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EDITORIAL

Shawn Godin

Hello ***Crux Mathematicorum*** readers and welcome to Volume 38! The last volume was a little rocky with my getting a late start and facing a few delays along the way, but we finally got through it. This volume will be one of change and transition as we look to develop a few more features and, hopefully, win some new readers.

Over the next volume you will see a number of changes. Some of them, little changes to the format and layout, will already be noticeable. Some larger changes will also be apparent to the seasoned *Crux* reader.

Mathematical Mayhem and *Skoliad* will be leaving the journal to form a new on-line journal later in 2013 or early in 2014. Solutions to problems that appeared in these columns in the last volume will appear in volume 38, but there will be no new problems. The CMS is still working on the exact format, and staff, of the new journal. Information about the new journal will be shared with readers of ***Crux Mathematicorum*** when they are known to the editor. Since *Mathematical Mayhem* is not longer part of the journal, you will notice that we have reverted back to ***Crux Mathematicorum***.

To fill in the void left by *Mathematical Mayhem* and *Skoliad*, a new column *Contest Corner* appears this issue. Like *Skoliad*, *The Olympiad Corner* and the old *Academy Corner*, *Contest Corner* will be made up of problems from senior high school and undergraduate mathematics competitions. The problems will be from contests below the Olympiad and Putnam levels. Readers will be invited to send in their solutions. Only problems will appear this volume with solutions starting to appear next volume.

Another major change is the number of issues. With rising mailing costs, the CMS was looking at lowering the number of issues of the journal being printed. We finally settled on a compromise. The journal will go back to appearing 10 times per year, monthly with the exception of July and August, but the issues will be printed in pairs, so that only 5 printed editions (of 2 issues each) will appear each year. This way you will have access to a new issue 10 months of the year in electronic form. We think it is a good compromise.

We are also in the process of developing a number of regular features. Our hope is to have at least one or two articles or columns per issue. These new features will appear as we get them developed; as a result, the number of pages per issue will vary during this volume until we get our format finalized for the next volume.

We hope that you will enjoy the new features and that you will bear with us as we go through the process of building a "New Crux". As always, your feedback is welcome. Enjoy the issue!

Shawn Godin

SKOLIAD No. 138

Lily Yen and Mogens Hansen

Skoliad has joined **Mathematical Mayhem** *which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time Skoliad will be discontinued in* **Crux**. *New Skoliad problems, and their solutions, will appear in* **Mathematical Mayhem** *when it is relaunched in 2013.*

In this issue we present the solutions to the Maritime Mathematics Competition, 2010, given in Skoliad 132 at [2011:130–132].

1. The Valhalla Winter Games are held in February, and the closing ceremonies are on the last day of the month. The first Valhalla Winter Games were held in the year 750, and since that year, they have been held every five years. How many times have the closing ceremonies been held on February 29^{th}? Note that year Y is a leap year if exactly one of the following conditions is true:

(a) Y is divisible by 4 but Y is not divisible by 100.

(b) Y is divisible by 400.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The Valhalla Winter Games have been held in years that end in a 0 or a 5. Leap years are divisible by 4, so the number formed by the last two digits of leap years must be divisible by 4. Therefore you only need to consider years that end in 00, 20, 40, 60, or 80. Now list these:

								760	780
800	820	840	860	880	~~900~~	920	940	960	980
~~1000~~	1020	1040	1060	1080	~~1100~~	1120	1140	1160	1180
1200	1220	1240	1260	1280	~~1300~~	1320	1340	1360	1380
~~1400~~	1420	1440	1460	1480	~~1500~~	1520	1540	1560	1580
1600	1620	1640	1660	1680	~~1700~~	1720	1740	1760	1780
~~1800~~	1820	1840	1860	1880	~~1900~~	1920	1940	1960	1980
2000									

Scratch the full centuries that are not leap years after all. Then 54 years remain in which the closing ceremonies were held on February 29th.

Also solved by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

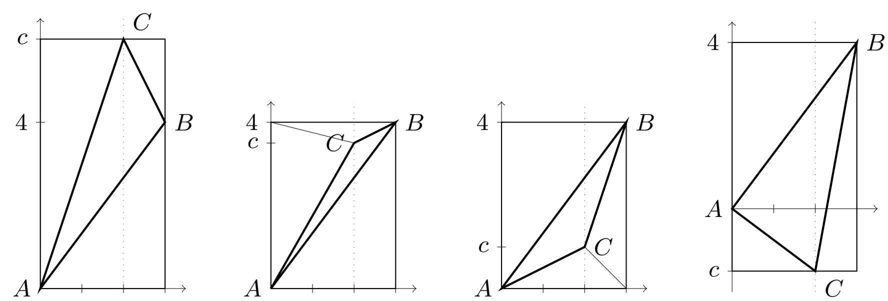
Since $\frac{2011-750}{20} = 63.05$, *you do not need to list the years in the table to see that the table must contain* 63 *years. Then subtract the whole centuries that are not leap years.*

2. A triangle with vertices $A(0,0)$, $B(3,4)$, and $C(2,c)$ has area 5. Find all possible values of the number c.

Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

The line through A and B has the equation $y = \frac{4}{3}x$, so if C were on that line, $c = \frac{8}{3}$.

Consider four cases for C: above B (that is, $c \geq 4$), between B and the line through A and B (that is, $\frac{8}{3} \leq c < 4$), between the x-axis and the line through A and B (that is, $0 \leq c < \frac{8}{3}$), and below the x-axis (that is, $c < 0$).



If $c \geq 4$, then the area of $\triangle ABC$ (that is, 5) can be found by subtracting the areas of the three other triangles in the leftmost figure from the area of the rectangle. That is, $5 = 3c - \frac{3\cdot4}{2} - \frac{2c}{2} - \frac{1(c-4)}{2} = 3c - 6 - c - \frac{1}{2}c + 2$, so $9 = \frac{3}{2}c$, so $c = 6$.

If $\frac{8}{3} \leq c < 4$, then the area of $\triangle ABC$ can be found by subtracting the areas of the three other triangles in the second figure from the area of the rectangle. That is, $5 = 3\cdot4 - \frac{3\cdot4}{2} - \frac{3(4-c)}{2} - \frac{4\cdot2}{2} = 12 - 6 - 6 + \frac{3}{2}c - 4$, so $9 = \frac{3}{2}c$, so $c = 6$, which contradicts the assumption that $c < 4$.

If $0 \leq c < \frac{8}{3}$, then the area of $\triangle ABC$ can be found by subtracting the areas of the three other triangles in the third figure from the area of the rectangle. That is, $5 = 3\cdot4 - \frac{3c}{2} - \frac{4\cdot1}{2} - \frac{3\cdot4}{2} = 12 - \frac{3}{2}c - 2 - 6$, so $\frac{3}{2}c = -1$, which contradicts the assumption that $c \geq 0$.

Finally, if $c < 0$, then the area of $\triangle ABC$ can be found by subtracting the areas of the three other triangles in the rightmost figure from the area of the rectangle. That is, $5 = 3(4-c) - \frac{1(4-c)}{2} - \frac{3\cdot4}{2} - \frac{2(-c)}{2} = 12 - 3c - 2 + \frac{1}{2}c - 6 + c$, so $\frac{3}{2}c = -1$, so $c = -\frac{2}{3}$.

Thus $c = 6$ or $c = -\frac{2}{3}$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

3. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers. Assume that $f(0)$, $f(1)$, and $f(2)$ are all integers.

(a) Prove that $f(2010)$ is also an integer.

(b) Decide if $f(2011)$ is an integer.

Solution based on contributions from Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC; Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.

Since $f(0)$, $f(1)$, and $f(2)$ are all integers, $f(2) - 2f(1) + f(0)$ is also an integer. Now,

$$f(2) - 2f(1) + f(0) = (4a + 2b + c) - 2(a + b + c) + (c) = 2a,$$

so $2a$ is an integer. Since $f(0)$ and $f(1)$ are both integers, $f(1) - f(0)$ is also an integer. Now, $f(1) - f(0) = (a+b+c) - (c) = a+b$, so $a+b$ is an integer. Finally, $f(0) = c$, so c is an integer.

Now

$$\begin{aligned} f(2010) &= 2010^2 a + 2010b + c \\ &= 4038090a + 2010a + 2010b + c \\ &= 2019045(2a) + 2010(a + b) + c, \end{aligned}$$

which is an integer because $2a$, $a + b$, and c are all integers. Also

$$\begin{aligned} f(2011) &= 2011^2 a + 2011b + c \\ &= 4042110a + 2011a + 2011b + c \\ &= 2021055(2a) + 2011(a + b) + c, \end{aligned}$$

which is an integer because $2a$, $a + b$, and c are all integers.

4. If x is a real number, let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x. For example, $\lfloor 7.012 \rfloor = 7$. If n is any positive integer, find a (simple) formula for

$$\left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2(n+1)}{3} \right\rfloor + \left\lfloor \frac{2(n+2)}{3} \right\rfloor .$$

Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

When you divide the integer n by 3, the remainder is either 0, 1, or 2. That is, $n = 3a$, $n = 3a + 1$, or $n = 3a + 2$ for some integer a.

If $n = 3a$, then

$$\begin{aligned} \left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2(n+1)}{3} \right\rfloor + \left\lfloor \frac{2(n+2)}{3} \right\rfloor &= \left\lfloor \frac{6a}{3} \right\rfloor + \left\lfloor \frac{2(3a+1)}{3} \right\rfloor + \left\lfloor \frac{2(3a+2)}{3} \right\rfloor \\ &= \lfloor 2a \rfloor + \left\lfloor 2a + \frac{2}{3} \right\rfloor + \left\lfloor 2a + \frac{4}{3} \right\rfloor \\ &= (2a) + (2a) + (2a + 1) = 6a + 1 = 2n + 1. \end{aligned}$$

If $n = 3a + 1$, then

$$\begin{aligned}\left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2(n+1)}{3} \right\rfloor + \left\lfloor \frac{2(n+2)}{3} \right\rfloor &= \left\lfloor \frac{2(3a+1)}{3} \right\rfloor + \left\lfloor \frac{2(3a+2)}{3} \right\rfloor + \left\lfloor \frac{2(3a+3)}{3} \right\rfloor \\ &= \left\lfloor 2a + \frac{2}{3} \right\rfloor + \left\lfloor 2a + \frac{4}{3} \right\rfloor + \lfloor 2a + 2 \rfloor \\ &= (2a) + (2a+1) + (2a+2) = 6a + 3 \\ &= 2(3a+1) + 1 = 2n + 1.\end{aligned}$$

If $n = 3a + 2$, then

$$\begin{aligned}\left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2(n+1)}{3} \right\rfloor + \left\lfloor \frac{2(n+2)}{3} \right\rfloor &= \left\lfloor \frac{2(3a+2)}{3} \right\rfloor + \left\lfloor \frac{2(3a+3)}{3} \right\rfloor + \left\lfloor \frac{2(3a+4)}{3} \right\rfloor \\ &= \left\lfloor 2a + \frac{4}{3} \right\rfloor + \lfloor 2a + 2 \rfloor + \left\lfloor 2a + \frac{8}{3} \right\rfloor \\ &= (2a+1) + (2a+2) + (2a+2) = 6a + 5 \\ &= 2(3a+2) + 1 = 2n + 1.\end{aligned}$$

Thus

$$\lfloor \frac{2n}{3} \rfloor + \lfloor \frac{2(n+1)}{3} \rfloor + \lfloor \frac{2(n+2)}{3} \rfloor = 2n + 1$$

for all integers n.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

5. (a) If a is a positive number, prove that

$$a + \frac{1}{a} \geq 2\,.$$

(b) If a and b are both positive numbers, prove that

$$a + \frac{1}{a} + b + \frac{1}{b} + \frac{1}{ab} \geq 4.5\,.$$

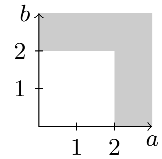
You may assume without proof that $f(x) = x + \frac{1}{x}$ is an increasing function for $x \geq 1$.

