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EDITORIAL

Shawn Godin

Welcome to another issue of ***Crux Mathematicorum***! We hope that you are enjoying our new features. We have heard some favorable feedback about some of these new additions and we thank you for your input.

On the other hand, the first *Department Highlight* column was not well received. Readers felt that the column didn't really fit into the scope of **Crux**. As such, the column will be discontinued. We value all the feedback we receive as it helps us to deliver the best possible journal to you, our readers.

Due to some technical difficulties at work, last year was a trying time for me as editor of **Crux**. Unfortunately, with a non-ideal work environment, a few readers solutions were overlooked. I am truly sorry for the oversight. Equipment has now been repaired or replaced and things are settling down, I am hopeful that things will run smoother. I would like to recognize the following solutions that were received, yet never acknowledged: DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA (3660); G. R. A. 20 Problem Solving Group, Roma, Italy (3655); OLIVER GEUPEL, Brühl, NRW, Germany (3650, 3652, 3653, 3654, 3655, 3656, 3657, 3659, 3660) PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (3642, 3645, 3646, 3649); PETER Y. WOO, Biola University, La Mirada, CA, USA (3641, 3644, 3647, 3650, 3656, 3659, 3660).

You may notice that the deadlines for submitting solutions are a bit shorter this issue. In the past, we would give 6 months from the date of publication, which gave us enough time to receive them, assess them, typeset the featured solution, acknowledge the other correct solutions and provide any other comments. As **Crux** has fallen behind publication schedule, we will be working over the next while to get back on track. In order to do this, we need your solutions a little quicker. Hopefully this is still enough time to get the great solutions that we usually do.

Now, as I have finished my rambling, you may get on with the contents of this issue, enjoy! As always, we always welcome your feedback.

Shawn Godin

SKOLIAD No. 141

Lily Yen and Mogens Hansen

Skoliad has joined **Mathematical Mayhem** *which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time Skoliad will be discontinued in* **Crux**. *New Skoliad problems, and their solutions, will appear in* **Mathematical Mayhem** *when it is relaunched.*

In this issue we present the solutions to the Calgary Mathematical Association 35$^{\text{th}}$ Junior High School Mathematics Contest, Part B, 2011, given in Skoliad 135 at [2011 : 337–339].

1. Ariel purchased a certain amount of apricots. 90% of the apricot weight was water. She dried the apricots until just 60% of the apricot weight was water. 15 kg of water was lost in the process. What was the original weight of the apricots (in kg)?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let m be the original weight (in kg) of the apricots. Then, originally, the weight of the water was 90% of m, so $\frac{9}{10}m$, and the weight of the dry matter was $\frac{1}{10}m$.

After drying, the weight (in kg) of the apricots is $m - 15$, the weight of the water is $\frac{9}{10}m - 15$, and the weight of the dry matter is still $\frac{1}{10}m$. Moreover, the weight of the water is now 60% of the weight of the apricots, so

$$\frac{\frac{9}{10}m - 15}{m - 15} = 60\% = \frac{3}{5}\,.$$

Therefore, solving for m we get

$$\begin{aligned} 5\left(\frac{9}{10}m - 15\right) &= 3(m - 15) \\ \frac{9}{2}m - 75 &= 3m - 45 \\ \frac{9}{2}m - 3m &= -45 + 75 \\ \frac{3}{2}m &= 30 \\ m &= \frac{30}{3/2} = 20\,. \end{aligned}$$

Thus, the original weight of the apricots was 20 kg.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

2. A group of ten friends all went to a movie together. Another group of nine friends also went to the same movie together. Fourteen of these nineteen people each bought a regular bag of popcorn as well. It turned out that the total cost of the movie plus popcorn for one of the two groups was the same as for the other group. A movie ticket costs \$6. Find all possible costs of a regular bag of popcorn.

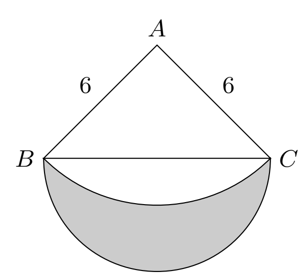
Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Since the larger group needs to buy just one ticket more than the smaller group, the price difference for tickets between the groups is \$6. Hence the smaller group paid \$6 more for popcorn than the larger group to compensate for the difference. Therefore the smaller group must have bought more than half of the 14 bags of popcorn, so at least 8 bags. On the other hand, they can buy at most one each, so at most 9 bags. The possibilities are now

Larger group	Smaller group	Difference	Price per bag
5 bags	9 bags	4 bags	$\frac{\$6}{4} = \$1.50$
6 bags	8 bags	2 bags	$\frac{\$6}{2} = \$3.00$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

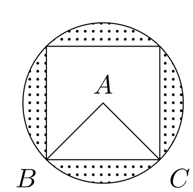
3. In the diagram, $|AB| = 6$, $|AC| = 6$, and $\angle BAC$ is a right angle. Two arcs are drawn: a circular arc with centre A and passing through B and C, and a semi-circle with diameter BC. What is the area of $\triangle ABC$? What is the length of BC? Find the area between the two arcs; that is, find the area of the shaded region in the diagram.



Solution by Justine Hansen, student, Burnaby North Secondary School, Burnaby, BC.

By the Pythagorean Theorem, $|BC|^2 = 6^2+6^2 = 72$, so $|BC| = \sqrt{72} = 6\sqrt{2}$. Therefore the radius of the semicircle is $3\sqrt{2}$, so the area of the semicircle is $\frac{1}{2}\pi(3\sqrt{2})^2 = \frac{1}{2}\pi \cdot 9 \cdot 2 = 9\pi$.

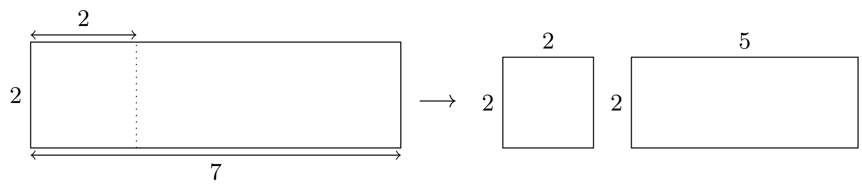
Ignore the shaded part of the figure for the moment and extend the remaining part by rotating it by right angles four times as in the figure on the right. Here the area of the circle is $\pi|AB|^2 = \pi 6^2 = 36\pi$, and the area of the square is $|BC|^2 = (6\sqrt{2})^2 = 72$, so the area of each of the four dotted regions is $\frac{1}{4}(36\pi - 72) = 9\pi - 18$.



Hence, the area of the shaded region in the original figure is the difference between the area of the semicircle in the original figure and the area of one of the dotted regions in the second figure, so $9\pi - (9\pi - 18) = 18$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

4. Given a non-square rectangle, a *square-cut* is a cutting-up of the rectangle into two pieces, a square and a rectangle (which may or may not be a square). For example, performing a square-cut on a 2×7 rectangle yields a 2×2 square and a 2×5 rectangle, as shown.

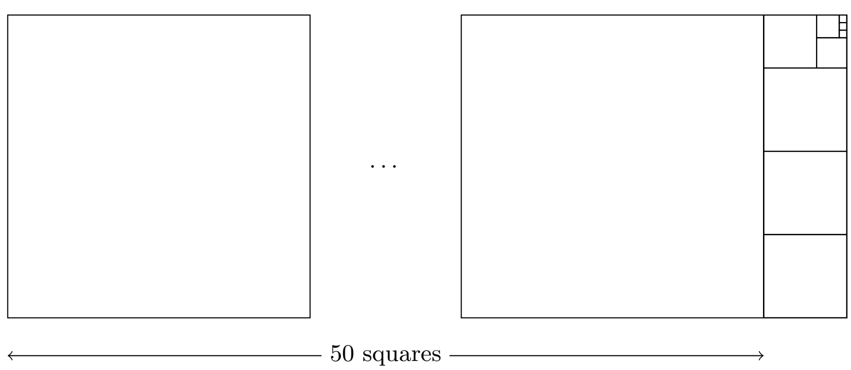


You are initially given a 40×2011 rectangle. At each stage, you make a square-cut on the non-square piece. You repeat this until all pieces are squares. How many square pieces are there at the end?

Solution by Justine Hansen, student, Burnaby North Secondary School, Burnaby, BC.

Since $2011 = 50 \cdot 40 + 11$, repeated square cuts slice the 40×2011 rectangle into fifty 40×40 squares and a 40×11 rectangle. Likewise, the 40×11 rectangle is sliced into three 11×11 squares and an 11×7 rectangle. The 11×7 rectangle is then sliced into a single 7×7 square and a 7×4 rectangle. The 7×4 rectangle is sliced into a 4×4 square and a 4×3 rectangle, which is sliced into a 3×3 square and a 3×1 rectangle. This last rectangle is finally sliced into three 1×1 squares.

In total, you now have $50 + 3 + 1 + 1 + 1 + 3 = 59$ squares.



Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

5. Five teams, A, B, C, D, and E, participate in a hockey tournament where each team plays against each other team exactly once. Each game either ends in a win for one team and a loss for the other, or ends in a tie for both teams. The table originally showed all of the results of the tournament, but some of the entries in the table have been erased. The result of each game played can be uniquely determined. For each of the ten games, determine who won or if it was a tie.

Team	Wins	Losses	Ties
A	3		
B	1		1
C	1		
D	0		
E			4

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

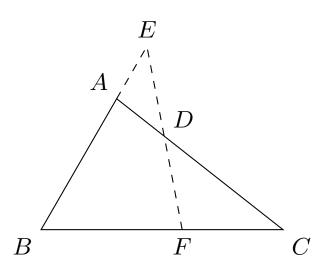
Each team played exactly four games.

- Since E has four ties, E simply tied all its games.
- Since A won three games and had one tie (against E), A won against each of B, C, and D.
- Team B has one win, one tie, and therefore two losses. Since B lost to A and tied with E, this means that one of B's games against C and D resulted in a win and the other in a loss.
- However, D has no wins, so B must have won against D and lost to C.
- Finally, C has one win (against B) and D has no wins, so C and D must have tied their game.

A won against B	A won against C	A won against D	A tied with E
	B lost to C	B won against D	B tied with E
		C tied with D	C tied with E
			D tied with E

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

6. A triangle ABC has sides $|AB| = 5$, $|AC| = 7$, and $|BC| = 8$. Point D is on side AC such that $|AB| = |CD|$. We extend the side BA past A to a point E such that $|AC| = |BE|$. Let the line ED intersect side BC at a point F.



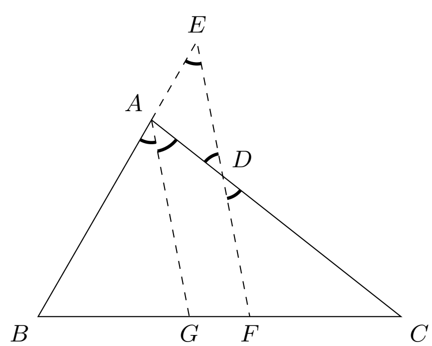
Find the lengths of AD, AE, BF, and FC.

Solution by Justine Hansen, student, Burnaby North Secondary School, Burnaby, BC.

Since $|CD| = |AB| = 5$ and $|AC| = 7$, $|AD| = |AC| - |CD| = 2$. Similarly, $|AE| = |BE| - |AB| = |AC| - |AB| = 2$. Therefore $\triangle DAE$ is isosceles and $\angle AED = \angle ADE = \angle CDF$. Moreover,

$$\angle BAC = 180^\circ - \angle DAE = \angle AED + \angle ADE = 2\angle AED\,.$$

Construct the angle bisector of $\angle BAC$, and let G be the intersection of the angle bisector and BC as in the figure. The argument above then shows that all the measures of the indicated angles in the figure are indeed equal. Hence $AG \parallel EF$.



Since AG and EF are parallel, $\triangle ABG \sim \triangle EBF$, so $|BG| : |GF| = |BA| : |AE| = 5 : 2$. Similarly, $\triangle CAG \sim \triangle CDF$, so $|GF| : |FC| = |AD| : |DC| = 2 : 5$. Thus $|BG| : |GF| : |FC| = 5 : 2 : 5$, so for some number a, $|BG| = 5a$, $|GF| = 2a$, and $|FC| = 5a$. Then $|BC| = |BG| + |GF| + |FC| = 12a$, but $|BC| = 8$, so $a = \frac{8}{12} = \frac{2}{3}$.

It follows that $|FC| = 5a = \frac{10}{3}$ and that $|BF| = |BG| + |GF| = 7a = \frac{14}{3}$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

The solution presented here is, perhaps, the most elementary. Our other two solvers used, respectively, Menelaus' Theorem (look it up on the internet) and Sine Law & Cosine Law.

