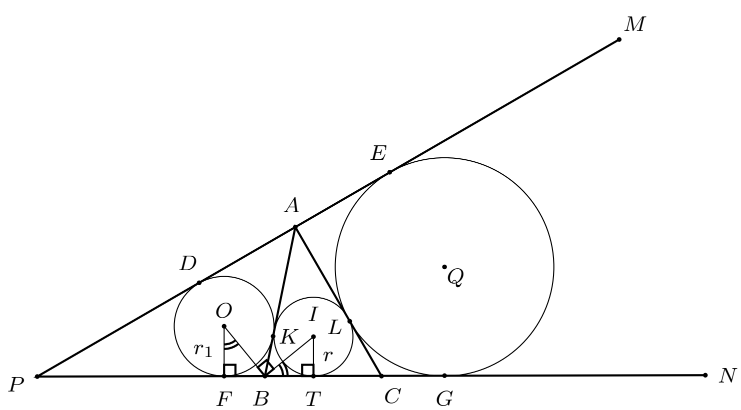
(because $[APC] = r_2(PE - AC) = r_2(PG - AL - CL) = r_2(PC - AL)$ and $[APB] = r_1(AD + PF + FB) = r_1(AD + PB)$). Thus, it is sufficient to prove

$$r_1 TC\frac{AD}{PB} = r_2 TB\frac{AL}{PC}. \tag{3}$$

But (3) is successively equivalent to

$$\begin{aligned}
(PG - CG)\cdot TC \cdot r_1 \cdot AK &= (PF + BF)\cdot TB \cdot r_2 \cdot AL\\
PG \cdot TC \cdot r_1 \cdot AK - PF \cdot TB \cdot r_2 \cdot AL &= rr_1r_2(AK + AL) \text{ (using (1))}\\
\frac{PG \cdot AK}{CG} - \frac{PF \cdot AL}{BF} &= AK + AL\\
\frac{(PC + CG)\cdot AK}{CG} - \frac{(PB - BF)\cdot AL}{BF} &= AK + AL\\
\frac{PC \cdot AK}{CG} &= \frac{PB \cdot AL}{BF}
\end{aligned}$$

which is true by (2).



Proof of (1): Let I be the incentre of ΔABC. Since BO and BI are the bisectors of $\angle PBA$, we have $\angle OBI = 90^\circ$ and so $\angle BOF = \angle TBI$. It follows that $\dfrac{OF}{FB} = \dfrac{TB}{TI}$ and so $BF \cdot BT = rr_1$. Similarly, $CT \cdot CG = rr_2$.

Proof of (2): the circle with centre Q, radius r_2 is the P-excircle of ΔAPC, hence

$$AL = \frac{PC + AC - PA}{2} \quad \text{and} \quad CG = CL = \frac{PA + AC - PC}{2}.$$

Therefore

$$\begin{aligned}\frac{AL \cdot CG}{PC} &= \frac{AC^2 - (PC - PA)^2}{4PC}\\ &= \frac{AC^2 - PC^2 - PA^2 + 2PC \cdot PA}{4PC}\\ &= \frac{-2PC \cdot PA\cos(P) + 2PC \cdot PA}{4PC}\\ &= \frac{PA(1 - \cos(P))}{2}\end{aligned}$$

where $P = \angle APC = \angle APB$.

Similarly, the circle with centre O, radius r_1 is the incircle of ΔAPB, hence

$$AK = \frac{PA + AB - PB}{2} \quad \text{and} \quad BF = \frac{PB + AB - PA}{2},$$

and an analogous calculation gives

$$\frac{AK \cdot BF}{PB} = \frac{PA(1 - \cos(P))}{2}.$$

CC14. Evaluate $\int_0^\pi \ln(\sin x)dx$.
(*Originally question # 2 from the 2001 University of Waterloo Big E Contest.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Richard I. Hess, Rancho Palos Verdes, CA, USA; R. Laumen, Deurne, Belgium; Henry Ricardo, Tappan, NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution of Wang.

Note that this is an improper integral. However, to avoid the notational complication caused by the formal limit definition we will treat it as a proper integral. The validity of the argument will not be affected.

Let I denote the integral. Then

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx + \int_{\frac{\pi}{2}}^{\pi} \ln(\sin x)dx.$$

Let $y = x - \frac{\pi}{2}$, then

$$\int_{\frac{\pi}{2}}^{\pi} \ln(\sin x)dx = \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(\frac{\pi}{2} + y\right)\right) dy = \int_0^{\frac{\pi}{2}} \ln(\cos y)dy.$$

Hence,

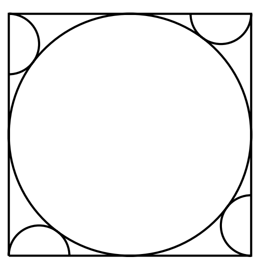
$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln(\sin x)dx + \int_0^{\frac{\pi}{2}} \ln(\cos x)dx \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin x \cdot \cos x)dx \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin 2x}{2}\right) dx \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin 2x)dx - \int_0^{\frac{\pi}{2}} (\ln 2)dx \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin 2x)dx - \frac{\pi}{2}\ln 2 \qquad (1)
\end{aligned}$$

Now let $z = 2x$. Then

$$\int_0^{\frac{\pi}{2}} \ln(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \ln(\sin z)dz = \frac{1}{2}I. \qquad (2)$$

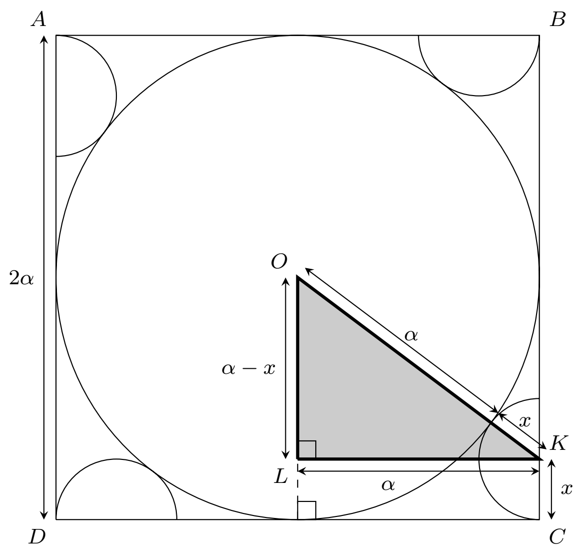
Substituting (2) into (1) we obtain $I = \frac{1}{2}I - \frac{\pi}{2}\ln 2$ so $I = -\pi \ln 2$.

CC15. A circle is inscribed in a square. Four semicircles with their flat sides along the edge of the square and tangent to the circle are inscribed in each of the four spaces between the square and circle. What is the ratio of the area of the circle to the total area of the four semicircles?



(*Originally question # 10 from the 2007 W.J. Blundon Mathematics Contest.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; R. Laumen, Deurne, Belgium; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We present the solution of Apostolopoulos.



Label the square $ABCD$, with centre O. Let $AD = 2\alpha$ and let the small circles have radius x. Let K be the centre of a small circle. From ΔKLO we get

$$OK^2 = OL^2 + LK^2$$

so

$$(\alpha + x)^2 = (\alpha - x)^2 + \alpha^2.$$

Solving gives $x = \dfrac{\alpha}{4}$, hence the ratio of the areas is

$$\frac{\pi\alpha^2}{2\pi\left(\frac{\alpha}{4}\right)^2} = 8\,.$$

THE OLYMPIAD CORNER

No. 311

Nicolae Strungaru

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-olympiad@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

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The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC121. Prove that for all positive real numbers x, y, z we have

$$\sum_{\text{cyclic}} (x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx).$$

OC122. We define a sequence $f_n(x)$ of functions by

$$f_0(x) = 1, f_1(x) = x,\ (f_n(x))^2 - 1 = f_{n-1}(x)f_{n+1}(x),\ \text{for } n \geq 1\,.$$

Prove that for every $n, f_n(x)$ is a polynomial with integer coefficients.

OC123. Let p be prime. Find all positive integers n for which, whenever x is an integer such that $x^n - 1$ is divisible by p, then $x^n - 1$ is also divisible by p^2.

OC124. Find all triples (a, b, c) of positive integers with the following property: for every prime p, if n is a quadratic residue (mod p), then $an^2 + bn + c$ is also a quadratic residue (mod p).

OC125. ABC is an acute angle triangle with $\angle A > 60^\circ$ and H as its orthocenter. M, N are two points on AB, AC respectively, such that $\angle HMB = \angle HNC = 60^\circ$. Let O be the circumcenter of triangle HMN. Let D be a point on the same side of BC as A such that $\triangle DBC$ is an equilateral triangle. Prove that H, O, D are collinear.

. . . .

OC121. Démontrer l'inégalité suivante, pour tout x, y, z nombres réels positifs :

$$\sum_{\text{cyclique}} (x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx).$$

OC122. Une suite de fonctions $f_n(x)$ est définie par

$$f_0(x) = 1, f_1(x) = x,\ (f_n(x))^2 - 1 = f_{n-1}(x)f_{n+1}(x),\ \text{pour } n \geq 1\,.$$

Démontrer que pour tout n, $f_n(x)$ est un polynôme avec coefficients entiers.

OC123. Soit p un nombre premier. Déterminer tous les entiers positifs n pour lesquels, aussitôt que x est un entier tel que $x^n - 1$ est divisible par p, il en découle que $x^n - 1$ est divisible par p^2.

OC124. Déterminer tous les triplets (a, b, c) formés d'entiers positifs et vérifiant la propriété suivante : pour tout nombre premier p, si n est un résidu quadratique (mod p), alors $an^2 + bn + c$ est aussi un résidu quadratique (mod p).

OC125. ABC est un triangle à angles aïgus avec $\angle A > 60^\circ$, dont l'orthocentre est H. M et N sont deux points sur AB et AC respectivement, tels que $\angle HMB = \angle HNC = 60^\circ$. Soit O le centre du cercle circonscrit du triangle HMN. Soit aussi D un point qui se trouve sur le même côté de BC que A et tel que $\triangle DBC$ est un triangle équilatéral. Démontrer que H, O et D sont colinéaires.

