Crux Mathematicorum

VOLUME 38, NO. 8

OCTOBER / OCTOBRE 2012

IN THIS ISSUE / DANS CE NUMÉRO

Published by
Canadian Mathematical Society
209 - 1725 St. Laurent Blvd.
Ottawa, Ontario, Canada K1G 3V4
FAX: 613–733–8994
email: subscriptions@cms.math.ca

Publié par
Société mathématique du Canada
209 - 1725 boul. St. Laurent
Ottawa (Ontario) Canada K1G 3V4
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EDITORIAL

Shawn Godin

As was pointed out last month, I experienced some equipment problems last year. This led to some **Mayhem** solutions being overlooked. I apologize for the oversight, and would like to acknowledge the following solutions that were received, but never acknowledged: DANIEL LOPEZ AGUAYO, Institute of Mathematics, UNAM, Morelia, Mexico (M489, M491, M493); AGAUSILIA DINDA ASMARA, student, SMPN 8, Yogyakarta, Indonesia (M482); GHINA ZHAFIRA ASTRIDIANTI, student, SMPN 8, Yogyakarta, Indonesia (M482); FLORENCIO CANO VARGAS, Inca, Spain(M481); ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia(M482, M483, M484); GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina(M500); DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA(M482, M484, M485, M486, M487); MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia(M483, M484); CARLOS TORRES NINAHUANCA, Lima, Perú(M484); AARON PERKINS and ADRIENNA BINGHAM, students, Angelo State University, San Angelo, TX, USA(M483, M484, M486); LAURENTIA ROSA RENATA, student, SMPN 8 ,Yogyakarta, Indonesia (M482); and MIHAÏ STOËNESCU, Bischwiller, France(M470, M471, M472, M473, M474, M475, M495, M497, M498, M499).

If you don't usually read the **Mayhem** section, you may want to check out problem **M506**. We received over 200 "solutions" from nine different solvers. The readers found solutions that were beautiful for their simplicity, like

$$343 = (3+4)^3 \qquad \text{or} \qquad 144 = (1+4)! + 4!,$$

as well as solutions were some creative thinking was used like

$$152 = \sum_{n=1}^{5} n! - \phi(2) \qquad \text{or} \qquad 512 = \sqrt[.\bar{1}]{\sqrt{2 \div .5}},$$

and even a couple of infinite families like

$$121 = 11^2,\ 12100 = 110^2 + 0,\ 1210000 = 1100^2 + 0 + 0,\ \ldots .$$

It was hoped that **Mayhem** and **Skoliad** would return in 2013 in an online journal aimed at high school students. Unfortunately, at this time, it doesn't look like that will happen in the near future. There is still some hope that it will return in one form or another, but the project is on hold at this point in time. We will let you know when any **Mayhem** plans are known.

Shawn Godin

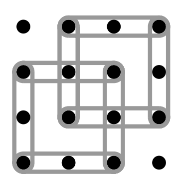
MAYHEM SOLUTIONS

Mathematical Mayhem *is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time* **Mathematical Mayhem** *will be discontinued in* **Crux**. *New* **Mayhem** *problems will appear when the journal is relaunched in 2013.*

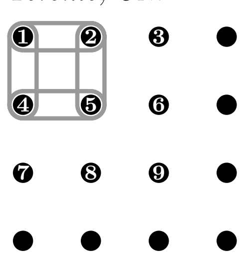
Due to a filing error the following solutions were received, but never acknowledged: DANIEL LOPEZ AGUAYO, Institute of Mathematics, UNAM, Morelia, Mexico (M489, M491, M493); AGAUSILIA DINDA ASMARA, student, SMPN 8, Yogyakarta, Indonesia (M482); GHINA ZHAFIRA ASTRIDIANTI, student, SMPN 8, Yogyakarta, Indonesia (M482); FLORENCIO CANO VARGAS, Inca, Spain(M481); ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia (M482, M483, M484); GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina(M500); DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA (M482, M484, M485, M486, M487); MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia(M483, M484); CARLOS TORRES NINAHUANCA, Lima, Perú(M484); AARON PERKINS and ADRIENNA BINGHAM, students, Angelo State University, San Angelo, TX, USA(M483, M484, M486); LAURENTIA ROSA RENATA, student, SMPN 8 ,Yogyakarta, Indonesia (M482); and MIHAÏ STOËNESCU, Bischwiller, France(M470, M471, M472, M473, M474, M475, M495, M497, M498, M499). The editor apologizes for the oversight.

M501. *Proposed by the Mayhem Staff.*

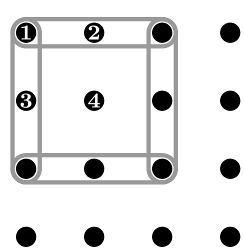
A 4 by 4 square grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form squares, two different 2 by 2 squares are shown in the diagram. How many different squares are possible?



Solution by Gloria (Yuliang) Fang, University of Toronto Schools, Toronto, ON.



Type 1



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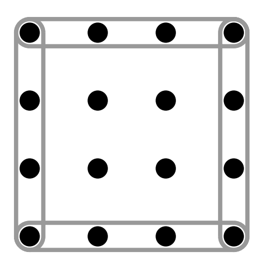
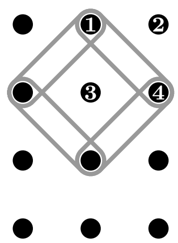
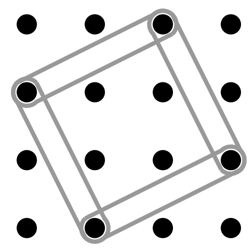


Type 3

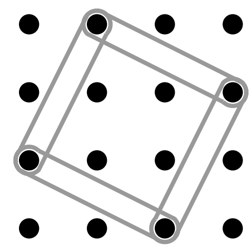
We can classify the types of solution, there are $3 \times 3 = 9$ squares of type 1, $2 \times 2 = 4$ squares of type 2, one square of type 3, $2 \times 2 = 4$ squares of type 4 and 2 squares of type 5 (labeled 5a and 5 b). In the diagrams with numbers, the numbers indicate the top or top left vertex for the other squares in that case. Thus there are $9 + 4 + 1 + 4 + 2 = 20$ possible squares.



Type 1



Type 5a



Type 5b

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; and MIHAÏ STOËNESCU, Bischwiller, France. Two incorrect solutions were received.

Salgueiro Fanego also pointed out that if the 4×4 grid was replaced by a $n \times n$ grid, there would be $\dfrac{n^4 - n^2}{12}$ *possible squares. He points to the web page* `http://www.arrakis.es/ mcj/prb141.htm` *where two solutions (in Spanish) are given.*

M502. *Proposed by the Mayhem Staff.*

At their last basketball game Alice, Bob and Cindy scored a total of 23 points between them. Each player got at least 1 point, and Cindy scored at least 10. How many different ways could the 23 points been awarded to satisfy the conditions? For example: 5 points for Alice, 3 points for Bob, 15 for Cindy; and 3 points for Alice, 5 points for Bob, 15 for Cindy; are two different possibilities.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

There are a number of cases.

- If $c = 10$ then there are 12 possibilities given by $(a, b) = (a, 13 - a)$ with $1 \leq a \leq 12$.
- If $c = 11$ then there are 11 possibilities given by $(a, b) = (a, 12 - a)$ with $1 \leq a \leq 11$.
- If $c = 12$ then there are 10 possibilities given by $(a, b) = (a, 11 - a)$ with $1 \leq a \leq 10$.

$$\vdots$$

- If $c = 20$ then there are 2 possibilities given by $(a, b) = (a, 3 - a)$ with $1 \leq a \leq 2$.

- If $c = 21$ then there is only 1 possibility, $(a, b) = (1, 1)$.

It total, there are $12 + 11 + 10 + \cdots + 2 + 1 = 78$ possibilities.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MARDATILLA NUR JUWITA, student, SMP N 8 YOGYAKARTA, Indonesia; CARL LIBIS, Department of Mathematics, Community College of Rhode Island, Warwick, RI, USA; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; MIHAÏ STOËNESCU, Bischwiller, France; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incorrect solutions were received.

M503. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Write any number and then follow that number by adjoining its reversal. For example, if you write 13 then you would get 1331. Show that the resulting number is always divisible by 11.

Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

Let $N = d_{n-1} \cdot 10^{n-1} + d_{n-2} \cdot 10^{n-2} + \cdots + d_1 \cdot 10 + d_0$ be an n digit number, where $0 \le d_i \le 9$ for each $i = 0, 1, \ldots, n-1$ are the numbers digits. Let M be the number created by adjoining the reversal of N to N, that is

$$\begin{aligned} M &= d_{n-1} \cdot 10^{2n-1} + d_{n-2} \cdot 10^{2n-2} + \cdots + d_1 \cdot 10^{n+1} + d_0 \cdot 10^n \\ &\qquad + d_0 \cdot 10^{n-1} + d_1 \cdot 10^{n-2} + \cdots + d_{n-2} \cdot 10 + d_{n-1} \\ &= d_{n-1}(10^{2n-1} + 1) + d_{n-2}(10^{2n-2} + 10) + \\ &\qquad \cdots + d_1(10^{n+1} + 10^{n-2}) + d_0(10^n + 10^{n-1}) \\ &= \sum_{i=1}^{n} d_{n-i}(10^{2n-i} + 10^{i-1}) \\ &= \sum_{i=1}^{n} d_{n-i} 10^{i-1}(10^{2n-2i+1} + 1) \end{aligned}$$

Now, $10 \equiv -1 \pmod{11}$, so

$$\begin{aligned} M &\equiv \sum_{i=1}^{n} d_{n-i}(-1)^{i-1}((-1)^{2n-2i+1} + 1) \\ &\equiv \sum_{i=1}^{n} d_{n-i}(-1)^{i-1}([(-1)^{n-i}]^2(-1) + 1) \\ &\equiv \sum_{i=1}^{n} d_{n-i}(-1)^{i-1}((1)(-1) + 1) \\ &\equiv 0 \pmod{11}, \end{aligned}$$

completing the proof that when a number is created as in the statement of the problem, it is always divisible by 11.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; IOAN VIOREL CODREANU, Secondary School, Satulung, Maramureş, Romania; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CARL LIBIS, Department of Mathematics, Community College of Rhode Island, Warwick, RI, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HENRY RICARDO, Tappan, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; MIHAÏ STOËNESCU, Bischwiller, France; and the proposer.

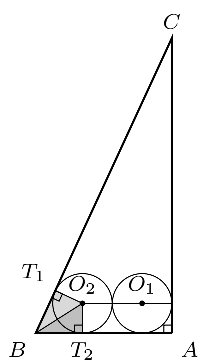
Several solvers used the divisibility by 11 *criterion, that is, a number is divisible by* 11 *if the alternating sum of its digits is divisible by* 11*. So, for example, when considering* 1331*, we would look at the sum* $1 - 3 + 3 - 1 = 0$ *which is divisible by* 11*, thus so is* 1331*. Salgueiro Fanego pointed out that the problem is a generalization of problem* **M297** *[2007 : 201, 202; 2008 : 205].*

M504. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides 3, 4, 5, two equal circles are drawn that are tangent to one another and to one leg. One circle of the pair is tangent to the hypotenuse. The other circle of the pair is tangent to the other leg. Determine the radii of the circles in both cases.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Consider the general case shown for arbitrary angle $\angle ABC = \theta$. Let the circles' radii be r. As the circle with centre O_2 is tangent to both AB and BC, radii to the points of tangency, T_1 and T_2, are perpendicular to the sides. Thus, as right triangle ΔO_2BT_1 and ΔO_2BT_2 share their hypotenuse and have another equal side, which are radii of the circle with centre O_2, then they are congruent, hence $\angle O_2BT_2 = \angle O_2BT_2 = \frac{1}{2}\angle ABC = \frac{\theta}{2}$. Hence, we see that $BT_2 = r\cot\frac{\theta}{2}$, then



$$AB = 3r + r\cot\frac{\theta}{2} = r\left(3 + \cot\frac{\theta}{2}\right) \Rightarrow r = \frac{AB}{3 + \cot\frac{\theta}{2}}. \qquad (1)$$

Using the identity $\cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}$, (1) can be rewritten

$$r = \frac{AB\sin\theta}{3\sin\theta + \cos\theta + 1}. \qquad (2)$$

In the problem posed, the first position has $AB = 3$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$. Using (2) we get $r = \frac{3}{5}$.

Similarly, in the second position we have $AB = 4$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$. Using (2) we get $r = \frac{2}{3}$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; and the proposer. One incomplete solution and one incorrect solution were received.

M505. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that, for all positive integers n, the quantities $A = 5n + 7$ and $B = 6n^2 + 17n + 12$ are coprime (i.e. have no common factors other than 1).

Solution by Ioan Viorel Codreanu, Secondary School, Satulung, Maramureş, Romania.

Let $d = \gcd(A, B)$. Then

$$d \mid \Big[5(6n^2 + 17n + 12) - 6n(5n + 7)\Big]$$

namely $d \mid 43n + 60$.

By $d \mid 5n + 7$ and $d \mid 43n + 60$ we deduce that

$$d \mid [43(5n + 7) - 5(43n + 60)]$$

namely $d \mid 1$, hence $d = 1$ which concludes the solution.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

M506. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

We are trying to create a set of positive integers, that each can be formed using their own digits only, along with any mathematical operations and/or symbols that are familiar to you. Each expression must in3ude at least one symbol/operation; the number of times a digit appears is the same as in the number itself. For example, $1 = \sqrt{1}$, $36 = 6 \times 3!$ and $121 = 11^2$. All valid contributions will be acknowledged.

Solutions by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany(1); Richard I. Hess, Rancho Palos Verdes, CA, USA(2); Carl Libis, Cumberland University, Lebanon, TN, USA(3); Ricard Peiró, IES "Abastos", Valencia, Spain(4); Bruno Salgueiro Fanego, Viveiro, Spain(5); Cássio dos Santos Sousa, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil(6); Titu Zvonaru, Cománeşti, Romania(7); UNB-ESSO-CMS Math Camp 2012 (8); and the proposers (9).

