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IN THIS ISSUE / DANS CE NUMÉRO

THE CONTEST CORNER

No. 20

Robert Bilinski and Kseniya Garaschuk

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-contest@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er avril 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

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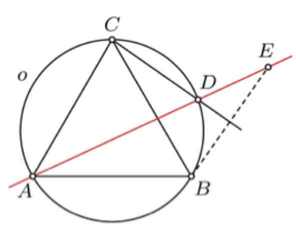
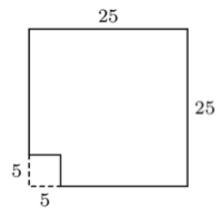
CC96. Un triangle équilatéral ABC est inscrit dans un cercle o. Le point D est sur l'arc BC de o. Le point E est le symétrique de B par rapport à CD. Prouver que A, D et E sont alignés.



CC97. Trouver la plus petite valeur de l'expression $a + b^3$ où a et b sont des nombres positifs dont le produit vaut 1.

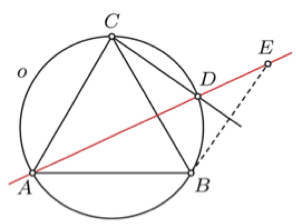
CC98. Existe-t-il des nombres réels x et y pour lesquels $\sqrt{x^2+1}+\sqrt{y^2+1} = x + y$?

CC99. Si on enlève un carré de côté 5 du coin d'un carré de côté 25, peut-on découper la partie restante en 100 rectangles de dimension soit 1×6 ou 2×3 ?



CC100. Six équipes participent à un tournoi où chaque équipe joue avec chaque autre exactement un match. Une victoire vaut 3 point alors qu'un match nul en vaut 1 et une défaite donne zéro point. Après le tournoi, on remarque que la somme des points marqués par toutes les équipes est 41. Prouver qu'il y a quatre équipes, dont chacune a au moins un match nul.

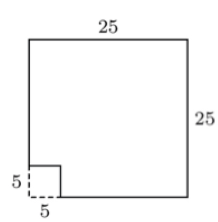
CC96. An equilateral triangle ABC is inscribed in a circle o. Point D is on arc BC of o. Point E is the symmetric of B with respect to line CD. Prove that A, D and E are collinear.



CC97. Find the smallest value of the expression $a+b^3$ where a and b are positive numbers whose product is 1.

CC98. Are there real numbers x and y such that $\sqrt{x^2+1}+\sqrt{y^2+1} = x+y$?

CC99. We cut off a square of side 5 from the corner of a square of side 25. Can we cut the remaining part into 100 rectangles of dimension either 1×6 or 2×3 ?



CC100. In a 6 team tournament, each team played with each other team exactly once. A team gets 3 points for a victory, 1 point for a draw and 0 for a defeat. After the tournament, the sum of the scores by all the teams is 41. Prove that the exists a group of 4 teams where each team tied at least once.

CONTEST CORNER SOLUTIONS

CC46. Starting with the input (m, n), Machine A gives the output (n, m). Starting with the input (m, n), Machine B gives the output $(m+3n, n)$. Starting with the input (m, n), Machine C gives the output $(m-2n, n)$. Natalie starts with the pair $(0, 1)$ and inputs it into one of the machines. She takes the output and inputs it into any one of the machines. She continues to take the output that she receives and inputs it into any one of the machines. (For example, starting with $(0, 1)$, she could use machines B, B, A, C, B in that order to obtain the output $(7, 6)$.) Is it possible for her to obtain $(20132013, 20142014)$ after repeating this process any number of times ?

This problem was inspired by 2009 Fermat Contest, number 24.

Solved by G. Geupel ; R. Hess ; and S. Muralidharan. We present the solution by S. Muralidharan.

For any machine M and input (x, y) let us write $M(x, y)$ for the output. Given that

$$A(m, n) = (n, m)$$

$$B(m, n) = (m + 3n, n)$$

$$C(m, n) = (m - 2n, n)$$

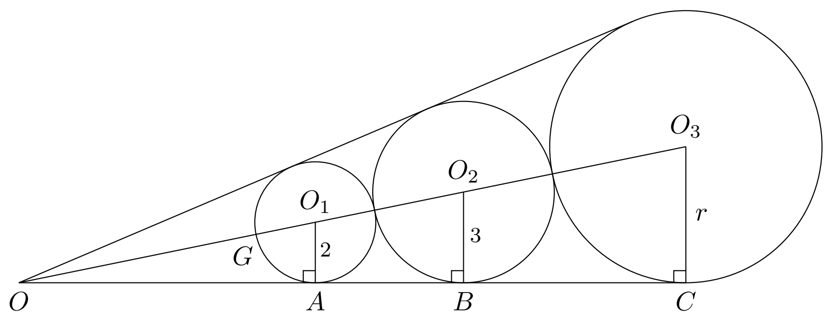
we clearly have $GCD(M(m, n)) = GCD(m, n)$ for any machine $M = A, B, C$ and any pair of integers (where GCD denotes the greatest common divisor). Since $GCD(0, 1) = 1$ and $GCD(20132013, 20142014) = 10001$, it follows that Natalie can not obtain $(20132013, 20142014)$ starting with $(0, 1)$.

CC47. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.

Originally 2005 W.J. Blundon Contest, problem 10.

Solved by Š. Arslanagić ; M. Bataille ; M. Coiculescu ; M. Amengual Covas ; G. Geupel ; J. G. Heuver ; S. Muralidharan ; and T. Zvonaru. We present solution by Matei Coiculescu with figure by Miguel Amengual Covas.

Denote by R the radius of the third circle. Let O be the vertex of the angle and let O_1, O_2 and O_3 be the centres of the respective circles with points of tangency A, B and C, respectively. The bisector OO_3 passes through the centres of the circles and intersects the circle with radius 2 at G. We denote the length OG by x.



It is easily seen that $\Delta OAO_1 \sim \Delta OBO_2 \sim \Delta OCO_3$. Thus $\frac{2}{3} = \frac{2+x}{7+x}$, or $x = 8$. Then $\frac{3}{r} = \frac{15}{18+r}$, which gives us $r = 4.5$.

CC48. Determine whether there exist two real numbers a and b such that both $(x-a)^3 + (x-b)^2 + x$ and $(x-b)^3 + (x-a)^2 + x$ contain only real roots.

Originally 2012 Sun Life Financial Repéchage Competition, problem 6.

Solved by M. Bataille ; R. Hess ; S. Muralidharan ; P. Perfetti ; and T. Zvonaru. We give a composite solution.

Suppose $(x-a)^3 + (x-b)^2 + x = x^3 - (3a-1)x^2 + (3a^2 - 2b + 1)x - (a^3 - b^2)$ has only real roots. Let r, s, t be these roots. Then

$$(x-r)(x-s)(x-t) = x^3 - (3a-1)x^2 + (3a^2 - 2b + 1)x - (a^3 - b^2).$$

Comparing coefficients we have

$$r + s + t = 3a - 1, \quad rs + st + tr = 3a^2 - 2b + 1.$$

Now observe that

$$\begin{aligned}(r+s+t)^2 &= r^2 + s^2 + t^2 + 2(rs + st + tr) \\ &= \left(\frac{r^2+s^2}{2} + \frac{s^2+t^2}{2} + \frac{t^2+r^2}{2}\right) + 2(rs + st + tr) \\ &\geq (rs + st + tr) + 2(rs + st + tr) \\ &= 3(rs + st + tr).\end{aligned}$$

Therefore,

$$(3a-1)^2 \geq 3(3a^2-2b+1).$$

Rearranging this we get $6a-6b \leq -2$. Similarly, since $(x-b)^3+(x-a)^2+x$ has only real roots, $6b-6a \leq -2$. Adding these two inequalities we deduce $0 \leq -4$, a contradiction. Thus it is impossible for the two given polynomials to have only real roots.

Editor's Comment : Many submitted solutions involved analyzing the derivatives of the cubics.

CC49. Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

Originally 2009 University of Waterloo Big E Contest, Question 4.

One solution was received, which managed to place 34 coins on the board, which is not the minimum possible number.

CC50. Show that the square root of a natural number of five or fewer digits never has a decimal part starting 0.1111, but that there is an eight-digit number with this property.

Originally 2005 APICS Math Competition, Question 7.

One incorrect solution was received.

THE OLYMPIAD CORNER

No. 318

Nicolae Strungaru

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-olympiad@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

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La rédaction souhaite remercier d'avoir traduit les problèmes.

OC156. Soit $ABCD$ un tétraèdre. Démontrer que le sommet D, le centre de la sphère inscrite et le centroïde de $ABCD$ sont colinéaires si et seulement si les surfaces des triangles ABD, BCD et CAD sont égales.

OC157. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f(f(x)^2 + f(y)) = xf(x) + y,\ \forall x, y \in \mathbb{R}\,.$$

OC158. Démonter qu'un graphe fini, simple et planaire possède une orientation telle que tout sommet ait un degré extérieur au plus égal à 3.

OC159. Soit p un nombre premier impair. Démontrer qu'il existe un nombre naturel x tel que x et $4x$ sont toutes deux des racines primitives modulo p.

OC160. Le cercle inscrit du triangle ABC est tangent aux côtés BC, CA et AB à D, E et F respectivement. Soit T la réflexion de F par rapport à B et soit S la réflexion de E par rapport à C. Démontrer que le centre du cercle inscrit du triangle AST est à l'intérieur ou sur le cercle inscrit du triangle ABC.

. . . .

OC156. Let $ABCD$ be a tetrahedron. Prove that the vertex D, the center of the insphere and the centroid of $ABCD$ are collinear if and only if the areas of triangles ABD, BCD and CAD are equal.

OC157. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y, \ \forall x, y \in \mathbb{R}\,.$$

OC158. Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

OC159. Let p be an odd prime number. Prove that there exists a natural number x such that x and $4x$ are both primitive roots modulo p.

OC160. The incircle of triangle ABC is tangent to sides BC, CA and AB at D, E respectively F. Let T be the reflection of F with respect to B and S the reflection of E with respect to C. Prove that the incenter of triangle AST is inside or on the incircle of triangle ABC.

OLYMPIAD SOLUTIONS

OC96. Let $a, b > 1$ be two relatively prime integers. We define $x_1 = a, x_2 = b$ and

$$x_n = \frac{{x_{n-1}}^2 + {x_{n-2}}^2}{x_{n-1} + x_{n-2}} \quad \forall n \geq 3\,.$$

Prove that x_n is not an integer for all $n \geq 3$.

Originally question 4 from Croatia, Team Selection Test 2011, Day 1.

There were no solutions received to this problem.

OC97. Let A be a set with 225 elements. Suppose that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

Originally question 6 from 2011 USA Math Olympiad.

Solved by Oliver Geupel and we present his solution.

Without loss of generality we can assume that $A_1 \cup A_2 \cup \ldots \cup A_{11} = \{1, 2, \ldots, n\}$, where n is the cardinality of $A_1 \cup A_2 \cup \ldots \cup A_{11}$. We have to prove that $n \geq 165$.