Solution based on work by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

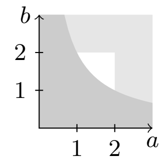
Let f be the function defined by $f(x) = x + \frac{1}{x}$. Then f is increasing on the interval $[1, \infty)$ according to the hint. Therefore, if $a \geq 1$, $f(a) \geq f(1) = 2$. If $0 < a < 1$, then $\frac{1}{a} > 1$, so $f(\frac{1}{a}) \geq f(1) = 2$, but $f(\frac{1}{a}) = f(a)$. Thus $f(a) \geq 2$ for all positive a.

To prove that $f(a) + f(b) + \frac{1}{ab} \geq 4.5$ one must consider several (partially overlapping) cases.

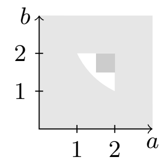
If $a \geq 2$, then $f(a) \geq f(2) = 2.5$. Since $f(b) \geq 2$ by Part (a), $f(a) + f(b) + \frac{1}{ab} \geq 2.5 + 2 + 0 = 4.5$. Similarly, if $b \geq 2$, then $f(a) + f(b) + \frac{1}{ab} \geq 2 + 2.5 + 0 = 4.5$. This proves the inequality for all (a, b) in the shaded region in the figure.



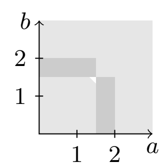
If $ab \leq 2$, then $\frac{1}{ab} \geq \frac{1}{2}$. Since $f(a) \geq 2$ and $f(b) \geq 2$ by Part (a), $f(a) + f(b) + \frac{1}{ab} \geq 2 + 2 + 0.5 = 4.5$. This proves the inequality for all (a, b) in the darker region in the figure, where the curve is given by $ab = 2$ (or $b = \frac{2}{a}$).



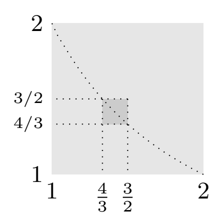
If $\frac{3}{2} \leq a \leq 2$ and $\frac{3}{2} \leq b \leq 2$, then $f(a) \geq f(\frac{3}{2}) = \frac{13}{6}$ (since f is increasing) and $f(b) \geq f(\frac{3}{2}) = \frac{13}{6}$. Moreover, $ab \leq 2 \cdot 2 = 4$, so $\frac{1}{ab} \geq \frac{1}{4}$. Therefore $f(a) + f(b) + \frac{1}{ab} \geq \frac{13}{6} + \frac{13}{6} + \frac{1}{4} = \frac{55}{12} > 4.5$. This proves the inequality in the darker region of the new figure.



If $a \leq \frac{3}{2}$ and $\frac{3}{2} \leq b \leq 2$, then $f(a) \geq 2$ and $f(b) \geq f(\frac{3}{2}) = \frac{13}{6}$. Moreover, $ab \leq \frac{3}{2} \cdot 2 = 3$, so $\frac{1}{ab} \geq \frac{1}{3}$. Therefore $f(a) + f(b) + \frac{1}{ab} \geq 2 + \frac{13}{6} + \frac{1}{3} = 4.5$. Similarly, if $\frac{3}{2} \leq a \leq 2$ and $b \leq \frac{3}{2}$, then $f(a) + f(b) + \frac{1}{ab} \geq \frac{13}{6} + 2 + \frac{1}{3} = 4.5$. This proves the inequality in the darker region of the newest figure. Note that a small white region still remains.



The curve given by $ab = 2$ passes through $(1, 2)$, $(\frac{3}{2}, \frac{4}{3})$, $(\frac{4}{3}, \frac{3}{2})$, and $(2, 1)$. The square given by $\frac{4}{3} \leq a \leq \frac{3}{2}$ and $\frac{4}{3} \leq b \leq \frac{3}{2}$ therefore completely covers the outstanding region. In that square, $f(a) \geq f(\frac{4}{3}) = \frac{25}{12}$, and $f(b) \geq f(\frac{4}{3}) = \frac{25}{12}$. Moreover, $ab \leq \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}$, so $\frac{1}{ab} \geq \frac{4}{9}$. Therefore $f(a)+f(b)+\frac{1}{ab} \geq \frac{25}{12}+\frac{25}{12}+\frac{4}{9} = \frac{83}{18} > 4.5$. This proves the inequality in the darker region in the figure and, thus, completes the proof for all positive a and b.



One can easily solve part (a) without using the hint: for any positive number a, $(a-1)^2 \geq 0$, so $a^2 - 2a + 1 \geq 0$, so $a^2 + 1 \geq 2a$. Dividing by the positive number a yields that $a + \frac{1}{a} \geq 2$ as desired.

A similar argument proves the hint:

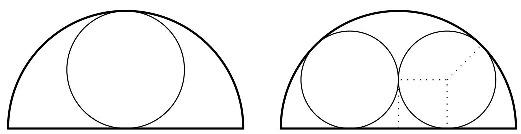
$$\begin{aligned} f(x) - f(y) &= x + \frac{1}{x} - y - \frac{1}{y} \\ &= \frac{x^2y + y - xy^2 - x}{xy} \\ &= \frac{xy(x-y) - (x-y)}{xy} \\ &= \frac{(xy-1)(x-y)}{xy}. \end{aligned}$$

Therefore, if $x \geq y \geq 1$, then $f(x) - f(y) \geq 0$, so $f(x) \geq f(y)$.

6. A hole in a concrete wall has the shape of a semi-circle with a radius of $\sqrt{2}$ metres. A utility company wants to place one large circular pipe or two smaller circular pipes of equal radius through the hole to supply water to Watertown. If they want to maximize the amount of water that could flow to Watertown, should they use one pipe or two pipes, and what size pipe(s) should they use?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

If you use a single pipe, the radius is at most $\frac{\sqrt{2}}{2}$. The area of a cross section of the pipe is therefore $\pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2}$.



If you use two pipes of radius r as in the right-hand figure, the side length of the dotted square in figure is r. By the Pythagorean Theorem, the diagonal of the square is $\sqrt{2}r$. The radius of the semi-circle is, thus, $\sqrt{2}r + r$. That is, $(\sqrt{2}+1)r = \sqrt{2}$, so

$$\begin{aligned} r &= \frac{\sqrt{2}}{\sqrt{2}+1} = \frac{\sqrt{2}}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} \\ &= \frac{2-\sqrt{2}}{\left(\sqrt{2}\right)^2 - 1^2} = 2-\sqrt{2}. \end{aligned}$$

A cross section of the two pipes therefore has area

$$\begin{aligned} 2\pi r^2 &= 2\pi(2-\sqrt{2})^2 = 2\pi(4-4\sqrt{2}+2) \\ &= (12-8\sqrt{2})\pi \approx 0.69\pi > \frac{\pi}{2}. \end{aligned}$$

Hence two pipes would allow more water to flow.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

We hope that more student readers will submit their solutions to one or more of the problems on the featured contest.

THE CONTEST CORNER

No. 1

Shawn Godin

The Contest Corner *is a new feature of* ***Crux Mathematicorum****. It will be filling the gap left by the movement of* Mathematical Mayhem *and* Skoliad *to a new on-line journal in 2013. The column can be thought of as a hybrid of* Skoliad, The Olympiad Corner *and the old* Academy Corner *from several years back. The problems featured will be from high school and undergraduate mathematics contests with readers invited to submit solutions. Readers' solutions will begin to appear in the next volume.*

Solutions can be sent to:

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K1C 2Z5

or by email to

crux-contest@cms.math.ca.

The solutions to the problems are due to the editor by **1 July 2013***.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of Université de Saint-Boniface, Winnipeg, MB for translating the problems from English into French.

CC1. A circle has centre O, diameter AC, and radius 1. A chord is drawn from A to an arbitrary point B (different from A) on the circle and extended to the point P with $BP = 1$. Thus P can take many positions. Let S be the set of points P. Determine whether or not there is a circle on which all points of S lie.

CC2. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that f is continuous and that $\int_0^1 f(a+tu)dt = 0$ for every point $a \in \mathbb{R}^2$ and every vector $u \in \mathbb{R}^2$ with $\|u\| = 1$. Show that f is constant.

CC3. All three sides of a right triangle are integers. Prove that the area of the triangle: is also an integer; is divisible by 3; and is even.

CC4. Suppose that $n \geq 3$. A sequence $a_1, a_2, a_3, \ldots, a_n$ of n integers, the first m of which are equal to -1 and the remaining $p = n - m$ of which are equal to 1, is called an MP sequence. Consider all of the products $a_i a_j a_k$ (with $i < j < k$) that can be calculated using the terms from an MP sequence $a_1, a_2, a_3, \ldots, a_n$. Determine the number of pairs (m, p) with $1 \leq m \leq p \leq 1000$ and $m + p \geq 3$ for which exactly half of these products are equal to 1.

CC5. Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P. Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides and a circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

. . . .

CC1. Un cercle de centre O et de diamètre AC a un rayon de 1. Au point A, on trace une corde jusqu'à un point quelconque B (différent de A) sur le cercle et on la prolonge jusqu'à un point P de manière que $BP = 1$. Ainsi P peut prendre un grand nombre de positions. Soit S l'ensemble des points P. Déterminer s'il existe ou non un cercle sur lequel sont situés tous les points de S.

CC2. Soit $f : \mathbb{R}^2 \to \mathbb{R}$. Supposons que f est continue et que $\int_0^1 f(a + tu)dt = 0$ pour tout point $a \in \mathbb{R}^2$ et tout vecteur $u \in \mathbb{R}^2$ tel que $\|u\| = 1$. Démontrer que f est constante.

CC3. Les trois côtés d'un triangle rectangle sont de longueurs entières. Démontrer que la surface du triangle est entière, puis que cet entier est divisible par 3 et qu'il est pair.

CC4. On considère une suite de n entiers, $a_1, a_2, a_3, \ldots, a_n$, $n \geq 3$, dont les m premiers termes égalent tous -1 et les p autres termes, $p = n - m$, égalent tous 1. Une telle suite est appelée une suite MP. On considère tous les produits $a_i a_j a_k$, $i < j < k$, que l'on peut former à partir d'une suite MP $a_1, a_2, a_3, \ldots, a_n$. Déterminer combien il y a de couples (m, p), $1 \leq m \leq p \leq 1000$ et $m + p \geq 3$, pour lesquels exactement la moitié de ces produits égalent 1.

CC5. Soit $ABCD$ un parallélogramme. On trace la diagonale AC. Un cercle est inscrit à lintérieur du ΔABC tangent aux trois côtés et touchant le côté AC en un point P. Tracer le segment de droite DP. Un cercle de rayon r_1 est inscrit à lintérieur du ΔDAP tangent aux trois côtés. Un cercle de rayon r_2 est inscrit à lintérieur du ΔDCP tangent aux trois côtés. Démontrer que

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

THE OLYMPIAD CORNER

No. 299

Nicolae Strungaru

The solutions to the problems are due to the editor by **1 July 2013***.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

OC61. 46 squares of a 9×9 grid are coloured red. Prove that we can find a 2×2 square on the grid which contains at least 3 red squares.

OC62. Let A, B, C, D be four non-coplanar points in space. The segments AB, BC, CD and DA are tangent to the same sphere. Prove that their four points of tangency are coplanar.

OC63. Prove that there exists a perfect square so that the sum of its digits is 2011.

OC64. Find all integer solutions of the equation

$$n^3 = p^2 - p - 1$$

where p is prime.

OC65. Let ABC be a triangle. F and L are two points on the side AC such that $AF = LC < AC/2$. If $AB^2 + BC^2 = AL^2 + LC^2$ find $\angle FBL$.

. . . .

OC61. 46 carrés d'une grille de format 9×9 sont coloriés en rouge. Montrer qu'on peut trouver, dans cette grille, un carré de format 2×2 pouvant contenir au moins 3 carrés rouges.

OC62. Soit A, B, C, D quatre points non coplanaires dans l'espace. Les segments AB, BC, CD et DA sont tangents à une même sphère. Montrer que leur quatre points de tangence sont coplanaires.

OC63. Montrer qu'il existe un carré parfait tel que la somme de ses chiffres est 2011.

OC64. Trouver toutes les solutions de l'équation

$$n^3 = p^2 - p - 1$$

avec n entier et p premier.

OC65. Soit ABC un triangle. F et L sont deux points sur le côté AC tel que $AF = LC < AC/2$. Si $AB^2 + BC^2 = AL^2 + LC^2$ trouver $\angle FBL$.