OLYMPIAD SOLUTIONS

OC61. 46 squares of a 9×9 grid are coloured red. Prove that we can find a 2×2 square on the grid which contains at least 3 red squares.
(*Originally question 2 from the 2011 Singapore National Olympiad.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

Let $a_{i,j}$, for $1 \leq i, j \leq 9$, represent the squares of the grid. We consider $a_{ij} = 1$ if the square is red and $a_{ij} = 0$ otherwise. For each $1 \leq i \leq 9$ we denote

$$r_i = \sum_{j=1}^{9} a_{ij} \,.$$

If $r_1 \leq 5$, we have

$$(r_2 + r_3) + (r_4 + r_5) + (r_6 + r_7) + (r_8 + r_9) = 46 - r_1 \geq 41 \,,$$

hence, there exists a $i \in \{2, 4, 6, 8\}$ such that

$$r_i + r_{i+1} \geq 11 \,.$$

If $r_3 \leq 5$, we have

$$(r_1 + r_2) + (r_4 + r_5) + (r_6 + r_7) + (r_8 + r_9) = 46 - r_3 \geq 41 \,,$$

hence, there exists a $i \in \{1, 4, 6, 8\}$ such that

$$r_i + r_{i+1} \geq 11 \,.$$

Repeating the argument, we can deduce that if one of r_1, r_3, r_5, r_7 or r_9 is less or equal than 5, then there exists some $1 \leq i \leq 8$ such that

$$r_i + r_{i+1} \geq 11 \,.$$

Moreover, since $r_1, r_9 \leq 9$ we have

$$(r_1 + r_2) + (r_2 + r_3) + ... + (r_8 + r_9) = 2 \cdot 46 - r_1 - r_9 \geq 74 \,.$$

Hence, there exists some $1 \leq i \leq 8$ such that $r_i + r_{i+1} \geq 10$. Moreover, either one of r_i or r_{i+1} is at least 6, or $r_i = r_{i+1} = 5$. In the second case by the first part of the solution, there exists some $1 \leq j \leq 8$ such that

$$r_j + r_{j+1} \geq 11 \,.$$

Thus, it suffices to prove the problem under the assumption that there exists an i so that

$$r_i + r_{i+1} \geq 10 \,\text{and}\, r_i \geq 6\,.$$

If $r_i + r_{i+1} \geq 11$, then the conclusion is clear, so without loss of generality we can assume that $r_i + r_{i+1} = 10$.

For convenience, we assume that $r_1 + r_2 = 10$ and $r_1 \geq 6$. As $r_2 \leq 4$, there exists some $i \in \{1,3,5,7,9\}$ such that

$$a_{1i} + a_{2i} \leq 1\,.$$

Removing the squares a_{1i} and a_{2i} from the first two columns, we are left with four 2×2 squares. As these 4 squares have $r_1 + r_2 - (a_{1i} + a_{2i}) \geq 9$ red squares, we are done.

OC62. Let A, B, C, D be four non-coplanar points in space. The segments AB, BC, CD and DA are tangent to the same sphere. Prove that their four points of tangency are coplanar.
(*Originally question 3 from the 2011 Spanish Olympiad, Day 1.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

Let segments AB, BC, CD, DA touch the sphere at R, S, T, U, respectively, and let

$$x = AR = AU; \quad y = BR = BS; \quad z = CS = CT \quad \text{and} \quad t = DT = DU.$$

Denoting by $\overrightarrow{\mathbf{M}}$ the vector from a fixed point to M, let I be the point determined by

$$m\,\overrightarrow{\mathbf{I}} = yzt\overrightarrow{\mathbf{A}} + ztx\overrightarrow{\mathbf{B}} + txy\overrightarrow{\mathbf{C}} + xyz\overrightarrow{\mathbf{D}}\,,$$

where

$$m = yzt + ztx + txy + xyz\,.$$

Then

$$m\,\overrightarrow{\mathbf{I}} = zt\left(y\overrightarrow{\mathbf{A}} + x\overrightarrow{\mathbf{B}}\right) + xy\left(t\overrightarrow{\mathbf{C}} + z\overrightarrow{\mathbf{D}}\right) = zt\,(y+x)\,\overrightarrow{\mathbf{R}} + xy\,(t+z)\,\overrightarrow{\mathbf{T}}\,.$$

Because $zt\,(y+x)$ and $xy\,(t+z)$ are positive and sum to m, it follows that I lies on the segment RT. Similarly,

$$m\,\overrightarrow{\mathbf{I}} = yz\left(t\overrightarrow{\mathbf{A}} + x\overrightarrow{\mathbf{D}}\right) + tx\left(z\overrightarrow{\mathbf{B}} + y\overrightarrow{\mathbf{C}}\right) = yz\,(t+x)\,\overrightarrow{\mathbf{U}} + tx\,(z+y)\,\overrightarrow{\mathbf{S}}\,.$$

showing that I lies on the segment US as well. Thus, the lines US and RT are concurrent at I, and determine a plane containing R, S, T, U. The result follows.

OC63. Prove that there exists a perfect square so that the sum of its digits is 2011.
(*Originally question 4 from 2011 Finland Math Olympiad.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; R. Laumen, Deurne, Belgium; Daniel Văcaru, Piteşti, Romania; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru.

We denote by $s(m)$ the sum of digits of m. It is easy to see that $s(m^2) = 0, 1, 4, 7 \pmod 9$. We will prove now that if $n = 0, 1, 4, 7 \pmod 9$ there exists a perfect square with the sum of digits n.

We have

$$\begin{aligned}
a &= (\underbrace{999..9}_{k})^2 = \underbrace{999..9}_{k-1}8\underbrace{000..0}_{k-1}1 & &\Rightarrow s(a) = 9k,\\
a &= (\underbrace{999..9}_{k-1}1)^2 = \underbrace{999..9}_{k-2}82\underbrace{000..0}_{k-2}81 & &\Rightarrow s(a) = 9k+1,\\
a &= (\underbrace{999..9}_{k-2}2)^2 = \underbrace{999..9}_{k-2}84\underbrace{000..0}_{k-2}64 & &\Rightarrow s(a) = 9k+4,\\
a &= (\underbrace{999..9}_{k-2}4)^2 = \underbrace{999..9}_{k-2}88\underbrace{000..0}_{k-1}36 & &\Rightarrow s(a) = 9k+7.
\end{aligned}$$

For $n = 2011$ we have

$$(\underbrace{999..9}_{222}2)^2 = \underbrace{999..9}_{221}84\underbrace{000..0}_{221}64$$

and thus

$$s(\underbrace{999..9}_{221}84\underbrace{000..0}_{221}64) = 221 \times 9 + 8 + 4 + 221 \times 0 + 6 + 4 = 2011.$$

Văcaru's, Geupel's and Laumen's solution is

$$\underbrace{999..9}_{223}7^2 = \underbrace{999..9}_{222}4\underbrace{000..0}_{222}9\,.$$

Several readers noted that the more general problem

Determine all the possible values for the sum of digits of a perfect square.

and variations of it have appeared in several other competitions and have been discussed in books, such as T. Andreescu, R.Gelca - "Mathematical Olympiad Challenges".

OC64. Find all integer solutions of the equation

$$n^3 = p^2 - p - 1$$

where p is prime.
(*Originally question 5 from the 2011 Italy Math Olympiad.*)

Solved by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comǎneşti, Romania. There were also one incomplete and one incorrect solution. We give the solution of Geupel.

It is easy to verify that $(n, p) = (1, 2)$ and $(n, p) = (11, 37)$ are solutions of the equation. We prove that there are no other solutions.

Suppose that the integer n and the prime p are solution to the given equation. Then we have $0 < n < p$ and

$$p(p-1) = (n+1)(n^2 - n + 1), \tag{1}$$

which implies that the prime number p divides either $n + 1$ or $n^2 - n + 1$. We consider both cases in succession.

First assume that p divides $n + 1$. Then there is a positive integer a such that $n + 1 = ap$. Since $n < p$, we have $a = 1$. Using (1), we obtain

$$n = p - 1 = n^2 - n + 1\,,$$

that is

$$(n-1)^2 = 0\,.$$

It follows that $(n, p) = (1, 2)$.

Second assume that p divides $n^2 - n + 1$. Then there is a positive integer b such that $n^2 - n + 1 = bp$. Applying (1), we obtain

$$p - 1 = b(n+1)\,. \tag{2}$$

It follows that

$$n^2 - n + 1 = bp = b(b(n+1) + 1)\,.$$

We successively obtain $n^2 - (b^2 + 1)n - (b^2 + b - 1) = 0$ and

$$n = \frac{1}{2}(b^2 + 1 \pm \sqrt{b^4 + 6b^2 + 4b - 3}). \tag{3}$$

Thus, the number $b^4 + 6b^2 + 4b - 3$ is a perfect square. This is not valid for $b = 1$ or $b = 2$. For $b = 3$, the equation (3) and (2) yield $(n, p) = (11, 37)$. For $b \geq 4$, we have

$$\begin{aligned}(b^2+3)^2 = b^4 + 6b^2 + 9 &< b^4 + 6b^2 + 4b - 3\\ &< b^4 + 6b^2 + 4b - 3 + 2(b-1)^2 + 17 = (b^2+4)^2.\end{aligned}$$

This is a contradiction, which completes the second case.

OC65. Let ABC be a triangle. F and L are two points on the side AC such that $AF = LC < AC/2$. If $AB^2 + BC^2 = AL^2 + LC^2$ find $\angle FBL$.
(*Originally question 4 from 2011 Morocco National Olympiad, Grade 11.*)

Solved by Michel Bataille, Rouen, France; Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Daniel Văcaru, Piteşti, Romania; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

The median from vertex B in ΔABC is also the median from B in ΔFBL. It follows that

$$2(AB^2 + BC^2) - AC^2 = 2(BL^2 + BF^2) - FL^2. \qquad (1)$$

Let $\theta = \angle FBL$. Since

$$BL^2 + BF^2 - FL^2 = 2BF \cdot BL \cdot \cos(\theta),$$

and

$$AB^2 + BC^2 = AL^2 + LC^2,$$

(1) can be rewritten as

$$2(AL^2 + LC^2) - (AL + LC)^2 = 4BF \cdot BL \cdot \cos(\theta) + FL^2,$$

or

$$(AL - LC)^2 = 4BF \cdot BL \cdot \cos(\theta) + FL^2.$$

Since $AL - LC = AL - AF = FL$ we finally get

$$4BF \cdot BL \cdot \cos(\theta) = 0.$$

Thus $\theta = 90°$.