$1 = \sqrt{1}$	8,9
$1 = 1!$	2, 6,8
$2 = 2!$	1, 2, 6,8
$2 = (2!)! = ((2!)!)! = \cdots$	6
$2 = -\lfloor -\sqrt{2} \rfloor$	7
$3 = -\lfloor -\sqrt{3!} \rfloor$	7
$3 = \left\lceil \sqrt{\sqrt{\sqrt{(3!)!}}} \right\rceil$	2
$3 = \left\lceil \sqrt{3!} \right\rceil$	8
$4 = \lfloor \sqrt{4!} \rfloor$	2, 7
$5 = \left\lceil \sqrt{\left\lceil \sqrt{\sqrt{5!}} \right\rceil !} \right\rceil$	2
$5 = \lceil \ln 5! \rceil$	8
$6 = \lfloor \ln 6! \rfloor$	8
$6 = -\left\lfloor -\sqrt{\left\lfloor \sqrt{6!} \right\rfloor} \right\rfloor$	7
$6 = \left\lceil \sqrt{\sqrt{6!}} \right\rceil$	2
$15 = \sum_{n=1}^{5} n$	6
$24 = 4! \lfloor \sqrt{2} \rfloor$	7
$24 = 4! \times \phi(2)$	6
$24 = \sqrt{(4!)^2}$	8
$25 = 5^2$	1, 3, 4, 5, 8, 9
$26 = -\lfloor -\sqrt{6!} \rfloor - \lfloor \sqrt{2} \rfloor$	7
$28 = 2\left\lfloor \sqrt{\left\lfloor \sqrt{8!} \right\rfloor} \right\rfloor$	7
$36 = 6 \times 3!$	8, 9
$48 = 4! + (\phi(8))!$	6
$48 = 4! \lfloor \sqrt{8} \rfloor$	7
$55 = 5(-\lfloor -\sqrt{5!} \rfloor)$	7
$64 = (\sqrt{4})^6$	7, 8, 9
$70 = \lfloor \sqrt{7!} \rfloor + 0$	7, 8
$71 = \sqrt{7! + 1}$	7, 8
$72 = \lfloor \sqrt{7!} \rfloor + 2$	7, 8
$73 = \lfloor \sqrt{7!} \rfloor + 3$	7, 8
$74 = \lfloor \sqrt{7!} \rfloor + 4$	7, 8
$75 = \lfloor \sqrt{7!} \rfloor + 5$	7, 8
$76 = \lfloor \sqrt{7!} \rfloor + 6$	7, 8
$77 = \lfloor \sqrt{7!} \rfloor + 7$	7, 8
$78 = \lfloor \sqrt{7!} \rfloor + 8$	7, 8
$79 = \lfloor \sqrt{7!} \rfloor + 9$	7, 8
$81 = 1 + \left\lceil \sqrt{\sqrt{\lceil \ln(8!) \rceil !}} \right\rceil$	8
$119 = [(\sqrt{9})! - 1]! - 1$	2
$120 = (1 \div .2)! + 0$	2
$120 = (10 \div 2)!$	8
$121 = (1 \div .2)! + 1$	2
$121 = 11^2$	8, 9
$122 = (1 \div .2)! + 2$	2
$123 = (1 \div .2)! + 3$	2
$124 = (1 \div .2)! + 4$	2
$125 = (1 \div .2)! + 5$	2
$125 = 5^{2+1}$	3, 5, 7, 8, 9
$126 = 6 \times 21$	8, 9
$126 = (1 \div .2)! + 6$	2
$127 = (1 \div .2)! + 7$	2
$128 = (1 \div .2)! + 8$	2
$128 = 2^{8-1}$	4, 7, 8, 9
$129 = (1 \div .2)! + 9$	2
$135 = 3 \times 5 \div .\bar{1}$	2
$143 = 3! \times 4! - 1$	2
$144 = 3! \times 4! \times 1$	2
$144 = (1 + 4)! + 4!$	6
$144 = 4! \times (4 - 1)!$	8, 9
$145 = 3! \times 4! + 1$	2
$145 = 1! + 4! + 5!$	7, 9
$146 = \left(4! + \sqrt{.\bar{1}}\right) \times 6$	2
$147 = 7^{\sqrt{4}} \div \sqrt{.\bar{1}}$	2
$150 = 50 \div \sqrt{.\bar{1}}$	2
$152 = \sqrt[.1]{\sqrt{2}} + 5!$	2
$152 = \sum_{n=1}^{5} n! - \phi(2)$	6
$153 = 3 \times 51$	2
$154 = \sum_{n=1}^{5} n! + \phi\left(\sqrt{4}\right)$	6
$162 = \sqrt[-.2]{\sqrt{.\bar{1}}} \times .\bar{6}$	2
$168 = \sqrt{8! \times (.6 + .1)}$	2
$184 = (4! - 1) \times 8$	2
$192 = \sqrt[.1]{\sqrt{2}} \times (\sqrt{9})!$	2
$214 = 4! \div .\bar{1} - 2$	2
$216 = 6^{1+2}$	2, 3, 5, 8, 9
$225 = 5 \div (.\bar{2} - .2)$	2
$240 = (4 + 0!)! \times 2$	2
$241 = \sqrt[-.2]{\sqrt{.\bar{1}}} - \sqrt{4}$	2
$243 = 3^{\frac{2}{.4}}$	2
$242 = 3^{4 + \lfloor \sqrt{1} \rfloor}$	8
$245 = \left(\sqrt{\sqrt{\sqrt{.2}}}\right)^{-(4!)} + 5!$	2
$250 = 50 \div .2 = 5 \div .02$	2
$256 = .5^{-2-6}$	2
$256 = (\lfloor \sqrt{5} \rfloor)^{6+2}$	7
$258 = .5^{-8} + 2$	2
$288 = 2^8 \div .\bar{8}$	2
$289 = (8 + 9)^2$	2, 9
$295 = 59 \div .2$	2
$315 = 35 \div .\bar{1}$	2
$324 = (4! - 3!)^2$	2, 9
$337 = 7^3 - 3!$	2

$343 = (3+4)^3$	2, 8, 9
$344 = (3!)! \times .\overline{4} + 4!$	2
$347 = 7^3 + 4$	2
$351 = (5! - 3) \div \sqrt{.\overline{1}}$	2
$354 = (5! - \sqrt{4}) \div \sqrt{.\overline{1}}$	2
$355 = (3!)! \times .5 - 5$	2
$359 = 3 \times 5! - .\overline{9}$	2
$360 = 6! \div (\lfloor\sqrt{3}\rfloor + 0!)$	7
$360 = 6! \div (3 - 0!)$	2
$384 = 8^{\sqrt{4}} \times 3!$	2
$395 = [(3!)! - 9] \times .5$	2
$432 = (3!)! \times (.4 + .2)$	2
$436 = (3!)! \times .6 + 4$	2
$456 = (5! - 6) \times 4$	2
$464 = (6! - 4!) \times \sqrt{.\overline{4}}$	2
$473 = (3!)! \times \sqrt{.\overline{4}} - 7$	2
$480 = (\sqrt{8 + 0!})! \times \sqrt{.\overline{4}}$	2
$484 = \left(\sqrt{\sqrt{4! - \sqrt{4}}}\right)^8$	2
$496 = (4! + ((\sqrt{9})!)!) \times .\overline{6}$	2
$512 = \sqrt[.\overline{1}]{\sqrt{2 \div .5}}$	2
$514 = \sqrt[-.\overline{1}]{.5} + \sqrt{4}$	2
$542 = (5! + .\overline{4}) \div .\overline{2}$	2
$584 = \sqrt[.5]{4!} + 8$	2
$595 = ((\sqrt{9})!)! - 5! - 5$	2
$599 = ((\sqrt{9})!)! - 5! - .\overline{9}$	2
$624 = 26 \times 4!$	2
$625 = 5^{6-2}$	2, 3, 7, 9
$640 = 6! - (4 + 0!)!$	9
$648 = (\sqrt{6})^8 \div \sqrt{4}$	2
$656 = 6! - .5^{-6}$	2
$660 = 6! - 60$	2
$672 = 7! \times .6 \times .\overline{2}$	2
$675 = 5! \div (.\overline{7} - .6)$	2
$688 = 86 \times 8$	2
$693 = (3!)! - \left(\sqrt{\sqrt{9}}\right)^6$	2
$696 = ((\sqrt{9})!)! - (6 \times .\overline{6})!$	2
$713 = (3!)! - 7 \times 1$	2
$715 = (7 - 1)! - 5$	2
$719 = ((\sqrt{9})!)! - 1^7$	2
$720 = (7 + 0! - 2)!$	2, 8
$720 = (7 - 0!)!\lfloor\sqrt{2}\rfloor$	7
$721 = ((\sqrt{2+7})!)! + 1$	2
$722 = ((\sqrt{2+7})!)! + 2$	2
$723 = ((\sqrt{2+7})!)! + 3$	2
$724 = ((\sqrt{2+7})!)! + 4$	2
$725 = ((\sqrt{2+7})!)! + 5$	2
$726 = ((\sqrt{2+7})!)! + 6$	2
$727 = ((\sqrt{2+7})!)! + 7$	2
$728 = ((\sqrt{2+7})!)! + 8$	2
$729 = ((\sqrt{2+7})!)! + 9$	2
$729 = 9^{\sqrt{7+2}}$	8, 9
$733 = (3!)! + 7 + 3!$	2
$734 = (3!)! + 7 \times \sqrt{4}$	2, 9
$736 = 3^6 + 7$	2
$744 = (4! + 7) \times 4!$	2
$784 = \left(\sqrt{\sqrt{4 \times 7}}\right)^8$	2
$790 = 70 + ((\sqrt{9})!)!$	2
$791 = 71 + ((\sqrt{9})!)!$	2
$792 = 72 + ((\sqrt{9})!)!$	2
$793 = 73 + ((\sqrt{9})!)!$	2
$794 = 74 + ((\sqrt{9})!)!$	2
$795 = 75 + ((\sqrt{9})!)!$	2
$796 = 76 + ((\sqrt{9})!)!$	2
$797 = 77 + ((\sqrt{9})!)!$	2
$798 = 78 + ((\sqrt{9})!)!$	2
$799 = 79 + ((\sqrt{9})!)!$	2
$809 = ((\sqrt{9})!)! \div .\overline{8} - 0!$	2
$810 = ((\sqrt{0! \div .\overline{1}})!)! \div .\overline{8}$	2
$816 = 6! \times (.8 + \sqrt{.\overline{1}})$	2
$834 = (3!)! \div .\overline{8} + 4!$	2
$864 = (\sqrt{6})^8 \times \sqrt{.\overline{4}}$	2
$895 = ((\sqrt{9})!)! \div .8 - 5$	2
$899 = ((\sqrt{9})!)! \div .8 - .\overline{9}$	2
$936 = (3!)^{\sqrt{9}} - 6!$	2
$991 = .1^{-\sqrt{9}} - 9$	2
$1024 = 2^{\lfloor\sqrt{104}\rfloor}$	7
$1024 = 2^{\sqrt{\sqrt{10^4}}}$	9
$1206 = 201 \times 6$	1
$1296 = 6^{\sqrt{9}+2-1}$	9
$1331 = 11^3\lfloor\sqrt{3}\rfloor$	7
$1331 = 11^{\sqrt{3\times 3}}$	9
$1440 = (4 + 1 + 0!)!\sqrt{4}$	7
$2048 = 2^{4+8-0!}$	8, 9
$2187 = (8 \div 2 - 1)^7$	4
$2401 = [(2 + 0!)! + 1]^4$	9
$2500 = 50^2 + 0$	1
$2592 = 2^5 \times 9^2$	9
$3125 = 5^{3\cdot 1+2}$	7
$3125 = 5^{3\times 2-1}$	3, 9
$3125 = 5^{(3+2)\times 1}$	3
$4096 = (\sqrt{4})^{6\times(\sqrt{9}-0!)}$	9
$5040 = (5 + 4 - 0! - 0!)!$	7, 8

$10201 = 101^2 + 0$	1
$11025 = 105^2 \times 1$	1
$12006 = 2001 \times 6$	1
$12100 = 110^2 + 0$	1
$12321 = (113 - 2)^2$	8
$14641 = 11^4 \lfloor 6 \div 4 \rfloor$	7
$14641 = 11^{4 \div .4 - 6}$	9
$15625 = 5^6 \lfloor \sqrt{5 - 2 - 1} \rfloor$	7
$15625 = 5^{6 \times 2 - 5 - 1}$	3
$15625 = 5^6 + \sqrt{5 - 1} - 2$	9
$32768 = 8^{\frac{2+6+7}{3}}$	4
$40320 = (40 - 32)!0!$	7
$40320(4 + 3 + 2 - 0!)! + 0$	8
$65536 = (5 - 6 \div 6)^{5+3}$	4
$65536 = (6 \div 3)^{5+5+6}$	8
$78125 = 5^{7 \times 2 + 1 - 8}$	4
$78125 = 5^{\frac{8+7-1}{2}}$	4
$78125 = 5^{8-1}(\lfloor \sqrt{7} \rfloor - \lfloor \sqrt{2} \rfloor$	7
$120006 = 20001 \times 6$	1
$161051 = 11^5 \cdot 16^0$	7
$250000 = 500^2 + 0 + 0$	1
$362880 = (8 + 8 + 2 - 3 - 6 - 0)!$	7
$362880 = [(8 + 8 + 6 + 2) \div 3 + 0!]!$	8
$390625 = 5^{(6 \times 2 - 9 \div 3 - 0!)}$	3
$531441 = (5 + 4)^{\frac{3+4+1}{1}}$	4
$1002001 = 1001^2 + 0 + 0$	1
$1048576 = (1 + 0!)^{8 \div 4 + 5 + 6 + 7}$	8
$1048576 = (\sqrt{4})^{6+7+8-1+0 \times 5}$	8
$1048576 = (\sqrt[4]{16})^{5+7+8+0}$	8
$1200006 = 200001 \times 6$	1
$1210000 = 1100^2 + 0 + 0$	1
$1771561 = 11^6(15 - 7 - 7)$	7
$1953125 = 5^{9+5+1-1 \times 2 \times 3}$	3
$3628800 = (8 + 8 + 3 - 2 - 6 - 0! - 0)!$	7
$3628800 = [(8 + 8) \div 2 + 6 \div 3]! + 0 + 0$	8
$19487171 = 11^7(8 + 7 - 1 - 9 - 4)$	7
$25000000 = 5000^2 + 0 + 0 + 0$	1
$121000000 = 11000^2 + 0 + 0 + 0$	1

The editors were impressed by the ingenuity of the solutions. Hess conjectured that all numbers could be reached using combinations of the factorial, square root, floor and ceiling functions. He gives as an example 1 through 6 (in the list above). Geupel gives several infinite families: $10201, 1002001, 100020001, \ldots$*;* $2500, 250000, 25000000, \ldots$*;* $12100, 1210000, 121000000, \ldots$*; and* $1206, 12006, 120006, \ldots$.