For each $1 \leq i \leq n$ and $1 \leq j \leq 11$, we define

$$e_{ij} = \begin{cases} 0 & \text{if } i \notin A_j \\ 1 & \text{if } i \in A_j\,. \end{cases}$$

Then,

$$495 = 11 \cdot 45 = \sum_{j=1}^{11} |A_j| = \sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij}^2\,.$$

By the Arithmetic-Quadratic Means inequality, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij} \right)^2 \leq n \sum_{i=1}^{n} \left(\sum_{j=1}^{11} e_{ij} \right)^2 &= n \left(\sum_{j=1}^{11} \sum_{i=1}^{n} e_{ij}^2 + 2 \sum_{1 \leq j < k \leq 11} \sum_{i=1}^{n} e_{ij} e_{ik} \right) \\ &= n \left(495 + 2 \sum_{1 \leq j < k \leq 11} |A_j \cap A_k| \right) \\ &= n \cdot \left(495 + 2 \binom{11}{2} \cdot 9 \right) = 1485n\,. \end{aligned}$$

We conclude $n \geq \dfrac{495^2}{1485} = 165$, which completes the proof.

It remains to give an example for which $n = 165$ holds. The set $\{1, 2, \ldots, 11\}$ has $\binom{11}{3} = 165$ subsets of cardinality 3. Let $B_1,\ B_2,\ \ldots,\ B_{165}$ be any list of these subsets. Define $A_1, \ldots, A_{11}$ by the condition

$$i \in A_j \Leftrightarrow j \in B_i, \qquad 1 \le i \le 165, \quad 1 \le j \le 11\,.$$

Every number $j \in \{1, 2, \ldots, 11\}$ is included in exactly $\binom{10}{2} = 45$ of the subsets B_i. Thus, $|A_j| = 45$. Moreover, for any fixed distinct values $j, k \in \{1, 2, \ldots, 11\}$, we have $i \in A_j \cap A_k$ if and only if $j, k \in B_i$, which is satisfied for nine subsets B_i. Therefore, $|A_j \cap A_k| = 9$. This completes our example.

OC98. Let ABC be a triangle with $\angle BAC = 60^\circ$. Let B_1 and C_1 be the feet of the bisectors from B and C. Let A_1 be the symmetrical of A with respect to the line B_1C_1. Prove that A_1, B and C are colinear.

Originally question 3 from Moldova's Team Selection Test 2011, Day 2.

Solved by Š. Arslanagić; M. Bataille; and T. Zvonaru and N. Stanciu. We give the solution of Šefket Arslanagić.

Let I denote the intersection of BB_1 and CC_1, that is I is the incenter of ABC. Let $\angle B = \beta, \angle C = \gamma$. We know that $\beta + \gamma = 120^\circ$. Then, we have

$$\angle B_1IC_1 = \angle BIC = 180^\circ - \frac{\beta}{2} - \frac{\gamma}{2} = 180^\circ - 60^\circ = 180^\circ - \angle A\,.$$

This implies that the quadrilateral AB_1IC_1 is cyclic, and hence

$$\angle AB_1C_1 = \angle AIC_1\,.$$

As AIC_1 is an exterior angle for the triangle AIC, it follows that

$$\angle AB_1C_1 = \angle AIC_1 = \angle IAC + \angle ICA = 30^\circ + \frac{\gamma}{2}\,.$$

As A_1 is the symmetric of A with respect to B_1C_1, we have $\Delta AB_1C_1 \equiv \Delta A_1B_1C_1$ and hence

$$\angle A_1B_1C_1 = \angle AB_1C_1 = 30^\circ + \frac{\gamma}{2},$$
$$\angle B_1A_1C_1 = \angle B_1AC = 60^\circ.$$

Since AB_1IC_1 is cyclic, we also get

$$\angle IB_1C_1 = \angle IAC_1 = 30^\circ,$$
$$\angle IC_1B_1 = \angle IAB_1 = 30^\circ.$$

As $\angle CB_1A_1 + \angle A_1B_1A = 180^\circ$, it follows that

$$\angle C_1B_1A = 180^\circ - \angle AB_1C_1 - \angle A_1B_1C_1 = 180^\circ - 2(30^\circ + \frac{\gamma}{2}) = \beta\,.$$

By the bisector theorem we have

$$\frac{BA}{BC} = \frac{AB_1}{B_1C} = \frac{A_1B_1}{B_1C}\,.$$

Therefore, as

$$\frac{BA}{BC} = \frac{A_1B_1}{B_1C} \quad \text{and} \quad \angle ABC = \angle A_1B_1C,$$

the triangles ΔBAC and ΔB_1A_1C are similar. This implies that

$$\angle B_1A_1C = \angle BAC = 60^\circ\,.$$

Similarly, we can show that $\angle BA_1C_1 = 60^\circ$. Therefore,

$$\angle BA_1C = \angle BA_1C_1 + \angle C_1A_1B_1 + \angle B_1A_1C = 60^\circ + 60^\circ + 60^\circ = 180^\circ\,.$$

This proves that B, A_1, C are colinear.

OC99. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ so that, for all $x \in \mathbb{Q}^+$ we have

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1} \text{ and } f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}\,.$$

Originally question 1 from Turkey Team Selection Test 2011.

Solved by Michel Bataille and we give his solution.

We show that the solutions are the functions $f_a : \mathbb{Q}^+ \to \mathbb{Q}^+$ (where $a \in \mathbb{Q}^+$) defined by

$$f_a(x) = a \cdot \frac{n^2}{d} \text{ if } x = \frac{n}{d} \text{ with } n, d \in \mathbb{N},\ \gcd(n, d) = 1\,.$$

Such a function f_a is a solution. Indeed, the equation $f_a\left(\frac{1}{x}\right) = \frac{f_a(x)}{x^3}$ is quickly verified and noticing that $\gcd(n, n+d) = 1$ if $\gcd(n, d) = 1$, we have

$$f_a\left(\frac{x}{x+1}\right) = f_a\left(\frac{n}{n+d}\right) = a \cdot \frac{n^2}{n+d} = \frac{a \cdot (n^2/d)}{1+\frac{n}{d}} = \frac{f_a(x)}{x+1}\,.$$

whenever $x = \frac{n}{d}$ with $n, d \in \mathbb{N},\ \gcd(n, d) = 1$.

Conversely, let f be an arbitrary solution and let $a = f(1)$. We show that $f\left(\frac{n}{d}\right) = f_a\left(\frac{n}{d}\right)$ for all $n, d \in \mathbb{N}$, $\gcd(n, d) = 1$ by induction on the positive integer $\max(n, d)$.

If $\max(n, d) = 1$, then $n = d = 1$ and $f(1) = a = a \cdot \frac{1^2}{1} = f_a(1)$.

Let $k \geq 2$ and assume that $f\left(\frac{\nu}{\delta}\right) = f_a\left(\frac{\nu}{\delta}\right)$ whever $\nu, \delta \in \mathbb{N}$, $\gcd(\nu, \delta) = 1$ with $\max(\nu, \delta) < k$. Let $n, d \in \mathbb{N}$ $\gcd(n, d) = 1$ with $\max(n, d) = k$. Note that $n = d$ cannot occur, so we distinguish the two cases $n < d$ and $d < n$. In the former case, since $\gcd(n, d - n) = 1$ and $\max(n, d - n) < d = k$, we may write

$$f\left(\frac{n}{d}\right) = f\left(\frac{n/(d-n)}{1 + (n/(d-n))}\right) = \frac{f(n/(d-n))}{d/(d-n)} = \frac{an^2/(d-n)}{d/(d-n)} = a \cdot \frac{n^2}{d}.$$

In the latter case, using the previous case for the calculation of $f(d/n)$,

$$f\left(\frac{n}{d}\right) = f\left(\frac{1}{d/n}\right) = \frac{f(d/n)}{d^3/n^3} = a \cdot \frac{d^2}{n} \cdot \frac{n^3}{d^3} = a \cdot \frac{n^2}{d}\,.$$

In any event, $f\left(\frac{n}{d}\right) = a \cdot \frac{n^2}{d} = f_a\left(\frac{n}{d}\right)$ and the induction step is complete.

OC100. Let a_n be the sequence defined by

$$a_0 = 1,\, a_1 = -1, \text{ and } a_n = 6a_{n-1} + 5a_{n-2}\, \forall n \geq 2\,.$$

Prove that $a_{2012} - 2010$ is divisible by 2011.

Originally question 2 from Vietnam National Olympiad 2011, Day 2.

No solution was received to this problem. We give a solution by the editor .

Consider the quadratic equation

$$X^2 - 6X - 5 \equiv 0 \pmod{2011}\,.$$

The discriminant of this equation is $\Delta = 56$ which is a quadratic residue modulo 2011. [This can be seen either by calculating the Legendre symbol, or more easely by seeing that $56 \equiv 56 + 4 * 2011 = 8100 \pmod{2011}$.]

Therefore, as $p = 2011$ is a prime, the equation has two roots $s \not\equiv t \pmod{2011}$. Moreover, it is clear that neither of them is $0 \pmod{2011}$. It follows that s, t are invertible modulo 2011.

As the characteristic equation of the recursion $\pmod{2011}$ has two roots, and we work over a field, it follows that there exists some integers C_1, C_2 so that

$$a_n \equiv C_1 s^n + C_2 t^n \pmod{2011}\;;\; \forall n\,.$$

By Fermat Little Theorem, we have

$$s^{2010} \equiv t^{2010} \equiv 1 \pmod{2011}$$

Therefore

$$\begin{aligned} a_{2012} &\equiv C_1 s^{2012} + C_2 t^{2012} \equiv C_1 s^{2010} s^2 + C_2 t^{2010} t^2 \\ &\equiv C_1 s^2 + C_2 t^2 \equiv a_2 \pmod{2011} . \end{aligned}$$

As $a_2 = -6 + 5 = -1$, the claim of the problem follows.

BOOK REVIEWS

John McLoughlin

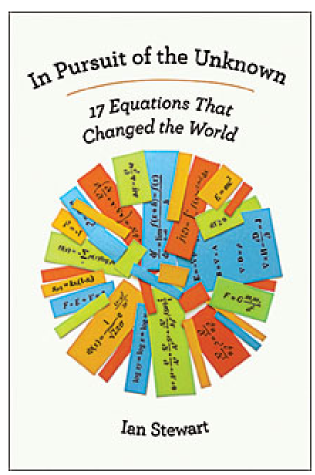
In Pursuit of the Unknown: 17 Equations That Changed the World by Ian Stewart
ISBN: 978-0-465-08598-9, Softcover, 342 + x pages
Basic Books, 2012, US$16.99

Reviewed by ***Shawn Godin****, Cairine Wilson S.S., Orleans, ON*

Mention the word equations to a random person on the street and you are likely to get a tale of woe. Yet to a mathematician, scientist or engineer, equations are ways to communicate relationships and a medium for solving problems and making discoveries. In the introduction to his book *In Pursuit of the Unknown: 17 Equations That Changed the World*, Ian Stewart tells the story of how Stephen Hawking was told by the publishers of *A Brief History of Time* that every equation included in his book would probably halve the sales. Stewart, fortunately, doesn't heed this advice, his excuse being that since the book is about equations, they are going to have to play a large part in the contents.

