

# *CruX Mathematicorum*

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# DEPARTMENT HIGHLIGHT

No. 1

*The Department Highlight provides a forum for mathematics departments at Canadian universities to outline some of their department's programs of study, community outreach programs and other departmental activities that may be of interest to the readers of **Cruæ Mathematicorum**.*

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## The School of Mathematics and Statistics Carleton University

The School of Mathematics and Statistics at Carleton is a strong, collegial and vibrant academic unit with a long standing tradition of excellence both in research and teaching. The School draws upon diverse and dedicated faculty and staff complements to provide students with a wide range of comprehensive and challenging programs across a number of sub-disciplines. In contrast to most other institutions in Ontario, Carleton offers a Bachelors of Mathematics degree which involves courses tailored for students in first year. In addition, the School provides a number of core first and second year courses to students in other disciplines of Science, Social Sciences and Engineering. While the School shares many common elements with like units at other institutions, there are also aspects which are unique and which draw on the particular characteristics of the School, the University, its location within the Capital region, and the strengths of the faculty.

The School of Mathematics and Statistics offers a broad range of programs, built around a common core which allows students to branch into many different avenues of specialization. In general the core offerings compose roughly the first three to four terms of the program and provide students with a balanced exposure to the fundamental areas of mathematics and statistics, from both theoretical and applied perspectives. Subsequently, students are then able to focus on any one of these aspects, possibly in combination with advanced studies in another discipline. Starting in September 2013, the School will offer a Concentration in Actuarial Science. The concentration will incorporate a targeted sequence of courses in Business and Economics that will provide students with the necessary background to satisfy all six undergraduate requirements set out by the Society of Actuaries (SOA) for professional designation.

Students from the School have participated in the Putnam Mathematics Competition that occurs in December each year. One of our faculty members serves as a coach in order to assist students in their preparation for the competition. Recently, the School has been heavily involved in outreach activities aimed at high school students. *Math for Success!* is a popular enrichment course targeted for students in grades 11 and 12. It provides enrichment in a broad array of mathematical topics beyond the Ontario High School curriculum, including algebra,

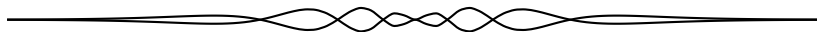
trigonometry, and combinatorics, giving students a head-start on their university studies. Classes meet for 90 minutes, one evening per week from September to March. The School has also offered *MathLink* since 2006, a second enrichment course aimed at students in grades 9 and 10. Classes run in a similar fashion to *Math for Success!*, and cover topics such as sequences, probability, geometry, modular arithmetic, and logic.

We are pleased to have a recently-created partnership with the SCM (School of Competitive Math, [www.competitivemath.org](http://www.competitivemath.org)) that was established through an adjunct professor with the School, Dragos Calitoiu. Weekly evening courses are offered in intermediate topics for students in grades 7 and 8, and advanced topics for students in grade 9. The School and SCM will organize an SCM Math Contest for students in grades 7 and 8 to be held on the evening of Tuesday, April 16, 2013, from 17:30 to 19:30 pm. Through our partnership with SCM, a new weekly enrichment course in geometry for students in grades 7 and 8 is scheduled to begin in 2013-14.

In addition to the above enrichment activities, the School has also contributed to the development of a student engagement program called *Math Matters*. This program offers incoming students to Carleton a one week intensive review at the end of the summer. A review of high school math concepts and techniques that are essential to the first year calculus and linear algebra courses is given. Due to steadily increasing enrolment, three sections are now offered, targeted for incoming students in Science, Engineering, and Business/Economics.

On January 23, 2013, the School officially opened its newly-created Consulting Centre, CQADS (the Centre for Quantitative Analysis and Decision Support [www.carleton.ca/math/cqads/](http://www.carleton.ca/math/cqads/)). Through the Centre, our objectives are to provide consulting services and share our expertise in solving real world problems, facilitate collaborative cross-disciplinary research in mathematics and statistics, stimulate research in mathematics and statistics, disseminate research to the wider research community by offering short courses, and to enhance the training and experience of our students and postdoctoral fellows through work experience and short courses in the Consulting Centre.

In summary, the School of Mathematics and Statistics at Carleton engages in a broad range of outreach activities, not only to foster and enrich the learning of new mathematical and statistical techniques, but also to highlight the connections of these approaches to a plethora of different applications and career paths.



# SKOLIAD No. 140

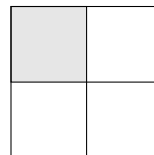
Lily Yen and Mogens Hansen

*Skoliad has joined **Mathematical Mayhem** which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of **Crux** will appear in this volume, after which time Skoliad will be discontinued in **Crux**. New Skoliad problems, and their solutions, will appear in **Mathematical Mayhem** when it is relaunched in 2013.*



In this issue we present the solutions to the National Bank of New Zealand Junior Mathematics Competition, 2010. given in Skoliad 131 at [2011:65–71].

**1.** Rebecca is holding a seminar at the place at which she works. She wants to create an unbroken ring of tables, using a set of identical tables shaped like regular polygons. (In a regular polygon, all sides have the same length, and all angles are equal. Squares and equilateral triangles are regular.) Each table must have two sides which completely coincide with the sides of other tables, such as the shaded square table seen to the right. Rebecca plans to put items on display inside the ring where everyone can see them.

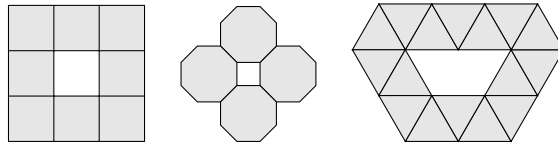


(If you cannot name a shape in this question, just give the number of sides. For example, if you think the shape has 235 sides, but don't know the name, just call it a 235-gon—that isn't an answer to any of the parts.)

1. Rebecca first decides to use identical square tables. What is the minimum number of square tables placed beside each other so that there is an empty space in the middle?
2. If Rebecca uses the minimal number of square tables, what shape is left bare in the middle?
3. Rebecca considers using octagon (eight sides) shaped tables.
  - (a) What is the minimum number of octagonal tables which Rebecca must have in order for there to be a bare space in the middle so that the tables form an enclosure?
  - (b) What is the name given to the bare shape in the middle? If you can't name it, giving the number of sides will be sufficient.
4. Apart from squares and octagons, are there any other shaped tables possible? If there are any, name one. If there isn't, say so.

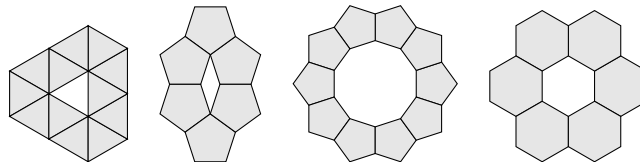
*Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.*

Rebecca needs at least eight square tables arranged as in the left-hand figure below. This leaves a square in the middle. If Rebecca uses regular octagons, she needs four arranged as in the middle figure below. This also leaves a square in the middle. Rebecca could also use equilateral triangles arranged as in the right-hand figure below.

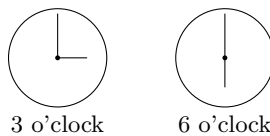


*Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha Secondary School, Burnaby, BC; KATIE PINTER, student, École Capitol Hill Elementary School, Burnaby, BC; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC; and NELSON TAM, student, John Knox Christian School, Burnaby, BC.*

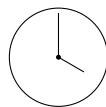
*Our solver uses sixteen equilateral triangles, but Rebecca could make do with twelve arranged as in the left-hand figure below. (Minimality was, of course, not part of the challenge.) Other solvers found more possible arrangements:*



**2.** An analogue clock displays the time with the use of two hands. Every hour the minute hand rotates 360 degrees, while the hour hand (which is shorter than the minute hand) rotates 360 degrees over a 12-hour period. Two example times are shown below:



1. Draw a clock face which shows 9 o'clock. Make sure the hour hand is shorter than the minute hand.
2. What is the angle between the two hands at both 3 o'clock and 9 o'clock?
3. What time to the closest hour (and minute) does the following clock face show?



4. What is the angle between the two hands at the following times?
- 1 o'clock.
  - 2 o'clock.
  - Half past one.
5. At what time (to the nearest minute) between 7 and 8 o'clock do the hands meet?

*Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.*

At 9 o'clock, an analogue clock looks like  $\ominus$ , and the angle between the two hands is  $90^\circ$ . At 3 o'clock, the angle is also  $90^\circ$ . The clock  $\odot$  shows 4:00.

At 1 o'clock,  $\odot$ , the minute hand points straight up while the hour hand has moved  $\frac{1}{12}$  of a turn. Therefore the angle between the hands is  $\frac{1}{12} \cdot 360^\circ = 30^\circ$  at 1 o'clock. Likewise, at 2 o'clock the angle between the hands is  $60^\circ$ . At half past one,  $\ominus$ , the minute hand points straight down while the minute hand is exactly half way between 12 and 3, so half way between  $0^\circ$  and  $90^\circ$  from up, so  $45^\circ$  from up. Therefore the angle between the hands is  $180^\circ - 45^\circ = 135^\circ$  at 1:30.

The minute hand makes one turn in 60 minutes, so it moves  $360^\circ$  in 60 minutes, so it moves  $6^\circ$  per minute. Likewise, the hour hand makes one turn in 12 hours, so it moves  $360^\circ$  in 720 minutes, so it moves  $\frac{1}{2}^\circ$  per minute. If the time is  $x$  minutes past 7 o'clock, then the minute hand has moved  $6x$  degrees from up, and the hour hand has moved  $(7 \cdot 60 + x) \frac{1}{2}$  degrees from up. Therefore  $6x = (7 \cdot 60 + x) \frac{1}{2}$  when the two hands meet, so  $12x = 420 + x$ , so  $11x = 420$ , so  $x = \frac{420}{11} \approx 38.2$ . Thus the two hands meet at approximately 7:38.

*Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha Secondary School, Burnaby, BC; KATIE PINTER, student, École Capitol Hill Elementary School, Burnaby, BC; and NELSON TAM, student, John Knox Christian School, Burnaby, BC.*

**3.** A six-digit number “ $abcdef$ ” is formed using each of the digits 1, 2, 3, 4, 5, and 6 once and only once so that “ $abcdef$ ” is a multiple of 6, “ $abcde$ ” is a multiple of 5, “ $abcd$ ” is a multiple of 4, “ $abc$ ” is a multiple of 3, and “ $ab$ ” is a multiple of 2.

- Find a solution for “ $abcdef$ .” Show key working.
- Is the solution you found unique (the only possible one)? If it is, briefly explain why. If it isn't, give another solution.

*Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.*

Since “ $abcde$ ” is divisible by 5,  $e = 5$ . Since “ $ab$ ”, “ $abcd$ ”, and “ $abcdef$ ” are all divisible by even numbers, “ $ab$ ”, “ $abcd$ ”, and “ $abcdef$ ” must themselves be even, so  $b, d$ , and  $f$  are even. Thus  $\{b, d, f\} = \{2, 4, 6\}$  and  $\{a, c\} = \{1, 3\}$ .

If  $a = 1$  and  $c = 3$ , the digit sum of “ $abc$ ” is  $4 + b$ . Since “ $abc$ ” is divisible by 3, this digit sum must be divisible by 3, so  $b = 2$ . Since “ $abcd$ ” is divisible by 4, “ $cd$ ” is divisible by 4. Since  $c = 3$  and  $d$  is either 4 or 6,  $d = 6$ . Then  $f$  must be 4, and “ $abcdef$ ” equals 123654.

If  $a = 3$  and  $c = 1$ , the digit sum of “ $abc$ ” is still  $4 + b$ , so again  $b = 2$ . The argument in the previous paragraph now leads to the solution 321654. Thus the problem admits two solutions.

*Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.*

**4.** A  $3 \times 2$  rectangle is divided up into six equal squares, each containing a bug. When a bell rings, the bugs jump either horizontally or vertically (they cannot jump diagonally and they stay within the rectangle) into a square adjacent to their previous square in any direction, although you cannot know in advance which exact square they will jump into. Every bug changes square; no bug stays put.

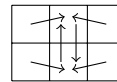
As an example, the ordered sextuplet  $(1, 1, 1, 1, 1, 1)$  (where this represents the result, not the movement) represents the situation where every bug jumped so that each square still had one bug in it (it could happen). Alternatively, two bugs could also land in the same square. An example (not the only way this could happen) of this might be represented by  $(2, 2, 1, 0, 0, 1)$  —see the diagram to the right. The first number in the sextuplet represents a corner square, the second represents a square on the middle of a side, and so on.

2	2	1
0	0	1

1. What is the average number of bugs per square in the 3 by 2 rectangle no matter how the bugs jump?
2. From the initial situation of one bug in every square, is it possible for three bugs to end up in the same square if the bell rings only once? If you think it is, write an ordered sextuplet like the two above where this could happen. If you think it can't happen, briefly explain why not.
3. From the initial situation of one bug in every square, it is certainly not possible in a  $3 \times 2$  rectangle for four bugs to end up in the same square if the bell rings only once. Write down the dimensions of the smallest rectangle for which it would be possible.
4. From the initial situation of one bug in every square, five bugs can never end up in the same square if the bell rings only once, no matter the size of the rectangle. In a few words, explain why not.
5. In the  $3 \times 2$  case, how many non-unique sextuplets (like  $(1, 1, 1, 1, 1, 1)$ ) are possible from the initial situation of one bug in every square, if the bell rings only once? You do not have to list them, although you might like to.

*Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.*

With six bugs and six squares, the average number of bugs per square is one. If the six bugs jump as in the figure to the right, there will be three bugs in each of the two middle squares after the first bell. To have four bugs in a square after the first bell, that square must have four neighbours (and all the neighbouring bugs must jump in, since the bug originally in any square must jump out). The smallest rectangle is therefore  $3 \times 3$ .



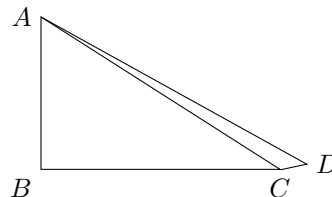
Each square has at most four neighbours, so at most four bugs can jump in at the first bell. The bug originally in any square must jump out at the first bell. Thus no square can contain five (or more) bugs after the first bell.

The bugs in the corner squares have two choices each for where to jump to. The bugs in the middle squares have three choices each. The figure lists the number of choices. The total number of jump-patterns is therefore  $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^4 \cdot 3^2 = 144$ .

2	3	2
2	3	2

*Also solved by ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.*

**5.** Pania and Rangi exercise weekly by running around two paddocks on their father's farm near Kakanui from  $A$  to  $B$  to  $C$  to  $D$  and then back to  $A$  (see the diagram). In a direct line from  $A$  to  $C$ , the distance is 6250 m.  $AB$  is shorter than  $BC$ .



1. If  $\triangle ABC$  is a right angled triangle in the ratio of  $3 : 4 : 5$ , with  $B$  at the right angle, find the lengths of the sides.
2. If  $\triangle ABC$  is a right angled triangle in the ratio of  $3 : 4 : 5$ , with  $B$  at the right angle, find the size of  $\angle CAB$  to one decimal place.
3. The angle at  $B$  is in fact a right angle, and  $AB$  and  $BC$  are whole metres in length, but the sides are not in the ratio of  $3 : 4 : 5$ . Find possible lengths for  $AB$  and  $BC$ .
4. The angle at  $D$  is not a right angle but is  $40^\circ$ , and  $CD$  is 600 m. Use this information to find the length of  $AD$ .

**Hint:** In any triangle  $XYZ$ , the following rules apply:

$$\text{Sine Law: } \frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z},$$

$$\text{Cosine Law: } x^2 = y^2 + z^2 - 2yz \cos X,$$

where side  $x$  is opposite to angle  $X$ , side  $y$  is opposite to angle  $Y$ , and side  $z$  is opposite to angle  $Z$ .



*Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.*

If  $|AB| : |BC| : |AC| = 3 : 4 : 5$ , then  $\frac{|AB|}{3} = \frac{|BC|}{4} = \frac{|AC|}{5} = \frac{6250}{5} = 1250$ . Therefore  $|AB| = 3 \cdot 1250 = 3750$  and  $|BC| = 4 \cdot 1250 = 5000$ . Moreover,  $\tan \angle CAB = \frac{|BC|}{|AB|} = \frac{5000}{3750} = \frac{4}{3}$ , so  $\angle CAB = \tan^{-1}(\frac{4}{3}) \approx 53.1^\circ$ .

If  $\angle ABC = 90^\circ$ , then  $|AB|^2 + |BC|^2 = |AC|^2$  by the Pythagorean Theorem. The task therefore is to find a Pythagorean triple where the largest part is a divisor of  $|AC| = 6250 = 2 \cdot 5^5$  other than 5. To that end, note that  $(n^2 + m^2)^2 = n^4 + 2n^2m^2 + m^4$  and that  $(n^2 - m^2)^2 = n^4 - 2n^2m^2 + m^4$ . Thus  $(n^2 + m^2)^2 - (n^2 - m^2)^2 = 4n^2m^2 = (2nm)^2$ , so

$$(n^2 + m^2)^2 = (2nm)^2 + (n^2 - m^2)^2.$$

Substituting integers for  $n$  and  $m$  in this equation will clearly generate many Pythagorean triples.

$n$	$m$	$n^2 + m^2$	$2nm$	$n^2 - m^2$
2	1	5	4	3
3	1	10	6	8
3	2	13	12	5
4	1	17	8	15
4	2	20	16	12
4	3	25	24	7
5	1	26	10	24
5	2	29	20	21
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

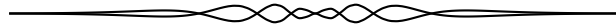
The triples  $10 : 6 : 8$  and  $20 : 16 : 12$  both reduce to the  $3 : 4 : 5$  triple (in some order), but  $25 : 24 : 7$  works out:  $|AC| = 25 \cdot 250 = 6250$ ,  $|AB| = 7 \cdot 250 = 1750$ , and  $|BC| = 24 \cdot 250 = 6000$ .

Cosine Law in  $\triangle ACD$  yields that

$$|AC|^2 = |AD|^2 + |CD|^2 - 2|AD||CD|\cos \angle ADC,$$

so if  $\angle ADC = 40^\circ$  and  $|CD| = 600$ , then  $6250^2 = |AD|^2 + 600^2 - 2|AD|600 \cos 40^\circ$ , so  $0 = |AD|^2 - 919.25333|AD| - 38702500$ . Solving this with the Quadratic Formula yields that  $|AD| = -5778.46$  (impossible) or  $|AD| = 6697.72$ .

*Our solver's formula for generating Pythagorean triples is called Euclid's Formula. This very useful formula generates all the interesting Pythagorean triples. The Wikipedia explains what we here mean by interesting.*



This issue's prize of one copy of *CruX Mathematicorum* for the best solutions goes to Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

# THE CONTEST CORNER

No. 5

Shawn Godin

The Contest Corner is a new feature of *Cru $x$  Mathematicorum*. It will be filling the gap left by the movement of Mathematical Mayhem and Skoliad to a new on-line journal in 2013. The column can be thought of as a hybrid of Skoliad, The Olympiad Corner and the old Academy Corner from several years back. The problems featured will be from high school and undergraduate mathematics contests with readers invited to submit solutions. Readers' solutions will begin to appear in the next volume.

Solutions can be sent to:

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Orleans, ON, CANADA  
K1C 2Z5

or by email to

[cru \$x\$ -contest@cms.math.ca](mailto:cru<math>x</math>-contest@cms.math.ca).

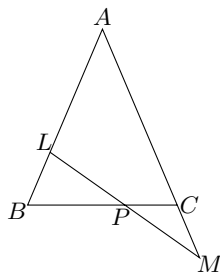
The solutions to the problems are due to the editor by **1 November 2013**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of Université de Saint-Boniface, Winnipeg, MB for translating the problems from English into French.



**CC21.** In the diagram  $\triangle ABC$  is isosceles with  $AB = AC$ . Prove that if  $LP = PM$ , then  $LB = CM$ .



**CC22.** Points  $A_1, A_2, \dots, A_{2k}$  are equally spaced around the circumference of a circle and  $k \geq 2$ . Three of these points are selected at random and a triangle is formed using these points as its vertices. Determine the probability that the triangle is acute.

**CC23.** The three-term geometric progression  $(2, 10, 50)$  is such that

$$(2 + 10 + 50) \times (2 - 10 + 50) = 2^2 + 10^2 + 50^2.$$

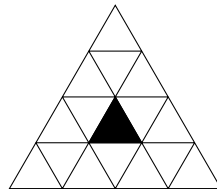
- (a) Generalize this (with proof) to other three-term geometric progressions.
- (b) Generalize (with proof) to geometric progressions of length  $n$ .

**CC24.** Given the equation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24.$$

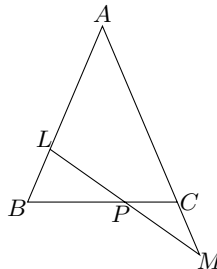
- (a) Prove that the equation has no integer solutions.
- (b) Does this equation have rational solutions? If yes, give an example. If no, prove it.

**CC25.** Alphonse and Beryl are playing a game, starting with the geometric shape shown. Alphonse begins the game by cutting the original shape into two pieces along one of the lines. He then passes the piece containing the black triangle to Beryl, and discards the other piece. Beryl repeats these steps with the piece she receives; that is to say, she cuts along the length of a line, passes the piece containing the black triangle back to Alphonse, and discards the other piece. This process continues, with the winner being the player who, at the beginning of his or her turn, receives only the black triangle. Is there a strategy that Alphonse can use to be guaranteed that he will win?



.....

**CC21.** Dans le diagramme,  $\triangle ABC$  est isocèle tel que  $AB = AC$ . Démontrer que si  $LP = PM$ , alors  $LB = CM$ .



**CC22.** Les points  $A_1, A_2, \dots, A_{2k}$  sont distribués à distances égales sur la circonférence d'un cercle ; aussi,  $k \geq 2$ . Si trois de ces points sont choisis au hasard et si un triangle est formé avec ces points comme sommets, déterminer la probabilité que ce triangle est aigu.

**CC23.** Trois termes d'une progression géométrique  $(2, 10, 50)$  sont tels que

$$(2 + 10 + 50) \times (2 - 10 + 50) = 2^2 + 10^2 + 50^2.$$

(a) Généraliser ce résultat à d'autres progressions géométriques dont on donne trois termes et en fournir une preuve.

(b) Généraliser ce résultat à d'autres progressions géométriques dont on donne  $n$  termes et en fournir une preuve.

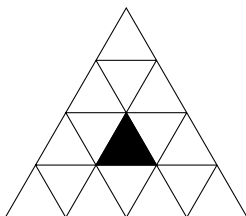
**CC24.** Soit l'équation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24.$$

(a) Démontrer que cette équation n'a aucune solution entière.

(b) Cette équation a-t-elle solution(s) rationnelle(s) ? Si oui, en fournir une. Si non, démontrer qu'il n'y en a pas.

**CC25.** Alphonse et Bernard s'amuse à un jeu qui démarre avec la forme géométrique indiquée. Alphonse commence le jeu en taillant la forme en deux suivant une de ses lignes. Il donne alors à Bernard le morceau qui contient le triangle noir, mettant l'autre morceau à la poubelle. Bernard répète la taille suivant une ligne et remet le morceau avec le triangle noir à Alphonse, mettant l'autre à la poubelle. Le processus continue ainsi. Le joueur gagnant est celui qui reçoit un simple triangle noir au début de son jeu. Alphonse a-t-il une stratégie qui assure qu'il va gagner ?



# THE OLYMPIAD CORNER

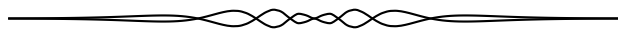
No. 303

Nicolae Strungaru

*The solutions to the problems are due to the editor by 1 November 2013.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.*



**OC81.** Find all triplets  $(x, y, z)$  of integers that satisfy

$$x^4 + x^2 = 7^z y^2.$$

**OC82.** The area and the perimeter of the triangle with sides 6, 8, 10 are equal. Find all triangles with integral sides whose area and perimeter are equal.

**OC83.** On a semicircle with diameter  $|AB| = d$  we are given points  $C$  and  $D$  such that  $|BC| = |CD| = a$  and  $|DA| = b$ , where  $a, b, d$  are different positive integers. Find the minimum possible value of  $d$ .

**OC84.** Let  $m, n$  be positive integers. Prove that there exist infinitely many pairs of relatively prime positive integers  $(a, b)$  such that

$$a + b \mid am^a + bn^b.$$

**OC85.** For any positive integer  $d$ , prove there are infinitely many positive integers  $n$  such that  $d(n!) - 1$  is a composite number.

.....

**OC81.** Trouver tous les triplets d'entiers  $(x, y, z)$  satisfaisant

$$x^4 + x^2 = 7^z y^2.$$

**OC82.** Dans le triangle dont les côtés mesurent 6, 8, 10, l'aire et le périmètre sont égaux. Trouver tous les triangles dont les côtés sont entiers et dont l'aire et le périmètre sont égaux.

**OC83.** Sur un demi-cercle de diamètre  $|AB| = d$ , on donne deux points  $C$  et  $D$  tels que  $|BC| = |CD| = a$  et  $|DA| = b$ , où  $a, b, d$  sont trois entiers positifs distincts. Trouver le minimum possible de la valeur de  $d$ .

**OC84.** Soit  $m$  et  $n$  deux entiers positifs. Montrer qu'il existe une infinité de couples d'entiers positifs relativement premiers  $(a, b)$  tels que

$$a + b \mid am^a + bn^b.$$

**OC85.** Montrer que pour tout entier positif  $d$ , il existe une infinité d'entiers positifs  $n$  tels que  $d(n!) - 1$  est un nombre composé.

## OLYMPIAD SOLUTIONS

**OC21.** A sequence of real numbers  $\{a_n\}$  is defined by  $a_0 \neq 0, 1$ ,  $a_1 = 1 - a_0$ , and  $a_{n+1} = 1 - a_n(1 - a_n)$  for  $n = 1, 2, \dots$ . Prove that for any positive integer  $n$ , we have

$$a_0 a_1 \cdots a_n \left( \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1.$$

(Originally question #1 from the 2008 China Western Mathematical Olympiad.)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Comănești, Romania. We give the solution of Krimker.

Since the equation  $x^2 - x + 1 = 0$  doesn't have any real solutions, it is clear that for all  $n$  we have  $a_n \neq 0$ . It is also easy to prove by induction that  $a_n \neq 1$ .