The editors would be pleased to received more (new) representations.

THE CONTEST CORNER

No. 8

Shawn Godin

The Contest Corner *est une nouvelle rubrique offerte pas* ***Crux Mathematicorum****, comblant ainsi le vide suite à la mutation en 2013 de* Mathematical Mayhem *et* Skoliad *vers une nouvelle revue en ligne. Il s'agira d'un amalgame de* Skoliad*,* The Olympiad Corner *et l'ancien* Academy Corner *d'il y a plusieurs années. Les problèmes en vedette seront tirés de concours destinés aux écoles secondaires et au premier cycle universitaire ; les lecteurs seront invités à soumettre leurs solutions ; ces solutions commenceront à paraître au prochain numéro.*

Les solutions peuvent être envoyées à : Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5 *ou par couriel à* `crux-contest@cms.math.ca`.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 février 2014**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC36. Pour chaque entier positif n, définissons $f(n)$ comme étant le plus petit entier positif s tel que $1+2+3+\cdots+(s-1)+s$ est divisible par n. Par exemple, $f(5)=4$ car $1+2+3+4$ est divisible par 5, tandis qu'aucun de 1, $1+2$, puis $1+2+3$ est divisible par 5. Déterminer, avec preuve, la plus petite valeur positive entière k pour laquelle l'équation $f(c)=f(c+k)$ possède au moins une solution positive entière impaire c.

CC37. $ABCD$ est un quadrilatère cyclique avec $AD=d$, d étant le diamètre du cercle. Soit aussi $AB=a$, $BC=a$ et $CD=b$. Si a, b et d sont des entiers tels que $a\neq b$,

(a) démontrer que d ne peut pas être un nombre premier ;

(b) déterminer la valeur minimale de d.

CC38. Chaque sommet d'un polygone régulier à 11 côtés est coloré soit noir soit or. Tous les triangles possibles sont formés à partir de ces sommets. Démontrer qu'il existe deux triangles congrus à sommets colorés noir ou qu'il existe deux triangles congrus à sommets colorés or.

CC39. Charles s'amuse avec une variante du Sudoku. À chaque point du treillis (x, y) tel que $1 \leq x, y < n$, il assigne un entier entre 1 et n, (inclusivement), où n. est un entier positif. Ces entiers doivent satisfaire la propriété que dans la rangée où $y = k$, les $n - 1$ entiers sont distincts et ne peuvent pas égaler k ; les colonnes doivent respecter un principe analogue. Maintenant, Charles choisit un des points du treillis, la probabilité du choix étant proportionnelle à l'entier qu'il y a assigné. Calculer la valeur espérée de $x + y$ pour le point choisi (x, y).

CC40. Définissons $P(1) = P(2) = 1$ et $P(n) = P(P(n-1)) + P(n - P(n-1))$ pour $n \geq 3$. Démontrer que $P(2n) \leq 2P(n)$ pour tout entier positif n.

. . . .

CC36. For each positive integer n, define $f(n)$ to be the smallest positive integer s for which $1 + 2 + 3 + \cdots + (s - 1) + s$ is divisible by n. For example, $f(5) = 4$ because $1 + 2 + 3 + 4$ is divisible by 5 and none of 1, $1 + 2$, or $1 + 2 + 3$ is divisible by 5. Determine, with proof, the smallest positive integer k for which the equation $f(c) = f(c + k)$ has an odd positive integer solution for c.

CC37. $ABCD$ is a cyclic quadrilateral with side $AD = d$, where d is the diameter of the circle. $AB = a$, $BC = a$ and $CD = b$. If a, b and d are integers $a \neq b$,

(a) prove that d cannot be a prime number.

(b) determine the minimum value of d.

CC38. Each vertex of a regular 11-gon is coloured black or gold. All possible triangles are formed using these vertices. Prove that there are either two congruent triangles with three black vertices or two congruent triangles with three gold vertices.

CC39. Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \leq x, y < n$, he assigns an integer between 1 and n, inclusive, for some positive integer n. These integers satisfy the property that in any row where $y = k$, the $n - 1$ values are distinct and are never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability proportional to the integer value he assigned to it. Compute the expected value of $x + y$ for the chosen point (x, y).

CC40. Define $P(1) = P(2) = 1$ and $P(n) = P(P(n-1)) + P(n - P(n-1))$ for $n \geq 3$. Prove that $P(2n) \leq 2P(n)$ for all positive integers n.

THE OLYMPIAD CORNER

No. 306

Nicolae Strungaru

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 février 2014**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC96. Soient $a, b > 1$ deux entiers relativement premiers. On pose $x_1 = a$, $x_2 = b$ et

$$x_n = \frac{{x_{n-1}}^2 + {x_{n-2}}^2}{x_{n-1} + x_{n-2}}$$

pour $n \geq 3$. Démontrer que x_n est entier pour aucune valeur $n \geq 3$.

OC97. Soit A un ensemble de 225 éléments. Supposons qu'il existe onze sous-ensembles $A_1, \ldots, A_{11}$ de A tels que $|A_i| = 45$ pour $1 \leq i \leq 11$ et $|A_i \cap A_j| = 9$ pour $1 \leq i < j \leq 11$. Dmontrer que $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, et donner un exemple où l'égalité tient.

OC98. Soit ABC un triangle tel que $\angle BAC = 60°$. Soient B_1 et C_1 les pieds des bissectrices partant de B et C. Soit A_1 symétrique à A par rapport à la ligne B_1C_1. Démontrer que A_1, B et C sont colinéaires.

OC99. Soit $\mathbb{Q}^+$ l'ensemble des nombres rationnels positifs. Déterminer toutes les fonctions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ telles que pour tout $x \in \mathbb{Q}^+$ les égalités

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1}$$

et

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}$$

tiennent.

OC100. Soit a_n la suite définie par $a_0 = 1$, $a_1 = -1$, et

$$a_n = 6a_{n-1} + 5a_{n-2}$$

pour $n \geq 2$. Démontrer que $a_{2012} - 2010$ est divisible par 2011.

OC96. Let $a, b > 1$ be two relatively prime integers. We define $x_1 = a$, $x_2 = b$ and

$$x_n = \frac{{x_{n-1}}^2 + {x_{n-2}}^2}{x_{n-1} + x_{n-2}}$$

for all $n \geq 3$. Prove that x_n is not an integer for all $n \geq 3$.

OC97. Let A be a set with 225 elements. Suppose that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

OC98. Let ABC be a triangle with $\angle BAC = 60^\circ$. Let B_1 and C_1 be the feet of the bisectors from B and C. Let A_1 be the symmetrical of A with respect to the line B_1C_1. Prove that A_1, B and C are collinear.

OC99. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ so that, for all $x \in \mathbb{Q}^+$ we have

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1}$$

and

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}\,.$$

OC100. Let a_n be the sequence defined by $a_0 = 1$, $a_1 = -1$, and

$$a_n = 6a_{n-1} + 5a_{n-2}$$

for all $n \geq 2$. Prove that $a_{2012} - 2010$ is divisible by 2011.

OLYMPIAD SOLUTIONS

OC36. The obtuse-angled triangle ABC has sides of length a, b, and c opposite the angles $\angle A$, $\angle B$ and $\angle C$ respectively. Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc\,.$$

(*Originally question #6 from the 2008/9 British Mathematical Olympiad, Round 1.*)

Similar solutions by Michel Bataille, Rouen, France and Titu Zvonaru, Comăneşti, Romania. No other solution was received .

By the law of cosines we have

$$\cos(A) = \frac{b^2+c^2-a^2}{2bc}\ ;\ \cos(B) = \frac{a^2+c^2-b^2}{2ac}\ ;\ \cos(C) = \frac{a^2+b^2-c^2}{2ab}\ .$$

Then the inequality reduces to

$$a^4b^2 + a^2b^4 + b^4c^2 + b^2c^4 + c^4a^2 + c^2a^4 - (a^6+b^6+c^6+2a^2b^2c^2) < 0$$

or equivalently

$$(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2) < 0$$

which is true since ABC is obtuse angled, and thus exactly one of $\cos(A)$, $\cos(B)$, $\cos(C)$ is negative.

Zvonaru observed that using the same argument we can prove that if ABC is acute angled then

$$a^3\cos(A) + b^3\cos(B) + c^3\cos(C) > abc$$

and if ABC is right triangle then

$$a^3\cos(A) + b^3\cos(B) + c^3\cos(C) = abc.$$

OC37. Find all integers n such that we can colour all the edges and diagonals of a convex n-gon by n given colours satisfying the following conditions:

(i) Every one of the edges or diagonals is coloured by only one colour;

(ii) For any three distinct colours, there exists a triangle whose vertices are vertices of the n-gon and the three edges are coloured by the three colours, respectively.

(*Originally question #5 from the 2009 Chinese Mathematical Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We prove that an integer $n \geq 3$ has the desired property if and only if n is odd.

Suppose that n has the required property. We show that n is odd. There are $\binom{n}{3}$ choices of three distinct colours out of n colours and also $\binom{n}{3}$ triangles out of n vertices. Hence, any two distinct such triangles have distinct sets of edge colours.

As there are exactly $\binom{n-1}{2}$ choices of three colours containing a fixed given color, each colour occurs in exactly $\binom{n-1}{2}$ triangles. On the other hand, every line segment belongs to $n-2$ triangles. Therefore, the number of segments of each colour is $\frac{1}{n-2}\binom{n-1}{2} = \frac{n-1}{2}$. We conclude that n is odd.

Next suppose that $n \geq 3$ is odd. We prove that n has the desired property. Suppose that there are n colours $C_0, C_1, \ldots, C_{n-1}$ and n vertices $P_0, P_1, \ldots, P_{n-1}$. Consider indices of colours and indices of vertices modulo n. Let us colour the edge with end points P_i and P_j with colour C_{i+j}. Let C_i, C_j, and C_k be any three distinct colours. It is now straightforward to verify that the triangle $P_{(i+j-k)/2}P_{(i+k-j)/2}P_{(j+k-i)/2}$ has edges with colours C_i, C_j, and C_k. (Here the calculation of indices is within the additive ring $\mathbb{Z}_n$. As n is odd, 2 is invertible in $\mathbb{Z}_n$.) This completes the proof.

OC38. Let a, b, c be positive real numbers such that $ab + bc + ca = \frac{1}{3}$. Prove the inequality:

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}\,.$$

(*Originally question #4 from the 16*$^{\text{th}}$ *Macedonian Mathematical Olympiad.*)

Solved by John Asmanis, Chalkida, Greece; Michel Bataille, Rouen, France; Marian Dincă, Bucharest, Romania; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Henry Ricardo, Tappan, NY, USA and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

First, we observe that $a^2 - bc + 1 = a^2 + ab + ac - (ab + ac + bc) + 1 = a(a + b + c) + \frac{2}{3}$.

We denote by $s := a + b + c$. Then, the inequality becomes equivalent to

$$\frac{1}{s^2} \leq \frac{a}{s} \cdot \frac{1}{as + \frac{2}{3}} + \frac{b}{s} \cdot \frac{1}{bs + \frac{2}{3}} + \frac{c}{s} \cdot \frac{1}{cs + \frac{2}{3}}\,. \tag{1}$$

Since $f(x) = \frac{1}{x}$ is convex on $(0, \infty)$ and $\frac{a}{s} + \frac{b}{s} + \frac{c}{s} = 1$ by the Jensen inequality we get

$$\begin{aligned}\frac{a}{s} \cdot \frac{1}{as + \frac{2}{3}} + \frac{b}{s} \cdot \frac{1}{bs + \frac{2}{3}} + \frac{c}{s} \cdot \frac{1}{cs + \frac{2}{3}} &\geq \frac{1}{\frac{a}{s}(as + \frac{2}{3}) + \frac{b}{s}(bs + \frac{2}{3}) + \frac{c}{s}(cs + \frac{2}{3})} \\ &= \frac{1}{a^2 + b^2 + c^2 + \frac{2}{3}} \\ &= \frac{1}{a^2 + b^2 + c^2 + 2(ab + ac + bc)} = \frac{1}{s^2}\,,\end{aligned}$$

which proves (1).

OC39. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.
(*Originally question #4 from the 2008/9 British Mathematical Olympiad, Round 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Cománeşti, Romania. We give a combination of their similar solutions.

For any positive integers ℓ and m and any bit string $x \in \{0,1\}^m$ (i.e., a word of length m over the alphabet $\{0,1\}$), let $B(x,\ell)$ denote the number of distinct bit strings of length ℓ and with leading bit 1 that occur as consecutive blocks in x. Let $B(x) = \sum_{\ell=1}^{\infty} B(x,\ell)$. Moreover, let $\beta(n)$ denote the binary (bit string) representation of the integer n. The identity $B(\beta(n)) = b(n)$ clearly holds.

Claim 1: If $2^{11} \leq n \leq 2050$ then $b(n) \leq 37$.

Indeed, $x = \beta(n)$ has the form $100x_9$ with $x_9 \in \{0,1\}^9$. By counting the possible bit strings, we obtain $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$. We have $B(x,4) \leq 7$, because there are not more than 6 blocks inside x_9 and one additional block starting at the leftmost bit of x. Similarly, we obtain $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, and $B(x,10) = B(x,11) = B(x,12) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37,$$

which completes the proof of Claim 1.

Now we assume that $n < 2048$.

Claim 2: If $n \leq 2^{10}$ then $b(n) \leq 36$.

By a similar counting to the one in Claim 1, we obtain $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 8$.

As a block of length 5 in $x = x_1x_2...x_{10}$ can only start at x_1, x_2, x_3, x_4, x_5 or x_6 we get $B(x,5) \leq 6$ and similarly $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, and $B(x,10) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 36,$$

which completes the proof of Claim 2.

Claim 3: If $n \leq 2^{11}$ and $x = \beta(n)$, it holds $B(x,1) \leq 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 8$, $B(x,5) \leq 7$, $B(x,6) \leq 6$, $B(x,7) \leq 5$, $B(x,8) \leq 4$, $B(x,9) \leq 3$, $B(x,10) \leq 2$, and $B(x,11) \leq 1$.

The first four relations follow immediately from the observation that, by counting all the binary strings of length k starting with 1 we have

$$B(x,k) \leq 2^{k-1}.$$

For the others it suffices to observe that there are exactly $11 - k + 1$ possible starting positions for any block of length k in x. Thus

$$B(X, k) \leq 12 - k\,.$$

This proves the claim.

Now we assume that $1024 \leq n < 2048$. Then $x := \beta(n) = 1x_2x_3...x_{11}$.

