Crux Mathematicorum with Mathematical Mayhem is a problem-solving journal at the secondary and university undergraduate levels, published by the Canadian Mathematical Society eight times per year (February, March, April, May, September, October, November, and December). Its aim is primarily educational; it is not a research journal.

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Canada	(CDN)	$ 5.00	$ 1.00	Please contact our
USA	(US)	$ 7.00	$ 2.00	office for Air Mail
Other	(US)	$ 8.00	$ 3.00	delivery quotes

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PUBLISHED BY THE CANADIAN MATHEMATICAL SOCIETY.
PRINTED IN CANADA BY THISTLE PRINTING LIMITED.

ISSN 1706-8142 (Print) 1496-4309 (Online)

See inside back cover for more information

Voir les numéros 2 (mars), 4 (mai), 6 (octobre) ou 8 (décembre) pour la version française.

37: No 7 November / Novembre 2011

Published by:

Canadian Mathematical Society
Société mathématique du Canada
209 - 1725 St. Laurent Blvd.
Ottawa, ON K1G 3V4, Canada
Fax/Téléc. : 613 733 8994

SYNOPSIS

This new, occasionally appearing column, highlights situations that reappear in *Crux* problems. In this issue problem editor J. Chris Fisher examines triangles for which $2B = C + A$. Enjoy!

This month's "free sample" is:

3687. *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\left(k+1-\frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

where k^n is taken to be 1 for $k = n = 0$ and $S(n,k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m) = S(n-1,m-1)+mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1\,.$$

3687. *Proposé par Albert Stadler, Herrliberg, Suisse.*

Soit n un entier non négatif. Montrer que

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\left(k+1-\frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

où l'on pose $k^n = 1$ pour $k = n = 0$ et où $S(n,k)$ sont les nombres de Striling du second ordre, définis par la récursion

$$S(n,m) = S(n-1,m-1)+mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1\,.$$

SKOLIAD No. 136

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **October 1, 2012**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

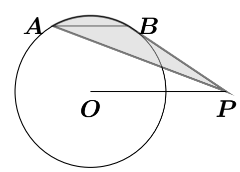
Our contest this month is the Final Round of the Swedish Junior High School Mathematics Contest 2009/2010. Our thanks go to Paul Vaderlind, Stockholm University, Sweden for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the contest.

Swedish Junior High School Mathematics Contest Final Round, 2010/2011

3 hours allowed

1. The year **2010** is divisible by three consecutive primes. Find the last year before that with this property.

2. Draw a line from the centre, O, of a circle with radius r to a point, P, outside the circle. Then choose two points, A and B, on the circle such that AB has length r and is parallel with OP. Find the area of the shaded region.



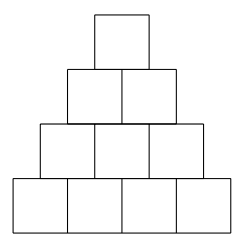
3. Five distinct positive numbers are given. No matter which two of them you choose, one divides the other. The sum of the five numbers is a prime. Show that one of the five numbers is **1**.

4. A large cube consists of eight identical smaller cubes. The faces of each of the smaller cubes bear the numbers **3**, **3**, **4**, **4**, **5**, and **5** such that opposite faces bear the same number. Assign to each face of the large cube the sum of the four visible numbers. Show that the numbers assigned to the faces of the large cube cannot be six consecutive integers.

5. The parallelogram $ABCD$ has area **12**. The point P is on the diagonal AC. The area of $\triangle ABP$ is one third of the area of $ABCD$. Find the area of $\triangle CDP$.

6. Place ten numbers in the grid subject to the following rules:

1. For neighbours in the bottom row, the number on the right must be twice as large as the number on the left.
2. Other than in the bottom row, each number is the sum of the two numbers immediately below it.

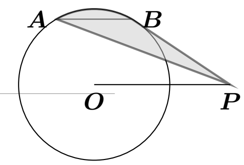


Find the smallest positive integer that you can place in the bottom left position such that the sum of all ten numbers is a square.

Concours mathématique suédois
Niveau école intermédiaire
Ronde finale 2010/2011
Durée : 3 heures

1. En 2010, on constate que ce nombre est divisible par trois nombres premiers consécutifs. Déterminer la dernière année avant ça, où cette propriété tenait.