BOOK REVIEWS

John McLoughlin

Calculus and Its Origins by David Perkins
The Mathematical Association of America, 2012
ISBN: 978-0-88385-575-1 (print), 978-1-61444-408-1 (electronic), Hardcover/e-book, 165 + xiv pages, US$60.00 (print), US$28.00 (electronic)

Reviewed by ***Brenda Davison****, Simon Fraser University, Burnaby, BC*

Calculus and Its Origins, by David Perkins, is a slim volume of 165 pages intended to illuminate college level calculus as "the culmination of an intellectual pursuit that lasted two thousand years". This approach has the distinct advantage that non–rigourous and intuitive ideas about how to solve many of the problems of calculus can be used — presenting the ideas with brevity and clarity and without the accompanying difficulty of rigour.

The book is not flashy. Despite the author's explanation for his choice, I did not find the cover photo or design appealing. In contrast, however, the diagrams throughout are extremely effective. They appear on most pages and have a hand drawn appearance. In the same way that the most effective bird identification books contain illustrations rather than photographs, the diagrams here are very effective at conveying the pertinent information.

Perkins starts his story by considering some of the patterns investigated by the Ancients, notably Archimedes. This allows him to state early in the first chapter that series are "arguably the most important tool in the calculus kit". This topic introduces the central and difficult concepts of both infinitely many and infinitely small. The series chosen and the geometric illustrations that allow you to visualize to what those series must converge are brilliant.

Chapter two, now nearly a millennium later, details the, perhaps, lesser known work of Ibn al Haytham finding the sum of the 4^{th} powers and using that to compute a volume of revolution by a method that we would now call the disc (or washer) method.

The first half of the book is a really elegant blend of geometry and the infinite and I found it motivated me to work through the examples. This does require some work on the part of the reader. The examples and the exercises are woven into the text and definitely aid in the learning of the material if they are done while reading through the chapter. While I could see using this book in a small seminar style class on calculus, the lack of a large set of exercises of varying difficulty would make it difficult to use this book for a traditional large freshman calculus course.

After reading this book, you are likely to have a greater appreciation for how geometric figures played an important role in the development of calculus. In the latter half of the book, starting with the Fundamental Theorem of Calculus, a reader who is algebraically orientated (most students) may well get bogged down in the geometry. A lot of effort is required here for perhaps less reward than in the first half of the book. There is a reason why the notation and algebraic methods of Leibniz prevailed over the geometric reasoning of Newton. As a result, I found this middle section of the book less compelling.

The last two chapters discuss the addition of rigour to, or the removal of ambiguity from, calculus. This discussion starts with d'Alembert and convergence of the geometric series. It rapidly touches upon Lagrange and the importance of a function, power series and Taylor series in particular, then Bolzano, Cauchy, convergence criteria and pathological functions. The whirlwind tour continues with Cantor and the infinite, Weierstrass, Riemann and Lebesgue integration and stops at non–standard analysis. Here, the topic list should have been reduced and a couple of simpler ideas explored in more detail. The feeling for the ideas that the book earlier established are lost with this rapid introduction of much more difficult, and sometimes counterintuitive, concepts.

I would recommend this book for small classes, for calculus students with a particular interest in geometry, and for instructors who want to find some less obvious and really interesting examples for their lectures.

FOCUS ON ...

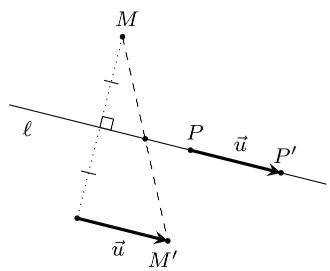
No. 6

Michel Bataille

Glide Reflections in the Plane

Introduction

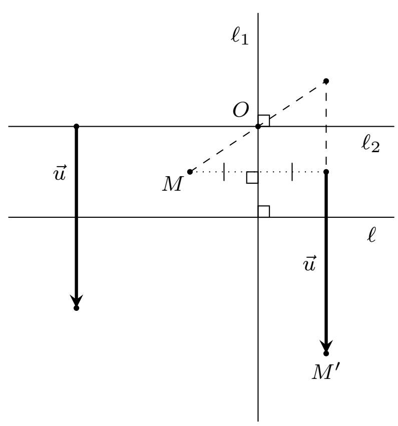
A glide reflection $\mathbf{g}$ is a composition $\mathbf{r}\circ\mathbf{t} = \mathbf{t}\circ\mathbf{r}$ where $\mathbf{r}$ is a reflection in a line ℓ (the axis of $\mathbf{g}$) and $\mathbf{t}$ is a translation whose nonzero vector $\overrightarrow{u}$ (the vector of $\mathbf{g}$) is parallel to ℓ. See the figure; note in passing that the midpoint of MM', where $M' = \mathbf{g}(M)$, is on ℓ for all points M.



If glide reflections are always cited (as they should be) in the list of plane isometries, they do not frequently appear in geometric properties or problems. The purpose of this number is to show a few situations where these transformations are involved.

Composition of isometries

Composing a finite number of isometries, an odd number of which are negative, gives rise to a glide reflection (or a reflection). Here is a very simple example: What is $\mathbf{r}\circ\mathbf{h}$ if $\mathbf{r}$ is the reflection in a line ℓ and $\mathbf{h}$ is the half-turn about a point O not on ℓ?

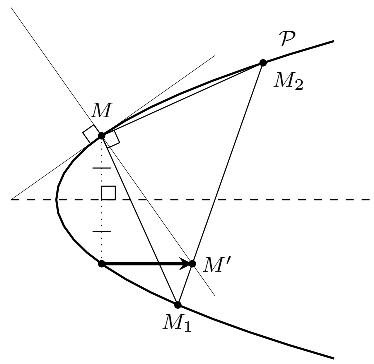


The idea is to write $\mathbf{h}$ as $\mathbf{r_2}\circ\mathbf{r_1}$ where $\mathbf{r_1}$, $\mathbf{r_2}$ are the reflections in perpendicular lines ℓ_1, ℓ_2 through O, respectively, with ℓ_2 parallel to ℓ. Then, $\mathbf{r}\circ\mathbf{h} = (\mathbf{r}\circ\mathbf{r_2})\circ\mathbf{r_1}$ and $\mathbf{r}\circ\mathbf{r_2}$ is a translation $\mathbf{t}$ whose vector $\overrightarrow{u}$ is parallel to ℓ_1. Thus, $\mathbf{r}\circ\mathbf{h}$ is the glide reflection $\mathbf{t}\circ\mathbf{r_1}$.

An unexpected appearance

Let M be an arbitrary point on a parabola $\mathcal{P}$. If two perpendicular lines m_1, m_2 through M meet $\mathcal{P}$ again at M_1, M_2, the line M_1M_2 passes through a point M' independent of the chosen lines m_1, m_2. This nice property is called the Frégier Theorem and M' is the Frégier point associated with M.

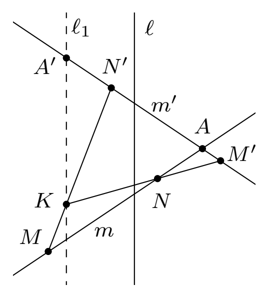
The following proof of the theorem brings out a surprising property: there exists a glide reflection $\mathbf{g}$ such that $M' = \mathbf{g}(M)$ for all points M of $\mathcal{P}$.



Let $M(\frac{y_0^2}{2p}, y_0)$ be a point of the parabola $\mathcal{P} : y^2 = 2px$. Since the pair (tangent, normal) to $\mathcal{P}$ at M is the limit-position of pairs (m_1, m_2), the point M' must be on the normal, say $M'(\frac{y_0^2}{2p} + pk, y_0 - y_0k)$ for some real number k. Similarly, considering the pair formed by the parallel and the perpendicular to the axis of $\mathcal{P}$ through M, we see that the ordinate of M' must be $-y_0$. It follows that $k = 2$ and the only candidate is $M'(\frac{y_0^2}{2p} + 2p, -y_0)$. Conversely, for an arbitrary pair m_1, m_2, we have $M_1(\frac{y_0^2}{2p} + \lambda\alpha, y_0 + \lambda\beta)$, $M_2(\frac{y_0^2}{2p} + \mu\beta, y_0 - \mu\alpha)$ where α, β are real numbers, not both zero, and λ, μ are the nonzero solutions to $(y_0 + \lambda\beta)^2 = 2p(\frac{y_0^2}{2p} + \lambda\alpha)$, $(y_0 - \mu\alpha)^2 = 2p(\frac{y_0^2}{2p} + \mu\beta)$. It is readily found that $\lambda\beta^2 = 2(p\alpha - \beta y_0)$, $\mu\alpha^2 = 2(\alpha y_0 + p\beta)$ and a routine verification confirms that our candidate M' is on the line M_1M_2. Thus, the Frégier point of M is $M'(\frac{y_0^2}{2p} + 2p, -y_0)$, that is, $\mathbf{g}(M)$ where $\mathbf{g}$ is the glide reflection with the same axis as the parabola and vector $2p\overrightarrow{i}$ for a well-chosen unit vector $\overrightarrow{i}$ along this axis. Clearly, the locus of M' is a parabola congruent to the given one. The following supplement is interesting: besides the identity, $\mathbf{g}$ is the only isometry mapping each point M of $\mathcal{P}$ to a point located on the normal at M (the proof is left as an exercise).

A locus and a construction

Let m be a line distinct from the axis ℓ of a glide reflection $\mathbf{g}$ and let $m' = \mathbf{g}(m)$ intersect m at A. For each pair of distinct points M, N on m, let K be the point of intersection of MN' and $M'N$ where $M' = \mathbf{g}(M)$, $N' = \mathbf{g}(N)$. The locus of K is the parallel ℓ_1 to ℓ through $A' = \mathbf{g}(A)$.



To prove this, first consider a point K of the locus. We may suppose $M, N \neq A$ (otherwise, $K = A'$). Then m' is a transversal of the triangle MNK with A, M', N' on the lines MN, NK, MK, respectively. From a well-known theorem, the midpoints of AK, NN', MM' are collinear and so the midpoint of AK is on ℓ. Since the midpoint of AA' is on ℓ as well, K is on ℓ_1.

Conversely, if K is any point on ℓ_1, take M on m such that MK meets m' at N'. If $N = \mathbf{g}^{-1}(N')$, then N is on m and the direct part of the proof shows that NM' passes through K. Thus, K is a point of the locus.