Ian Stewart is a well-known mathematical writer. He spent ten years writing the *Mathematical Recreations* column in *Scientific American*. He has written numerous books on puzzles and mathematical recreations like *Professor Stewart's Cabinet of Mathematical Curiosities* and *How to Cut a Cake*; mathematical expositions on topics such as Chaos in *Does God Play Dice? The New Mathematics of Chaos*; as well as textbooks such as *Galois Theory* and *Algebraic Number Theory and Fermat's Last Theorem* (with D. Tall). In *In Pursuit of the Unknown* Stewart takes us on a trip through the history and interconnectedness of mathematics.

The seventeen chapters, each devoted to a particular equation, give us the background of where an equation came from, how it is used, why it is important and how it is tied in with some of the other equations that he explores. The equations and topics explored, illustrated by the subtitles of the chapters, are: Pythagoras's Theorem, Logarithms, Calculus, Newton's Law of Gravity, The Square Root of Minus One, Euler's Formula for Polyhedra, Normal Distribution, Wave Equation, Fourier Transform, Navier–Stokes Equation, Maxwell's Equations, Second Law of Thermodynamics, Relativity, Schrödinger's Equation, Information Theory, Chaos Theory and Black–Scholes Equation.



The opening chapter *The squaw on the hippopotamus*, discussing Pythagoras's Theorem, gives the reader the sense of what is to come. We are led through the stories of Pythagoras (c. 570 BC - c. 495 BC) and Euclid (fl. 300 BC) and told how Babylonian tablets a thousand years before Pythagoras showed evidence that the relation was known at that time. The chapter continues with how the theorem is related to the development of trigonometry, and how trigonometry and triangulation transformed surveying and map making in the sixteenth century. We are shown how the theorem is central to analytic geometry and the chapter concludes with a generalization of the theorem by examining non-Euclidean geometries. Later, in Chapter 13 *One Thing is Absolute*, about the theory of relativity, Pythagoras's Theorem is used in a thought experiment whose result shows, paradoxically, that observers moving relative to each other would measure different time intervals between the same events. Yet again in Chapter 15 *Codes, Communications, and Computers*, about information theory, Pythagoras's Theorem, used as a way to calculate distance in Euclidean space, is generalized to give the *Hamming distance* which measures the distance between two bit–strings, or binary "words". Using the Hamming distance, codes can be created that can detect and correct errors that occur in transmission, a necessary tool in this increasingly digital world.

This book is a wonderful snapshot of some of the history of mathematics and science and how it is all tied together. It is accessible to a wide audience. I would highly recommend this book to teachers, motivated high school students, or any admirer of mathematics and science.

PROBLEM SOLVER'S TOOLKIT

No. 9

Gerhard J. Woeginger

The Problem Solver's Toolkit contains short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.

If it is not prime, it must be composite: part 2

Part 1 of this article appears on pages 409–412 of the previous issue (Volume 39, number 9).

3 Products of small numbers

No prime p divides a positive integer that is strictly smaller than p. Consequently: if we can show that a number q divides a product $f_1 f_2$ with $1 \le f_1, f_2 < q$, then q must be composite. This trivial observation is surprisingly useful.

Problem 11 *Let a, b, c, d, e, f be positive integers, for which the sum $S = a + b + c + d + e + f$ divides both $Q = ab + ac + bc - de - df - ef$ and $R = abc + def$. Prove that S is composite.*

The occurrence of the terms $ab+ac+bc$, $de+df+ef$, abc and def in the problem statement urges us to consider the polynomial $P(x) = (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f) = Sx^2 + Qx + R$. Then S divides $P(x)$ for all integers x, and in particular S divides $P(d) = (a+d)(b+d)(c+d)$. As S is larger than each of the three factors, we conclude that S is composite.

Problem 12 *Consider the Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$. Prove that $F_n + 1$ is composite for all $n \ge 4$.*

The Fibonacci numbers satisfy the well-known Catalan identity $F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}$. For odd n, we use $r = 1$ so that the identity becomes $(F_n+1)(F_n-1) = F_{n-1}F_{n+1}$. If $p = F_n + 1$ is prime, then it cannot divide the factor F_{n-1} which is smaller than p. But the other factor $F_{n+1} = F_n + F_{n-1}$ satisfies $p < F_{n+1} < 2p$, and hence cannot be divisible by p either. The argument for even n is very similar. We set $r = 2$ and use $(F_n+1)(F_n-1) = F_{n-2}F_{n+2}$. If $p = F_n + 1$ is prime, then the larger factor $F_{n+2} = 2F_n + F_{n-1}$ satisfies $2p < F_{n+2} < 3p$, and hence cannot be divisible by p. All in all, $F_n + 1$ must be composite.

The following three exercises can all be settled via products of small numbers.

Problem 13 *Let a, b, c, d be positive integers such that $a^2+ab+b^2 = c^2+cd+d^2$. Prove that $a+b+c+d$ is composite.*

Problem 14 *Let a, b, c be positive integers such that $a^2 - bc$ is a square. Prove that $2a + b + c$ is composite.*

Problem 15 *Let $a > b > c > d$ be positive integers with $a^2-ac+c^2 = b^2+bd+d^2$. Prove that $ab + cd$ is composite.*

4 Stepping stones

If you are stuck in a swamp of primes, you may sometimes use one of these primes as a stepping stone; you jump to a composite from it and reach the safe shore. The following folklore example illustrates this idea.

Problem 16 *Let $P(n)$ be a non-constant polynomial with integer coefficients. Prove that there exists an integer n for which $P(n)$ is composite.*

Pick an integer n for which $p = P(n)$ is prime as your stepping stone (if no such n exists, the proof is already complete). Consider the values $P(n+kp)$ for $k \geq 1$. As $(n+kp)-n$ divides $P(n+kp)-P(n)$, every such value $P(n+kp)$ is a multiple of p. But as the polynomial is non-constant, only a finite number of these infinitely many multiples can be equal to p. Hence you have successfully pulled yourself out of the swamp and jumped to a composite number $P(n+kp)$.

Problem 17 *Prove that the sequence $A(n) = n!^2 - n! + 1$ contains infinitely many composite numbers.*

Consider an odd integer $n \geq 3$ for which $p = A(n)$ is prime; note that $p > 2n + 1$. This prime p will be our stepping stone. Since $k \equiv -(p-k) \bmod p$ holds for $1 \leq k \leq n$, we derive the following chain of equivalences modulo p:

$$\begin{aligned} (p-n-1)!\; n! &\equiv (p-n-1)! \cdot (p-n)(p-n+1)\cdots(p-2)(p-1) \cdot (-1)^n \\ &\equiv (p-1)! \cdot (-1)^n \equiv 1 \pmod{p}. \end{aligned}$$

In the last step of this chain, we have used Wilson's theorem together with the fact that n is odd. Next, we use this to show

$$\begin{aligned} n!^2 \cdot A(p-n-1) &\equiv n!^2 \cdot (p-n-1)!^2 - n!^2 \cdot (p-n-1)! + n!^2 \\ &\equiv 1 - n! + n!^2 \equiv A(n) \equiv 0 \pmod{p}. \end{aligned}$$

Since $n < p$, we conclude that p divides $A(p-n-1)$ and that $A(p-n-1)$ hence is composite. Now it is time to wrap things up: Whenever $A(n)$ is prime for some odd integer $n \geq 3$, then there exists another integer $m = p-n-1 > n$ for which $A(m)$ is composite. Hence the sequence indeed contains infinitely many composites.

We challenge the reader to settle the following problems along the lines indicated above.

Problem 18 *Let r and s be positive integers. Prove that the sequence $B(n) = r2^n + s3^n$ contains infinitely many composites.*

Problem 19 *Prove that there exist infinitely many odd integers n, for which $n!+1$ is composite.*

Problem 20 *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a non-constant polynomial with integer coefficients. Prove that there exists a positive integer m for which $P(m!)$ is composite.*

Hints, comments, and references

11. This is problem N3 from the IMO 2005 shortlist, proposed by Mongolia.

13. Use $(b+c)(b+d) = (a+b+c+d)(c+d-a)$.

14. This is problem 1 from ELMO'2009, proposed by Evan O'Dorney. Let $a^2 - bc = x^2$, and use $(2a+b+c)(2a-b-c) = (2x+b-c)(2x-b+c)$ with $a > x$.

15. This is problem 6 from IMO'2001, proposed by Aleksander Ivanov (Bulgaria). Show that $ab+cd > ac+bd > ad+bc$, and show that $ac+bd$ divides $(ab+cd)(ad+bc)$.

18. If $p = B(n)$ is prime with $p \notin \{2,3\}$, then $B(n+(p-1)k)$ is a multiple of p.

19. As n must be odd, Wilson does not help us here. Show that whenever $p = n!+1$ is prime for some odd $n \geq 3$, then p divides $(p-n-1)!+1$.

20. This is problem N7 from the IMO 2005 shortlist, proposed by Russia. Our problem 17 settles the special case with $P(x) = x^2 - x + 1$. The solution for the general case uses an additional trick: study the auxiliary polynomial $Q(x) = a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n$ and show that if a prime p divides $Q(m!)$ then p also divides $P((p-m-1)!)$.

Another Application of the Harmonic Mean

Valcho Milchev

These lines were motivated by the Problem Solver's Toolkit No. 6 ("The Harmonic Mean File" [2013 : 262 - 265]), where J. Chris Fisher provided constructions involving the harmonic mean. Here we see how the construction of a harmonic sequence can be modified to produce the reciprocals of the numbers in three familiar sequences that are defined by linear recurrences—the Fiboncci, Lucas, and Pell numbers.

In Figure 1, we recall one way to construct harmonic means.

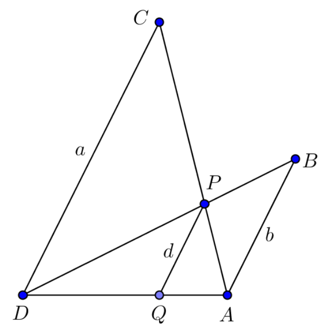


Figure 1: PQ is half the harmonic mean of CD and AB: $\frac{1}{d} = \frac{1}{a} + \frac{1}{b}$.

Specifically, let $ABCD$ be a trapezoid with $DC||AB$ and the diagonals intersecting in P; let the line through P that is parallel to those parallel sides meet AD in Q. If $a = DC$, $b = AB$, and $d = PQ$, then d is half the harmonic mean of the lengths a and b:

$$\frac{1}{d} = \frac{1}{a} + \frac{1}{b}.$$

The Fibonacci Numbers.

The nth term of the Fibonacci sequence $F_1, F_2, F_3, \ldots$ satisfies the linear recurrence, $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. To construct the reciprocals of these numbers, consider the rectangle $ABDA'$ in Figure 2, whose diagonals intersect in the point labeled D_3. For each point D_n with $n \geq 3$ on the diagonal AD we define A_n and B_n to be the projections of D_n on the sides AA' and AB, respectively. D_{n+1} can then be defined recursively to be the intersection of AD with A_nB_{n-1}. Starting with $A'D = f_1$, $AB = f_2$, and setting $f_3 = A_3D_3 = AB_3$, then we have $\frac{1}{f_3} = \frac{1}{f_2} + \frac{1}{f_1}$ (cf. Figure 1). With $f_n := A_nD_n = AB_n$ we get a sequence of segments having lengths f_n that satisfy

$$\frac{1}{f_n} = \frac{1}{f_{n-1}} + \frac{1}{f_{n-2}}.$$

Setting $f_1 = f_2 = 1$, we conclude that $\frac{1}{f_n} = F_n$ are elements of the Fibonacci sequence. In other words, $\{f_n\} = \left\{1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \dots\right\}$ is the sequence of reciprocals of the Fibonacci numbers.