Now, using

$$a_n = \frac{1 - a_{n+1}}{1 - a_n},$$

we get

$$a_0 a_1 \cdots a_n = (1 - a_1) \frac{1 - a_2}{1 - a_1} \frac{1 - a_3}{1 - a_2} \cdots \frac{1 - a_{n+1}}{1 - a_n} = 1 - a_{n+1}$$

We are now ready to prove the statement by induction. Since

$$a_0 a_1 \left( \frac{1}{a_0} + \frac{1}{a_1} \right) = a_0 + a_1 = 1,$$

the statement is true for  $n = 1$ . Next assume that the statement is true for some value of  $n$ , then

$$\begin{aligned} a_0 a_1 \cdots a_n a_{n+1} & \left( \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}} \right) \\ & = a_0 a_1 \cdots a_n a_{n+1} \left( \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) + a_0 a_1 \cdots a_n \\ & = a_{n+1} + 1 - a_{n+1} = 1 \end{aligned}$$

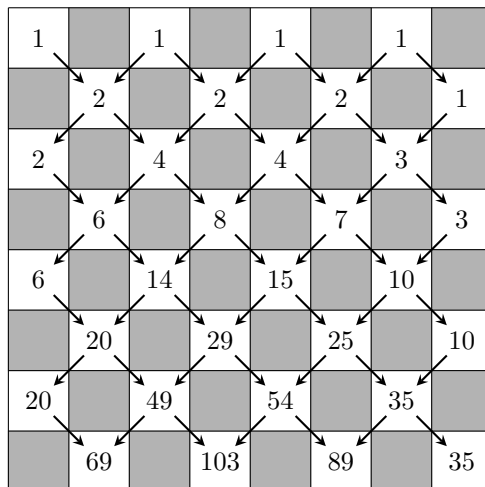
which completes the proof.

**OC22.** Consider a standard  $8 \times 8$  chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

(Originally question #1 from the 2008/9 British Mathematical Olympiad, Round 1.)

Solved by Geneviève Lalonde, Massey, ON.

We can label each white square with the number of (distinct) paths that reach the square from the top. As a result, the number of paths to each white square is the sum of the number of paths to the two white squares above it, or equal to the number of paths to the only white square above it.



Thus there are a total of 296 zig-zag paths.

**OC23.** Determine all nonnegative integers  $n$  such that

$$n(n - 20)(n - 40)(n - 60) \cdots r + 2009$$

is a perfect square where  $r$  is the remainder when  $n$  is divided by 20.

(Originally question #2 from the 40th Austrian Mathematical Olympiad, National Competition, Final Round (G. Baron, Vienna).)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA and Titu Zvonaru, Comănești, Romania. We give the solution of Curtis, modified by the editor.

We claim that  $n = 16$  is the only solution.

First let's observe that if  $n \geq 40$  then

$$n(n - 20)(n - 40) \equiv n(n + 1)(n + 2) \equiv 0 \pmod{3} .$$

Thus

$$n(n-20)(n-40)\cdots r + 2009 \equiv 2 \pmod{3}.$$

Hence, no  $n \geq 40$  produces a perfect square.

If  $20 \leq n < 39$  then

$$n(n-20) + 2009 = k^2 \Leftrightarrow (n-10)^2 = k^2 - 1909.$$

But then

$$10^2 \leq k^2 - 1909 \leq 19^2 \Rightarrow 2009 \leq k^2 \leq 2270 \Rightarrow 44.8 \leq k \leq 47.6.$$

If  $k = 45$  then  $(n-10)^2 = 116$  which is not possible.

If  $k = 46$  then  $(n-10)^2 = 207$  which is not possible.

If  $k = 47$  then  $(n-10)^2 = 300$  which is not possible.

Last, if  $n < 20$  then we have

$$n + 2009 = k^2.$$

Thus

$$2009 \leq k^2 < 2029 \Rightarrow k = 45 \Rightarrow n = 16.$$

**OC24.** Let  $O$  be the circumcentre of the triangle  $ABC$ . Let  $K$  and  $L$  be the intersection points of the circumcircles of the triangles  $BOC$  and  $AOC$  with the bisectors of the angles at  $A$  and  $B$  respectively. Let  $P$  be the midpoint of  $KL$ ,  $M$  symmetrical to  $O$  relative to  $P$  and  $N$  symmetrical to  $O$  relative to  $KL$ . Prove that  $KLMN$  is cyclic.

(Originally question #2 from the 16th Macedonian Mathematical Olympiad.)

Similar solutions by Michel Bataille, Rouen, France; Mihai-Ioan Stoënescu, Bischwiller, France and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

We will solve the more general problem:

Let  $KOL$  be any triangle and let  $P$  be the midpoint of  $KL$ . Let  $M$  be the symmetrical image of  $O$  relative to  $P$  and  $N$  the symmetrical image of  $O$  relative to  $KL$ . Then  $KLMN$  is cyclic.

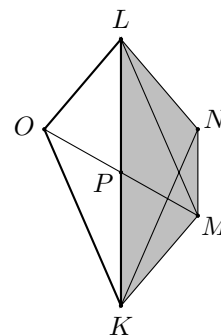
If  $LO = OK$ , then  $M = N$  and  $KLMN$  is a triangle.

Otherwise, the diagonals of  $LMKO$  halve each other, and hence  $LMKO$  is a parallelogram. Thus

$$\angle LMK = \angle LOK. \quad (1)$$

By symmetry we also have

$$\angle LNK = \angle LOK. \quad (2)$$





Combining (1) and (2) we get  $\angle LMK = \angle LNK$  which proves the desired result.

**OC25.** Show that the inequality  $3^{n^2} > (n!)^4$  holds for all positive integers  $n$ .  
(Originally question #1 from the 40th Austrian Mathematical Olympiad, National Competition, Final Round (G. Baron, Vienna).)

We provide the similar solutions by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA and Titu Zvonaru, Comănești, Romania.

We first prove by induction the proposition  $P_1(n)$  that  $3^{2n+1} > (n+1)^4$  for all  $n \geq 1$ .

As  $3^3 > 2^4$ ,  $P_1(1)$  is true. Suppose  $P_1(n)$  is true for some  $n \geq 1$ . Since  $\left(\frac{n+2}{n+1}\right)^2 = \left(1 + \frac{1}{n+1}\right)^2 \leq \left(1 + \frac{1}{n+1}\right)^{n+1} < e < 3$ , we have  $9 > \frac{(n+2)^4}{(n+1)^4}$ , so

$$\begin{aligned} 3^{2n+3} &= 3^2 3^{2n+1} > 9(n+1)^4 \\ &> \frac{(n+2)^4}{(n+1)^4} (n+1)^4 = (n+2)^4, \end{aligned}$$

hence  $P_1(n+1)$  is true and thus  $P_1(n)$  is true for all  $n \geq 1$ .

Now we can prove the original problem by induction. Let  $P_2(n)$  be the proposition that  $3^{n^2} > (n!)^4$ , then  $P_2(1)$  is clearly true since:  $3^1 > 1^4$ . Suppose  $P_2(n)$  is true for some  $n \geq 1$ , then

$$\begin{aligned} 3^{(n+1)^2} &= 3^{n^2} 3^{2n+1} \\ &> (n!)^4 (n+1)^4 = [(n+1)!]^4. \end{aligned}$$

Thus  $P_2(n+1)$  is true. This solves the problem.

*Editor's Note:* The inequality  $3^{2n+1} > (n+1)^4$  has a very simple combinatorial proof. Let  $A = \{1, 2, 3, \dots, n+1\} \times \{1, 2, 3, \dots, n+1\} \times \{1, 2, 3, \dots, n+1\} \times \{1, 2, 3, \dots, n+1\}$  and  $B = \{f : \{1, 2, 3, \dots, 2n+1\} \rightarrow \{0, 1, 2\}\}$ . Then it is easy to construct an injective function from  $A$  to  $B$ .

For example:  $(a, b, c, d) \rightarrow f$  where  $f : \{1, 2, 3, \dots, 2n+1\} \rightarrow \{0, 1, 2\}$  is defined by

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 2 & \text{if } x = b \\ 1 & \text{if } x = n+1+c \\ 2 & \text{if } x = n+1+d \\ 0 & \text{otherwise} \end{cases}$$

is such a function. [Note that  $c = n+1$  if and only if  $f$  only takes the value 1 once, and  $d = n+1$  if and only if  $f$  only takes the value 2 once.]

## BOOK REVIEWS

Amar Sodhi

*Voltaire's Riddle: Micromégas and the Measure of All Things* by Andrew Simoson  
Mathematical Association of America, 2010

ISBN: 978-0-88385-345-0, Hardcover, 377 + xvii pages, US\$58.95

Reviewed by **T. Archibald**, *Simon Fraser University, Burnaby, BC*

Voltaire (1694-1778) is not a name that often comes up in connection with mathematics, except for the fact that he was the lover of Émilie de Châtelet, who wrote about Newton. Best known as a satirist and a quick-witted society figure who occasionally found himself at odds with authority, he is likely now most remembered for his book *Candide*, about the difficulties of an innocent young man in understanding the hypocritical ways of the world. In this unexpected volume, Voltaire's wit is taken as the jumping-off point for a dozen studies in mathematics, physics and astronomy that are accompanied by exercises both mathematical and general. The result is a work that puzzles and sometimes fascinates, representing broad learning and showing connections between mathematics and history that are often illuminating. If it occasionally reaches a bit too far for the beginner, it is on the whole accessible both as a work of mathematics and as one of general learning.

The central conceit of the book arises in trying to understand Voltaire's story *Micromégas*, about a visitor to Earth from Sirius of enormous size. This being, on departing from Earth, offered to the Paris Academy of Sciences a book that contained the answers to "all things", that is, all questions about the nature of the universe. The riddle of Simoson's title arises from the fact that the book, in Voltaire's story, is found to be blank, and one thread connecting the diverse material in the book links possible ways of figuring out why this is so.

But even without that unifying thread, the many different topics – about the shape of the earth, the scale of living beings, trajectories in flatland, the precession of the Earth's poles – share an eighteenth-century concern with the importance of mathematics in understanding the universe. Each topic is introduced by a "vignette" in which historical or literary figures are characters in discussions ranging from the zodiac to space travel. The topics work well for the mathematically inclined exactly because they aren't typical textbook problems. Often the reader will be pushed much farther than in a typical calculus book, and one skill that the book will develop is a patience for working one's way through some complicated (yet mostly elementary) calculations to get at a question that isn't just a "math book problem". The exercises develop this, and the comments in the back will guide the reader. The exercises go beyond this, however, and pose lots of fresh problems that the *CruX* reader may enjoy. As a sample, find the regular  $n$ -gon of radius 5 for which the perimeter is closest to 30. The overall level is roughly second year university, with a certain amount quite accessible to a good high school student. One of the MAA's Dolciani Mathematical Expositions, Simoson's unusual and entertaining book does what many books in that series do: take mathematics in an unusual direction that broaden our notions of what the subject can be.

# FOCUS ON . . .

No. 2

Michel Bataille

## The Geometry Behind the Scene

### Introduction

The algebraic presentation of some problems (and their solutions) can mask the connection with a geometric problem. For familiar examples, think of a diophantine equation, such as  $x^2 - 4xy + 6y^2 - 2x - 20y = 29$  from Problem 2850 [2003 : 242 ; 2004 : 302], which can be simplified by introducing the centre of the corresponding ellipse; or recall, in the proof of an inequality, the link between a constraint such as  $abc = a + b + c$  and the angles of a triangle. We will consider in detail two less obvious examples and unveil their geometric background.

### First example: A hidden hyperboloid

Virgil Nicula's Problem 3309 [2008 : 45,48 ; 2009 : 59], asks for a necessary and sufficient condition on nonzero real numbers  $\alpha, \beta, \gamma$  for the system

$$\begin{aligned}\alpha x + \beta y + \gamma z &= 1 \\ xy + yz + zx &= 1\end{aligned}$$

to have a unique solution. The featured solution (by G. Tsapakidis) is nice and elementary, resting on the properties of quadratic equations. However, faced with this problem, I, and probably many other solvers, cannot help seeing a plane  $\mathcal{P}$  in  $\alpha x + \beta y + \gamma z = 1$  and a quadric  $\mathcal{H}$  in  $xy + yz + zx = 1$  (namely a hyperboloid with two sheets). I suspect that this is also the origin of the problem! Once this observation has been made, a less elementary but more illuminating solution can follow: the uniqueness of a solution just means that  $\mathcal{P}$  is tangent to the hyperboloid at some point  $(x_0, y_0, z_0)$ . Since the equation of the plane tangent to  $\mathcal{H}$  at  $(x_0, y_0, z_0)$  is

$$(x - x_0)(y_0 + z_0) + (y - y_0)(z_0 + x_0) + (z - z_0)(x_0 + y_0) = 0,$$

$\mathcal{P}$  is tangent to  $\mathcal{H}$  if and only if

$$\frac{y_0 + z_0}{\alpha} = \frac{z_0 + x_0}{\beta} = \frac{x_0 + y_0}{\gamma} \quad (1)$$

for some  $(x_0, y_0, z_0)$  such that

$$x_0 y_0 + y_0 z_0 + z_0 x_0 = 1 \quad (2)$$

and

$$\alpha x_0 + \beta y_0 + \gamma z_0 = 1. \quad (3)$$

Now, the common value of the ratios in (1) is

$$\frac{x_0(y_0 + z_0) + y_0(z_0 + x_0) + z_0(x_0 + y_0)}{\alpha x_0 + \beta y_0 + \gamma z_0} = 2,$$

and solving (1) for  $x_0, y_0, z_0$ , we obtain

$$x_0 = \gamma + \beta - \alpha, \quad y_0 = \alpha + \gamma - \beta, \quad z_0 = \alpha + \beta - \gamma. \quad (4)$$

Plugging these values in either of the conditions (2) or (3) yields the desired relation

$$\alpha^2 + \beta^2 + \gamma^2 + 1 = 2(\alpha\beta + \beta\gamma + \gamma\alpha).$$

### A sphere in a cylinder

Our second example is Problem 3455 [2009 : 326, 328 ; 2010 : 344], a problem that I made up (here I am quite sure of its geometric origin!). Kee-Wai Lau's featured solution is short and elegant but does not convey the geometrical flavor. We offer the following generalization and a solution based on the geometrical ideas that underlie the problem:

*Given real numbers  $a, b, c$ , not all zero, and  $k > 0$ , find the minimal value of  $x^2 + y^2 + z^2$  over all real numbers  $x, y, z$  subject to  $f(x, y, z) \geq k^2$  where*

$$f(x, y, z) = (b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 - 2abxy - 2bcyz - 2cazx.$$

Noticing that

$$f(x, y, z) = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 = \|\vec{\Omega} \times \overrightarrow{OM}\|^2 \quad (5)$$

where  $\vec{\Omega}(a, b, c)$  and  $M(x, y, z)$  in a system of axes with origin  $O$  in space, we see that  $f(x, y, z) = k^2$  is the equation of a cylinder  $\mathcal{C}$  whose generators are parallel to  $\vec{\Omega}$ . Thus, the problem can be interpreted as the search for the minimal distance between  $O$  and a point  $M$  exterior to or on the cylinder  $\mathcal{C}$  (alternatively for the biggest sphere with centre  $O$  entirely contained in  $\mathcal{C}$ ). Clearly, this minimal distance  $OM$  is attained when  $M$  is on the normal section through  $O$ , that is, the circle intersection of  $\mathcal{C}$  with the plane  $ax + by + cz = 0$ .