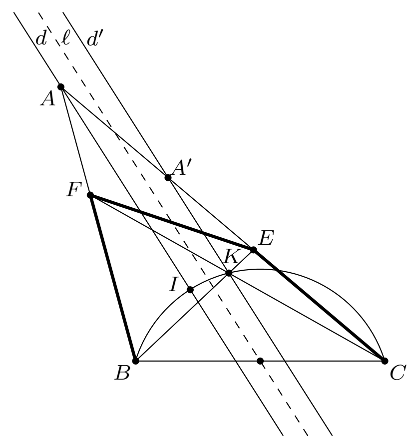
I used this interesting property in my solution of a problem of the 2006 Zhautykov Olympiad (see [2011 : 213]). As another application, consider the following problem found in the book [1]: Given triangle ABC, construct point E on side AC and F on side AB such that $BF = FE = EC$.

The author's solution rests upon a suitable rotation. Using a glide reflection instead leads to a new construction that seems simpler to achieve. Here are the steps:

(1) locate A' on the ray $[CA)$ such that $CA' = BA$ and through A' draw the line d' parallel to the bisector d of $\angle BAC$;

(2) draw the arc of circle subtending $\frac{1}{2}(180^\circ + A)$ on the line segment BC and passing through the incentre I of ΔABC. This arc meets d' at K.

The line BK intersects CA at E and CK intersects BA at F.



We will just examine the case when $BA \neq CA$ and BC is the shortest side of the triangle (as a more thorough study would show, the latter ensures that the problem has a unique solution). We introduce the glide reflection $\mathbf{g}$ whose axis ℓ is the parallel to the bisector d of $\angle BAC$ through the midpoint of BC and such that $\mathbf{g}(B) = C$. This isometry $\mathbf{g}$ transforms F into E, A into A', and from the property above, the point of intersection K of BE and CF is on the parallel d' to d through A'.

In addition, the triangles BFE and CEF are isosceles, hence if $\alpha = \angle FEB = \angle FBE$, $\beta = \angle EFC = \angle ECF$, then $\angle BKC = 180^\circ - (\alpha + \beta)$ as well as $180^\circ - (B - \alpha) - (C - \beta) = A + \alpha + \beta$. Therefore $\angle BKC = \frac{1}{2}(180^\circ + A)$ and the construction follows.

Two exercises

To conclude, we propose two problems:

(a) Let A, B, C, D be four points in the plane such that $AB = CD$ and let M, N be the midpoints of AD, BC, respectively. Show that the angle MN makes with the line AB equals the angle it makes with the line CD. [This problem was set by P. Jullien in the bulletin of the APMEP (French math teachers' association) in 2005, but no solution ever appeared.]

(b) If ABC is a triangle, find the axis and the vector of the glide reflection $\mathbf{r}_{AC} \circ \mathbf{r}_{BC} \circ \mathbf{r}_{AB}$ where $\mathbf{r}_{XY}$ denotes the reflection in the line XY.

References

[1] I. M. Yaglom, *Geometric Transformations I*, MAA, 1962, p. 68 and pp. 132-3.

PROBLEM OF THE MONTH

No. 5

Shawn Godin

*This column is dedicated to the memory of former **CRUX with MAYHEM** Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

The problem that we will consider is problem 40 from [1]:

A function $f(x)$ is periodic with period p if and only if $f(x+p) = f(x)$ for each x. Prove that $\sin x^2$ is not periodic with any non-zero period.

I find problems like this appealing because they seem apparent, yet we must be careful to prove the seemingly obvious. Since $\sin x$ is periodic with period 2π, it would seem that for $\sin x^2$ each period of $\sin x$ gets mapped to regions that are increasingly more compressed as x grows, so repetition seems out of the question. Solutions 2 through 4 will follow the ideas presented in [1].

Solution 1: We use some properties of periodicity along with algebraic manipulation.

Note that the zeros of f are $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \cdots$, we will set $x_n = \sqrt{n\pi}$. If f is periodic, then there must be infinitely many pairs of consecutive zeros that are the same distance apart, since if x_n and x_{n+1} are consecutive zeros then so are $(x_n + kp)$ and $(x_{n+1} + kp)$, where p is the period and k is any integer. Suppose x_n and x_{n+1} and x_m and x_{m+1} are two pairs of positive, distinct, consecutive zeros that are the same distance apart, then $\sqrt{(n+1)\pi} - \sqrt{n\pi} = \sqrt{(m+1)\pi} - \sqrt{m\pi}$ and hence $\sqrt{n+1} - \sqrt{n} = \sqrt{m+1} - \sqrt{m}$. Squaring and rearranging makes this equivalent, successively, to

$$\begin{aligned}
n + 1 + n - 2\sqrt{n(n+1)} &= m + 1 + m - 2\sqrt{m(m+1)} \\
n - m &= \sqrt{n(n+1)} - \sqrt{m(m+1)} \\
n^2 + m^2 - 2nm &= n^2 + n + m^2 + m - 2\sqrt{nm(n+1)(m+1)} \\
2\sqrt{nm(n+1)(m+1)} &= n + m + 2nm \\
4n^2m^2 + 4n^2m + 4nm^2 + 4nm &= n^2 + m^2 + 4n^2m^2 + 4n^2m + 4nm^2 + 2nm \\
(n-m)^2 &= 0
\end{aligned}$$

and hence $n = m$, a contradiction. Thus no pairs of consecutive zeros are the same distance apart, and hence f is not periodic.

Solution 2: We will prove the statement by contradiction.

Assume that the function does have a non-zero period p, so that $f(x+p) = f(x)$ for each x. Then $f(2p) = f(p) = f(0) = 0$. Thus, since the zeros of $\sin x$ are $x = k\pi$ for $k \in \mathbb{Z}$ then as $f(p) = 0$ we must have $\sin p^2 = 0$, so $p = \sqrt{n\pi}$ for some $n \in \mathbb{Z}^+$.

Now, consider the zeros of f on the interval $0 \leq x \leq 2p = 2\sqrt{n\pi} = \sqrt{4n\pi}$. These values satisfy $x^2 = k\pi$, hence the zeros are

$$x = 0, \sqrt{\pi}, \sqrt{2\pi}, \cdots, \sqrt{4n\pi}.$$

Thus $f(x) = 0$ has $n-1$ solutions on the interval $0 < x < p$ and $4n - n - 1 = 3n - 1$ solutions on the interval $p < x < 2p$.

But, if f is periodic with period p, each interval of length p will have the same number of zeros, a contradiction. Therefore f is not periodic.

Solution 3: We will use some trigonometric identities and properties of trigonometric functions to prove the statement, again by contradiction.

We will make use of the identity

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right). \tag{1}$$

Let p be a period for f. Thus, using (1) we get

$$\begin{aligned} \sin((x+p)^2) - \sin((x-p)^2) &= \sin(x^2 + 2px + p^2) - \sin(x^2 - 2px + p^2) \\ &= 2\cos(x^2 + p^2)\sin(2px). \end{aligned}$$

But, as p is a period of f, $f(x+p) = f(x-p)$, and hence $f(x+p) - f(x-p) = 0$ for all $x \in \mathbb{R}$. Thus

$$2\cos(x^2 + p^2)\sin(2px) = 0 \tag{2}$$

for all $x \in \mathbb{R}$. Thus, for any interval $I = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, $\cos(p^2 + x^2)$ vanishes at only finitely many $x \in I$. But, for (2) to be true for all real x, we must have $\sin(2px) = 0$ for all $x \in I$, and so $p = 0$, a contradiction.

Solution 4: We reproduce the pretty proof sent to Professor Barbeau by Adrian Tang, which uses a little calculus and insight.

Let $f(x) = \sin(x^2)$, then $f'(x) = 2x\cos(x^2)$. Thus f has local extreme values for $x = 0$ or $x = \sqrt{\frac{(2k-1)\pi}{2}}$, for $k \in \mathbb{Z}$. But $f(0) = 0$ while $f\left(\sqrt{\frac{(2k-1)\pi}{2}}\right) = \pm 1$. Thus the only local minimum of f where $f(x) = 0$ occurs when $x = 0$. If f was periodic, this minimum would occur infinitely often (once each period). Hence, $\sin x^2$ cannot be periodic.

References

[1] Edward J. Barbeau, *Mathematical Olympiads' Correspondence Program (1995-96)*, A Taste Of Mathematics Volume 1, CMS, 1997

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Electronic submissions are preferable, with each solution contained in a separate file. Solution files should be named using the convention LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions to the editor at* `crux-editors@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Original problems are particularly sought, but other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 July 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3808. *Replacement. Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with area Δ; circumradius R; exradii r_a, r_b, r_c; and excenters I_a, I_b, I_c. The excircle with centre I_a touches the sides of ABC at K, L, and M. Let Δ_1 represent the area of triangle KLM and let Δ_2 and Δ_3 be similarly defined. Prove that

$$\frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = \frac{r_a + r_b + r_c}{2R} .$$

3821. *Proposed by Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan.*

Prove that any triangle can be divided into five triangles such that one of the triangles is equilateral, one is isosceles, one is right angled, one is acute and one is obtuse.

3822. *Proposed by M. N. Deshpande, Nagpur, India.*

Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = \alpha$. Further, let G be its centroid and circle Γ passes through B, C and G. Point D is on the circle, different from G, such that $BD = CD$ and let $\angle BDC = \delta$. Show that

(i) $\alpha + \delta \geq 120^\circ$.

(ii) $\left(\dfrac{\cos\alpha + \cos\delta}{1 + \cos\alpha\cos\delta}\right)$ does not depend on α.

3823. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania; and Titu Zvonaru, Comǎneşti, Romania.*

Let ABC be a triangle with height AD, where E and F are the midpoints of sides AC and AB respectively. For any point P in the plane of the triangle ABC, let Y and Z be its symmetric from the points E and F, respectively. If P' is the midpoint of DP and $M = BY \cap CZ$, then prove that the line through M and P' passes through a fixed point.

3824. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Dexter Wei, University of Waterloo, Waterloo, ON.*

Let

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + + \frac{1}{\sqrt{n}}$$

where $n \in \mathbb{N}$. It is well known that $S_n \geq 2(\sqrt{n+1} - 1)$. Prove or disprove the stronger inequality that

$$S_n \geq \frac{2n}{1 + \sqrt{n}}\,.$$

3825. *Proposed by Brian Brzycki, Troy High School, Whittier, CA, USA.*

Triangle ABC is acute. Points X and Y trisect side BC, with X closer to B. Semicircles centred at X and Y and tangent to AB and AC are drawn, respectively.