2. Tracer une ligne du centre, O, d'un cercle ayant rayon r jusqu'à un point, P, en dehors du cercle. Choisir alors deux points, A et B, sur le cercle et tels que AB est de longueur r et est parallèle à OP. Déterminer la surface ombrée.

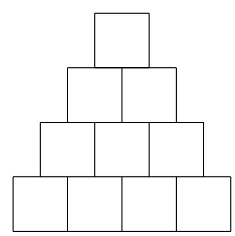


3. Cinq nombres positifs distincts vous sont donnés. Qu'importe lesquels deux vous choisissez, l'un d'eux divise l'autre. La somme des cinq nombres est un nombre premier. Démontrer que l'un des cinq nombres est **1**.

4. Un gros cube est formé de huit petits cubes identiques. Les faces des petits cubes portent les nombres **3**, **3**, **4**, **4**, **5** et **5**, de façon à ce que les faces opposées portent le même nombre. Assigner à chaque face du gros cube la somme de ses quatre nombres visibles. Démontrer que les nombres assignés aux faces du gros cube ne peuvent pas être six entiers consécutifs.

5. Le parallelogramme $ABCD$ a une surface de **12**. Le point P se trouve sur la diagonale AC. La surface de $\triangle ABP$ est le tiers de la surface de $ABCD$. Déterminer la surface de $\triangle CDP$.

6. Placer dix nombres sur la grille, sujet aux règles suivantes.



1. Pour des voisins dans la rangée du bas, le nombre à droite doit être deux fois celui à gauche.
2. Pour les rangées autres que celle du bas, chaque nombre est la somme des deux nombres immédiatement sous lui.

Déterminer le plus petit entier positif qu'on peut placer en bas à l'extrême gauche, si la somme des dix nombres est un nombre carré.

Next follow solutions to the Niels Henrik Abel Mathematics Contest, 2009–2010, 2$^{\text{nd}}$ Round, given in Skoliad 130 at [2011 : 3–5].

1. A four-digit whole number is *interesting* if the number formed by the leftmost two digits is twice as large as the number formed by the rightmost two digits. (For example, **2010** is interesting.) Find the largest whole number, d, such that all interesting numbers are divisible by d.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The first interesting numbers are $\mathbf{1005}$ and $\mathbf{1206}$. Since $\mathbf{1005 = 3 \cdot 5 \cdot 67}$ and $\mathbf{1206 = 2 \cdot 3^2 \cdot 67}$, their greatest common divisor is $\mathbf{3 \cdot 67 = 201}$. Thus, $\boldsymbol{d}$ is a divisor of $\mathbf{201}$.

On the other hand, if $\boldsymbol{x}$ is an interesting number whose last two digits form the number $\boldsymbol{k}$, then the first two digits of $\boldsymbol{x}$ form the number $\boldsymbol{2k}$, so $\boldsymbol{x = 200k + k = 201k}$. Thus every interesting number is divisible by $\mathbf{201}$. Consequently, $\boldsymbol{d = 201}$.

Also solved by DAVID GOU, student, Burnaby North Secondary School, Burnaby, BC; and NELSON TAM, student, John Knox Christian School, Burnaby, BC.

2. A calculator performs this operation: It multiplies by $\mathbf{2.1}$, then erases all digits to the right of the decimal point. For example, if you perform this operation on the number $\mathbf{5}$, the result is $\mathbf{10}$; if you begin with $\mathbf{11}$, the result is $\mathbf{23}$. Now, if you begin with the whole number $\boldsymbol{k}$ and perform the operation three times, the result is $\mathbf{201}$. Find $\boldsymbol{k}$.

Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

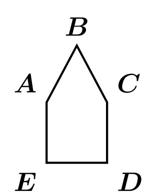
To undo the calculator's operation, divide by $\mathbf{2.1}$ and round up. That is, $\mathbf{201/2.1 \approx 95.7 \to 96}$, $\mathbf{96/2.1 \approx 45.7 \to 46}$, and $\mathbf{46/2.1 \approx 21.9 \to 22}$. Then verify: $\mathbf{22 \cdot 2.1 = 46.2 \to 46}$, $\mathbf{46 \cdot 2.1 = 96.6 \to 96}$, and $\mathbf{96 \cdot 2.1 \approx 201.6 \to 201}$. Thus $\boldsymbol{k = 22}$ is a possible solution.