We split the problem in four cases:

Case 1: $x_2 = 0$. Then, $x := \beta(n) = 10x_3...x_{11}$, and exactly as in Claim 1 we get $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 7$, $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, and $B(x,10) = B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 36\,.$$

Case 2: $x_2 = 1, x_3 = 0$. Then $x := \beta(n) = 110x_4...x_{11}$, and exactly as in Claim 1 we get $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 7$, $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 2$, $B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37\,.$$

Case 3: $x_2 = 1, x_3 = 1, x_4 = 0$. Then $x := \beta(n) = 1110x_5...x_{11}$, and exactly as in Claim 1 we get $B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4$, $B(x,4) \leq 7$, $B(x,5) \leq 6$, $B(x,6) \leq 5$, $B(x,7) \leq 4$, $B(x,8) \leq 3$, $B(x,9) \leq 3$, $B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 38\,.$$

Case 4: $x_2 = 1, x_3 = 1, x_4 = 1$. We will also split this case in 7 subcases:

SubCase 4a: $x_5 = x_6 = 0$. Then $x := \beta(n) = 111100x_7...x_{11}$.

Then, $B(x,5) \leq 5$, $B(x,6) \leq 5, B(x,7) \leq 4, B(x,8) \leq 4$, and hence from Claim 3 we get

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 38\,.$$

SubCase 4b: $x_5 = 0, x_6 = 1, x_7 = 0$. Then $x := \beta(n) = 1111010x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \leq 2$, $B(x,3) \leq 4, B(x,4) \leq 6, B(x,5) \leq 5$, $B(x,6) \leq 5$, $B(x,7) \leq 4, B(x,8) \leq 4, B(x,9) \leq 3, B(x,10) \leq 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \leq 37\,.$$

SubCase 4c: $x_5 = 0, x_6 = 1, x_7 = 1$. Then $x := \beta(n) = 1111011x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \le 2$, $B(x,3) \le 4, B(x,4) \le 7, B(x,5) \le 6$, $B(x,6) \le 5$, $B(x,7) \le 4, B(x,8) \le 4, B(x,9) \le 3, B(x,10) \le 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \le 39\,.$$

To get equality we must have $B(x,1) = 1$, $B(x,2) = 2$, $B(x,3) = 4, B(x,4) = 7, B(x,5) = 6$, $B(x,6) = 5$, $B(x,7) = 4, B(x,8) = 4, B(x,9) = 3, B(x,10) = 2$, and $B(x,11) = 1$.

If $x_8 = 0$, we get $B(x,4) \le 6$, thus $b(n) \le 38$.

Thus we can only get equality if $x_8 = 1$. Then $x = 11110111x_9x_{10}x_{11}$. Since $B(x,3) = 4$ we must have all 4 blocks $100, 101, 110, 111$ in the binary representation of n. Hence either $x_9 = x_{10} = 0$ or $x_{10} = x_{11} = 0$, which shows that equality can only occur for

$$x \in \{11110111000, 11110111001, 11110111100\} = X\,.$$

As 11110111100 doesn't contain 1000 and 1001, $B(11110111100, 4) \ne 7$, and hence $B(11110111100) \ne 39$.

As 11110111001 doesn't contain 1010 and 1000, $B(11110111100, 4) \ne 7$, and hence $B(11110111100) \ne 39$.

As 11110111100 doesn't contain 1010 and 1000, $B(11110111100, 4) \ne 7$, and hence $B(11110111100) \ne 39$.

Thus, there is no solution in this case.

SubCase 4d: $x_5 = 1, x_6 = 0, x_7 = 0$. Then $x := \beta(n) = 1111100x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \le 2$, $B(x,3) \le 4, B(x,4) \le 5, B(x,5) \le 5$, $B(x,6) \le 5$, $B(x,7) \le 4, B(x,8) \le 4, B(x,9) \le 3, B(x,10) \le 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \le 37\,.$$

SubCase 4e: $x_5 = 1, x_6 = 0, x_7 = 1$. Then $x := \beta(n) = 1111101x_8x_9x_{10}x_{11}$.

$B(x,1) = 1$, $B(x,2) \le 2$, $B(x,3) \le 4, B(x,4) \le 6, B(x,5) \le 6$, $B(x,6) \le 5$, $B(x,7) \le 5, B(x,8) \le 4, B(x,9) \le 3, B(x,10) \le 2$, and $B(x,11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x,\ell) \le 39\,.$$

To get equality we must have $B(x,1) = 1$, $B(x,2) = 2$, $B(x,3) = 4, B(x,4) = 6, B(x,5) = 6$, $B(x,6) = 5$, $B(x,7) = 5, B(x,8) = 4, B(x,9) = 3, B(x,10) = 2$, and $B(x,11) = 1$.

If $x_8 = 0$, then $x = 11111010x_9x_{10}x_{11}$ we get $B(x,4) \le 5$, thus $b(n) \le 38$.

Thus we can only get equality if $x_8 = 1$. Then $x = 11111011x_9x_{10}x_{11}$. Since $B(x,3) = 4$ we must have all 4 blocks $100, 101, 110, 111$ in the binary representation of n. Hence either $x_9 = x_{10} = 0$ or $x_{10} = x_{11} = 0$, which shows that equality can only occur for

$$x \in \{11111011000, 11111011001, 11111011100\}\,.$$

It is easy to check that both $x = 11111011000$ and $x = 11111011001$ work. Thus, in this case, we get equality for $n = 2008$ and $n = 2009$.

SubCase 4f: $x_5 = 1, x_6 = 1, x_7 = 0$. Then $x := \beta(n) = 1111110x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1$, $B(x, 2) \le 2$, $B(x, 3) \le 4, B(x, 4) \le 5, B(x, 5) \le 5$, $B(x, 6) \le 6$, $B(x, 7) \le 5, B(x, 8) \le 4, B(x, 9) \le 3, B(x, 10) \le 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \le 38\,.$$

SubCase 4g: $x_5 = 1, x_6 = 1, x_7 = 1$. Then $x := \beta(n) = 1111111x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1$, $B(x, 2) \le 2$, $B(x, 3) \le 4, B(x, 4) \le 5, B(x, 5) \le 5$, $B(x, 6) \le 6$, $B(x, 7) \le 5, B(x, 8) \le 4, B(x, 9) \le 3, B(x, 10) \le 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \le 38\,.$$

This proves that for all $n \le 2500$ we have $b(n) \le 39$ with equality if and only if

$$n \in \{2008, 2009\}\,.$$

OC40. Let M and N be the intersection of two circles, Γ_1 and Γ_2. Let AB be the line tangent to both circles closer to M, say $A \in \Gamma_1$ and $B \in \Gamma_2$. Let C be the point symmetrical to A with respect to M, and D the point symmetrical to B with respect to M. Let E and F be the intersections of the circle circumscribed around DCM and the circles Γ_1 and Γ_2, respectively.

Show that the circles circumscribed around the triangles MEF and NEF have radii of the same length.

(*Originally question #5 from the 2009 Italian Team Selection Test.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

Let P be the intersection point of AB and MN. Using the power of a point with respect to a circle we get

$$PA^2 = PM \cdot PN = PB^2\,.$$

Thus P is the midpoint of AB. As M is the midpoint of BD we get

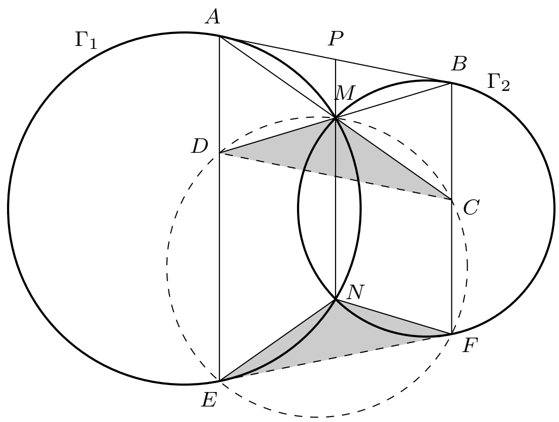
$$AD \parallel PM \parallel BC\,.$$

Combining the fact that M, D, E, F, C are on the same circle with the fact that $ABCD$ is a parallelogram and AB tangent to Γ_1, we get

$$\angle DEM = \angle DCM = \angle MAB = \angle AEM\,,$$

$$\angle CFM = \angle MDC = \angle MBA = \angle BFM\,.$$

Hence A, E, M are collinear and B, C, F are also collinear.



It results that the trapezoids $DEFC$, $AENM$, $BFNM$ are cyclic, hence isosceles. It follows that

$$DC = EF\,;\; EN = AM = MC\,;\; NF = BM = MD\,.$$

Thus the triangles NEF and MCD are congruent. Since the triangles MEF and MCD are inscribed in the same circle, the statement of the problem follows.

BOOK REVIEWS

Amar Sodhi

La Balade de la Médiane et le Thórème de Pythagoron par Jean-Claude Pont
Editions du Tricorne, Genève, 2012
ISBN : 978-2-940450-03-9, Softcover, 93 pages, 14,35 €
Reviewed by ***Michel Bataille****, Rouen, France*

Dans les années 1970, l'auteur enseignait à des élèves non-spécialistes de mathématiques. Dans le but de développer chez eux curiosité et goût de la recherche, il proposa à un groupe d'élèves intéressés le thème qui est à l'origine de cet ouvrage : l'étude de certaines familles de triangles caractérisés par une relation portant sur les côtés (comme les triangles rectangles le sont par la relation de Pythagore). Le coeur du livre, présentation détaillée de ce sujet en une vingtaine de courts chapitres mélangeant agréablement algèbre et géométrie, constitue un exposé mathematique intéressant et de bon niveau. Il est introduit par des considérations d'ordre philosophique sur la géométrie (Jean-Claude Pont est maintenant professeur de philosophie à l'université de Genève) et suivi de trois annexes. Je m'empresse de signaler que la clarté, le ton particulièrement plaisant, voire joyeux, et l'humour de l'auteur rendent la lecture facile et agréable. Ceci est sensible dès le titre, avec ce Pythagoron, mot-valise formé sur Pythagore, que tout un chacun connaît, et sur Goron, sans doute moins familier : c'est le nom d'un vin du Valais (un des cantons suisses). Je ne dévoilerai pas la raison (amusante) de ce rapprochement... Des phrases provoquant le sourire émaillent le texte, comme celle-ci, que j'ai particulièrement appréciée : *Ces triangles offrent un champ de recherche étendu, dont les résultats se distinguent aussi par une notable inutilité.*

Les triangles pythagoroniens, point de départ du thème, sont les triangles dont les côtés a, b, c satisfont $a^2 + b^2 = 2c^2$ et leur étude est menée jusqu'à une caractérisation en termes de médianes (le théorème de Pythagoron !). Bien sûr, les lecteurs auront reconnu le sujet d'une chronique de Chris Fisher ([2011 : 304]) et pourront ajouter le vocable "pythagoronien" aux adjectifs déjà indiqués par C. Fisher comme "automédian" ou "quasi-isocèle". Mais Jean-Claude Pont ne va pas en rester à ces triangles et généralise rapidement aux triangles *rituels d'ordre* n, ceux qui vérifient $a^2 + b^2 = nc^2$ (n entier positif). Les propriétés des triangles rectangles (rituels d'ordre 1) servent de fil conducteur pour leur étude, suggérant des voies de recherche ou des figures associées. C'est ainsi que l'on voit apparaître des cercles rituels et des parallélogrammes rituels. Pour le lecteur de ***Crux Mathematicorum***, tous ces chapitres constituent d'excellents exercices d'entraînement, avec souvent des surprises stimulantes, comme cette élégante construction des décagones réguliers (convexes ou étoilés) au chapitre 8. L'auteur, poursuivant sur sa lancée, s'intéresse ensuite aux triangles *pararituels d'ordre* n caractérisés par la relation $a^2 - b^2 = nc^2$. Cette fois, une droite rituelle et une hyperbole rituelle vont s'introduire naturellement. Tout aussi naturellement arrive ensuite l'examen des triangles à la fois rituels et pararituels... En conclusion, Jean-Claude Pont propose deux questions plus difficiles comme directions de recherche dans le prolongement de son travail. Il me semble qu'une troisième question aurait pu concerner les triangles rituels à côtés entiers (pour faire suite à une remarque sur les triangles pythagoroniens au chapitre 4). Par exemple, des formules existent pour les côtés entiers de triangles rituels d'ordre 5 (voir problème 3353, [2009 : 334]) ; qu'en est-il pour les autres ordres ?

Trois annexes viennent conclure l'ouvrage. La première introduit le quadrilatère des trimédianes, un autre point de départ pour une étude dans l'esprit de celle qui s'achève. Les quatre propriétés données sans démonstration constituent un très bon exercice de géométrie élémentaire. Les deux autres annexes sont des *monologues imaginaires*, réflexions à la fois poétiques et philosophiques portant sur le cercle (*Pensées encerclées*), puis sur l'espace (*L'espace d'un instant*).

Pour reprendre ses mots, Jean-Claude Pont nous offre *une promenade champêtre, un morceau de géométrie joyeuse et fraîche.* J'y ajouterais l'adjectif un peu désuet qui m'est venu à l'esprit pendant la lecture : délectable.

A Mathematician Comes of Age by Steven G. Krantz
Mathematical Association of America, Spectrum Series, 2012
ISBN: 978-0-88385-578-2 (print); 978-1-61444-511-1 (electronic), 137 + xvii pages, US$27.50(ebook) and US$60.95(print)
Reviewed by ***Jeff Hooper****, Acadia University, Wolfville, N.S.*

What is mathematical maturity? How can we recognize it? How can we foster it?

Stephen Krantz is a widely published Professor at Washington University in St Louis, and in *A Mathematician Comes of Age* he takes on these themes, from the perspective of an experienced mathematician and professor, generously sharing his years' worth of experiences, insights and opinions. At approximately 120 pages plus addenda it is not a long book, but Krantz still finds time to tackle a wide variety of topics.

The notion of mathematical maturity is a vague one, a concept which is not clearly understood within the mathematics community. It is certainly not a topic that is discussed much outside of mathematics. Many of us might say we "know it when we see it" or be able to identify examples, such as the ability to analyze and create proofs. Yet as the author points out, if we want to be able to teach mathematics better and more effectively, then we need to improve our understanding of mathematical maturity.

Krantz's main theme is to suggest what this thing we call mathematical maturity is all about: what its identifying features are, what the stages are in its development (from the early grades through to graduate schools), what its place is in wider society, and what roles (positive and negative) social and cognitive forces play in its development. He creates a picture of this maturity as both a model of mathematical reasoning, extending to fields beyond mathematics, as well as a model for what rigorous and precise reasoning should be.