With the help of (5), this approach can be specified as follows:

Since  $\|\vec{\Omega} \times \overrightarrow{OM}\|^2 = \|\vec{\Omega}\|^2 \|\overrightarrow{OM}\|^2 \sin^2 \theta$  where  $\theta$  is the angle of  $\vec{\Omega}$  and  $\overrightarrow{OM}$ , we have

$$x^2 + y^2 + z^2 = \|\overrightarrow{OM}\|^2 \geq \frac{\|\vec{\Omega} \times \overrightarrow{OM}\|^2}{\|\vec{\Omega}\|^2} = \frac{f(x, y, z)}{a^2 + b^2 + c^2}$$

for all  $x, y, z$  and so

$$x^2 + y^2 + z^2 \geq \frac{k^2}{a^2 + b^2 + c^2}$$

when  $f(x, y, z) \geq k^2$  with equality if  $M$  is on  $\mathcal{C}$  and  $\overrightarrow{OM}$  is orthogonal to  $\overrightarrow{\Omega}$ . Thus, the desired minimum is  $\frac{k^2}{a^2 + b^2 + c^2}$  (and the biggest sphere with centre  $O$  interior to  $\mathcal{C}$  is the one with radius  $\frac{k}{\sqrt{a^2 + b^2 + c^2}}$ ).

### An exercise

Perhaps the reader would like to interpret the following problem (slightly adapted from problem 11301 in [1]) in plane geometry and discover a variant of solution:

*Show that for any complex numbers  $a, b, c$ ,*

$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq \frac{9}{16}(|a|^2 + |b|^2 + |c|^2)^2.$$

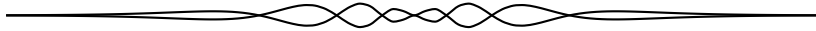
**Hint:** note that

$$ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) = (b - a)(a - c)(c - b)(a + b + c)$$

and considering  $a, b, c$  as the complex affixes of  $A, B, C$ , express  $OA^2 + OB^2 + OC^2$  with the help of the isobarycentre of  $A, B, C$ .

## References

- [1] *Amer. Math. Monthly*, Vol. 116, No 1, January 2009, p. 85-6.



# PROBLEM OF THE MONTH

No. 1

Shawn Godin

*This column is dedicated to the memory of former **CRUX with MAYHEM** Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students; other people's students, through his work on mathematics contests and outreach programs; and the readers of **CRUX with MAYHEM**. The problem of the month is a short article that features a problem and its solution that we know Jim would have liked.*

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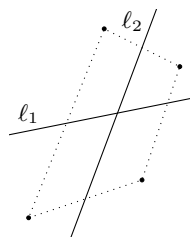
As an undergraduate student, I was fortunate to have Ross Honsberger as a professor a couple of times. One of the classes was on problem solving, and, in the first class, Professor Honsberger presented us with 100 problems. The remainder of the semester was spent solving and discussing the problems and the techniques used to solve them. One that remains in my memory is the following.

*One million points are drawn in a plane such that no three are collinear. Prove that you can draw a line such that half the points are on one side of the line and half are on the other.*

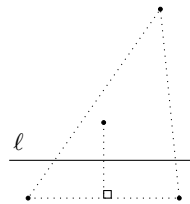
I remember being perplexed by this when I first saw it. I probably started looking at easier cases. The easiest case, with two points, is easy, we can choose the perpendicular bisector of the segment joining the two points.

Let's consider the next case with 4 points. We have two cases:

**Case 1:** The four points are vertices of a convex quadrilateral. In this case, note that if we draw the line through the midpoints of opposite sides, we get the desired result (as a matter of fact, a line through any two points on opposite sides works, as long as you don't use one of the original points). In the diagram  $\ell_1$  and  $\ell_2$  are lines that satisfy the condition.



**Case 2:** The four points are not the vertices of a convex quadrilateral, thus one of the points would be inside the triangle with vertices at the other three points. In this case, if we drop a perpendicular from the point on the inside to one of the sides, and then create the perpendicular bisector of the segment from the inside point to the base of the perpendicular, it has the desired property. In the diagram below  $\ell$  is one of three such lines.



It doesn't take long to see that things will get very complex very quickly. In the case of six points, we will get 4 cases corresponding to when the *convex hull* (the smallest convex shape that contains all the points) being a hexagon, a pentagon, a quadrilateral or a triangle. As the number of points grows, so will the number of cases. We need a new strategy.

Since we have showed that it works for 2 and 4 points, we may try an inductive proof. Assuming that we have shown that the process is possible for  $2n$  points, for some value of  $n$ , now we will look at the case of  $2n + 2$  points. By the induction hypothesis, each collection of  $2n$  points has a line that satisfies the desired property. If, for some collection of  $2n$  points, with line  $\ell$  that satisfies the property, the extra 2 points are on opposite sides of  $\ell$ , we are done. Unfortunately, there is no guarantee that this can be done in general.

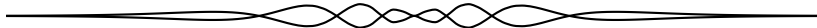
We may be able to complete the induction proof with an "elementary" argument, but that could be quite long and complex. The solution to the problem, which I didn't produce at the time, still brings tears to my eyes because of its beauty.

**Solution:** Construct a line so that all points are on one side and the line is skew to all possible lines through every possible pair of points. Then, if we translate this line towards the points, it will encounter them one at a time. Thus we can move the line until it has passed through 500 000 points. After the line has passed through 500 000 points, but before it hits the 500 001<sup>st</sup> point, it satisfies the condition of the problem.

So there it is, a seemingly "obvious" property shown to be true by looking at it from the right perspective. Thank you Professor Honsberger!

You may want to try your hand at a similar problem given to me by Dr. Robert Craigen from the University of Manitoba.

*One hundred planets in a solar system are moving in some complex pattern that keeps them from crashing into each other. Each planet has radius exactly 1000 km. Planets are lit by the following process: whenever a point on the surface of one planet can be seen from from a point on the surface of another, both points are lit. Conversely it is dark at any point on any planet from which none of the other planets are visible. Prove that the total dark area on all the planets together is a constant, and determine that constant.*



# About the Japanese theorem

Nicușor Minculete, Cătălin Barbu and Gheorghe Szöllösy

Dedicated to the memory of the great professor,  
Laurențiu Panaitopol

## Abstract

The aim of this paper is to present three new proofs of the Japanese Theorem and several applications.

## 1 Introduction

A *cyclic quadrilateral* (or *inscribed quadrilateral*) is a convex quadrilateral whose vertices all lie on a single circle. Given a cyclic quadrilateral  $ABCD$ , denote by  $O$  the circumcenter,  $R$  the circumradius, and by  $a, b, c, d, e$ , and  $f$  the lengths of the segments  $AB, BC, CD, DA, AC$  and  $BD$  respectively. Recall Ptolemy's Theorems [4, pages 62 and 85] for a cyclic quadrilateral  $ABCD$ :

$$ef = ac + bd \quad (1)$$

and

$$\frac{e}{f} = \frac{ad + bc}{ab + cd}. \quad (2)$$

Another interesting relation for cyclic quadrilaterals is given by the Japanese Theorem ([4]). This relates the radii of the incircles of the triangles  $BCD$ ,  $CDA$ ,  $DAB$  and  $ABC$ , denoted by  $r_a, r_b, r_c$ , and  $r_d$  respectively, in the following way:

$$r_a + r_c = r_b + r_d. \quad (3)$$

In [8], W. Reyes gave a proof of the Japanese Theorem using a result due to the French geometer Victor Thébault. Reyes mentioned that a proof of this theorem can be found in [3, Example 3.5(1), p. 43, 125-126]. In [9, p. 155], P. Yiu found a simple proof of the Japanese Theorem. In [5], D. Mihalca, I. Chițescu and M. Chiriță demonstrated (3) using the identity  $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ , which is true in any triangle  $ABC$ , where  $r$  is the inradius of  $ABC$ , and in [7], M. E. Panaitopol and L. Panaitopol show that

$$r_a + r_c = R(\cos x + \cos y + \cos z + \cos u - 2) = r_b + r_d,$$

where  $m(\widehat{AB}) = 2x$ ,  $m(\widehat{BC}) = 2y$ ,  $m(\widehat{CD}) = 2z$  and  $m(\widehat{AD}) = 2u$ . In this paper, we will give three new proofs.



## 2 MAIN RESULTS

**Lemma 1** *If  $ABCD$  is a cyclic quadrilateral, then  $\frac{e}{f} = \frac{(a+b+e)(c+d+e)}{(b+c+f)(a+d+f)}$ .*

Proof. From (2), we deduce the equality  $abe + cde = adf + bcf$ . Adding the same terms in both parts of this equality, we have

$$abe + cde + e^2 f + aef + def + bef + cef + ef^2 = adf + bcf + ef^2 + aef + def + bef + cef + e^2 f.$$

But, from equation (1), we have  $e^2 f = e(ac + bd) = ace + bde$  and  $ef^2 = f(ac + bd) = acf + bdf$ . Therefore, we obtain

$$\begin{aligned} &abe + cde + ace + bde + aef + def + bef + cef + ef^2 \\ &= adf + bcf + acf + bdf + aef + def + bef + cef + e^2 f, \end{aligned}$$

which means that  $e(b+c+f)(a+d+f) = f(a+b+e)(c+d+e)$ , and the Lemma follows.  $\square$

In the following we give a property of a cyclic quadrilateral which we use in proving the Japanese Theorem.

**Theorem 1** *In any cyclic quadrilateral there is the following relation:*

$$r_a \cdot r_c \cdot e = r_b \cdot r_d \cdot f \tag{4}$$

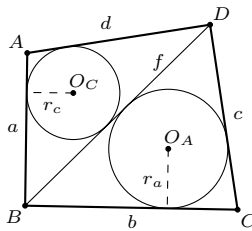


Figure 1

Proof. For triangles  $BCD$  and  $ABD$ , we write the equations [2, p. 11]

$$r_a = \frac{b+c-f}{2} \tan \frac{C}{2}, \quad r_c = \frac{a+d-f}{2} \tan \frac{A}{2}.$$

But  $\tan \frac{A}{2} \cdot \tan \frac{C}{2} = 1$ , because  $A + C = \pi$ . Therefore, we obtain

$$4r_a r_c = ab + cd + ac + bd - f(a+b+c+d) + f^2,$$

so from (1), we deduce

$$4r_a r_c = ab + cd + f(e + f - a - b - c - d).$$

Multiplying by  $e$ , we obtain

$$4r_a r_c e = e(ab + cd) + ef(e + f - a - b - c - d). \tag{5}$$

Similarly, we deduce that

$$4r_b r_d f = f(ad + bc) + ef(e + f - a - b - c - d). \quad (6)$$

Combining (2), (5) and (6) we obtain (4).  $\square$

G. Szöllösy, [6], proposed (7) below for a cyclic quadrilateral. We provide two new proofs of this relation.

**Theorem 2** *In a cyclic quadrilateral, the identity*

$$\frac{abe}{a+b+e} + \frac{cde}{c+d+e} = \frac{bcf}{b+c+f} + \frac{adf}{a+d+f}, \quad (7)$$

holds.

Proof I. Let  $m(\widehat{AB}) = 2x$ ,  $m(\widehat{BC}) = 2y$ ,  $m(\widehat{CD}) = 2z$  and  $m(\widehat{AD}) = 2t$ . Then  $x + y + z + t = \pi$ . By definition,  $a = 2R \sin x$ ,  $b = 2R \sin y$ ,  $c = 2R \sin z$ ,  $d = 2R \sin t$ ,  $e = 2R \sin(x + y) = 2R \sin(z + t)$ ,  $f = 2R \sin(x + t) = 2R \sin(y + z)$ . Equation (7) now follows from the trigonometric identity

$$\begin{aligned} \frac{\sin \alpha \sin \beta \sin(\alpha + \beta)}{\sin \alpha + \sin \beta + \sin(\alpha + \beta)} &= 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\alpha + \beta}{2} \\ &= \left( \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \right) \cos \frac{\alpha + \beta}{2}, \end{aligned}$$

for any  $\alpha, \beta \in \mathbb{R}$ , with  $\sin \alpha + \sin \beta + \sin(\alpha + \beta) \neq 0$ .  $\square$

Proof II. From Lemma 1, we have

$$\frac{e}{(a+b+e)(c+d+e)} = \frac{f}{(b+c+f)(a+d+f)}. \quad (8)$$

From (2),  $abe + cde = adf + bcf$ , we obtain

$$ab(c+d+e) + cd(a+b+e) = bc(a+d+f) + ad(b+c+f). \quad (9)$$

Combining (8) and (9), we deduce

$$\frac{abe(c+d+e) + cde(a+b+e)}{(a+b+e)(c+d+e)} = \frac{bcf(a+d+f) + adf(b+c+f)}{(b+c+f)(a+d+f)}.$$

Consequently, we obtain (7).  $\square$

Next, we present three new proofs of the Japanese Theorem.

**Theorem 3** *(The Japanese Theorem) Let  $ABCD$  be a convex quadrilateral inscribed in a circle. Denote by  $r_a, r_b, r_c$ , and  $r_d$  the inradii of the triangles  $BCD$ ,  $CDA$ ,  $DAB$ , and  $ABC$  respectively. Then  $r_a + r_c = r_b + r_d$ .*

Proof I. Recall [4, Section 298i, p. 190] that for any triangle  $ABC$  with circumradius  $R$  and inradius  $r$ , we have the relation

$$r = \frac{abc}{2R(a+b+c)}.$$

In particular, for our four triangles we have

$$r_a = \frac{bcf}{2R(b+c+f)}, r_b = \frac{cde}{2R(c+d+e)}, r_c = \frac{adf}{2R(a+d+f)} \text{ and } r_d = \frac{abe}{2R(a+b+e)}.$$

The theorem then follows immediately from (7). □

Proof II. Applying the equations for the inradii that we used in the first proof to triangles  $BCD$  and  $ABD$ , we obtain

$$\begin{aligned} r_a + r_c &= r_a r_c \cdot \left( \frac{1}{r_a} + \frac{1}{r_c} \right) = \frac{r_a r_c}{f} \cdot \left( \frac{f}{r_a} + \frac{f}{r_c} \right) \\ &= \frac{r_a r_c}{f} \cdot \frac{2R}{abcd} \cdot [abc + abd + acd + bcd + f(ad + bc)]. \end{aligned} \quad (10)$$

Similarly, for triangles  $CDA$  and  $ABC$ , we deduce

$$r_b + r_d = \frac{r_b r_d}{e} \cdot \frac{2R}{abcd} \cdot [abc + abd + acd + bcd + e(ab + cd)]. \quad (11)$$

From Equation (2),  $e(ab + cd) = f(ad + bc)$ . Plug this together with Equation (4) into equations (10) and (11), and the theorem follows. □

Proof III. In the cyclic quadrilateral  $ABCD$  we let  $I_a; I_b; I_c$ , and  $I_d$  denote the incenters of triangles  $BCD; DAC; ABD$ , and  $ABC$  respectively (see Figure 2).

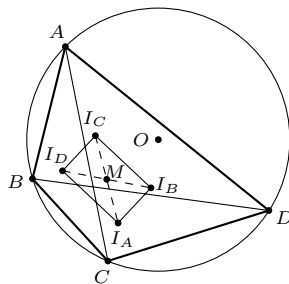


Figure 2

A theorem attributed to Fuhrmann [4, Section 422, p. 255] says that the quadrilateral  $I_a I_b I_c I_d$  is a rectangle. See also [9, p. 154] for a neat proof. Let  $M$  be a point so that  $I_a I_c \cap I_b I_d = \{M\}$ , so  $M$  is the midpoint of the diagonals  $I_a I_c$  and  $I_b I_d$ . The following theorem has been attributed to Apollonius [2, p. 6]: *In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects*

*the third side.* We apply Apollonius's Theorem to the triangles  $I_aOI_c$  and  $I_bOI_d$ , where  $O$  is the circumcenter of the cyclic quadrilateral  $ABCD$ , and we obtain the relations  $4OM^2 = 2(OI_a^2 + OI_c^2) - I_aI_c^2$  and  $4OM^2 = 2(OI_b^2 + OI_d^2) - I_bI_d^2$ , whence, and because  $I_aI_c = I_bI_d$ ,

$$OI_a^2 + OI_c^2 = OI_b^2 + OI_d^2. \quad (12)$$

Euler's formula for the distance  $d$  between the circumcentre ( $O$ ) and incentre ( $I$ ) of a triangle is given by  $d^2 = R^2 - 2Rr$ , where  $R$  and  $r$  denote the circumradius and inradius respectively [2, p. 29]. For a proof using complex numbers we mention the book of T. Andreescu and D. Andrica [1]. In our case, the triangles  $ABC$ ,  $BCD$ ,  $CDA$ ,  $DAB$  have the same circumcircle. In these triangles we apply Euler's relation. Hence, (12) becomes  $R^2 - 2Rr_a + R^2 - 2Rr_c = R^2 - 2Rr_b + R^2 - 2Rr_d$ , and the theorem follows.  $\square$

### 3 APPLICATIONS

If for a triangle  $ABC$  the points  $A'$ ,  $B'$ , and  $C'$  are the points of contact between the sides  $BC$ ,  $AC$ , and  $AB$  and the three excircles, respectively, then the segments  $AA'$ ,  $BB'$ , and  $CC'$  meet at one point, which is called the *Nagel point*. Denote by  $O$  the circumcenter,  $I$  the incenter,  $N$  the Nagel point,  $R$  the circumradius, and  $r$  the inradius of  $ABC$ . An important distance is  $ON$  and it is given by

$$ON = R - 2r. \quad (13)$$

Equation (13) gives the geometric difference between the quantities involved in Euler's inequality  $R \geq 2r$ . A proof using complex numbers is given in the book of T. Andreescu and D. Andrica [1].

**Application 1.** *Let  $ABCD$  be a convex quadrilateral inscribed in a circle with the center  $O$ . Denote by  $N_a, N_b, N_c, N_d$  the Nagel points of the triangles  $BCD$ ,  $CDA$ ,  $DAB$ , and  $ABC$ , respectively. Then the relation  $ON_a + ON_c = ON_b + ON_d$  holds.*

*Proof.* From the Japanese Theorem, we have  $r_a + r_c = r_b + r_d$ . Therefore we obtain  $R - 2r_a + R - 2r_c = R - 2r_b + R - 2r_d$ . The statement of the Theorem now follows from (13).  $\square$

Our final application follows quickly from (3) and (4).

**Application 2.** *In any cyclic quadrilateral there are the following relations:*

$$f\left(\frac{1}{r_a} + \frac{1}{r_c}\right) = e\left(\frac{1}{r_b} + \frac{1}{r_d}\right)$$

and

$$e(r_a^2 + r_c^2) = f(r_b^2 + r_d^2).$$

**Acknowledgements.** We would like to thank the anonymous reviewer for providing valuable comments to improve the manuscript.

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# PROBLEMS

*Solutions to problems in this issue should arrive no later than 1 November 2013. An asterisk (\*) after a number indicates that a problem was proposed without a solution.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.*

**Note:** *Due to an editorial mix-up, problem 3724 [2012 : 105, 106] already appeared as problem M504 [2011 : 346, 347]. In this issue we replace 3724 with a new problem.*

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**3724.** *Replacement. Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

The edge-lengths of a cyclic quadrilateral are 7, 8, 4, 1, in that order. What are the lengths of the diagonals?

**3741.** *Proposed by Péter Ivády, Budapest, Hungary.*

Find the largest value of  $a$  and the smallest value of  $b$  for which the inequalities

$$\frac{ax}{a+x^2} < \sin x < \frac{bx}{b+x^2},$$

hold for all  $0 < x < \frac{\pi}{2}$ .

**3742.** *Proposed by Michel Bataille, Rouen, France.*

In a scalene triangle  $ABC$ , let  $K, L, M$  be the feet of the altitudes from  $A, B, C$ , and  $P, Q, R$  be the midpoints of  $BC, CA, AB$ , respectively. Let  $LM$  and  $QR$  intersect at  $X$ ,  $MK$  and  $RP$  at  $Y$ ,  $KL$  and  $PQ$  at  $Z$ . Show that  $AX, BY, CZ$  are parallel.

**3743.** *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Two equal circles are tangent to the parabola  $y = x^2$  at the same point. One of the circles is also tangent to the  $x$ -axis, while the other is tangent to the  $y$ -axis. Find the radius of the circles. *This problem was inspired by problem 3732 [2012 : 149, 151].*

**3744.** *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let  $a, b, c$  be positive real numbers with sum 4. Prove that

$$\frac{a^8 + b^8}{(a^2 + b^2)^2} + \frac{b^8 + c^8}{(b^2 + c^2)^2} + \frac{c^8 + a^8}{(c^2 + a^2)^2} + abc \geq a^3 + b^3 + c^3.$$

**3745.** *Proposed by Abdilkadir Altıntaş, mathematics teacher, Emirdağ, Turkey.*

In the square  $ABCD$  the semicircle with diameter  $AD$  intersects the quarter circle with centre  $C$  and radius  $CD$  in the point  $P$ . Show that  $PB = \sqrt{2}AP$ .

**3746.** *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Let  $Q(n)$  denote the sum of the digits of the positive integer  $n$ . Prove that there are infinitely many positive integers  $n$  such that

$$Q(n) + Q(n^2) + Q(n^3) = [Q(n)]^2.$$

*This is an extension of problem 3506 [2010 : 45, 47; 2011 : 57, 58].*

**3747.** *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let  $a, b, c$  be real numbers with  $a + b + c = 0$  and  $c \geq 1$ . Prove that

$$a^4 + b^4 + c^4 - 3abc \geq \frac{3}{8}.$$

**3748★.** *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given three mutually external circles in general position, there will exist six distinct lines that are common internal tangents to pairs of the circles. Prove that if three of those common tangents, one to each pair of the circles, are concurrent, then the other three common tangents are also concurrent.

**3749.** *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let  $D$  and  $E$  be arbitrary points on the sides  $BC$  and  $AC$  of a triangle  $ABC$ . Prove that

$$\sqrt{[ADE]} + \sqrt{[BDE]} \leq \sqrt{[ABC]},$$

where  $[XYZ]$  denotes the area of triangle  $XYZ$ .

**3750.** *Proposed by Michel Bataille, Rouen, France.*

Let  $T_k = 1 + 2 + \dots + k$  be the  $k^{\text{th}}$  triangular number. Find all positive integers  $m, n$  such that  $T_m = 2T_n$ .

.....

**3724.** *Remplacement. Proposé par Richard K. Guy, Université de Calgary, Calgary, AB.*

Les longueurs des côtés d'un quadrilatère cyclique sont, dans l'ordre, 7, 8, 4, 1. Quelles sont les longueurs des diagonales ?

**3741.** *Proposé par Péter Ivády, Budapest, Hongrie.*

Trouver la valeur maximale de  $a$  et la valeur minimale de  $b$  pour lesquelles les inégalités

$$\frac{ax}{a+x^2} < \sin x < \frac{bx}{b+x^2},$$

sont satisfaites pour tout  $0 < x < \frac{\pi}{2}$ .

**3742.** *Proposé par Michel Bataille, Rouen, France.*

Dans un triangle scalène  $ABC$ , soit respectivement  $K, L, M$  les pieds des hauteurs issues de  $A, B, C$ , et  $P, Q, R$  les points milieu de  $BC, CA, AB$ . Soit respectivement  $X, Y$  et  $Z$  les intersections de  $LM$  avec  $QR$ ,  $MK$  avec  $RP$ , et  $KL$  avec  $PQ$ . Montrer que  $AX, BY, CZ$  sont parallèles.

**3743.** *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Deux cercles égaux sont tangents à la parabole  $y = x^2$  au même point. L'un d'eux est aussi tangent à l'axe des  $x$ , tandis que l'autre est tangent à l'axe des  $y$ . Trouver le rayon de ces cercles. *Ce problème s'inspire du problème 3732 [2012 : 149, 151].*

**3744.** *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit  $a, b, c$  trois nombres réels positifs dont la somme est 4. Montrer que

$$\frac{a^8 + b^8}{(a^2 + b^2)^2} + \frac{b^8 + c^8}{(b^2 + c^2)^2} + \frac{c^8 + a^8}{(c^2 + a^2)^2} + abc \geq a^3 + b^3 + c^3.$$

**3745.** *Proposé par Abdilkadir Altıntaş, mathematics teacher, Emirdağ, Turkey.*

Dans le carré  $ABCD$  le demi-cercle de diamètre  $AD$  coupe le quart de cercle de centre  $C$  et de rayon  $CD$  au point  $P$ . Montrer que  $PB = \sqrt{2}AP$ .

**3746.** *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

On note  $Q(n)$  la somme des chiffres du nombre entier positif  $n$ . Montrer qu'il existe une infinité d'entiers positifs  $n$  tels que

$$Q(n) + Q(n^2) + Q(n^3) = [Q(n)]^2.$$

*Ceci est une extension du problème 3506 [2010 : 45, 47; 2011 : 57, 58].*

**3747.** *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit  $a, b, c$  trois nombres réels avec  $a + b + c = 0$  and  $c \geq 1$ . Montrer que

$$a^4 + b^4 + c^4 - 3abc \geq \frac{3}{8}.$$



**3748\***. *Proposé par Nguyen Thanh Binh, Hanoi, Vietnam.*

Étant donné trois cercles mutuellement extérieurs en position générale, il va exister six droites distinctes, à savoir les tangentes internes communes aux paires de cercles. Montrer que si trois de ces tangentes communes, une pour chaque paire de cercles, sont concourantes, les trois autres sont aussi concourantes.

**3749.** *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

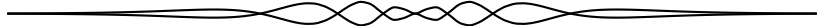
Soit  $D$  et  $E$  deux points arbitraires sur les côtés  $BC$  et  $AC$  d'un triangle  $ABC$ . Montrer que

$$\sqrt{[ADE]} + \sqrt{[BDE]} \leq \sqrt{[ABC]},$$

où  $[XYZ]$  dénote l'aire du triangle  $XYZ$ .

**3750.** *Proposé par Michel Bataille, Rouen, France.*

Soit  $T_k = 1 + 2 + \dots + k$  le  $k^{\text{e}}$  nombre triangulaire. Trouver tous les entiers positifs  $m, n$  tels que  $T_m = 2T_n$ .



# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Due to a filing error, a number of readers' solutions got misplaced and were never acknowledged. The following solutions were received by the editor-in-chief: ARKADY ALT, San Jose, CA, USA(3624); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3626, 3634, 3635, 3636); MICHEL BATAILLE, Rouen, France (3624); VÁCLAV KONEČNÝ, Big Rapids, MI, USA (3542); PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (3639); EDMUND SWYLAN, Riga, Latvia (3620, 3634, 3635, 3638); and PETER Y. WOO, Biola University, La Mirada, CA, USA (3626, 3627, 3628, 3629, 3632, 3634, 3635, 3638). The editor apologizes sincerely for the oversight.*



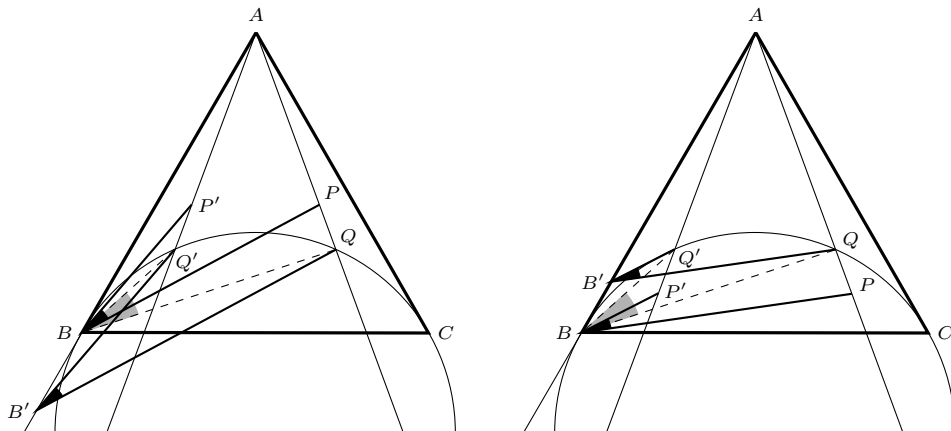
**2597.** [2000 : 499;2001 : 559] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let  $P$  be an arbitrary interior point of an equilateral triangle  $ABC$ . Prove that  $|\angle PBC - \angle PCB| \leq \arcsin\left(2 \sin \frac{|\angle PAB - \angle PAC|}{2}\right) - \frac{|\angle PAB - \angle PAC|}{2} \leq |\angle PAB - \angle PAC|$ . Show that the left inequality cannot be improved in the sense that there is a position  $Q$  of  $P$  on the ray  $AP$  giving an equality. (Thus the inequality in **2255** [1997: 300; 1998: 378-379; 1999: 113-114] is improved.)

*Solution by Tomasz Cieřla, student, University of Warsaw, Poland.*

When  $\angle BAP = \angle PAC$  the given relations hold because the three quantities being compared are all zero; therefore, without loss of generality we shall assume that  $\angle BAP > \angle PAC$  (and, consequently,  $\angle PCB \geq \angle PBC$ ). Define  $\ell$  to be that portion of the line  $AP$  in the interior of  $\triangle ABC$ . We first will prove that the position of  $P$  on  $\ell$  that maximizes the quantity on the left is where  $\angle BPC = \frac{2\pi}{3}$ .

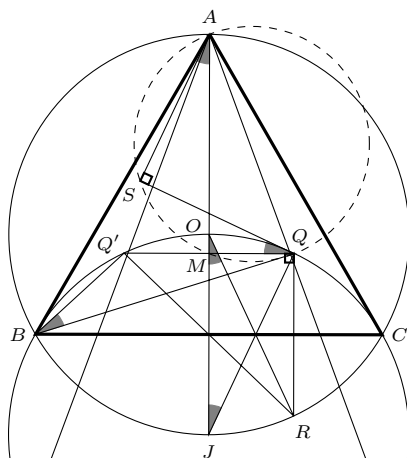
On  $\ell$  choose point  $Q$  such that  $\angle BQC = \frac{2\pi}{3}$ . Denote by  $P'$  and  $Q'$  the reflections of  $P$  and  $Q$  in the bisector of angle  $BAC$ . Then we have  $\angle PCB - \angle PBC = \angle PBP'$  and  $\angle QCB - \angle QBC = \angle QBQ'$ . Our claim is that  $\angle QBQ' \geq \angle PBP'$  for all positions of  $P$  on  $\ell$ . Consider homothety centered at  $A$  which sends  $P$  into  $Q$ . Then  $P'$  is sent to  $Q'$ , and  $B$  is sent to some point  $B'$  lying on line  $AB$ . We have  $\angle PBP' = \angle QB'Q'$ . Since circle  $(BQ'QC)$  is tangent to line  $AB$  at  $B$  (because  $\angle BQC = \frac{2\pi}{3}$ ), we see that point  $B'$  lies outside the circle on the same side of line  $QQ'$  as  $B$ . This implies that  $\angle QBQ' \geq \angle QB'Q' = \angle PBP'$  as claimed.



Next we will see that

$$\angle QBQ' = \arcsin \left( 2 \sin \frac{\angle QAB - \angle QAC}{2} \right) - \frac{\angle QAB - \angle QAC}{2}. \quad (1)$$

Because the difference  $\arcsin \left( 2 \sin \frac{|\angle PAB - \angle PAC|}{2} \right) - \frac{|\angle PAB - \angle PAC|}{2}$  is constant for all points  $P$  on  $\ell$ , this will prove that the left inequality holds and, moreover, it cannot be improved.



Denote the circumcentre of  $\triangle ABC$  by  $O$ , and the reflections of  $Q$  and  $O$  in  $BC$  by  $R$  and  $J$ . Because triangle  $ABC$  is equilateral, points  $R, J$  lie on the circumcircle of  $\triangle ABC$  and  $J$  is the circumcenter of trapezoid  $BCQ'Q'$ . Note that  $O$  is midpoint of arc  $QQ'$  of circle  $(BQQ'C)$ . Angle chasing gives us

$$\angle Q'OQ = \pi - \angle QBQ' = \pi - \frac{1}{2}\angle QJQ' = \pi - \angle QJO = \pi - \angle JOR = \angle ROA.$$

In addition,  $OQ' = OQ$  and  $OR = OA$ . Thus there exists a rotation about  $O$  which maps  $Q'$  to  $Q$  and  $R$  to  $A$ ; denote by  $S$  the image of  $Q$  under this rotation.

Then  $\angle QSA = \angle Q'QR = \frac{\pi}{2}$ . Since  $O$  is the circumcenter of isosceles triangle  $Q'QS$ ,

$$\angle SQQ' = 2\angle OQQ' = 2\angle OBQ' = \angle QBQ'. \quad (2)$$

Let  $M$  be midpoint of  $QQ'$ . Points  $Q, M, S, A$  lie on the circle with diameter  $QA$ , because  $\angle QMA = \frac{\pi}{2} = \angle QSA$ . Thus

$$\angle SQQ' = \angle SQM = \angle SAM. \quad (3)$$

Observe that  $\sin \angle SAQ = \frac{SQ}{AQ} = \frac{Q'Q}{AQ} = \frac{2MQ}{AQ} = 2 \sin \angle MAQ = 2 \sin \angle OAQ$ . From that we get

$$\angle SAQ = \arcsin(2 \sin \angle OAQ). \quad (4)$$

From  $\angle OAQ = \frac{\angle QAB - \angle QAC}{2}$  and equations (2) through (4),

$$\angle QBQ' = \angle SQQ' = \angle SAM = \angle SAQ - \angle OAQ,$$

which is equation (1), as claimed.

For the inequality on the right, simply note that we have proved that the middle difference is the maximum of  $|\angle PCB - \angle PBC|$  over all points  $P \in \ell$ , while Problem 2255 established that this difference is at most  $|\angle PAB - \angle PAC|$ . This observation concludes the proof.

*Also solved by the proposer; no solution was published before now.*

*For an alternative proof of the right inequality, let  $x = |\angle PAB - \angle PAC|$ ,  $0 \leq x < \frac{\pi}{3}$ .*

*The inequality to prove reduces to  $\arcsin(2 \sin \frac{x}{2}) \leq \frac{3x}{2}$ , for  $0 \leq x < \frac{\pi}{3}$ , which is an elementary exercise. It is interesting to note that according to the solution of Problem 2255, the inequality there, namely  $|\angle PAB - \angle PAC| \geq |\angle PCB - \angle PBC|$ , holds for all isosceles triangles  $ABC$  for which  $\angle A \geq \frac{\pi}{3}$  (and  $\angle B = \angle C \leq \frac{\pi}{3}$ ), while the inequality fails for some positions of  $P$  in isosceles triangles with  $\angle A < \frac{\pi}{3}$ . Note that  $\arcsin(2 \sin \frac{x}{2})$  is no longer real for  $x > \frac{\pi}{3}$ , so that there are positions of  $P$  for which the right inequality of the present problem fails for isosceles triangles with  $\angle A > \frac{\pi}{3}$ .*

**3641.** [2011 : 234, 237] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let  $0 \leq x_1, x_2, \dots, x_n < \pi/2$  be real numbers. Prove that

$$\left( \frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left( 1 - \left( \frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} \geq 1.$$

*I. Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let  $f(x) = \sec x$ ,  $g(x) = \sin x$  and set  $\bar{x} = \frac{1}{n} \sum_{k=1}^n x_k$ . Since  $f''(x) = \frac{1 + \sin^2 x}{\cos^3 x} > 0$  and  $g''(x) = -\sin x < 0$  for  $0 < x < \frac{\pi}{2}$ ,  $f$  is convex and  $g$  is concave on the interval  $(0, 1)$ .

Hence Jensen's Inequality ensures that

$$\frac{1}{n} \sum_{k=1}^n \sec x_k \geq \sec(\bar{x}) \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^n \sin x_k \leq \sin(\bar{x}).$$

Therefore we have

$$\begin{aligned} \left( \frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left( 1 - \left( \frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} &\geq \sec(\bar{x})(1 - \sin^2(\bar{x}))^{1/2} \\ &= \sec(\bar{x}) \cos(\bar{x}) = 1. \end{aligned}$$

*II. Composite of virtually identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.*

By Cauchy-Schwarz Inequality we have

$$n \left( \sum_{k=1}^n \sin^2(x_k) \right) = \left( \sum_{k=1}^n 1^2 \right) \left( \sum_{k=1}^n \sin^2(x_k) \right) \geq \left( \sum_{k=1}^n \sin(x_k) \right)^2$$

so

$$\left( \frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \leq \frac{1}{n} \sum_{k=1}^n \sin^2(x_k).$$

Hence,

$$\begin{aligned} \left( \frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left( 1 - \left( \frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} &\geq \left( \frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left( 1 - \frac{1}{n} \sum_{k=1}^n \sin^2(x_k) \right)^{1/2} \\ &= \left( \frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left( \frac{1}{n} \sum_{k=1}^n (1 - \sin^2(x_k)) \right)^{1/2} \\ &= \left( \frac{1}{n} \sum_{k=1}^n \frac{1}{\cos(x_k)} \right) \left( \frac{1}{n} \sum_{k=1}^n \cos^2(x_k) \right)^{1/2} \\ &\geq \left( \prod_{k=1}^n \frac{1}{\cos(x_k)} \right)^{1/n} \left( \left( \prod_{k=1}^n \cos^2(x_k) \right)^{1/n} \right)^{1/2} = 1 \end{aligned}$$

by the AM-GM Inequality.

Clearly, equality holds if and only if  $x_1 = x_2 = \dots = x_n$ .

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herriberg, Switzerland; and the proposer.*

**3642.** [2011 : 235, 237] *Proposed by Michel Bataille, Rouen, France.*

Evaluate

$$\lim_{n \rightarrow \infty} \frac{\int_0^1 (2x^2 - 5x - 1)^n dx}{\int_0^1 (x^2 - 4x - 1)^n dx}.$$

*Solution by Paul Bracken, University of Texas, Edinburg, TX, USA; modified by the editor.*

Write the limit as  $\lim_{n \rightarrow \infty} I_1(n)/I_2(n)$ , where  $I_1(n) = \int_0^1 (1 + 5x - 2x^2)^n dx$  and  $I_2(n) = \int_0^1 (1 + 4x - x^2)^n dx$ , and let  $\phi(x) = \ln(1 + 5x - 2x^2)$ . Integration by parts gives

$$\begin{aligned} I_1(n) &= \int_0^1 e^{n\phi(t)} dt = \frac{1}{n} \int_0^1 \frac{1}{\phi'(t)} \frac{d}{dt} [e^{n\phi(t)}] dt \\ &= e^{n\phi(t)} \Big|_0^1 - \frac{1}{n} \int_0^1 \frac{d}{dt} \left[ \frac{1}{\phi'(t)} \right] e^{n\phi(t)} dt \\ &= \frac{4^{n+1}}{n} - \frac{1}{5n} + \frac{1}{n} \int_0^1 \frac{d}{dt} \left[ \frac{1}{\phi'(t)} \right] e^{n\phi(t)} dt. \end{aligned}$$

The function  $d/dt[1/\phi'(t)] = 1/2 + (33/2)(5 - 4t)^{-2}$  increases on  $[0, 1]$ , taking the value  $29/25$  at  $t = 0$  and the value  $17$  at  $t = 1$ . It follows that

$$\frac{29}{25} \int_0^1 e^{n\phi(t)} dt \leq \int_0^1 \frac{d}{dt} \left[ \frac{1}{\phi'(t)} \right] e^{n\phi(t)} dt \leq 17 \int_0^1 e^{n\phi(t)} dt,$$

and hence there is a constant  $C_1$  with  $29/25 \leq C_1 \leq 17$  and such that

$$\int_0^1 \frac{d}{dt} \left[ \frac{1}{\phi'(t)} \right] e^{n\phi(t)} dt = C_1 \int_0^1 e^{n\phi(t)} dt.$$

Thus,

$$I_1(n) = \frac{4^{n+1}}{n} - \frac{1}{5n} + \left( \frac{C_1}{n} \right) I_1(n)$$

and solving for  $I_1(n)$  gives

$$I_1(n) = \left( 1 - \frac{C_1}{n} \right)^{-1} \cdot \frac{1}{n} \cdot \left( 4^{n+1} - \frac{1}{5} \right).$$

By similar calculations there is a constant  $C_2$  with  $9/8 \leq C_2 \leq 3$  such that

$$I_2(n) = \left( 1 - \frac{C_2}{n} \right)^{-1} \cdot \frac{1}{n} \cdot \left( 2 \cdot 4^n - \frac{1}{4} \right).$$

Finally,

$$\lim_{n \rightarrow \infty} \frac{I_1(n)}{I_2(n)} = \lim_{n \rightarrow \infty} \frac{\left( 1 - \frac{C_2}{n} \right) \left( 2 - \frac{1}{10 \cdot 4^n} \right)}{\left( 1 - \frac{C_1}{n} \right) \left( 1 - \frac{1}{2 \cdot 4^{n+1}} \right)} = 2.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer. One incorrect solution and one incomplete solution were received.

**3643.** [2011 : 235, 238] Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.

Let  $u$  and  $v$  be positive real numbers. Prove that

$$\frac{1}{8} \left( 17 - \frac{2uv}{u^2 + v^2} \right) \leq \sqrt[3]{\frac{u}{v}} + \sqrt[3]{\frac{v}{u}} \leq \sqrt{(u+v) \left( \frac{1}{u} + \frac{1}{v} \right)}$$

For each inequality, determine when equality holds.

*Editor's note:* Perfetti pointed out that the very same problem (by the same proposer) has appeared in Mathematics Magazine (Vol. 82, No. 3, 2009) and a solution was published in Vol. 83, No. 3, pp. 229 – 230. However, we decide to publish a different solution which is completely elementary.

*Solution by Titu Zvonaru, Comănești, Romania.*

Let  $x = \sqrt[3]{\frac{u}{v}}$ . Then  $x > 0$  and  $x^3 = \frac{u}{v}$ .

The left inequality is equivalent, in succession, to

$$\begin{aligned} \frac{1}{8} \left( 17 - \frac{2x^3}{x^6 + 1} \right) &\leq x + \frac{1}{x} \\ \frac{17x^6 - 2x^3 + 17}{x^6 + 1} &\leq \frac{8x^2 + 8}{x} \\ 8x^8 - 17x^7 + 8x^6 + 2x^4 + 8x^2 - 17x + 8 &\geq 0 \\ (x-1)^2(8x^6 - x^5 - 2x^4 - 3x^3 - 2x^2 - x + 8) &\geq 0 \\ (x-1)^2((8x^6 - x^5 - 2x^4 - 10x^3 - 2x^2 - x + 8) + 7x^3) &\geq 0 \\ (x-1)^2((x-1)^2(8x^4 + 15x^3 + 20x^2 + 15x + 8) + 7x^3) &\geq 0 \end{aligned}$$

which is clearly true.

To establish the right inequality note that  $(u+v) \left( \frac{1}{u} + \frac{1}{v} \right) = x^3 + \frac{1}{x^3} + 2$  and

$$\begin{aligned} x + \frac{1}{x} \leq \sqrt{x^3 + \frac{1}{x^3} + 2} &\Leftrightarrow x^2 + \frac{1}{x^2} + 2 \leq x^3 + \frac{1}{x^3} + 2 \\ &\Leftrightarrow x^6 - x^5 - x + 1 \geq 0 \Leftrightarrow (x-1)(x^5 - 1) \geq 0 \\ &\Leftrightarrow (x-1)^2(x^4 + x^3 + x^2 + x + 1) \geq 0 \end{aligned}$$

which clearly holds. Note that equality holds in either inequality if and only if  $x = 1$ ; that is, if and only if  $u = v$ .

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

**3644.** [2011 : 235, 238] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

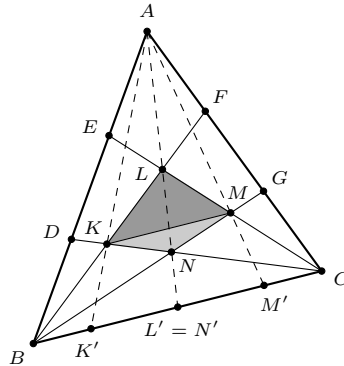
We trisect the sides  $AB$  and  $AC$  of triangle  $ABC$  with the points  $D, E$  and  $F, G$  respectively such that  $AE = ED = DB$  and  $AF = FG = GC$ . The line  $BF$  intersects  $CD, CE$  in the points  $K, L$  respectively, while  $BG$  intersects  $CD, CE$  in  $N, M$  respectively.

Prove that:

- (a)  $KM$  is parallel to  $BC$ ;  
 (b)  $\text{Area}(KLM) = \frac{5}{7}\text{Area}(KLMN)$ .

*Composite of complementary solutions by Edmund Swylan, Riga, Latvia; and Titu Zvonaru, Comănești, Romania.*

Let  $K', L', M'$ , and  $N'$  be the points where the side  $BC$  meets the lines joining  $A$  to  $K, L, M$ , and  $N$ , respectively. (One can easily show that, in fact,  $L' = N'$  is the midpoint of  $BC$ , but this is not relevant to our work here.)



By applying Van Aubel's theorem (see, for example, F. G.-M., *Exercices de géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, quatrième édition, J. De Gigord, Paris (1907), paragraph 1242j, page 542) four times, we have

$$\begin{aligned} \frac{AK}{KK'} &= \frac{AD}{DB} + \frac{AF}{FC} = 2 + \frac{1}{2}, \quad \text{or} \quad \frac{KK'}{AK'} = \frac{2}{7}; \\ \frac{AL}{LL'} &= \frac{AE}{EB} + \frac{AF}{FC} = \frac{1}{2} + \frac{1}{2}, \quad \text{or} \quad \frac{LL'}{AL'} = \frac{1}{2}; \\ \frac{AM}{MM'} &= \frac{AE}{EB} + \frac{AG}{GC} = \frac{1}{2} + 2, \quad \text{or} \quad \frac{MM'}{AM'} = \frac{2}{7}; \\ \frac{AN}{NN'} &= \frac{AD}{DB} + \frac{AG}{GC} = 2 + 2, \quad \text{or} \quad \frac{NN'}{AN'} = \frac{1}{5}. \end{aligned}$$

From the first and third of these equations we get  $\frac{AK}{KK'} = \frac{AM}{MM'}$ , whence  $KM \parallel K'M'$ , and part (a) follows immediately. For part (b) we assume without loss



of generality that the altitude from  $A$  to  $BC$  has length 1. The above equations then imply that the line segments perpendicular to  $BC$  from  $K, L, M, N$  equal  $\frac{2}{7}, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}$ , respectively. Thus

$$\frac{\text{Area}(KLM)}{\text{Area}(KMN)} = \frac{\frac{1}{2} - \frac{2}{7}}{\frac{2}{7} - \frac{1}{5}} = \frac{5}{2},$$

and therefore

$$\frac{\text{Area}(KLM)}{\text{Area}(KLMN)} = \frac{5}{7}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); and MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL.

**3645.** [2011 : 235, 238] Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let  $a, b$ , and  $c$  be positive numbers such that  $a^2 + b^2 + c^2 + 2abc = 1$ . Prove that

$$\sum_{\text{cyclic}} \sqrt{a \left( \frac{1}{b} - b \right) \left( \frac{1}{c} - c \right)} > 2.$$

*I. Solution by Arkady Alt, San Jose, CA, USA.*

Observe that  $0 < a, b, c < 1$  so that  $abc \neq 1$ . The inequality is equivalent to

$$a\sqrt{(1-b^2)(1-c^2)} + b\sqrt{(1-c^2)(1-a^2)} + c\sqrt{(1-a^2)(1-b^2)} > 2\sqrt{abc}. \quad (1)$$

Since

$$(1-b^2)(1-c^2) = 1 - b^2 - c^2 + b^2c^2 = a^2 + 2abc + (bc)^2 = (a+bc)^2,$$

and, similarly,  $(1-c^2)(1-a^2) = (b+ca)^2$  and  $(1-a^2)(1-b^2) = (c+ab)^2$ , the left side is equal to

$$\begin{aligned} a(a+bc) + b(b+ca) + c(c+ab) &= a^2 + b^2 + c^2 + 3abc \\ &= 1 + abc > 2\sqrt{abc} \end{aligned}$$

by the Arithmetic-Geometric Means Inequality.

*II. Solution using ideas from Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; Salem Malikić, student, Simon Fraser University, Burnaby, BC; and Albert Stadler, Herrliberg, Switzerland.*

We can select acute angles  $A, B, C$  for which  $a = \cos A, b = \cos B, c = \cos C$ . Then

$$\begin{aligned}\cos(A + B + C) &= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C \\ &\quad - \sin A \sin B \cos C \\ &= abc - a(a + bc) - b(b + ca) - c(c + ab) = -1,\end{aligned}$$

so that  $A + B + C = \pi$ . Therefore

$$\begin{aligned}2a\sqrt{(1-b^2)(1-c^2)} &= \cos A[\cos(B-C) - \cos(B+C)] \\ &= -\cos(B+C)\cos(B-C) + \cos^2 A \\ &= -\frac{1}{2}(\cos 2B + \cos 2C) + \cos^2 A \\ &= -\cos^2 B - \cos^2 C + 1 + \cos^2 A \\ &= 1 + a^2 - b^2 - c^2,\end{aligned}$$

with similar equations for the other two terms of the left side of (1) in the first solution. Therefore, the left side of (1) is equal to

$$\frac{1}{2}(3 - a^2 - b^2 - c^2) = 1 + abc > 2\sqrt{abc}.$$

*Also solved by the proposer.*

**3646.** [2011 : 235, 238] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let  $\alpha \geq 0$  and let  $\beta$  be a positive number. Find the limit

$$L(\alpha, \beta) = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \left( 1 + \frac{k^\alpha}{n^\beta} \right)^k - n \right).$$

*Solution by Anastasios Kotrononis, Athens, Greece; modified slightly by the editor.*

We will prove that

$$L(\alpha, \beta) = \begin{cases} \infty & \text{if } \beta - \alpha < 2 \\ 0 & \text{if } \beta - \alpha > 2 \\ \frac{1}{\beta} & \text{if } \beta - \alpha = 2 \end{cases}.$$

by considering three cases separately.

**Case (i)**  $\beta - \alpha < 2$ . By Bernoulli's Inequality, we have

$$\left( 1 + \frac{k^\alpha}{n^\beta} \right)^k \geq 1 + \frac{k^{\alpha+1}}{n^\beta}$$

so

$$\begin{aligned} \sum_{k=1}^n \left(1 + \frac{k^\alpha}{n^\beta}\right)^k - n &\geq \sum_{k=1}^n \left(1 + \frac{k^{\alpha+1}}{n^\beta}\right) - n = \sum_{k=1}^n \frac{k^{\alpha+1}}{n^\beta} \\ &= n^{\alpha-\beta+2} \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\alpha+1}\right). \end{aligned} \quad (1)$$

Note that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\alpha+1} = \int_0^1 x^{\alpha+1} dx = \frac{1}{\alpha+2}. \quad (2)$$

Since  $\alpha - \beta + 2 > 0$  and  $\alpha + 2 > 0$ , we conclude from (1) and (2) that  $L(\alpha, \beta) = \infty$ .

**Case (ii)**  $\beta - \alpha > 2$ . Note first that for all  $k = 1, 2, \dots, n$  and  $i = 0, 1$ , we have

$$0 < \frac{k^{\alpha+i}}{n^\beta} \leq \frac{1}{n^{\beta-\alpha-i}} < \frac{1}{n^{2-i}}.$$

In particular,

$$0 < \frac{k^\alpha}{n^\beta} < \frac{1}{n^2} \quad \text{and} \quad 0 < \frac{k^{\alpha+1}}{n^\beta} < \frac{1}{n}. \quad (3)$$

It is well known that as  $x \rightarrow 0^+$  we have

$$\ln(1+x) = x + O(x^2) \quad (4)$$

and

$$e^x = 1 + x + O(x^2). \quad (5)$$

Using (3), (4) and (5), we have, as  $n \rightarrow \infty$ , that

$$\begin{aligned} \left(1 + \frac{k^\alpha}{n^\beta}\right)^k &= \exp\left(k \ln\left(1 + \frac{k^\alpha}{n^\beta}\right)\right) \\ &= \exp\left(k \left(\frac{k^\alpha}{n^\beta} + O\left(n^{2(\alpha-\beta)}\right)\right)\right) \\ &= \exp\left(\frac{k^{\alpha+1}}{n^\beta} + O\left(n^{2(\alpha-\beta)+1}\right)\right) \\ &= 1 + \frac{k^{\alpha+1}}{n^\beta} + O\left(n^{2(\alpha-\beta)+2}\right) \end{aligned}$$

so

$$\begin{aligned} \sum_{k=1}^n \left(1 + \frac{k^\alpha}{n^\beta}\right)^k - n &= \sum_{k=1}^n \frac{k^{\alpha+1}}{n^\beta} + O\left(n^{2(\alpha-\beta)+3}\right) \\ &= \frac{1}{n^{\beta-\alpha-2}} \left(\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\alpha+1}\right) + O\left(n^{2(\alpha-\beta)+3}\right). \end{aligned} \quad (6)$$

Since  $\beta - \alpha - 2 > 0$  and  $2(\alpha - \beta) + 3 < -1$ , it follows from (2) and (6) that  $L(\alpha, \beta) = 0$ .

**Case (iii)**  $\beta - \alpha = 2$ . We proceed as in case (ii). Since  $\beta - \alpha - 2 = 0$ , it follows from (2) and (6) again that  $L(\alpha, \beta) = \frac{1}{\alpha+2} = \frac{1}{\beta}$

This completes our proof.

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer.*

**3647.** [2011 : 236, 238] *Proposed by Panagiotis Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Show that in triangle  $ABC$  with exradii  $r_a$ ,  $r_b$  and  $r_c$ ,

$$\sum_{\text{cyclic}} \frac{(r_a + r_b)(r_b + r_c)}{ac} \geq 9,$$

where  $AB = c$ ,  $BC = a$ , and  $CA = b$ .

*Similar solutions by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; and Kee-Wai Lau, Hong Kong, China.*

We know that  $r_a = \frac{\Delta}{s-a}$ ,  $r_b = \frac{\Delta}{s-b}$ , and  $r_c = \frac{\Delta}{s-c}$ , where  $s = \frac{a+b+c}{2}$  is the semiperimeter of  $ABC$  and  $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$  is its area. It follows that

$$r_a + r_b = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = \frac{c\Delta}{(s-a)(s-b)}$$

and

$$r_b + r_c = \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{a\Delta}{(s-b)(s-c)},$$

whence

$$\frac{(r_a + r_b)(r_b + r_c)}{ac} = \frac{\Delta^2}{(s-a)(s-b)^2(s-c)} = \frac{s}{s-b}.$$

Similarly,

$$\frac{(r_b + r_c)(r_c + r_a)}{ba} = \frac{s}{s-c} \quad \text{and} \quad \frac{(r_c + r_a)(r_a + r_b)}{cb} = \frac{s}{s-a}.$$

These last three equations give us

$$\sum_{\text{cyclic}} \frac{(r_a + r_b)(r_b + r_c)}{ac} = \frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c}. \quad (1)$$

But by the AM-HM inequality,

$$\frac{1}{3} \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) \geq \frac{3}{s-a+s-b+s-c} = \frac{3}{s},$$

which, when multiplied by  $3s$ , yields

$$\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} \geq 9. \quad (2)$$

The desired inequality follows from (1) and (2). Equality holds if and only if the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comănești, Romania; and the proposer.

**3648.** [2011 : 236, 239] Proposed by Michel Bataille, Rouen, France.

Find all real numbers  $x, y, z$  such that  $xyz = 1$  and  $x^3 + y^3 + z^3 = \frac{S(S-4)}{4}$  where  $S = \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}$ .

*I. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

We assume the equation

$$x^3 + y^3 + z^3 = \frac{S(S-4)}{4}(xyz),$$

without the restriction on  $xyz$ . This is equivalent to

$$\sum(x^4y^2 + x^3y^3 + x^2y^2z^2) = \sum(x^4yz + 2x^3y^2z),$$

where both sums, taken over the six permutations of the variables, are symmetric.

By the Arithmetic-Geometric Means Inequality,

$$\begin{aligned} \sum x^4y^2 &= x^4(y^2 + z^2) + y^4(z^2 + x^2) + z^4(x^2 + y^2) \\ &\geq 2x^4yz + 2y^4zx + 2z^4xy = \sum x^4yz, \end{aligned}$$

with equality if and only if  $x = y = z$ .

Recall Schur's Inequality that, for  $a, b, c \geq 0$ ,

$$\begin{aligned} (a^3 + b^3 + c^3) + 3abc - (a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \\ = a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0. \end{aligned}$$

Setting  $(a, b, c) = (yz, zx, xy)$ , we obtain that

$$\sum(x^3y^3 + x^2y^2z^2) \geq 2 \sum x^3y^2z,$$

where again each sum is symmetric with six terms.

Therefore

$$\sum(x^4y^2 + x^3y^3 + x^2y^2z^2) \geq \sum(x^4yz + 2x^3y^2z).$$

Since we are assuming that equality holds and that  $xyz = 1$ , the given equation is satisfied if and only if  $x = y = z = 1$ .

II. *Solution by the proposer.*

The given conditions imply that  $S = xy^2 + yx^2 + yz^2 + zy^2 + zx^2 + xz^2$ . Without loss of generality, let  $x$  be the maximum of  $x, y, z$ . Define

$$T = 4(xy^2 + yx^2 + yz^2)(zx^2 + xz^2 + zy^2).$$

Then

$$T = S^2 - (y - z)^2(x^2 + xy + xz - yz)^2$$

and also

$$T = 4xyz(x^3 + y^3 + z^3 + S) + 4y^2z^2(x - y)(x - z),$$

whence

$$S^2 = 4(x^3 + y^3 + z^3 + S) + 4y^2z^2(x - y)(x - z) + (y - z)^2(x^2 + xy + xz - yz)^2.$$

Since, by hypothesis,  $4(x^3 + y^3 + z^3 + S) = S^2$ , we deduce that

$$4y^2z^2(x - y)(x - z) + (y - z)^2(x^2 + xy + xz - yz)^2 = 0.$$

Since  $x \geq y, z$ , both terms on the left are nonnegative and therefore must vanish. If, say,  $x = y$ , then  $x^2 + xy + xz - yz = 2x^2 \neq 0$ , so that  $y = z$ . Since  $xyz = 1$ , we must have that  $x = y = z = 1$ .

*No other solutions were received.*

**3649.** [2011 : 236, 239] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let  $a, b$ , and  $c$  be three positive real numbers and let

$$k = (a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Prove that

$$(a^3 + b^3 + c^3) \left( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) \geq \frac{k^3 - 15k^2 + 63k - 45}{4},$$

and equality holds if and only if  $(a, b, c) = \left( \frac{k - 5 \pm \sqrt{k^2 - 10k + 9}}{4}, 1, 1 \right)$  or any of its permutations.

*Solution by Oliver Geupel, Brühl, NRW, Germany.*

All sums shall be cyclic. Let  $x = \sum \frac{a}{b}$ ,  $y = \sum \frac{b}{a}$ ,  $m = \sum \frac{a^2}{bc}$ , and  $n = \sum \frac{bc}{a^2}$ . We have  $x + y = k - 3$ , hence  $4xy \leq (k - 3)^2$ . Using the relations

$$\begin{aligned} \sum \frac{a^3}{b^3} &= x^3 - 3(m + n) - 6, \\ \sum \frac{b^3}{a^3} &= y^3 - 3(m + n) - 6, \end{aligned}$$

and  $m + n = xy - 3$ , we deduce that

$$\begin{aligned}
 (a^3 + b^3 + c^3) \left( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) &= x^3 + y^3 + 9 - 6xy \\
 &= (x + y)^3 - 3(x + y)xy + 9 - 6xy \\
 &= (k - 3)^3 + 9 - 3(k - 1)xy \\
 &\geq (k - 3)^3 + 9 - 3(k - 1) \cdot \frac{(k - 3)^2}{4} \\
 &= \frac{k^3 - 15k^2 + 63k - 45}{4}.
 \end{aligned}$$

This proves the inequality.

Equality holds if and only if  $0 = x - y = (a - b)(b - c)(c - a)/abc$ , that is, if two of  $a$ ,  $b$ ,  $c$  coincide. Without loss of generality, suppose that  $b = c$ . The equality is then equivalent to  $(k - 3)/2 = x = 1 + a/b + b/a$ . However, this holds if and only if  $p = a/b$  is a root of the quadratic  $p^2 - \left(\frac{k-5}{2}\right)p + 1$ . The condition for equality (up to permutation) therefore needs to be corrected to

$$(a, b, c) = \left( \lambda \cdot \frac{k - 5 \pm \sqrt{k^2 - 10k + 9}}{4}, \lambda, \lambda \right),$$

where  $\lambda > 0$ .

*Also solved by the proposers. Our featured solver said his solution was similar to and inspired by the solution to problem 75 in Secrets in Inequalities (Vol. 1) by Pham Kim Hung, GIL Publishing House, Zalău, 2007, pp. 214-215.*

**3650.** [2011 : 318, 320] *Replacement. Proposed by Michel Bataille, Rouen, France.*

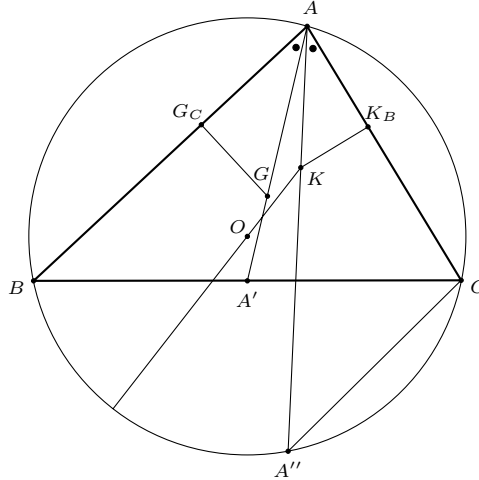
Let  $ABC$  be a triangle and  $R$ ,  $O$ ,  $G$  and  $K$  its circumradius, circumcentre, centroid and Lemoine point, respectively. Prove that

$$BC \cdot \frac{KA}{GA} = CA \cdot \frac{KB}{GB} = AB \cdot \frac{KC}{GC} = \sqrt{3(R^2 - OK^2)}.$$

Recall that a symmedian of a triangle is the reflection of the median from a vertex in the angle bisector of the same vertex. The Lemoine point of a triangle is the point of intersection of the three symmedians.

*Solution by Edmund Swylan, Riga, Latvia.*

Let the length of  $BC$  be denoted by  $a$  and its midpoint by  $A'$ . Let the symmedian to  $BC$  meet the circumcircle again at  $A''$  and, finally, let  $G_c$  be the foot of the perpendicular from  $G$  to  $AB$  and  $K_b$  be the foot of the perpendicular from  $K$  to  $AC$ .



Using, in turn, the similar right triangles  $AG_cG$  and  $AK_bK$ , the fact that  $GG_c$  equals a third of the altitude from  $C$ , and the sine law, we deduce that

$$\frac{KA}{GA} = \frac{KK_b}{GG_c} = \frac{KK_b}{\frac{1}{3}a \sin B} = \frac{KK_b \cdot 2R \cdot 3}{ab},$$

with analogous formulas for  $\frac{KB}{GB}$  and  $\frac{KC}{GC}$ .

We take

$$\frac{KK_b}{b} = \frac{abc}{2R(a^2 + b^2 + c^2)}$$

to be a known property of the Lemoine point (see, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, Dover reprint (1960), page 214, paragraph 342) and obtain

$$a \frac{KA}{GA} = b \frac{KB}{GB} = c \frac{KC}{GC} = \frac{3abc}{a^2 + b^2 + c^2}. \quad (1)$$

On the other hand,

$$R^2 - OK^2 = (R - OK)(R + OK) = KA \cdot KA'' = KA(AA'' - KA). \quad (2)$$

Because  $\triangle ABA' \sim \triangle AA''C$ ,

$$AA'' = \frac{bc}{A'A}. \quad (3)$$

From (1) we have

$$KA = \frac{bc \cdot 3GA}{a^2 + b^2 + c^2} = \frac{bc \cdot 2A'A}{a^2 + b^2 + c^2}. \quad (4)$$

Next, Stewart's theorem says that

$$A'A^2 = \frac{1}{4}(-a^2 + 2b^2 + 2c^2). \quad (5)$$



Putting together equations (2) through (5), we deduce that

$$\begin{aligned} R^2 - OK^2 &= \frac{2b^2c^2}{a^2 + b^2 + c^2} - \frac{4b^2c^2AA'^2}{(a^2 + b^2 + c^2)^2} \\ &= \frac{2b^2c^2}{a^2 + b^2 + c^2} - \frac{4b^2c^2}{(a^2 + b^2 + c^2)^2} \cdot \frac{1}{4}(-a^2 + 2b^2 + 2c^2) \\ &= \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2}, \end{aligned}$$

as desired.

*Also solved by TITU ZVONARU, Comănești, Romania; and the proposer.*

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