If you begin with $\mathbf{23}$, the result after three steps is $\mathbf{210}$, and if you begin with $\mathbf{21}$, the result after three steps is $\mathbf{193}$. Thus $\boldsymbol{k = 22}$ is the only integer solution.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

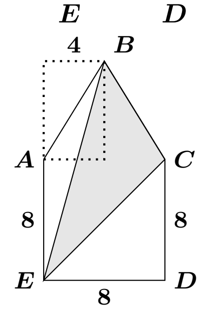
Our solver's claim that the inverse operation is to divide by $\mathbf{2.1}$ *and round up is not quite correct. The calculator sends* any *number in the interval* $\left[\frac{\mathbf{201}}{\mathbf{2.1}}, \frac{\mathbf{202}}{\mathbf{2.1}}\right) \approx \mathbf{[95.72, 96.19)}$ *to* $\mathbf{201}$. *However, chasing such intervals is much more work than estimating a solution, as our solver does, verifying it, and checking uniqueness.*

3. The pentagon $\boldsymbol{ABCDE}$ consists of a square, $\boldsymbol{ACDE}$, with side length $\mathbf{8}$, and an isosceles triangle, $\boldsymbol{ABC}$, such that $\boldsymbol{AB = BC}$. The area of the pentagon is $\mathbf{90}$. Find the area of $\triangle \boldsymbol{BEC}$.



Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

Note that $\boldsymbol{EB}$ and $\boldsymbol{EC}$ slice the pentagon into three triangles. You can therefore find the area of $\triangle \boldsymbol{BEC}$ by subtracting the areas of $\triangle \boldsymbol{ABE}$ and $\triangle \boldsymbol{CDE}$ from the area of the pentagon.



Since $\boldsymbol{ACDE}$ is a square with side $\mathbf{8}$, $\angle \boldsymbol{CDE} = \mathbf{90}^\circ$, and the area of $\triangle \boldsymbol{CDE}$ is $\frac{\mathbf{8 \cdot 8}}{\mathbf{2}} = \mathbf{32}$. Since $\triangle \boldsymbol{ABC}$ is isosceles with base $\mathbf{8}$, the distance to $\boldsymbol{B}$ from the line through $\boldsymbol{A}$ and $\boldsymbol{E}$ is $\mathbf{4}$. Therefore $\triangle \boldsymbol{ABE}$ has base $|\boldsymbol{AE}| = \mathbf{8}$, height $\mathbf{4}$, and, thus, area $\frac{\mathbf{8 \cdot 4}}{\mathbf{2}} = \mathbf{16}$.

The area of $ABCDE$ is 90, so the area of $\triangle BEC$ is $90 - 32 - 16 = 42$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

4. In how many ways can one choose three different integers between 0.5 and 13.5 such that the sum of the three numbers is divisible by 3?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

When adding integers and checking divisibility (by 3), the only relevant property of the integers is the remainder (after division by 3). The table lists the remainders of the integers in question.

Rem.	1	2	0
	1	2	3
	4	5	6
	7	8	9
	10	11	12
	13		

If all three integers leave remainder 0, so will the sum. You can choose three of the four numbers 3, 6, 9, and 12 in four ways. (Choosing three out of four is the same as choosing the one to leave behind. Surely, you can choose one of four in four ways.)

If all three numbers leave remainder 1, their sum will leave remainder $1 + 1 + 1 = 3 \equiv 0 \pmod 3$. You must now choose three of the five numbers 1, 4, 7, 10, and 13. You can choose the first number in 5 ways, the second in 4 ways, and the third in 3 ways. Thus you can choose the three numbers in a specific order in $5 \cdot 4 \cdot 3 = 60$ ways. Three numbers can be arranged in 6 ways, so you can choose three of five numbers in $60/6 = 10$ ways without order.

If all three numbers leave remainder 2, their sum will leave remainder $2 + 2 + 2 = 6 \equiv 0 \pmod 3$. You can choose three of the four numbers 2, 5, 8, and 11 in four ways.