The book begins with an introductory chapter that highlights many of the ideas and threads to be taken up later. The author introduces mathematical maturity as a developed understanding of mathematics and how mathematics works. He considers mathematics teaching at the university level and some guiding principles for teaching in an effective way. As witnessed in a number of Krantz's other books, his profound concern for the manner in which mathematics is taught comes through very strongly here. The chapter continues with some history of mathematics teaching and examples of maturity at different levels, as well as a discussion about how our view of mathematical maturity has evolved over time and how it in fact continues to change.

The heart of the book are the next four chapters. The first of these, on Mathematical Concepts, examines mathematics problems that can be used to foster mathematical maturity, looks at what parts of the mathematics curriculum play important roles in this (and which do not), as well as the role that can be played in this development by computers, by proofs, and by making mistakes. A chapter on Teaching Techniques comes next, picking up the thread of teaching improvement from the first chapter. Here Krantz considers a number of current trends, such as online learning, capstone experiences, and student research. He also discusses some new approaches to mathematics teaching.

Any consideration of developing mathematical maturity must consider external influences and the next two chapters discuss Social and Cognitive issues. There are of course many societal influences on studying mathematics. Krantz discusses a number of these, including whether various syndromes (Asperger's, schizophrenia, depression) are related to the learning of mathematics and whether various standardized tests can adequately help teachers track the development of mathematical maturity. Similarly, looking at cognitive issues, Krantz delves into the nature vs nurture question, considers learning styles and whether any of these are more relevant to maturity, and discusses the roles played in this process by the motivation and goals of the learner. He also looks at the psychology of learning, and at types of intelligence.

The author ends by revisiting the mathematical maturity concept, in light of the previous chapters, looking at what makes a mathematician, and in particular at a list of key attributes a mathematically mature person should have, and revisits the place of mathematical maturity in our world.

Despite the lengthy list of topics listed above, the book is very readable, written in a conversational style. It is also very funny in places. This book will be of interest to high school students who enjoy mathematics and are (or should be) considering mathematical studies at university, to undergraduates considering further graduate work, and to professional mathematicians with an interest in developing mathematical maturity in their own students.

PROBLEM SOLVER'S TOOLKIT

No. 2

Shawn Godin

*The Problem Solver's Toolkit is a new feature in **Crux Mathematicorum**. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Counting With Care

Counting problems usually involve quite elementary mathematics and techniques, yet can prove to be quite complex. In many cases the correct way of looking at the problem can simplify or greatly illuminate the situation.

As an example, I was recently visiting my family and we decided to play a game. There were six of us and we had to break into three groups of two. Someone asked me off hand "how many different ways could the groups be formed?" Caught off guard (it was my holidays) I gave a wrong answer: 90. I knew as soon as it was out of my mouth that it was wrong, but they would never know so I let it be and decided to think about it later.

The solution that I did in my head was: first we need to form the first team from the six, then another team from the four and the final group from the two. As such, I calculated

$$\binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} = 15 \times 6 \times 1 = 90.$$

The result seemed a little big, but it impressed them and I was sure they wouldn't think about it again ..., but I would. My mistake, a typical one with counting problems, was that my method counts the groups many times. That is, if I label our players A, B, C, D, E and F, my method considers choosing A and B as team #1, C and D as team #2 and E and F as team #3 as different from choosing C and D as team #1 , and E and F as team #2, and A and B as team #3. As a result, each of my groupings of teams has been counted 6 times, thus the real (and less impressive) answer to the question is

$$\frac{\binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{3!} = 15.$$

Now we could leave it at that and just take away the message "be careful when you are counting", but, like many counting problems, we can attack this from another point of view.

Consider one of the players, A for example, and consider him "fixed". That is, he needs a partner one way or the other, and there are 5 ways to pick one for him. Next, pick one of the remaining players to consider as "fixed" and she needs a partner as well, which we can do in 3 ways. All that remains is 2 people who become the last team. Thus, the number of ways to make the teams is just

$$5 \times 3 = 15$$

which is much easier to see.

The next problem is a favourite of mine.

> **Problem** (question S3, 2001 Interprovincial Mathematics Olympiad, senior team portion, [2002 : 42, 392]) *Eight boxes, each a unit cube, are packed in a $2 \times 2 \times 2$ crate, open at the top. The boxes are taken out one by one. In how many ways can this be done? (Remember that a box in the bottom layer can only be removed after the box above it has been removed.)*

If it suits you, take a few minutes and try this problem before you read on.

This problem was from back when I was doing the Skoliad Corner, before we started printing solutions from readers. I remember quite clearly receiving a solution from a reader that, although correct, was quite long and complicated. The reader had basically broken the problem down into all the possible cases. Their solution started like this:

> We only have 4 choices for the first box, as it must come from the top. For the second box, we have four choices, but we must break them up into two cases:
>
> **Case 1:** We choose another box from the top (3 possible choices), then our next choice we will have four boxes to choose from, two from the top and two from the bottom, which leads to two more cases.
>
> **Case 2:** We chose the box from the bottom (1 possible choice), so for our next choice we only have 3 boxes to choose from (all from the top).
>
> ⋮

Since choosing a box from the top or choosing a box from the bottom lead to different scenarios for the next choice, the problem broke down into a large number of cases. As such, the solution was very long and hard to follow (to make sure nothing was double counted, my sin from the first problem, or nothing was left out). In the end it was correct.

I love this problem because it is so simple to state and understand, yet it can be tricky to solve. I have given this problem to high school teachers in a number of workshops and it always causes a stir.

The key to solving the problem, is choosing the correct point of view. I know that a bottom box cannot be removed until the one above it has been removed. Start by numbering the top boxes 1, 2, 3, 4 and numbering each bottom box with the same number as the box above it. Now, when I pull the boxes out, I will just record the number on them. When you see something like 2, 3, 1, 2, 4, 1, 3, 4 there is no ambiguity, the first 2 has to be the top box, and the second 2 has to be the bottom box. As a result, our problem is equivalent to finding the number of arrangements of the eight numbers 1, 1, 2, 2, 3, 3, 4, 4 which is

$$\frac{8!}{2!2!2!2!} = 2520.$$

The 8! in the solution is there since we are arranging 8 objects, but this includes all possible arrangements, including when two identical numbers are "switched". For example the 8! would count both 1**1**223344 and **1**1223344 (where I have bolded and enlarged one of the ones so that you can see the "switch"). For each distinct digit, there are 2! ways that the two copies can be interchanged, hence we divide by all of these factors to eliminate duplicate arrangements from the original total. This solution is much more illuminating and satisfying than two pages of cases.

Here are a couple of related problems to try:

1. Twenty-four boxes, each a unit cube, are packed in a crate 2 units long, 3 units wide and 4 units deep, open at the top. The boxes are taken out one by one. In how many ways can this be done?

2. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe? (problem 16, 2001 AMC12 contest)

So, when you tackle a counting problem, take some time and think of different ways it can be interpreted. A good practice would be to try to find multiple solutions to any given problem, it will help to count with care.

RECURRING CRUX CONFIGURATIONS 8

J. Chris Fisher

Heronian Triangles

We call a triangle *Heronian* if its sides and area are all integers; it is called *rational* when the sides and area are rational numbers. (Unfortunately, not even **Crux** has been consistent with this terminology; the reader should note that some sources require only rational sides and area for a triangle to be Heronian.) It is clear that any rational triangle is similar to a Heronian triangle—because the altitudes of a rational triangle are necessarily rational, it suffices to multiply each side and one of the altitudes by twice the common denominator. While the literature devoted to this topic is extensive, Heronian triangles have appeared in these pages in only four problems and one short note. The note [1982: 206] was "An Heronian Oddity" by Leon Bankoff. The author discussed the 13-14-15 Heronian triangle, which is formed by placing a 5-12-13 right triangle beside a 9-12-15 right triangle so that they share the leg of length 12. The common leg then serves as the altitude to the side of length $5+9$ of the 13-14-15 triangle, while extending the string of consecutive integers. Bankoff remarked further that the inradii of the 5-12-13, the 9-12-15, and the 13-14-15 triangles are also measured by consecutive integers: 2, 3, and 4, respectively. This observation may not be very deep, but it becomes important if you run out of other things to talk about while on a date.

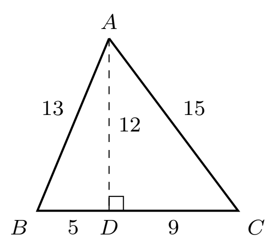


Figure 1: The juxtaposition of integer-sided right triangles ABD and ADC to form a Heronian triangle ABC.

It turns out that *all* Heronian triangles and *only* Heronian triangles are obtained either by the juxtaposition of two integer-sided right triangles as in the figure, or by the reduction of such a juxtaposition by a common factor. Remarkably, according to David Singmaster [1985: 222-223] this result was not completely proved until 1978 in [3]; see also [2]. That proof depends on the following theorem.

Theorem. Let k, a, b, c be positive integers. Then a, b, c are the sides of a Heronian triangle if and only if ka, kb, kc form a Heronian triangle.

As a consequence, a triangle is Heronian if and only if its sides can be represented either as

$$a = m(u^2 - v^2) + n(r^2 - s^2), \quad b = m(u^2 + v^2), \quad c = n(r^2 + s^2), \tag{1}$$

for positive integers m, n, r, s, u, v satisfying $muv = nrs$, or as a reduction by a common factor of a triangle given by (1). The altitude to the side a of the triangle in (1) is the common product $muv = nrs$. Singmaster's comments were his contribution to the solution of the following problem, the problem that inspired Bankoff's musings on consecutive-integer triangles that we reproduced above.

Problem 290 [1977: 251; 1978: 142-147; 1985: 222-223] (proposed by R. Robinson Rowe). Find a 9-digit integer representing the area of a triangle of which the three sides are consecutive integers.

The solution by Clayton W. Dodge characterized those Heronian triangles whose sides are consecutive integers. Specifically, they are those triangles whose middle side and area are

$$a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n, \qquad \text{Area } = \frac{3}{4\sqrt{3}}\left((2+\sqrt{3})^{2n} - (2-\sqrt{3})^{2n}\right).$$

Examples can be calculated by hand using the recurrence relations

$$\begin{array}{llll} x_1 = 2, & x_2 = 7, & x_{n+2} = 4x_{n+1} - x_n, & n = 1, 2, \cdots \; ; \quad \text{and} \\ y_1 = 1, & y_2 = 4, & y_{n+2} = 4y_{n+1} - y_n, & n = 1, 2, \cdots \; . \end{array}$$

The middle side of the nth triangle will be $a_n = 2x_n$, and its area $A_n = 3x_n y_n$. The first four Heronian triangles whose sides are consecutive integers have middle side and area as follows:

n	a_n	A_n
1	4	6
2	14	84
3	52	1170
4	194	16296

For the record, the 9-digit area requested by Rowe is $A_8 = 613,283,664$. Dodge's solution was followed by two pages of comments by the editor, most of which summarized the information related to this problem found in the 11 pages of [1] that Dickson devoted to rational triangles.

Problem 1148 [1987: 108; 1988: 83; 1991: 302-303] (proposed by Stanley Rabinowitz). Find the triangle of smallest area that has integral sides and integral altitudes.

The smallest such triangle is the right triangle with sides $15, 20, 25$, altitudes $20, 15, 12$, and area 150. A nice proof was supplied in 1991 by Sam Maltby, who as a bonus proved that

- The smallest non-right triangles with integral sides and altitudes are the 25-25-30 (acute) and 25-25-40 (obtuse) triangles with areas of 300 each and altitudes 24,24,20 and 24,24,15, respectively.

- The triangle of smallest area having integral sides, area, circumradius, and inradius (described as an open problem in [1, p. 200]) is the 6-8-10 triangle with area 24, $R = 5$, and $r = 2$. If we add the restriction that the altitudes must also be integral, we get the 30-40-50 triangle.

Problem 2764 [2002: 397; 2003: 348-349] (proposed by Christopher J. Bradley). Find an integer-sided scalene triangle in which the lengths of the internal angle bisectors all have integer lengths.

I found four typographical errors in the last paragraph of the featured solution. The example there is a Heronian triangle whose sides are 10-digit numbers; perhaps with numbers so large the introduction of errors should come as no big surprise. A more appealing problem would have called for examples with rational sides and rational bisectors, allowing the reader to multiply by a common denominator to satisfy his own curiosity. That, in fact, describes problem E 418 in [4]. The solution by E.P. Starke included the comment that *if the sides and internal angle bisectors are rational, so also are the external angle bisectors, the altitudes, the area, R, r, and the three exradii.* His solution began with two rational solutions (x_1, y_1) and (x_2, y_2) of $x^2 + y^2 = 1$, with $x_1x_2 > y_1y_2$. Here x_1 plays the role of $\cos\frac{B}{2}$, x_2 of $\cos\frac{C}{2}$. Then any numbers a, b, c proportional to

$$(x_1x_2 - y_1y_2)(x_1y_2 + y_1x_2), \quad x_1y_1, \quad x_2y_2$$

are the sides of a triangle whose angle bisectors are

$$\frac{2bc(x_1y_2 + y_1x_2)}{b + c}, \quad \frac{2cax_1}{c + a}, \quad \frac{2abx_2}{a + b}.$$

The characterization in (1) corresponds to Starke's by replacing the integers m, u, v by the rational numbers x_1y_1, x_2, y_2, respectively, as well as n, r, s by x_2y_2, x_1, y_1. For example, the solutions $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\frac{12}{13}, \frac{5}{13}\right)$ produce a scalene triangle with sides 84, 169, 125, and angle bisectors $\frac{975}{7}, \frac{12600}{209}, \frac{26208}{253}$. If you wish, you may multiply through by the common denominator $7 \cdot 11 \cdot 19 \cdot 23$ to get a triangle whose sides and bisectors are 7-digit integers.

The fourth **Crux** problem dealing with Heronian triangles, number 2765 [2002: 397; 2003: 349-351], involves triangles whose nine-point centre lies on side BC; it will be discussed in the ninth essay of our series.

References

[1] Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Volume II, pp. 191-201.

[2] David Singmaster, Some Corrections to Carlson's "Determination of Heronian Triangles", *The Fibonacci Quarterly*, **11**:2 (April 1973) 157-158.

[3] David Singmaster, Letter to the Editor, *Mathematical Spectrum*, **11** (1978/1979) 58-59.