OLYMPIAD SOLUTIONS

OC1. Find all positive integers w, x, y and z which satisfy $w! = x! + y! + z!$.
(*Originally question # 1 from the 1983 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Curtis.

By symmetry we may assume that $x \leq y \leq z \leq w$.

If $y > x$ then $(x+1)!$ divides $y!, z!$ and $w!$, thus $(x+1)!$ must divide $x!$, a contradiction. Thus we get $x = y$.

We now break the problem in two cases:

Case 1: $y < z$. In this case $(y+1)!$ divides $z!$ and $w!$ thus it must divide $x! + y! = 2y!$. This implies that $(y+1)$ divides 2, and hence $y = 1$. Our equation becomes then

$$w! = z! + 2\,.$$

From here we get $w > z$, and since $z!$ divides $w!$ it must also divide 2. Thus, as $z > y = 1$, we get that $z = 2$, and hence $w! = 4$, which is not possible.

We get no solution in this case.

Case 2: $y = z$. In this case our equation becomes

$$w! = 3x!\,.$$

Then $w > x$ and hence

$$(x+1)!|w! = 3x! \Rightarrow x+1|3\,.$$

Since $x \geq 1$, we get $x = 2$ which yields the solution $x = y = z = 2$ and $w = 3$.

Hence the only solution is $(2, 2, 2, 3)$.

[*Ed.: It is clear that $x, y, z < w$ thus $x!, y!, z! \leq (w-1)!$. Hence we get*

$$w! = x! + y! + z! \leq 3(w-1)! \Rightarrow w \leq 3$$

with equality if and only if $x = y = z = w - 1$. Thus, either $x = y = z = 2$ and $w = 3$ or $w \leq 2$, in which case $x! + y! + z! \geq 3 > 2! \geq w!$ which yields no new solutions.]

OC2. Suppose that f is a real-valued function for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real numbers x and y.

(a) Give a nonconstant polynomial that satisfies the condition.

(b) Prove that $f(x) \geq 0$ for all real x.

(*Originally question # 3 from the 2007 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Curtis.

(a) Let $P(x) = x^2 + 4$. Then

$$\begin{aligned} P(xy) + P(y-x) - P(y+x) & \\ &= x^2y^2 + 4 + y^2 - 2xy + x^2 + 4 - x^2 - y^2 - 2xy - 4 \\ &= x^2y^2 - 4xy + 4 = (xy-2)^2 \geq 0. \end{aligned}$$

Thus

$$P(xy) + P(y - x) \geq P(y + x)\,.$$

(b) Let α be any real number. Solving $x + y = xy$ and $y - x = \alpha$ yields

$$x = \frac{2 + \sqrt{4 + \alpha^2} - \alpha}{2}\,;\; y = \frac{2 + \sqrt{4 + \alpha^2} + \alpha}{2}\,.$$

Setting these values in our inequality we get

$$f(2 + \sqrt{\alpha^2 + 4}) + f(\alpha) \geq f(2 + \sqrt{\alpha^2 + 4})\,,$$

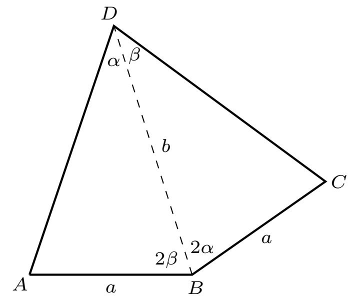
hence $f(\alpha) \geq 0$ for any real number α.

OC3. Let $ABCD$ be a convex quadrilateral with

$$\begin{aligned} \angle CBD &= 2\angle ADB, \\ \angle ABD &= 2\angle CDB \\ \text{and} \qquad AB &= CB. \end{aligned}$$

Prove that $AD = CD$.
(*Originally question # 4 from the 2000 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.



Let $\alpha = \angle ADB$, $\beta = \angle CDB$, $a = AB = CB$ and $b = DB$. By the Law of Sines in the triangles ABD and CBD we get

$$\frac{\sin(\alpha)}{\sin(\alpha+2\beta)} = \frac{a}{b} = \frac{\sin(\beta)}{\sin(2\alpha+\beta)}\,.$$

It follows that

$$\sin(\alpha)\sin(2\alpha+\beta) = \sin(\beta)\sin(\alpha+2\beta)\,,$$

and hence

$$\cos(\alpha+\beta) - \cos(3\alpha+\beta) = \cos(\alpha+\beta) - \cos(\alpha+3\beta)\,.$$

Thus

$$\cos(3\alpha+\beta) - \cos(\alpha+3\beta) = 0\,,$$

or

$$\sin(2\alpha+2\beta)\sin(\alpha-\beta) = 0\,.$$

Since $ABCD$ is convex we have $0 < \angle ABC = 2\alpha + 2\beta < 180^\circ$ and thus $\sin(2\alpha+2\beta) \neq 0$. This implies that $\sin(\alpha-\beta) = 0$, and hence, using $0 < \alpha+\beta < 90^\circ$, we get $\alpha = \beta$. Then, the triangles ABC and CBD are congruent, which yields the desired equality.

[*Ed.: The only place we use the fact that $ABCD$ is convex is to deduce that* $\sin(2\alpha+2\beta) \neq 0$. *If we drop the convexity requirement, the only other possibility is* $\angle ABC = 180^\circ$. *In this case* ΔDAC *is a right angle triangle, and* B *is the midpoint of* AC. *Such a triangle is always a non-convex solution.*]

OC4. Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.
(*Originally question # 1 from the 2011 Canadian Mathematical Olympiad.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We provide the similar solutions of Curtis and Zvonaru.

For a positive integer n, let $s(n)$ denote the sum of the digits of n. It is well known that $n \equiv s(n) \pmod 9$.

Suppose by contradiction that there are two numbers $A > B$ as in the problem so that $B \mid A$. Then, there exists a positive integer C so that $A = BC$. Then, since $A < 8 \cdot 10^{69}$ and $B > 10^{69}$ we get $1 < C < 8$.

We have $s(A) = s(B) = 10 \cdot (1+2+3+4+5+6+7) = 280$, and hence $A \equiv B \equiv 1 \pmod 9$. Thus $1 \equiv A \equiv BC \equiv C \pmod 9$. But this contradicts $1 < C < 8$. Since we got a contradiction, our assumption is wrong, and thus the claim of the problem is true.

OC5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0,$$
$$a_1 + a_2 \leq 100$$
$$\text{and} \qquad a_3 + a_4 + \cdots + a_{100} \leq 100.$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.
(*Originally question # 5 from the 2000 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Bataille.

Let $Q = a_1^2 + a_2^2 + \cdots a_{100}^2$. We show that the maximum value of Q is 10 000, attained exactly when $(a_1, a_2, \ldots, a_{100}) = (100, 0, 0, \ldots, 0)$ or $(50, 50, 50, 50, 0, \ldots, 0)$. First, we calculate:

$$\begin{aligned} 100a_2 - (a_3^2 + \cdots + a_{100}^2) &\geq a_2(a_3 + \cdots + a_{100}) - (a_3^2 + \cdots + a_{100}^2) \\ &= a_3(a_2 - a_3) + \cdots + a_{100}(a_2 - a_{100}) \geq 0. \end{aligned}$$

Thus,

$$100a_2 \geq a_3^2 + a_4^2 + \cdots + a_{100}^2. \qquad (1)$$

On the other hand, since $0 \leq a_1 \leq 100 - a_2$, we have

$$a_1^2 + a_2^2 \leq (100 - a_2)^2 + a_2^2. \qquad (2)$$

By addition, (1),(2) give

$$Q \leq (100 - a_2)^2 + a_2^2 + 100a_2 = 10\,000 + 2a_2(a_2 - 50).$$

Now, the conditions $a_1 \geq a_2$ and $a_1 + a_2 \leq 100$ call for $a_2 \leq 50$, hence $2a_2(a_2 - 50) \leq 0$. It follows that $Q \leq 10\,000$ and, if equality holds, then $a_2 = 0$ or $a_2 = 50$. In the former case, we have $a_3 = a_4 = \cdots = a_{100} = 0$ and since we must also have $a_1 = 100 - a_2$, we obtain $(a_1, a_2, \ldots, a_{100}) = (100, 0, 0, \ldots, 0)$. In the latter case, $a_1 = 100 - a_2 = 50$ and since equality holds in (1), $a_3 + \cdots + a_{100} = 100$ and for each $k \in \{3, 4, , \ldots, 100\}$, $a_k = 0$ or $a_k = a_2 = 50$. Recalling that $a_3 \geq a_4 \cdots \geq a_{100}$, this implies that $a_3 = a_4 = 50$ and $a_k = 0$ for $k \geq 5$. Thus, $(a_1, a_2, \ldots, a_{100}) = (50, 50, 50, 50, 0, \ldots, 0)$. Conversely, the two 100-tuples $(100, 0, 0, \ldots, 0)$ and $(50, 50, 50, 50, 0, \ldots, 0)$ satisfy the conditions and give $Q = 10\,000$.

Several solvers noted that some of the problems have appeared in past issues of the journal. As this first problem set came at a transition period for the *Olympiad Corner*, the Editor-in-Chief put the problem set together using CMO problems from the not so recent past (in most cases). Future problem sets will consist of problems that haven't appeared in the journal before (we hope!).

BOOK REVIEWS

Amar Sodhi

History of Mathematics: Highways and Byways by Amy Dahan-Dalmedico and Jeanne Peiffer (translated by Sanford Segal)
Mathematics Association of America, 2010
ISBN13: 978-0-88385-562-1, Hardcover, 330 + xi pp., US$52.95
Reviewed by ***Brenda Davison****, Simon Fraser University, Burnaby, BC*

Anyone writing a survey book on the History of Mathematics is forced by the sheer volume of material and by the mathematical sophistication of the intended audience to make countless choices about what material to include. As well, a choice about the order of presentation of the topics must be made; the most natural choice is often chronological.

In 'History of Mathematics, Highways and Byways', Amy Dahan-Dalmedico and Jeanne Peiffer navigate these choices with aplomb and have produced a book which, at 326 pages, contains an enormous amount of information, but which is presented so as not to overwhelm the reader.

This book has been translated from the French by Sanford Segal who attempts "to remain faithful to the original French while maintaining fluent English". The fact that someone was willing to translate this book is itself a strong recommendation in its favour.

The first chapter of the book is a whirlwind tour of three thousand years of mathematical history in thirty-three pages. In each of the remaining seven chapters, the authors have chosen a single mathematical topic and followed that topic from its birth to, typically, the end of the 19th century. This naturally means that you revisit a given time period on many occasions through the lens of the different topics discussed. I found that once I was used to this, it worked very well and that the book does a good job of incorporating the broader historical and cultural contexts in which the particular mathematics was developing.

The topics chosen for each of the chapters are: Greece — a moment of rationality, classical algebra, geometry and space, limits, the function concept and analysis, complex numbers, and the emergence of algebraic structures.

The book is demanding of its readers — both in terms of the language used and the mathematical knowledge required. One gets a sense, correctly or not, from this book that French students are introduced to higher-level mathematics sooner than their Canadian counterparts. For example, at the opening of the final chapter on the emergence of algebraic structures, the authors state that "For an advanced secondary school or university student at the end of the twentieth century, set theoretic ideas and simple algebraic structures (group, ring, field, vector space) were part of the mathematical baggage they had encountered as long ago as their first years of secondary instruction." It does, however, repay the reader's effort amply.

My criticisms are few and minor. The main flaw with the book is its nearly complete lack of a bibliography. At just ten items, two of which are basic survey books of less sophistication than 'Highways and Byways', the reader is left without direction on where to turn for more information about the many interesting topics presented in the text.

The authors also use "etc." and ellipsis frequently and in a manner that leaves you wondering what or whom they have left out for you to include. An example of this comes during a discussion of algebra during 19th century Germany: "The German school presented an impressive group of brilliant algebraists (Kummer, Kronecker, Dedekind, Weber, . . .)".

Anyone interested in how some of the main themes of mathematics have developed over time should read this book. You will learn a lot of history and will probably learn some mathematics as well.

Train Your Brain: A Year's Worth of Puzzles by George Grätzer
CRC Press, 2011
ISBN: 978-1-56881-710-1, softcover, 235 + xiii pp. US$29.95
Reviewed by ***Amar Sodhi****, Grenfell Campus, MUN, Corner Brook, NL*

In my formative years I stumbled on a Reader's Digest Anthology. Much of that volume held little interest to me, but there was an article on "The Magic of Lewis Carroll" and a selection of riddles and brain teasers. This was my first introduction into the world of recreational mathematics and I became hooked on puzzles which taxed one's skills in logic and/or arithmetic and sometimes required a touch of the "Aha!" insight in order to obtain a satisfactory solution. It is little wonder, therefore, that I found it a delight to review "Train Your Brain: A Year's Worth of Puzzles" by George Grätzer.

George Grätzer is a name which may be familiar to many CRUX readers. He is currently a Professor at the University of Manitoba and has books published in two areas of mathematics (Universal Algebra and Lattice Theory). Users of LaTeX are probably aware of the many handbooks he has published on the subject. Chess enthusiasts may have come across some of the prize winning chess studies that the youthful Grätzer composed prior to leaving his native Hungary in 1963. A selection of these compositions, including a "mate in 45", can be found at `http://server.math.umanitoba.ca/homepages/gratzer/chess.pdf`.

However, in Hungary at least, it is George's father, József (1897-1945), who is renowned as a communicator of puzzles. In 2007, a Hungarian publisher approached George with the aim of re-publishing two collections of József's brain teasers, Rébusz and Sicc, which were originally published in 1935. George agreed to this, but suggested that his only collection of teasers, "Elmesport egy esztendöre" (published in 1959) also be re-published to form a "Grätzer trilogy". This was done; the second edition of "Elmesport egy esztendöre" was subsequently translated (by Tim Artin) into English as "Train Your Brain: A Year's Worth of Puzzles".

"Train Your Brain" is a wonderfully eclectic collection of problems; each one contained within a story so as to either pique the interest of the reader or to merely provide a conduit to efficiently pose the teaser. The story lines used may be original, but Grätzer does not claim that any of the puzzles are his creation and acknowledges that a major source of the brain teasers in the first edition "relies heavily on the beautiful book "Mathematical Recreations" by M. Kraitchik (1955)". He also cites the Hungarian "Mathematical and Physical Journal for Secondary Schools" as another source of problems. Many teasers reminded me of the time I struggled through H. E. Dudeney's "More Puzzles and Curious Problems" as a teenager. Standard logic and number problems (which include age computation conundrums) abound, but the diligent reader will also encounter combinatorial games, a rewording of the Tower of Hanoi puzzle and Russell's Paradox.

The puzzles are arranged by weeks; for the intent is for the reader to use the problems, over the course of the year, to achieve a sharper mind. This being the case, the (subjectively) more challenging puzzles appear towards the end of the book. Also, each of the first 36 weeks contains three problems, whereas only two problems per week feature in the last 16 weeks of "Black Belt" conundrums. Part 2 of the book provides useful hints for many of the problems. Detailed solutions for each problem are given in part 3. The hints and solutions allow Grätzer to introduce concepts, such as arithmetic and geometric mean, binomial coefficients, the Fibonacci sequence and Pascal's triangle. A description of mathematical induction and a table of primes and prime factorizations are included in the appendices. Although this may seem intimidating to the younger audience, the author claims that part of the audience he wishes to reach are "high-school students who want to learn deductive reasoning". This audience will certainly not be disappointed in Grätzer's book.

In summary, "Train Your Brain: A Year's Worth of Puzzles" is an excellent book to give to an aunt, uncle, niece, nephew, Mayhem reader or anyone who likes to relax to a brain teaser. However, if Grätzer really wanted this book to last a year, then he should have provided at least one puzzle a day. Restricting oneself to just two or three teasers a week is far too challenging; at least for me.

On sums and differences of powers of rational numbers

Luis H. Gallardo and Philippe Goutet

Abstract

Given two nonzero integers $a, b \in \mathbb{Z}^*$, we characterize the rational numbers x, y such that $ax^n - by^n \in \mathbb{Z}$ for all non-negative integers $n \in \mathbb{N}$.

1 Introduction

If a rational number $x \in \mathbb{Q}$ has a power which is an integer, then x itself is forced to be an integer by the fundamental theorem of arithmetic. In other words if we have $x = \frac{N}{D}$ with a positive integer D and an integer N, and both satisfy $(N/D)^r = K$ (where K is an integer), for some positive integer r, where N/D is a reduced fraction (N and D have no common factors, we also say that N and D are *coprime*); then by comparing exponents of each prime number appearing in both sides of the equality

$$N^r = D^r K$$

we get $D = 1$ so that $x = N$ is indeed an integer.

A natural generalization of this problem consists in looking at $c_n = ax^n - by^n$ where a, $b \in \mathbb{Z}^*$ are two nonzero integers and x, $y \in \mathbb{Q}$ are two rational numbers, and asking if the existence of some values of n such that c_n is an integer, i.e., $c_n \in \mathbb{Z}$ implies that x and y are indeed integers, i.e., x, $y \in \mathbb{Z}$.

The existence of only one n such that $c_n \in \mathbb{Z}$ is not sufficient, as shown, for example (check it !), by the relation $(\frac{13}{2})^5 + (\frac{19}{2})^5 = 88981 \in \mathbb{Z}$. However, the result becomes true with the stronger assumption that all the c_n are in $\mathbb{Z}$.

Theorem 1 *Consider two nonzero integers $a, b \in \mathbb{Z}^*$ and two rational numbers $x, y \in \mathbb{Q}$. If, for all $n \in \mathbb{N}$, $ax^n - by^n \in \mathbb{Z}$, then x and y are both integers unless $a = b$ and $x = y$.*

Robert Israel (University of British Columbia), gives a direct proof [3] of the case $a = b = 1$. At the end of the present note, we look at how to weaken the assumption that *all* the c_n are in $\mathbb{Z}$ when $a \neq b$.

We recall some classical notation used in the proof: If a and b are two integers such that there exists an integer m such that $ma = b$ then we say that a divides b and we write: $a \mid b$. As usual, we write $d = \gcd(a, b)$ their greatest common divisor, so that, for example, $\gcd(17, 51) = 17$, while $\gcd(a, b) = 1$ is equivalent to a, b are coprime. Now, we fix a positive integer $n \in \mathbb{N}$. First of all, Euler's totient function computed on n, denoted $\varphi(n)$ gives us the number of positive integers h in between 1 and n that are coprime with n. Second, and this is a little more complicated object we consider here: the n-th cyclotomic polynomial $\Phi_n(t)$ is a one variable polynomial in the indeterminate t with integral coefficients that has the property that it is the polynomial, with integer coefficients, of minimal degree that vanishes when $t = w$ where the complex, but non-real, number $w \in \mathbb{C}$ is a n-th root of unity; this means that $w^n = 1$. For example, $\Phi_3(t) = t^2 + t + 1$, since $\Phi_3(t) = \frac{t^3-1}{t-1}$ shows that $\Phi_3(w) = 0$ for $w = \frac{-1+i\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}$ and also for $w^2 = \frac{-1-i\sqrt{3}}{2} = e^{\frac{-2\pi i}{3}}$, where w, w^2 are the, non-real, 3-roots of unity in the field of complex numbers $\mathbb{C}$; while any polynomial of degree 1 with integer coefficients cannot vanish simultaneously in w and in w^2. A nice result of Gauss is that the degree of $\Phi_n(t)$ is precisely $\varphi(n)$.

2 The proof

We write x and y as irreducible fractions $x = \frac{N}{D}$ and $y = \frac{M}{E}$ with $D, E > 0$. In order to show that both x and y are integers, we proceed in two steps, first showing that $D = E$ and then showing that $D = 1$.

Lemma 1 *$D = E$.*

Proof. As $c_n = ax^n - by^n \in \mathbb{Z}$, we have $aN^nE^n - bM^nD^n = c_nE^nD^n$. Since D and N are coprime, we deduce that $D^n \mid aE^n$. Similarly, $E^n \mid bD^n$.

Consider a prime number p and write $a = p^\alpha a'$, $b = p^\beta b'$, $D = p^d D'$, and $E = p^e E'$ with a', b', D', and E' coprime to p. Because $E^n \mid bD^n$, we have $ne \leq nd + \beta$ and, similarly, $D^n \mid aE^n$ gives $nd \leq ne + \alpha$. By taking $n > \max(\alpha, \beta)$, we deduce that $e \leq d$ and $d \leq e$ and so $d = e$. As this is valid for any prime p, we conclude that $D = E$.

Lemma 2 *$D = 1$.*

Proof. As $D = E$, we can rewrite $ax^n - by^n = c_n$ as $aN^n - bM^n = c_nD^n$ and so $D^n \mid aN^n - bM^n$ for all $n \in \mathbb{N}$. We consider two cases, depending on whether $a = b$ or not.

FIRST CASE: $a \neq b$. We have $D^n \mid aN^n - bM^n$ and $D^n \mid D^{2n} \mid aN^{2n} - bM^{2n}$. Hence, $D^n \mid (aN^n - bM^n)(aN^n + bM^n) = a^2N^{2n} - b^2M^{2n}$ and thus $D^n \mid (a^2N^{2n} - b^2M^{2n}) - a(aN^{2n} - bM^{2n}) = b(a-b)M^{2n}$. Because $D = E$ and M are coprime, we deduce that $D^n \mid b(a-b)$. The number $b(a-b)$ is $\neq 0$ because $b \neq 0$ and $a \neq b$, hence $D = 1$.

SECOND CASE: $a = b$. This case is a bit more difficult. As mentioned in the Theorem, we exclude the case $x = y$ or else $c_n = 0 \in \mathbb{Z}$ for all n, independently of the value of x. Let $R = \gcd(M, N)$ and write $N = RN_1$ and $M = RM_1$. Because D is coprime to both N and M, D is coprime to R. As $D^n \mid a(N^n - M^n)$, we deduce that $D^n \mid a(N_1^n - M_1^n)$ and we write $a(N_1^n - M_1^n) = a(N_1 - M_1)C_n$ where $C_n = (N_1^n - M_1^n)/(N_1 - M_1)$. Since $D \mid a(N_1 - M_1)$, we deduce, for each n such that C_n is coprime to a and $N_1 - M_1$, that $D^n \mid a(N_1 - M_1)$. If this is true for infinitely many n, we will have $D = 1$ as $a(N_1 - M_1) \neq 0$ since $a \neq 0$ and $x \neq y$.

We are thus reduced to showing that C_n is coprime to both a and $N_1 - M_1$ for infinitely many n. We do so for n a prime number which divides neither $N_1 - M_1$ nor a nor $\varphi(a)$. For such an n, Lemma 3 below applies to show that $N_1 - M_1$ and C_n are coprime and Lemma 5 applies to show that a and C_n are coprime, hence the result.

3 Auxiliary lemmas

We recall the notion of *order* of an integer n modulo a prime number p, say $o_p(n)$: it is the minimal positive integer r such that $n^r \equiv 1 \pmod p$. One knows that $o_p(n)$ divides $\varphi(p) = p - 1$; for example (check it !) $o_{1093}(2) = 364$.

In the previous proof, we have used the following lemmas. The first two are classical results, but we recall their proofs for the convenience of the reader.

Lemma 3 *Consider $n \geq 1$. If N_1 and M_1 are two coprime integers such that n and $N_1 - M_1$ are coprime, then $N_1 - M_1$ and $C_n = (N_1^n - M_1^n)/(N_1 - M_1)$ are coprime.*

In fact, one can show [1, Exercise 71, p. 20] that, if $d = \gcd(a, b)$, then

$$\gcd(\frac{a^n - b^n}{a - b}, a - b) = \gcd(nd^{n-1}, a - b).$$

Proof. Let p be a prime number dividing $N_1 - M_1$. As $N_1 \equiv M_1 \mod p$, we have $M_1^i N_1^j \equiv N_1^{i+j} \mod p$ and thus $C_n \equiv nN_1^{n-1} \mod p$. As n and p are coprime and N_1 and p are also coprime (because p divides $N_1 - M_1$ with N_1 coprime to M_1), we deduce that C_n and p are coprime. This is true for each prime p dividing $N_1 - M_1$, thus C_n and $N_1 - M_1$ are coprime.

Lemma 4 *If $n \neq p$ are two prime numbers, then the existence of $x \in \mathbb{Z}$ such that $\Phi_n(x) \equiv 0 \mod p$ implies that $n \mid \varphi(p) = p - 1$.*

Although it simplifies the proof, the fact that n is prime is not necessary as long as n and p are coprime; see [2, Theorem 94, p. 164].

Proof. As $\Phi_n(x) \equiv 0 \mod p$, we have $x^n \equiv 1 \mod p$. Because $\Phi_n(1) = n \not\equiv 0 \mod p$, we deduce that $x \not\equiv 1 \mod p$ and so x is of order n as n is prime. Hence, $n \mid \varphi(p)$.

Lemma 5 *Let n be a prime number, N_1 and M_1 two coprime integers and $C_n = (N_1^n - M_1^n)/(N_1 - M_1)$. If n is coprime to both a and $\varphi(a)$, then a is coprime to C_n.*

Proof. As n is a prime number, we can write $C_n = M_1^{n-1}\Phi_n(\frac{N_1}{M_1}) = N_1^{n-1}\Phi_n(\frac{M_1}{N_1})$. Let p be a prime number dividing a; as n and a are coprime, so are n and p; similarly, n and $\varphi(p)$ are coprime since $\varphi(p) \mid \varphi(a)$. Because M_1 and N_1 are coprime, one of them, let's say M_1, is not divisible by p. Denote by M_1' the inverse of M_1 mod p so that $C_n \equiv M_1^{n-1}\Phi_n(N_1 M_1') \mod p$. Since $M_1 \not\equiv 0 \mod p$ and $\Phi_n(N_1 M_1') \not\equiv 0 \mod p$ by Lemma 4, we have $C_n \not\equiv 0 \mod p$. As this is true for every prime dividing a, we deduce that a and C_n are coprime.

4 Strengthening of the theorem

In the previous theorem, it is not necessary to assume that all the c_n are in $\mathbb{Z}$: we only need $c_n \in \mathbb{Z}$ for $1 \le n \le N$ with N sufficiently large. How large depends on a and b, as we will now see in the case $a \neq b$. Before that, we introduce the notation $\epsilon(1) = 0$ and, if $m \ge 2$, $\epsilon(m) = \max_{1 \le i \le r} \alpha_i$ where $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ is the prime decomposition of m (all the p_i being distinct).

Proposition 1 *Consider $a \neq b$ in $\mathbb{Z}^*$, $x, y \in \mathbb{Q}$, and $N \in \mathbb{N}$. Assume that $N > \epsilon(a)$, $N > \epsilon(b)$ and $N > 2\epsilon(b(a-b))$. If $ax^n - by^n \in \mathbb{Z}$ for $1 \le n \le N$, then x and y are both integers.*

Proof. We only need to show that the proofs of Lemma 1 and Lemma 2 stay valid. In the proof of Lemma 1, we need to be able to take $n > \max(\alpha, \beta)$, which is allowed by the conditions $N > \epsilon(a)$ and $N > \epsilon(b)$. In the case $a \neq b$ of Lemma 2, we need $n > \epsilon(b(a-b))$ for the condition $D^n \mid b(a-b)$ to imply that $D = 1$; but as this condition is obtained by considering $D^{2n} \mid aN^{2n} - bM^{2n}$, we need to be able to take $n > 2\epsilon(b(a-b))$, which is allowed by the condition $N > 2\epsilon(b(a-b))$.

Example: If $a = 2$ and $b = 1$, the minimal N satisfying the assumptions of the previous proposition is $N = 2$ since $\epsilon(a) = 1$ and $\epsilon(b) = \epsilon(b(a-b)) = 0$. By considering $(x, y) = (\frac{1}{2}, 3)$, we see that this value of N is optimal.

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PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 July 2013****. An asterisk* (⋆) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3690. *Correction. Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be three distinct positive real numbers with $a+b+c=1$. Show that

$$(5x^2-6xy+5y^2)(a^3+b^3+c^3)+12(x^2-3xy+y^2)abc>(x-y)^2$$

for all real numbers x and y, not both zero.

3693. *Correction. Proposed by Michel Bataille, Rouen, France.*

Given $k \in \left(-\frac{1}{4},0\right)$, let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0=2$, $a_1=1$ and the recursion $a_{n+2}=a_{n+1}+ka_n$. Evaluate

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2}\,.$$

3701. *Proposed by R. F. Stöckli, Buenos Aires, Argentina.*

Let S be the set of all real continuous functions f defined on the closed interval $[0,1]$ with $f(0)=f(1)=0$. Find all numbers $0<r<1$ such that for every f belonging to S there exists $c=c(f)$ and $d=d(f)$ belonging to $[0,1]$ such that $d-c=r$ and $f(c)=f(d)$.

3702. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the incircle (with centre I) of triangle ABC touch the sides BC at D, CA at E, and AB at F. Define M and P to be the points where BC intersects the lines IE and IF, N and Q to be the second intersections of the incircle with IE and IF, and R and S to be the second intersections of the incircle with MQ and PN. If H and K are the points where NQ intersects ER and FS, prove that $\angle KDH=\angle BAC$.

3703. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b, and c be the sides, and r the inradius of a triangle ABC. Prove that

$$\frac{a\sqrt[3]{abc}}{bc(a^2+bc)}+\frac{b\sqrt[3]{abc}}{ac(b^2+ca)}+\frac{c\sqrt[3]{abc}}{ab(c^2+ab)} \leq \frac{1}{8r^2}\,.$$

3704. *Proposed by Richard McIntosh, University of Regina, Regina, SK.*

Let $p \equiv 1 \pmod 3$ be a prime and let n be an integer satisfying $n^2+n+1 \equiv 0 \pmod p$. Prove that $(n+1)^p \equiv n^p + 1 \pmod{p^3}$.

3705. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a scalene triangle and G its centre of gravity. Let the perpendicular to BC through G meet the internal bisector of $\angle BAC$ at A'.

(a) Show that G and the orthogonal projections of A' onto the lines AB and AC are collinear.

(b) If B' and C' are defined similarly to A', prove that

$$\frac{GA'\cdot GB'}{CA\cdot CB}+\frac{GB'\cdot GC'}{AB\cdot AC}+\frac{GC'\cdot GA'}{BC\cdot BA}=\frac{1}{9}\,.$$

3706. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Prove that for all positive real numbers a, b, c, and d which satisfy $a,b,c \geq 1$ and $abcd = 1$,

$$\sum_{\text{cyclic}} \frac{1}{(a^2-a+1)^2} \leq 4\,.$$

3707. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let k and m be positive integers. Prove that

$$\int_0^\infty \frac{\sin^{2k} x}{(\pi^2-x^2)((2\pi)^2-x^2)\cdots((k\pi)^2-x^2)}dx = 0\,.$$

3708. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Construct the isosceles trapezoid with three equal sides a, with a straightedge and a compass alone, provided that its base $b = AB$ and the angle α $(0^\circ < \alpha < 90^\circ)$ at the base are given.

3709. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be nonnegative real numbers, k and $l \geq 0$ and define

$$\frac{a+b}{2} - \sqrt{ab} \;=\; k^2\,, \;\frac{a+b+c}{3} - \sqrt[3]{abc} \;=\; l^2\,.$$

Prove that

$$\max(a,b,c) \;\geq\; \min(a,b,c) + \frac{3}{2}(k-l)^2\,.$$

3710. *Proposed by Billy Jin, Waterloo Collegiate Institute and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Given $n \in \mathbb{N}$, show that there exists a $k \in \mathbb{N}$ such that for all $m \geq k$, there exists a sequence of m consecutive natural numbers which contains exactly n primes.

. . . .

3690. *Correction. Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c trois nombres réels positifs distincts avec $a+b+c = 1$. Montrer que

$$(5x^2 - 6xy + 5y^2)(a^3 + b^3 + c^3) + 12(x^2 - 3xy + y^2)abc > (x-y)^2$$

pour tous les nombres réels x et y, non simultanément nuls.

3693. *Correction. Proposé par Michel Bataille, Rouen, France.*

Etant donné $k \in \left(-\frac{1}{4}, 0\right)$, soit $\{a_n\}_{n=0}^{\infty}$ la suite définie par $a_0 = 2$, $a_1 = 1$ et la récursion $a_{n+2} = a_{n+1} + ka_n$. Evaluer

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}\,.$$

3701. *Proposé par R. F. Stöckli, Buenos Aires, Argentina.*

Soit S l'ensemble de toutes les fonctions réelles, définies dans l'intervalle fermé $[0,1]$, avec $f(0) = f(1) = 0$. Trouver tous les nombres $0 < r < 1$ tels que, pour tout f élément de S, il existe $c = c(f)$ et $d = d(f)$ appartenant à $[0,1]$ tels que $d - c = r$ and $f(c) = f(d)$.

3702. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

Dans un triangle ABC le cercle inscrit (de centre I) touche les côtés BC en D, CA en E et AB en F. Soit respectivement M et P les points où BC coupe les droites IE et IF, N et Q de même que R et S les deuxièmes intersections du cercle inscrit avec les droites MQ et PN. Si H et K sont les d'intersection de NQ avec ER et FS, montrer que $\angle KDH = \angle BAC$.

3703. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci,
Italie.*

Soit a, b et c les côtés et r le rayon du cercle inscrit d'un triangle ABC. Montrer que

$$\frac{a\sqrt[3]{abc}}{bc(a^2+bc)} + \frac{b\sqrt[3]{abc}}{ac(b^2+ca)} + \frac{c\sqrt[3]{abc}}{ab(c^2+ab)} \ \leq\ \frac{1}{8r^2}\,.$$

3704. *Proposé par Richard McIntosh, Université de Regina, Regina, SK.*

Soit $p \equiv 1 \pmod 3$ un nombre premier et n un entier satisfaisant $n^2+n+1 \equiv 0 \pmod p$. Montrer que $(n+1)^p \equiv n^p + 1 \pmod{p^3}$.

3705. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle scalène et G son centre de gravité. Désignons par A' le point d'intersection de la perpendiculaire à BC passant par G avec la bissectrice intérieure de $\angle BAC$.

(a) Montrer que G et les projections orthogonales de A' sur les droites AB et AC sont colinéaires.

(b) Si B' et C' sont définis de manière analogue à celle de A', montrer que

$$\frac{GA'\cdot GB'}{CA\cdot CB} + \frac{GB'\cdot GC'}{AB\cdot AC} + \frac{GC'\cdot GA'}{BC\cdot BA} = \frac{1}{9}\,.$$

3706. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Montrer que pour tous les nombres réels positifs a, b, c et d satisfaisant $a,b,c \geq 1$ et $abcd = 1$, on a

$$\sum_{\text{cyclique}} \frac{1}{(a^2-a+1)^2} \ \leq\ 4\,.$$

3707. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit k et m deux entiers positifs. Montrer que

$$\int_0^\infty \frac{\sin^{2k} x}{(\pi^2-x^2)((2\pi)^2-x^2)\cdots((k\pi)^2-x^2)}dx = 0\,.$$

3708. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

A l'aide de la règle et du compas, construire un trapézoïde avec trois côtés égaux sachant que sa base $b = AB$ est donnée, de même que l'angle α $(0^\circ < \alpha < 90^\circ)$ à la base.

3709. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit a, b et c trois nombres réels non négatifs, k et $l \geq 0$ et définissons

$$\frac{a+b}{2} - \sqrt{ab} \;=\; k^2\,,\; \frac{a+b+c}{3} - \sqrt[3]{abc} \;=\; l^2\,.$$

Montrer que

$$\max(a,b,c) \;\geq\; \min(a,b,c) + \frac{3}{2}(k-l)^2\,.$$

3710. *Proposé par Billy Jin, Waterloo Collegiate Institute et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit donné $n \in \mathbb{N}$, montrer qu'il existe un $k \in \mathbb{N}$ tel que pour tout $m \geq k$, il existe une suite de m nombres naturels consécutifs contenant exactement n premiers.

Just a reminder to readers sending problems or solutions to ***Crux Mathematicorum***. Electronic submissions are preferred and can be emailed to `crux-editors@cms.math.ca`. Submissions should be in LaTeX and/or pdf format (some other formats are acceptable, please contact the editor if submitting in another format). Each solution should be in a separate file and the problem number, your name, school (if relevant), city and country should appear on **each page**, preferably in the header or footer.

Readers submitting problems to be considered for publication in ***Crux Mathematicorum*** should submit the problem and solution in an acceptable format to the editor. It is preferred that the problems are original, but the editors will consider other problems as long as they are not well known and have novel solutions. Any notes or references on the origin or inspiration of the problem would be helpful to the editors. ***Crux Mathematicorum*** is a problems solving journal at the upper high school and beginning undergraduate level and material should be appropriate for students and teachers at this level. As such, problems requiring highly specialized knowledge or techniques will not be accepted for publication.

Readers can still submit problems and solutions through regular mail to the address on the inside cover.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

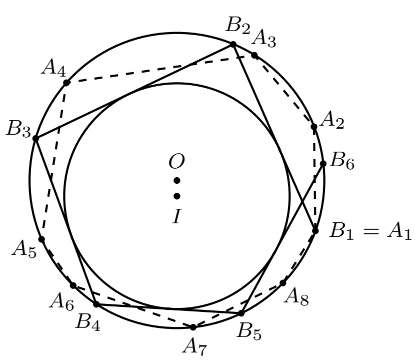
1580★. [1990 : 240;1991 : 308; 1992 : 76-80] *Proposed by Ji Chen, Ningbo University, China.*

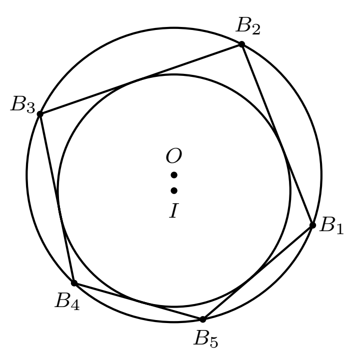
For every convex n-gon, if one circle with center O and radius R contains it and another circle with center I and radius r is contained in it, prove or disprove that $R^2 \geq r^2 \sec^2 \frac{\pi}{n} + OI^2$.

Solution by Tomasz Cieśla, student, University of Warsaw, Poland.

We will prove that the inequality holds.

Let Γ be the outer circle, γ be the inner circle, and $\mathcal{P}$ be the given n-gon that contains γ and is contained in Γ. Label the vertices of $\mathcal{P}$ counterclockwise by $A_1, A_2, \cdots A_n$. We may as well assume that $\mathcal{P}$ is inscribed in Γ because we can project from I the vertices of the given polygon onto Γ to obtain to obtain a new convex n-gon that contains γ, is contained in Γ, and has the same R, r, O, and I. We can simplify the problem further by assuming without loss of generality that γ is inscribed in $\mathcal{P}$, as follows. Set $B_1 = A_1$ and call B_2 the point where the tangent from B_1 to γ in the counterclockwise direction intersects Γ again. Note that A_2 lies on the counterclockwise arc of Γ from B_1 to B_2 . Define B_3 to be the point where the second tangent to γ from B_2 meets Γ. We continue this procedure until we produce a segment B_kB_{k+1} that intersects the initial segment B_1B_2 (or $B_{k+1} = B_1$). Because each B_i $(1 \leq i \leq k+1)$ is further than A_i around the circumference of Γ (in the counterclockwise direction), we deduce that $k \leq n$. Now we enlarge γ so that B_{k+1} coincides with B_1: for $R - r > \varepsilon \geq 0$ define γ_ε to be a circle with center I and radius $r + \varepsilon$. Define $B_1^\varepsilon = B_1$ and construct points $B_2^\varepsilon, B_3^\varepsilon, ..., B_{k+1}^\varepsilon$ similar to our construction of the B_i, but instead of γ, use γ_ε. By continuity, we conclude that there exists $\lambda \geq 0$ such that $B_{k+1}^\lambda = B_1$. Then $B_1^\lambda B_2^\lambda ... B_k^\lambda$ is a k-gon inscribed in Γ and circumscribed about γ_λ. Recalling that $k \leq n$, we note that $r^2 \sec \frac{\pi}{n} \leq (r+\lambda)^2 \sec^2 \frac{\pi}{k}$; thus, we need only to show that $R^2 \geq (r+\lambda)^2 \sec^2 \frac{\pi}{k} + OI^2$.



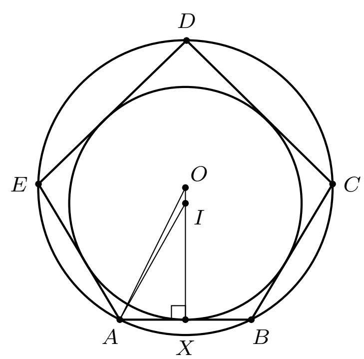


Take the point $X \in \gamma_\lambda$ such that X lies on ray OI beyond I. Poncelet's porism (called Poncelet's great theorem in [1, Section 16.6, pages 203-204], where a proof can be found), asserts that if there exists a k-gon inscribed in one circle and circumscribed about another, then each point of the inner circle serves as the point where an edge of one such k-gon is tangent.

In particular, we know that there exists the k-gon $\mathcal{Q}$ inscribed in Γ and circumscribed about γ_λ that has a side tangent to γ_λ at X. Let's call this side AB. Because X is the point of γ_λ that is farthest from O, we know that AB is the shortest side of $\mathcal{Q}$, whence $\angle XIA \le \frac{\pi}{k}$. It follows that

$$\begin{aligned} R^2 = OA^2 = OX^2 + XA^2 &= (OI + IX)^2 + IA^2 - IX^2 \\ &= OI^2 + 2\cdot OI\cdot IX + IX^2 + IA^2 - IX^2 \\ &= OI^2 + IA^2 + 2OI\cdot IX \\ &\ge OI^2 + IA^2 \\ &= OI^2 + (r+\lambda)^2\sec^2\angle XIA \\ &\ge OI^2 + (r+\lambda)^2\sec^2\frac{\pi}{k} \;\ge\; r^2\sec^2\frac{\pi}{n} + OI^2, \end{aligned}$$

as desired.



References

[1] Marcel Berger, *Geometry*, Vol. 2, Springer-Verlag, 1987.

No other complete solutions have been received.

3601. [2011 : 46, 48] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Suppose that b is a positive real number such that there are exactly two integers strictly between b and $2b$, and exactly two integers strictly between $2b$ and b^2. Find all possible values of b.

Composite of several solutions.

The possible values of b are 3 and all the numbers in the open interval $(\sqrt{6}, 5/2)$. If $0 < b < 1$, then $(b, 2b)$ contains at most one integer. If $1 < b \le 2$, then $1 < b < b^2 \le 2b < 4$. Thus we must have $b^2 < 2 < 3 < 2b$, which leads to the false conclusion that $(3/2)^2 < b^2 < 2$. Hence $b > 2$.

First assume that $2b$ is not an integer. For the conditions to be satisfied, there must be an integer $k \ge 2$ for which

$$k < b < k+1 < k+2 < 2b < k+3 < k+4 < b^2 \le k+5.$$

Since $k^2 < b^2 \le k+5$, then $k \le 2$. Therefore $k = 2$ and so $2 < b < 3 < 4 < 2b < 5 < 6 < b^2 \le 7$. Therefore $\sqrt{6} < b < 5/2$.

If $2b$ is an integer, then we have for some integer k, $k \le b < k+1 < k+2 < 2b = k+3 < k+4 < k+5 < b^2 \le k+6$. Therefore $4(k+5) < (k+3)^2 \le 4(k+6)$, which reduces to $12 < (k+1)^2 \le 16$. Thus $k = 3$ and so $b = 3$.

Solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; LUCIMAR MYERS and DARTH VAN NOORT, California State University, Fresno, CA, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Konečný observed that if you ask for three integers instead of two in each interval $(b, 2b)$ and $(2b, b^2)$, then $\sqrt{8} < b < 3$ or $3 < b < \sqrt{10}$. For integers larger than three, there are no solutions.

There were five incorrect solutions.

3602. [2011 : 46, 48] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $a_i > 0$ for $i = 1, 2, 3, 4$, then

$$\sum_{\text{cyclic}} \frac{1}{a_i^2 + a_{i+1}^2 + a_{i+2}^2} \ge \frac{12}{(a_1 + a_2 + a_3 + a_4)^2}$$

Composite of similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and Michel Bataille, Rouen, France.

By symmetry, we may assume that $a_1 \ge a_2 \ge a_3 \ge a_4$. Let $x = a_1 + \frac{a_3+a_4}{2}$ and $y = a_2 + \frac{a_3+a_4}{2}$. Since $a_3 + a_4 \ge 2a_4$ and $a_1(a_3 + a_4) \ge a_3^2$, we have

$$x^2 = a_1^2 + \frac{(a_3 + a_4)^2}{4} + a_1(a_3 + a_4) \ge a_1^2 + a_4^2 + a_3^2.$$

Similarly,

$$y^2 = a_2^2 + \frac{(a_3 + a_4)^2}{4} + a_2(a_3 + a_4) \ge a_2^2 + a_4^2 + a_3^2.$$

Furthermore it is clear that $a_1^2+a_2^2+a_3^2 \le x^2+y^2$ and $a_1^2+a_2^2+a_4^2 \le x^2+y^2$. Using these together with the AM-GM inequality, if we let $S = \sum_{\text{cyclic}} \frac{1}{a_i^2+a_{i+1}^2+a_{i+2}^2}$ we then have

$$\begin{aligned} S &\ge \frac{2}{x^2+y^2}+\frac{1}{x^2}+\frac{1}{y^2} = \frac{2}{x^2+y^2}+\frac{x^2+y^2}{x^2y^2} \ge \frac{2}{x^2+y^2}+\frac{2}{xy} \\ &= \left(\frac{2}{x^2+y^2}+\frac{1}{xy}\right)+\frac{1}{xy} \ge 2\sqrt{\frac{2}{xy(x^2+y^2)}}+\frac{1}{xy} \\ &\ge 2\sqrt{\frac{2}{xy(x^2+y^2)}}+\frac{4}{(x+y)^2} = \frac{4}{\sqrt{2xy(x^2+y^2)}}+\frac{4}{(x+y)^2} \\ &\ge \frac{4}{\frac{1}{2}(2xy+x^2+y^2)}+\frac{4}{(x+y)^2} = \frac{12}{(x+y)^2} = \frac{12}{(a_1+a_2+a_3+a_4)^2}, \end{aligned}$$

completing the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Both Bataille and Geupel remarked that the method they used in their proof, like the one featured above, is exposed in the book "Secrets in Inequalities" (Vol. 1), GIL Publ., Zalău, Romania, 2007, by Pham Kim Hung. From the proof featured above, it is clear that the inequality is always strict. However, Stadler, Zvonaru and AN-anduud Problem Solving Group all pointed out that if we weaken the condition to $a_i \ge 0$ such that $a_i+a_{i+1}+a_{i+2} > 0$ for all $i = 1,2,3,4$, then equality holds in the given inequality if and only if two of the a_i's are zeros and the other two are positive and equal to each other. This was also mentioned by the proposer in his original submission in which he assumed that $a_i \ge 0$ for all i but did not consider the possibility that some $a_i^2+a_{i+1}^2+a_{i+2}^2$ could be zero.

3603. [2011 : 46, 49] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

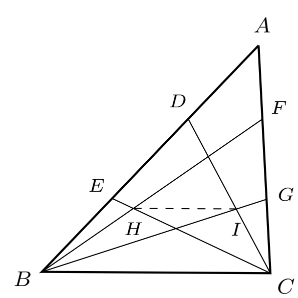
Let ABC be a given triangle and $0 < \lambda < \frac{1}{2}$. Let D and E be points on AB such that $AD = BE = \lambda \cdot AB$, and F, G points on AC such that $AF = CG = \lambda \cdot AC$. Let $BF \cap CE = H$ and $BG \cap CD = I$. Show that

i) $HI \parallel BC$ and

ii) $HI = \dfrac{1-2\lambda}{\lambda^2-\lambda+1} BC$.

Solution by Albert Stadler, Herrliberg, Switzerland.