Finally, if you choose one number in each column in the table, the sum leaves remainder $1 + 2 + 0 = 3 \equiv 0 \pmod 3$. You have 5 choices in the first column, 4 in the second, and 4 in the last. Thus you can choose the three numbers in $5 \cdot 4 \cdot 4 = 80$ ways.

All in all, you can choose three numbers whose sum leaves remainder 0 when divided by 3 in $4 + 10 + 4 + 80 = 98$ ways.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

5. If a and b are positive integers such that $a^3 - b^3 = 485$, find $a^3 + b^3$.

Solution by the editors.

Note that $485 = 5 \cdot 97$, so 485 can only be written as a product of two integers in two ways, $5 \cdot 97$ and $1 \cdot 485$. Now, $485 = a^3 - b^3 = (a-b)(a^2+ab+b^2)$, and both $a - b$ and $a^2 + ab + b^2$ are integers. Also $a - b < a^2 + ab + b^2$ since both a and b are positive integers. This leaves two cases:

If $a - b = 1$ and $a^2 + ab + b^2 = 485$, then $a = b + 1$ and $(b+1)^2 + (b+1)b + b^2 = 485$, hence $b^2 + 2b + 1 + b^2 + b + b^2 = 485$, so $3b^2 + 3b - 484 = 0$, thus $3(b^2 + b) = 484$. Since b is an integer and 484 is not a multiple of 3, this is not possible.

If $a-b=5$ and $a^2+ab+b^2=97$, then $a=b+5$ and $(b+5)^2+(b+5)b+b^2=97$, so $b^2+10b+25+b^2+5b+b^2=97$, thus $3b^2+15b-72=0$, hence $3(b+8)(b-3)=0$. Since b is positive, $b=3$ and $a=b+5=8$.

Hence the only solution is that $a=8$ and $b=3$. Thus $a^3+b^3=8^3+3^3=512+27=539$.

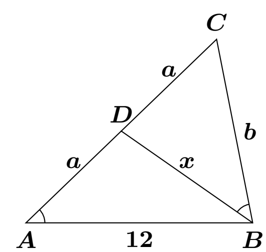
6. If a and b are positive integers such that $a^3+b^3=2ab(a+b)$, find $a^{-2}b^2+a^2b^{-2}$.

Solution by the editors.

Note that $a^3+b^3=(a+b)(a^2-ab+b^2)$. Since $a^3+b^3=2ab(a+b)$ and $a+b\neq 0$, $a^2-ab+b^2=2ab$, so $a^2+b^2=3ab$. Now divide through by ab; then $\frac{a}{b}+\frac{b}{a}=3$. Thus $3^2=(\frac{a}{b}+\frac{b}{a})^2=\frac{a^2}{b^2}+2\frac{a}{b}\cdot\frac{b}{a}+\frac{b^2}{a^2}=a^2b^{-2}+2+a^{-2}b^2$, and hence $a^{-2}b^2+a^2b^{-2}=3^2-2=7$.

7. Let D be the midpoint of side AC in $\triangle ABC$. If $\angle CAB=\angle CBD$ and the length of AB is 12, then find the square of the length of BD.

Solution by the editors.



Let a be $|AD|$ (which equals $|CD|$ since D is the midpoint), let b be $|BC|$, and let x be $|BD|$, as labeled in the diagram. Since $\angle CAB=\angle CBD$ and $\angle ACB=\angle BCD$ then $\triangle ABC\sim\triangle BDC$. Therefore $\frac{|AC|}{|BC|}=\frac{|BC|}{|CD|}$ and $\frac{|BC|}{|AB|}=\frac{|CD|}{|BD|}$, so $\frac{2a}{b}=\frac{b}{a}$ and $\frac{b}{12}=\frac{a}{x}$, hence $2a^2=b^2$ and $bx=12a$. Thus $b^2x^2=144a^2=72\cdot 2a^2=72b^2$, so $|BD|^2=x^2=72$.

8. If x, y, and z are whole numbers and $xyz+xy+2yz+xz+x+2y+2z=28$ find $x+y+z$.

Solution by the editors.

Note: The question should have said that x, y, and z are positive integers.