[4] E.P. Stark, Solution to problem E 418, *American Mathematical Monthly*, **48**:1 (Jan. 1941) 67-68. (Problem proposed by W.E. Buker, 1940, p. 240.)

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 février 2014**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7, et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8, et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal d'avoir traduit les problèmes.

3771. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

(a) Trouver une infinité de paires (a, b) de nombres rationnels positifs telles que $\sqrt{a} - \sqrt{b}$ est une racine de $x^2 + ax - b$.

(b) Trouver deux entiers positifs a, b tels que $\sqrt{a} - \sqrt{b}$ est une racine de $x^2 + ax - b$.

3772. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

On donne un carré $ABCD$ de côté a. Choisir respectivement des points K et L sur BC et CD de sorte que le périmètre de ΔKCL soit $2a$. Déterminer les mesures des angles de ΔAKL qui minimisent son aire.

3773. *Proposé par Michel Bataille, Rouen, France.*

Soit respectivement R et r les rayons des cercles circonscrit et inscrit d'un triangle de côtés a, b, c. Sous quelle condition sur les angles du triangle l'inégalité

$$a + b + c \le 2\sqrt{3}(R + r)$$

est-elle respectée ?

3774. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} + \frac{a(b^2+c^2)}{\sqrt{b^3+c^3}} + \frac{b(c^2+a^2)}{\sqrt{c^3+a^3}} \le \frac{3}{4}\left(a^2+b^2+c^2+a+b+c\right).$$

3775. *Proposé par Marcel Chirita, Bucharest, Romainia.*

Soit $ABCD$ un quadrilatère avec $AC \perp BD$. Montrer que $ABCD$ est cyclique si et seulement si $BC \cdot AD = IA \cdot IB + IC \cdot ID$, où I est le point d'intersection des diagonales.

3776. *Proposé par Wei-Dong Jiang, Collège Professionnel de Weihai, Weihai, Province de Shandong, Chine.*

Montrer que dans le triangle ABC, on a

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \geq \frac{1}{2}\left(\sec\left(\frac{A}{2}\right) + \sec\left(\frac{B}{2}\right) + \sec\left(\frac{C}{2}\right)\right).$$

3777. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit x, y et z trois nombres réels positifs tels que $xyz = 1$ et $\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} = 3$. Trouver toutes les valeurs possibles de $x^4 + y^4 + z^4$.

3778. *Proposé par Michel Bataille, Rouen, France.*

Soit O le centre du cercle circonscrit d'un triangle $A_1A_2A_3$, γ son cercle inscrit, de centre I et de rayon r. Pour $i = 1, 2, 3$, soit A'_i sur le côté A_iA_{i+1} et A''_i sur le côté A_iA_{i+2} deux points tels que $A'_iA''_i \perp OA_i$ et que γ soit le cercle exinscrit du triangle $A_iA'_iA''_i$ correspondant à A_i, où $A_4 = A_1$, $A_5 = A_2$. Montrer que

(a) $A'_1A''_1 \cdot A'_2A''_2 \cdot A'_3A''_3 = \dfrac{4a_1a_2a_3}{(a_1 + a_2 + a_3)^2} \cdot r^2$

(b) $A'_1A''_1 + A'_2A''_2 + A'_3A''_3 = \dfrac{a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3} \cdot IK^2 + \dfrac{3a_1a_2a_3}{a_1^2 + a_2^2 + a_3^2}$

où a_1, a_2 et a_3 sont les longueurs des côtés du triangle $A_1A_2A_3$ et K est son point symédian.

3779$\star$. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Dans un triangle ABC de demi-périmètre s, soit x, y et z les distances respectives du centre de gravité aux côtés BC, CA et AB. Démontrer ou réfuter la validité de l'inégalité

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \leq \frac{s}{\sqrt{3}}.$$

3780. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $f : [0, 1] \to \mathbb{R}$ une fonction continuement différentiable et soit

$$x_n = f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right).$$

Calculer $\lim\limits_{n\to\infty}(x_{n+1} - x_n)$.

. . . .

3771. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

(a) Find infinitely many pairs (a, b) of positive rational numbers so that $\sqrt{a} - \sqrt{b}$ is a root of $x^2 + ax - b$.

(b) Find two positive integers a, b so that $\sqrt{a} - \sqrt{b}$ is a root of $x^2 + ax - b$.

3772. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Given a square $ABCD$ with side length a. Points K and L are on BC and CD, respectively, such that the perimeter of ΔKCL is $2a$. Determine the measures of the angles of ΔAKL which minimize its area.

3773. *Proposed by Michel Bataille, Rouen, France.*

Let R and r be the circumradius and the inradius of a triangle with sides a, b, c. Under which condition on the angles of the triangle does the inequality

$$a + b + c \leq 2\sqrt{3}(R + r)$$

hold?

3774. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let a, b, c be positive real numbers. Prove that

$$\frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} + \frac{a(b^2+c^2)}{\sqrt{b^3+c^3}} + \frac{b(c^2+a^2)}{\sqrt{c^3+a^3}} \leq \frac{3}{4}(a^2+b^2+c^2+a+b+c)\,.$$

3775. *Proposed by Marcel Chirita, Bucharest, Romainia.*

Let $ABCD$ be a quadrilateral with $AC \perp BD$. Show that $ABCD$ is cyclic if and only if $BC \cdot AD = IA \cdot IB + IC \cdot ID$, where I is the point of intersection of the diagonals.

3776. *Proposed by Wei-Dong Jiang, Weihai Vocational College, Weihai, Shandong Province, China.*

In ΔABC prove that

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \geq \frac{1}{2}\left(\sec\left(\frac{A}{2}\right) + \sec\left(\frac{B}{2}\right) + \sec\left(\frac{C}{2}\right)\right)\,.$$

3777. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let x, y, and z be positive real numbers such that $xyz = 1$ and $\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} = 3$. Determine all possible values of $x^4 + y^4 + z^4$.

3778. *Proposed by Michel Bataille, Rouen, France.*

Let $\Delta A_1A_2A_3$ be a triangle with circumcentre O, incircle γ, incentre I, and inradius r. For $i = 1, 2, 3$, let A_i' on side A_iA_{i+1} and A_i'' on side A_iA_{i+2} be such that $A_i'A_i'' \perp OA_i$ and γ is the A_i–excircle of $\Delta A_iA_i'A_i''$ where $A_4 = A_1$, $A_5 = A_2$. Prove that

(a) $A_1'A_1'' \cdot A_2'A_2'' \cdot A_3'A_3'' = \dfrac{4a_1a_2a_3}{(a_1 + a_2 + a_3)^2} \cdot r^2$

(b) $A_1'A_1'' + A_2'A_2'' + A_3'A_3'' = \dfrac{a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3} \cdot IK^2 + \dfrac{3a_1a_2a_3}{a_1^2 + a_2^2 + a_3^2}$

where a_1, a_2, a_3 are the side lengths of $\Delta A_1A_2A_3$ and K is its symmedian point.

3779★. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let ΔABC have semi-perimeter s and let x, y, z be the distances from the centroid to the sides BC, CA, AB, respectively. Prove or disprove that

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \leq \frac{s}{\sqrt{3}}\,.$$

3780. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function and let

$$x_n = f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\,.$$

Calculate $\lim\limits_{n\to\infty}(x_{n+1} - x_n)$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

210. [1977 : 10, 160-164, 196-198; 1978 : 13-16, 193-194] *Proposed by the late Murray S. Klamkin, University of Alberta, Edmonton, AB.*

P, Q, R denote points on the sides BC, CA, and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \quad \left(\neq 0, \frac{1}{2}, 1\right) \tag{1}$$

then PQR (in some order) is similar to ABC.

VII. Comment by Grégoire Nicollier, University of Applied Sciences of Western Switzerland, Sion, Switzerland.

Previous discussions of this problem have dealt with a fixed triangle ABC and determined those values of k for which there exists an inscribed triangle PQR that is similar to it. As stated, however, problem 210 calls for the converse: Given a real number k, determine ΔABC with the desired property. It turns out that there is a family of solution triangles for every value of k. Before making that precise, let us review the facts that have already been established.

First, note that in the excluded cases ($k = 0, 1, \frac{1}{2}$), for all triangles ABC the triangle PQR would be BCA, CAB, and the medial triangle of ABC, respectively, which are always directly similar to ABC. Of course, when ABC is equilateral, every real value of k produces an equilateral triangle PQR; conversely, solutions II [1977 : 163] and III [1977 : 197-198] proved that if ΔPQR is *directly* similar to ΔABC and $k \neq 0, 1, \frac{1}{2}$, then those triangles are necessarily equilateral. In general (see [1]),

(a) if one excepts the equilateral case, the solutions k exist only for scalene triangles ABC, and these solutions are the signed ratios

$$k_1 = \frac{a^2 - b^2}{2a^2 - b^2 - c^2}, \quad k_2 = \frac{b^2 - c^2}{2b^2 - c^2 - a^2}, \quad k_3 = \frac{c^2 - a^2}{2c^2 - a^2 - b^2}. \tag{2}$$

The similarity ratio is then $\sqrt{1 - 3k_\ell + 3k_\ell^2}$ and the similarity is always opposite with $PQ : QR : RP = b : a : c$ for k_1, $a : c : b$ for k_2, and $c : b : a$ for k_3 (these formulae are incorrect in [1] and in [2]).

For the present paragraph only, we adopt the convention that our triangle ABC has been labeled so that $a > b > c$. Triangles for which $2b^2 = c^2 + a^2$ (in which case the denominator of k_2 is zero) were called *Root-Mean-Square* triangles in [1978 : 14-16; 2010 : 304-307], but Europeans seem to prefer the terminology *automedian* (since these triangles are characterized by being similar to the triangle formed by the three medians, a triangle that is sometimes called the *median triangle* [1978 : 14]). Triangle ABC with $a > b > c$ is automedian if and only if the vertex opposite to the middle side b lies on the circle of radius $\frac{\sqrt{3}}{2}b$ centered at the midpoint of b (as does the apex of the equilateral triangle erected on b) [1978 : 13]. Continuing with the results from [1],

(b) one has $k_{\ell+1} = \frac{1-2k_\ell}{1-3k_\ell}$ for $\ell = 1, 2, 3$ by setting $k_4 = k_1$. With sides labeled so that $a > b > c$, we have $0 < k_1 < \frac{1}{2} < k_3 < 1$, while the value k_2 lies outside $[0, 1]$, except that it fails to exist when ΔABC is automedian.

An alternative approach to these and related results that combines the discrete Fourier transformation, convolution products, and a shape function for triangles, can be found in [3]. We turn now to the promised solution to problem 210.

We first look at the behavior of PQR for $k \to \pm\infty$: The angles of PQR tend to the angles of the triangle whose vertices are the tips of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ when their common origin is the centroid of ABC. A homothety of ratio $1/2$ about the centroid transforms this triangle into a triangle that is directly congruent to that formed by the medians of ABC. The triangle PQR is thus directly similar to the median triangle of ABC in the three cases $k = \frac{1}{3}$, $k = \frac{2}{3}$, and $k = \infty$ (the point at infinity of the extended real line). We can therefore assert that automedian scalene triangles also have three solutions: $k = \frac{1}{3}$, $\frac{2}{3}$, and ∞.

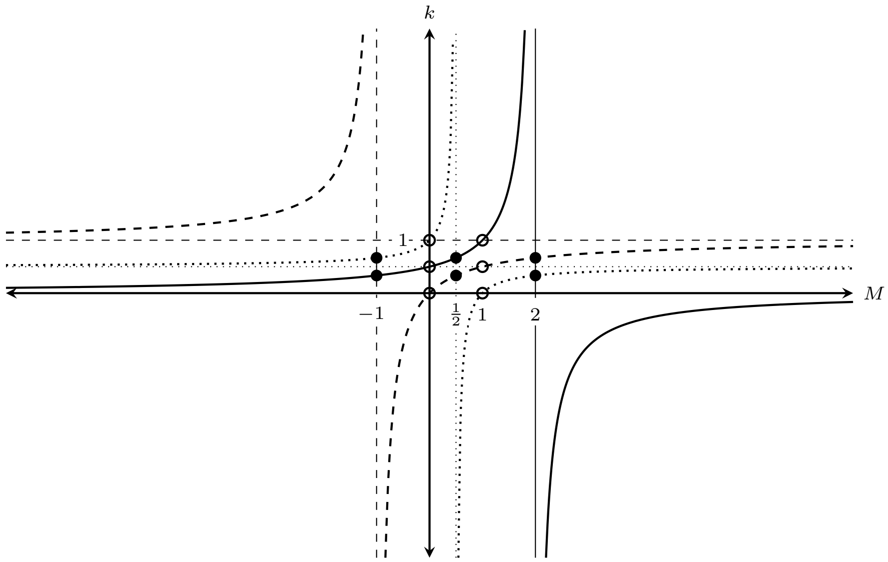
From now on, the symbols A, B, and C will denote complex numbers as well as the vertices of the triangle that they represent. Because we will not require multiplication, we will continue to denote our triangles by ABC (which is not a product!). The *normalized triangle* $\Delta(z)$ corresponding to ΔABC is a triangle with vertices 0, 1, and $z := \frac{C-A}{B-A}$. It is the unique normalized triangle directly similar to ABC with 0 corresponding to A, 1 to B, and z to C.

Theorem 1 *When ΔABC is scalene with normalized triangle $\Delta(z)$, the circle through $z = \frac{C-A}{B-A}$, $e^{i\pi/3}$, and $e^{-i\pi/3}$ is centered at $M_1 = \frac{b^2-c^2}{b^2-a^2}$. BCA corresponds to a normalized triangle with center $M_2 = \frac{c^2-a^2}{c^2-b^2}$, and CAB to $M_3 = \frac{a^2-b^2}{a^2-c^2}$.*

Proof. Note that $e^{\pm i\pi/3}$ are the apices of the normalized equilateral triangles. With $z = x + iy$, the center M of the circle corresponding to ABC is given by $(x - M)^2 + y^2 = r^2 = M^2 - M + 1$, i.e., $M = \frac{x^2+y^2-1}{2x-1} = M_1$. ∎

The solutions k_ℓ given by (2) are cyclically related by $k \mapsto \frac{1-2k}{1-3k}$, and the three circle centers M_ℓ by $M \mapsto 1 - \frac{1}{M}$. Note that $M_1M_2M_3 = -1$. The center M_ℓ beginning (in Theorem 1) with the middle side is negative. If one has $a > b > c$, $b > c > a$, or $c > a > b$, the center M_ℓ beginning with the longest side lies between 0 and 1, and the center beginning with the shortest side is greater than 1. The situation is reversed in the other arrangements of a, b, c.