Let $\vec{c} := \overrightarrow{AB}$, $\vec{b} := \overrightarrow{AC}$. Then $\overrightarrow{AD} = \lambda\vec{c}$, $\overrightarrow{AE} = (1-\lambda)\vec{c}$, $\overrightarrow{AF} = \lambda\vec{b}$, and $\overrightarrow{AG} = (1-\lambda)\vec{b}$. For some real numbers u and v we have



$$\overrightarrow{AH} = \overrightarrow{AB} + u\overrightarrow{BF} = \vec{c} + u(\lambda\vec{b} - \vec{c})$$

and

$$\overrightarrow{AH} = \overrightarrow{AC} + v\overrightarrow{CE} = \vec{b} + v((1-\lambda)\vec{c} - \vec{b}),$$

therefore
$$\overrightarrow{b}(\lambda u + v - 1) + \overrightarrow{c}(-u - (1-\lambda)v + 1) = \overrightarrow{0}.$$
Since $\overrightarrow{b}$ and $\overrightarrow{c}$ are linearly independent, we have
$$\begin{aligned} u + (1-\lambda)v &= 1 \\ \lambda u + v &= 1, \end{aligned}$$
which solves to $u = \dfrac{\lambda}{\lambda^2 - \lambda + 1}$, $v = \dfrac{1-\lambda}{\lambda^2 - \lambda + 1}$. Hence,
$$\overrightarrow{AH} = \overrightarrow{c} + u(\lambda\overrightarrow{b} - \overrightarrow{c}) = (1-u)\overrightarrow{c} + u\lambda\overrightarrow{b} = \frac{(\lambda-1)^2}{\lambda^2-\lambda+1}\overrightarrow{c} + \frac{\lambda^2}{\lambda^2-\lambda+1}\overrightarrow{b}$$
and similarly
$$\overrightarrow{AI} = \frac{(\lambda-1)^2}{\lambda^2-\lambda+1}\overrightarrow{b} + \frac{\lambda^2}{\lambda^2-\lambda+1}\overrightarrow{c}.$$
So we have
$$\overrightarrow{HI} = \overrightarrow{AI} - \overrightarrow{AH} = \frac{(\lambda-1)^2}{\lambda^2-\lambda+1}(\overrightarrow{b} - \overrightarrow{c}) + \frac{\lambda^2}{\lambda^2-\lambda+1}(\overrightarrow{c} - \overrightarrow{b}) = \frac{1-2\lambda}{\lambda^2-\lambda+1}\overrightarrow{BC},$$
which proves i) and ii).

Also solved by MICHEL BATAILLE, Rouen, France; MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Tucson, AZ, USA; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3604. [2011 : 46, 49] *Proposed by Michel Bataille, Rouen, France.*

Evaluate
$$\lim_{n\to\infty} \frac{\int_0^1 (x^2 - x - 2)^n dx}{\int_0^1 (4x^2 - 2x - 2)^n dx}.$$

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

Let $I_n = \int_0^1 (2 + x - x^2)^n dx$ and $J_n = \int_0^1 (2 + 2x - 4x^2)^n dx$. The limit we seek is $\lim\limits_{n\to\infty} \dfrac{I_n}{J_n}$ (this reformulation is to avoid negative integrands). Using the change of variable $u = 2x$ we have $J_n = \frac{1}{2}\int_0^2 (2 + u - u^2)^n du = \frac{1}{2}(I_n + K_n)$ where $K_n = \int_1^2 (2 + u - u^2)^n du$.

Since $2 + x - x^2 = 2 + x(1-x) \geq 2$ for $0 \leq x \leq 1$ we have $I_n \geq 2^n$. On the other hand, since $u^2 - 2u + 1 \geq 0$, and $0 \leq u \leq 2$, we have $0 \leq 2 - u(u-1) = 2 + u - u^2 \leq 3 - u$ whence $0 \leq K_n \leq \int_1^2 (3-u)^n du = \frac{1}{n+1}(2^{n+1} - 1)$.

Hence
$$0 \le \frac{K_n}{I_n} \le \frac{2^{n+1}-1}{2^n(n+1)} = \frac{2-2^{-n}}{n+1}$$
so $\lim\limits_{n\to\infty} \dfrac{K_n}{I_n} = 0$. Therefore,
$$\lim_{n\to\infty} \frac{I_n}{J_n} = \lim_{n\to\infty} \frac{2I_n}{I_n+K_n} = \lim_{n\to\infty} \frac{2}{1+\frac{K_n}{I_n}} = 2.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ANASTASIOS KOTRONIS, Athens, Greece; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3605. [2011 : 47, 49] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ be a real polynomial with positive coefficients and having all its zeros real. Prove that
$$\sqrt[n]{A(1)A(2)\cdots A(n)} \ge (n+1)!$$

Composite of essentially the same solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Oliver Geupel, Brühl, NRW, Germany; and Evangelos Mouroukos, Agrinio, Greece.

Since the coefficients of $A(x)$ are positive, the roots r_i's of $A(x)$ are all negative. Hence we can write $A(x) = \prod_{i=1}^n (x+s_i)$ where $s_i = -r_i > 0$ such that $\prod_{i=1}^n s_i = 1$.

For each $k = 1, 2, \ldots, n$, we have, by the AM-GM inequality that
$$\begin{aligned} A(k) &= \prod_{i=1}^n (k+s_i) = \prod_{i=1}^n (1+1+\cdots+1+s_i) \ \ \text{(with } k \text{ 1's)} \\ &\ge \prod_{i=1}^n (k+1)s_i^{\frac{1}{k+1}} = (k+1)^n. \end{aligned}$$
Hence,
$$\prod_{k=1}^n A(k) \ge \prod_{k=1}^n (k+1)^n = ((n+1)!)^n$$
and the result follows.

Note that the equality holds if and only if $s_i = 1$ for all i, that is, if and only if $A(x) = (x+1)^n$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HENRY RICARDO, Tappan, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer.

Though it is easy to see that equality holds if and only if $A(x) = (x+1)^n$, Mouroukos was the only one to point this out explicitly.

3606. [2011 : 47, 49] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let ABC be a triangle with $\angle A = 20^\circ$. Let BD be the angle bisector of $\angle ABC$ with D on AC. If $AD = DB + BC$, determine $\angle B$.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that $\angle B = \frac{4\pi}{9}$ or $\frac{2\pi}{3}$. Let $\angle B = 2\theta$ so that $\angle ABD = \angle DBC = \theta$. Since $\angle CAB + \angle ABC < \pi$, so $\theta < \frac{4\pi}{9}$; and since $AD > DB$, so $\theta > \frac{\pi}{9}$; that is,

$$\frac{\pi}{9} < \theta < \frac{4\pi}{9}.$$

Without loss of generality suppose that $DB = 1$. Applying the sine law to triangles ABD and BCD respectively, we obtain

$$AD = \frac{\sin\theta}{\sin\frac{\pi}{9}} \quad \text{and} \quad BC = \frac{\sin\frac{9\theta+\pi}{9}}{\sin\frac{18\theta+\pi}{9}}.$$

The requirement $AD = 1 + BC$ is equivalent, in turn, to

$$\begin{aligned}
\sin\frac{18\theta+\pi}{9}\sin\theta &= \sin\frac{\pi}{9}\left(\sin\frac{9\theta+\pi}{9} + \sin\frac{18\theta+\pi}{9}\right)\\
\sin\frac{18\theta+\pi}{9}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) &= \sin\frac{\pi}{9}\cdot 2\sin\left(\frac{1}{2}\cdot\frac{27\theta+2\pi}{9}\right)\cos\left(\frac{1}{2}\cdot\frac{9\theta}{9}\right)\\
\sin\frac{18\theta+\pi}{9}\sin\frac{\theta}{2} &= \sin\frac{\pi}{9}\sin\frac{27\theta+2\pi}{18}\\
\cos\frac{27\theta+2\pi}{18} + \cos\frac{27\theta+4\pi}{18} &= \cos\frac{45\theta+2\pi}{18} + \cos\frac{3\theta}{2}\\
\cos\frac{9\theta+\pi}{6}\cos\frac{\pi}{18} &= \cos\frac{36\theta+\pi}{18}\cos\frac{9\theta+\pi}{18}\\
\left(4\cos^2\frac{9\theta+\pi}{18} - 3\right)\cos\frac{\pi}{18} &= \cos\frac{36\theta+\pi}{18}
\end{aligned}$$

and finally,

$$2\cos\frac{9\theta+\pi}{9}\cos\frac{\pi}{18} - \cos\frac{\pi}{18} - \cos\frac{36\theta+\pi}{18} = 0.$$

Setting $f(\theta)$ equal to the expression on the left of the last equation, we see that

$$f\left(\frac{2\pi}{9}\right) = 2\cos\frac{\pi}{3}\cos\frac{\pi}{18} - \cos\frac{\pi}{18} - \cos\frac{\pi}{2} = 0,$$

and

$$f\left(\frac{\pi}{3}\right) = 2\sin\frac{\pi}{18}\cos\frac{\pi}{18} + \left(\cos\frac{5\pi}{18} - \cos\frac{\pi}{18}\right) = \sin\frac{\pi}{9} - 2\sin\frac{\pi}{6}\sin\frac{\pi}{9} = 0.$$

In summary, if $\angle B = \frac{4\pi}{9}$ or $\frac{2\pi}{3}$, then $AD = DB + BC$, as claimed.

It remains to show that $f(\theta)$ can have no further zeros between $\frac{\pi}{9}$ and $\frac{4\pi}{9}$. For this we use derivatives,

$$\begin{aligned} f'(\theta) &= -2\sin\frac{9\theta+\pi}{9}\cos\frac{\pi}{18} + 2\sin\frac{36\theta+\pi}{18}, \\ f''(\theta) &= -2\cos\frac{9\theta+\pi}{9}\cos\frac{\pi}{18} + 4\cos\frac{36\theta+\pi}{18} \end{aligned}$$

For $\frac{\pi}{9} < \theta < \frac{2\pi}{9}$ we have

$$f'(\theta) > -2\sin\frac{9\theta+\pi}{9} + 2\sin\frac{36\theta+\pi}{18} = 4\sin\frac{18\theta+\pi}{12}\cos\frac{18\theta-\pi}{36} > 0.$$

For $\frac{3\pi}{9} < \theta < \frac{4\pi}{9}$ we have

$$\begin{aligned} f'(\theta) &< -2\sin\frac{4\pi}{9}\cos\frac{\pi}{18} + 2\sin\frac{5\pi}{18} \\ &= -\left(\sin\frac{9\pi}{18} + \sin\frac{7\pi}{18}\right) + 2\sin\frac{5\pi}{18} < -2\sin\frac{7\pi}{18} + 2\sin\frac{5\pi}{18} < 0. \end{aligned}$$

For $\frac{2\pi}{9} < \theta < \frac{3\pi}{9}$ we have $\cos\frac{36\theta+\pi}{18} \le 0$, whence

$$f''(\theta) < 0.$$

It follows that for $\frac{\pi}{9} < 0 < \frac{4\pi}{9}$, $f(\theta)$ vanishes if and only if $\theta = \frac{2\pi}{9}, \frac{\pi}{3}$, as desired.

*Also solved by *GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; *MICHEL BATAILLE, Rouen, France; *RICHARD I. HESS, Rancho Palos Verdes, CA, USA; *GEORGES MELKI, Fanar, Lebanon; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; *ALBERT STADLER, Herrliberg, Switzerland; *EDMUND SWYLAN, Riga, Latvia; and the proposer. An asterisk indicates a solution which includes a proof that there are exactly two possible values for $\angle B$. In addition we received two incomplete solutions that found only one value for $\angle B$.*

3607. [2011 : 47, 49] *Proposed by George Miliakos, Sparta, Greece.*

Let $c_1 = 9$, $c_2 = 15$, $c_3 = 21$, $c_4 = 25$, $\ldots$, where c_n is the n^{th} composite odd integer. Evaluate

$$\lim_{n\to\infty}\frac{c_n}{c_{n+1}}.$$

Composite of similar solutions by almost all the solvers (indicated by an asterisk beside their names in the list below).

Note first that among any three consecutive odd integers one must be divisible by 3 and thus is composite if it is greater than 3.

Since $c_n \geq 9$ for all n, one of the three odd integers c_n+2, c_n+4 and c_n+6 must be composite. This implies $c_n < c_{n+1} \leq c_n+6$ so $1-\dfrac{6}{c_{n+1}} \leq \dfrac{c_n}{c_{n+1}} < 1$.

Since $\lim\limits_{n\to\infty} c_{n+1} = \infty$, it follows from the squeeze theorem that $\lim\limits_{n\to\infty} \dfrac{c_n}{c_{n+1}} = 1$.

*Solved by *AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; *ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; *MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; *ROY BARBARA, Lebanese University, Fanar, Lebanon; *MICHEL BATAILLE, Rouen, France; *CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; *OLIVER GEUPEL, Brühl, NRW, Germany; *RICHARD I. HESS, Rancho Palos Verdes, CA, USA; *GERHARDT HINKLE, student, Central High School, Springfield, MO, USA; *KEE-WAI LAU, Hong Kong, China; *KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; *SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; *ALBERT STADLER, Herrliberg, Switzerland; *EDMUND SWYLAN, Riga, Latvia; *STAN WAGON, Macalester College, St. Paul, MN, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

3608. [2011 : 47, 50] *Proposed by Michel Bataille, Rouen, France.*

Let

$$f(x) \;=\; \frac{e^{1/x}-1}{e^{1/(x+1)}-1}\,.$$

(a) Show that for all $x \in (0,\infty)$,

$$f(x) \;>\; \sqrt{\frac{x+1}{x}}\,.$$

(b) ★ Prove or disprove:

$$f(x) \;<\; \sqrt{\frac{x+1}{x-1}}$$

for all $x \in (1,\infty)$.

(a) I. Solution by Kee-Wai Lau, Hong Kong, China.

Suppose that $x = \cot^2\theta$, where $0 < \theta < \frac{\pi}{2}$. The inequality becomes

$$\frac{e^{\tan^2\theta}-1}{e^{\sin^2\theta}-1} > \frac{1}{\cos\theta}.$$

This is clear since

$$\cos\theta(e^{\tan^2\theta}-1) = \sum_{n=1}^{\infty}\frac{\cos\theta(\tan^{2n}\theta)}{n!} = \sum_{n=1}^{\infty}\frac{\sin^{2n}\theta}{n!\cos^{2n-1}\theta}$$
$$> \sum_{n=1}^{\infty}\frac{\sin^{2n}\theta}{n!} = e^{\sin^2\theta}-1.$$

(a) II. Similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Václav Konečný, Big Rapids, MI, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway.

We have that

$$\begin{aligned}\sqrt{x}(e^{1/x}-1) &= \frac{1}{\sqrt{x}}+\frac{1}{2!x\sqrt{x}}+\frac{1}{3!x^2\sqrt{x}}+\cdots\\ &> \frac{1}{\sqrt{x+1}}+\frac{1}{2!(x+1)\sqrt{x+1}}+\frac{1}{3!(x+1)^2\sqrt{x+1}}+\cdots\\ &= \sqrt{x+1}(e^{1/(x+1)}-1).\end{aligned}$$

for $x > 0$, from which the result follows.

(a) III. Composite of solutions by Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; Albert Stadler, Herrliberg, Switzerland; and Roberto Tauraso and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

We must show that $g(x) \geq g(x+1)$ where $g(x) = \sqrt{x}(e^{1/x}-1)$. We have that

$$\begin{aligned}g'(x) &= \frac{e^{1/x}}{2\sqrt{x}}\left[\left(1-\frac{2}{x}\right)-e^{-(1/x)}\right]\\ &< \frac{e^{1/x}}{2\sqrt{x}}\left[\left(1-\frac{2}{x}\right)-\left(1-\frac{1}{x}\right)\right] = -\frac{e^{1/x}}{2x^{3/2}} < 0.\end{aligned}$$

for $x > 0$. (Note that $e^t > 1+t$ for all $t \neq 0$.) Thus $g(x)$ is decreasing and the result follows.

(a) IV. Solution by Joe Howard, Portales, NM, USA.

By examining the graph of $y = 1/x$ on the interval $[u, u+1]$, we see that $\log(1+u) - \log u > 1/(u+1)$ for $u > 0$. Taking $t = 1+u$ yields $e^{\frac{1}{t}} - 1 < 1/(t-1)$. However, when $t > 2$, we have that $1/(t-1) < 2/(t-2)$, whence

$$e^{\frac{1}{t}} < \frac{t}{t-2}. \tag{1}$$

To solve the problem, it suffices to show that the function $t(e^{1/t}-1)^2$ is decreasing for $t > 0$. Its derivative is equal to

$$\left(\frac{e^{\frac{1}{t}}-1}{t}\right)(e^{1/t}(t-2)-t).$$

This is clearly negative when $0 < t \leq 2$ and is negative for $t > 2$ by (1). The desired result follows.

V. Solution by Oliver Geupel, Brühl, NRW, Germany.

(a) For $0 < y < 1$,

$$y < 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{y}{2}\right)^{2n+1} = \log\frac{2+y}{2-y},$$

whence

$$e^y < \frac{2+y}{2-y}.$$

Substituting $y = (x+1)^{-1}$, we obtain

$$\frac{1}{e^{1/(x+1)} - 1} > \frac{2x+1}{2}.$$

Combining this with the inequality $e^{1/x} - 1 > 1/x$, we see that, for $x > 0$,

$$f(x) > 1 + \frac{1}{2x} > \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}}.$$

(b) The conjecture is true. Let $0 < y < \frac{1}{2}$. Using an initial segment of the McLaurin expansion of e^{2y}, we have that

$$\begin{aligned} e^{2y}(1-y)^4 + 2y - 1 &> (1 + 2y + 2y^2)(1-y)^4 + 2y - 1 = 2y^6 - 6y^5 + 5y^4 \\ &= 2y^4\left(\left(y - \frac{3}{2}\right)^2 + \frac{1}{4}\right) > 0, \end{aligned}$$

whence

$$e^y\sqrt{1-2y} > \frac{1-2y}{(1-y)^2} = 1 - \frac{y^2}{(1-y)^2}.$$

Therefore

$$\begin{aligned} &(1-y)^2(e^y\sqrt{1-2y} - 1) + y^2e^{y/(1-y)} \\ &\qquad > (1-y)^2(e^y\sqrt{1-2y} - 1) + y^2 > 0. \end{aligned}$$

Let

$$g(y) = e^y - 1 - \sqrt{1-2y}(e^{y/(1-y)} - 1).$$

Then $g(0) = 0$ and

$$g'(y) = \frac{(1-y)^2(e^y\sqrt{1-2y} - 1) + y^2e^{y/(1-y)}}{\sqrt{1-2y}(1-y)^2} > 0.$$

Consequently $g(y) > 0$. Now let $y = (x+1)^{-1}$. Then $x > 1$ and

$$e^{1/(x+1)} - 1 - \sqrt{1 - \frac{2}{x+1}}(e^{1/x} - 1) = g\left(\frac{1}{x+1}\right) > 0$$

and the conclusion follows.

Part (a) also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; C.R. PRANESACHAR, Indian Institute of Science, Bangalore, India; and the proposer. There was one incomplete solution to (a) and one incorrect solution to (b) received.

Apostolopoulos took $x = (t^2-1)^{-1}$ *and showed that* $e^{t^2-1} - 1 - t(e^{(t^2-1)t^{-2}} - 1)$ *increases from 0 for* $t > 1$. *Curtis took* $x = 1/s$ *and showed that* $s^{-1}(e^s - 1)^2$ *increases from 0 for* $s > 0$. *Pranesachar compared the terms in the McLaurin expansion of* $e^y - 1$ *and* $(e^{y/(y+1)} - 1)\sqrt{1+y}$.

Part (b) also solved by VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; C.R. PRANESACHAR, Indian Institute of Science, Bangalore, India; ALBERT STADLER, Herrliberg, Switzerland; and ROBERTO TAURASO and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The approach of Pranesachar was to compare the coefficients in the McLaurin expansions of the two sides of the equivalent inequality $e^{y/(1-y)} < (e^y-1)(1-2y)^{-1/2}$. *These were calculated numerically up to the terms of the 11th degree and the following inequalities were used to compare the rest:*

$$\frac{1}{2^n}\binom{2n}{n} > \frac{2^{n+1}}{n+1}$$

for $n \geq 12$, *and*

$$\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1} < \frac{2^n}{n}$$

for all n. *If the first inequality is weakened by replacing* 2^{n+1} *by* 2^n *and the second strengthened by replacing the summands by* $\frac{1}{(k+1)!}\binom{n}{k}$, *then the number of numerical verifications is reduced to 4.*

3609. [2011 : 47, 50] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let r be a real number. and let D, E, and F be points on the sides BC, CA, and AB of a triangle ABC with

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = r.$$

The cevians AD, BE, and CF bound a triangle PQR whose area we denote by $[PQR]$. Find the value of r for which the ratio of the areas, $\frac{[DEF]}{[PQR]}$ equals 4.

Composite of the submitted solutions.

The solution follows quickly from two formulas that are familiar to *Crux* readers:

$$\frac{[DEF]}{[ABC]} = \frac{r^3+1}{(r+1)^3} = \frac{r^2-r+1}{(r+1)^2} \quad \text{and} \quad \frac{[PQR]}{[ABC]} = \frac{(r^3-1)^2}{(r^2+r+1)^3} = \frac{(r-1)^2}{r^2+r+1}.$$

These appeared in the solution to problem 2752 [2003: 331-332]; see [1979: 191-192] for complete proofs. The second formula is a special case of Routh's theorem; you can find three proofs in [1981: 199-203] and yet another in [1982: 228-232]. Other treatments are readily found on the internet and in textbooks. Returning to our problem, we divide the first quotient by the second to get

$$\frac{[DEF]}{[PQR]} = \frac{(r^2-r+1)(r^2+r+1)}{(r^2-1)^2}.$$

A routine calculation shows that this ratio equals 4 if and only if $r^4 - 3r^2 + 1 = 0$, which holds if and only if

$$r^2 = \frac{3 \pm \sqrt{5}}{2} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^2.$$

Because the problem requires the given points to lie on the sides of the given triangle, we must take the positive roots, whence the desired values of r are the golden section $\frac{\sqrt{5}+1}{2}$ and its reciprocal $\frac{\sqrt{5}-1}{2}$. Several correspondents observed that there is no need to restrict the given points to the interior of the sides: the area formulas also hold should the given points lie on the extension of the sides, and we can conclude that the ratio of areas equals 4 also when $r = -\frac{\sqrt{5}\pm 1}{2}$.

Solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect submission.

The submitted solutions differed only in the amount of detail included; a few readers established the area formulas rather than invoke them.

3610. [2011 : 48, 50] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $S = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, \ldots\}$ be the set of positive integers whose only prime divisors are 2 or 3. Let $a_1 = 2, a_2 = 3, \ldots,$ be the elements of S, with $a_1 < a_2 < \ldots$.

(i) Determine $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{a_i}\right)$.

(ii) ★ For each positive integer n, let $s(n)$ be the sum of all its divisors including 1 and n itself. Prove $\frac{s(n)}{n} < 3$ for all members of S.

Composite of essentially the same solution by all solvers.

(i) Since each a_i is of the form $2^j 3^k$ for some $j, k \geq 0$ with $j^2 + k^2 \neq 0$, we have

$$\sum_{i=1}^{\infty}\left(\frac{1}{a_i}\right) = \left(\sum_{j=0}^{\infty}\frac{1}{2^j}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right) - 1 = \left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right) - 1 = 2.$$

(ii)★ Let $n = 2^a 3^b$ where $a, b \geq 0$ such that $a^2 + b^2 \neq 0$. Since it is well known that s is a multiplicative function, we have

$$\begin{aligned} s(n) &= s(2^a 3^b) = s(2^a)s(3^b) \\ &= (1 + 2 + 2^2 + \cdots + 2^a)(1 + 3 + 3^2 + \cdots + 3^b) = (2^{a+1} - 1)\left(\frac{3^{b+1} - 1}{2}\right) \end{aligned}$$

so

$$\frac{s(n)}{n} = \frac{2^{a+1} - 1}{2^{a+1}} \cdot \frac{3^{b+1} - 1}{3^b} = \left(1 - \frac{1}{2^{a+1}}\right)\left(3 - \frac{1}{3^b}\right) < 3.$$

Solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GERHARDT HINKLE, student, Central High School, Springfield, MO, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway(part (i) only); MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer.

Note that in the proof of (i) above the representation of $\sum_{i=1}^{\infty} \frac{1}{a_i}$ *as the product of two geometric series by rearranging the terms of the series is justified since the series are absolutely convergent. However, this was explicitly or implicitly pointed out only by Geupel, Manes, and Modak.*