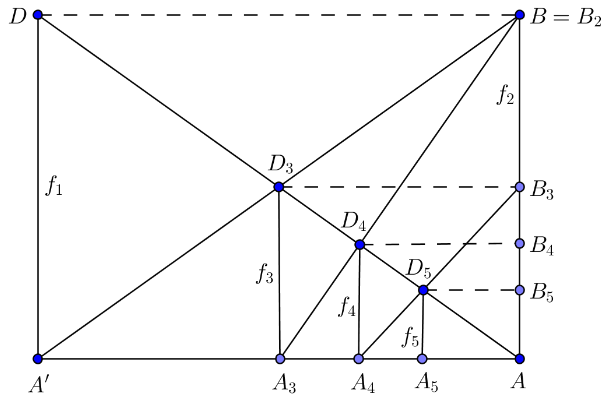


Figure 2: Sequence of segments whose lengths are reciprocals of the Fibonacci numbers.

The Lucas Numbers.

Of course, there was no reason to require $AB = A'D$ in the previous argument. We modify Figure 2 by allowing $\ell_1 = A'D$ to have a length different from that of $\ell_2 = AB$. Here $\ell_n = A_nD_n = AB_n$ satisfies

$$\frac{1}{\ell_n} = \frac{1}{\ell_{n-1}} + \frac{1}{\ell_{n-2}}.$$

Figure 3 shows the case $\ell_1 = \frac{1}{2}$ and $\ell_2 = 1$, so that $\{\ell_n\} = \left\{\frac{1}{2}, 1, \frac{1}{3}, \frac{1}{4}, \frac{1}{7}, \frac{1}{11}, \dots\right\}$. We recognize the elements of this sequence to be reciprocals of the Lucas numbers L_n: $L_1 = 2, L_2 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

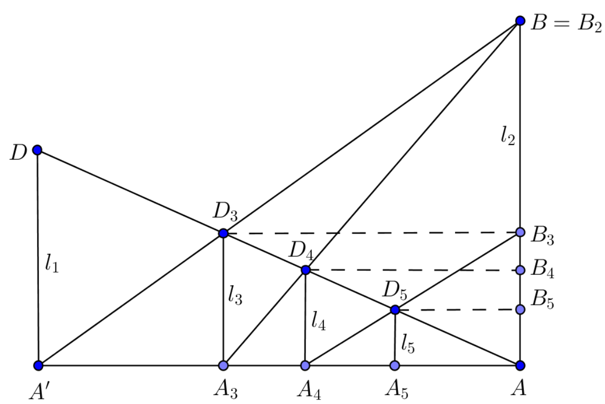


Figure 3: Sequence of segments whose lengths ℓ_n are reciprocals of the Lucas numbers.

The Pell Numbers.

The Pell numbers $\{P_n\} = \{1, 2, 5, 12, 29, \ldots\}$ satisfy the recursion $P_n = 2P_{n-1} + P_{n-2}$ with initial values $P_1 = 1$ and $P_2 = 2$. This sequence is closely related to the Pell equation $x^2 - 2y^2 = 1$. (See, for example, *The On-Line Encyclopedia of Integer Sequences*, www.oeis.org, sequence number A000129.)

Figure 4 on the next page shows a square AB_1CA_1 with D the midpoint of the side A_1C (and therefore $A_1D = \frac{1}{2}AA_1$). Compared to the previous constructions, the Pell sequence requires an extra step, namely, reflect the points B_n (which lie along the side AB_1) in the diagonal AC to the points A_n (which lie along the side AA_1). Note that the recursion begins with the segment $AA_3 = AB_3 = B_3'D_3$; we set $D_n = AD \cap A_{n-1}B_{n-2}$, while B_n and B_n' are the projections of D_n on the sides AB_1 and AA_1, respectively. The reflected image of the line D_nB_n in the diagonal AC intersects the line AD in the point labeled D_n', and the base AA_1 in A_n. It follows that for $n \geq 3$,

$$B_n'D_n = AB_n = AA_n \quad \text{and} \quad A_nD_n' = \frac{1}{2}AA_n. \tag{1}$$

We begin with $D_3 = AD \cap A_2B_1$ so that $B_3'D_3$ is half the harmonic mean of A_2D_2' and AB_1. But $B_3'D_3 = AB_3 = AA_3$, while $A_2D_2' = \frac{1}{2}AA_2$, whence

$$\frac{1}{AA_3} = \frac{1}{B_3'D_3} = \frac{1}{A_2D_2'} + \frac{1}{AB_1} = 2\frac{1}{AA_2} + \frac{1}{AA_1}.$$

From (1) we see that in general,

$$\frac{1}{AA_n} = \frac{1}{B_n'D_n} = \frac{1}{A_{n-1}D_{n-1}'} + \frac{1}{AB_{n-2}} = 2\frac{1}{AA_{n-1}} + \frac{1}{AA_{n-2}}.$$

Setting $p_1 = AA_1 = 1$ and $p_2 = AA_2 = \frac{1}{2}$, we have $p_3 = AA_3 = \frac{1}{5}, p_4 = AA_4 = \frac{1}{12}$, and, in general, $\frac{1}{p_n} = 2\frac{1}{p_{n-1}} + \frac{1}{p_{n-2}}$. Define $P_n = \frac{1}{p_n}$ for $n \geq 1$. Then $P_1 = 1, P_2 = 2$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 3$, so that our constructed lengths p_n are reciprocals of the Pell numbers. Note, finally, that if D had been defined to be the point on the segment A_1C for which $A_1D = \frac{1}{k}A_1C$, then the numbers p_n would satisfy the more general Pell recursion $\frac{1}{p_n} = k\frac{1}{p_{n-1}} + \frac{1}{p_{n-2}}$.

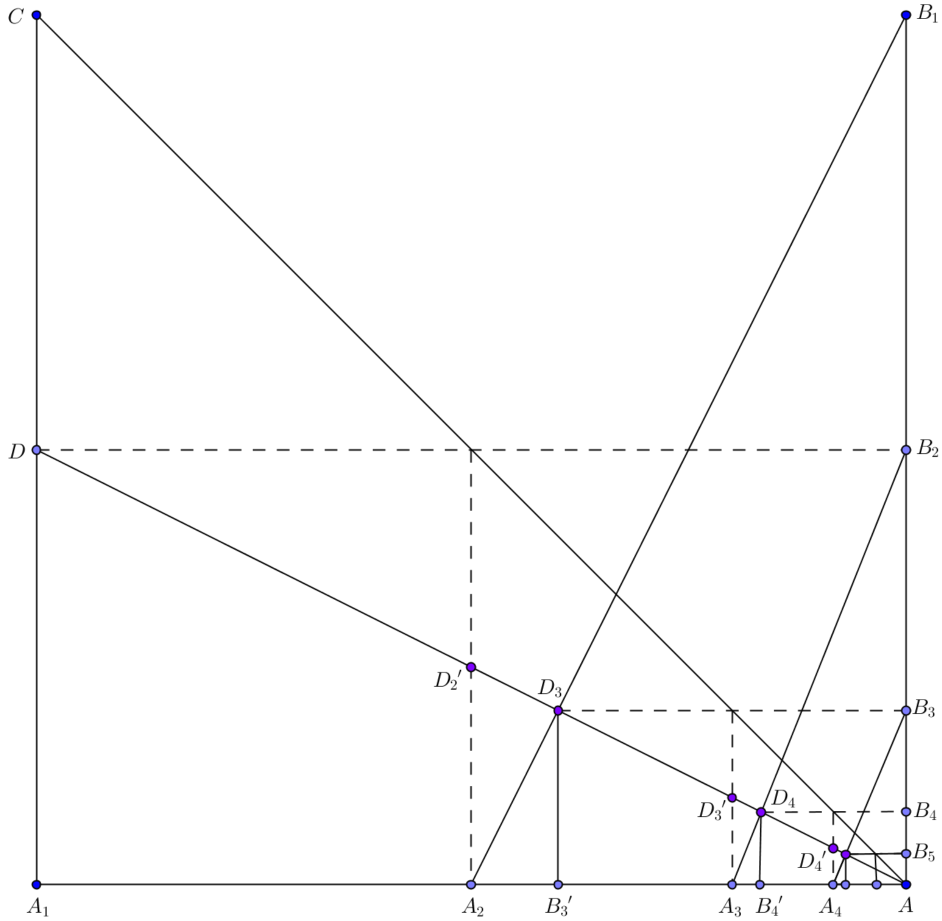


Figure 4: Sequence of segments whose lengths p_n are reciprocals of the Pell numbers.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-psol@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page.*

Comment soumettre un problème. *Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er avril 2015***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction remercie André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

3891. *Proposé par Michel Bataille.*

Soit ABC un triangle qui n'est ni équilatéral, ni rectangle et soit O le centre de son cercle circonscrit. Les hauteurs issues des sommets A, B et C coupent le cercle de nouveau aux points respectifs A', B' et C'. Soit U, V et W les centres respectifs des cercles circonscrits aux triangles OBC, OCA et OAB. Démontrer que UA', VB' et WC' sont concourants et identifier leur point commun.

3892. *Proposé par George Apostolopoulos.*

Soit a, b et c les longueurs des côtés d'un triangle ABC. Soit R le rayon du cercle circonscrit au triangle et r le rayon du cercle inscrit dans le triangle. Démontrer que

$$\frac{R}{r} \geq \frac{2}{3}(\cos A + \cos B + \cos C) + \frac{a^3+b^3+c^3}{3abc}.$$

3893. *Proposé par Ovidiu Furdui.*

Soit n un entier supérieur ou égal à 1. La partie décimale $\{a\}$ d'un nombre réel a est définie par l'expression $\{a\} = a - \lfloor a \rfloor$. Évaluer

$$\int_0^{\frac{\pi}{2}} \sin 2x \left\{ \ln^{2n-1} \tan x \right\} \mathrm{d}x.$$

3894. *Proposé par Paul Bracken.*

a) Démontrer que pour tout n, $n \in \mathbb{N}$, on a

$$1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

b) Évaluer la limite du membre de droite de l'équation de la partie a) lorsque $n \to \infty$ et l'exprimer par une formule de forme fermée.

3895. *Proposé par Neculai Stanciu et Titu Zvonaru.*

Soit ABC un triangle acutangle tel que $AB \neq AC$. Soit A' le pied de la hauteur issue du sommet A. La bissectrice de l'angle A coupe BC en D et le cercle circonscrit au triangle en M. Étant donné un point T sur le segment AD, soit P et N les projections respectives de T sur AA' et BC. Démontrer que si M, N et P sont alignés, alors T est le centre du cercle inscrit dans le triangle.

3896. *Proposé par Dao Thanh Oai et Nguyen Minh Ha.*

Soit $[WXYZ]$ l'aire algébrique du quadrilatère $WXYZ$ (W, X, Y et Z étant n'importe quels points du plan), c'est-à-dire la moitié de l'aire algébrique du parallélogramme formé par les vecteurs $\overrightarrow{WY}$ et $\overrightarrow{XZ}$:

$$[WXYZ] = \frac{1}{2}|\overrightarrow{WY}||\overrightarrow{XZ}|\sin(\overrightarrow{WY}, \overrightarrow{XZ}).$$

Soit $A_1A_2\ldots A_{2n}$ et $B_1B_2\ldots B_{2n}$ deux $2n$-gones réguliers ayant la même orientation dans un plan. Démontrer que $[A_iA_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]$ est une constante pour n'importe quel i, $1 \leq i \leq 2n$, les indices étant réduits modulo $2n$.

3897. *Proposé par Yakub Aliyev.*

Soit BB_1 et CC_1 deux céviennes du triangle ABC qui se coupent au point O. On considère une droite passant au point O et qui coupe le segment BC_1 en X et le segment B_1C en Y. Démontrer que

$$\frac{|BX|}{|XC_1|} > \frac{|B_1Y|}{|YC|}.$$

3898. *Proposé par Dragoljub Milošević.*

Soit un pentagone régulier $ABCDE$. On place les points F et G sur le prolongement du côté AB du pentagone, dans l'ordre F, A, B, G, de manière que $AG = BF = AC$. Comparer l'aire du triangle FGD à celle du pentagone $ABCDE$.

3899. *Proposé par George Apostolopoulos.*

Soit a, b et c des réels strictement positifs tels que $a + b + c = 3$. Démontrer que

$$\left(\frac{a^3+1}{a^2+1}\right)^2 + \left(\frac{b^3+1}{b^2+1}\right)^2 + \left(\frac{c^3+1}{c^2+1}\right)^2 \geq ab + bc + ca.$$

Quand y a-t-il égalité ?

3900. *Proposé par Abdilkadir Altintaş et Halit Çelik.*

Dans un triangle ABC, on a $AB = AC$ et $m(BAC) = 20°$. D est le point sur AC tel que $m(DBC) = 25°$ et E est le point sur AB tel que $m(BCE) = 65°$. Déterminer la mesure de l'angle CED.

. . . .

3891. *Proposed by Michel Bataille.*

Let ABC be a triangle that is neither equilateral nor right-angled and let O be its circumcentre. Let the altitudes from A, B and C meet the circumcircle again at A', B' and C', respectively. If U, V and W denote the circumcentres of triangles OBC, OCA and OAB, respectively, prove that the lines UA', VB' and WC' are concurrent and identify their common point.

3892. *Proposed by George Apostolopoulos.*

Let a, b and c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that

$$\frac{R}{r} \geq \frac{2}{3}(\cos A + \cos B + \cos C) + \frac{a^3+b^3+c^3}{3abc}.$$

3893. *Proposed by Ovidiu Furdui.*

Let $n \geq 1$ be an integer and let the decimal part of a real number a be defined by $\{a\} = a - \lfloor a \rfloor$. Evaluate

$$\int_0^{\frac{\pi}{2}} \sin 2x \left\{\ln^{2n-1} \tan x\right\} \mathrm{d}x.$$

3894. *Proposed by Paul Bracken.*

a) Prove that for $n \in \mathbb{N}$,

$$1 + 2\sum_{k=1}^{n} \frac{1}{(3k)^3 - 3k} = \sum_{k=1}^{2n+1} \frac{1}{k+n}.$$

b) For the equation in part a), evaluate the limit of the right-hand side as $n \to \infty$ and compute the sum in closed form.

3895. *Proposed by Neculai Stanciu and Titu Zvonaru.*

In the acute triangle ABC with $AB \neq AC$, let A' be the foot of the altitude from A, and let the bisector of the angle at A meet BC at D and the circumcircle at M. Finally, for a point T on the segment AD, let P and N be its projections on AA' and BC, respectively. Prove that if M, N, and P are collinear, then T is the incentre of the triangle.

3896. *Proposed by Dao Thanh Oai and Nguyen Minh Ha.*

Let $[WXYZ]$ represent the signed area of the quadrilateral $WXYZ$ (where W, X, Y, Z can be any four points in the plane), namely half the signed area of the parallelogram formed by the vectors $\overrightarrow{WY}$ and $\overrightarrow{XZ}$:

$$[WXYZ] = \frac{1}{2}|\overrightarrow{WY}||\overrightarrow{XZ}|\sin(\overrightarrow{WY}, \overrightarrow{XZ}).$$

If $A_1A_2\ldots A_{2n}$ and $B_1B_2\ldots B_{2n}$ are two similarly oriented regular $2n$-gons in the plane, prove that $[A_iA_{i+1}B_{i+1}B_i] + [A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}]$ is constant for any i, $1 \leq i \leq 2n$, where the indices are reduced modulo $2n$.

3897. *Proposed by Yakub Aliyev.*

Let the cevians BB_1 and CC_1 of a triangle ABC intersect at the point O. Prove that if a line is drawn through O meeting line segment BC_1 at X and line segment B_1C at Y, then

$$\frac{|BX|}{|XC_1|} > \frac{|B_1Y|}{|YC|}.$$

3898. *Proposed by Dragoljub Milošević.*

On the extension of the side AB of the regular pentagon $ABCDE$, let the points F and G be placed in the order F, A, B, G so that $AG = BF = AC$. Compare the area of triangle FGD to the area of pentagon $ABCDE$.

3899. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a + b + c = 3$. Prove that

$$\left(\frac{a^3+1}{a^2+1}\right)^2 + \left(\frac{b^3+1}{b^2+1}\right)^2 + \left(\frac{c^3+1}{c^2+1}\right)^2 \geq ab + bc + ca.$$

When does equality hold?

3900. *Proposed by Abdilkadir Altintaş and Halit Çelik.*

In a triangle ABC, $AB = AC$, $m(BAC) = 20^\circ$, D is the point on AC such that $m(DBC) = 25^\circ$ and E is the point on AB such that $m(BCE) = 65^\circ$. Find the measure of the angle CED.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3791. [2012 : 420, 422] *Proposed by John Lander Leonard.*

The numbers 0 through 12 are randomly arranged around a circle.

(a) Show that there must exist a trio of three adjacent numbers which sum to at least 18.

(b) Determine the maximum n such that there must exist a trio of three adjacent numbers which sum to at least n.

Solved by E. G. Goodaire and M. Parmenter; R. Hess; S. Malikić; Skidmore College Problem Group; P. Y. Woo (generalization of part (a)); T. Zvonaru; and the proposer. We present a composite of similar solutions submitted by Edgar G. Goodaire and Michael Parmenter (together) and by Titu Zvonaru (independently).

Set $a_0 = 0$ and let $a_1, a_2, \ldots, a_{12}$ be a permutation of the integers from 1 to 12. Since

$$(a_1+a_2+a_3)+(a_4+a_5+a_6)+(a_7+a_8+a_9)+(a_{10}+a_{11}+a_{12}) = 1+2+\cdots+12 = 78,$$

we deduce that there exists an index $i \in \{1, 4, 7, 10\}$ such that

$$a_i + a_{i+1} + a_{i+2} \geq \left\lceil \frac{78}{4} \right\rceil = 20.$$

Taking $(a_0, a_1, a_2, \ldots, a_{12}) = (0, 12, 7, 1, 11, 6, 3, 10, 5, 4, 9, 2, 8)$, we see that

$$\max\{a_0+a_1+a_2, a_1+a_2+a_3, \ldots, a_{10}+a_{11}+a_{12}, a_{11}+a_{12}+a_0, a_{12}+a_0+a_1\} = 20.$$

Thus, $n = 20$ must be the maximum value of n satisfying the requirement of part (b), namely, there must exist a trio of three adjacent numbers that sum to at least 20. Of course, this argument also provides a solution to part (a).

3792. [2012 : 420, 422; 2013 : 31, 34] *Proposed by Marcel Chiriţiţă.*

Solve the following system

$$\begin{aligned} 2^x + 2^y &= 12 \\ 3^x + 3^y &= 36 \end{aligned}$$

for $x, y \in \mathbb{R}$.

Solved by Š. Arslanagić (2 solutions); R. Barbara; M. Bataille; B. D. Beasley; R. Boukharfane; P. Deiermann; J. Hawkins and D. R. Stone; R. Hess; N. Hodžić and S. Malikić; V. Konečný; O. Kouba; J. Ling; D. E. Manes; P. Perfetti; V. Sadaphal; D. Smith; H. Wang and J. Wojdylo; and the proposer. We present the solution by Roy Barbara.

One easily confirms that $x = 2, y = 3$ is a solution to the given system of equations, as is $x = 3, y = 2$. We shall see that these are the only solutions. The claim is based on the following observation: Fix positive constants a and b and let $k > 1$; then the equation in t,

$$t^k + (a-t)^k = b,\ 0 \le t \le a,$$

can have at most two solutions. Indeed, consider the real continuous function in t (with $0 \le t \le a$), $f(t) = t^k + (a-t)^k - b$. Since $k - 1 > 0$, the derivative $f'(t) = k\left(t^{k-1} - (a-t)^{k-1}\right)$ is negative for $t < \frac{a}{2}$, it vanishes at $t = \frac{a}{2}$, and is positive for $t > \frac{a}{2}$. Hence, $f(t)$ strictly decreases form 0 to $\frac{a}{2}$, and strictly increases from $\frac{a}{2}$ to a. The observation follows.

Set $t = 2^x, u = 2^y, k = \frac{\ln 3}{\ln 2}$, and note that $k > 1$. The given system becomes

$$\begin{aligned} t + u &= 12 \\ t^k + u^k &= 36. \end{aligned}$$

Hence, $t^k + (12-t)^k = 36$. As t and u are positive, we have $0 < t < 12$. By the observation our equation in t has at most two solutions, namely $t = 4, 8$. Hence $(t, u) = (4, 8), (8, 4)$ are all the solutions of our transformed system of equations, yielding (as $2^x = t, 2^y = u$) all the solutions $(x, y) = (2, 3), (3, 2)$ of the given system.

Editors Comment. Deiermann's submission dealt with the more general system of equations,

$$\begin{aligned} a^x + a^y &= a^2(a+1) \\ b^x + b^y &= b^2(b+1), \end{aligned}$$

for $1 < a < b$. Using arguments similar to our featured solution, he proved that the only solutions are $\{x, y\} = \{2, 3\}$. Hawkins and Stone provided a similar generalization.

3793. [2012 : 420, 422; ; 2013 : 89, 91] *Proposed by George Apostolopoulos.*

Let a, b, and c be positive real numbers such that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = 1007\sqrt{2}\ .$$

Find the minimum value of the expression

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}\ .$$

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; N. Evgenidis; O. Geupel; R. Hess; J. G. Heuver; O. Kouba; A. Li; S. Malikić; M. R. Modak; C. Mortici; P. Perfetti; R. Peiro; A. Plaza; C. M. Quang (found the maximum only); D. Smith; I. Stallion; E. Swylan; I. Uchiha; S. Wagon; H. Wang and J. Wojdylo; T. Zvonaru; and the proposer. We present 2 solutions.

Solution 1 by Ang Li.

By using the root-mean-square inequality we find that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} = \sqrt{2}\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right)$$

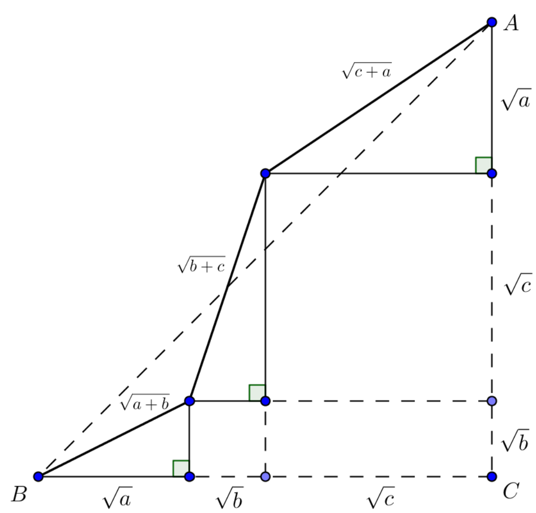
$$= \sqrt{2}\left(\sqrt{\frac{(\sqrt{a})^2+(\sqrt{b})^2}{2}}+\sqrt{\frac{(\sqrt{b})^2+(\sqrt{c})^2}{2}}+\sqrt{\frac{(\sqrt{c})^2+(\sqrt{a})^2}{2}}\right)$$

$$\geq \sqrt{2}\left(\frac{\sqrt{a}+\sqrt{b}}{2}+\frac{\sqrt{b}+\sqrt{c}}{2}+\frac{\sqrt{c}+\sqrt{a}}{2}\right)$$

$$= \sqrt{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}) = \sqrt{2}(1007\sqrt{2}) = 2014.$$

Equality holds if and only if $a=b=c=2\left(\frac{1007}{3}\right)^2$, which implies that the required minimum value is 2014.

Solution 2 by Itachi Uchiha.



In the diagram AB is the hypotenuse of the right triangle ABC, which shows that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \geq AB = \sqrt{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}) = 2014.$$

Equality holds if and only if the slopes of the thick lines are equal, namely $\frac{\sqrt{b}}{\sqrt{a}} = \frac{\sqrt{c}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{c}}$; that is, if and only if $a=b=c$, and then the desired minimum is 2014.

Editor's Comment. Most of the submitted solutions were quite similar to the first of the featured solutions. Due to a transcription error, the problem before the correction ([2012: 420, 422]) called for the maximum value of $\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$. Several solvers observed that because $\sqrt{x+y}<\sqrt{x}+\sqrt{y}$ when x and y are both positive,

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}<2(\sqrt{a}+\sqrt{b}+\sqrt{c})=2014\sqrt{2}.$$

As the sum on the left equals $2014\sqrt{2}$ when $\sqrt{a}=1007\sqrt{2}$ while $b=c=0$, we deduce that $2014\sqrt{2}$ is the least upper bound (which cannot be attained using positive values for a, b, and c). Also many correspondents observed that the quantity $1007\sqrt{2}$ in the statement of the problem could be replaced by any constant k, in which case we would have

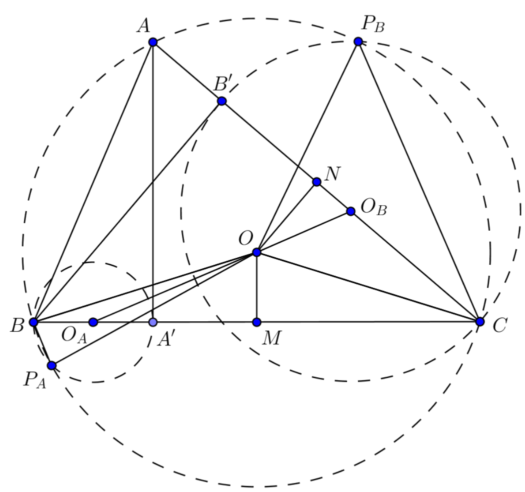
$$\sqrt{2}\,k \le \sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}<2k.$$

3794. [2012 : 420, 422] *Proposed by Václav Konečný.*

Let an acute triangle ABC be inscribed in the circle Γ, and A' and B' be the feet of the altitudes from A and B respectively. Let the circle on diameter BA' intersect Γ again at P_A, and the circle on diameter CB' intersect Γ again at P_B. Determine the angle between the lines AP_A and BP_B.

Solved by T. Zvonaru; and the proposer. We present the solution by Titu Zvonaru.

As usual, let $a=BC$, $b=CA$, and $c=AB$. Let O be the center of Γ and R the radius of Γ. Let M and N be the midpoints of of BC and CA, respectively, and let O_A and O_B be the midpoints of BA' and CB', respectively.



Since $OB=OP_A$, and OO_A is perpendicular to BP_A, we deduce that OO_A is the bisector of $\angle P_AOB$. Similarly, OO_B is the bisector of $\angle P_BOC$.

Denote $x = \tan A$, $y = \tan B$, $z = \tan C$. It is known that $x + y + z = xyz$.

We have
$$O_AM = BM - O_AB = \frac{a - c\cos B}{2} = \frac{b\cos C}{2}$$
and $OM = R\cos A$; hence
$$\tan\angle O_AOM = \frac{O_AM}{OM} = \frac{b\cos C}{2R\cos A} = \frac{\sin B\cos C}{\cos A} = \frac{\sin B\cos C}{-\cos(B+C)}$$
$$= \frac{\sin B\cos C}{\sin B\sin C - \cos B\cos C} = \frac{\tan B}{\tan B\tan C - 1} = \frac{y}{yz-1}.$$
Similarly $\tan\angle O_BON = \frac{z}{zx-1}$. Since $\angle BOM = A$, it follows that
$$\tan\angle BOO_A = \tan(A - \angle O_AOM) = \frac{x - \frac{y}{yz-1}}{1 + \frac{xy}{yz-1}} = \frac{xyz - x - y}{xy + yz - 1} = \frac{z}{xy + yz - 1}.$$
We denote by ϕ the angle between the lines AP_A and BP_B. We have
$$\phi = \frac{BP_A + AP_B}{2} = \frac{BP_A}{2} + \frac{AC}{2} - \frac{CP_B}{2} = \angle BOO_A + B - \angle COO_B$$
$$= \angle BOO_A + B - (B - \angle O_BON) = \angle BOO_A + \angle O_BON.$$
Then we have:
$$\tan\phi = \frac{\frac{z}{xy+yz-1} + \frac{z}{zx-1}}{1 - \frac{z^2}{(xy+yz-1)(zx-1)}} = \frac{z(xy + yz + zx - 2)}{(xy + yz - 1)(zx - 1) - z^2}.$$
Since
$$\begin{aligned}&(xy + yz - 1)(zx - 1) - z^2\\ &= x^2yz - xy + xyz^2 - yz - xz + 1 - z^2\\ &= x(x + y + z) + z(x + y + z) - xy - yz - zx + 1 - z^2\\ &= 1 + zx + x^2,\end{aligned}$$
it follows that
$$\tan\phi = \frac{z(xy + yz + zx - 2)}{1 + zx + x^2}.$$

3795. [2012 : 421, 422] *Proposed by José Luis Díaz-Barrero.*

Let a, b, c be the lengths of the sides of a triangle ABC with altitudes h_a, h_b, h_c and circumradius R. Prove that
$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(R + \frac{h_a + h_b + h_c}{6}\right) > 3\ .$$

Solved by AN-anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; C. Curtis; O. Geupel; K. Lau; S. Malikić; P. Perfetti; I. Uchiha;

P. Y. Woo; T. Zvonaru; D. Văcaru; and the proposer. We present 2 solutions and an extension.

Solution 1 by Itachi Uchiha.

Let $\triangle$ denote the area of the triangle. By the well-known formulae

$$\triangle = \frac{abc}{4R} = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

Now, using this fact, together with the AM-GM Inequality, we have

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(R+\frac{h_a+h_b+h_c}{6}\right) &= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{abc}{4\triangle}+\frac{\triangle}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right)\\ &\geq 3\frac{1}{\sqrt[3]{abc}}\left(\frac{abc}{4\triangle}+\frac{\triangle}{\sqrt[3]{abc}}\right)\\ &\geq 3\frac{1}{\sqrt[3]{abc}}\sqrt[3]{abc} = 3.\end{aligned}$$

Equality holds if and only if $a = b = c$ and $\dfrac{abc}{4\triangle} = \dfrac{\triangle}{\sqrt[3]{abc}}$, or in other words, when $4\triangle^2 = (abc)^{\frac{4}{3}}$ or $\triangle = \frac{1}{2}(abc)^{\frac{2}{3}} = \frac{1}{2}a^2$. However, when $a = b = c$, we have $\triangle = \frac{\sqrt{3}}{4}a^2$. Hence, the inequality is strict.

Solution 2 by P. Y. Woo modified slightly by the editor.

Since $c = 2R\sin C$, we have $\frac{R}{c} = \frac{1}{2\sin C}$ and, similarly, $\frac{R}{a} = \frac{1}{2\sin A}$ and $\frac{R}{b} = \frac{1}{2\sin B}$. Also, $h_a = c\sin B = 2R\sin B\sin C$ and, similarly, $h_b = 2R\sin C\sin A$ and $h_c = 2R\sin A\sin B$.

Let $k = (\sin A\sin B\sin C)^{\frac{1}{3}}$ and let L denote the left side of the given inequality. Then using the AM-GM Inequality, we have

$$\begin{aligned}L &= \frac{1}{2}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\left(1+\frac{\sin B\sin C+\sin C\sin A+\sin A\sin B}{3}\right)\\ &\geq \frac{1}{2}\left(\frac{1}{k}\right)(1+k^2)\\ &= \frac{3}{2}\left(\frac{1}{k}+k\right) \geq 3.\end{aligned}$$

If equality holds, then $\frac{1}{k} = k$ implies $k = 1$, so $\sin A = \sin B = \sin C = 1$, which is clearly impossible. Hence the inequality is strict.

Extension by AN-anduud Problem Solving Group.

Let L be as in Solution 2 above. We prove the stronger result that $L \geq \frac{7\sqrt{3}}{4}$.

Since $ab = 2Rh_c$, $bc = 2Rh_a$, $ca = 2Rh_b$ and $a+b+c \leq 3\sqrt{3}R$, by applying the AM-GM Inequality several times, we then have

$$
\begin{aligned}
L &= \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(4R+\frac{2}{3}h_a+\frac{2}{3}h_b+\frac{2}{3}h_c\right)\\
&\geq \frac{1}{4}\cdot 3\cdot\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}\cdot 7\sqrt[7]{R^4\left(\frac{2}{3}h_a\right)\left(\frac{2}{3}h_b\right)\left(\frac{2}{3}h_c\right)}\\
&= \frac{7}{4}\cdot\frac{3}{\sqrt[3]{abc}}\cdot\sqrt[7]{\frac{R}{27}(abc)^2}\\
&\geq \frac{7}{4}\cdot\frac{3}{\sqrt[3]{abc}}\cdot\sqrt[7]{\frac{1}{27}\cdot\frac{a+b+c}{3\sqrt{3}}\cdot(abc)^2}\\
&\geq \frac{7}{4}\cdot\frac{3}{\sqrt[3]{abc}}\cdot\sqrt[7]{\frac{1}{27}\cdot\frac{1}{\sqrt{3}}(abc)^{\frac{7}{3}}}\\
&= \frac{7}{4}\sqrt{3},
\end{aligned}
$$

and we are done. Note that this is in fact sharp, as equality holds for the equilateral triangle.

3796. [2012 : 421, 423] *Proposed by Michel Bataille.*

Show that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^{\binom{2n}{2k-1}}}{(2k+1)^{\binom{2n}{2k}}}=1\ .$$

Solved by A. Kotronis; O. Kouba; and the proposer. We present two solutions.

Solution 1 by Omran Kouba.

We will use the following lemma:

Lemma 1 *For $a>0$,*

$$\int_0^\infty \frac{e^{-t}-e^{-at}}{t}dt=\ln a.$$

Proof. For $0<\epsilon<X$, we have that

$$
\begin{aligned}
\int_\epsilon^X \frac{e^{-t}-e^{-at}}{t}dt &= \int_\epsilon^X \frac{e^{-t}}{t}dt-\int_{a\epsilon}^{aX}\frac{e^{-u}}{u}du\\
&= \int_\epsilon^{a\epsilon}\frac{e^{-t}}{t}dt-\int_X^{aX}\frac{e^{-t}}{t}dt\\
&= \ln a-\int_\epsilon^\infty \frac{1-e^{-t}}{t}dt-\int_X^{aX}\frac{e^{-t}}{t}dt.
\end{aligned}
$$

Since the integrals $\int_0^1(1-e^{-t})t^{-1}dt$ and $\int_1^\infty e^{-t}t^{-1}dt$ are convergent, we can let ϵ tend to 0 and X tend to infinity to obtain the result. ■

Let u_n denote the logarithm of the product in the statement of the problem. Then

$$\begin{aligned} u_n &= \sum_{k=0}^{2n} (-1)^{k-1} \binom{2n}{k} \ln(k+1) \\ &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_0^\infty \frac{e^{-(k+1)t} - e^{-t}}{t} dt \\ &= \int_0^\infty \frac{e^{-t}}{t} \left(\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{-kt} - \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \right) dt \\ &= \int_0^\infty \frac{e^{-t}}{t} \left((1-e^{-t})^{2n} - 0 \right) dt = \int_0^\infty \frac{(1-e^{-t})^{2n} e^{-t}}{t} dt. \end{aligned}$$

For $t > 0$, the integrand decreases to 0 with n, so that

$$\lim_{n\to\infty} u_n = \int_0^\infty \lim_{n\to 0} \left(\frac{(1-e^{-t})^{2n} e^{-t}}{t} \right) dt = 0,$$

from which we see that the desired limit is 1.

Solution 2 by Michel Bataille.

Define u_n as in the first solution. We first show that $u_n = (2n)! I_{2n}$, where

$$I_n = \int_0^\infty \frac{dx}{(x+1)(x+2)\cdots(x+n+1)}$$

for $n \geq 2$. From the partial fraction decomposition

$$\frac{1}{(x+1)(x+2)\cdots(x+n+1)} = \sum_{k=0}^{n} (-1)^k \frac{1}{n!} \binom{n}{k} \frac{1}{x+k+1},$$

we have, for $X > 0$ and $n \geq 2$,

$$\begin{aligned} &n! \int_0^X \frac{dx}{(x+1)(x+2)\cdots(x+n+1)} \\ &= \left[\sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(x+k+1) \right]_0^X \\ &= (\ln X) \sum_{k=0}^{n} (-1)^k \binom{n}{k} + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln\left(1 + \frac{k+1}{X}\right) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(k+1). \end{aligned}$$

Since $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ and $\lim_{X\to\infty} \ln(1 + (k+1)X^{-1}) = 0$, it follows that

$$(n!) I_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \ln(k+1).$$

It suffices to show that $\lim_{n\to\infty}(n!)I_n = 0$. We make use of the following inequality: *if a_i and b_i $(1 \le i \le m)$ are nonnegative, then*

$$(a_1+b_1)(a_2+b_2)\cdots(a_m+b_m) \ge ((a_1a_2\cdots a_m)^{1/m} + (b_1b_2\cdots b_m)^{1/m})^m.$$

(To see this, note that the $\binom{m}{j}$ terms on the left side with exactly j factors a_i and $m-j$ factors b_i together involve each a_i recurring $(j/m)\binom{m}{j}$ times and each b_i recurring $(m-j)/m\binom{m}{j}$ times; by the arithmetic-geometric means inequality, their sum is not less than $\binom{m}{j}(a_1a_2\cdots a_m)^{j/m}(b_1b_2\cdots b_m)^{(m-j)/m}$.) Thus

$$(x+1)(x+2)\cdots(x+n+1) \ge (x + \sqrt[n+1]{(n+1)!})^{n+1},$$

so that

$$0 \le (n!)I_n \le n!\int_0^\infty \frac{dx}{\left(x+\sqrt[n+1]{(n+1)!}\right)^{n+1}} = \frac{\sqrt[n+1]{(n+1)!}}{n(n+1)}.$$

Since $\sqrt[n]{n} \sim n/e$ as $n \to \infty$, then the right side of this inequality tends to 0 and with it $(n!)I_n$. The desired result follows.

3797. [2012 : 421, 423] *Proposed by Panagiote Ligouras.*

Let m_a, m_b, m_c be the medians and k_a, k_b, k_c be the symmedians of a triangle ABC. If n is a positive integer, prove that

$$\left(\frac{m_a}{k_a}\right)^n + \left(\frac{m_b}{k_b}\right)^n + \left(\frac{m_c}{k_c}\right)^n \ge 3\ .$$

Solved by A. Alt; AN-Anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; M. Amengual Covas; C. Curtis; O. Geupel; O. Kouba; S. Malikić; C. M. Quang; P. Y. Woo; T. Zvonaru; and the proposer. We present the solution that is a composite of nearly all received solutions.

By standard formulas,

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4} \text{ and } k_a^2 = \frac{b^2c^2\left(2b^2+2c^2-a^2\right)}{\left(b^2+c^2\right)^2},$$

so that

$$\frac{m_a}{k_a} = \frac{b^2+c^2}{2bc} \ge 1,$$

by the AM-GM inequality. Similarly, $\frac{m_b}{k_b} \ge 1$ and $\frac{m_c}{k_c} \ge 1$. Thus,

$$\sum_{\text{cyclic}}\left(\frac{m_a}{k_a}\right)^n \ge \sum_{\text{cyclic}}\frac{m_a}{k_a} \ge 3.$$

3798. [2012 : 421, 423] *Proposed by Albert Stadler.*

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty} k^n \left(k+1-\frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

where k^n is taken to be 1 for $k=n=0$ and $S(n,k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1\,.$$

Editor's Comment. Note, this is the corrected version of problem 3687.

Solved by AN-Anduud Problem Solving Group; M. Bataille; O. Kouba; and the proposer. We present two solutions.

Solution 1 by Omran Kouba.

We will use the following lemma:

Lemma 1 *Let $f : (0,\infty) \longrightarrow \mathbb{R}$ be an $n-$times differentiable function, and let $g : \mathbb{R} \to \mathbb{R}$ be defined by the formula $g(t) = f(e^t)$. Then*

$$g^{(n)}(t) = \sum_{k=0}^{n} S(n,k) f^{(k)}(e^t)e^{kt}.$$

Proof. The result holds for $n=0$ and $n=1$. Assume that it holds for $n-1 \geq 1$ and consider an $n-$times differentiable function f. Since

$$g^{(n-1)}(t) = \sum_{k=0}^{n-1} S(n-1,k) f^{(k)}(e^t)e^{kt},$$

we obtain that

$$\begin{aligned} g^{(n)}(t) &= \sum_{k=0}^{n-1} S(n-1,k)(f^{(k+1)}(e^t)e^{(k+1)t} + kf^{(k)}(e^t)e^{kt}) \\ &= \sum_{k=1}^{n} S(n-1,k-1)f^{(k)}(e^t)e^{kt} + \sum_{k=0}^{n-1} kS(n-1,k)f^{(k)}(e^t)e^{kt} \\ &= \sum_{k=1}^{n} (S(n-1,k-1) + kS(n-1,k))f^{(k)}(e^t)e^{kt} \\ &= \sum_{k=1}^{n} S(n,k)f^{(k)}(e^t)e^{kt}. \blacksquare \end{aligned}$$

To solve the problem let $f(x) = e^x$. From the lemma

$$(\exp(e^t))^{(n)} = \exp(e^t) \sum_{k=0}^{n} S(n,k)e^{kt}.$$

Since $\exp(e^t) = \sum_{k=0}^{\infty} e^{kt}/k!$, by taking the nth derivative, we obtain that

$$\exp(-e^t) \sum_{k=0}^{\infty} \frac{k^n e^{kt}}{k!} = \sum_{k=0}^{n} S(n,k)e^{kt}.$$

Set $t = \ln x$ to obtain for $x > 0$ that

$$e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!} = \sum_{k=0}^{n} S(n,k)x^k.$$

Since $m! = \int_0^\infty e^{-t}t^m dt$ for each nonegative integer m, we see that

$$(k+1)! - \int_1^\infty e^{-t}t^{k+1}dt = \int_0^1 e^{-t}t^{k+1}dt.$$

Thus

$$\begin{aligned}
\sum_{k=0}^{\infty} k^n \left(k + 1 - \frac{1}{k!}\int_1^\infty e^{-t}t^{k+1}dt\right) &= \sum_{k=0}^{\infty} \frac{k^n}{k!}\int_0^1 e^{-t}t^{k+1}dt \\
&= \int_0^1 te^{-t}\left(\sum_{k=0}^{\infty} \frac{k^n t^k}{k!}\right)dt \\
&= \int_0^1 \left(\sum_{k=0}^{n} S(n,k)t^{k+1}\right)dt \\
&= \sum_{k=0}^{n} \frac{S(n,k)}{k+2}
\end{aligned}$$

as desired.

Solution 2 by Michel Bataille.

First, we show that for all nonnegative integers k, we have that

$$k + 1 - \frac{1}{k!}\int_1^\infty e^{-t}t^{k+1}dt = \frac{(k+1)R_{k+1}}{e}$$

where $R_m = \sum_{j\geq m+1} 1/j!$.