The points (M_1, k_1), (M_2, k_2), (M_3, k_3) lie in the accompanying figure on the plain hyperbola $k = f_1(M) = \frac{1}{2-M}$, i.e., $M = 2 - \frac{1}{k}$. The points (M_1, k_2),



(M_2, k_3), (M_3, k_1) lie on the dashed hyperbola $k = f_2(M) = \frac{M}{M+1}$, i.e., $M = \frac{k}{1-k}$. And the points (M_1, k_3), (M_2, k_1), (M_3, k_2) lie on the dotted hyperbola $k = f_3(M) = \frac{1-M}{1-2M}$, i.e., $M = \frac{1-k}{1-2k}$. These three hyperbolas are cyclically related by $f_{\ell+1}(M) = f_\ell\Big(1 - \frac{1}{M}\Big)$.

The six sets

$$\{f_1(M_\ell), f_2(M_\ell), f_3(M_\ell)\} \quad \text{and} \quad \{f_\ell(M_1), f_\ell(M_2), f_\ell(M_3)\}, \quad \ell = 1, 2, 3,$$

are all equal to the solution set $\{k_1, k_2, k_3\}$. One can retrieve all solutions k_ℓ from a single center M', and all circle centers M_ℓ from a single solution k' by intersecting the three hyperbolas with the lines $M = M'$ and $k = k'$. When the triangle is not automedian, one circle center lies between $\frac{1}{2}$ and 2, one between -1 and $\frac{1}{2}$, and one outsides $[-1, 2]$. Note that $M = -1$, $\frac{1}{2}$, 2 correspond to normalized automedian triangles and $M = 0$, 1, ∞ to isosceles ones.

Theorem 2 *The scalene ΔABC is a k-triangle (i.e., ΔABC is similar to ΔPQR given by (1) for that value of k) if and only if the circle through $e^{\pm i\pi/3}$ and the apex of the normalized triangle of ABC is centered at $2 - \frac{1}{k}$, $\frac{k}{1-k}$, or $\frac{1-k}{1-2k}$.*

Proof. The given circle is centered at $M_1 = \frac{b^2-c^2}{b^2-a^2}$ by Theorem 1. By equation (2), $M_1 = 2 - \frac{1}{k}$, $\frac{k}{1-k}$, and $\frac{1-k}{1-2k}$ if and only if $k = k_1$, k_2, and k_3, respectively. ■

Theorem 2 can also be formulated as follows: the scalene ΔABC is a k-triangle if and only if the circle through $e^{i\pi/3}$ centered at $2 - \frac{1}{k}$ contains the apex of a normalized triangle similar (in some order) to ABC. One can replace $2 - \frac{1}{k}$ by $\frac{k}{1-k}$ and by $\frac{1-k}{1-2k}$.

References

[1] W. J. Blundon and D. Sokolowsky, Solution to Problem 210 (Klamkin), *Crux Math.* 3 (1977) 160–164 and 196-197.

[2] A. Liu and B. Shawyer (eds.), *Problems from Murray Klamkin: The Canadian Collection*, The Mathematical Association of America, Washington, DC, 2008.

[3] G. Nicollier, Convolution filters for triangles, *Forum Geom.* 13 (2013) 61–85. http://forumgeom.fau.edu/FG2013volume13/FG201308index.html

3653. [2011 : 318, 321; 2012 : 247] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let O be the centre of a sphere S circumscribing a tetrahedron $ABCD$. Prove that:

(i) there exists tetrahedra whose four faces are obtuse triangles; and

(ii) $\star$ if O is inside or on $ABCD$, then at least two faces of $ABCD$ are acute triangles.

II. Combination of solutions by Tomasz Cieśla, student, University of Warsaw, Poland; and the proposer.

The statement of part (ii) is not quite correct. To repair it, we will revise both statements. In the solution to the new part (i) we will extend the solution that appeared before in [2012 : 247].

(i) (Revised) *If O is outside $ABCD$ or at the midpoint of one of its edges, then it is possible that no faces are acute triangles.*

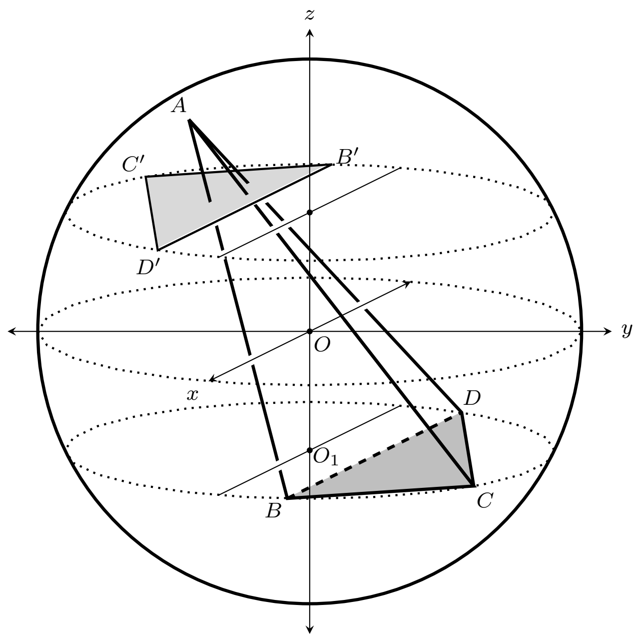
Start with a trapezoid $ABCD$ for which $AD||BC$ and the angles $\angle BAC$, $\angle BDC$ are both obtuse. Then triangles BAD and CDA are also obtuse. Now lift B a short distance above the plane so that lines AC and BD are skew. Then $ABCD$ is no longer a trapezoid, but a tetrahedron whose four faces are obtuse triangles. The location of its circumcentre will be a consequence of the argument in part (ii). For an example that has O on an edge, take the convex hull of the diagonal of a cube and a disjoint edge (for example, the tetrahedron with vertices $(0,0,0),(1,0,0),(1,1,0),(0,0,1)$). All four of its faces will be right triangles.

(ii) (Revised) *If O is inside the tetrahedron $ABCD$ or in the interior of one of its faces, then at least two faces are acute triangles.*

For our proof we require two lemmas that follow immediately from the corresponding plane theorems.

Lemma 1. For three points P,Q,R on a sphere with centre O, the plane triangle PQR is acute if and only if the foot O' of the perpendicular from O to the plane lies inside the triangle. (Note that O' is the circumcentre of ΔPQR, so that this result reduces to the familiar plane theorem.)

Lemma 2. For three points PQR on a sphere with centre O, the plane containing the segment QR and perpendicular to the line joining O to its midpoint will intersect the sphere in a circle whose diameter is QR; $\angle QPR$ will be right, acute, or obtuse according as the point P lies on that circle, in the larger cap determined by that circle, or in the smaller cap.



We turn now to the proof of (ii). We are given $ABCD$ inscribed in a sphere whose centre lies inside the tetrahedron or in the interior of one of its faces; we are to prove that at least two of its four faces are acute triangles. To this end we assume that $\angle BCD$ is an obtuse or right angle and will show that there can be at most one further angle that is not acute. In the accompanying figure we display the circumsphere with its centre at the origin of a cartesian coordinate system, with the x-axis parallel to BD, and the y- and z-axes oriented so that the xy-plane is parallel to and above the plane of the face BCD. By Lemma 1, the projection O_1 of O into the plane of BCD must lie outside the triangle or on the edge BD. Moreover, the line AO must strike that plane inside the triangle or at the midpoint of BD. Reflecting that triangle in O to $\Delta B'C'D'$, we see that A will be in the portion of the sphere that is separated from ΔBCD by the vertical and horizontal planes through $B'D'$. Consequently, A lies in the larger cap determined by the spherical circle on the diameter DB. By Lemma 2 $\angle DAB$ must be acute. Next, observe that A lies with D in the cap determined by the circle on diameter BD'; because $\angle DBX = 90^\circ$ for all points X on that circle, we deduce that $\angle DBA$ is acute. The same argument with the roles of B and D interchanged shows that $\angle BDA$ is also acute; in summary,

$$\angle DAB, \angle DBA, \text{ and } \angle BDA \text{ are all acute angles.}$$

In other words, we have shown that if a face of our tetrahedron has a nonacute angle at one vertex, the face of the tetrahedron opposite that vertex must be an acute triangle. It follows that for our tetrahedron to have three nonacute face angles, they would necessarily have C as a vertex. But it is easily seen that the circumcentre of a tetrahedron having three nonacute angles at C would necessarily be exterior to the tetrahedron: Returning to the figure, we see that the line through C perpendicular to the plane BCD (and therefore parallel to OO_1) would intersect the sphere at a point, call it P that is separated from the vertex A by the plane through BD that is parallel to CP. The faces BCP and DCP of the tetrahedron $PBCD$ have right angles at P. To make those angles obtuse, P would have to be chosen on the sphere further from A; that is, at least one of the angles BCA and DCA would have to be acute.

3671★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a tetrahedron and let M be a point in its interior. Prove or disprove that

$$\frac{[BCD]}{AM^2} = \frac{[ACD]}{BM^2} = \frac{[ABD]}{CM^2} = \frac{[ABC]}{DM^2} = \frac{2}{\sqrt{3}},$$

if and only if the tetrahedron is regular and M is its centroid. Here $[T]$ denotes the area of T.

No solutions have been received. This problem remains open.

3672. [2011 : 389, 392] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2 + y^2 = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy} \;\geq\; \frac{3}{1+\left(\frac{x+y}{2}\right)^2}.$$

When does this inequality occur?

Solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Marian Dincă, Bucharest, Romania; Dimitrios Koukakis, Kato Apostoloi, Greece; Kee-Wai Lau, Hong Kong, China; Salem Malikić, student, Simon Fraser University, Burnaby, BC; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Cománeşti, Romania; and the proposer.

Let $t = xy$. Since $2|xy| \leq x^2 + y^2 = 1$, $|t| \leq \frac{1}{2}$. The difference between the two sides of the proposed inequality is

$$\begin{aligned}\frac{2+x^2+y^2}{1+x^2+y^2+x^2y^2} + \frac{1}{1+xy} - \frac{12}{4+x^2+y^2+2xy} \\ &= \frac{3}{2+t^2} + \frac{1}{1+t} - \frac{12}{5+2t} \\ &= \frac{(1-2t)(1+3t+5t^2)}{(2+t^2)(1+t)(5+2t)} = \frac{(1-2t)(1+t^2+(1+3t)^2)}{2(2+t^2)(1+t)(5+2t)} \\ &\geq 0\end{aligned}$$

with equality if and only if t $= 1/2$. With the given condition, this implies that equality occurs if and only if $x = y = \pm 1/\sqrt{2}$.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; STAN WAGON, Macalester College, St. Paul, MN, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA;

Wagon used mathematical software to find that, when $x^2+y^2=1$,

$$3 \leq \left(\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy}\right)\left(1 + \left(\frac{x+y}{2}\right)^2\right) \leq \frac{10}{3}$$

with equality on the left if and only if $x = y = \pm 1/\sqrt{2}$. and equality on the right if and only if $x = -y = \pm 1/\sqrt{2}$.

3673. [2011 : 390, 392] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the product

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)^{(-1)^{n-1}} .$$

I. Composite of solutions by Paul Bracken, University of Texas, Edinburg, TX, USA; Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; and AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $\pi^2/8$. Recall the Wallis formula:

$$\lim_{m\to\infty} \frac{1}{2m+1} \prod_{k=1}^{m} \frac{(2k)^2}{(2k-1)^2} = \frac{\pi}{2}.$$

For $n \geq 2$, let

$$P(n) = \prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)^{(-1)^{k-1}} .$$

Then

$$\begin{aligned}
P(2m) &= \prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)(2k+1)}\right]\prod_{k=1}^{m-1}\left[\frac{(2k)(2k+2)}{(2k+1)^2}\right]\\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\prod_{k=1}^{m-1}\left[\frac{(2k)^2}{(2k+1)^2}\right]\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\prod_{k=1}^{m-1}\left[\frac{2k+2}{2k+1}\right]\\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\left(\frac{2m}{2m-1}\right)\frac{1}{2}\frac{1}{(2m)^2}\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\\
&\qquad\qquad \times\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\left(\frac{2m}{2m-1}\right)\frac{1}{2}\frac{1}{(2m)^2}\left[\frac{(2m-1)^2}{(2m)^2}\right]\\
&\qquad\qquad \times\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\\
&= \frac{2m+1}{2(2m)}\left[\frac{1}{2m+1}\prod_{k=1}^{m}\frac{(2k)^2}{(2k-1)^2}\right]^2,
\end{aligned}$$

and

$$P(2m+1) = \frac{(2m)(2m+2)}{(2m+1)^2}P(m).$$

Hence

$$\begin{aligned}
\lim_{m\to\infty} P(2m+1) &= \lim_{m\to\infty}\frac{(2m)(2m+2)}{(2m+1)^2}\lim_{m\to\infty} P(2m)\\
&= \lim_{m\to\infty} P(2m) = \lim_{m\to\infty}\left[\frac{2m+1}{4m}\right]\frac{\pi^2}{4} = \frac{\pi^2}{8}.
\end{aligned}$$

II. Composite of solutions by Michel Bataille, Rouen, France; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Recall Stirling's formula:

$$\lim_{n\to\infty}\frac{(e^n)n!}{n^n\sqrt{2\pi n}} = 1.$$

Defining $P(n)$ as in Solution I, we have that

$$\begin{aligned} P(2m+1) &= \frac{1}{2}\cdot\frac{[2\cdot 4\cdot 6\cdots(2m)]^4(2m+2)(2m+1)}{[3\cdot 5\cdot 7\cdots(2m+1)]^4} \\ &= \frac{[2\cdot 4\cdots 6\cdots(2m)]^8(m+1)(2m+1)}{[(2m+1)!]^4} \\ &= \frac{2^{8m}(m+1)(m!)^8}{(2m+1)^3[(2m)!]^4}. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{m\to\infty} P(2m+1) &= \lim_{m\to\infty}\frac{2^{8m}(m+1)(2\pi m)^4 m^{8m}e^{-8m}}{(2m+1)^3(4\pi m)^2(2m)^{8m}e^{-8m}} \\ &= \lim_{m\to\infty}\frac{(m+1)m^2\pi^2}{(2m+1)^3} = \frac{\pi^2}{8} \end{aligned}$$

and

$$\lim_{m\to\infty} P(2m) = \lim_{m\to\infty}\frac{(2m+1)^2}{(2m)(2m+2)}P(2m+1) = \frac{\pi^2}{8}.$$

III. Composite of solution by Richard I. Hess, Rancho Palos Verdes, CA, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; Missouri State University Problem Solving Group, Springfield, MO; Skidmore College Problem Group, Saratoga Springs, NY; Albert Stadler, Herrliberg, Switzerland; and the proposer.