This issue's prize for the best solutions goes to Justine Hansen, student, Burnaby North Secondary School, Burnaby, BC.

We hope that more readers will share their joy of mathematical problem solving by submitting their solutions to our featured contest to

`crux-skoliad@cms.math.ca`

or the postal address listed inside the back cover.

THE CONTEST CORNER

No. 7

Shawn Godin

The Contest Corner *is a new feature of* ***Crux Mathematicorum****. It will be filling the gap left by the movement of* Mathematical Mayhem *and* Skoliad *to a new on-line journal in 2013. The column can be thought of as a hybrid of* Skoliad, The Olympiad Corner *and the old* Academy Corner *from several years back. The problems featured will be from high school and undergraduate mathematics contests with readers invited to submit solutions. Readers' solutions will begin to appear in the next volume.*

Solutions can be sent to:

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or by email to

crux-contest@cms.math.ca.

The solutions to the problems are due to the editor by **1 January 2014**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of Université de Saint-Boniface for translating the problems from English into French.

CC31. Triangle ABC is right angled with its right angle at A. The points P and Q are on the hypotenuse BC such that $BP = PQ = QC$, $AP = 3$ and $AQ = 4$. Determine the length of each side of ΔABC.

CC32. Four boys and four girls each bring one gift to a Christmas gift exchange. On a sheet of paper, each boy randomly writes down the name of one girl, and each girl randomly writes down the name of one boy. At the same time, each person passes their gift to the person whose name is written on their sheet. Determine the probability that both of these events occur:

(i) Each person receives exactly one gift;

(ii) No two people exchanged presents with each other (i.e., if A gave his gift to B, then B did not give her gift to A).

CC33. The abundancy index $I(n)$ of a positive integer n is $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all of the positive divisors of n, including 1 and n itself. For example, $I(12) = \frac{1+2+3+4+6+12}{12} = \frac{7}{3}$. Determine, with justification, the smallest odd positive integer n such that $I(n) > 2$.

CC34. At the Mathville Dim Sum restaurant, all dishes come in three sizes: small, medium, and large. Small dishes cost \$$x$, medium dishes cost \$$y$, and large dishes cost \$$z$, where x, y, z are positive integers with $x < y < z$. At this restaurant, there is no tax on any dish and the prices haven't changed for a long time.

Margaret, Art and Edgar had dinner there last night, and together, they ordered 9 small dishes, 6 medium dishes and 8 large dishes. When the bill came, the following conversation ensued:

Margaret: "This bill is exactly twice as much as when I last came here."

Art: "This bill is exactly three times as much as when I last came here."

Edgar: "Oh, that was a delicious meal, and very reasonably priced too. Even if we give the waiter a 10% tip, the total is still less than \$100."

Determine the values of x, y and z, and prove that your answer is unique.

CC35. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}\,.$$

. . . .

CC31. Soit le triangle rectangle ABC avec l'angle droit à A. Les points P et Q se trouvent sur l'hypoténuse BC de façon à ce que $BP = PQ = QC$, $AP = 3$ et $AQ = 4$. Déterminer les longueurs des côtés de ΔABC.

CC32. Quatre garçons et quatre filles apportent chacun un cadeau lors de l'échange des cadeaux de Noël. Sur une feuille de papier, chaque garçon inscrit, au hasard, le nom d'une des filles, tandis que chaque fille inscrit, au hasard, le nom d'un des garçons. Déterminer la probabilité que les deux événements suivants aient lieu :

(i) chaque personne reçoit exactement un cadeau ;

(ii) aucun couple échange des cadeaux l'un avec l'autre (c'est-à-dire si A donne son cadeau à B, alors B ne peut pas donner son cadeau à A).

CC33. L'indice d'abondance $I(n)$ pour un entier positif n est défini comme étant $I(n) = \frac{\sigma(n)}{n}$, où $\sigma(n)$ est la somme de tous les diviseurs positifs de n, incluant 1 et n. Par exemple, $I(12) = \frac{1+2+3+4+6+12}{12} = \frac{7}{3}$. Déterminer, avec justification, la plus petite valeur entière positive impaire n telle que $I(n) > 2$.

CC34. Au restaurant Dim Sum des Matheux, toutes les entrées sont disponibles en trois tailles : petite, moyenne et grande. Les petites entrées coûtent x \$ chacune, les moyennes y \$, et les grandes z \$, où x, y, et z sont des entiers positifs tels que $x < y < z$. À ce restaurant, il n'y a aucune taxe et les prix n'ont pas changé depuis belle lurette.

Marguerite, Arthur et Edgar ont mangé là hier soir ; au total, ils ont commandé 9 petites entrées, 6 moyennes et 8 grandes. L'addition étant arrivée, la conversation se poursuit ainsi :

Marguerite : ≪ L'addition est exactement le double de celle ma dernière fois ici. ≫

Arthur : ≪ L'addition est exactement le triple de celle ma dernière fois ici. ≫

Edgar : ≪ Quel délicieux repas, à prix modique en plus ! Si nous laissions un pourboire de 10%, le total serait toujours inférieur à 100\$. ≫

Dterminer les valeurs de x, y et z, et démontrer que la réponse est unique.

CC35. Évaluer

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}\,.$$

THE OLYMPIAD CORNER

No. 305

Nicolae Strungaru

The solutions to the problems are due to the editor by **1 January 2014**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of Université de Saint-Boniface for translations of the problems.

OC91. Prove that no integer consisting of one 2, one 1 and the rest of digits 0 can be written either as the sum of two perfect squares or the sum of two perfect cubes.

OC92. Let $ABCD$ be a convex quadrilateral. Let P be the intersection of external bisectors of $\angle DAC$ and $\angle DBC$. Prove that $\angle APD = \angle BPC$ if and only if $AD + AC = BC + BD$.

OC93. For every positive integer n, determine the maximum number of edges a simple graph with n vertices can have if it contains no cycles of even length.

OC94. Let $x_1, x_2, \cdots, x_{25}$ be real numbers such that for all $1 \leq i \leq 25$ we have $0 \leq x_i \leq i$. Find the maximum value of

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{25}x_1x_2)\,.$$

OC95. Can we find three relatively prime integers a, b, c so that the square of each number is divisible by the sum of the other two?

. . . .

OC91. Démontrer qu'aucun entier dont la représentation décimale consiste d'un 2 et d'un 1, les autres positions étant 0, peut être écrit comme somme de deux carrés ou comme somme de deux cubes.

OC92. Soit $ABCD$ un quadrilatère convexe et soit P le point d'intersection des bissectrices externes des angles $\angle DAC$ et $\angle DBC$. Démontrer que $\angle APD = \angle BPC$ si et seulement si $AD + AC = BC + BD$.

OC93. Pour n un entier positif, déterminer le nombre maximum d'arêtes que peut avoir un graphe simple à n sommets si le graphe ne contient aucun cycle de longueur paire.

OC94. Soient $x_1, x_2, \cdots, x_{25}$ des nombres réels tels que pour tout $1 \le i \le 25$, on a $0 \le x_i \le i$. Déterminer la valeur maximale de

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{25}x_1x_2)\,.$$

OC95. Est-il possible de construire trois entiers relativement premiers, a, b, et c tels que le carré de chacun d'entre eux est divisible par la somme des deux autres ?

OLYMPIAD SOLUTIONS

OC31. Find all pairs (p, q) of prime numbers such that $pq \mid (5^p + 5^q)$.
(*Originally question #2 from the 2009 Chinese Mathematical Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the similar solutions of Manes and Zelator.

Note that if (p, q) is a solution so is (q, p) by symmetry. We show that the solutions are $(2, 3), (2, 5), (5, 5), (5, 313)$ and the pairs with swapped components. Because of this, we will seek only the solutions with $p \le q$.

We break the problem in few cases:

Case 1: $p = 2$. It is easy to see that $q = 2$ is not a solution in this case. Thus q must be an odd prime in this case. Then $2q \mid (25 + 5^q)$ and hence

$$25 + 5^q \equiv 0 \pmod{q}\ .$$

By Fermat's little theorem $5^q \equiv 5 \pmod{q}$. Thus

$$0 \equiv 25 + 5^q \equiv 25 + 5 \equiv 30 \pmod{q}\ ,$$

and hence $q \mid 30$. As q is an odd prime, we get $q \in \{3, 5\}$. It is straightforward to check that these are solutions, thus in this case we get $(2, 3)$ and $(2, 5)$ as solutions.

Case 2: $p = q$ is an odd prime. Then $p^2 \mid 2 \cdot 5^p \Rightarrow p \mid 2 \cdot 5^p$. Since p is an odd prime, we get $p = 5$, and it is easy to see that $p = q = 5$ is a solution. In this case we get the solution $(5, 5)$.

Case 3: $p < q$ are odd primes. We split this case in three subcases.

Subcase 3a: $p = 5$. Then

$$5^5 + 5^q \equiv 0 \pmod{q}\ .$$

Since $\gcd(5, q) = 1$ we can cancel 5 modulo q and thus

$$5^4 + 5^{q-1} \equiv 0 \pmod{q}\ .$$

By Fermat's little theorem we also get

$$0 \equiv 5^4 + 5^{q-1} \equiv 5^4 + 1 \equiv 626 \pmod{q} .$$

Thus $q \mid 626 = 2 \cdot 313$. As q is an odd prime, the only possible solution is $q = 313$. We check now that this is indeed a solution.

By Fermat's little theorem we have

$$5^4 + 5^{312} \equiv 5^4 + 1 \equiv 0 \pmod{313} ,$$

and hence

$$5^5 + 5^{313} \equiv 0 \pmod{5 \cdot 313} .$$

Thus $(5, 313)$ is the only solution in this subcase.

Subcase 3b: $q = 5$. As $p < 5$ is an odd prime we get that $p = 3$, and it is easy to check that $(3, 5)$ is not a solution. Thus there is no solution in this subcase.

Subcase 3c: $p \neq 5$ and $q \neq 5$. We show there is no solution in this subcase. Assume by contradiction that (p, q) is a solution where $p \neq q, p \neq 5, q \neq 5$, with p and q odd.

As $pq \mid 5^p + 5^q$ we get $p \mid 5^p + 5^q$ and $q \mid 5^p + 5^q$. Therefore, by Fermat's little theorem we get

$$5 + 5^q \equiv 0 \pmod{p} ,$$

$$5^p + 5 \equiv 0 \pmod{q} ,$$

hence

$$5^{q-1} \equiv -1 \pmod{p} ,$$

and

$$5^{p-1} \equiv -1 \pmod{q} .$$

Let e, f be the orders of 5 modulo p respectively q. Then

$$e \mid 2(q-1)\,;\; e \nmid (q-1)\,,$$

$$f \mid 2(p-1)\,;\; f \nmid (p-1)\,.$$

By Fermat's little theorem we also have

$$e \mid (p-1)\,;\; f|(q-1)\,.$$

Let a, b be the powers of 2 in $p-1$ respectively $q-1$. As $e \mid 2(q-1)$ but $e \nmid (q-1)$ we get $2^{b+1} \mid e \mid p-1$. Hence $b+1 \leq a$. Similarly, as $f \mid 2(p-1)$ but $f \nmid (p-1)$ we get $2^{a+1} \mid f \mid q-1$. Hence $a+1 \leq b$.

Thus

$$b + 2 \leq a + 1 \leq b\,,$$

a contradiction.

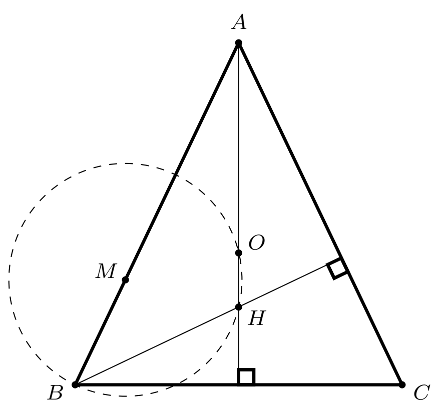
Since we reached a contradiction, our assumption is wrong, and hence there is no solution in this subcase.

Thus all the solutions are

$$\{(2,3),(3,2),(2,5),(5,2),(5,5),(5,313),(313,5)\}.$$

OC32. Let ABC be an acute-angled triangle with $\angle B = \angle C$. Let the circumcentre be O and the orthocentre be H. Prove that the centre of the circle BOH lies on the line AB.
(*Originally question #2 from the 2008/9 British Mathematical Olympiad, Round 2.*)

Solved by Michel Bataille, Rouen, France ; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain ; Oliver Geupel, Brühl, NRW, Germany ; Mihaï-Ioan Stoënescu, Bischwiller, France ; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.