Noting that many of the terms in the given equation contain an x, one may try to factor out that x: $x(yz+y+z+1)+2yz+2y+2z=28$. Noting that the x-free terms almost equal twice the expression in the brackets, one may try to reconstruct the expression in the brackets: $x(yz+y+z+1)+2yz+2y+2z+2=30$, so $x(yz+y+z+1)+2(yz+y+z+1)=30$, so $(x+2)(yz+y+z+1)=30$. Encouraged by this success, try the idea again: $(x+2)\Big(y(z+1)+z+1\Big)=30$, so $(x+2)(y+1)(z+1)=30=2\cdot 3\cdot 5$. Since x, y, and z are all positive integers, $x+2\geq 3$, $y+1\geq 2$, and $z+1\geq 2$. Therefore the only possible solutions are $(x+2,y+1,z+1)=(3,2,5)$, $(x+2,y+1,z+1)=(3,5,2)$, $(x+2,y+1,z+1)=(5,2,3)$, and $(x+2,y+1,z+1)=(5,3,2)$. That is, $(x,y,z)=(1,1,4)$, $(x,y,z)=(1,4,1)$, $(x,y,z)=(3,1,2)$, and $(x,y,z)=(3,2,1)$. In all four cases, $x+y+z=6$.

If you allow x, y, and z to be whole numbers, you have to consider a few more cases for (x,y,z), namely: $(0,0,14)$, $(0,2,4)$, $(0,4,2)$, $(0,14,0)$, $(1,0,9)$, $(1,9,0)$, $(3,0,5)$, $(3,5,0)$, $(4,0,4)$, $(4,4,0)$, $(8,0,2)$, $(8,2,0)$, $(13,0,1)$, $(13,1,0)$, *and* $(28,0,0)$. *This adds* 8, 10, 14, *and* 28 *as possible values of* $x+y+z$.

9. Henrik's math class needs to choose a committee consisting of two girls and two boys. If the committee can be chosen in 3630 ways, how many students are there in Henrik's math class?

Solution by the editors.

Say Henrik's class has x girls and y boys. You can choose the first girl for the committee in x ways and the second girl in $x-1$ ways. Thus you can choose the two girls in a particular order in $x(x-1)$ ways. However, the order the two girls were chosen in is irrelevant, so the number of ways to choose the girls for the committee is $\frac{1}{2}x(x-1)$.

Likewise, you can choose the boys in $\frac{1}{2}y(y-1)$ ways. Thus you can choose the committee in $\frac{1}{2}y(y-1)\cdot\frac{1}{2}x(x-1)$ ways, so

$$\frac{1}{4}x(x-1)y(y-1) = 3630,$$

so

$$x(x-1)y(y-1) = 14520 = 2^3\cdot 3\cdot 5\cdot 11^2.$$

The left-hand side includes two pairs of consecutive integers. To match those on the right-hand side, one pair must be 11 and $2\cdot 5 = 10$, while the other is 11 and $2^2\cdot 3 = 12$. Thus $x = 11$ and $y = 12$ or the other way around. In either case, Henrik's class has 23 students.

10. Let S be $1!(1^2+1+1) + 2!(2^2+2+1) + 3!(3^2+3+1) + \cdots + 100!(100^2+100+1)$. Find $\dfrac{S+1}{101!}$. (As usual, $k! = 1\cdot 2\cdot 3\cdot\cdots\cdot(k-1)\cdot k$.)

Solution by the editors.

Each term in the sum has the form

$$\begin{aligned} k!(k^2+k+1) &= k!\Big((k^2+2k+1)-k\Big) = k!\Big((k+1)^2-k\Big) \\ &= k!(k+1)^2 - k!\,k = (k+1)!\,(k+1) - k!\,k \end{aligned}$$

Therefore the sum is telescoping:

$$\begin{aligned} S &= (2!\cdot 2 - 1!\cdot 1) + (3!\cdot 3 - 2!\cdot 2) \\ &\qquad + \cdots + (100!\cdot 100 - 99!\cdot 99) + (101!\cdot 101 - 100!\cdot 100) \\ &= 101!\cdot 101 - 1!\cdot 1 = 101!\cdot 101 - 1\,, \end{aligned}$$

so $S+1 = 101!\cdot 101$, thus $\frac{S+1}{101!} = 101$.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

We hope that the very low number of reader solutions this time was caused by the irregular production schedule that ***Crux Mathematicorum*** has suffered and the due date in September.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

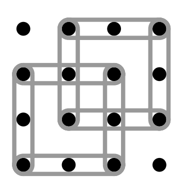
*Please send your solutions to the problems in this edition by **15 October 2012**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.

M507. *Proposed by the Mayhem Staff.*

A $\mathbf{4}$ by $\mathbf{4}$ square grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form squares, two different $\mathbf{2}$ by $\mathbf{2}$ squares are shown in the diagram. What is the least number of pegs that must be removed so that **no** squares can be formed?



M508. *Proposed by the Mayhem Staff.*

In 1770, Joseph Louis Lagrange proved that **every** non-negative integer can be expressed as the sum of the squares of four integers. For example $\mathbf{6 = 2^2 + 1^2 + 1^2 + 0^2}$ and $\mathbf{27 = 5^2+1^2+1^2+0^2 = 4^2+3^2+1^2+1^2 = 3^2+3^2+3^2+0^2}$ (in the theorem it is acceptable to use $\mathbf{0^2}$, or to use a square more than once). Notice that $\mathbf{27}$ had several different representations. Show that there is a number, not greater than $\mathbf{1\ 000\ 000}$ that can be represented as a sum of four **distinct non-negative** integers in more than $\mathbf{100}$ ways. (Note that rearrangements are not considered different, so $\mathbf{4^2 + 3^2 + 2^2 + 1^2 = 1^2 + 2^2 + 3^2 + 4^2}$ are the same representation of $\mathbf{30}$.)

M509. *Proposed by Titu Zvonaru, Comănești, Romania.*

Let ABC be a triangle with angles B and C acute. Let D be the foot of the altitude from vertex A. Let E be the point on AC such that $DE \perp AC$ and let M be the midpoint of DE. Show that if $AM \perp BE$, then $\triangle ABC$ is isosceles.

M510. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

If $a, b, c \in \mathbb{C}$ such that $|a| = |b| = |c| = r > 0$ and $a + b + c \neq 0$, compute the value of the expression

$$\frac{|ab + bc + ca|}{|a + b + c|}$$

in terms of r.

M511. *Proposed by Gili Rusak, student, Shaker High School, Latham, NY, USA.*

Pens come in boxes of 48 and 61. What is the smallest number of pens that can be bought in two ways if you must buy at least one box of each type?

M512. *Selected from a mathematics competition.*

A class of 20 students was given a three question quiz. Let x represent the number of students that answered the first question correctly. Similarly, let y and z represent the number of students that answered the second and the third questions correctly, respectively. If $x \geq y \geq z$ and $x+y+z \geq 40$, determine the smallest possible number of students who could have answered all three questions correctly in terms of x, y and z.

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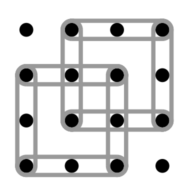
M507. *Proposé par l'Équipe de Mayhem.*



Un grillage 4 par 4 est formé de chevilles amovibles se situant à un centimètre l'une de l'autre, tel qu'illustré dans le schéma. Des élastiques sont attachés aux chevilles de façon à former des carrés ; deux carrés différents de taille 2 par 2 sont illustrés dans le schéma. Quel est le plus petit nombre de chevilles qui doivent être enlevées de façon à ce que **pas un seul carré** puisse être formé ?

M508. *Proposé par l'Équipe de Mayhem.*

En 1770, Joseph Louis Lagrange a démontré que **tout** entier non négatif pouvait s'écrire comme somme de carrés de quatre entiers. Par exemple, $\mathbf{6 = 2^2 + 1^2 + 1^2 + 0^2}$ et $\mathbf{27 = 5^2 + 1^2 + 1^2 + 0^2 = 4^2 + 3^2 + 1^2 + 1^2}$. (Dans le théorème de Lagrange il est permis d'utiliser $\mathbf{0^2}$ et de répéter un carré.) Remarquer que **27** a plusieurs représentations différentes. Démontrer qu'il y a un nombre inférieur ou égal à **1 000 000** qui peut être représenté comme somme de carrés de quatre entiers **non négatifs distincts**, de plus de **100** manières. (Noter que les réarrangements ne sont pas considérés distincts, c'est-à-dire que $\mathbf{4^2 + 3^2 + 2^2 + 1^2 = 1^2 + 2^2 + 3^2 + 4^2}$ constituent la même représentation de **30**.)

M509. *Proposé par Titu Zvonaru, Comăneşti, Roumanie.*

Soit $\boldsymbol{ABC}$ un triangle avec angles aigus $\boldsymbol{B}$ et $\boldsymbol{C}$. Soit $\boldsymbol{D}$ le pied de l'altitude à partir du sommet $\boldsymbol{A}$. Soit $\boldsymbol{E}$ le point sur $\boldsymbol{AC}$ tel que $\boldsymbol{DE \perp AC}$; soit $\boldsymbol{M}$ le mipoint de $\boldsymbol{DE}$. Démontrer que si $\boldsymbol{AM \perp BE}$ alors $\boldsymbol{\triangle ABC}$ est isocèle.

M510. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Si $\boldsymbol{a, b, c \in \mathbb{C}}$ tels que $\boldsymbol{|a| = |b| = |c| = r > 0}$ et $\boldsymbol{a + b + c \neq 0}$, calculer la valeur de l'expression

$$\frac{|\boldsymbol{ab + bc + ca}|}{|\boldsymbol{a + b + c}|}$$

en termes de $\boldsymbol{r}$.

M511. *Proposé par Gili Rusak, étudiant, Shaker High School, Latham, NY, É-U.*

Des plumes nous viennent en boîtes de **48** et de **61**. Quel est le plus petit nombre de plumes qui peuvent être achetées de deux manières différentes si on doit acheter au moins une boîte de chaque type ?

M512. *Sélectionné à partir de concours mathématiques.*

Une classe de **20** étudiants a subi un examen à trois questions. Soit $\boldsymbol{x}$ le nombre d'étudiants ayant répondu correctement à la première question. De même, soit $\boldsymbol{y}$ et $\boldsymbol{z}$ les nombres d'étudiants ayant répondu correctement à la deuxième puis à la troisième question respectivement. Si $\boldsymbol{x \geq y \geq z}$ et $\boldsymbol{x + y + z \geq 40}$, déterminer, en termes de $\boldsymbol{x}$, $\boldsymbol{y}$ et $\boldsymbol{z}$, le plus petit nombre possible d'étudiants ayant pu répondre correctement aux trois questions.

Mayhem Solutions

M470. *Proposed by the Mayhem Staff*

Vazz needs to buy desks and monitors for his new business. A desk costs $\$250$ and a monitor costs $\$260$. Determine all possible ways that he could spend exactly $\$10\,000$ on desks and monitors.

Solution by Gili Rusak, student, Shaker High School, Latham, NY, USA.

Let d and m represent the number of desks and monitors Vazz buys, respectively. Using the information from the problem we get $250d + 260m = 10\ 000$ which reduces to $25d + 26m = 1000$ or $26m = 1000 - 25d$. Looking at this modulo 25 we get $m \equiv 0 \pmod{25}$. Since $25d + 26m = 1000$, we must have $26m \leq 1000$, so $m < 50$.

Therefore, m, being a non-negative integer, can only equal 0 or 25. When $m = 0$, $d = 40$ and when $m = 25$, $d = 14$.

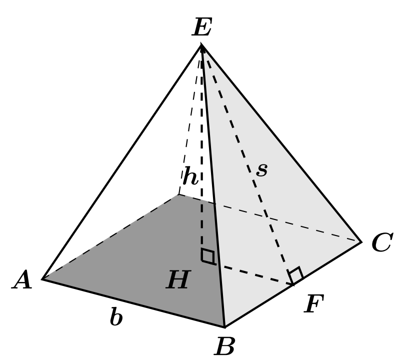
There are only two possibilities for Vazz to buy: 14 desks and 25 monitors or 40 desks and 0 monitors.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; JEREMY COOPER, student, Angelo State University, San Angelo, TX, USA; A. WIL EDIE, student, Missouri State University, Springfield, MO, USA; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; MUHAMMAD ROIHAN MUNAJIH, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.

M471. *Proposed by the Mayhem Staff*

Square based pyramid $ABCDE$ has a square base $ABCD$ with side length 10. Its other four edges AE, BE, CE, and DE each have length 20. Determine the volume of the pyramid.