(a) Prove that the two semicircles must intersect.

(b) If the semicircles intersect at Z, and $\angle XZY = \theta$, prove that

$$\cos(2B) + \cos(2C) + 4\sin(B)\sin(C)\cos(\theta) = 0\,.$$

3826. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \mapsto [0,\infty)$ and let $g : [0,1] \mapsto [0,\infty)$ be two continuous functions. Find the value of

$$\lim_{n\to\infty} \sqrt[n]{f\left(\frac{1}{n}\right) g\left(\frac{n}{n}\right) + f\left(\frac{2}{n}\right) g\left(\frac{n-1}{n}\right) + \cdots + f\left(\frac{n}{n}\right) g\left(\frac{1}{n}\right)}\,.$$

3827. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

For integer k, let $f(k)$ be the largest prime factor of k. The sequences $\{a_n\}$, $\{b_n\}$ are defined by $a_0 = b_0 = pq$, $a_{n+1} = a_n + pf(a_n)$, $b_{n+1} = b_n + qf(b_n)$ for $n \geq 1$, for given positive integers p and q. Prove that there are infinitely many pairs of integers (c, d) that satisfy

$$\frac{a_c}{p} = \frac{b_d}{q}\,.$$

3828. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be an acute angled triangle with $\angle B = 2\angle C$ and altitude AD. Drop perpendiculars DK and DL from D to the sides AB and AC respectively.

(a) Prove that $\sin A > \dfrac{2\sin^2 C}{1+\cos C}$.

(b) If $\frac{AD}{KL} = \sqrt{5}-1$, find the angles of the triangle ABC.

3829. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, c be positive real numbers and $\Delta = a^2 + b^2 + c^2 - (ab + bc + ca)$. Improve the well known inequality $\Delta \geq 0$ by proving that

$$\Delta \geq \left(\frac{a(a-b)^2(a-c)^2 + b(b-c)^2(b-a)^2 + c(c-a)^2(c-b)^2}{a+b+c}\right)^{\frac{1}{2}}\,.$$

3830. *Proposed by Tigran Hakobyan, Yerevan State University, Yerevan, Armenia.*

Let $a > 0$. Define the sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers by

$$a_1 = a, a_{n+1} = a_n + \{a_n\}, n \geq 1$$

where $\{x\}$ is the fractional part of x. Find all $a > 0$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined above is bounded.

. . . .

3808. *Remplacement. Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle d'aire Δ, R le rayon de son cercle circonscrit, r_a, r_b et r_c les rayons des cercles exinscrits, I_a, I_b et I_c leurs centres respectifs. Le cercle exinscrit de centre I_a touche les côtés de ABC en K, L et M. Soit Δ_1 l'aire du triangle KLM et soit Δ_2 et Δ_3 définies de manière analogue. Montrer que

$$\frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = \frac{r_a + r_b + r_c}{2R}\,.$$

3821. *Proposé par Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan.*

Montrer que tout triangle peut être divisé en cinq triangles, soit un triangle équilatéral, un isocèle, un rectangle, un acutangle et un cinquième obtusangle.

3822. *Proposé par M. N. Deshpande, Nagpur, Inde.*

Soit ABC un triangle isocèle avec $AB = AC$ et $\angle A = \alpha$. De plus, soit G son centre de gravité et Γ le cercle passant par les points B, C et G. Soit D un point du cercle, différent de G, tel que $BD = CD$ et soit $\angle BDC = \delta$. Montrer que

(i) $\alpha + \delta \geq 120°$.

(ii) $\left(\dfrac{\cos\alpha + \cos\delta}{1 + \cos\alpha\cos\delta}\right)$ ne dépend pas de α.

3823. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie ; et Titu Zvonaru, Comăneşti, Roumanie.*

Soit ABC un triangle de hauteur AD, E et F les points milieu respectifs des côtés AC et AB. Pour un point P quelconque dans le plan du triangle ABC, soit respectivement Y et Z ses symétriques par rapport aux points E et F. Si P est le point milieu de DP et $M = BY \cap CZ$, montrer alors que la droite passant par M et P passe par un point fixe.

3824. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON ; et Dexter Wei, Université de Waterloo, Waterloo, ON.*

Soit

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + + \frac{1}{\sqrt{n}}$$

où $n \in \mathbb{N}$. On sait très bien que $S_n \geq 2(\sqrt{n+1} - 1)$. Montrer si oui ou non l'inégalité plus forte

$$S_n \geq \frac{2n}{1 + \sqrt{n}}\,.$$

est encore valide.

3825. *Proposé par Brian Brzycki, Troy High School, Whittier, CA, USA.*

Soit ABC un triangle acutangle et X et Y deux points distincts sur le côté BC, X le plus proche de B. On dessine respectivement deux demi-cercles, centrés en X et Y, tangents à AB et AC.

(a) Montrer que les deux demi-cercles doivent se couper.

(b) Si les demi-cercles se coupent en Z, et $\angle XZY = \theta$, montrer que

$$\cos(2B) + \cos(2C) + 4\sin(B)\sin(C)\cos(\theta) = 0\,.$$

3826. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $f : [0,1] \mapsto [0,\infty)$ et soit $g : [0,1] \mapsto [0,\infty)$ deux fonctions continues. Trouver la valeur de

$$\lim_{n\to\infty} \sqrt[n]{f\left(\frac{1}{n}\right) g\left(\frac{n}{n}\right) + f\left(\frac{2}{n}\right) g\left(\frac{n-1}{n}\right) + \cdots + f\left(\frac{n}{n}\right) g\left(\frac{1}{n}\right)}.$$

3827. *Proposé par Jung In Lee, École Secondaire Scientifique de Séoul, Séoul, République de Corée.*

Pour un entier k, soit $f(k)$ le plus grand facteur premier de k. On définit les suites $\{a_n\}$ et $\{b_n\}$ par $a_0 = b_0 = pq$, $a_{n+1} = a_n + pf(a_n)$, $b_{n+1} = b_n + qf(b_n)$ pour $n \geq 1$, p et q étant deux entiers positifs donnés. Montrer qu'il existe une infinité de paires d'entiers (c, d) satisfaisant

$$\frac{a_c}{p} = \frac{b_d}{q}.$$

3828. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle acutangle avec $\angle B = 2\angle C$ et hauteur AD. Tirer deux perpendiculaires DK et DL de D sur les côtés AB et AC respectivement.

(a) Montrer que $\sin A > \dfrac{2\sin^2 C}{1+\cos C}$.

(b) Si $\frac{AD}{KL} = \sqrt{5} - 1$, trouver les angles du triangle ABC.

3829. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c trois nombres réels positifs et $\Delta = a^2 + b^2 + c^2 - (ab + bc + ca)$. Améliorer l'inégalité bien connue $\Delta \geq 0$ en montrant que

$$\Delta \geq \left(\frac{a(a-b)^2(a-c)^2 + b(b-c)^2(b-a)^2 + c(c-a)^2(c-b)^2}{a+b+c}\right)^{\frac{1}{2}}.$$

3830. *Proposé par Tigran Hakobyan, Université d'État de Yerevan , Yerevan, Arménie.*

Soit $a > 0$. On définit la suite $\{a_n\}_{n=0}^{\infty}$ de nombres réels par

$$a_1 = a, a_{n+1} = a_n + \{a_n\}, n \geq 1$$

où $\{x\}$ est la partie fractionnaire de x. Trouver tous les $a > 0$ tels que la suite définie plus haut soit bornée.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

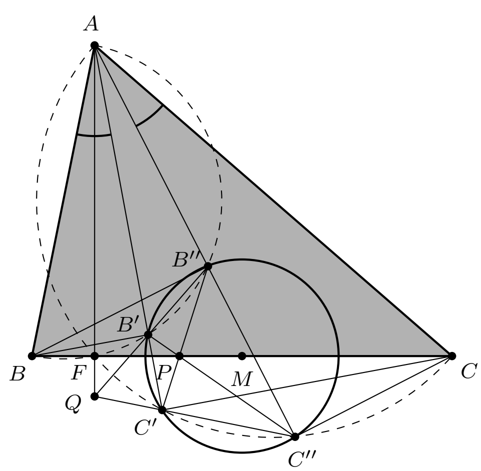
The editor would like to acknowledge PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; whose solution to problem 3718 was overlooked. The editor apologizes sincerely for the oversight.

3721. [2012 : 104, 106] *Proposed by Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain.*

Given the triangle ABC and two isogonal cevians AA', AA'', call B', C' the orthogonal projections of B, C on AA' and B'', C'' the orthogonal projections of B, C on AA''. If $P = B'C'' \cap C'B''$ and $Q = B'B'' \cap C'C''$, show that P lies on line BC and Q lies on the altitude through A. *Dedicated to the memory of Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Solution by Michel Bataille, Rouen, France.

(a) We first show that Q is on the altitude from A.



Let F be the foot of the altitude from A. Because $AF \perp FB$, $AB' \perp B'B$, and $AB'' \perp B''B$, the circle Γ_b with diameter AB passes through F, B', and B''. Similarly, the circle Γ_c with diameter AC passes through F, C', and C''. If we use $\angle(\ell, \ell')$ to denote the angle directed from ℓ to ℓ' (through which ℓ must be rotated in the positive direction in order to become parallel to, or to coincide with ℓ'), we have, modulo π,

$$\angle(B'A, B'B'') = \angle(BA, BB''), \quad \text{that is,} \quad \angle(B'C', B'B'') = \angle(AB, AA'') + \frac{\pi}{2}$$

[where the angles on the left of each equality are identical while those on the right are oppositely oriented angles in the same right triangle ABB'']; in the same way

$$\angle(C''B'', C''C') = \angle(CA, CC') = \angle(AC, AA') + \frac{\pi}{2}.$$

Thus,

$$\angle(B'C', B'B'') - \angle(C''C', C''B'') = \angle(AB, AA'') + \angle(AC, AA') = 0,$$

where the last equality follows from the assumption that AA' and AA'' are isogonal. As a consequence, B', C', B'', C'' are concyclic, say on the circle Γ. The power of Q with respect to Γ is $\overrightarrow{QB'} \cdot \overrightarrow{QB''} = \overrightarrow{QC'} \cdot \overrightarrow{QC''}$, which means that the power of Q with respect to Γ_b equals the power of Q with respect to Γ_c. Thus, Q is on the radical axis of the two circles, which is just the altitude AF.

(b) We now prove that P lies on the line BC.

Because $B'C'$ intersects $B''C''$ at A, while $B''C', B'C''$ intersect at P and $C'C'', B'B''$ intersect at Q, the triangle APQ is self-conjugate with respect to Γ. In particular, AQ, as the polar of P with respect to Γ is perpendicular to MP, where M is the centre of Γ. Since $AQ \perp BC$, the lines PM and BC would coincide (in which case P would lie on BC), if M were on BC. To prove the latter, consider the inversion $\mathbf{I}$ with centre A such that $\mathbf{I}(B') = C'$. We then have $\mathbf{I}(B'') = C''$ and the lines $B'B'', C'C''$ invert into Γ_c, Γ_b, respectively. It follows that $\mathbf{I}$ exchanges Q and F, and that the circle with diameter AQ inverts into BC, the perpendicular to AQ at F. Now, A and Q being conjugate with respect to Γ, the circle with diameter AQ is orthogonal to Γ and so its image BC under $\mathbf{I}$ is a diameter of Γ; that is, $M \in BC$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany (property of point P only); JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Geupel observed that much of our problem follows from paragraphs 631 and 633 of Nathan Altshiller Court's College Geometry*, 2nd ed., 1952; in particular, Court provides an alternative proof that the four orthogonal projections B', B'', C', C'' of B and C upon the sides of $\angle A'AA''$ lie on a common circle whose centre is the midpoint of BC. Here is Court's elementary proof that the centre M of Γ is the midpoint of BC: Because $B'C'$ is a chord of Γ, its perpendicular bisector contains M; moreover, because this perpendicular bisector along with BB' and CC' are three parallel lines that intercept equal segments on the transversal $B'C'$, these lines must likewise intersect the transversal BC in B, C, and the midpoint of the segment BC. The same argument applied to the segment $B''C''$ yields a second line (not parallel to the first) that also contains both M and the midpoint of BC, which implies that these two points must coincide.*

3722. [2012 : 104, 106] *Proposed by Michel Bataille, Rouen, France.*

Prove that

$$\left(\frac{1}{4} - 4\cos^2\frac{2\pi}{17}\cos^2\frac{8\pi}{17}\right)\left(\frac{1}{4} - 4\cos^2\frac{3\pi}{17}\cos^2\frac{5\pi}{17}\right) + 4\cos\frac{2\pi}{17}\cos\frac{3\pi}{17}\cos\frac{5\pi}{17}\cos\frac{8\pi}{17} = 0.$$

I. Solution by Itachi Uchiha, Hong Kong, China.

Let
$$x = \cos\frac{2\pi}{17}\cos\frac{8\pi}{17} \quad \text{and} \quad y = \cos\frac{3\pi}{17}\cos\frac{5\pi}{17}.$$
It is required to show that
$$(1-16x^2)(1-16y^2) = -64xy.$$

We begin by showing that
$$1+4x-4y+16xy = 0.$$

Repeated use of the product-to-sum conversion formula and the relation $\cos\frac{(34-2k)\pi}{17} = \cos\frac{2k\pi}{17}$ yields that
$$\begin{aligned} 1+4x-4y+16xy &= 1+2\left(\cos\frac{6\pi}{17}+\cos\frac{10\pi}{17}\right)-2\left(\cos\frac{2\pi}{17}+\cos\frac{8\pi}{17}\right) \\ &\quad +4\left(\cos\frac{6\pi}{17}+\cos\frac{10\pi}{17}\right)\left(\cos\frac{2\pi}{17}+\cos\frac{8\pi}{17}\right) \\ &= 1+2\sum_{k=1}^{8}\cos\frac{2k\pi}{17} = 1+\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{17}+\cos\frac{(34-2k)\pi}{17}\right) \\ &= \sum_{k=0}^{16}\cos\frac{2k\pi}{17} = \mathrm{Re}\sum_{k=0}^{16}\zeta^k = 0, \end{aligned}$$

where $\zeta = \cos\frac{2\pi}{17} + i\sin\frac{2\pi}{17}$ is a primitive seventeenth root of unity.

Therefore
$$(1-4x)(1+4y) = 1-4x+4y-16xy = 2-(1+4x-4y+16xy) = 2$$
and
$$(1+4x)(1-4y) = (1+4x-4y+16xy)-32xy = -32xy.$$
Multiplying these two equations yields the desired result.

II. Solution by the proposer.

Let $\zeta = \cos\frac{2\pi}{17} + i\sin\frac{2\pi}{17}$,
$$a = 4\cos\frac{2\pi}{17}\cos\frac{8\pi}{17} = 2\left(\cos\frac{6\pi}{17}+\cos\frac{10\pi}{17}\right) = \zeta^3+\zeta^{14}+\zeta^5+\zeta^{12}$$
and
$$b = 4\cos\frac{3\pi}{17}\cos\frac{5\pi}{17} = 2\left(\cos\frac{2\pi}{17}+\cos\frac{8\pi}{17}\right) = \zeta+\zeta^{16}+\zeta^4+\zeta^{13}.$$
It is required to show that
$$(1-a^2)(1-b^2)+4ab = 0.$$

For easier computation, introduce $c = \zeta^6 + \zeta^{11} + \zeta^7 + \zeta^{10}$ and $d = \zeta^2 + \zeta^{15} + \zeta^8 + \zeta^9$. Since $a + b + c + d = -1$ and $ab = 2b + c + d = b - a - 1$, therefore

$$\begin{aligned}(1 - a^2)(1 - b^2) + 4ab &= (1 - a)(1 - b)(1 + a)(1 + b) + 4ab \\ &= (1 + ab - a - b)(1 + ab + a + b) + 4ab \\ &= (b - a - a - b)(b - a + a + b) + 4ab \\ &= (-2a)(2b) + 4ab = 0.\end{aligned}$$

III. Solution using elements of solutions from Brian D. Beasley, Presbyterian College, Clinton, SC, USA; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Let $a_k = \cos(k\pi/17)$ and $x = a_2 = \cos(2\pi/17)$. The left side of the identity is equal to

$$\begin{aligned}&\frac{1}{16}[(1 + 16a_2a_8a_3a_5)^2 - 16(a_2a_8 - a_3a_5)^2] \\ &\quad = \frac{1}{16}(1 + 16a_2a_8a_3a_5 - 4a_2a_8 + 4a_3a_5)(1 + 16a_2a_8a_3a_5 + 4a_2a_8 - 4a_3a_5).\end{aligned}$$

Since

$$a_2a_8 = a_2[2a_4^2 - 1] = a_2[2(2a_2^2 - 1)^2 - 1] = x(8x^4 - 8x^2 + 1)$$

and

$$2a_3a_5 = a_2 + a_8 = 8x^4 - 8x^2 + x + 1,$$

the final factor on the right side is equal to

$$\begin{aligned}&1 + 8x(8x^4 - 8x^2 + 1)(8x^4 - 8x^2 + x + 1) \\ &\qquad\qquad + 4x(8x^4 - 8x^2 + 1) - 2(8x^4 - 8x^2 + x + 1) \\ &\quad = 512x^9 - 1024x^7 + 64x^6 + 672x^5 - 80x^4 - 160x^3 + 24x^2 + 10x - 1 \\ &\quad = (2x - 1)f(x),\end{aligned}$$

where

$$f(x) = 256x^8 + 128x^7 - 448x^6 - 192x^5 + 240x^4 + 80x^3 - 40x^2 - 8x + 1.$$

With $T_{17}(x)$ denoting the Chebyshev polynomial of degree 17, we have that

$$\begin{aligned}1 &= \cos 2\pi = T_{17}(x) \\ &= 65536x^{17} - 278528x^{15} + 487424x^{13} - 452608x^{11} \\ &\qquad\qquad + 239360x^9 - 71808x^7 + 11424x^5 - 816x^3 + 17x \\ &= 1 + (x - 1)f(x)^2,\end{aligned}$$

so that $f(x) = 0$ and the desired result follows.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, *Germany; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and TITU ZVONARU, Comăneşti, Romania.*

Geupel and the Angelo State University trio showed that the left side was equal to

$$\frac{17}{16}\left(1-2\sum_{k=1}^{8}(-1)^{k+1}a_k\right),$$

which vanished because the sum was shown to be equal to 1/2. Lau followed the strategy of the third solution and showed that the left side is equal to $(1/16)(4a_1^2-3)f(a_1)f(-a_1)g(a_1)$*, where* g *is a polynomial of degree 18.*

3723. [2012 : 104, 106] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $a+b+c=1$. If n is a positive integer, prove that

$$\frac{(3a)^n}{(b+1)(c+1)}+\frac{(3b)^n}{(c+1)(a+1)}+\frac{(3c)^n}{(a+1)(b+1)}\geq\frac{27}{16}.$$

Solution by Itachi Uchiha, Hong Kong, China.

By the Power Mean Inequality followed by the AM-GM Inequality, we have

$$\begin{aligned}
&(3a)^n(a+1)+(3b)^n(b+1)+(3c)^n(c+1)\\
&\qquad=\frac{1}{3}\left[(3a)^{n+1}+(3b)^{n+1}+(3c)^{n+1}\right]+3\cdot\frac{1}{3}\left[(3a)^n+(3b)^n+(3c)^n\right]\\
&\qquad\geq\left(\frac{3a+3b+3c}{3}\right)^{n+1}+3\left(\frac{3a+3b+3c}{3}\right)^n\\
&\qquad=1+3=4=\frac{27}{16}\left(\frac{4}{3}\right)^3=\frac{27}{16}\left(\frac{a+1+b+1+c+1}{3}\right)^3\\
&\qquad\geq\frac{27}{16}(a+1)(b+1)(c+1)\,.
\end{aligned}$$

Divide both sides by $(a+1)(b+1)(c+1)$ and the result follows. Clearly we have equality if and only if $a=b=c=\frac{1}{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RADOUAN BOUKHARFANE, Polytechnique de Montréal, QC; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3724. [2012 : 105, 106, 194, 195; 2013 : 55, 56] *Replacement. Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

The edge-lengths of a cyclic quadrilateral are 7, 8, 4, 1, in that order. What are the lengths of the diagonals?

[*Ed.: The original problem 3724 was a reproduction of Mayhem problem M504. Some solutions to the first version of 3724 were received by the editor and a new generalization was published in the Mayhem section of the previous issue [2013 : 55, 56].*]

I. Solution by George Apostolopoulos, Messolonghi, Greece.

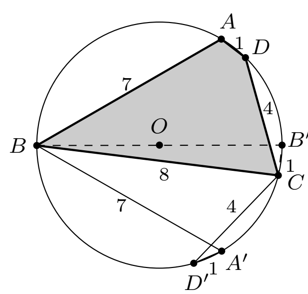
Label the sides of our quadrilateral $ABCD$ by $AB = a = 7$, $BC = b = 8$, $CD = c = 4$, $DA = d = 1$, and the unknown diagonals $AC = x$, $BD = y$. Ptolemy's theorem tells us that the diagonals satisfy

$$xy = ac + bd, \quad \text{and} \quad \frac{y}{x} = \frac{ab + cd}{ad + bc};$$

that is

$$xy = 7 \cdot 4 + 8 \cdot 1 = 36 \quad \text{and} \quad \frac{y}{x} = \frac{7 \cdot 8 + 4 \cdot 1}{7 \cdot 1 + 8 \cdot 4} = \frac{20}{13}.$$

Multiplying gives us $y = 12\sqrt{\frac{5}{13}} = \frac{12}{13}\sqrt{65} \approx 7.442$;
division yields $x = 3\sqrt{\frac{13}{5}} = \frac{3}{5}\sqrt{65} \approx 4.837$.



A cyclic quadrilateral is, by definition, convex; on the other hand, it turns out to be interesting to find the lengths of the diagonals of a crossed quadrilateral inscribed in a circle with edge lengths $7, 8, 4, 1$, in that order; this is the quadrilateral $A'BCD'$ in the accompanying figure. The vertices lie clockwise in the order $A'D'BC$, so that the resulting *convex* quadrilateral has sides $A'D' = 1, D'B = y, BC = 8, CA' = x$, and diagonals $CD = 4, AB = 7$. Ptolemy's theorem now implies that

$$4 \cdot 7 = xy + 8 \cdot 1 \quad \text{and} \quad \frac{4}{7} = \frac{1 \cdot y + x \cdot 8}{1 \cdot x + y \cdot 8},$$

so that the diagonals of $A'BCD'$ are

$$x = 5 \cdot \sqrt{\frac{5}{13}} = \frac{5}{13}\sqrt{65} \approx 3.101, \quad y = 4\sqrt{\frac{13}{5}} = \frac{4}{5}\sqrt{65} \approx 6.450.$$

II. Solution by Mihaï-Ioan Stoënescu, Bischwiller, France.

Le théorème des cosinus dans les triangles DAB et BCD nous dit que:

$$BD^2 = DA^2+AB^2-2\cdot DA\cdot AB\cos\angle A,\ \text{ et } BD^2 = BC^2+CD^2-2\cdot BC\cdot CD\cos\angle C.$$

Or les angles A et C sont supplémentaires donc $\cos\angle A = -\cos\angle C = t$. En remplaçant par les valeurs données, on tire que

$$1+49-14t = 64+16+64t,$$

d'où $t = -\frac{5}{13}$. Par conséquent, $BD^2 = 50+\frac{70}{13} = \frac{720}{13}$.
Il vient que $BD = \sqrt{\frac{720}{13}}$.
Le même théorème dans les triangles ABC et CDA donne que $\cos\angle D = -\frac{4}{5}$ et $AC = \sqrt{\frac{117}{5}}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; DIMITRIOS KOUKAKIS, Kilkis, Greece; KEE-WAI LAU, Hong Kong, China; PANAGIOTE LIGOURAS, Leonardo da Vinci High School, Noci, Italy; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

Almost all submissions resembled one of the featured solutions; some used directly the known formulas for the diagonals of cyclic quadrilateral (which follow immediately from the two forms of Ptolemy's theorem used in the first solution). Those formulas are readily found on the internet (just google "cyclic quadrilateral") or in college geometry texts. Compare the related problem 3751 [2012 : 241, 243] which calls for the circumradius of an inscribed quadrilateral in terms of its side lengths. Although there is a familiar formula also for this, Woo observed that the proposer chose the lengths (in both problems) quite carefully: note that $7^2+4^2 = 8^2+1^2 = 65$. In other words, the edges can be rearranged to form a quadrilateral composed of a pair of right triangles that share a hypotenuse of length $\sqrt{65}$. It is an easy exercise to show (without resorting to algebra) that any convex cyclic quadrilateral with sides of length $1, 4, 7,$ and 8 in any order would have the same value $\frac{\sqrt{65}}{2}$ as its circumradius. As a bonus in our problem, because one of those quadrilaterals has a diagonal that is a diameter of the circle (the line BB' in the figure) , the crossed quadrilateral from the first solution is also inscribed in the same circle.

3725. [2012 : 105, 107] *Proposed by Cîrnu Mircea, Bucharest, Romania.*

Prove that the sequence of nonzero real numbers, $x_1, x_2, \ldots$, is a geometric progression if and only if it satisfies the recurrence relation

$$nx_1x_n = \sum_{k=1}^{n} x_kx_{n+1-k},\ n = 1, 2, \ldots.$$

Composite of similar solutions by all solvers.

If $x_1, x_2, \ldots$ is a geometric progression, then for some constant $r > 0$, we have $x_n = r^{n-1}x_1$ for all $n \in \mathbb{N}$. Hence

$$\sum_{k=1}^{n} x_k x_{n+1-k} = \sum_{k=1}^{n} r^{k-1}x_1 r^{n-k}x_1 = \sum_{k=1}^{n} x_1\left(r^{n-1}x_1\right) = nx_1x_n\,.$$

Conversely, suppose $\sum_{k=1}^{n} x_k x_{n+1-k} = nx_1x_n$ for all $n \in \mathbb{N}$. We prove by induction that for all $n \in \mathbb{N}$,

$$x_n = r^{n-1}x_1, \quad \text{where} \quad r = \frac{x_2}{x_1}\,. \tag{1}$$

Since (1) is clearly true for $n = 1, 2$, we assume that it holds for $i = 1, 2, \ldots, n$ for some $n \geq 2$, Then we have

$$(n+1)x_1x_{n+1} = \sum_{k=1}^{n+1} x_k x_{n+2-k} = 2x_1x_{n+1} + \sum_{k=2}^{n} x_k x_{n+2-k}$$

so

$$(n-1)x_1x_{n+1} = \sum_{k=2}^{n} r^{k-1}x_1 r^{n+1-k}x_1 = \sum_{k=2}^{n} r^n x_1^2 = (n-1)r^n x_1^2$$

from which it follows that $x_{n+1} = r^n x_1$ completing the induction and the proof.

Solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; D. M. BĂTINEŢU-GIURGIU, Bucharest, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; RADOUAN BOUKHARFANE, Polytechnique de Montréal, QC; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3726. [2012 : 105, 107] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let A, B, C, s, r, R represent the angles (measured in radians), the semiperimeter, the inradius and the circumradius of a triangle, respectively. Prove that

$$\left(\frac{A}{B} + \frac{B}{C} + \frac{C}{A}\right)^3 \geq \frac{2s^2}{Rr}.$$

Composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and the proposer.

By the AM-GM Inequality, we have

$$\frac{A}{B}+\frac{A}{B}+\frac{B}{C}\geq 3\sqrt[3]{\frac{A^2}{BC}}=\frac{3A}{\sqrt[3]{ABC}}\,.$$

Similarly, $\dfrac{B}{C}+\dfrac{B}{C}+\dfrac{C}{A}\geq\dfrac{3B}{\sqrt[3]{ABC}}$ and $\dfrac{C}{A}+\dfrac{C}{A}+\dfrac{A}{B}\geq\dfrac{3C}{\sqrt[3]{ABC}}$. Adding the three inequalities above we have

$$\frac{A}{B}+\frac{B}{C}+\frac{C}{A}\geq\frac{A+B+C}{\sqrt[3]{ABC}}=\frac{\pi}{\sqrt[3]{ABC}}\,. \tag{1}$$

In [1], it was proved that

$$\frac{abc}{ABC}\geq\left(\frac{2s}{\pi}\right)^3 \tag{2}$$

where a, b, c denote the lengths of the sides of the given triangle. Since it is well known that $abc=4Rrs$, it follows from (1) and (2) that

$$\left(\frac{A}{B}+\frac{B}{C}+\frac{C}{A}\right)^3\geq\frac{\pi^3}{ABC}\geq\frac{(2s)^3}{abc}=\frac{8s^3}{4Rrs}=\frac{2s^2}{Rr}\,,$$

and the proof is complete. It is easy to see that equality holds if and only if the triangle is equilateral.

References

[1] J. Sandor and D. M. Milošević, *Some inequalities for the elements of a triangle*, OCTOGON (Romania) Mathematical Magazine, Vol. 6, No. 1, April, 1998, pp. 42–43.

Also solved by MARIAN DINCĂ, Bucharest, Romania; EDMUND SWYLAN, Riga, Latvia; and PETER Y. WOO, Biola University, La Mirada, CA, USA;

3727. [2012 : 105, 107] *Proposed by J. Chris Fisher, University of Regina, Regina, SK.*

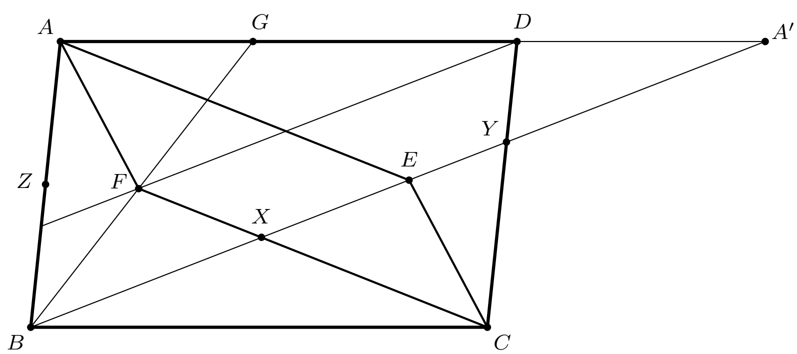
Let $ABCD$ and $AECF$ be two parallelograms with vertices E and F inside the region bounded by $ABCD$. Prove that line BE bisects segment CF if and only if BF meets AD in a point G that satisfies

$$\frac{DA}{DG}=\frac{BF}{FG}.$$

I. Solution by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India.

Observe that if either of E or F were to lie on the diagonal BD, then so would the other, in which case we would have neither BE bisecting CF nor $\frac{DA}{DG}=\frac{BF}{FG}$.

Thus let us assume that E does not lie on BD, so that BE will intersect AD in a point, call it A', for which GBA' is a proper triangle. Further, define the points X and Y to be the intersections of BE with CF and CD, respectively. From the symmetry of the given parallelograms about their common centre we conclude that DF and BE are parallel, whence in triangle CFD we see that X is the midpoint of CF if and only if Y is the midpoint of CD. Moreover, in triangle GBA' we have $\frac{BF}{FG} = \frac{A'D}{DG}$. It follows that the desired equality $\frac{DA}{DG} = \frac{BF}{FG}$ would hold if and only if $DA = DA'$. Our assumption that the point E is inside $ABCD$ implies that $A \neq A'$, and we could have $DA = DA'$ if and only if D were the midpoint of AA'. This in turn is equivalent to the triangles $DA'Y$ and CBY being symmetric about their common vertex Y, which occurs if and only if Y is the midpoint of CD, which (as we remarked earlier) is equivalent to BE intersecting CF in its midpoint X.



II. Composite of similar solutions by George Apostolopoulos, Messolonghi, Greece; Edmund Swylan, Riga, Latvia; Itachi Uchiha, Hong Kong, China; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

In contrast to the first solution we start with X the midpoint of CF and Y the midpoint of CD, and we define Z to be the midpoint of AB. Again by symmetry $BE||FD$, which implies (in ΔCFD) that BE bisects CF if and only if it bisects CD. With the help of Menelaus's theorem applied to ΔGAB and points D, F, Z we therefore have

$$\begin{aligned} BE \text{ bisects } CF \Leftrightarrow X \in BE \Leftrightarrow Y \in BE \Leftrightarrow Z \in DF \text{ (by symmetry)} \\ \Leftrightarrow \frac{BF}{FG} \cdot \frac{GD}{DA} \cdot \frac{AZ}{ZB} = -1 \\ \Leftrightarrow \frac{DA}{DG} = \frac{BF}{FG} \text{ (as directed distances).} \end{aligned}$$

Observe that if we interpret notation such as PQ to represent the length of the line segment joining P to Q (instead of representing the distance directed from P to Q), and if D were between G and A, then $\frac{DA}{DG} = \frac{BF}{FG}$ would imply that F must be situated between G and B on a side of ΔGAB with $DF||AB$. But that would place F on the line CD and, thus, not inside the parallelogram $ABCD$ as prescribed.

We conclude that the result holds whether or not we interpret distances as being directed.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; and the proposer. There was one incomplete submission.

With a little care, either solution shows that the result continues to hold when F is chosen anywhere in the plane of $ABCD$ as long as it is not on the line CD. Note further that this comment together with solution II implies that our problem is equivalent to the theorem,

If F is any point in the plane of ΔABD that is not on the parallel to AB through D, and G is the point where BF meets AD, then F lies on the median through D if and only if G satisfies $\frac{DA}{DG} = \frac{BF}{FG}$.

3728. [2012 : 105, 107] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Given a continuous function $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ that satisfies

$$\int_0^{\frac{\pi}{2}} \left([f(x)]^2 - 2f(x)(\sin x - \cos x)\right) dx = 1 - \frac{\pi}{2},$$

show that

$$\int_0^{\frac{\pi}{2}} f(x)dx = 0.$$

Composite of many submitted solutions.

Observe that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} &[f(x) - (\sin x - \cos x)]^2 dx \\ &= \int_0^{\frac{\pi}{2}} [f(x)^2 - 2f(x)(\sin x - \cos x)]dx + \int_0^{\frac{\pi}{2}} (\sin x - \cos x)^2 dx \\ &= \left(1 - \frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} (1 - \sin 2x)dx = \left(1 - \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 1\right) = 0,\end{aligned}$$

whence $f(x) = \sin x - \cos x$ and $\int_0^{\frac{\pi}{2}} f(x) = 0$.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; D.M. BATINETU-GIURGIU, Bucharest and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; RADOUAN BOUKHARFANE, Polytechnique de Montréal, QC; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JAYSON K.T. SMITH, Southeastern Missouri State University, Cape Griardeau, MO; ALBERT STADLER, Herrliberg, Switzerland; ITACHI UCHIHA, Hong Kong, China; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

The main variant in the solutions was due to Boukharfane who replaced $\sin x - \cos x$ *by* $\sqrt{2}\sin(x - \pi/4)$.

3729. [2012 : 105, 107] *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

If a, b, c are the side lengths of a triangle, prove that

$$\frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab} \le \frac{3(a+b+c)}{ab+bc+ca}.$$

Solution by the proposer, expanded slightly by the editor.

The given inequality is equivalent, in succession, to

$$\sum_{\text{cyclic}} \left(\frac{1}{a} - \frac{b+c}{a^2+bc}\right) \ge \left(\sum_{\text{cyclic}} \frac{1}{a}\right) - \frac{3(a+b+c)}{ab+bc+ca},$$

$$\sum_{\text{cyclic}} \frac{a^2+bc-ab-ac}{a(a^2+bc)} \ge \frac{ab+bc+ca}{abc} - \frac{3(a+b+c)}{ab+bc+ca},$$

$$\begin{aligned}\sum_{\text{cyclic}} \frac{(a-b)(a-c)}{a(a^2+bc)} &\ge \frac{(ab+bc+ca)^2 - 3abc(a+b+c)}{abc(ab+bc+ca)} \\ &= \frac{\displaystyle\sum_{\text{cyclic}} (b^2c^2 + bca^2 - b^2ca - bc^2a)}{abc(ab+bc+ca)} \\ &= \frac{\displaystyle\sum_{\text{cyclic}} bc(a-b)(a-c)}{abc(ab+bc+ca)},\end{aligned}$$

$$\sum_{\text{cyclic}} (a-b)(a-c)\left(\frac{1}{a(a^2+bc)} - \frac{1}{a(ab+bc+ca)}\right) \ge 0,$$

$$\sum_{\text{cyclic}} \frac{(a-b)(a-c)(b+c-a)}{(a^2+bc)(ab+bc+ca)} \ge 0,$$

$$\sum_{\text{cyclic}} \frac{(a-b)(a-c)(b+c-a)}{a^2+bc} \ge 0. \qquad (1)$$

Now, without loss of generality, we can assume that $a \ge b \ge c$. Since a, b, and c are the side lengths of a triangle, we have

$$ca - c^2 - ab + b^2 = (b^2 - c^2) - a(b-c) = (b-c)(b+c-a) \ge 0$$

so $c(a-c) \ge ab - b^2 = b(a-b)$. Hence

$$a - c \ge \frac{b(a-b)}{c}. \qquad (2)$$

Using (2) we have, since $a-b\ge 0$ and $a-c\ge 0$,

$$\begin{aligned}\sum_{\text{cyclic}}\frac{(a-b)(a-c)(b+c-a)}{a^2+bc} &\ge \frac{(b-c)(b-a)(c+a-b)}{b^2+ca}+\frac{(c-a)(c-b)(a+b-c)}{c^2+ab}\\ &\ge \frac{(b-c)(b-a)(c+a-b)}{b^2+ca}+\frac{b(a-b)(b-c)(a+b-c)}{c(c^2+ab)}\\ &= (a-b)(b-c)\left(\frac{b(a+b-c)}{c(c^2+ab)}-\frac{c+a-b}{b^2+ca}\right)\\ &\ge (a-b)(b-c)\left(\frac{b(c+a-b)}{c(c^2+ab)}-\frac{c+a-b}{b^2+ca}\right)\\ &= \frac{(a-b)(b-c)(c+a-b)(b^3-c^3)}{c(c^2+ab)(b^2+ca)}\ge 0\,,\end{aligned}$$

which establishes (1) and completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Three of these solutions were either computer aided or by complicated argument using Schur's Inequality. Lau gave a proof using calculus together with Euler's Theorem ($2r \le R$) and some other known results. The solution given by the proposer and featured above is the only elementary one. There was also an incorrect solution and a solution making claims with no justifications.

3730. [2012 : 106, 107] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Points D, E and F are the feet of the perpendiculars from some point P in the plane to the lines BC, CA and AB determined by the sides of an equilateral triangle ABC. Prove that the cevians AD, BE, CF are concurrent (or parallel) if and only if at least one of D, E or F is a midpoint of its side.

Composite of solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and by Titu Zvonaru, Cománeşti, Romania.

Without loss of generality we assume that the sides of the equilateral triangle have length 2 so that if $BD=\alpha$, $CE=\beta$, and $AF=\gamma$, then $DC=2-\alpha$, $EA=2-\beta$, and $FB=2-\gamma$. Consequently,

$$\begin{aligned}\alpha^2-(2-\alpha)^2&=PB^2-PC^2,\\ \beta^2-(2-\beta)^2&=PC^2-PA^2,\quad\text{and}\\ \gamma^2-(2-\gamma)^2&=PA^2-PB^2.\end{aligned}$$

Adding them, we get

$$\alpha+\beta+\gamma=3. \tag{1}$$

Ceva's theorem tells us that AD, BE, CF are concurrent or parallel if and only if

$$\alpha\beta\gamma=(2-\alpha)(2-\beta)(2-\gamma),$$

or

$$2\alpha\beta\gamma=8-4(\alpha+\beta+\gamma)+2(\alpha\beta+\beta\gamma+\gamma\alpha).$$

By (1) this is equivalent to $\alpha\beta\gamma - (\alpha\beta + \beta\gamma + \gamma\alpha) + 2 = 0$, or

$$(\alpha - 1)(\beta - 1)(\gamma - 1) = 0.$$

Hence, AD, BE, CF are concurrent or parallel if and only if either $\alpha = 1$, or $\beta = 1$, or $\gamma = 1$; in other words, if and only if D is the midpoint of BC, or E is the midpoint of CA, or F is the midpoint of AB.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; EDMUND SWYLAN, Riga, Latvia; and the proposer.

Geupel reminds us that problem 2508 [2000 : 46; 2001 : 58-61; 2003 : 402] dealt with points having the property of P; in particular, any point D on the line joining the vertices B and C of an arbitrary triangle ABC determines 0, 1, 2, or infinitely many positions of a point P whose pedal triangle DEF has vertices with the property that AD, BE, CF are concurrent or parallel. The current problem shows that when the triangle is equilateral, there are always exactly two candidates for P for each position of D on the line BC except for the midpoint of the segment BC (in which case P could be any point on the perpendicular to BC through D).