Since AC is perpendicular to the altitude BH, we have

$$\angle ABH = 90^\circ - \angle A.$$

Let M be the centre of the circle BOH. Then

$$\angle MBH = 90^\circ - \frac{1}{2}\angle BMH = 90^\circ - \angle BOH.$$

Since $\angle B = \angle C$, it holds

$$\angle BOH = \frac{1}{2}\angle BOC = \angle A.$$

We deduce that

$$\angle ABH = \angle MBH,$$

that is, the points A, B, and M are collinear.

OC33. Let n and k be integers such that $n \geq k \geq 1$. There are n light bulbs placed in a circle. They are all turned off. Each turn, you can change the state of any set of k consecutive light bulbs.

How many of the 2^n possible combinations can be reached

(a) if k is an odd prime?

(b) if k is an odd integer?

(c) if k is an even integer?

(*Originally question #1 from 2009 Italian Team Selection Test.*)

No solution to this problem was received.

OC34. Let m, n be integers with $4 < m < n$, and $A_1A_2\cdots A_{2n+1}$ be a regular $2n+1$-gon. Let $P = \{A_1, A_2, \cdots, A_{2n+1}\}$. Find the number of convex m-gons with exactly two acute internal angles whose vertices are all in P.
(*Originally question #3 from the 2009 Chinese Mathematical Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We claim that the required number is $(2n+1)\left[\binom{n}{m-1}+\binom{n+1}{m-1}\right]$.

If $\angle A_iA_jA_k$ is acute then the arc A_iA_k of the circumcircle that contains the point A_j is more than a half of the circle. Hence, if a convex m-gon contains two acute angles, then their vertices are neighbouring vertices of the m-gon.

Let $\angle ABC$ and $\angle BCD$ be the two acute angles. Assume for the moment that $A = A_1$. Also suppose that the smaller arc AD contains k points from P in its interior. There are $\binom{k}{m-4}$ choices for $m-4$ points out of these k points. The longer arc AD (the one that contains the points B and C) contains $2n-1-k$ points from P in its interior. At least n points from P are between D and B on the arc that contains C. Hence, there are $n-1-k$ choices for B and, analogously, $n-1-k$ choices for C.

The required number of m-gons with $A = A_1$ is therefore

$$\sum_{k=1}^{n-2}(n-1-k)^2\binom{k}{m-4} = \sum_{j=1}^{n-2} j^2\binom{n-1-j}{m-4}.$$

Dropping the hypothesis $A = A_1$, we get

$$(2n+1)\left[\sum_{j=1}^{n-2} j^2\binom{n-1-j}{m-4}\right]$$

regular polygons. To complete the proof, we prove that

$$\sum_{j=1}^{n-2} j^2\binom{n-1-j}{m-4} = \binom{n}{m-1}+\binom{n+1}{m-1}. \qquad (1)$$

We prove it by mathematical induction. It holds for $n = m+1$, because

$$\sum_{j=1}^{m-1} j^2\binom{m-j}{m-4} = \binom{m-1}{3}+4\binom{m-2}{2}+9(m-3)+16 = \binom{m+1}{m-1}+\binom{m+2}{m-1}.$$

The relation (1) also holds for $m = 5$:

$$\begin{aligned}\sum_{j=1}^{n-2} j^2(n-1-j) &= \left[(n-1)\sum_{j=1}^{n-2} j^2\right] - \left[\sum_{j=1}^{n-2} j^3\right]\\ &= \frac{(n-2)(n-1)^2(2n-3)}{6} - \frac{(n-2)^2(n-1)^2}{4}\\ &= \frac{(n-2)(n-1)^2}{24}\left((8n-12)-(6n-12)\right)\\ &= \frac{2n(n-2)(n-1)^2}{24}\\ &= \frac{n(n-2)(n-1)}{24}(2n-2)\\ &= \frac{n(n-2)(n-1)}{24}\left((n-3)+(n+1)\right)\\ &= \binom{n}{4} + \binom{n+1}{4}.\end{aligned}$$

The induction step from $(m-1,\ n)$ and $(m,\ n)$ to $(m,\ n+1)$ is

$$\begin{aligned}\sum_{j=1}^{n-2} j^2\binom{n-j}{m-4} &= \sum_{j=1}^{n-2} j^2\binom{n-1-j}{m-4} + \sum_{j=1}^{n-2} j^2\binom{n-1-j}{(m-1)-4}\\ &= \binom{n}{m-1} + \binom{n+1}{m-1} + \binom{n}{m-2} + \binom{n+1}{m-2}\\ &= \binom{n+1}{m-1} + \binom{n+2}{m-1}\end{aligned}$$

This completes the induction and therefore the proof of (1).

OC35. Find all pairs of integers (x, y) such that

$$y^3 = 8x^6 + 2x^3y - y^2\,.$$

(*Originally question #3 from 2009 Italian Team Selection Test.*)

Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany ; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA and Titu Zvonaru, Comǎneşti, Romania. We give the solution of Manes.

We show that the only solutions are $(0,-1), (0,0)$ and $(1,2)$. It is easy to check that these are solutions, we show now that there is no other solution.

Solving the quadratic equation in x^3 we get

$$x^3 = \frac{-y \pm y\sqrt{9+8y}}{8}\,.$$

From here we get $y \geq -1$. If $y = -1$ then $x = 0$ while if $y = 0$ we get that $x = 0$.

Assume $y \geq 1$. Since x, y are integers, it follows that $\sqrt{9+8y}$ is rational thus integer. Moreover, as $9+8y$ is odd, $\sqrt{9+8y}$ must be an odd integer.

Write

$$\sqrt{9+8y} =: 2n+3\,.$$

Then $n \geq 1$,

$$y = \frac{n(n+3)}{2}\,,$$

and either

$$x^3 = \frac{n(n+1)(n+3)}{8}$$

or

$$x^3 = \frac{-n(n+2)(n+3)}{8}\,.$$

If $n = 1$ then $y = 2$ and $x = 1$, while for $n \geq 2$ we have

$$(n+1)^3 < n(n+1)(n+3) < (n+2)^3 \text{ and}$$

$$-(n+2)^3 < -n(n+2)(n+3) < -(n+1)^3\,.$$

Thus

$$\frac{n+1}{2} < \sqrt[3]{\frac{n(n+1)(n+3)}{8}} < \frac{n+2}{2}\,,$$

$$-\frac{n+2}{2} < \sqrt[3]{\frac{-n(n+2)(n+3)}{8}} < -\frac{n+1}{2}\,,$$

and hence x cannot be an integer. This shows that there are no solutions for $n \geq 2$.

BOOK REVIEWS

Amar Sodhi

Mathematics Galore! The First Five Years of the St. Mark's Institute of Mathematics by James Tanton
The Mathematical Association of America, 2012
ISBN: 978-0-88385-776-2, hardcover, xv+271 pages, US$50
Reviewed by ***Ed Barbeau****, University of Toronto, Toronto, ON*

This is not the first book with the title *Mathematics Galore!* that I have reviewed. The earlier one, by C. J. Budd and C. J. Sangwin, was British and based on master classes for secondary students sponsored by the Royal Institution of Great Britain. The book under review consists of newsletters sent out to students in the United States along with commentaries. Both books demonstrate that there is an abundance of really interesting and sometimes deep mathematics that is accessible to high school students.

Having achieved his doctorate at Princeton in 1994, James Tanton taught at the tertiary level before taking the plunge (as he put it) in 2004 to teach at the secondary level in St. Mark's School in Southbrook, MA. With a missionary's zeal to share the beauty of mathematics with the young, he founded the St. Mark's Institute of Mathematics. Currently, as the first recipient of the Mathematical Association of America Dolciani Visiting Mathematician Award, he is based at the MAA headquarters in Washington.

The author's enthusiasm, breadth of interest, background in higher mathematics and experience with adolescents have resulted in a collection of topics to engage amateurs, teachers and budding mathematicians. There are twenty-six newsletters. These generally open with a puzzle followed by interesting facts and research suggestions. Subsequent comments carry the discussion further, providing solutions, making connections, suggesting further investigations and describing known results. The reader is strongly encouraged to pause and work on his own. The topics, including such things as lattice polygons, fractions, Benford's Law, tilings, braids, partitions, numerate triangles and reflections, will likely be largely familiar to Crux readers who may nonetheless find something new. The only section that seemed artificial was the sixteenth on Lagrange's polynomial interpolation formula, but this is still useful for secondary students to know about.

Newsletter 12 illustrates the approach. It begins with three puzzlers:

(1) can there be an equilateral lattice triangle?

(2) what are possible dimensions of lattice squares?

(3) construct equilateral lattice polygons with particular even numbers of sides, and decide whether there is such a polygon with an odd number of sides.

The "tidbit" provides a proof that the area of a lattice polygon is either an integer or a half-integer and a statement of Pick's theorem. The puzzlers are solved in the commentary where references are supplied for Pick's theorem and the reader is challenged to prove that equiangular lattice polygons must have four or eight sides.

The appendices are quite substantial. Appendix I on the numbers expressible as the sums of two squares uses an ingenious parity argument by Benjamin and Zeilberger that involves continued fractions; it also touches on a bit of geometric number theory. Appendix II relates how a research group of middle and high school students handled Pick's theorem. Appendix III explores the relationship between the locker problem and the Möbius inversion formula. Appendix IV treats the Borsuk-Ulam and ham sandwich theorems. Appendix V shows how students devised "proofs without words" for the result

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11}$$

and generalizations. Appendix VI discusses a problem on the final state of candies possessed by children in a ring after a passing regime. Appendix VII looks at Buffon's needle problem when the needle is deformed into a closed geometric figure, and the eighth appendix shows how $2N$ red and $2N$ blue dots on either a plane or a sphere can be divided by a geodesic line so that there are equal numbers of dots of both colours on each side.

This book is useful for anyone looking for extracurricular material to present to students, as well as for students wishing to get involved in mathematical exploration. In particular, I strongly recommend it for secondary teachers of mathematics for their own professional development.

References

[1] C.J. Budd and C.J. Sangwin, *Mathematics Galore! Masterclasses, Workshops and Team Projects in Mathematics and its Applications.* Oxford University Press, 2001. ISBN 0198507690 (Hbk), 0198507704 (Pbk). Reviewed in *Notices of the American Mathematical Society* 49:8 (September, 2005), 905-910.

[2] Author's website: `http://www.jamestanton.com`.

[3] St. Mark's Institute: `http://www.stmarksschool.org/academics/mathematics/math-institute/index.aspx`.

FOCUS ON . . .

No. 3

Michel Bataille

From Linear Recurrences to a Polynomial Identity

Introduction

Linear recurrences of the form $U_{n+2} = xU_{n+1} - yU_n$ frequently turn up in problems. To determine the set $\mathcal{F}_{x,y}$ of all sequences $\{U_n\}$ satisfying the recurrence, the classical method rests on the so-called characteristic equation $q^2 - xq + y = 0$, whose solutions are the values of q for which the geometric sequence $\{q^n\}$ belongs to $\mathcal{F}_{x,y}$. Confronting the result with a more direct approach will lead us to a very general polynomial identity in $\mathbb{C}[X, Y, Z, T]$, a good starting point for solvers and posers.

The classical method and further

Let z be a complex number such that $z^2 = x^2 - 4y$. We will assume that $z \neq 0$. Then, the characteristic equation has two distinct solutions $q_1 = \frac{x+z}{2}$ and $q_2 = \frac{x-z}{2}$ and any sequence $\{U_n\}$ of $\mathcal{F}_{x,y}$ is given by $U_n = \lambda q_1^n + \mu q_2^n$. The initial conditions $U_0 = a,\ U_1 = b$ provide the values of λ and μ: $\lambda = \frac{b-aq_2}{z},\ \mu = \frac{aq_1-b}{z}$ and a short calculation readily yields that for $n \geq 2$,

$$U_n = \frac{1}{2^n z}\left[b\left((x+z)^n - (x-z)^n\right) - 2ay\left((x+z)^{n-1} - (x-z)^{n-1}\right)\right].$$

An obvious way to go further is to invoke the binomial theorem. First, we easily obtain

$$U_n = \frac{1}{2^n z}\left[2b\sum_{k\geq 0}\binom{n}{2k+1}x^{n-2k-1}z^{2k+1} - 4ay\sum_{k\geq 0}\binom{n-1}{2k+1}x^{n-2k-2}z^{2k+1}\right]$$

and then, recalling the relation $z^2 = x^2 - 4y$,

$$U_n = \frac{1}{2^n}\left[2b\sum_{k\geq 0}\binom{n}{2k+1}\,x^{n-2k-1}(x^2-4y)^k \right.$$
$$\left. -4ay\sum_{k\geq 0}\binom{n-1}{2k+1}x^{n-2k-2}(x^2-4y)^k\right]. \qquad (1)$$

A direct approach

Two remarks will lead us in another direction. First, because of the linearity of the recurrence, we only need to know the sequences $\{P_n\}$ and $\{Q_n\}$ of $\mathcal{F}_{x,y}$

with the initial values $P_0 = 1,\ P_1 = 0$ and $Q_0 = 0,\ Q_1 = 1$: the general sequence $\{U_n\}$ with $U_0 = a,\ U_1 = b$ is just $\{aP_n + bQ_n\}$. Second, it is actually sufficient to determine $\{Q_n\}$. Indeed, the sequence $\{-yQ_n\}$ satisfies $-yQ_0 = 0 = P_1$, $-yQ_1 = -y = P_2$ and, of course, is in $\mathcal{F}_{x,y}$. Hence, $\{-yQ_n\} = \{P_{n+1}\}$.

A general expression of Q_{n+1} for $n \geq 0$ is given by the following formula that is easily proved by induction using the well-known $\binom{n-k}{k} + \binom{n-k}{k-1} = \binom{n+1-k}{k}$:

$$Q_{n+1} = \sum_{k\geq 0}(-1)^k \binom{n-k}{k} x^{n-2k} y^k.$$

Taking the two previous remarks into account, we deduce the following expression for U_{n+1} $(n \geq 1)$:

$$U_{n+1} = \sum_{k\geq 0}(-1)^k \left(bx\binom{n-k}{k} - ay\binom{n-1-k}{k}\right) x^{n-2k-1} y^k. \qquad (2)$$

Conclusion

Comparing the results (1) and (2), we conclude that we have for all complex numbers x, y, a, b such that $x^2 - 4y \neq 0$ and for all positive integer n, $A_n(x, y, a, b) = B_n(x, y, a, b)$ where

$$A_n(X,Y,Z,T) = \sum_{k\geq 0}(-1)^k \left(XT\binom{n-k}{k} - YZ\binom{n-1-k}{k}\right) X^{n-2k-1} Y^k$$

and

$$B_n(X,Y,Z,T) = \frac{1}{2^n}\sum_{k\geq 0} \left(XT\binom{n+1}{2k+1} - 2YZ\binom{n}{2k+1}\right) X^{n-2k-1} (X^2-4Y)^k.$$

Note that the sums $\sum_{k\geq 0}$ can be replaced by $\sum_{k=0}^{\lfloor n/2 \rfloor}$.

The polynomials A_n and B_n take the same value on all quadruplets (x, y, a, b) of $\mathbb{C}^4$, except for the roots of the nonzero polynomial $X^2 - 4Y$. A consequence is the formal equality in $\mathbb{C}[X, Y, Z, T]$

$$A_n(X,Y,Z,T) = B_n(X,Y,Z,T), \qquad (3)$$

a beautiful polynomial identity.

In recent years, I used the above method to compose some problems. See for example problems 3430 [2009 : 173, 175 ; 2010 : 181] and 3217 [2007 : 110, 114 ; 2008 : 112]. For the reader's enjoyment, here is a result that can be obtained starting from (3).

For integers m, n such that $0 \leq m \leq n$, the following equality holds.

$$\sum_{j=0}^{m} \binom{n-j}{m-j}\binom{2n+1}{2j} = 2^{2m}\binom{m+n}{2m}.$$

(*Hint: consider first* $A_{2n}(1, -\frac{Y}{4}, 0, 1) = B_{2n}(1, -\frac{Y}{4}, 0, 1)$.)

PROBLEM OF THE MONTH

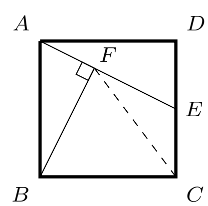
No. 2

Shawn Godin

This column is dedicated to the memory of former ***CRUX with MAYHEM*** *Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

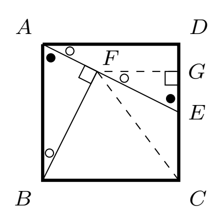
As this column is dedicated to the memory of Jim Totten, what can be more fitting than using one of his own problems? When Jim was a graduate student, he was assigned an office near Ross Honsberger's office. As a result, professor Honsberger often shared his "gems" with Jim. When Jim began teaching he wanted to share some great problems with his students, which he did through a "Problem of the Week". Jim continued this practice throughout his career. This issue's featured problem comes from the CMS ATOM series (**A** **T**aste **O**f **M**athematics), Volume VII, "Problems of the Week" by Jim Totten.

Given a square $ABCD$ with E the mid-point of side CD. Join A to E and drop a perpendicular from B to AE at F. Prove $CF = CD$.



What makes a problem beautiful? As with anything, beauty is in the eye of the beholder. I tend to be drawn to problems that submit to many different solutions, which is the case with this problem.

Solution #1: Since $ABCD$ is a square, then $\angle FAB$ and $\angle DAE$ are complementary, hence $\angle FAB = \angle AED$ and thus right angled triangles ΔFAB and ΔDEA are similar. So, if we let the side length of the square be s, then $DA = s$, $ED = \frac{s}{2}$ and $AE = \frac{\sqrt{5}s}{2}$ by the Pythagorean theorem. Thus, by similarity we have



$$\frac{FA}{DE} = \frac{AB}{EA} \Rightarrow \frac{2FA}{s} = \frac{2s}{\sqrt{5}s} \Rightarrow FA = \frac{s}{\sqrt{5}}.$$

Drop a perpendicular from F to CD at G. Now the homothetic triangles GFE and DAE are similar and $FE = \frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}$, hence

$$\frac{GF}{DA} = \frac{FE}{AE} \Rightarrow \frac{GF}{s} = \frac{\frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}}{\frac{\sqrt{5}s}{2}} \Rightarrow GF = \frac{3s}{5}$$

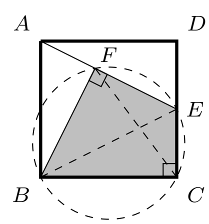
and similarly, $EG = \frac{3s}{10}$.

Finally, we can apply the Pythagorean theorem to ΔCGF, with $CG = \frac{s}{2} + \frac{3s}{10} = \frac{4s}{5}$ to get

$$\begin{aligned} CF^2 &= \left(\frac{3s}{5}\right)^2 + \left(\frac{4s}{5}\right)^2 \\ &= \left(\frac{9}{25} + \frac{16}{25}\right) s^2 \\ &= s^2 \end{aligned}$$

hence $CF = s = CD$ completing the proof.

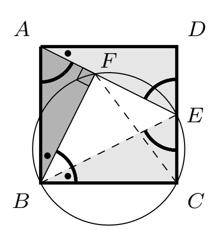
Solution #2: Since $\angle BFE = \angle ECB = 90°$, quadrilateral $BFEC$ is cyclic. As in solution # 1 we can determine $BC = s$, $CE = \frac{s}{2}$, $EF = \frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}$, $FB = \frac{2s}{\sqrt{5}}$ and $BE = \frac{\sqrt{5}s}{2}$. Thus, by Ptolemy's theorem we have



$$\begin{aligned} CF \cdot BE &= BC \cdot EF + CE \cdot FB \\ CF\left(\frac{\sqrt{5}s}{2}\right) &= s\left(\frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}\right) + \left(\frac{s}{2}\right)\left(\frac{2s}{\sqrt{5}}\right) \\ CF &= s - \frac{2}{5}s + \frac{2}{5}s, \end{aligned}$$

hence $CF = s = CD$.

Solution #3: Continuing with the setup to the last solution, note that right-triangles ΔADE and ΔBCE are congruent, and both similar to ΔBFA. Thus,



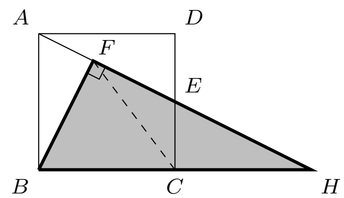
$$\angle AED = \angle BEC = \angle BAF.$$

Since $\angle ABC$ is a right angle, angle $\angle ABF$ and $\angle FBC$ are complements, hence

$$\angle AED = \angle BEC = \angle BAF = \angle FBC.$$

Thus as $\angle BEC = \angle FBC$ are equal inscribed angles, then the chords they subtend are equal, that is $CF = BC$. Thus, since $ABCD$ is a square $CD = BC = CF$ and we are done.

Solution #4: A reflection in the point E takes ΔAED to ΔHEC, where H is the point where the extension of AE meets the line BC.

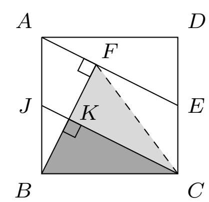


Now ΔBFH is a right angled triangle so we can inscribe it in a circle with BH as a diameter. Hence, the radius of the circle is s and $CF = BC = CH = s$, thus $CF = CD$.

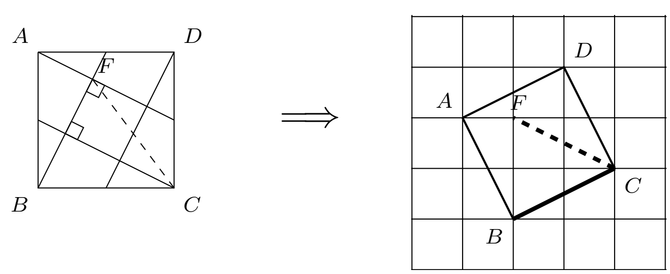
Solution #5: Let J be the midpoint of AB and let CJ meet BF at K. Then CJ and EA are parallel and thus BF and CJ are perpendicular and

$$\frac{BK}{KF} = \frac{BJ}{JA} = 1,$$

whence $BK = KF$. Now we have enough information to see that triangles ΔCKF and ΔCKB are congruent yielding $CF = BC$



Solution #6: If we continue with the idea of the last solution, we see that if we start with a square $ABCD$ and connect A to the midpoint of CD, B to the midpoint of AD, C to the midpoint of AB and D to the midpoint of BC, then pairs of these segments intersect at the corner of a smaller square. It doesn't take much to see that we can create a "proof without words" creating a lattice of the little squares as in the diagrams below.



The obvious moral of the story is that when you have "solved" a problem, it is often worthwhile to take a second look. In some cases, like this month's problem, a variety of solutions might be possible, some being more elegant or insightful or even more beautiful than others. Then again, beauty is in the eye of the beholder.

On a problem in divisibility

Amitabha Tripathi

The following problem appeared at the first level of the Indian Mathematical Olympiad in December 2012.

> Let a, b, c be positive integers such that a divides b^5, b divides c^5 and c divides a^5. Prove that abc divides $(a+b+c)^{31}$.

A first step towards solving this problem is to note that a, b, c have the same set of prime divisors. This includes the possibility $a = b = c = 1$. For prime p and positive integer n, let $\mathsf{e}_p(n)$ denote the highest power of p that divides n. For each common prime divisor p of a, b, c, write

$$\mathsf{e}_p(a) = \alpha, \quad \mathsf{e}_p(b) = \beta, \quad \mathsf{e}_p(c) = \gamma.$$

To complete the proof, one can show that

$$\mathsf{e}_p(abc) = \alpha + \beta + \gamma \leq (1 + 5 + 5^2)\min\{\alpha, \beta, \gamma\} \leq 31 \cdot \mathsf{e}_p(a+b+c).$$

There is another method to approach this problem. If one expands $(a+b+c)^{31}$, the only terms that do not involve all three of a, b, c are terms of the type $a^r b^{31-r}$ with $r \in \{0, 1, 2, \ldots, 31\}$, together with those obtained by replacing a, b with a, c and with b, c. The divisibility conditions now show that abc divides each of these exceptional terms.

The purpose of this note is to extend this problem to any number of variables, with the divisibility condition extended to any uniform positive integer power. We have the following theorem.

Theorem 1 *For any collection $a_1, a_2, \ldots, a_k, n$ of positive integers, with $k > 1$, that satisfy*

$$a_1 \mid a_2^n, \;\; a_2 \mid a_3^n, \;\; \ldots \;\; , \;\; a_{k-1} \mid a_k^n, \;\; a_k \mid a_1^n, \tag{1}$$

the least positive integer N such that

$$a_1 \cdot a_2 \cdots a_k \text{ divides } (a_1 + a_2 + \cdots + a_k)^N \tag{2}$$

is given by $N_0 = 1 + n + n^2 + \cdots + n^{k-1}$.

Proof. We treat the cases $n = 1$ and $n > 1$ separately.

1.8Case I: ($n = 1$) Suppose $a_1, a_2, \ldots, a_k$ is a collection of positive integers that satisfy (1) with $n = 1$. Then $a_1 \leq a_2 \leq \ldots \leq a_k \leq a_1$. Hence $a_i = a$ for each $i \in \{1, 2, \ldots, k\}$, and (2) reduces to the condition $a^k \mid (ka)^N$. This condition is met for all a if $N \geq k$, but not met if $\gcd(a, k) = 1$ for $N < k$. Hence $N_0 = k$ when $n = 1$.

1.8Case II: $(n > 1)$ Suppose $a_1, a_2, \ldots, a_k$ is a collection of positive integers that satisfy (1) with $n > 1$. To show that (2) holds for some positive integer N, we must show that

$$\mathsf{e}_p\Big(a_1 \cdot a_2 \cdots a_k\Big) \le N \cdot \mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big) \tag{3}$$

for every prime p. Fix a prime p and write $\mathsf{e}_p(a_i) = e_i$ for $i \in \{1, 2, \ldots, k\}$. Since $a_i \mid a_{i+1}^n$ for $i \in \{1, 2, \ldots, k\}$ (where $a_{k+1} = a_1$), we have

$$e_i = \mathsf{e}_p(a_i) \le \mathsf{e}_p\Big(a_{i+1}^n\Big) = n \cdot e_{i+1}$$

for $i \in \{1, 2, \ldots, k\}$. It follows that

$$e_i \le n^{k-i} \cdot e_k$$

for $i \in \{1, 2, \ldots, k\}$, from which we have

$$e_1 + e_2 + \cdots + e_k \le (n^{k-1} + n^{k-2} + \cdots + n + 1)\, e_k.$$

By symmetry, we must also have

$$e_1 + e_2 + \cdots + e_k \le (n^{k-1} + n^{k-2} + \cdots + n + 1)\, e_i$$

for each $i \in \{1, 2, \ldots, k\}$. Hence

$$e_1 + e_2 + \cdots + e_k \le (n^{k-1} + n^{k-2} + \cdots + n + 1) \cdot \min\{e_1, e_2, \ldots, e_k\}. \tag{4}$$

This establishes (3) with $N = 1 + n + n^2 + \cdots + n^{k-1}$, since $\mathsf{e}_p\Big(a_1 \cdot a_2 \cdots a_k\Big) = e_1 + e_2 + \cdots + e_k$ and $\mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big) \ge \min\{e_1, e_2, \ldots, e_k\}$. Hence, we have both the existence of the positive integer N in (2) as well as the inequality $N_0 \le 1 + n + n^2 + \cdots + n^{k-1}$.

For the reverse inequality, consider the collection of positive integers p, p^n, p^{n^2}, $\ldots, p^{n^{k-1}}$ where p is any prime. If we take $a_i = p^{n^{k-i}}$, then $a_{i+1}^n = \Big(p^{n^{k-i-1}}\Big)^n = p^{n^{k-i}} = a_i$, thereby meeting the requirement in (1). Now

$$\begin{aligned}
&\mathsf{e}_p\Big(a_1 \cdot a_2 \cdots a_k\Big) = e_1 + e_2 + \cdots + e_k = 1 + n + n^2 + \cdots + n^{k-1},\\
&\mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big) = 1
\end{aligned}$$

imply $N_0 \ge 1 + n + n^2 + \cdots + n^{k-1}$. Therefore $N_0 = 1 + n + n^2 + \cdots + n^{k-1}$ when $n > 1$. ■

Theorem 1 gives the least positive integer N_0 for which (2) holds for each sequence of positive integers satisfying (1). Therefore there must exist at least one sequence of positive integers satisfying (1) for which (2) fails to hold when $N = N_0 - 1$. We follow the argument in Theorem 1 to characterize such extremal sequences.

For $n = 1$, (1) implies $a_i = a$ for $i \in \{1, 2, \ldots, k\}$. So (2) fails to hold for the collection $a_1, a_2, \ldots, a_k$ with $N = N_0 - 1 = k - 1$ is equivalent to the condition $a^k \nmid (ka)^{k-1}$. Thus the only such collections are those in which each term equals a positive integer a, where $a \nmid k^{k-1}$.

Let $n > 1$. Suppose the sequence $a_1, a_2, \ldots, a_k$ satisfies (1) and is such that

$$a_1 \cdot a_2 \cdots a_k \text{ does not divide } (a_1 + a_2 + \cdots + a_k)^{N_0 - 1}.$$

Then there must exist a prime p for which

$$\mathsf{e}_p\Big(a_1 \cdot a_2 \cdots a_k\Big) = e_1 + e_2 + \cdots + e_k > (N_0 - 1) \cdot \mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big). \quad (5)$$

On the other hand, every sequence $a_1, a_2, \ldots, a_k$ that satisfies (1) also satisfies (4) for each prime p. Combining (4) and (5) gives

$$(N_0 - 1) \cdot \mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big) < e_1 + e_2 + \cdots + e_k \le N_0 \cdot \min\{e_1, e_2, \ldots, e_k\} \quad (6)$$

for at least one prime p. This characterizes extremal sequences $a_1, a_2, \ldots, a_k$ when $n > 1$.

When the set $\{e_1, e_2, \ldots, e_k\}$ has a *unique* least element, $\mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big) = \min\{e_1, e_2, \ldots, e_k\}$. This may also hold when the set $\{e_1, e_2, \ldots, e_k\}$ does not have a unique least element, but the equality is not guaranteed. In case of the unique least element, and consequently of the equality $\mathsf{e}_p\Big(a_1 + a_2 + \cdots + a_k\Big) = \min\{e_1, e_2, \ldots, e_k\} = e$, (6) reduces to

$$e(N_0 - 1) < e_1 + e_2 + \cdots + e_k \le eN_0.$$

When $e = 1$, this further reduces to $e_1 + e_2 + \cdots + e_k = N_0$, as is the case with the collection $p, p^n, p^{n^2}, \ldots, p^{n^{k-1}}$ for prime p.

References

[1] David M. Burton, *Number Theory*, Cambridge University Press, 1994.

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PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 January 2014****. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3757. *Correction. Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let A, B, C be the angles (measured in radians), R the circumradius and r the inradius of a triangle. Prove that

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\le\frac{9}{2\pi}\cdot\frac{R}{r}.$$

3761. *Proposed by Peter Saltzman, Berkeley, CA, USA; and Stan Wagon, Macalester College, St. Paul, MN, USA.*

Let $B_{m,n}$ be a graph of possible moves by a white bishop on an $m\times n$ chessboard, where we assume $m\le n$ and that the lower-left square is white.

(a) For which pairs of positive integers (m,n) does $B_{m,n}$ have a Hamiltonian cycle?

(b) Show that the edges of $B_{m,n}$ can be coloured using Δ colours so that intersecting edges are coloured differently, where Δ is the maximum degree.

3762. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Three sides of a cyclic quadrilateral $ABCD$ have lengths $AB=1$, $BC=2$ and $CD=3$, and one of the angles of the quadrilateral equals 60°. Find all possible lengths of AD.

3763. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a,b,c be positive real numbers. Prove that

$$\frac{a}{2a+b+c}+\frac{b}{2b+c+a}+\frac{c}{2c+a+b}\le\frac{a}{2b+2c}+\frac{b}{2c+2a}+\frac{c}{2a+2b}\,.$$

3764. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania; and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1} - a_n}{n} = a \in \mathbb{R}^+$. Compute

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n} \right),$$

where $a_1! = a_1$ and $a_n! = a_n \cdot a_{n-1}!$ for $n > 1$.

3765. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with circumcircle Γ and orthocentre H and let the circle with diameter AH intersect Γ again at K. Prove that

(a) $KB \cdot HC = KC \cdot HB$.

(b) the lines KB, HC meet on the circle tangent to Γ at K and passing through H.

3766. *Proposed by Max A. Alekseyev, University of South Carolina, Columbia, SC, USA.*

Let $x_1 < x_2 < \cdots < x_n$ be positive integers such that

$$\left(\sum_{k=1}^{n} x_k \right)^2 = \sum_{k=1}^{n} x^3 .$$

Prove that $x_k = k$ for each $k = 1, 2, \ldots, n$.

3767. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let R, r be the circumradius and inradius of a right-angled triangle. Prove that

$$\frac{R}{r} + \frac{r}{R} \geq 2\sqrt{2} .$$

3768★. *Proposed by Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

In the equilateral triangle ABC, E and D lie on side AC such that $\angle EBD = 30^\circ$, $AE = x$, $ED = y$ and $DC = z$. Show that

$$y^2 = (x+z)^2 - xz .$$

3769. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b, and c be the sides, r the inradius and R the circumradius of a triangle ABC. Prove that

$$\frac{a^3 c}{a^2 + ab + b^2} + \frac{b^3 a}{b^2 + bc + c^2} + \frac{c^3 b}{c^2 + ca + a^2} \geq 6rR .$$

3770. *Proposed by William Gosnell, Amherst, MA, USA.*

Given a right-angled triangle with legs a, b and hypotenuse c. Assume that the square of the hypotenuse is equal to twice the triangle's area plus its perimeter. Also assume that $c - a = 1$. Find a, b and c in terms of $\varphi = \dfrac{1+\sqrt{5}}{2}$.

. . . .

3757. *Correction. Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Dénotons par A, B et C les angles d'un triangle, mesurés en radians, par R le rayon de son cercle circonscrit et par r le rayon de son cercle inscrit. Démontrer

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{9}{2\pi} \cdot \frac{R}{r}.$$

3761. *Proposé par Peter Saltzman, Berkeley, CA, É-U ; et Stan Wagon, Macalester College, St. Paul, MN, É-U.*

Soit $B_{m,n}$ un graphe des mouvements possibles du fou blanc sur un échiquier de dimensions $m \times n$, en supposant que $m \leq n$ et que la case inférieure gauche est blanche.

(a) Pour quelles paires d'entiers positifs (m, n) le graphe $B_{m,n}$ possède-t-il un cycle Hamiltonien ?

(b) Montrer que les arêtes de $B_{m,n}$ peuvent être colorées avec Δ couleurs de sorte que les arêtes qui se coupent sont de couleur différente, Δ étant le degré maximal.

3762. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Trois côtés d'un quadrilatère cyclique $ABCD$ sont de longueur $AB = 1$, $BC = 2$ et $CD = 3$, et un des angles du quadrilatère vaut 60°. Trouver toutes les longueurs possibles de AD.

3763. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c trois nombres réels positifs. Montrer que

$$\frac{a}{2a+b+c} + \frac{b}{2b+c+a} + \frac{c}{2c+a+b} \leq \frac{a}{2b+2c} + \frac{b}{2c+2a} + \frac{c}{2a+2b}\,.$$

3764. *Proposé par D. M. Bătineţu-Giurgiu, Collège National Matei Basarab, Bucarest, Roumanie ; et Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit $(a_n)_{n\geq 1}$ une suite réelle positive telle que $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{n} = a \in \mathbb{R}^+$. Calculer

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n} \right),$$

où $a_1! = a_1$ et $a_n! = a_n \cdot a_{n-1}!$ pour $n > 1$.

3765. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle, Γ son cercle circonscrit, H son orthocentre et soit K la deuxième intersection du cercle de diamètre AH avec Γ. Montrer que

(a) $KB \cdot HC = KC \cdot HB$.

(b) les droites KB, HC se coupent sur le cercle tangent à Γ en K et passant par H.

3766. *Proposé par Max A. Alekseyev, Université de la Caroline Sud, Columbia, SC, USA.*

Soit $x_1 < x_2 < \cdots < x_n$ des entiers positifs tels que

$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x^3 .$$

Montrer que $x_k = k$ pour chaque $k = 1, 2, \ldots, n$.

3767. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit respectivement R, r les rayons des cercles circonscrit et inscrit d'un triangle rectangle. Montrer que

$$\frac{R}{r} + \frac{r}{R} \geq 2\sqrt{2} .$$

3768$\star$. *Proposé par Abdilkadir Altintaş, enseignant en mathématiques, Emirdağ, Turquie.*

Dans le triangle équilatéral ABC, on suppose que les points E et D situés sur le côté AC de sorte que $\angle EBD = 30^\circ$, $AE = x$, $ED = y$ et $DC = z$. Montrer que

$$y^2 = (x+z)^2 - xz .$$

3769. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit a, b, et c les côtés, r le rayon du cercle inscrit et R celui du cercle circonscrit d'un triangle ABC. Montrer que

$$\frac{a^3c}{a^2+ab+b^2} + \frac{b^3a}{b^2+bc+c^2} + \frac{c^3b}{c^2+ca+a^2} \geq 6rR .$$

3770. *Proposé par William Gosnell, Amherst, MA, USA.*

On donne un triangle rectangle de côtés a et b et d'hypoténuse c. On suppose que le carré de l'hypoténuse est égal au double de l'aire du triangle plus son périmètre. De plus, on suppose que $c - a = 1$. Trouver a, b et c en fonction de $\varphi = \dfrac{1+\sqrt{5}}{2}$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Due to a filing error, a number of readers' solutions got misplaced and were never acknowledged. Several others were late or misnumbered. The following solutions were received by the editor-in-chief : DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA (3660) ; G. R. A. 20 Problem Solving Group, Roma, Italy (3655) ; OLIVER GEUPEL, Brühl, NRW, Germany (3650, 3652, 3653, 3654, 3655, 3656, 3657, 3659, 3660) PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (3642, 3645, 3646, 3649) ; PETER Y. WOO, Biola University, La Mirada, CA, USA (3641, 3644, 3647, 3650, 3656, 3659, 3660). The editor apologizes sincerely for the oversight.

3661. [2011 : 320, 322] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Consider a triangle ABC with the midpoints D, E, F of its sides BC, CA, AB. For an arbitrary point P, let X, Y, Z be the reflections of P in D, E, F respectively. Show that the lines AX, BY, CZ are concurrent.

Solution by Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA.

Assume a system of coordinates with the origin at P. Then

$$D = \frac{B+C}{2}, \quad E = \frac{A+C}{2}, \quad F = \frac{A+B}{2}.$$

Thus

$$X = 2D = B + C, \quad Y = 2E = A + C, \quad Z = 2F = A + B.$$

Since the point $K = \frac{A+B+C}{2}$ is the midpoint of the segments AX, BY, CZ, the concurrency is proven.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain ; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia ; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina ; MICHEL BATAILLE, Rouen, France ; MARIAN DINCĂ, Bucharest, Romania ; JOHN G. HEUVER, Grande Prairie, AB ; VÁCLAV KONEČNÝ, Big Rapids, MI, USA ; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; TITU ZVONARU, Cománeşti, Romania ; and the proposer.

3662. [2011 : 320, 322] *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{R}$ denote the set of positive integers whose base ten expression is a single repeated digit (e.g. $5 \in \mathcal{R}$, $222 \in \mathcal{R}$, $88888 \in \mathcal{R}$). Let $T(n) = (n-2)^2 + n^2 + (n+2)^2$ where n is a non-negative integer.

(a) Find all even integers n such that $T(n) \in \mathcal{R}$.

(b) Find one odd integer $n > 1$ such that $T(n) \in \mathcal{R}$. Extra credit will be given to anyone who finds more than one odd integer $n > 1$ such that $T(n) \in \mathcal{R}$.

(a) Composite of many of the submitted solutions.

Since $T(n) = 3n^2 + 8$, we need to find triples (n, k, r) with n an integer, r a positive integer and k a nonzero digit for which

$$3n^2 + 8 = k(1 + 10 + 10^2 + \cdots + 10^{r-1}).$$

When $r = 1$, the only possibility is $T(0) = 8$. When $r = 2$, then $3n^2 + 8 \equiv 0 \pmod{11}$ and the only possibiity is $T(\pm 1) = 11$.

Suppose that $r \geq 3$. Then $3(n^2 + 1) \equiv k \pmod 5$ and $3n^2 \equiv 7k \pmod 8$. The only values of k admitting both congruences are 5 and 8. But we can reject $k = 8$, as it would lead to $3n^2 = 80(1 + 10 + \cdots + 10^{r-2})$. This is not possible as 5 divides the right member to the first power and must divide the left member to an even power.

Hence $k = 5$ and

$$0 \equiv 3(n^2 + 1) = 50(1 + 10 + \cdots + 10^{r-2}) \equiv 2(r - 1)$$

modulo 3, so that $r = 3s + 1$ for some positive integer s. When $s = 1$, we find that $3n^2 + 8 = 5555$, which leads to $n = 43$. Therefore, five solutions to the problem are $n = 0, \pm 1, \pm 43$.

Solution to (b) by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Continuing the foregoing solution, let $s \geq 2$. Then

$$3n^2 + 8 = 5555 + (10^4)(555)(1 + 10^3 + \cdots + 10^{3(s-2)}),$$

whence

$$\begin{aligned} n^2 + 1 &= 1850 + (10^4)(185)(1 + 10^3 + \cdots + 10^{3(s-2)}) \\ &= (10)(185)(1 + 10^3 + \cdots + 10^{3(s-1)}) \\ &= \frac{(10)(185)(10^{3s} - 1)}{999} = \frac{2(5^2)(37)(10^s - 1)(10^{2s} + 10^s + 1)}{(27)(37)} \\ &= \frac{2(5^2)(10^s - 1)(10^{2s} + 10^s + 1)}{3^3}. \end{aligned}$$

Since $10^{2s} + 10^s + 1$ is congruent to 1 modulo 4 and is divisible by 3, but not by 9, it must be divisible to an odd power by a prime q exceeding 3 that is congruent to 3 modulo 4. This prime does not divide $10^s - 1$, and so it must divide $n^2 + 1$ to an odd power. But this contradicts the criterion for a number to be the sum of two squares. Thus, there are no further solutions to the problem other than the five given.

Part (a) was also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHARLES DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; EDMUND SWYLAN, Riga, Latvia; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Solvers used many strategies to narrow down the field, as exemplified in the featured solution. In particular, Hess and Yiu looked at the equation modulo 100. For $r \geq 2$, $(3n)^2 \equiv 33k - 24$; it can be checked that the right member is a quadratic residue (modulo 100) only if $k = 1$ or $k = 5$. The case $r = 2$ is quickly disposed of. When $r \geq 3$ and $k = 1$, then $3n^2 \equiv 111 \equiv 7 \pmod 8$, which is impossible. There was one incomplete solution.

Part (b) was solved only by Yiu.

3663. [2011 : 320, 322] *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[3]{\frac{2a}{4a+4b+c}} + \sqrt[3]{\frac{2b}{4b+4c+a}} + \sqrt[3]{\frac{2c}{4c+4a+b}} < 2\,.$$

I. Solution by the proposer, modified slightly by the editor.

We first establish the following lemma :

Lemma 1 *If x and y are positive real numbers, then*

$$\sqrt[3]{4(x+y)} \geq \sqrt[3]{x} + \sqrt[3]{y}$$

with equality if and only if $x = y$.

Proof. Since $4(x^3+y^3) - (x+y)^3 = 3(x^3 - x^2y - xy^2 + y^3) = 3(x-y)(x^2-y^2) = 3(x-y)^2(x+y) \geq 0$, we have $4(x^3+y^3) \geq (x+y)^3$ or $\sqrt[3]{4(x^3+y^3)} \geq x+y$. Replacing x and y with $\sqrt[3]{x}$ and $\sqrt[3]{y}$, respectively, the lemma follows.

The given inequality is equivalent to

$$\sqrt[3]{\frac{a}{16a+16b+4c}} + \sqrt[3]{\frac{b}{16b+16c+4a}} + \sqrt[3]{\frac{c}{16c+16a+4b}} < 1. \qquad (1)$$

Using the lemma twice we have

$$\sqrt[3]{\frac{a}{16a+16b+4c}} = \sqrt[3]{\frac{a}{4((4a+4b)+c)}} \leq \frac{\sqrt[3]{a}}{\sqrt[3]{4(a+b)} + \sqrt[3]{c}} \leq \frac{\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}.$$

Hence,

$$\sum_{\text{cyclic}} \sqrt[3]{\frac{a}{16a+16b+4c}} \leq \sum_{\text{cyclic}} \frac{\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}} = 1.$$

If equality holds, then we must have $4a+4b = c$, $4b+4c = a$, and $4c+4a = b$ which imply that $8(a+b+c) = a+b+c$, so $a+b+c = 0$, a contradiction.

Hence (1) holds and our proof is complete.

II. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified slightly by the editor.

We prove the stronger inequality that

$$\sum_{\text{cyclic}} \sqrt[3]{\frac{2a}{4a+4b+c}} \le 6^{\frac{1}{3}}.$$

Let S denote the summation on the left side of the given inequality. By the power-mean-inequality, we have, for $x_i \ge 0$, $i = 1, 2, 3$, that

$$\left(\frac{\sum_{i=1}^{3} x_i^{\frac{1}{3}}}{3}\right)^3 \le \left(\frac{\sum_{i=1}^{3} x_i^{\frac{1}{2}}}{3}\right)^2$$

or

$$\sum_{i=1}^{3} x_i^{\frac{1}{3}} \le 3^{\frac{1}{3}} \left(\sum_{i=1}^{3} x_i^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

which implies that

$$S \le 3^{\frac{1}{3}} \left(\sum_{\text{cyclic}} \sqrt{\frac{2a}{4a+4b+c}}\right)^{\frac{2}{3}} = 6^{\frac{1}{3}} \sum_{\text{cyclic}} \sqrt{\frac{a}{4a+4b+c}}\,.$$

We need to prove that

$$\sum_{\text{cyclic}} \sqrt[3]{\frac{a}{4a+4b+c}} \le 1. \qquad (2)$$

Since the function $\sqrt{x}$ is concave on $[0, \infty)$, Jensen's Inequality implies that

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{\frac{a}{4a+4b+c}} &= \sum_{\text{cyclic}} \frac{4a+4c+b}{9(a+b+c)} \sqrt{\frac{9^2 a(a+b+c)^2}{(4a+4b+c)(4a+4c+b)^2}} \\
&\le \sqrt{\sum_{\text{cyclic}} \frac{81a(4a+4c+b)(a+b+c)^2}{9(a+b+c)(4a+4b+c)(4a+4c+b)^2}} \\
&= \sqrt{\sum_{\text{cyclic}} \frac{9a(a+b+c)}{(4a+4b+c)(4a+4c+b)}}.
\end{aligned}$$

Hence (2) would follow if we show that

$$\sum_{\text{cyclic}} \frac{9a(a+b+c)}{(4a+4b+c)(4a+4c+b)} \le 1. \qquad (3)$$

By homogeneity, we may assume that $a + b + c = 1$. Then (3) reduces to

$$\sum_{\text{cyclic}} \frac{9a}{(4a-3b)(4-3c)} \leq 1$$

which is equivalent, in succession, to

$$9\sum_{\text{cyclic}} a(4-3a) \leq (4-3a)(4-3b)(4-3c)$$
$$36 - 27(a^2+b^2+c^2) \leq 64 - 48 + 36(ab+bc+ca) - 27abc$$
$$27(a^2+b^2+c^2) + 36(ab+bc+ca) \geq 27abc + 20. \qquad (4)$$

We now define $x \geq 0$ by $3(ab+bc+ca) = 1 - x^2$. Then $x < 1$, $ab+bc+ca \leq \frac{1}{3}$ and $a^2+b^2+c^2 = 1 - \frac{2}{3}(1-x^2) = \frac{1+2x^2}{3}$. We employ the following reslt in the paper "On a Class of Three Variable Inequalities" by Vo Quoc Ba Can in the book *Mathematical Reflection, the first two years*, by Titu Andreescu, XYZ press, p. 480

$$\frac{(1+x)^2}{27} \leq abc \leq \frac{(1-x)^2(1+2x)}{27}.$$

In particular, $27abc \leq (1-x)^2(1+2x)$. Hence to prove (4) it suffices to show that

$$9(1+2x^2) + 12(1-x^2) \geq (1-x)^2(1+2x) + 20$$

which upon simplifications, reduces to $21 + 6x^2 \geq 21 - 3x^2 + 2x^3$ or $2x^3 - 9x^2 \leq 0$ or $x^2(2x-9) \leq 0$ which is clearly true and the proof is complete.

[*Ed : By examining the above proof for equality cases, it is easy to see that equality holds if and only if* $a = b = c$.]

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; ALBERT STADLER, Herrliberg, Switzerland; and HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

Stadler used Hölder's Inequality together with some result which appeared in Crux in the past. He also obtained the sharper upper bound and pointed out the equality case.

3664. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$|(1-a^2b)(1-b^2c)(1-c^2a)| \ \leq \ 3|1-abc|\,.$$

[*Ed.* : The proposer's solution was flawed. STAN WAGON, Macalester College, St. Paul, MN, USA showed using Mathematica that the inequality is most likely true. Thus the problem remains open.]

3665. [2011 : 388, 390] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the diagonals AC and BD of the cyclic quadrilateral $ABCD$ intersect at M, and let the line joining M to the midpoint of BC meet AD at Q. Prove that MQ is perpendicular to AD if and only if the sides AD and BC are parallel (in which case $ABCD$ is an isosceles trapezoid), or the diagonals are perpendicular (and we have Brahmagupta's configuration).

Solution by John G. Heuver, Grande Prairie, AB.

When $AD||BC$ the cyclic quadrilateral $ABCD$ is symmetric about the line joining the centre to the midpoint of BC ; in particular, that line passes through M and is perpendicular to AD at its midpoint Q. On the other hand, when $AC \perp BD$ at M while P is the midpoint of BC (Brahmagupta's configuration shown in the left figure), then $MP = BP$ (because MP is the median to the hypotenuse of the right triangle MBC) and $\angle MBP = \angle BMP = \angle DMQ$. But $\angle MBP$ also equals the inscribed angles $\angle DBC = \angle DAC$ $(= \angle DAM)$, so that triangles DMQ and DAM are similar, and it follows that $\angle MQD = 90°$, as desired.

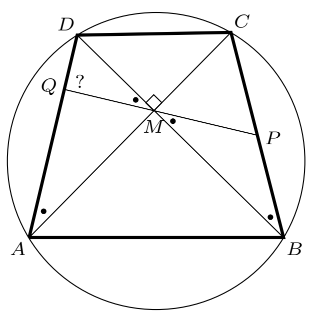


Figure 1 : Assume that $AC \perp BD$ at H

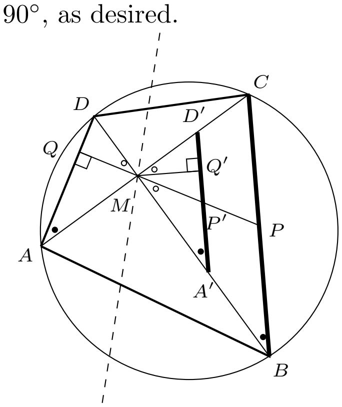


Figure 2 : Assume that $MQ \perp AD$

Conversely, let MQ be perpendicular to AD and bisect BC at P (as in the figure on the right). Reflection in the line that bisects the vertical angles $\angle AMB$ and $\angle CMD$ takes the collinear points A, Q, D to A', Q', D' (with A' and D' on the diagonals BD and AC, respectively, and $MQ' \perp A'D'$). Because the inscribed angles DAC and DBC are equal, we have $\angle DA'D' = \angle DBC$, whence $A'D'||BC$ and the triangles $MA'D'$ and MBC are homothetic ; the midpoint of $A'D'$, call it P', thus lies on the line QP through M. If $Q' \in QM$, then $AD||BC$ as follows : then $Q' = P'$ and, consequently, MQ is perpendicular to $A'D'$ as well as to AD and (because $A'D'||BC$) to BC. Otherwise, if Q' does not coincide with P', then we have

$$\angle A'MP' = \angle BMP = \angle DMQ = \angle Q'MD';$$

that is, the altitude MQ' of $\Delta A'MD'$ is the isogonal conjugate of the median MP'. As a consequence, we have the median passing through the circumcentre while the perpendicular bisector of $A'D'$ passes also through the midpoint P', whence P' must be the circumcentre and, therefore, $\angle A'MD' = 90°$. That is, the diagonals of $ABCD$ are perpendicular.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; BOB SERKEY, Tucson, AZ, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3666. [2011 : 388, 391] *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{R}$ denote the set of all pairs of relatively prime integers and

$$\mathcal{S} = \{(a,b) \in \mathcal{R} : 5a + 42b \equiv 0 \pmod{1789}\}.$$

Find an explicit bijection between $\mathcal{S}$ and $\mathcal{R} - S$.

Solution by the proposer.

Observe that 1789 is prime and is equal to $42^2 + 5^2$. Let $(u, v) \in \mathcal{R} - \mathcal{S}$ and define

$$(a, b) = (5u + 42v, 42u - 5v).$$

Then $5a + 42b = 1789u$ and $42a - 5b = 1789v$. Therefore d, the greatest common divisor of a and b must divide $1789u$ and $1789v$. Now, a is not a multiple of 1789, u and v are relatively prime and 1789 is prime, so d must be equal to 1. Hence $(a, b) \in \mathcal{S}$ and we define the mapping

$$\phi(u, v) = (5u + 42v, 42u - 5v)$$

from $\mathcal{R} - \mathcal{S}$ to $\mathcal{S}$.

We show that ϕ is a bijection. Let $(a, b) \in \mathcal{S}$. There exists an integer u for which $5a + 42b = 1789u$. Since 5 and 42 are coprime and since $5(a - 5u) = 42(42u - b)$, there exists an integer v for which $a - 5u = 42v$ Thus

$$(a, b) = (5u + 42v, 42u - 5v).$$

We show that $(u, v) \in \mathcal{R} - \mathcal{S}$. Any common divisor of u and v must divide both a and b, so that u and v are coprime. If, say, $5u + 42v = 1789w$, then because

$$5(42u - 5v) = 42(1789w - 42v) - 5^2 v = 1789(42w - v),$$

1789 would have to divide both a and b, contrary to hypothesis. As a result, $(u, v) \in \mathcal{R} - \mathcal{S}$ and $\phi(u, v) = (a, b)$. Since the system $5x + 42y = a; 42x - 5y = b$ has a unique solution, ϕ is the desired bijection.

There were no other solutions.

3667. *Proposed by Joe Howard, Portales, NM, USA.*

Suppose $b_i > 0$ for $i = 1, 2, \ldots, n$; $n \geq 3$; and $\prod_{i=1}^{n} b_i = 1$. Prove

$$\sum_{i=1}^{n} b_i^{n-2} \geq \sum_{i=1}^{n} b_i^{\frac{n-1}{2}}.$$

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia, expanded by the editor.

If $n=3$, then $n-2=\frac{n-1}{2}$ so we have equality.

Assume now that $n>3$ and let $\alpha=\frac{n^2-3}{n-3}$, then $\alpha>0$. By the weighted AM-GM Inequality, we have

$$\begin{aligned}
\sum_{i=1}^{n} b_i^{n-2} &= \sum_{i=1}^{n}\left(\frac{\alpha}{n-1+\alpha}+\frac{n-1}{n-1+\alpha}\right)b_i^{n-2} \\
&= \sum_{i=1}^{n}\frac{\alpha}{n-1+\alpha}b_i^{n-2}+\frac{1}{n-1+\alpha}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} b_j^{n-2} \\
&= \sum_{i=1}^{n}\left(\frac{\alpha}{n-1+\alpha}b_i^{n-2}+\frac{1}{n-1+\alpha}\sum_{\substack{j=1\\ j\neq i}}^{n} b_j^{n-2}\right) \\
&\geq \sum_{i=1}^{n}\left(b_i^{\frac{\alpha(n-2)}{n-1+\alpha}}\cdot\prod_{\substack{j=1\\ j\neq i}}^{n} b_j^{\frac{n-2}{n-1+\alpha}}\right) \\
&= \sum_{i=1}^{n}\left(b_i^{\alpha(n-2)}\cdot\prod_{\substack{j=1\\ j\neq i}}^{n} b_j^{n-2}\right)^{\frac{1}{n-1+\alpha}} \\
&= \sum_{i=1}^{n}\left(b_i^{(\alpha-1)(n-2)}\cdot\prod_{j=1}^{n} b_j^{n-2}\right)^{\frac{1}{n-1+\alpha}} \\
&= \sum_{i=1}^{n} b_i^{\frac{(\alpha-1)(n-2)}{n-1+\alpha}}\,, \qquad (1)
\end{aligned}$$

since $\prod_{j=1}^{n} b_j^{n-2}=1$.

Since $\alpha-1=\frac{n^2-3}{n-3}-1=\frac{n^2-n}{n-3}$ and $n-1+\alpha=n-1+\frac{n^2-3}{n-3}=\frac{2n^2-4n}{n-3}$, we have

$$\frac{(\alpha-1)(n-2)}{n-1+\alpha}=\frac{(n^2-n)(n-2)}{2n^2-4n}=\frac{(n-1)(n-2)}{2(n-2)}=\frac{n-1}{2}\,. \qquad (2)$$

The result follows by substituting (2) into (1).

II. Solution by Michel Bataille, Rouen, France.

Let $L = \sum_{i=1}^{n} b_i^{n-2}$ and $R = \sum_{i=1}^{n} b_i^{\frac{n-1}{2}}$. Furthermore, let $p = \dfrac{2n-4}{n-3}$ and $q = \dfrac{2n-4}{n-1}$. Then $p > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

[*Ed. : When* $n = 3$, *we have equality as pointed out in I above.*]

Hence, by Hölder's Inequality we have

$$n^{\frac{1}{p}} \cdot L^{\frac{1}{q}} = \left(\sum_{i=1}^{n} 1^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \left(b_i^{\frac{n-1}{2}}\right)^q\right)^{\frac{1}{q}} \geq \sum_{i=1}^{n} b_i^{\frac{n-1}{2}} = R$$

or raising to the $(2n-4)^{\text{th}}$ power, $n^{n-3} \cdot L^{n-1} \geq R^{2n-4}$, so

$$\left(\frac{L}{R}\right)^{n-1} \geq \left(\frac{R}{n}\right)^{n-3} . \tag{3}$$

By the AM-GM Inequality, we have

$$\frac{R}{n} \geq \left(\prod_{i=1}^{n} b_i^{\frac{n-1}{2}}\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} b_i\right)^{\frac{n-1}{2n}} = 1 . \tag{4}$$

From (3) and (4) we have $\left(\dfrac{L}{R}\right)^{n-1} \geq 1$ from which $L \geq R$ follows.

III. Generalization by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.

We prove the more general result that if $b_i \geq 0$ for $i = 1, 2, \ldots, n$; $n \geq 3$ with $\prod_{i=1}^{n} b_i = 1$ and $\alpha \geq \beta > 0$, then

$$\sum_{i=1}^{n} b_i^{\alpha} \geq \sum_{i=1}^{n} b_i^{\beta} .$$

By the power mean inequality we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} b_i^{\alpha}\right)^{\frac{1}{\alpha}} \geq \left(\frac{1}{n}\sum_{i=1}^{n} b_i^{\beta}\right)^{\frac{1}{\beta}} . \tag{5}$$

Since $\prod_{i=1}^{n} b_i = 1$, the AM-GM Inequality yields

$$\frac{1}{n}\sum_{i=1}^{n} b_i^{\beta} \geq \left(\prod_{i=1}^{n} b_i^{\beta}\right)^{\frac{1}{n}} = 1 .$$

Hence,

$$\left(\frac{1}{n}\sum_{i=1}^{n} b_i^{\beta}\right)^{\frac{\alpha}{\beta}} \geq \frac{1}{n}\sum_{i=1}^{n} b_i^{\beta}\,. \tag{6}$$

From (5) and (6) it follows that

$$\sum_{i=1}^{n} b_i^{\alpha} \geq n\left(\frac{1}{n}\sum_{i=1}^{n} b_i^{\beta}\right)^{\frac{\alpha}{\beta}} \geq n\left(\frac{1}{n}\sum_{i=1}^{n} b_i^{\beta}\right) = \sum_{i=1}^{n} b_i^{\beta}$$

which completes the proof.

The given inequality is the special case when $\alpha = n-2$ and $\beta = \frac{n-1}{2}$. Since $n \geq 3$, it is clear that $\alpha \geq \beta > 0$.

Also solved by MARIAN DINCĂ, Bucharest, Romania; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Zvonaru proved the generalization that if $b_i > 0$ for $i = 1, 2, \ldots, m$ where $m \geq 1$ such that $\prod_{i=1}^{m} b_i = 1$, then for $n \geq 3$, $\sum_{i=1}^{m} b_i^{n-2} \geq \sum_{i=1}^{m} b_i^{\frac{n-1}{2}}$.

3668. [2011 : 389, 391] *Proposed by Neven Jurič, Zagreb, Croatia.*

Suppose p, q and r are distinct prime numbers. How many positive integer solutions has the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ where $y = pqr$?

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA; modified by the editor.

More generally, we will show that for a fixed $y \in \mathbb{N}$, the number of positive integer solutions (x, z) to the given equation is $\frac{\tau(y^2)-1}{2}$ where $\tau(n)$ is the number of positive divisors of $n \in \mathbb{N}$.

It is well known that if the prime power factorization of n is $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then $\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$ from which we see that $\tau(n)$ is odd if and only if n is a perfect square.

Throughout the rest of our proof, we use "divisors" to mean positive divisors.

Suppose (x, z) is a solution to the given equation. Then $yz + xz = xy$ so $(x+y)(y-z) = y^2$. Hence, if we let $d = x + y$, then both d and $\frac{y^2}{d} = y - z$ are divisors of y^2 with $d > y > \frac{y^2}{d}$.

Note also that $d \neq y$ since $x > 0$. Hence, (x, z) induces a pair (d_1, d_2) of divisors of y^2 where $d_1 > y > d_2$ and $d_1 d_2 = y^2$. Conversely, if (d_1, d_2) is such a pair of divisors of y^2, then by setting $x = d_1 - y$ and $z = y - d_2$ we have $x > 0$, $z > 0$ and $(x+y)(y-z) = d_1 d_2 = y^2$ showing that (x, z) is a solution to the given equation.

Since $\tau(y^2)$ is odd, by deleting y, we have an even number of divisors of y^2 which can be grouped into $\frac{\tau(y^2)-1}{2}$ pairs of divisors (d_1, d_2) where $d_1 > d_2$ with $d_1 d_2 = y^2$, and this is the number we are seeking.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA ; MICHEL BATAILLE, Rouen, France ; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA ; OLIVER GEUPEL, Brühl, NRW, Germany ; RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece ; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia ; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL ; ALBERT STADLER, Herrliberg, Switzerland ; and the proposer. There were also two incorrect solutions.

3669. [2011 : 389, 391] *Proposed by Michel Bataille, Rouen, France.*

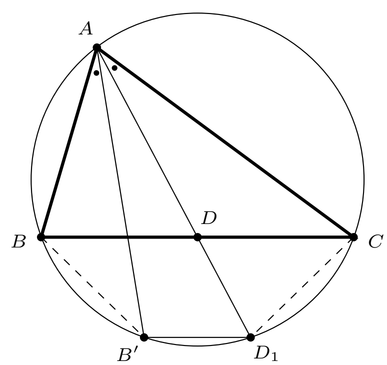
Let ABC be a triangle inscribed in a circle Γ and let r be a real number ($r \neq 0, 1$). Points D, E, F are defined by $\overrightarrow{BD} = r\overrightarrow{BC}$, $\overrightarrow{CE} = r\overrightarrow{CA}$, $\overrightarrow{AF} = r\overrightarrow{AB}$. The circle Γ meets again DA at D_1 and the parallel to BC through D_1 at B'. Points C' and A' are constructed in a similar way. For which r are AA', BB', CC' concurrent lines ? What is then their point of concurrency ?

Solution by Titu Zvonaru, Comăneşti, Romania.

[*Editor's comment* : It sometimes happens that a mathematician misreads a question and gets lucky ; he not only gains insight into the original question, but he discovers a new result. The error in this case was to denote the point generated from vertex A by A' instead of by B'. We begin with Zvonaru's discovery, but will maintain the notation of the original problem.]

Theorem 1(Zvonaru). The lines AB', BC', and CA' are concurrent if and only if $r = \frac{1}{2}$, in which case they meet in the symmedian point.

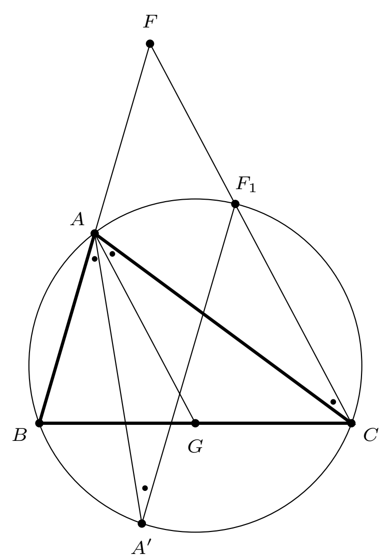
Theorem 2 (Bataille). The lines AA', BB', and CC' are concurrent if and only if $r = -1$, in which case they meet in the symmedian point.



Proof. Since $D_1B' || BC$, the chords D_1C and BB' have the same length and, therefore, subtend the equal angles $\angle D_1AC = \angle BAB'$; this implies that AB' is the isogonal conjugate of AD. Similarly, BC' and BE are isogonal conjugates, as are CA' and CF. As a consequence, AB', BC', CA' are concurrent if and only if AD, BE, CF are ; this happens by Ceva's theorem if and only if

$$1 = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \left(\frac{r}{1-r}\right)^3 ;$$

that is, if and only if $r = \frac{1}{2}$, in which case AD, BE, CF are the medians of ΔABC and they are concurrent at the centroid. Because the symmedian point is the isogonal conjugate of the centroid, the first theorem is proved.



For the second theorem (and the solution to the original problem), we define the points $G \in BC$, $H \in CA$, and $J \in AB$ so that

$$AG||CF, \quad BH||AD, \quad \text{and} \quad CJ||BE.$$

Because $BA||A'F_1$ (given), we have $\angle BAA' = \angle AA'F_1$; the last angle equals $\angle ACF_1$ (because they are inscribed angles subtended by the same arc), and these equal $\angle GAC$ (because $AG||F_1C$). We conclude that for any position of F on AB, the point G on BC satisfies $\angle BAA' = \angle GAC$; that is, AA' and AG are isogonal conjugates. With similar arguments for BH and CJ we deduce that AA', BB', and CC' are concurrent if and only if AG, BH, and CJ are. Because $GA||CF$ we have $\frac{GC}{GB} = \frac{AF}{AB} = r$, with similar results for H and J. By Ceva's theorem,

$$1 = \frac{BG}{GC} \cdot \frac{CH}{HA} \cdot \frac{AJ}{JB} = \left(-\frac{1}{r}\right)^3,$$

or, $r = -1$. We conclude, finally, that AA', BB', and CC' are concurrent if and only if $r = -1$. Again, concurrency occurs at the symmedian point because $AB = -AF$ implies that G is the midpoint of side BC (and similarly for H and J).

Also solved by MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3670. [2011 : 389, 391] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $n \geq 2$ be an integer. Calculate

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{x+y+z}.$$

I. Solution by Michel Bataille, Rouen, France ; Brian D. Beasley, Presbyterian College, Clinton, SC, USA ; Paul Bracken, University of Texas, Edinburg, TX, USA ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; Richard I. Hess, Rancho Palos Verdes, CA, USA ; Anastasios Kotronis, Athens, Greece ; Mitch Kovacs, St. Bonaventure University, St. Bonaventure, NY, USA ; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The integral is equal to

$$\begin{aligned}
&\int_0^1 \int_0^1 [\ln(1+y+z) - \ln(y+z)]dydz \\
=&\int_0^1 [(1+y+z)\ln(1+y+z) - (y+z)\ln(y+z) - 1]_0^1 dz \\
=&\int_0^1 [(2+z)\ln(2+z) - 2(1+z)\ln(1+z) + z\ln z]dz \\
=&\Big[\frac{1}{2}(2+z)^2\ln(2+z) - \frac{1}{4}(2+z)^2 - (1+z)^2\ln(1+z) \\
&\qquad\qquad + \frac{1}{2}(1+z)^2 + \frac{1}{2}z^2\ln z - \frac{1}{4}z^2\Big]_0^1 \\
=&\frac{9}{2}\ln 3 - 6\ln 2 = \frac{3}{2}\ln\left(\frac{27}{16}\right) = \ln\left(\frac{81\sqrt{3}}{64}\right),
\end{aligned}$$

where we have used the fact that $\lim\limits_{\epsilon \to 0^+} \epsilon^2 \ln \epsilon = 0$.

II. Solution by Mohammed Aassila, Strasbourg, France.

We show that, when $n \geq 2$,

$$I_n \equiv \int_0^1 \int_0^1 \cdots \int_0^1 \frac{dx_1 dx_2 \cdots dx_n}{x_1 + x_2 + \cdots + x_n} = \frac{(-1)^n}{(n-1)!} \sum_{k=2}^{n} \binom{n}{k} (-1)^k k^{n-1} \ln k.$$

Observe that

$$\begin{aligned}
I_n &= \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^\infty \exp\{-t(x_1 + x_2 + \cdots + x_n)\} dt dx_1 dx_2 \cdots dx_n \\
&= \int_0^\infty \left(\int_0^1 \exp(-tx)dx\right)^n dt = \int_0^\infty \left(\frac{1-e^{-t}}{t}\right)^n dt.
\end{aligned}$$

Let $0 \le m \le n-1$. Then $(1-e^{-t})^n = (t - \frac{1}{2}t^2 + \cdots)^n = t^n + \cdots$ so that $D_t^m(1-e^{-t})^n = n(n-1)\cdots(n-m+1)t^{n-m} + \cdots = O(t^{n-m})$. Also

$$D_t^m(1-e^{-t})^n = \sum_{k=0}^{n}(-1)^{m+k}\binom{n}{k}k^m e^{-kt}.$$

Integrating by parts n times and making the substitution $s = kt$, we find that

$$\begin{aligned}
I_n &= \frac{1}{(n-1)!}\int_0^\infty \left(\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n-k-1}k^n e^{-kt}\right)\ln t dt \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k-1}k^n \int_0^\infty e^{-kt}\ln t dt \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{n-1}\int_0^\infty e^{-s}(\ln k - \ln s)ds \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n-k}k^{n-1}\ln k \\
&\qquad \times \int_0^\infty e^{-s}ds - \frac{1}{(n-1)!}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{n-k}k^{n-1}\int_0^\infty e^{-s}\ln s ds.
\end{aligned}$$

The quantity $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{n-1}$ in the second term vanishes since it is the nth order difference at 0 of the polynomial x^{n-1}. The first two terms of the sum $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^{(n-1)}\ln k$ also vanish, since $n \ge 2$ and $\lim_{\epsilon\to 0^+}\epsilon^{n-1}\ln\epsilon = 0$. The desired result follows.

III. Solution by the Missouri State University Problem Solving Group, Springfield, MO, USA.

For $n \ge 1$ and $t > 0$, define

$$F_n(t) = \int_0^1\int_0^1\cdots\int_0^1 \frac{dx_1 dx_2 \cdots dx_n}{x_1 + x_2 + \cdots + x_n + t}.$$

We show that, for $n \ge 1$,

$$F_n(t) = \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n-k)^{n-1}\ln(t+n-k).$$

Since $F_1(t) = \ln(t+1) - \ln t$, this holds for $n = 1$. Assume that it holds up to the index $n \geq 1$. Then $F_{n+1}(t)$ is given by

$$\begin{aligned}
\int_0^1 & F_n(x_{n+1}+t)dx_{n+1} \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1 (x_{n+1}+t+n-k)^{n-1}\ln(x_{n+1}+t-n-k)dx_{n+1} \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{(x_{n+1}+t+n-k)^n\ln(x_{n+1}+t+n-k)}{n}\right. \\
&\qquad\qquad\qquad\qquad\left. - \frac{(x_{n+1}+t+n-k)^n}{n^2}\right]_0^1 \\
&= \frac{1}{n!}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n\ln(t+n+1-k)\right. \\
&\qquad \left. - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n-k)^n\ln(t+n-k)\right] \\
&\qquad - \frac{1}{n(n!)}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n - \sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n-k)^n\right] \\
&= \frac{1}{n!}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n\ln(t+n+1-k)\right. \\
&\qquad \left. + \sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}(t+n+1-k)^n\ln(t+n+1-k)\right] \\
&\qquad - \frac{1}{n(n!)}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+n+1-k)^n\right. \\
&\qquad \left. + \sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}(t+n+1-k)^n\right] \\
&= \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(t+n+1-k)^n\ln(t+n+1-k) \\
&\qquad - \frac{1}{n(n!)}\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(t+n+1-k)^n.
\end{aligned}$$

The latter sum, being the $(n+1)^{\text{th}}$ difference of the polynomial $(t+n+1-k)^n$ in k of degree n vanishes, and the desired result follows by induction.

When $n \geq 2$, $F_n(0)$ is defined and equal to $\lim\limits_{t\to 0^+} F_n(t)$ and we obtain the representation obtained in Solution 2.

One incorrect solution was received. The generalization to an $n-$fold integral was also established by ALBERT STADLER, Herrliberg, Switzerland; and the proposer. Using computer software, STAN WAGON, Macalester College, St. Paul, MN, USA; discovered that, when a, b, c

are positive,

$$\int_0^a \int_0^b \int_0^c \frac{1}{x+y+z} dxdydz$$
$$= \frac{1}{2}\left[-c^2 \ln \frac{c}{a+c} - a\left(c + a\ln\frac{a+c}{a}\right) + c(2b+c)\ln\frac{b+c}{a+b+c}\right.$$
$$+ b^2 \ln \frac{(a+b)(b+c)}{b(a+b+c)} + a\left(c + a\ln\left(1+\frac{c}{a+b}\right)\right)$$
$$+ 2c\left(-c\ln\frac{b+c}{c} + (a+c)\ln\left(1+\frac{b}{a+c}\right) + b\ln\left(1+\frac{a}{b+c}\right)\right)$$
$$\left.+2b\left(-b\ln\frac{b+c}{b} + (a+b)\ln\left(1+\frac{c}{a+b}\right) + c\ln\left(1+\frac{a}{b+c}\right)\right)\right].$$