Let $I_k = \int_1^\infty e^{-t}t^k dt$. Then $I_0 = 1/e$. Integrating by parts, we find that $I_{k+1} = (1/e) + (k+1)I_k$, so that

$$\frac{I_{k+1}}{(k+1)!} - \frac{I_k}{k!} = \frac{1}{e} \cdot \frac{1}{(k+1)!}$$

for $k \geq 0$. Therefore

$$\frac{I_{k+1}}{(k+1)!} - I_0 = \sum_{j=0}^{k} \left(\frac{I_{j+1}}{(j+1)!} - \frac{I_j}{j!} \right) = \frac{1}{e} \sum_{j=0}^{k} \frac{1}{(j+1)!}.$$

Finally, we obtain that

$$\begin{aligned} k+1-\frac{1}{k!}I_{k+1} &= (k+1)\left(1-\frac{I_{k+1}}{(k+1)!}\right) \\ &= \frac{k+1}{e}\left(e-\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(k+1)!}\right)\right) = \frac{(k+1)R_{k+1}}{e}. \end{aligned}$$

Recall that the factorial powers are defined by $x^{(0)} = 1$ and $x^{(m)} = x(x-1)\cdots(x-m+1)$ for positive integer m and that $x^n = \sum_{k=0}^{n} S(n,k)x^{(k)}$. The left side L of the desired equality satisfies

$$L = \sum_{k=0}^{\infty} \frac{k^n(k+1)R_{k+1}}{e} = \frac{1}{e}\sum_{i=0}^{n} S(n,i) \sum_{k=0}^{\infty} (k+1)^{(i+1)} R_{k+1}.$$

Since the summands are nonnegative, we have that

$$\begin{aligned} \sum_{k=0}^{\infty}(k+1)^{(i+1)}R_{k+1} &= \sum_{k=i}^{\infty}\sum_{j=k+2}^{\infty}\frac{(k+1)^{(i+1)}}{j!} = \sum_{j=1}^{\infty}\frac{1}{(i+j+1)!}\sum_{l=1}^{j}(i+l)^{(i+1)} \\ &= \sum_{j=1}^{\infty}\frac{1}{(i+j+1)!}\sum_{l=1}^{j} l(l+1)\cdots(l+i) \\ &= \sum_{j=1}^{\infty}\frac{1}{(i+j+1)!}\left(\frac{(j+i+1)!}{(i+2)(j-1)!}\right) \\ &= \sum_{j=1}^{\infty}\frac{1}{(i+2)(j-1)!} = \frac{e}{i+2}. \end{aligned}$$

Therefore,

$$L = \frac{1}{e}\sum_{i=0}^{n} S(n,i)\frac{e}{i+2} = \sum_{k=0}^{n}\frac{S(n,k)}{k+2}.$$

3799. [2012 : 421, 423] *Proposed by Constantin Mateescu.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s for which we denote $K = \sum_{\text{cyclic}} \sin\frac{A}{2}$. Prove that

$$s^2 = 4R(K-1)^2\left[R(K+1)^2 + r\right].$$

Solved by A. Alt; M. Bataille; S. Malikić; P. Y. Woo; T. Zvonaru; and the proposer. We present the solution by Michel Bataille.

First, we recall some well-known formulas involving the elements of a triangle:

$$\begin{aligned}
&\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\\
&\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},\\
&ab+bc+ca = s^2+r^2+4rR,\\
&\cos A+\cos B+\cos C = 1-\frac{r}{R}.
\end{aligned}$$

From the latter, we have

$$\sum_{\text{cyclic}}\sin^2\frac{A}{2} = \sum_{\text{cyclic}}\frac{1-\cos A}{2} = 1-\frac{r}{2R}. \qquad (1)$$

We will also prove the following formula

$$\sum_{\text{cyclic}}\sin^2\frac{A}{2}\sin^2\frac{B}{2} = \frac{s^2+r^2-8rR}{16R^2}. \qquad (2)$$

Proof. We have

$$\begin{aligned}
\sin^2\frac{A}{2}\sin^2\frac{B}{2} &= \frac{(s-b)(s-c)}{bc}\cdot\frac{(s-c)(s-a)}{ca}\\
&= \frac{s-c}{c}\cdot\frac{F^2}{s}\cdot\frac{1}{abc} = \frac{s-c}{c}\cdot r^2 s\cdot\frac{1}{4Rrs} = \frac{s-c}{c}\cdot\frac{r}{4R}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{\text{cyclic}}\sin^2\frac{A}{2}\sin^2\frac{B}{2} &= \frac{r}{4R}\sum_{\text{cyclic}}\frac{s-c}{c}\\
&= \frac{r}{4R}\left(\frac{s(s^2+r^2+4rR)}{4rsR}-3\right) = \frac{s^2+r^2-8rR}{16R^2}. \blacksquare
\end{aligned}$$

Turning back to the problem, we need to prove

$$4R^2(K^2-1)^2+4rR(K-1)^2 = s^2. \qquad (3)$$

Let $Q = \sum_{\text{cyclic}}\frac{1}{\sin\frac{A}{2}}$. Then, using (1), we have

$$K^2 = \sum_{\text{cyclic}}\sin^2\frac{A}{2}+2\sum_{\text{cyclic}}\sin\frac{A}{2}\sin\frac{B}{2} = 1-\frac{r}{2R}+2\cdot\frac{r}{4R}\cdot Q = 1+\frac{r}{2R}(Q-1).$$

Therefore,

$$K^2-1 = \frac{r}{2R}(Q-1)$$

and

$$(K-1)^2 = 1 + \frac{r}{2R}(Q-1) - 2K + 1 = \frac{r}{2R}(Q-1) - 2(K-1).$$

As a result,

$$\begin{aligned} 4R^2(K^2-1)^2 + 4rR(K-1)^2 &= r^2(Q-1)^2 + 2r^2(Q-1) - 8rR(K-1) \\ &= (r^2Q^2 - 8rRK) - r^2 + 8rR \qquad (4). \end{aligned}$$

Now,

$$Q^2 = \sum_{\text{cyclic}} \frac{1}{\sin^2 \frac{A}{2}} + 2\sum_{\text{cyclic}} \frac{1}{\sin \frac{A}{2} \sin \frac{B}{2}} = \left(\frac{4R}{r}\right)^2 \sum_{\text{cyclic}} \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} + 2 \cdot \frac{4R}{r} \cdot K.$$

Using (2), we have $r^2Q^2 = s^2 + r^2 - 8rR + 8rRK$. Back to (4), we finally get

$$4R^2(K^2-1)^2 + 4rR(K-1)^2 = s^2 + r^2 - 8rR - r^2 + 8rR$$

and (3) follows.

3800. [2012 : 422, 423] *Proposed by Ovidiu Furdui.*

Let $n \geq 2$ be an integer. Calculate

$$\int_0^\infty \int_0^\infty \left(\frac{e^{-x} - e^{-y}}{x-y}\right)^n dxdy \ .$$

Solved by M. Bataille; O. Kouba; and the proposer. We present the solution by Omran Kouba.

Let the considered integral be denoted by I_n. We will prove that

$$I_n = \frac{2}{n!} \sum_{k=2}^{n} (-1)^k \binom{n}{k} k^{n+1} \ln k.$$

The function $t \mapsto e^{-t}$ is decreasing, so $\frac{e^{-y}-e^{-x}}{x-y} > 0$ for every $x \neq y$. Consequently, the integrand in the following integral is positive, and we can make the change of variables $(x, y) \leftarrow (t+u, t-u)$, as follows:

$$\begin{aligned}
(-1)^n I_n &= \int_0^\infty \int_0^\infty \left(\frac{e^{-y}-e^{-x}}{x-y}\right)^n dxdy \\
&= 2\int_{-\infty}^\infty \int_{|u|}^\infty \left(\frac{e^{-t+u}-e^{-t-u}}{2u}\right)^n dtdu \\
&= 2\int_{-\infty}^\infty \left(\frac{e^{u}-e^{-u}}{2u}\right)^n \left(\int_{|u|}^\infty e^{-nt}dt\right) du \\
&= \frac{4}{n}\int_0^\infty \left(\frac{e^{u}-e^{-u}}{2u}\right)^n e^{-nu}du \\
&= \frac{4}{n}\int_0^\infty \left(\frac{1-e^{-2u}}{2u}\right)^n du = \frac{2}{n}J_n, \qquad (1)
\end{aligned}$$

where

$$J_n = \int_0^\infty \left(\frac{1-e^{-v}}{v}\right)^n dv \qquad (2)$$

Now, the integral J_n was calculated in Solution 2 of Problem 3670 by Mohammed Aassila [see ***Crux Mathematicorum***, Vol 38, No 7 (2013), p.300-301]. It was proven that

$$J_n = \frac{(-1)^n}{(n-1)!}\sum_{k=2}^{n}(-1)^k\binom{n}{k}k^{n-1}\ln k. \qquad (3)$$

The announced result follows immediately from (1), (2), and (3).

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(Bold font indicates featured solution.)

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YEAR-END FINALE

The past year was a year of changes at ***Crux***. First of all, ***Mathematical Mayhem*** left ***Crux Mathematicorum*** (hopefully to come back soon as a separate publication) and ***Skoliad*** has been replaced by ***Contest Corner***. New columns ***Focus On ...***, ***The Problem Solver's Toolkit*** and ***Problem of the Month*** have been going strong. Finally, back files of ***Crux***, all the way from Volume 1 in 1975, have been released online.

A big change in the life of ***Crux*** has been the retirement of Shawn Godin, who has been the driving force behind ***Crux*** for a number of years. Shawn has done an amazing job in establishing a more electronic file system, clearing proposals backlog and introducing new features to ***Crux***. I am very happy that CMS has recognized his contributions to mathematics by presenting him with the CMS 2014 Graham Wright Award for Distinguished Service – what a great way to celebrate his achievements.

At this year end, I would like to thank all of the contributors, all the editors and the translators for their dedication to the journal as well as their passion for mathematics and the willingness to share it with the world. I would like to also especially thank all the editors for making my transition a smooth one and for establishing a supportive environment in the creation of each issue of ***Crux***. Without the awe-inspiring expertise and patience of each member of the editorial board, this journal simply would not be the highest quality internationally recognized problem solving publication that it is.

In this Volume, we are saying goodbye to editors John McLoughlin, Peter O'Hara, Mohamed Omar and Edna James – thank you for all your hard work and commitment to excellence. On the other hand, I would like to welcome guest editors Joseph Horan and Amanda Malloch who have been a great help to me already.

I would like to thank the readers for giving voice to ***Crux*** as well as for their patience as we try to work our way through the backlog. Your submissions and comments are essential in the existence of ***Crux*** because it is driven by its audiences.

The staff at the CMS office has been invaluable in ***Crux*** production. The CMS is present in every step of the process: the receiving and filing of proposals and solutions, the online and print support, corrections and updates and so on. Thank you for being there for ***Crux***.

In the words of Shawn from last Year-End Finale, "I look forward to another n years of your contributions, letters and email. Now I have to start working on issue 1 ..."

Kseniya Garaschuk

INDEX TO VOLUME 39, 2013

Skoliad *Lily Yen and Mogens Lemvig Hansen*

Mathematical Mayhem Solutions *Shawn Godin*

The Contest Corner *Shawn Godin, Kseniya Garaschuk, Robert Bilinski*

The Olympiad Corner *Nicolae Strungaru*

Book Reviews *Amar Sodhi, John Grant McLoughlin*

Focus On . . . *Michel Bataille*

Problem of the Month

Problem Solver's Toolkit

Crux Articles *Robert Dawson*

Problems

Solutions

Miscellaneous

Crux Mathematicorum

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