We use the infinite product representations for the sine and cosine functions:

$$\sin\pi x = \pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right);$$

$$\cos\pi x = (1-4x^2)\prod_{n=1}^{\infty}\left(1-\frac{4x^2}{(2n+1)^2}\right).$$

The product of the problem can be written as a fraction whose numerator is

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n+1)^2}\right) = \lim_{x\to\frac{1}{2}}\frac{\cos\pi x}{1-4x^2} = \lim_{x\to\frac{1}{2}}\frac{\pi\sin\pi x}{8x} = \frac{\pi}{4},$$

and whose denominator is

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n)^2}\right) = \frac{\sin\pi/2}{\pi/2} = \frac{2}{\pi}.$$

It follows from this that the answer is $\pi^2/8$.

There were variants on the third solution. Stadler and the Skidmore group avoided consideration of the cosine product by writing the product as a fraction with numerator $\prod(1-1/n^2)$ *and denominator* $\prod(1-1/4n^2)^2$. *Koukakis used the Wallis formula instead of the sine product to calculate the denominator.*

Stan Wagon generalized the product to

$$\prod_{n=2}^{\infty}\left(1-\frac{a}{n^2}\right)^{(-1)^n},$$

so that $a=1$ *gives the reciprocal of the given product. Using mathematical software, he then provided the answer for specific values of* a. *For example, when* $a=2$, *the product is* $-\sqrt{2}\tan(\pi/\sqrt{2})/\pi$; *when* $a=1/2$, *it is* $\sqrt{2}\tan(\pi/2\sqrt{2})/\pi$; *when* $a=4$, *it is* 0 *and when* $a=1/4$, *it is* $3/\pi$. *He concludes that, setting* $b=1/a$, *"indeed, there seems to be general formula here that looks like*

$$\frac{2(b-1)\tan(\pi/2\sqrt{b})}{\pi\sqrt{b}},$$

though I have not investigated the exact range of truth for it." When $a=b=1$, *we find through l'Hôpital's Rule that the limit as b tends to 1 is the expected* $8/\pi^2$.

Wagon's recourse to mathematical software raises two issues. Many of the Crux problems can be easily handled by machine, but are still worth posing when they can attract solutions that reveal underlying structure or when they draw attention to a particularly comely mathematical fact. The challenge is not to just solve the problem, but to do so in a way that is elegant, interesting or insightful. The efficiency of the software allows for experimentation that leads to conjectures not otherwise obtainable. This problem is a good example of these effects.

The proposer also asked for the values of the infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{(-1)^{n-1}}$$

and

$$\prod_{n=2}^{\infty}\left(\frac{n^2+1}{n^2-1}\right)^{(-1)^{n-1}}.$$

The value of the first is $\frac{\pi}{2}\tanh\frac{\pi}{2}$ *and of the second is* $\frac{2}{\pi}\tanh\frac{\pi}{2}$.

3674★. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let I denote the centre of the inscribed sphere of a tetrahedron $ABCD$ and let A_1, B_1, C_1, D_1 denote their symmetric points of point I about planes BCD, ACD, ABD, ABC respectively. Must the four lines AA_1, BB_1, CC_1, DD_1 be concurrent?

No solutions have been received. This problem remains open.

3675. [2011 : 390, 392] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle and let s be its semiperimeter. Let r and R denote its inradius and circumradius respectively. Prove that

$$6\ \le\ \sum_{\text{cyclic}}\frac{b(s-b)+c(s-c)}{a(s-a)}\ \le\ \frac{3R}{r}\,.$$

Solution by the proposer.

By regrouping the middle expression to $\sum_{\text{cyclic}}\left(\frac{a(s-a)}{b(s-b)}+\frac{b(s-b)}{a(s-a)}\right)$, the left-hand inequality follows immediately from the fact that $x+\frac{1}{x}\geq 2$ for positive x. The right-hand inequality rewrites as

$$\sum_{\text{cyclic}} bc(s-b)(s-c)\left[b(s-b)+c(s-c)\right]\leq \frac{3R}{r}\cdot abc\cdot(s-a)(s-b)(s-c). \quad (1)$$

Using Heron's formula and the fact that $rs=\frac{abc}{4R}$, the right-hand side of (1) becomes

$$\frac{3R}{r}\cdot abc\cdot\frac{(rs)^2}{s}=3R(abc)(rs)=\frac{3(abc)^2}{4}.$$

Now, since $(s-b)(s-c)\leq\left(\frac{s-b+s-c}{2}\right)^2=\frac{a^2}{4}$ for all cyclic permutations of a, b, and c, then the left-hand side L of (1) satisfies

$$L\leq\frac{abc}{4}\left(\sum_{\text{cyclic}} a\left[b(s-b)+c(s-c)\right]\right)=\frac{abc}{4}\left(\sum_{\text{cyclic}} ab(s-b+s-a)\right)$$
$$=\frac{abc}{4}(abc+abc+abc)=\frac{3(abc)^2}{4}.$$

It follows that (1) holds.

Note that equality (on either side) occurs if and only if the triangle is equilateral and that the result of the problem improves the classical inequality $R\geq 2r$.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and TITU ZVONARU, Comăneşti, Romania.

3677. [2011 : 454, 456] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n-1}(-1)^k\sin^n(k\pi/n)=\frac{(1+(-1)^n)n}{2^n}\cdot\cos\frac{n\pi}{2}\,.$$

Solution by Anastasios Kotronis, Athens, Greece.

[*Ed.: For the summation to make sense, we assume that* $n\geq 2$.]

Let $w=e^{\frac{i\pi}{n}}=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$. Then $w^n=\cos\pi=-1$, so for all $k=1,2,\ldots,n-1$ we have $w^{kn}=(-1)^k$ and $w^{2kn}=1$.

Since $\sin\frac{k\pi}{n} = \dfrac{w^k - w^{-k}}{2i}$, we have

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^k \sin^n\left(\frac{k\pi}{n}\right) &= \sum_{k=1}^{n-1}(-1)^k\left(\frac{w^k - w^{-k}}{2i}\right)^n \\
&= \sum_{k=1}^{n-1}\left(\frac{-i}{2}\right)^n w^{kn}(w^k - w^{-k})^n = \left(\frac{-i}{2}\right)^n \sum_{k=1}^{n-1}(w^{2k}-1)^n \\
&= \left(\frac{-i}{2}\right)^n \sum_{k=1}^{n-1}\left(\sum_{m=0}^{n}(-1)^m\binom{n}{m}w^{2km}\right) \\
&= \left(\frac{-i}{2}\right)^n \sum_{m=0}^{n}\left((-1)^m\binom{n}{m}\sum_{k=1}^{n-1}w^{2km}\right) \\
&= \left(\frac{-i}{2}\right)^n \left(\binom{n}{0}(n-1) + (-1)^n\binom{n}{n}(n-1)\right. \\
&\qquad\qquad \left. + \sum_{m=1}^{n-1}(-1)^m\binom{n}{m}\sum_{k=1}^{n-1}w^{2km}\right) \\
&= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1)\right. \\
&\qquad\qquad \left. + \sum_{m=1}^{n-1}(-1)^m\binom{n}{m}\left(\frac{w^{2mn}-w^{2m}}{w^{2m}-1}\right)\right) \\
&= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1)\right. \\
&\qquad\qquad \left. - \sum_{m=1}^{n-1}(-1)^m\binom{n}{m}\right),
\end{aligned}$$

since $w^{2mn} = 1$ for $m = 1, 2, \ldots, n-1$. Thus

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^k \sin^n\left(\frac{k\pi}{n}\right) &= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1)\right. \\
&\qquad\qquad \left. - \left(0 - (-1)^0\binom{n}{0} - (-1)^n\binom{n}{n}\right)\right) \\
&= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1) + 1 + (-1)^n\right) \\
&= \left(\frac{-i}{2}\right)^n (1+(-1)^n)\, n = \frac{1+(-1)^n}{2^n}\cos\frac{n\pi}{2}.
\end{aligned}$$

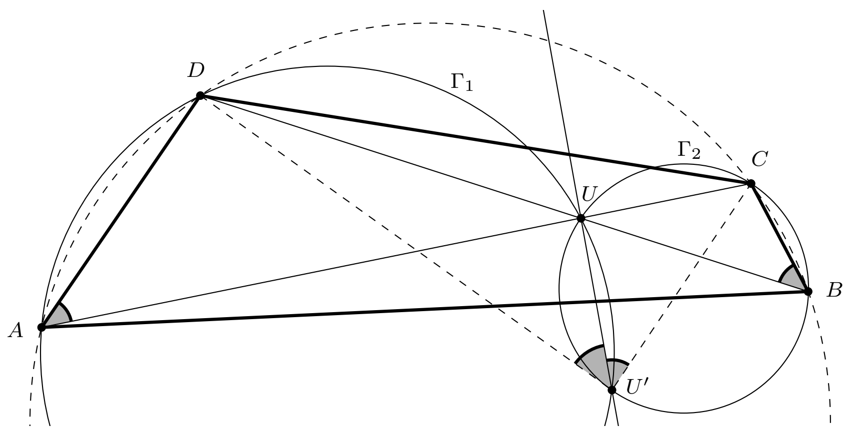
[*Ed.: If* n *is odd then* $\cos\frac{n\pi}{2} = 0 = 1 + (-1)^n$*; if* $n = 4k$*, then* $\cos\frac{n\pi}{2} = 1 = (-i)^n$*; and if* $n = 4k+2$*, then* $\cos\frac{n\pi}{2} = -1 = (-i)^2 = (-i)^n$.]

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; CALAH PAULHUS *and IRINA STALLION, students, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. All these solutions are similar to, or virtually the same as the one featured above.*

3678. [2011 : 454, 456] *Proposed by Michel Bataille, Rouen, France.*

Let Γ_1, Γ_2 be two intersecting circles and U one of their common points. Show that there exists infinitely many pairs of lines passing through U and meeting Γ_1 and Γ_2 in four concyclic points. Give a construction of such pairs.

Solution by John G. Heuver, Grande Prairie, AB.



Suppose the problem is solved and let $ABCD$ be the cyclic quadrilateral and let U' be the second point of intersection of the two circles. Consider diagonal AC passing through U with A on Γ_1 and C on Γ_2. Note that

$$\angle UU'C = \angle UBC = \angle DBC = \angle DAC = \angle DAU = \angle DU'U.$$

It follows that $\angle UU'C = \angle DU'U$. Hence, we can construct point D as the point of intersection of Γ_1 with the image of the ray $U'C$ in reflection in $U'U$. Then B is the intersection of DU with Γ_2. This construction produces cyclic quadrilateral $ABCD$, once AC through U is determined. Since diagonal AC through U was chosen arbitrarily, it follows that there are infinitely many cyclic quadrilaterals satisfying the condition of the problem.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ROY BARBARA and GEORGES MELKI, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; and the proposer.

3679. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$(a^2b+c)(b^2c+a)(c^2a+b) \;\leq\; 4(ab+bc+ca-abc)\,.$$

Ed.: It seems the proposer's solution was flawed and we received only one other solution. Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, submitted a computer assisted solution. Perfetti's solution transformed the original problem into one of showing that

$$\sum_{k=0}^{7} P_k(x,y)c^k \leq 0$$

where $x = a - c$, $y = b - c$ and each P_k is a homogeneous polynomial in x and y of degree $9 - k$. He then went on to show that each $P_k(x,y) \leq 0$ for any x, y, k.

As we have not received any other "nice" solutions it is possible that one does not exist. We will leave the problem open in case any ingenious **Crux** reader has a flash of insight.

3680. [2011 : 454, 457] *Proposed by Michel Bataille, Rouen, France.*

In a system of axes (Ox, Oy, Oz), let $U(1,1,1)$, $S(a,b,c)$ and $H(h_a, h_b, h_c)$ where a, b, c are the sides of a triangle ABC and h_a, h_b, h_c are the corresponding altitudes. Given that the lines OU and SH intersect at M such that $|HM| = \frac{1}{3}|HS|$, find the angles of ΔABC.

Solution by the proposer.

The angles of ΔABC are $\frac{\pi}{6}, \frac{\pi}{6}$, and $\frac{2\pi}{3}$. To see this we write $\overrightarrow{HM} = t\overrightarrow{HS}$ (where $|t| = \frac{1}{3}$) and obtain

$$M\left(ta + (1-t)h_a,\ tb + (1-t)h_b,\ tc + (1-t)h_c\right).$$

Furthermore, because M is on the line OU, we have

$$ta + (1-t)h_a = tb + (1-t)h_b = tc + (1-t)h_c. \tag{1}$$

Eliminating t from (1) yields $(h_b - h_c)(b - a + h_a - h_b) = (h_a - h_b)(c - b + h_b - h_c)$, whence

$$(h_b - h_c)(b - a) = (h_a - h_b)(c - b). \tag{2}$$

Since the area of ΔABC is $F = \dfrac{ah_a}{2} = \dfrac{bh_b}{2} = \dfrac{ch_c}{2}$, equation (2) easily reduces to

$$\frac{(c-b)(b-a)}{bc} = \frac{(b-a)(c-b)}{ab},$$

from which it follows that $(b-a)(a-c)(c-b) = 0$ and, thus, ΔABC is isosceles. Suppose that the triangle has been labeled so that $b = c$. Note that a can equal neither b nor c (otherwise the lines OU and SH would coincide), so that $h_a - h_b \neq 0$ and $b - a + h_a - h_b \neq 0$ (because $b - a$ and $h_a - h_b$ are either both positive or both negative). Now (1) gives

$$t = \frac{1}{1 + \dfrac{b-a}{h_a - h_b}} = \frac{1}{1 + \dfrac{b-a}{\frac{2F}{a} - \frac{2F}{b}}} = \frac{1}{1 + \dfrac{ab}{2F}} = \frac{1}{1 + \dfrac{1}{\sin C}}.$$

Thus $t > 0$, so that $t = \frac{1}{3}$ and, therefore, $\sin C = \frac{1}{2}$. Since a base angle C of an isosceles triangle cannot be obtuse, we conclude that $C = B = \frac{\pi}{6}$ and $a = \frac{2\pi}{3}$, as claimed.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA.