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SKOLIAD No. 142

Lily Yen and Mogens Hansen

Skoliad has joined **Mathematical Mayhem** *which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time Skoliad will be discontinued in* **Crux**. *New Skoliad problems, and their solutions, will appear in* **Mathematical Mayhem** *when it is relaunched.*

In this issue we present the solutions to the Swedish Junior High School Mathematics Contest, Final Round, 2010/2011, given in Skoliad 136 at [2011 : 409–410].

1. The year 2010 is divisible by three consecutive primes. Find the last year before that with this property.

Solution by Lucy Yuan, student, New Westminster Secondary School, New Westminster, BC.

The first few primes are 2, 3, 5, 7, 11, 13, and 17. The desired year cannot be divisible by the last three of these, since $11 \times 13 \times 17 = 2431$, which is too large. Likewise, any triple of larger primes can be discarded.

If the desired year is divisible by 2, 3, and 5, then it is divisible by $2 \times 3 \times 5 = 30$. Now $2010 \div 30 = 67$, so the latest such year prior to 2010 is $66 \times 30 = 1980$.

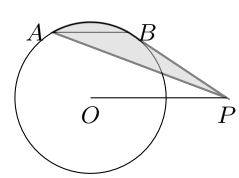
If the desired year is divisible by 3, 5, and 7, then it is divisible by $3 \times 5 \times 7 = 105$. Since $2010 \div 105 \approx 19.1$, the latest such year prior to 2010 is $19 \times 105 = 1995$ (which is closer than 1980).

Similarly, $5 \times 7 \times 11 = 385$ and $2010 \div 385 \approx 5.2$, so the latest year divisible by those three primes is $5 \times 385 = 1925$.

Finally, $7 \times 11 \times 13 = 1001$ and $2010 \div 1001 \approx 2.01$, so the latest year divisible by those three primes is $2 \times 1001 = 2002$.

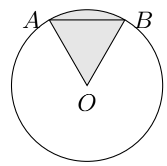
Of the four years found, 1980, 1995, 1925, and 2002, the one closest to 2010 is 2002.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; MAGGIE LIU, student, Burnaby Central Secondary School, Burnaby, BC; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.



2. Draw a line from the centre, O, of a circle with radius r to a point, P, outside the circle. Then choose two points, A and B, on the circle such that AB has length r and is parallel with OP. Find the area of the shaded region.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.



Using AB as the base for both triangles, it is easy to see that $\triangle ABP$ and $\triangle ABO$ have the same area. Therefore the shaded region in the problem has the same area as the shaded region in the figure on the right.

Since $|AB| = r$, the radius of the circle, $\triangle ABO$ is equilateral, so $\angle AOB = 60^\circ$, and the shaded sector is $\frac{60}{360} = \frac{1}{6}$ of the circle. Therefore the area of the shaded sector is $\frac{1}{6}\pi r^2$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

3. Five distinct positive numbers are given. No matter which two of them you choose, one divides the other. The sum of the five numbers is a prime. Show that one of the five numbers is 1.

Solution by Andrew Tao, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA.

Let x denote the smallest of the five positive integers. Since no number can divide a smaller number, x divides each of the other four numbers. Therefore x divides the sum of the five numbers. Since the sum is a prime, either x is the sum or $x = 1$. Since the numbers are positive, x cannot equal the sum, so $x = 1$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

4. A large cube consists of eight identical smaller cubes. The faces of each of the smaller cubes bear the numbers 3, 3, 4, 4, 5, and 5 such that opposite faces bear the same number. Assign to each face of the large cube the sum of the four visible numbers. Show that the numbers assigned to the faces of the large cube cannot be six consecutive integers.

Solution by Anna Verkhovskaya, student, John Ware Junior High, Calgary, AB.

Each of the smaller cubes sits at a corner of the large cube. Therefore three faces of each smaller cube are visible, and none of these faces are opposite each other. Thus, each of the smaller cubes shows the numbers 3, 4, and 5, so each of the smaller cubes contribute $3+4+5 = 12$ to the sum of the numbers on the faces of the large cube. Therefore the sum of the numbers on the faces of the large cube is $8 \times 12 = 96$.

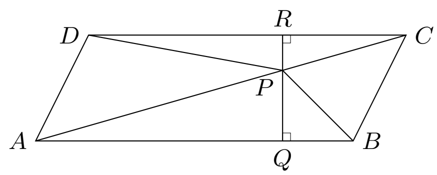
Since any six consecutive integers contain three odd and three even integers, their sum must be an odd number. As 96 is not odd, it cannot be the sum of six consecutive integers, and therefore the sides of the cubes cannot show six consecutive integers.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA.

5. The parallelogram $ABCD$ has area 12. The point P is on the diagonal AC. The area of $\triangle ABP$ is one third of the area of $ABCD$. Find the area of $\triangle CDP$.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let Q and R be the feet of the perpendiculars to P from AB and CD, respectively. The area of $ABCD$ is $|AB||QR| = 12$, and the area of $\triangle ABP$ is $\frac{1}{2}|AB||QP| = \frac{1}{3} \cdot 12 = 4$, so $|AB||QP| = 8$. Therefore,

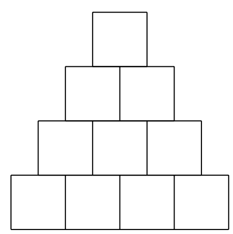


$$\frac{|QP|}{|QR|} = \frac{|AB||QP|}{|AB||QR|} = \frac{8}{12} = \frac{2}{3},$$

so $|QP| = \frac{2}{3}|QR|$. Since $|QP| + |PR| = |QR|$, it follows that $|PR| = \frac{1}{3}|QR| = \frac{1}{2}|QP|$. Thus, the area of $\triangle CDP$ is $\frac{1}{2}|CD||PR| = \frac{1}{2}|AB|\frac{1}{2}|QP| = \frac{1}{2}(\frac{1}{2}|AB||QP|)$, which is half the area of $\triangle ABP$, so half of 4, so 2.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

6. Place ten numbers in the grid subject to the following rules:

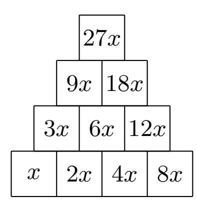


1. For neighbours in the bottom row, the number on the right must be twice as large as the number on the left.
2. Other than in the bottom row, each number is the sum of the two numbers immediately below it.

Find the smallest positive integer that you can place in the bottom left position such that the sum of all ten numbers is a square.

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

Say the number in the bottom left slot is x. Then the numbers in the bottom row must be x, $2x$, $4x$, and $8x$, and you can fill in the entire diagram as shown. The sum of all ten numbers in the diagram is then $90x$.



If $90x = 3^2 \cdot 2 \cdot 5 \cdot x$ is to be a square (and x a positive integer), then $x = 2 \cdot 5 \cdot n^2$, where n is an integer other than zero. The smallest possible value for x occurs, then, when $n = 1$, so $x = 10$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

This issue's prize for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

THE CONTEST CORNER

No. 9

Shawn Godin

The Contest Corner *is a new feature of* ***Crux Mathematicorum****. It will be filling the gap left by the movement of* Mathematical Mayhem *and* Skoliad *to a new on-line journal in 2013. The column can be thought of as a hybrid of* Skoliad, The Olympiad Corner *and the old* Academy Corner *from several years back. The problems featured will be from high school and undergraduate mathematics contests with readers invited to submit solutions. Readers' solutions will begin to appear in the next volume.*

Solutions can be sent to:

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or by email to

crux-contest@cms.math.ca.

The solutions to the problems are due to the editor by **1 March 2014**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translating the problems from English into French.

CC41. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

CC42. ΔABC has its vertices on a circle of radius r. If the lengths of two of the medians of ΔABC are equal to r, determine the side lengths of ΔABC.

CC43. A circle has diameter AB. P is a fixed point of AB lying between A and B. A point X, distinct from A and B, lies on the circumference of the circle. Prove that $\tan(\angle AXP) \div \tan(\angle XAP)$ is constant for all values of X.

CC44. Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}a_n^2$. Find $\lim\limits_{n\to\infty} na_n$, if it exists.

CC45. The *baseball sum* of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{a+c}{b+d}$. Starting with the rational numbers $\frac{0}{1}$ and $\frac{1}{1}$ as Stage 0, the baseball sum of each consecutive pair of rational numbers in a stage is inserted between the pair to arrive at the next stage. The first few stages of this process are shown below:

STAGE 0:	$\frac{0}{1}$								$\frac{1}{1}$
STAGE 1:	$\frac{0}{1}$				$\frac{1}{2}$				$\frac{1}{1}$
STAGE 2:	$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$
STAGE 3:	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

Prove that:

(i) no rational number will be inserted more than once,

(ii) no inserted fraction is reducible, and

(iii) every rational number between 0 and 1 will be inserted in the pattern at some stage.

. . . .

CC41. Laflèche court à une vitesse constante et Leclair court x fois plus vite, $x > 1$. Leclair donne une longueur d'avance de y mètres à Laflèche puis, au signal, les deux partent dans la même direction. Déterminer la distance que Leclair doit parcourir pour rattraper Laflèche.

CC42. Les sommets du triangle ABC sont situés sur un cercle de rayon r. Sachant que le triangle ABC a deux médianes de longueur r, déterminer la longueur de chacun des côtés du triangle ABC.

CC43. On considère un cercle de diamètre AB. P est un point fixe sur ce diamètre, autre que A et B. X est un point sur le cercle, autre que A et B. Démontrer que $\tan(\angle AXP) \div \tan(\angle XAP)$ est une constante, quelle que soit la position de X.

CC44. Soit $a_0 = 1$. Lorsque $n \geq 0$, soit $a_{n+1} = a_n - \frac{1}{2}a_n^2$. Déterminer $\lim\limits_{n\to\infty} na_n$ si elle existe.

CC45. La *somme naïve* de deux nombres rationnels, $\frac{a}{b}$ et $\frac{c}{d}$, est le nombre $\frac{a+c}{b+d}$. À l'étape 0, on écrit les nombres rationnels $\frac{0}{1}$ et $\frac{1}{1}$. Pour atteindre l'étape suivante, on considère chaque paire de fractions consécutives de l'étape précédente et on insère entre elles la somme naïve de ces deux fractions. Les premières étapes de ce processus sont indiquées ci-dessous :

ÉTAPE 0 :	$\frac{0}{1}$								$\frac{1}{1}$
ÉTAPE 1 :	$\frac{0}{1}$				$\frac{1}{2}$				$\frac{1}{1}$
ÉTAPE 2 :	$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$
ÉTAPE 3 :	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

Démontrer que :

(i) aucun nombre rationnel ne sera inséré plus d'une fois ;

(ii) chaque fraction insérée est irréductible ;

(iii) chaque nombre rationnel entre 0 et 1 sera inséré à une étape quelconque.

THE OLYMPIAD CORNER

No. 307

Nicolae Strungaru

The solutions to the problems are due to the editor by **1 March 2014**.

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The editor thanks Rolland Gaudet of Université de Saint-Boniface for translations of the problems.

OC101. Let n, k be positive integers so that $1 < k < n - 1$. Prove that the binomial coefficient $\binom{n}{k}$ is divisible by at least two distinct primes.

OC102. Let $\mathbb{N}$ denote the set of all nonnegative integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that both (1) and (2) are satisfied.

(1) $0 \le f(x) \le x^2$ for all $x \in \mathbb{N}$.

(2) $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}$ with $x > y$.

OC103. Let K and L be the points on the semicircle with diameter AB. Denote the intersection of AK and BL as T and let N be the foot of the perpendicular from T to AB. If U is the intersection of the perpendicular bisector of AB and KL and V is a point on KL such that angles UAV and UBV are equal, then prove that NV is perpendicular to KL.

OC104. Given a triangle ABC, let D be the midpoint of the side AC and let M be the point on the segment BD so that $BM : MD = 1 : 2$. The rays AM and CM intersect the sides BC and AB at E and F respectively. We know that $AM \perp CM$. Prove that the quadrilateral $AFED$ is cyclic if and only if the median from A in ΔABC meets the line EF at a point situated on the circumcircle of ΔABC.

OC105. Let $n > 1$ be an integer, and let k be the number of distinct prime divisors of n. Prove that there exists an integer a, $1 < a < \frac{n}{k} + 1$, such that $n \mid a^2 - a$.

. . . .

OC101. Soient n et k des entiers positifs tels que $1 < k < n - 1$. Démontrer que le coefficient binomial $\binom{n}{k}$ est divisible par au moins deux nombres premiers distincts.

OC102. Soit $\mathbb{N}$ l'ensemble de tous les entiers non négatifs. Déterminer toutes les fonctions $f : \mathbb{N} \to \mathbb{N}$ telles que les deux conditions suivantes soient satisfaitent.

(1) $0 \leq f(x) \leq x^2$ pour tout $x \in \mathbb{N}$.

(2) $x - y$ divise $f(x) - f(y)$ pour tout $x, y \in \mathbb{N}$ tels que $x > y$.

OC103. Soient K et L des points sur le demi cercle de diamètre AB. Dénoter par T le point d'intersection de AK et BL ; soit N le pied de la perpendiculaire de T vers AB. Soit U le point d'intersection de la bissectrice perpendiculaire de AB avec KL ; soit V un point sur KL tel que les angles UAV et UBV soient égaux. Démontrer que NV est perpendiculaire à KL.

OC104. Soit le triangle ABC. Soient aussi D le mipoint du côté AC et puis M le point sur le segment BD tel que $BM : MD = 1 : 2$. Les rayons AM et CM intersectent les côtés BC et AB aux points E et F respectivement. Nous savons que $AM \perp CM$. Démontrer que le quadrilatère $AFED$ est cyclique si et seulement si la médiane de A dans ΔABC rencontre la ligne EF à un point situé sur le cercle circonscrit de ΔABC.

OC105. Soit $n > 1$ entier et soit k le nombre de diviseurs premiers distincts de n. Démontrer qu'il existe un entier a, $1 < a < \frac{n}{k} + 1$, tel que $n \mid a^2 - a$.

OLYMPIAD SOLUTIONS

OC30. Let P be an interior point of a regular n-gon $A_1A_2\cdots A_n$. Each line A_iP meets the n-gon at another point B_i. Prove that

$$\sum_{i=1}^{n} PA_i \geq \sum_{i=1}^{n} PB_i\,.$$

(*Originally question 8 from the 2008 China Western Mathematical Olympiad.*)

Solved by George Apostolopoulos, Messolonghi, Greece.

Let $m = \left\lceil \frac{n}{2} \right\rceil$, then, with the convention $A_{n+k} = A_k$ for all $k = 1, 2, ..., n$, the diagonals A_kA_{k+m} are the longest diagonals in the polygon. Let d denote the length of these diagonals. For each A_i there exists some j so that B_i is a point on the edge A_jA_{j+1}, possibly one of the vertices A_j, A_{j+1}. Then

$$A_iB_i \leq \max\{A_iA_j, A_iA_{j+1}\} \leq d\,.$$

Thus,

$$PA_i + PB_i = A_iB_i \leq d\,.$$

As $A_iA_{i+m} = d$ we also get by the triangle inequality that

$$PA_i + PA_{i+m} \geq A_iA_{i+m} = d \geq PA_i + PB_i\,.$$

Hence,

$$PA_{i+m} \geq PB_i\,.$$

Adding these relations, we get the desired result.

OC41. Let P be a point in the interior of a triangle ABC. Show that

$$\frac{PA}{BC} + \frac{PB}{AC} + \frac{PC}{AB} \geq \sqrt{3}\,.$$

(*Originally question 10 from the 2009 India IMO selection test.*)

Similar solutions by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Marian Dincă, Bucharest, Romania; and David E. Manes, SUNY at Oneonta, Oneonta, NY, USA. We will give the solution of Dinca.

We start by proving the Hayashi inequality:

$$\frac{PA \cdot PB}{CA \cdot CB} + \frac{PA \cdot PC}{BA \cdot BC} + \frac{PB \cdot PC}{AB \cdot AC} \geq 1\,.$$

To prove this inequality we proceed as follows. We view A, B, C, P as points in the complex plane, and we denote by a, b, c, z their complex coordinates.

Let

$$P(z) = (z-a)(z-b)(a-b) + (z-b)(z-c)(b-c) + (z-c)(z-a)(c-a)\,.$$

Then $P(z)$ is a polynomial of degree at most two, and it is easy to see that $P(a) = P(b) = P(c) = (a-b)(b-c)(c-a)$. Thus, $P(z)$ must be the constant polynomial $(a-b)(b-c)(c-a)$. Hence

$$(z-a)(z-b)(a-b)+(z-b)(z-c)(b-c)+(z-c)(z-a)(c-a) = (a-b)(b-c)(c-a)\,.$$

Then

$$\begin{aligned} AB \cdot AC \cdot BC &= |(a-b)(b-c)(c-a)| \\ &= |(z-a)(z-b)(a-b) + (z-b)(z-c)(b-c) \\ &\qquad\qquad + (z-c)(z-a)(c-a)| \\ &\leq |(z-a)(z-b)(a-b)| + |(z-b)(z-c)(b-c)| \\ &\qquad\qquad + |(z-c)(z-a)(c-a)| \\ &= PA \cdot PB \cdot AB + PB \cdot PC \cdot BC + PC \cdot PA \cdot CA\,. \end{aligned}$$

Dividing the inequality by $AB \cdot AC \cdot BC$ we get the Hayashi inequality.

Now, using the well known $(x+y+z)^2 \geq 3(xy+yz+zx)$ with $\frac{PA}{BC} = x$, $\frac{PB}{AC} = y$, $\frac{PC}{AB} = z$ we get

$$\left(\frac{PA}{BC} + \frac{PB}{AC} + \frac{PC}{AB}\right)^2 \geq 3\left(\frac{PA \cdot PB}{CA \cdot CB} + \frac{PA \cdot PC}{BA \cdot BC} + \frac{PB \cdot PC}{AB \cdot AC}\right) \geq 3\,.$$

OC42. Find the smallest n for which $n!$ has at least 2010 different divisors.
(*Originally question 3 from the 2009-2010 Finish National Olympiad, Final round.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA ; Mihaï-Ioan Stoënescu, Bischwiller, France ; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Manes.

The smallest n is 14.

Let $\tau(n)$ denote the number of divisors of n. As τ is a multiplicative function, with $\tau(p^\alpha) = \alpha + 1$ when p is prime and $\alpha \geq 0$ is an integer, we get

$$\begin{aligned}\tau(13!) &= \tau(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)\\ &= (10+1)(5+1)(2+1)(1+1)(1+1)(1+1) = 1584\,,\\ \tau(14!) &= \tau(2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)\\ &= (11+1)(5+1)(2+1)(2+1)(1+1)(1+1) = 2592\,.\end{aligned}$$

As $k!$ divides 13! for all $k \leq 13$, we know that $\tau(k!) \leq \tau(13!) = 1584$ for all $k \leq 13$, and this shows that $n = 14$ is the smallest number with the desired property.

OC43. Find all functions $f : \mathbb{R} \to \mathbb{R}$ verifying

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)\,;\ \forall x, y \in \mathbb{R}\,.$$

(*Originally question 3 from the 2009 Romania National Olympiad, 10th grade.*)

Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany ; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.

The functions

$$f(x) = cx, \qquad c \in \mathbb{R}$$

satisfy the given functional equation, and we are going to prove that there are no other solutions.

Suppose that f is a solution. Setting $y = 0$, we see that $f(x^3) = xf(x^2)$. Hence, $f(x^3 + y^3) = xf(x^2) + yf(y^2) = f(x^3) + f(y^3)$, which implies the identity $f(x+y) = f(x)+f(y)$. Also, $f(-x^3) = -xf(x^2) = -f(x^3)$, from which we obtain the identity $f(-x) = -f(x)$. We conclude

$$\begin{aligned}0 &= f\left((x+1)^3\right) - (x+1)f\left((x+1)^2\right) + f\left((x-1)^3\right) - (x-1)f\left((x-1)^2\right)\\ &= f\left(x^3\right) + 3f\left(x^2\right) + 3f\left(x\right) + f(1) - (x+1)\left(f\left(x^2\right) + 2f(x) + f(1)\right)\\ &\qquad + f\left(x^3\right) - 3f\left(x^2\right) + 3f\left(x\right) - f(1) - (x-1)\left(f\left(x^2\right) - 2f(x) + f(1)\right)\\ &= 2(f(x) - f(1) \cdot x),\end{aligned}$$

that is, $f(x) = f(1) \cdot x$, which completes the proof.

OC44. In a scalene triangle ABC, we denote by α and β the interior angles at A and B. The bisectors of these angles meet the opposite sides of the triangle at points D and E, respectively. Prove that the acute angle between the lines D and E does not exceed $\frac{|\alpha-\beta|}{3}$.
(*Originally question 1 from the 2009 Serbia Mathematical Olympiad, first day.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

Fixing the typo in the problem statement, we are going to prove that the acute angle between the lines AB and DE does not exceed $\frac{|\alpha-\beta|}{3}$. Moreover, we show that the inequality is strict.

Let a, b, and c denote the lengths of the sides opposite to A, B, and C, respectively. Since the points D and E divide BC and CA in the ratios $c:b$ and $a:c$, respectively, we have

$$BD = \frac{ca}{b+c}, \quad CD = \frac{ba}{b+c}, \quad CE = \frac{ab}{c+a}, \quad AE = \frac{cb}{c+a}.$$

Let the lines AB and DE meet at point F. There is no loss of generality in assuming that $a > b$. Then, A lies between B and F. By Menelaus' theorem, it holds

$$\frac{AF}{AF+c} = \frac{AF}{BF} = \frac{CD}{BD} \cdot \frac{AE}{CE} = \frac{b}{a},$$

hence $AF = \frac{bc}{a-b}$ and $BF = \frac{ac}{a-b}$. [*Ed.* : CF is the external bisector of angle C.]

Let δ denote the acute angle AFE. By the law of sines, in the triangles AEF and BDF it holds

$$\frac{\sin(\alpha-\delta)}{\sin\delta} = \frac{AF}{AE} = \frac{a+c}{a-b}, \qquad \frac{\sin(\beta+\delta)}{\sin\delta} = \frac{BF}{BD} = \frac{b+c}{a-b}.$$

Thus,

$$\sin\delta = \sin(\alpha-\delta) - \sin(\beta+\delta) = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta-2\delta}{2}.$$

Since $\sin\delta$ and $\cos\dfrac{\alpha+\beta}{2}$ are positive, we see that $\sin\dfrac{\alpha-\beta-2\delta}{2}$ is also positive, that is, $0 < \dfrac{\alpha-\beta-2\delta}{2} < \dfrac{\alpha+\beta}{2}$. We obtain

$$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta-2\delta}{2} < 2\cos\frac{\alpha-\beta-2\delta}{2}\sin\frac{\alpha-\beta-2\delta}{2} = \sin(\alpha-\beta-2\delta).$$

Thus,

$$\sin\delta < \sin(\alpha-\beta-2\delta).$$

By the monotonicity of the sine function in the interval $[0,\ \pi/2]$ we deduce that $\delta < \alpha-\beta-2\delta$. The conclusion follows.

OC45. Let $a_1, a_2, a_3, ..., a_{15}$ be prime numbers forming an arithmetic progression with common difference $d > 0$. If $a_1 > 15$, prove that $d > 30,000$.
(*Originally question 3 from the 2009 Singapore Mathematical Olympiad, open section, round 2.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Alex Song, Phillips Exeter Academy, NH, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Song and Wang.

Let p_n denote the nth prime. We prove the following general result:

If $a_1, .., a_m$ are prime numbers in arithmetic progression, with common difference d and if $a_1 > m > p_n$ then d is divisible by $\prod_{k=1}^{n} p_k$.

Indeed, assume by contradiction that $p_k \nmid d$ for some $1 \leq k \leq n$. Then d is invertible modulo p_k, which implies that the equation

$$xd \equiv -a_1 \pmod{p_k}$$

has a solution $0 \leq r \leq p_k < m$. But then

$$a_r \equiv a_1 + rd \equiv 0 \pmod{p_k} ,$$

which implies $p_k \mid a_r$. As $a_r > a_1 > p_k$, we get that a_r is not prime, a contradiction.

In particular, in our problem $a_1 > 15 > 13 = p_6$, and hence d is divisible by $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$, so $d > 30000$.

BOOK REVIEWS

Amar Sodhi

X and the City: Modeling Aspects of Urban Life by John A. Adam
Princeton University Press, 2012
ISBN: 978-0-69115-464-0, Hardcover, 319+xviii pages, US$29.95
Reviewed by ***Shannon Patrick Sullivan***, *Memorial University of Newfoundland, St. John, NL*

X and the City is a spiritual successor to the author's previous work, *A Mathematical Nature Walk* (Princeton University Press, 2011). Whereas that text focussed on mathematical models inspired by the countryside, *X and the City* instead shifts its emphasis to the urban landscape. Over the course of twenty-five chapters (and twelve appendices), Adam uses mathematics to investigate a plethora of city-related phenomena; from the practical (gasoline consumption), to the trivial (counting leaves on trees), to the melodramatic (asteroid impacts).

Each chapter seizes upon a particular theme, the conceit being to redefine the *X* of the title for each new section. For instance, there are chapters about gardening, air pollution and — perhaps inevitably — sex and the city. In the latter (which actually deals with demographics), *X* variously represents the current population of the city, the number of bed bugs in the city, and the total number of people to have ever lived in the city.

Some chapters start by developing a basic mathematical formulation and are then content to study variations on, and consequences of, this formulation, thus giving such chapters the shape of an arc from beginning to end. Others are far more varied, and feel more like a manic zigzag from idea to idea. A chapter on life in the city, for example, takes in questions ranging from determining how many doctor's offices are needed in a city, to calculating the optimal viewing angle for a piece of museum statuary, to assessing the likelihood of a long wait in line at the post office. Despite such stylistic differences, however, these chapters can largely be read in any order, and tend to make only fleeting references to one another. The result is that *X and the City* tends to be an easy book to dip in and out of.

That being said, at times Adam does seek to explore topics in greater depth, and consequently some chapters — especially in the second half of the book — are more heavily interrelated. For example, there is a collection of chapters about traffic and vehicular congestion, and another dealing with questions involving problems of optics and light sources. Even in these chapters, however, the text is fairly modular, and consequently instructors should find *X and the City* a great source of ideas to implement in their own lectures or assignments.

The mathematics which Adam brings to bear in *X and the City* is delightfully varied. As might be expected from a text with "modeling" in its subtitle, differential equations play a prominent role. But Adam also incorporates everything from stochastic analysis to simple algebraic exploration. Adam also has a knack for taking familiar problems and considering them from an unusual angle. For instance, Adam gives a typical illustration of the Mean Value Theorem by considering a jogger running at an average speed of 8 miles per hour, and observing that there must therefore be some point in time at which she is running at a speed of exactly 8 miles per hour. But then he observes that this means the jogger covers a mile in an average of 7.5 minutes, and wonders whether this implies that there is a continuous mile that the jogger must run in exactly 7.5 minutes. The resulting mathematics are quite absorbing.

Indeed, it is a hallmark of *X and the City* that Adam writes with an easy, comfortable style. He frequently punctuates potentially dry passages with unexpected humour, and the result is that even models which might, at first blush, hold little appeal for the reader are well worth perusing. Adam does not dwell too long upon the gory details of his mathematics — and the appendices offer some additional background or analysis which would bloat the main text — but he typically has a good sense of how much to retain to preserve readability. There are exceptions to this, such that less mathematically-sophisticated readers may occasionally find themselves struggling to keep up. There is also a not-insignificant number of typographical errors in both text and mathematics that may pose a point of confusion for some. But, by and large, Adam has succeeded in crafting a text which offers something for a very wide audience. Bright high school students through to veteran mathematicians will find much in *X and the City* that is both fascinating and instructive.

Unsolved Crux Problem

Over the years, a number of the proposed problems have gone unsolved. Below is one of these unsolved problems. Note that the solution to part (a) has been published [1996 : 183-184] but (b) remains open.

2025. [1995 : 158; 1996 : 183-184] *Proposed by Federico Ardila, student, MIT, Cambridge, Massachusetts.*

(a) An equilateral triangle ABC is drawn on a sheet of paper. Prove that you can repeatedly fold the paper along the lines containing the sides of the triangle, so that the entire sheet of paper has been folded into a wad with the original triangle as its boundary. More precisely, let f_a be the function from the plane of the sheet of paper to itself defined by

$$f_a(x) = \begin{cases} x \text{ if } x \text{ is on the same side of } BC \text{ as } A, \\ \text{the reflection of } x \text{ about line } BC \text{ otherwise.} \end{cases}$$

(f_a describes the result of folding the paper along line BC), and analogously define f_b and f_c. Prove that there is a finite sequence $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$, with each $f_{i_j} = f_a, f_b$ or f_c, such that $f_{i_n}(\ldots(f_{i_2}(f_{i_1}(x)))\ldots)$ lies in or on the triangle for every point x on the paper.

(b)$\star$ Is the result true for arbitrary triangles ABC?

FOCUS ON ...

No. 4

Michel Bataille

The Barycentric Equation of a Line

Introduction

Barycentric coordinates relative to a triangle ABC constitute a common and convenient tool in plane geometry. A point P has coordinates (x, y, z) (with $x+y+z \neq 0$) if P is the barycentre of A, B, C with respective masses x, y, z, that is, if $(x+y+z)\overrightarrow{MP} = x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}$ for any point M. These coordinates are also called areal coordinates because x, y, z are proportional to the signed areas $[PBC], [PCA], [PAB]$, a nice geometric interpretation of x, y, z. In this context, the equation of a line is of the form $ux+vy+wz = 0$ for some real numbers u, v, w, not all zero, and this leads to systematic ways of solving problems of collinearity or concurrency. Stepping back, we would like to give a geometric look to the coefficients u, v, w and offer some applications.

Two simple results about u, v, w

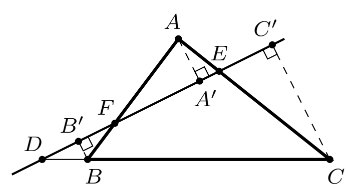
In this paragraph, we assume that u, v, w are not zero, leaving these special cases to the reader. Let ℓ be the line with equation $ux + vy + wz = 0$ and let ℓ intersect the sidelines BC, CA, AB at D, E, F respectively. Then we have

$$\frac{v}{w} = -\frac{BD}{DC}, \quad \frac{w}{u} = -\frac{CE}{EA}, \quad \frac{u}{v} = -\frac{AF}{FB} \tag{1}$$

and, if A', B', C' are the orthogonal projections of A, B, C onto ℓ,

$$\frac{AA'}{u} = \frac{BB'}{v} = \frac{CC'}{w} \tag{2}$$

where all distances are signed.



For example, if $(0, \beta, \gamma)$ are the coordinates of D, then $v\beta + w\gamma = 0$ and $\beta\overrightarrow{DB} + \gamma\overrightarrow{DC} = \overrightarrow{0}$, hence $w\overrightarrow{DB} = v\overrightarrow{DC}$. The first equality in (1) follows (alternatively, one can observe that $v[DCA] + w[DAB] = 0$ and $\frac{[DAB]}{[DCA]} = \frac{BD}{DC}$). As for (2), the homothety with centre D transforming B into C transforms B' into C', hence $\frac{DB}{DC} = \frac{BB'}{CC'}$. Similarly, $\frac{EC}{EA} = \frac{CC'}{AA'}$ and $\frac{FA}{FB} = \frac{AA'}{BB'}$ and (2) follows with the help of (1).

Another solution to a 2006 problem

The equalities (1) directly give Menelaus's relation for the transversal ℓ: indeed, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -\frac{v}{w} \cdot \frac{w}{u} \cdot \frac{u}{v} = -1$. Not surprisingly, the barycentric equation of a line can be a shortcut avoiding the use of Menelaus's theorem. For example, consider Virgil Nicula's problem **3156** ([2006 : 305 ; 2007 : 312]):

Let Γ be the circumcircle of ΔABC. Let M be an interior point on the side AB, and let N be an interior point on the side AC. Let D be an intersection point of MN with Γ. Prove that

$$\left|\frac{MB}{MA}\cdot\frac{AC}{DB}-\frac{NC}{NA}\cdot\frac{AB}{DC}\right|=\frac{BC}{DA}. \tag{3}$$

In the featured solution, Peter Y. Woo applies Menelaus's theorem twice. Here is a shorter proof: From (1), the equation of the line MN can be written as $x=\frac{MB}{MA}\,y+\frac{NC}{NA}\,z$ (not signed distances). Since the signed areas $[DCA]$ and $[DAB]$ are of opposite signs, we obtain $[DBC]=\left|\frac{MB}{MA}[DCA]-\frac{NC}{NA}[DAB]\right|$ if areas are no longer signed. Because A,B,C,D are concylic, we have $\sin A=\sin\angle BDC$, $\sin B=\sin\angle CDA$, $\sin C=\sin\angle ADB$. It follows that $DB\cdot DC\sin A=\left|\frac{MB}{MA}\,DA\cdot DC\sin B-\frac{NC}{NA}\,DA\cdot DB\sin C\right|$ which, using the proportionality of BC, CA, AB and $\sin A$, $\sin B$, $\sin C$, easily leads to (3).

A property of the tangents to the circumcircle

To illustrate (2), we prove the following:

Let ℓ be a tangent to the circumcircle Γ of ΔABC and let $BC=a, CA=b, AB=c, d_a=d(A,\ell), d_b=d(B,\ell), d_c=d(C,\ell)$. Then, one of the numbers $a\sqrt{d_a}, b\sqrt{d_b}, c\sqrt{d_c}$ is the sum of the other two.

Proof. Since the equation of Γ is $a^2yz+b^2zx+c^2xy=0$, the equation of ℓ is $x(b^2z_0+c^2y_0)+y(c^2x_0+a^2z_0)+z(a^2y_0+b^2x_0)=0$ where (x_0,y_0,z_0) are the coordinates of the point of tangency. Expressing that the coefficients of x,y,z are proportional to d_a,d_b,d_c (from (2)) and solving for x_0,y_0,z_0 give

$$x_0:y_0:z_0=a^2(c^2d_c+b^2d_b-a^2d_a):b^2(a^2d_a+c^2d_c-b^2d_b):c^2(b^2d_b+a^2d_a-c^2d_c).$$

Now, $d_ax_0+d_by_0+d_cz_0=0$ leads to

$$a^4d_a^2+b^4d_b^2+c^4d_c^2=2a^2b^2d_ad_b+2b^2c^2d_bd_c+2c^2a^2d_cd_a,$$

that is,

$$(a\sqrt{d_a}+b\sqrt{d_b}+c\sqrt{d_c})(a\sqrt{d_a}+b\sqrt{d_b}-c\sqrt{d_c})\\ \times(b\sqrt{d_b}+c\sqrt{d_c}-a\sqrt{d_a})(c\sqrt{d_c}+a\sqrt{d_a}-b\sqrt{d_b})=0$$

and the result follows. A synthetic proof of this (perhaps new) property would be interesting.

An exercise

To conclude, we propose the following problem to the reader: Let E and F be points on the sides AC and AB, respectively. Show that $[PBC]$ is the geometric mean of $[PAB]$ and $[PCA]$ for some point P on the line segment EF if and only if $AE\cdot AF\geq 4\,CE\cdot BF$.

PROBLEM OF THE MONTH

No. 3

Ross Honsberger

*This column is dedicated to the memory of former **CRUX with MAYHEM** Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

The integer nearest the real number x is often denoted by curly brackets, $\{x\}$. Unfortunately, in determining $\{x\}$, you never know whether x will need to be "rounded up" or to be "rounded down" until you have its numerical value in hand. This vagueness always makes me worry that a nearest integer function might behave erratically. Imagine my surprise and delight, then, in coming across the following engaging property.

For n a positive integer, prove that

$$\sum_{i=1}^{n^2+n} \{\sqrt{i}\} = 2(1^2 + 2^2 + 3^2 + \cdots + n^2).$$

The first thing we might notice is that, as n changes from n to $n+1$, the upper limit of the range of summation on the left side jumps from n^2+n to $(n+1)^2+(n+1)$, thus introducing

$$[(n+1)^2 + (n+1)] - (n^2+n) = 2(n+1)$$

new terms, namely

$$\{\sqrt{n^2+n+1}\}, \{\sqrt{n^2+n+2}\}, \ldots, \{\sqrt{(n+1)^2+(n+1)}\}.$$

On the other hand, the right side increases by $2(n+1)^2$, that is, by

$$2(n+1)(n+1).$$

Now, even the most optimistic among us might feel it is too much to hope that every one of these $2(n+1)$ new terms on the left side will turn out to be equal to $n+1$. Since it probably won't take us long to dispose of this outrageous notion, let's humor the dreamer who suggested it.

First of all, when is the integer nearest $\sqrt{i}$ equal to $n+1$? Since $\sqrt{i}$ is either an integer or is irrational, it is impossible for $\sqrt{i}$ to lie halfway between consecutive integers. Therefore

$$\{\sqrt{i}\} = n+1$$

if and only if

$$n + \frac{1}{2} < \sqrt{i} < n + \frac{3}{2}.$$

Squaring gives

$$n^2 + n + \frac{1}{4} < i < n^2 + 3n + \frac{9}{4},$$

and since i is an integer, this is equivalent to

$$n^2 + n + 1 \le i \le n^2 + 3n + 2,$$

that is, to

$$n^2 + n + 1 \le i \le (n+1)^2 + (n+1).$$

These values of i are precisely its $2(n+1)$ new values on the left side, and so we can only applaud the boldness of our intrepid dreamer and rest in the knowledge that we have discovered an inductive solution to the problem.

This problem was proposed almost sixty years ago in the September issue of *Mathematics Magazine* by Joseph Lambek (McGill University) and Leo Moser (University of Alberta), and their concise solution by induction was duly published in the March-April issue in 1955, page 237.

An Exercise: While we are on the subject of "nearest integer", here's a cute little challenge from Polya and Szego's famous Problems and Theorems in Analysis - vol. 2, problem 5, page 111 (1976 Springer edition).

> *Let x be a real number that is not halfway between two consecutive integers. Express the integer nearest x, $\{x\}$, in terms of the "integer part" symbol* $[\cdot]$.

Everything is so easy and straightforward when the solution is laid before you; so, if you want to have some fun with this problem, take a few moments to figure it out for yourself before reading on.

Solution: A real number x is the sum of its integer part and its fractional part:

$$x = [x] + f, \text{ where } 0 \le f < 1.$$

If $f < \frac{1}{2}$, then the nearest integer is just $[x]$, and if $f > \frac{1}{2}$, it is $[x] + 1$.

Now, for an integer n, any non-zero fractional part of $n+x$ must be contained in the number x, and so

$$[n + x] = n + [x].$$

Therefore

$$[2x] = [2[x] + 2f] = 2[x] + [2f]$$

and, transposing $[x]$, we have

$$[2x] - [x] = [x] + [2f].$$

But $[x] + [2f]$ is just $[x]$ when $f < \frac{1}{2}$ and $[x] + 1$ when $f > \frac{1}{2}$. Hence $[x] + [2f]$ always yields the integer nearest x and it follows that

$$\{x\} = [2x] - [x].$$

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Unsolved Crux Problem

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux* [2010 : 545, 547]. Below is one of these unsolved problems:

714*. [1982 : 48; 1983 : 58] *Proposed by Harry D. Ruderman, Hunter College, New York, NY, USA.*

Prove or disprove that for every pair (p, q) of non-negative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer.

(This problem was suggested by Problem 556 [1980 : 184; 1981 : 189, 241, 282] proposed by Paul Erdős.)

On a Problem Concerning Two Conics

Aleksander Simonic

Abstract

The paper describes a connection between concyclity of four distinct common points of two affine conics and their axes of symmetry.

1 Introduction

The aim of this paper is to solve the following problem:

> *On a plane we have two conics with four distinct common points. Does a necessary and sufficient condition on interrelations of conics exist for the common points to be concyclic?*

We show that the answer to our problem is in connection with an axis of symmetry of a conic and has the following form:

> *If conics $\mathcal{C}_1$ and $\mathcal{C}_2$ have four distinct common points then these points are concyclic if and only if for an arbitrary axis of symmetry s_1 of $\mathcal{C}_1$ there exists an axis of symmetry s_2 of $\mathcal{C}_2$ such that s_1 is perpendicular (or parallel) to s_2.*

The main step in the proof is Theorem 1, which is a criterion for points on a conic to be concyclic. Then we use this theorem in the case where a given conic and a circle have exactly three common points. Here we show how to construct a point C on the conic such that, given two points A, B on the conic, a circle through these points has the same tangent with the conic at C.

2 Preliminaries

The following lemma, which is a vector form of the well known *power of a point theorem* [2, p. 28] is needed.

Lemma 1 *Let A, B, C, D be four distinct points. Let lines AC and BD intersect at X. Then A, B, C, D are concyclic if and only if*

$$\overrightarrow{XA} \cdot \overrightarrow{XC} = \overrightarrow{XB} \cdot \overrightarrow{XD}.$$

To begin with, let us introduce some notation. We say that the perpendicular lines p_1, p_2 form ***a perpendicular lines system*** (or just ***a system***) with a notation $\{p_1, p_2\}$. If there are two systems $\{p_1, p_2\}$ and $\{q_1, q_2\}$ with a property that $p_1 \parallel q_1$ and $p_2 \parallel q_2$ they are called ***congruent*** and are denoted as $\{p_1, p_2\} \cong$

$\{q_1, q_2\}$. If congruent systems have the same intersection point, the notation will be $\equiv$. On the plane equipped with the coordinate system $(x, y) \in \mathbb{R}^2$, we have $\{x, y\}$ where x stands for the x-axis and y stands for the y-axis. We introduce a special system called ***an angle bisectors system*** of non-parallel lines p_1, p_2 denoted as $[p_1, p_2]$. Therefore $[p_1, p_2] = \{s_1, s_2\}$, where s_1 and s_2 are angle bisectors of the lines p_1, p_2. In connection with this system we have the following

Lemma 2 *Let p_1 and p_2 be non-parallel lines with the slopes k_1 and k_2. Then $k_1 = -k_2$ if and only if $\{x, y\} \cong [p_1, p_2]$.*

Proof. Let p_1 and p_2 intersect at $X(x_0, y_0)$. Set $A(x_0 + 1, y_1) \in p_1$ and $B(x_0 + 1, y_2) \in p_2$. From the definition of the slope of the line we get $k_1 = y_1 - y_0$ and $k_2 = y_2 - y_0$. Since $k_1 = -k_2$ if and only if $y_0 = (y_1 + y_2)/2$, we conclude that $k_1 = -k_2$ if and only if the angle bisector of $\angle AXB$ is the same as the perpendicular bisector of AB, which is parallel to the x-axis. ■

Because we will work with *non-degenerate conics* (*an ellipse*, *a parabola* and *a hyperbola*) we simply call them *conics*. By $\mathcal{C}$ we denote an element of the set of all conics that are not circles. We know that an ellipse and a hyperbola have two perpendicular axes of symmetry s_1 and s_2 so they can be considered a system $\{s_1, s_2\}$. The line perpendicular to s_1 with the common point at the vertex of a parabola is taken as another "axis of symmetry". If $\mathcal{C}$ is an arbitrary conic different from a circle, the corresponding ***axes of symmetry system*** is denoted by $\mathrm{Sim}(\mathcal{C})$.

3 The case of a conic and a circle

The following theorem brings up the connection between concyclity of four distinct points on a conic, their angle bisectors system and axes of symmetry system. What is noteworthy here is that the given four distinct points on a conic can always be given names A, B, C, D so that $AC \nparallel BD$.

Theorem 1 *Let A, B, C, D be four distinct points on $\mathcal{C}$. Then A, B, C, D are concyclic if and only if $[AC, BD] \cong \mathrm{Sim}(\mathcal{C})$, that is, the axes of $\mathcal{C}$ are parallel to the bisectors of the angles formed by the lines AC and BD.*

Proof. Set $\mathrm{Sim}(\mathcal{C}) \equiv \{x, y\}$. Then the equation of $\mathcal{C}$ may be written in the form $y^2 = \alpha x^2 + \beta x + \gamma$ with $\alpha \neq -1$. Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4)$ be four distinct points on $\mathcal{C}$ and let $X(x_0, y_0) := AC \cap BD$. Let the equation of the line AC be $y_0 - y = k_1(x_0 - x)$ and the equation of the line BD be $y_0 - y = k_2(x_0 - x)$. Then

$$\overrightarrow{XA} \cdot \overrightarrow{XC} = (x_0 - x_1)(x_0 - x_3)(1 + k_1^2),$$
$$\overrightarrow{XB} \cdot \overrightarrow{XD} = (x_0 - x_2)(x_0 - x_4)(1 + k_2^2).$$

Because $y = y_0 - k_1(x_0 - x)$ and $y^2 = \alpha x^2 + \beta x + \gamma$ we obtain the quadratic equation

$$(k_1^2 - \alpha)x^2 + (2k_1 y_0 - 2k_1^2 x_0 - \beta)x + (k_1^2 x_0^2 - 2k_1 x_0 y_0 + y_0^2 - \gamma) = 0 \qquad (1)$$

with solutions x_1 and x_3. By applying Vieta's formulas [3] we obtain

$$(x_0 - x_1)(x_0 - x_3) = -\frac{\alpha x_0^2 + \beta x_0 + \gamma - y_0^2}{k_1^2 - \alpha}. \qquad (2)$$

Similarly, replacing k_1 with k_2 in (1) yields a quadratic equation with solutions x_2 and x_4 and we obtain the formula for $(x_0 - x_2)(x_0 - x_4)$ after replacing k_1 with k_2 in (2). We always have $k_1^2 \neq \alpha$ and $k_2^2 \neq \alpha$ because otherwise $A = C$ and $B = D$.

Since $X \notin \mathcal{C}$ we have $\alpha x_0^2 + \beta x_0 + \gamma - y_0^2 \neq 0$ and since $\alpha \neq -1$ we can conclude $\overrightarrow{XA} \cdot \overrightarrow{XC} = \overrightarrow{XB} \cdot \overrightarrow{XD}$ if and only if $|k_1| = |k_2|$. We cannot have $k_1 = k_2$ and the only possibility is $k_1 = -k_2$. Now the theorem follows from Lemma 1 and Lemma 2.

In this proof, we made the assumption that k_1 and k_2 existed. But k_1 does not exist if and only if $x_1 = x_3$, so it remains to be proven that $x_1 = x_3$ implies that points A, B, C, D are not concyclic. When $x_1 = x_3$, $\alpha x_0^2 + \beta x_0 + \gamma = y_1^2 = y_3^2$ and therefore $\overrightarrow{XA} \cdot \overrightarrow{XC} = y_0^2 - y_1^2$ and $\overrightarrow{XB} \cdot \overrightarrow{XD} = (1 + k_2^2)(y_0^2 - y_1^2)/(k_2^2 - \alpha)$. Since $\alpha \neq -1$, the four points are not concyclic. ■

In the case of an ellipse $\mathcal{E}$, we can find another proof of Theorem 1 using *an affine transformation*. In what follows, the proof will only be briefly described, whilst the details are left to the reader. Following [1], every function $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an invertible 2×2 matrix and $\mathbf{b} \in \mathbb{R}^2$, is said to be an affine transformation of $\mathbb{R}^2$. Two of the properties of t are: it preserves the collinearity of points and it preserves the ratios of lengths along a given straight line. If $\mathrm{Sim}(\mathcal{E}) \equiv \{x, y\}$, then the equation of an ellipse is $(x/a)^2 + (y/b)^2 = 1$. Applying an affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

we get $t(\mathcal{E}) = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$. Let A, B, C, D be four distinct points on $\mathcal{E}$ and let $X := AC \cap BD$. Then $\lambda, \mu \in \mathbb{R}$ such that $\overrightarrow{XC} = \lambda \cdot \overrightarrow{XA}$ and $\overrightarrow{XD} = \mu \cdot \overrightarrow{XB}$ and therefore $\overrightarrow{XA} \cdot \overrightarrow{XC} = \overrightarrow{XB} \cdot \overrightarrow{XD}$ if and only if $(|XB|/|XA|)^2 = \lambda/\mu$. Let $A' = t(A), B' = t(B), C' = t(C), D' = t(D), X' = t(X)$. From Lemma 1 we get $(|X'B'|/|X'A'|)^2 = \lambda/\mu$. After combining and reducing these two equations in coordinate form, we apply Lemma 2.

We say that two intersecting smooth curves have ***a point of tangency*** if the tangent at that point on the first curve is the same as on the second. The following corollary deals with the case where a conic and a circle have exactly three distinct common points. In that case there exists only one point of tangency (for example in Figure 1b such point is C).

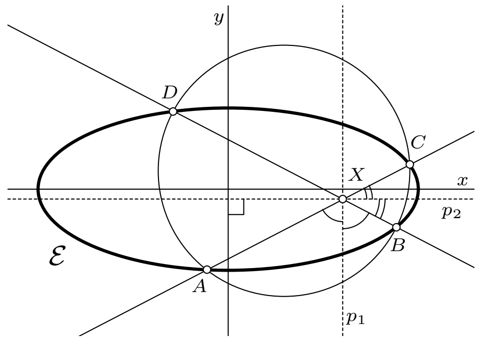

Figure 1a: Example in the sense of Theorem 1 on an ellipse $\mathcal{E}$.

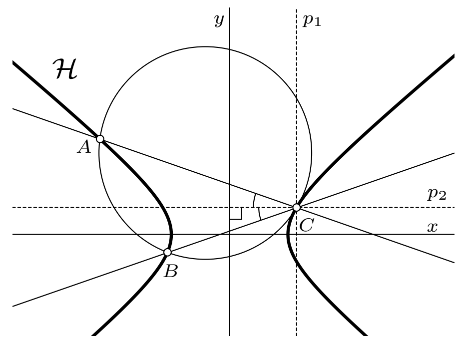

Figure 1b: Example in the sense of Corollary 1 on a hyperbola $\mathcal{H}$.

Given three non-collinear points A, B, C we denote by $\mathcal{K}[A, B, C]$ the circle through these points.

Corollary 1 *Let A, B, C be three distinct points on $\mathcal{C}$. Then C is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$ if and only if $[AC, BC] \cong \mathrm{Sim}(\mathcal{C})$.*

Proof. Let $[AC, BC] \cong \mathrm{Sim}(\mathcal{C})$. Denote by $B_\varepsilon(C)$ an open Euclidean ball with the radius $\varepsilon > 0$ at a point C. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive real numbers, such that $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$ points $C'_n, C''_n \in B_{\varepsilon_n}(C) \cap \mathcal{C}$ exist with $[AC'_n, BC''_n] \cong \mathrm{Sim}(\mathcal{C})$ and therefore A, B, C'_n, C''_n are concyclic by Theorem 1. Because the boundary of $\mathcal{C}$ is smooth, the sequence of the lines $C'_nC''_n$ converge to the tangent of $\mathcal{C}$ at C. The limit circle is therefore $\mathcal{K}[A, B, C]$ which has a common tangent with $\mathcal{C}$ at C.

On the other hand, let $[AC, BC] \ncong \mathrm{Sim}(\mathcal{C})$. Then a point $X \in AC$, $X \neq C$ exists such that $[AC, BX] \cong \mathrm{Sim}(\mathcal{C})$. If XB is not a tangent on $\mathcal{C}$ and $X \neq A$, then there exists a point $D \in \mathcal{C} \cap BX$. By Theorem 1 it follows that A, B, C, D are concyclic and therefore C is not the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$. If $X = A$, then A is the point of tangency by the first part of this proof, so C is not. If XB is a tangent, then for each $n \in \mathbb{N}$ points $B'_n, B''_n \in B_{\varepsilon_n}(B) \cap \mathcal{C}$ exist with $B'_nB''_n \parallel XB$ and therefore A, C, B'_n, B''_n are concyclic by Theorem 1. We conclude that B is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$. In any case C cannot be the point of tangency, this completes the proof. ∎

With the same method as in the proof of Corollary 1 we are able to prove another result concerning points of tangency of a circle and a conic:

> *Let A, B, C be three distinct points on $\mathcal{C}$. Then C is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$ if and only if $[AB, t] \cong \mathrm{Sim}(\mathcal{C})$ where t is a tangent on $\mathcal{C}$ at C.*

Given two points A, B on a conic $\mathcal{C}$, in general there are two different circles $\mathcal{K}_1$ and $\mathcal{K}_2$ through A, B where A and B are consecutive points of tangency. But a point $C \in \mathcal{C}$ exists such that C is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$. Define $C_1 \in \mathcal{K}_1 \cap \mathcal{C}$, $C_2 \in \mathcal{K}_2 \cap \mathcal{C}$ and t as a tangent on $\mathcal{C}$ at C (see Figure 2). Combining Corollary 1 and the above result we obtain $[BA, C_1A] \cong [AB, C_2B] \cong [AB, t]$ and therefore $AC_1 \parallel BC_2 \parallel t$. Using the *midpoint theorem for conics: given chord l of a conic, then the midpoints of the chords parallel to l are collinear* (see [1] for the proof in the case of an ellipse and a hyperbola) we are able to construct C as the point of tangency from the given points A and B. In Figure 2 there are two parallel chords AC_1 and BC_2 with consecutive midpoints M_1 and M_2. Then $C = M_1M_2 \cap \mathcal{P}$. When a conic is an ellipse or a hyperbola, the midpoint theorem further ensures that the line through midpoints of the chords goes through the center of a conic. In that case there exist two points $C_1, C_2 \in \mathcal{C}$ such that C_i is the point of tangency of $\mathcal{K}[A, B, C_i]$ and $\mathcal{C}$ for $i \in \{1, 2\}$. The reader is invited to draw some figures of these cases.

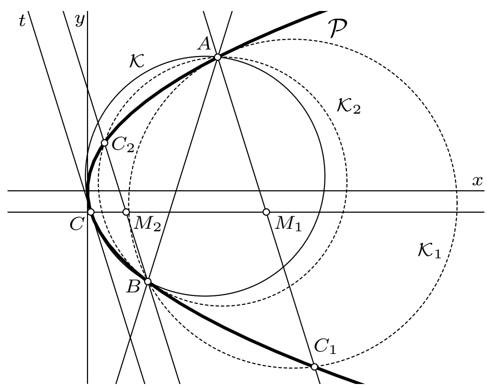

Figure 2: A parabola $\mathcal{P}$ with circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}$ and consecutive points of tangency A, B, C.

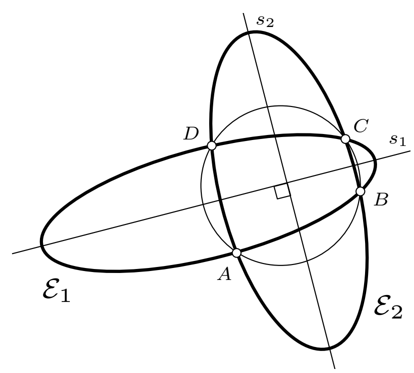

Figure 3: Two ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ with four concyclic common points A, B, C, D.

4 Solution

In this last section we will provide an answer to our problem. We need the following fact about conics: if $|\mathcal{C}_1 \cap \mathcal{C}_2| = 4$, then the conics $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have a point of tangency.

Corollary 2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conics with four distinct common points. These points are concyclic if and only if* $\operatorname{Sim}(\mathcal{C}_1) \cong \operatorname{Sim}(\mathcal{C}_2)$.

Proof. Let the common points be A, B, C, D. If they are concyclic, we have $\operatorname{Sim}(\mathcal{E}_1) \cong [AC, BD] \cong \operatorname{Sim}(\mathcal{E}_2)$.

Suppose $\operatorname{Sim}(\mathcal{C}_1) \cong \operatorname{Sim}(\mathcal{C}_2)$ and assume (for a contradiction) that A, B, C, D are not concyclic. Denote $\mathcal{K} := \mathcal{K}[A, B, C]$ and since points are not concyclic $D \notin \mathcal{K}$. There are three possibilities: $|\mathcal{K} \cap \mathcal{C}_1| = 3$ and $|\mathcal{K} \cap \mathcal{C}_2| = 3$, $|\mathcal{K} \cap \mathcal{C}_1| = 3$ or $|\mathcal{K} \cap \mathcal{C}_2| = 3$, $|\mathcal{K} \cap \mathcal{C}_1| = 4$ and $|\mathcal{K} \cap \mathcal{C}_2| = 4$. In the first case two points of tangency among A, B, C, say A and B, exist. From Corollary 1 we get $[BA, CA] \cong \operatorname{Sim}(\mathcal{C}_1)$

and $[AB, CB] \cong \mathrm{Sim}(\mathcal{C}_2)$. But then $[BA, CA] \cong [AB, CB]$, which is a contradiction. In the second case let us have $|\mathcal{K} \cap \mathcal{C}_1| = 3$ with C as the point of tangency and $|\mathcal{K} \cap \mathcal{C}_2| = 4$. Define $D \in \mathcal{K} \cap \mathcal{C}_2$. We assume without loss of generality that $AD \nparallel BC$. But then this is a contradiction with Theorem 1 and Corollary 1 because we get $AD \parallel AC$. In the third case define $D_1 \in \mathcal{K} \cap \mathcal{C}_1$ and $D_2 \in \mathcal{K} \cap \mathcal{C}_2$. Then $D_1 \neq D_2$ but from Theorem 1 we get $BD_1 \parallel BD_2$ which is a contradiction. ■

It is clear that the answer from the introduction of this paper is equivalent to Corollary 2. For example, in Figure 3 we have two ellipses with four concyclic common points. We observe that s_1 is perpendicular to s_2 where s_1 is an axis of symmetry of $\mathcal{E}_1$ and s_2 is an axis of symmetry of $\mathcal{E}_2$. This agrees with our findings.

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PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 March 2014****. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3781. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Solve the equation

$$3^{1-x} + 3^{\sqrt{3x-2x^2}} = 4\,.$$

3782. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Billy Jin, student, Waterloo Collegiate, Waterloo, ON.*

For $n \in \mathbb{N}$, let $S = \{1, 2, 3, \ldots, n\}$. For each nonempty $T \subseteq S$ define the "drop" of T by $d(T) = f(T) - g(T)$ where $f(T)$ and $g(T)$ denote the maximum and minimum elements of T, respectively. (e.g., $d(\{2\}) = 0$, $d(\{2,3,7\}) = 5$) Evaluate $D_n = \sum d(T)$, the total of the drops of S, where the summation is over all non-empty subsets T of S.

3783. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers. Prove that

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \geq 10abc\,.$$

3784. *Proposed by Constantin Mateescu, "Zinca Golescu" National College, Pitesti, Romania.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s for which we denote $Q = \sum_{\text{cyclic}} \cos\left(\frac{A}{2}\right)$. Prove that

$$s = 2Q\left(\sqrt{R^2Q^2 - Rr} - 2R\right)\ .$$

3785. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Consider an ellipse $\mathcal{E}$ given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$. Find the coordinates, in the first quadrant, of the point P on $\mathcal{E}$ such that the acute angle θ between the tangent t to $\mathcal{E}$ at P and the line OP is minimized.

3786. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with medians m_a, m_b and m_c, circumradius R and inradius r. Let P be the point of intersection of the bisector of $\angle A$ and the median from B, Q be the point of intersection of the bisector of $\angle B$ and the median from C, and R be the point of intersection of the bisector of $\angle C$ and the median from A. If $\angle APB = \alpha$, $\angle BQC = \beta$ and $\angle CRA = \gamma$, prove that

$$\frac{m_a m_b m_c \sin\alpha \sin\beta \sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{r}{32R} \ .$$

3787. *Proposed by Michel Bataille, Rouen, France.*

Let S be a finite set with cardinality $|S| = n \geq 1$ and let k be a positive integer. Calculate

$$\sum_{(A)} |A(1) \cap A(2) \cap \cdots \cap A(k)| \ \text{ and } \ \sum_{(A)} |A(1) \cup A(2) \cup \cdots \cup A(k)|$$

where the summation $\sum\limits_{(A)}$ is over all mappings A from $\{1, 2, \ldots, k\}$ to the power set $\mathcal{P}(S)$.

3788. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b and c be the sides of an acute-angled triangle ABC. Let H be the orthocentre, and let d_a, d_b and d_c be the distances from H to the sides BC, CA and AB, respectively. Prove that

$$\sum_{\text{cyclic}} \sqrt{\frac{1}{a^2b^2} + \frac{1}{b^2c^2} - \frac{1}{c^2a^2}} \leq \frac{9}{4(d_a + d_b + d_c)^2} \ .$$

3789. *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC be inscribed in a circle with centre O and radius R and P be any point in its plane. Let P' be such that $\Delta PBP'$ is directly similar to ΔCOA and P'' be the reflection of P in AC. Prove that

$$P'P'' \geq \frac{2F}{R}$$

where F is the area of ΔABC. For which P does equality hold?

3790. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $a, \alpha \geq 0$ be nonnegative real numbers and let β be a positive number. Determine the limit

$$L(\alpha, \beta) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^\alpha}{(n^2 + kn + a)^\beta} \ .$$

. . . .

3781. *Proposé par Marcel Chiriţă, Bucharest, Romania.*

Résoudre l'équation

$$3^{1-x}+3^{\sqrt{3x-2x^2}}=4\,.$$

3782. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON; et Billy Jin, étudiant, Waterloo Collegiate, Waterloo, ON.*

Soit $n\in\mathbb{N}$ et soit $S=\{1,2,3,\ldots,n\}$. Pour tout sous-ensemble non vide $T\subseteq S$, définissons la "chute" de T par $d(T)=f(T)-g(T)$ où $f(T)$ et $g(T)$ dénotent respectivement les éléments maximal et minimal de T. (p. ex, $d(\{2\})=0$, $d(\{2,3,7\})=5$). Evaluer $D_n=\sum d(T)$, le total des chutes de S, où la sommation s'étant sur tous les sous-ensembles non vides T de S.

3783. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a,b,c trois nombres réels positifs. Montrer que

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2}+(3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2}+(3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2}\geq 10abc\,.$$

3784. *Proposé par Constantin Mateescu, "Zinca Golescu" Collège National, Pitesti, Roumanie.*

Soit respectivement R, r et s le rayon du cercle circonscrit, celui du cercle inscrit et le demi-périmètre d'un triangle ABC et soit $Q=\sum_{\text{cyclique}}\cos\left(\frac{A}{2}\right)$. Montrer que

$$s=2Q\left(\sqrt{R^2Q^2-Rr}-2R\right)\ .$$

3785. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

On donne une ellipse $\mathcal{E}$ d'équation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ avec $a>b$. Trouver, dans le premier quadrant, les coordonnées du point P sur $\mathcal{E}$ rendant minimal l'angle aigu θ entre la tangente t à $\mathcal{E}$ en P et la droite OP.

3786. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit m_a, m_b et m_c les médianes d'un triangle ABC et respectivement R et r les rayons des cercles circonscrit et inscrit. Soit respectivement P le point d'intersection de la bissectrice de $\angle A$ et de la médiane issue de B, Q celui de la bissectrice de $\angle B$ et de la médiane issue de C, et R celui de la bissectrice de $\angle C$ et de la médiane issue de A. Si $\angle APB=\alpha$, $\angle BQC=\beta$ et $\angle CRA=\gamma$, montrer que

$$\frac{m_am_bm_c\sin\alpha\sin\beta\sin\gamma}{(a+2b)(b+2c)(c+2a)}=\frac{r}{32R}\ .$$

3787. *Proposé par Michel Bataille, Rouen, France.*

Soit S un ensemble fini de cardinalité $|S| = n \geq 1$ et soit k un entier positif. Calculer

$$\sum_{(A)} |A(1) \cap A(2) \cap \cdots \cap A(k)| \ \text{ et } \ \sum_{(A)} |A(1) \cup A(2) \cup \cdots \cup A(k)|$$

où la sommation $\sum\limits_{(A)}$ porte sur toutes les applications A de $\{1, 2, \ldots, k\}$ dans l'ensemble des parties de S, $\mathcal{P}(S)$.

3788. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit a, b et c les côtés d'un triangle acutangle ABC. Soit H son orthocentre, et d_a, d_b et d_c les distances respectives de H aux côtés BC, CA et AB. Montrer que

$$\sum_{\text{cyclique}} \sqrt{\frac{1}{a^2b^2} + \frac{1}{b^2c^2} - \frac{1}{c^2a^2}} \leq \frac{9}{4(d_a + d_b + d_c)^2} \ .$$

3789. *Proposé par Michel Bataille, Rouen, France.*

On considère un point quelconque P dans le plan d'un cercle de centre O et de rayon R contenant un triangle ABC. Soit P' un point tel que $\Delta PBP'$ est directement semblable à ΔCOA et soit P'' la réflexion de P par rapport à AC. Montrer que

$$P'P'' \geq \frac{2F}{R}$$

où F désigne l'aire de ΔABC. Pour quel P a-t-on égalité ?

3790. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $a, \alpha \geq 0$ deux nombres réels non négatifs et soit β un nombre positif. Calculer la limite

$$L(\alpha, \beta) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^\alpha}{(n^2 + kn + a)^\beta} \ .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge the solution to problem 3654 by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. The editor apologizes sincerely for the oversight.

3674★. [2011 : 390, 392; 2012 : 347] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let I denote the centre of the inscribed sphere of a tetrahedron $ABCD$ and let A_1, B_1, C_1, D_1 denote their symmetric points of point I about planes BCD, ACD, ABD, ABC respectively. Must the four lines AA_1, BB_1, CC_1, DD_1 be concurrent?

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

No. We show that in a rectangular coordinate system with $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$ and $D = (0, 0, 0)$ (a, b, c positive), the three lines AA_1, BB_1, CC_1 are concurrent only if $a = b = c$.

Let r be the radius of the inscribed sphere. Its center I is the point (r, r, r). The symmetric points A', B', C' are $(-r, r, r)$, $(r, -r, r)$, $(r, r, -r)$ respectively. If AA', BB', CC' are concurrent, we write the common point as convex combinations of A, A'; B, B'; C, C':

$$(1-x)(a, 0, 0)+x(-r, r, r) = (1-y)(0, b, 0)+y(r, -r, r) = (1-z)(0, 0, c)+z(r, r, r)$$

for real numbers x, y, z. This is equivalent to

$$(a-(a+r)x, rx, rx) = (ry, b-(b+r)y, ry) = (rz, rz, c-(c+r)z).$$

From the third components of the first two triplets, we have $x = y$. Similarly, $z = x$. Therefore, $x = y = z$. It follows that $a-(a+r)x = rx$, and $x = \frac{a}{a+2r}$. Similarly, $y = \frac{b}{b+2r}$ and $z = \frac{c}{c+2r}$. Now, $x = y = z$ again implies

$$\frac{a}{a+2r} = \frac{b}{b+2r} = \frac{c}{c+2r} \Rightarrow \frac{a}{2r} = \frac{b}{2r} = \frac{c}{2r} \Rightarrow a = b = c.$$

A stronger converse clearly holds: if $a = b = c$, then $D' = \left(\frac{2a-3r}{3}, \frac{2a-3r}{3}, \frac{2a-3r}{3}\right)$ and the four lines AA', BB', CC', DD' concur at $\left(\frac{ar}{a+2r}, \frac{ar}{a+2r}, \frac{ar}{a+2r}\right)$. A simple calculation shows that $r = \frac{a}{3+\sqrt{3}}$.

Except for one incomplete solution, we received no other submissions.

3681. [2011 : 455, 457] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let D, E, and F be the points where the incircle of ΔABC touches the sides. Let Z be the Gergonne point (where AD, BE, and CF concur), and let M be the midpoint of BC. Define T to be the tangency point of the incircle with the circle through B and C that is tangent to it, and let the common tangent line at that point intersect AC at S. Prove that AB, SZ, and ME are concurrent.

Solution by Titu Zvonaru, Comăneşti, Romania.

The statement of the result is not quite correct. We shall prove that the lines AB, SZ, and ME are *concurrent or parallel*. We use standard notation: r denotes the inradius of ΔABC, while $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a + b + c)$. Then $BF = BD = s - b$, $CD = CE = s - c$, $AE = AF = s - a$.

Van Aubel's theorem asserts that $\frac{CZ}{ZF} = \frac{CE}{EA} + \frac{CD}{DB}$, hence

$$\frac{CZ}{ZF} = \frac{s-c}{s-a} + \frac{s-c}{s-b} = \frac{c(s-c)}{(s-a)(s-b)}. \qquad (1)$$

If $b = c$ then $ST||BC$ and, because AM would be perpendicular to BC, $AM = \frac{2rs}{a}$ and

$$\frac{CS}{SA} = \frac{2r}{AM - 2r} = \frac{a}{s-a}. \qquad (2)$$

Otherwise, we can assume that $b > c$ and let $P = ST \cap BC$. Using the power of P with respect to the two circles we see that $PD^2 = PT^2 = PB \cdot PC$, or equivalently, $(PC - DC)^2 = PC(PC - BC)$, $(PC - (s - c))^2 = PC(PC - a)$, and finally,

$$PC = \frac{(s-c)^2}{b-c}.$$

Let s' be the semiperimeter of ΔPCS and $x = SE$; observe that $s' = PC + x$. Making use of the areas of ΔPCS and ΔABC we have $rs' = \frac{PC \cdot SC \cdot \sin C}{2}$, which is equivalent to

$$\left(\frac{(s-c)^2}{b-c} + x\right) \cdot r = \frac{(s-c)^2}{b-c}(x + s - c)\frac{\sin C}{2},$$
$$\left(\frac{(s-c)^2}{b-c} + x\right) \cdot abr = \frac{(s-c)^2}{b-c}(x + s - c)\frac{ab\sin C}{2}, \text{ and}$$
$$\left(\frac{(s-c)^2}{b-c} + x\right) \cdot ab = \frac{(s-c)^2}{b-c}(x + s - c)s.$$

Solving for x we obtain

$$SE = x = \frac{ab(s-c)^2 - s(s-c)^3}{s(s-c)^2 - ab(b-c)}.$$

It follows that

$$SC = SE + s - c = \frac{ab(s-c)^2 - ab(b-c)(s-c)}{s(s-c)^2 - ab(b-c)} = \frac{ab(s-c)(s-b)}{s(s-c)^2 - ab(b-c)},$$

and

$$SA = b - CS = \frac{b\left(s(s-c)^2 - ab(b-c) - a(s-c)(s-b)\right)}{s(s-c)^2 - ab(b-c)}.$$

With a few lines of algebra (and recalling that $2s = a+b+c$), one can show that the numerator of SA reduces to $b(s-a)(s-b)^2$. It follows that

$$\frac{CS}{SA} = \frac{a(s-c)}{(s-a)(s-b)}. \tag{3}$$

Finally, observe that if $a = c$ then E would be the midpoint of CA and, thus, ME would be parallel to AB. Using equation (1) with either (2) or (3), we find $a = c$ implies that $\frac{CZ}{ZF} = \frac{a}{s-b} = \frac{CS}{SA}$, whence we would have AB and ME also parallel to SZ. So, let us assume that $a \neq c$ and let Q be the point where AB and ME intersect. Applying Menelaus's theorem to ΔABC, we get $\frac{AQ}{QB}\cdot\frac{BM}{MC}\cdot\frac{CE}{EA} = -1$ if and only if

$$\frac{QA}{QA+c} = \frac{s-a}{s-c}, \quad \text{and, therefore,} \quad QA = \frac{c(s-a)}{a-c}.$$

Consequently, $QF = \frac{c(s-a)}{a-c} + s - a = \frac{a(s-a)}{a-c}$, and (because Q is necessarily outside the segment AB we require the negative sign for applying the converse of Menelaus's theorem using directed distances)

$$\frac{AQ}{QF} = -\frac{c}{a}. \tag{4}$$

We are now ready to apply the converse of Menelaus's theorem to ΔAFC. When $b \neq c$ we use (4), (1), and (3) to obtain

$$\frac{AQ}{QF}\cdot\frac{FZ}{ZC}\cdot\frac{CS}{SA} = -\frac{c}{a}\cdot\frac{(s-a)(s-b)}{c(s-c)}\cdot\frac{a(s-c)}{(s-a)(s-b)} = -1.$$

We conclude that the points Q, Z, S are collinear, which means that the lines AB, SZ, and ME are concurrent in Q. The conclusion continues to hold when $b = c$ if we replace (3) by (2).

Also solved by MICHEL BATAILLE, Rouen, France and the proposer.

Computer graphics strongly suggest that the result continues to hold when the incircle of ΔABC is replaced by the excircle opposite vertex A, and even in the intermediate case in which $BF||CE$. See the comments following problem 3684.

3682. [2011 : 455, 457] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 1$. Prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-cd} + \frac{1}{1-da} + \frac{1}{1-bd} + \frac{1}{1-ac} \le 8.$$

I. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Michel Bataille, Rouen, France; and the proposer.

We have the following inequality with its analogues for the other pairs of variables:

$$\begin{aligned}\frac{1}{1-ab} &= 1+\frac{ab}{1-ab} = 1+\frac{ab}{a^2+b^2+c^2+d^2-ab}\\ &= 1+\frac{2ab}{2(c^2+d^2)+a^2+b^2+(a-b)^2}\\ &\le 1+\frac{2ab}{(c^2+d^2+a^2)+(c^2+d^2+b^2)}\\ &\le 1+\frac{1}{2}\left[\frac{(a+b)^2}{(c^2+d^2+a^2)+(c^2+d^2+b^2)}\right]\\ &\le 1+\frac{1}{2}\left[\frac{a^2}{c^2+d^2+a^2}+\frac{b^2}{c^2+d^2+b^2}\right].\end{aligned}$$

The second inequality is a consequence of the Arithmetic Geometric Means Inequality and the third results from the identity

$$\frac{a^2}{u}+\frac{b^2}{v}-\frac{(a+b)^2}{u+v} = \frac{(av-bu)^2}{uv(u+v)}.$$

Adding the six inequalities yields the desired result. Equality occurs if and only if $a=b=c=d=\frac{1}{2}$.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

Observe first that $ab \le \frac{1}{2}(a^2+b^2) \le \frac{1}{2}$, with similar inequalities for the other pairs of variables. When $0 \le x \le \frac{1}{2}$, we have that $2(5+16x^2)(1-x)-9 = (1-2x)(1-4x)^2 \ge 0$ so that

$$\frac{1}{1-x} \le \frac{2}{9}(5+16x^2).$$

Replacing x by the products of pairs of the variables and setting $s = a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2$, we find that the left side of the inequality does not exceed

$$\frac{20}{3}+\frac{32}{9}s.$$

Since

$$\begin{aligned}3-8s &= 3(a^2+b^2+c^2+d^2)^2-8s\\ &= (a^2-b^2)^2+(a^2-c^2)^2+(a^2-d^2)^2+(b^2-c^2)^2+(b^2-d^2)^2+(c^2-d^2)^2,\end{aligned}$$

$s \le 3/8$ and the result follows. Equality occurs when each variable equals $\frac{1}{2}$.

One incorrect solution was received.

3683. [2011 : 455, 457] *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 2$ and z a complex number with $|z| \leq 1$. Prove that

$$\sum_{k=1}^{n} kz^{n-k} \neq 0\,.$$

I. Solution by Dimitrios Koukakis, Kato Apostoloi, Greece.

The sum does not vanish when $z = 1$. When $z \neq 1$, the sum vanishes if and only if

$$0 = z^n + z^{n-1} + \cdots + z - n = (z-1)(z^{n-1} + 2z^{n-2} + \cdots + (n-1)z + n).$$

When $|z| < 1$, then

$$|z^n + z^{n-1} + \cdots + z| \leq |z|^n + |z|^{n-1} + \cdots + |z| < n$$

so the sum does not vanish.

Suppose that $|z| = 1$ and that

$$z^n + z^{n-1} + \cdots + z = n.$$

Taking the complex conjugate and using the fact that $\bar{z} = 1/z$, we obtain the equations

$$1 + z + z^2 + \cdots + z^{n-1} = nz^n$$

and

$$z + z^2 + z^3 + \cdots + z^n = nz^{n+1}.$$

Subtracting each of these equations from the first leads to $z^n - 1 = n(1 - z^n)$ and $n(1 - z^{n+1}) = 0$. Hence $z^n = z^{n+1} = 1$, so that $z = 1$. The result follows.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

For $z \neq 1$, it is equivalent to show that $z^n + z^{n-1} + \cdots + z \neq n$ when $|z| \leq 1$. It is straightforward to establish the result when z is real and satisfies $-1 \leq z < 1$.

Suppose that $|z| \leq 1$ and z is nonreal. Then the arguments of z and z^2 are distinct, so that $|z^2 + z| < |z^2| + |z| \leq 2$. Hence

$$|z^n + z^{n-1} + \cdots + z^3 + z^2 + z| \leq |z|^n + |z|^{n-1} + \cdots + |z|^3 + |z^2 + z| < n$$

and the desired result follows.

III. Solution by Kee-Wai Lau, Hong Kong, China.

The case $|z| < 1$ can be handled as in the first solution. Observe that

$$z^{n+1} - (n+1)z + n = (z-1)^2(z^{n-1} + 2z^{n-2} + \cdots + (n-1)z + n).$$

Suppose that $|z| = 1$ and that

$$z^{n+1} = (n+1)z - n.$$

With $z = \cos\theta + i\sin\theta$, this leads to

$$\cos(n+1)\theta = (n+1)\cos\theta - n$$
$$\sin(n+1)\theta = (n+1)\sin\theta.$$

Squaring and adding these equations yields that

$$1 = (n+1)^2 - 2n(n+1)\cos\theta + n^2 = 2n(n+1)(1-\cos\theta) + 1 \geq 1.$$

Hence $\theta = 0$, so that $z = 1$. Since the sum of the problem does not vanish when $z = 1$, we see that it cannot vanish whenever $|z| \leq 1$.

IV. Solution by the proposer.

The case $|z| < 1$ can be handled as in the first solution. Suppose that $z = e^{i\theta}$ where $0 < \theta < 2\pi$ and that $z^{n+1} = (n+1)z - n$. Then

$$1 = |z|^{n+1} = |(n+1)z - n| = (n+1)\left|e^{i\theta} - \frac{n}{n+1}\right|.$$

But

$$\left|e^{i\theta} - \frac{n}{n+1}\right|^2 - \frac{1}{(n+1)^2} = \left(\cos\theta - \frac{n}{n+1}\right)^2 + \sin^2\theta - \frac{1}{(n+1)^2}$$
$$= \frac{2n}{n+1}(1-\cos\theta) > 0.$$

Hence $|e^{i\theta} - n/(n+1)| > 1/(n+1)$ and we get a contradiction. The result follows.

V. Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

The sum does not vanish when $z = 0$. Otherwise, set $z = 1/w$. It is equivalent to show that all the zeros of $g(w) = 1 + 2w + 3w^2 + \cdots + (n-1)w^{n-2} + nw^{n-1}$ lie inside the open unit disc $|w| < 1$. Now $g(z)$ is the derivative of the function $f(z) = 1 + z + z^2 + \cdots + z^n$ whose zeros are the $(n+1)$th roots of unity distinct from 1, and so are simple and lie on the unit circle $|w| = 1$.

By the Gauss-Lucas Theorem [1, 2], the zeros of $g(w)$ are contained within the closed polygon whose vertices are the zeros of $f(w)$. Since none of the zeros of $f(w)$ are zeros of $g(w)$, all the zeros of $f(w)$ are contained within the open unit disc.

References

[1] Peter Borwein and Tamás Erdelyi, *Polynomials and Polynomial Inequalities.* Springer, New York, 1995. page 18

[2] Victor V. Prasolov, *Polynomials.* Springer, Berlin, Heidelberg, 2004. page 13

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA. Three incomplete solutions were received.

3684. [2011 : 455, 457] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given two circles that are internally tangent at T, let the chord BC of the outer circle be tangent to the inner circle at D. Let the second tangents from B and C touch the inner circle at F and E respectively, and define $J = EF \cap DT$ and $Z = BE \cap CF$. Prove that

(a) JZ intersects BC at its midpoint, and

(b) TD bisects $\angle BTC$.

Comment. This result allows for a solution to a special case of the Problem of Apollonius: Construct a circle through two given points that is tangent to a given circle which, itself, is tangent to the line joining the given points.

Solution by Titu Zvonaru, Cománeşti, Romania.

If the common tangent at T is parallel to the chord BC, then the quadrilateral $EFBC$ is an isosceles trapezoid, and both parts of the problem are easily verified. We therefore assume that the tangent at T meets BC at P and EC at S. The argument also becomes easier should DT be parallel to EC (in which case E is the midpoint of SC). We therefore define

$$V = EF \cap BC \text{ and } Y = DT \cap EC.$$

Part (a). Let us now consider the case in which CE meets BF at a point A and, moreover, the inner circle is the incircle of ΔABC. Because the outer circle is the circle through B and C that is tangent to the incircle, we are working with the configuration of problem **3681**. That means we can use the results obtained there, specifically,

$$\begin{aligned} PC &= \frac{(s-c)^2}{b-c} \text{ (and, therefore, } PD = PC - (s-c) = \frac{(s-b)(s-c)}{b-c}); \\ SE &= \frac{ab(s-c)^2 - s(s-c)^3}{s(s-c)^2 - ab(b-c)} = \frac{(s-a)(s-b)(s-c)^2}{s(s-c)^2 - ab(b-c)}; \\ SC &= \frac{ab(s-b)(s-c)}{s(s-c)^2 - ab(b-c)}; \end{aligned}$$

hence,

$$\frac{ES}{SC} = \frac{(s-a)(s-c)}{ab}.$$

By van Aubel's sum theorem,

$$\frac{BZ}{ZE} = \frac{BF}{FA} + \frac{BD}{DC} = \frac{s-b}{s-a} + \frac{s-b}{s-c} = \frac{b(s-b)}{(s-a)(s-c)}. \tag{1}$$

We now apply Menelaus's theorem three times.
(i) To the transversal VEF of ΔBCA:

$$-1 = \frac{BV}{VC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{BV}{VC} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b};$$

thus, $\frac{VB}{VC}=\frac{s-b}{s-c}$. Because $VB=VM-\frac{a}{2}$ and $VC=VM+\frac{a}{2}$, we calculate that $VM=\frac{a^2}{2(b-c)}$, whence

$$\frac{BM}{MV}=\frac{c-b}{a}. \tag{2}$$

Furthermore, we have

$$VD=VB+(s-b)=\frac{a^2}{2(b-c)}-\frac{a}{2}+s-b=\frac{2(s-b)(s-c)}{b-c}.$$

(ii) To the transversal YDT of ΔSCP:

$$\frac{SY}{YC}\cdot\frac{CD}{DP}\cdot\frac{PT}{TS}=-1.$$

Because $DP=PT$, we deduce that $YS=\frac{YC\cdot TS}{DC}=\frac{(YS+SC)\cdot SE}{EC}$, whence

$$YS=\frac{SC\cdot SE}{EC-SE}.$$

It follows that

$$YE=\frac{SC\cdot SE}{EC-SE}+SE=\frac{SE\cdot(SC-SE+EC)}{EC-SE}=\frac{2SE\cdot EC}{EC-SE},$$

and

$$YC=YE+EC=\frac{EC\cdot(2SE+EC-SE)}{EC-SE}=\frac{EC\cdot SC}{EC-SE}.$$

Consequently,

$$\frac{EY}{YC}=\frac{2ES}{SC}=-\frac{2(s-a)(s-c)}{ab}.$$

And (iii) to the transversal YDJ of ΔECV:

$$-1=\frac{EY}{YC}\cdot\frac{CD}{DV}\cdot\frac{VJ}{JE}=-\frac{2(s-a)(s-c)}{ab}\cdot\frac{s-c}{\frac{2(s-b)(s-c)}{b-c}}\cdot\frac{VJ}{JE};$$

thus,

$$\frac{VJ}{JE}=\frac{ab(s-b)}{(s-a)(s-c)(b-c)}. \tag{3}$$

Using equations (3), (1), and (2) we deduce that

$$\frac{VJ}{JE}\cdot\frac{EZ}{ZB}\cdot\frac{BM}{MV}=\frac{ab(s-b)}{(s-a)(s-c)(b-c)}\cdot\frac{(s-a)(s-c)}{b(s-b)}\cdot\frac{c-b}{a}=-1.$$

The converse of Menelaus's theorem applied to ΔVEB and the points J,Z and M implies that J,Z, and M are collinear, which completes part (a).
Part (b). Because $\angle TPB=\angle TPC$ and $\angle PTB=\angle TCP$, the triangles PBT and PTC are similar, so that

$$\frac{TB}{CT}=\frac{PT}{PC}=\frac{PD}{PC}=\frac{(s-b)(s-c)}{b-c}\cdot\frac{b-c}{(s-c)^2}=\frac{s-b}{s-c}=\frac{BD}{DC}.$$

In words, D is the point that divides the side BC of ΔTBC internally in the ratio equal to that of the other two sides, so that it lies on the bisector of the angle between those two sides; that is, TD bisects $\angle BTC$.

Also solved by MICHEL BATAILLE, Rouen, France and the proposer.

Disclaimer: The statement of the problem allows for the inner circle to be the excircle that is tangent to side BC of the triangle ABC formed by the lines BC, CE, BF. Indeed, it even allows for the lines BF and CE to be parallel so that A would be at infinity. None of the submitted solutions addressed these possibilities. Although it would probably be easy in each case to modify the featured solution, the details seem to involve more than simply using directed distances and angles. It seems prudent, therefore, for us to follow Euclid's approach and present only a typical case of the problem, leaving the remaining cases to the reader. Of course, Euclid gave the special cases to his grad students to check the details. Unfortunately, this editor has no grad students, so I had to rely on the graphics program Cinderella *to confirm the claims. The computer turned up no surprises, so Problem* **3684** *is probably correct as stated; it has been proved, however, only for the arrangement described by the earlier Problem* **3681**.

3685. [2011 : 455, 458] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \to (0,\infty)$ be a bounded function which is continuous at 0. Find the value of

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f\left(\frac{1}{1}\right)}+\sqrt[n]{f\left(\frac{1}{2}\right)}+\cdots+\sqrt[n]{f\left(\frac{1}{n}\right)}}{n}\right)^n .$$

Solution by Michel Bataille, Rouen, France.

The answer is $f(0)$. We require two results, the power mean inequality for $0 < r < 1$, $a_k > 0$,

$$\left(\prod_{k=1}^n a_k\right)^{1/n} \le \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{1/r} \le \frac{a_1 + a_2 + \cdots + a_n}{n},$$

and the Cesaro limit result for a convergent sequence $\{b_n\}$,

$$\lim_{n\to\infty}\frac{b_1 + b_2 + \cdots + b_n}{n} = \lim_{n\to\infty} b_n.$$

Let

$$U_n = \left(\frac{\sqrt[n]{f(1/1)}+\sqrt[n]{f(1/2)}+\cdots+\sqrt[n]{f(1/n)}}{n}\right)^n,$$

$$A_n = \frac{f(1/1)+f(1/2)+\cdots+f(1/n)}{n},$$

and

$$G_n = (f(1/1)\cdot f(1/2)\cdot\cdots\cdot f(1/n))^{1/n}.$$

Using the power mean inequality with $r = 1/n$, we obtain $G_n \le U_n \le A_n$. Since $\lim\limits_{n\to\infty} f(1/n) = f(0) > 0$, we can apply the Cesaro result with $b_n = f(1/n)$ to obtain that $\lim_{n\to\infty} A_n = f(0)$ and $\lim_{n\to\infty} \ln G_n = \ln f(0)$. By the squeeze principle $\lim_{n\to\infty} U_n = f(0)$.

Also solved by DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. Schlosberg used the harmonic mean instead of the geometric mean.

3686. [2011 : 456, 458] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be real numbers such that $abc = 1$. Show that

$$\left(a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c}\right)^2 \;\le\; 2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)\,.$$

I. Solution by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Cománeşti, Romania(independently).

Let $x = a+b+c$ and $y = ab+bc+ca$. Since $abc = 1$, the difference between the two sides of the inequality is

$$\begin{aligned}
&2(a^2+1)(b^2+1)(c^2+1) - (a+b+c-bc-ca-ab)^2\\
&\quad= 2(2+a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2) - (a+b+c-ab-bc-ca)^2\\
&\quad= 2(2+y^2-2x+x^2-2y) - (x-y)^2 = (x^2+y^2+2xy-4x-4y+4)\\
&\quad= (x+y-2)^2 \ge 0.
\end{aligned}$$

Equality occurs if and only if $a+b+c+ab+bc+ca-2 = 0$.

II. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; Kee-Wai Lau, Hong Kong, China; Salem Malikić, student, Simon Fraser University, Burnaby, BC; and Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA (independently).

Let

$$f(a,b) = 2(a^2+1)(b^2+1)(a^2b^2+1) - (a^2b+ab^2+1-a-b-a^2b^2)^2.$$

Replacing c by $1/ab$, we find that the difference of the two sides of the inequality is

$$\begin{aligned}
a^{-2}b^{-2}f(a,b) &= a^{-2}b^{-2}(1+a+b-2ab+a^2b+ab^2+a^2b^2)^2\\
&= (c+ac+bc-2+a+b+ab)^2 \ge 0,
\end{aligned}$$

from which the result follows, with the foregoing condition for equality.

III. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy(independently).

We can determine nonzero real values u, v, w for which $a = v/w$, $b = w/u$ and $c = u/v$. Then

$$\begin{aligned} a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c} &= (uvw)^{-1}(uv^2 - uw^2 + vw^2 - vu^2 + wu^2 - wv^2) \\ &= (uvw)^{-1}(u-v)(v-w)(w-u) \end{aligned}$$

and

$$\begin{aligned} &2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) \\ =&(uvw)^{-2}[(1^2+1^2)(v^2+w^2)][(u^2+v^2)(u^2+w^2)] \\ =&(uvw)^{-2}[(v+w)^2+(v-w)^2][(u^2+vw)^2+u^2(v-w)^2] \\ =&(uvw)^{-2}[((v+w)(u^2+vw)+u(v-w)^2)^2+(u(v+w)(v-w)-(v-w)(u^2+vw))^2] \\ =&(uvw)^{-2}[(uv(u+v)+vw(v+w)+wu(u+v)-2uvw)^2+(v-w)^2(u-v)^2(u-w)^2]. \end{aligned}$$

The desired inequality follows; equality occurs if and only if

$$uv(u+v) + vw(v+w) + wu(u+w) - 2uvw = 0,$$

which is equivalent to $ac + a + ba + b + bc + c - 2 = 0$.

Also solved by DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; and the proposer. Several solvers pointed out that equality occurs for infinitely many triples. Since the condition can be written as $(a + a^{-1}) + (b + b^{-1}) + (c + c^{-1}) = 2$, it is clear that not all of a, b, c can be positive. McCartney found this in a more precise way by noting that

$$f(a, b) - 16 = (1 + a)(1 + b)(1 + ab)(1 + a + b - 6ab + a^2b + ab^2 + a^2b^2)$$

and observing that the last factor is positive for positive a, b, c since $6ab \le 1+a+b+a^2b+ab^2+a^2b^2$ by the Arithmetic-Geometric Means Inequality. Since $abc = 1$, the condition for equality can be written as

$$a(a + 1)b^2 + (a - 1)^2b + (a + 1) = 0.$$

This is a quadratic equation in b with discriminant $a^4-8a^3-2a^2-8a+1 = (a^2-1)^2-8a(a^2+1)$. Select any negative value of a to get a positive discriminant, solve the quadratic for b and set $c = a^{-1}b^{-1}$ to obtain equality.

3687. [2011 : 456, 458] *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\left(k + 1 - \frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

where k^n is taken to be 1 for $k = n = 0$ and $S(n, k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1\,.$$

[*Ed.: The proposer's original problem erroneously had an extra term* $k!$ *in the denominator that was not caught by the editorial board. As a result, no other* *solutions were received. The corrected version of the problem will appear in a future problem set.*]

3688. [2011 : 540, 542] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $T_n(x)$ be the Chebyshev polynomial of the first kind defined by the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Find all positive integers n such that

$$T_n(x) \ \geq \ (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}\,, \quad x \in [1, \infty)\,.$$

[*Ed.: Note problem* **3585** *was originally printed with the wrong inequality.*]

Solution by Michel Bataille, Rouen, France.

For $n \geq 1$, let $P_n(x) = (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}$. We will show that $T_n(x) \geq P_n(x)$ holds for all $x \in [1, \infty)$ if and only if $n \in \{2, 3, 4, 5, 6, 7\}$.

First, we notice that since $T_1(x) = x$ and $P_1(x) = \frac{3}{2}x - \frac{1}{2}$ then $T_1(x) < P_1(x)$ for $x \in (1, \infty)$, so we may assume $n \geq 2$ in what follows. It is well-known that $T_n(\cos\theta) = \cos(n\theta)$ for $\theta \in \mathbb{R}$. Using the fact that $\cos(n\theta)$ is the real part of $(\cos\theta + i\sin\theta)^n$ and the binomial theorem, it is readily obtained that

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k.$$

It follows that

$$T_n(x) - P_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k - 2^{n-2}x^{n-1}(x-1) = x^{n-1}(x-1)\delta_n(x),$$

where

$$\delta_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \left(1 + \frac{1}{x}\right)^k \left(1 - \frac{1}{x}\right)^{k-1} - 2^{n-2}.$$

Therefore, $T_n(x) - P_n(x)$ has the same sign as $\delta_n(x)$ for $x > 1$. By induction, it is easy to show that $\delta_n(1) < 0$ for all $n \geq 8$ and by continuity, $\delta_n(1) < 0$ for $x > 1$ sufficiently close to 1. Thus, $T_n(x) \geq P_n(x)$ for all $x \in [1, \infty)$ can hold only if $n \in \{2, 3, 4, 5, 6, 7\}$.

Now, since

$$x\delta_2(x) = 1, \qquad x\delta_3(x) = x + 3,$$
$$x^3\delta_4(x) = x(3x^2-1) + (7x^2-1), \qquad x^3\delta_5(x) = x(7x^2-5) + (15x^2-5),$$

and

$$\begin{aligned} x^5\delta_6(x) &= 15x^5 + 31x^4 - 17x^3 - 17x^2 + x + 1 \\ &\geq 46x^4 - 17x^3 - 17x^2 + x + 1 = 17x^3(x-1) + x^2(29x^2-17) + x + 1, \end{aligned}$$

then $\delta_n(x) > 0$ for all $x > 1$ and $n = 2, 3, 4, 5, 6$. Finally, if $n = 7$, we obtain $x^5\delta_7(x) = \phi(x)$ where $\phi(x) = 31x^5 + 63x^4 - 49x^3 - 49x^2 + 7x + 7$. Since $\phi^{'}(x) = 155x^4 + 252x^3 - 147x^2 - 98x + 7 > 0$ for $x \geq 1$, we have $\phi(x) > \phi(1) > 0$ for $x > 1$ and $\delta_7(x) > 0$ for all $x > 1$ again. This completes the proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3689. [2011 : 540, 543] *Proposed by Ivaylo Kortezov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In a group of n people, each one has a different book. We say that a pair of people performs a *swap* if they exchange the books they currently have. Find the least possible number $E(n)$ of swaps such that each pair of people has performed at least one swap and at the end each person has the book he or she had at the start.

Solution by M. A. Prasad, India; expanded slightly by the editor.

We show that

$$E(n) = \begin{cases} \frac{n(n-1)}{2}, & \text{if } n \equiv 0, 1 \pmod 4 \\ \frac{n(n-1)}{2} + 1, & \text{if } n \equiv 2, 3 \pmod 4 . \end{cases}$$

To avoid triviality, we assume that $n \geq 2$. Since there are $\binom{n}{2}$ pairs of people $E(n) \geq \binom{n}{2} = \frac{n(n-1)}{2}$. Furthermore, since everyone gets his/her book back at the end, E(n) must be even.

[*Ed.: Using the terminology and well known facts from the theory of permutation groups, E(n) is the minimum number of transpositions (2-cycles) performed on the set* $\{1, 2, 3, \ldots, n\}$ *such that the product of which yields the identity permutation* σ*, and every transposition* (i, j) *must appear at least once in the product for all* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$*. It is well known that a transposition is odd and* σ *is even. Hence there must be an even number of transpositions in the product.*]

Therefore, if $\frac{n(n-1)}{2}$ is odd, then $E(n) \geq \frac{n(n-1)}{2} + 1$.

Clearly $E(2) = 2$ and it is easy to see that $E(3) = 4$. We label the n people by $1, 2, \ldots, n$ and for $i, j = 1, 2, \ldots, n$ with $i \neq j$ we use (i, j) to denote the swap between i and j. For $n = 4$, the sequence of swaps $(1, 2)$, $(1, 3)$, $(2, 4)$, $(1, 4)$, $(2, 3)$, $(3, 4)$ (performed from left to right) shows that $E(4) = 6$ and for $n = 5$, the sequence $(1, 5)$, $(1, 2)$, $(2, 5)$, $(3, 5)$, $(3, 4)$, $(4, 5)$, $(2, 3)$, $(1, 4)$, $(1, 3)$, $(2, 4)$ shows that $E(5) = 10$. We now proceed by induction to prove our claim.

Suppose that $E(n) = \frac{n(n-1)}{2} + \mathcal{E}$ for some $n \geq 5$ where $\mathcal{E} = 0$ or 1 depending on whether $\frac{n(n-1)}{2}$ is even or odd. We show that

$$\begin{aligned} E(n+4) &= \frac{n(n-1)}{2} + \mathcal{E} + 4n + 6 \\ &= \frac{(n+4)(n+3)}{2} + \mathcal{E}. \end{aligned}$$

We denote the $n + 4$ books by $b_1, b_2, \ldots, b_{n+4}$ and break up the swaps into six steps as follows:

(i) Performing the $\frac{n(n-1)}{2} + \mathcal{E}$ swaps between $5, 6, \ldots, n+4$ to ensure that at the end each of these people has his/her own book.

(ii) Performing the swaps $(1,5)$, $(1,6)$, $\ldots$, $(1, n+4)$ so the books are permuted to $(b_{n+4}, b_2, b_3, b_4, b_1, b_5, b_6, \ldots, b_{n+3})$.

(iii) Performing the swap $(1,2)$ to get $(b_2, b_{n+4}, b_3, b_4, b_1, b_5, b_6, \ldots, b_{n+3})$.

(iv) Performing the swaps $(2, n{+}4)$, $(2, n{+}3)$, $\ldots$, $(2,5)$ so the books are permuted to $(b_2, b_1, b_3, b_4, b_5, \ldots, b_{n+4})$.

(v) Repeat the operations in (ii), (iii) and (iv) with 1 replaced by 3 and 2 replaced by 4 (that is, we perform $(3,5)$, $(3,6)$, $\ldots$, $(3, n+4)$ followed by $(3,4)$ and then $(4, n{+}4)$, $(4, n{+}3)$, $\ldots$, $(4,5)$.) The books are permuted as $(b_2, b_1, b_4, b_3, b_5, \ldots, b_{n+4})$.

(vi) Performing the swaps $(2,3)$, $(1,4)$, $(1,3)$ and $(2,4)$ would then bring all the books back to their original order.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and the proposer. There was an incorrect solution which was caused by misunderstanding the statement of the problem.

3690. [2011 : 540, 543; 2012 : 23, 25] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be three distinct positive real numbers with $a + b + c = 1$. Show that

$$(5x^2 - 6xy + 5y^2)(a^3 + b^3 + c^3) + 12(x^2 - 3xy + y^2)abc > (x - y)^2$$

for all real numbers x and y, not both zero.

Solution by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.

The inequality to be proven is equivalent to

$$f(x,y) \equiv \left(5\sum_{\text{cyclic}} a^3 + 12abc - 1\right)(x-y)^2 + \left(\sum_{\text{cyclic}} a^3 - 3abc\right)4xy > 0.$$

Let C denote the coefficient of $(x-y)^2$. Then

$$C = 5\sum_{\text{cyclic}} a^3 + 12abc - 1 = \left(\sum_{\text{cyclic}} a^3 - 3abc\right) + \left(4\sum_{\text{cyclic}} a^3 + 15abc - 1\right).$$

By the AM-GM inequality, we have $\sum_{\text{cyclic}} a^3 - 3abc > 0$ since a, b, and c are positive and distinct.

Furthermore, since

$$1 = \left(\sum_{\text{cyclic}} a\right)^3 = \sum_{\text{cyclic}} a^3 + 3\sum_{\text{cyclic}} (a^2b + ab^2) + 6abc,$$

we have

$$4\sum_{\text{cyclic}} a^3 + 15abc - 1 = 3\left(\sum_{\text{cyclic}} a^3 - \sum_{\text{cyclic}} (a^2b + ab^2) + 3abc\right) > 0$$

by Schur's Inequality. Hence

$$\begin{aligned} f(x,y) = \left[\left(\sum_{\text{cyclic}} a^3 - 3abc\right) + 3\left(\sum_{\text{cyclic}} a^3 - \sum_{\text{cyclic}} (a^2b + ab^2) + 3abc\right)\right](x-y)^2 \\ + \left(\sum_{\text{cyclic}} a^3 - 3abc\right) 4xy \\ = 3\left(\sum_{\text{cyclic}} a^3 - \sum_{\text{cyclic}} (a^2b + ab^2) + 3abc\right)(x-y)^2 \\ + \left(\sum_{\text{cyclic}} a^3 - 3abc\right)(x+y)^2 > 0 \end{aligned}$$

since if $x - y = x + y = 0$, then $x = y = 0$.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; RADOUAN BOUKHARFANE, Polytechnique de Montréal, Montréal, PQ; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; DANIEL VACARU, Pitesti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Almost all of the submitted solutions used Schur's Inequality and many of which are similar to the featured solution.