Solution by Scott Brown, Auburn University, Montgomery, AL, USA.



The volume of a square pyramid as shown in the diagram is $V = \frac{b^2h}{3}$. According to the information given, $b = 10$ and the edges $EA = EB = EC = ED = 20$.

To find h, we will first find the slant height s. Consider the triangle EBC and let $s = EF$, where F is the midpoint of side BC. Now, triangle EFB is a right triangle, where $EB = 20$ and $BF = 5$.

Using the Pythagorean Theorem yields $s^2 = 20^2 - 5^2 = 375$. So the slant height is $s = 5\sqrt{15}$.

Now let H be the point where the height meets the square base of the pyramid. Triangle EHF is a right triangle. Using the Pythagorean Theorem again yields $h^2 = (5\sqrt{15})^2 - 5^2 = 350$. So the height is $h = 5\sqrt{14}$.

Thus the volume is $V = \frac{b^2h}{3} = \frac{500\sqrt{14}}{3}$ square units.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; MUHAMMAD ROIHAN MUNAJIH, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GILI RUSAK, student, Shaker High School, Latham, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain(two solutions); AND NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.

The problem could also be solved by using the Pythagorean Theorem to find the length of the diagonal of the base, then using the Pythagorean Theorem a second time in a triangle such as $\triangle EHB$. *About half the solutions used this method while the rest were similar to the featured solution.*

M472. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Suppose that x is a real number. Without using calculus, determine the maximum possible value of $\dfrac{2x^2-8x+17}{x^2-4x+7}$ and the minimum possible value of $\dfrac{x^2+6x+8}{x^2+6x+10}$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

We have

$$\begin{aligned}\frac{2x^2-8x+17}{x^2-4x+7} &= \frac{2(x^2-4x+7)+3}{x^2-4x+7}\\ &= 2+\frac{3}{x^2-4x+7}\\ &= 2+\frac{3}{(x-2)^2+3}.\end{aligned}$$

Hence maximizing $\dfrac{2x^2-8x+17}{x^2-4x+7}$ is equivalent to maximizing $\dfrac{3}{(x-2)^2+3}$. This, in turn, is equivalent to minimizing $(x-2)^2+3$. Since $(x-2)^2 \geq 0$, the sum $(x-2)^2+3$ is a minimum when $x-2=0$, i.e. when $x=2$.

Thus the maximum value of $\dfrac{2x^2-8x+17}{x^2-4x+7}$ is $2+\frac{3}{3}=3$.

Similarly, we have

$$\frac{x^2+6x+8}{x^2+6x+10} = 1-\frac{2}{(x+3)^2+1}.$$

Hence minimizing $\dfrac{x^2+6x+8}{x^2+6x+10}$ is equivalent to maximizing $\dfrac{2}{(x+3)^2+1}$. Thus the minimum occurs when $x=-3$ and the minimum value is -1.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HENRY RICARDO, Tappan, NY, USA; GILI RUSAK, student, Shaker High School, Latham, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposer.

M473. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania*

Determine all pairs (a, b) of positive integers for which $a^2+b^2-2a+b = 5$.

I. Solution by George Apostolopoulos, Messolonghi, Greece.

Rearranging the equation we obtain $a^2 - 2a + b^2 + b - 5 = 0$ which we will treat as a quadratic equation in a with discriminant

$$D = (-2)^2 - 4 \cdot 1(b^2 + b - 5) = 4 - 4b^2 - 4b + 20 = -4b^2 - 4b + 24.$$

In order for our quadratic equation to have real solutions we must have $D \geq 0$, so

$$\begin{aligned} -4b^2 - 4b + 24 \geq 0 &\Leftrightarrow b^2 + b - 6 \leq 0 \\ &\Leftrightarrow (b-2)(b+3) \leq 0 \\ &\Leftrightarrow -3 \leq b \leq 2. \end{aligned}$$

But b is a positive integer, so $b = 1$ or $b = 2$.

For $b = 1$, we get $a = 3$ or $a = -1$. For $b = 2$, we get $a = 1$. Thus the solutions are the pairs $(a, b) = (3, 1)$ and $(a, b) = (1, 2)$.

II. Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

First observe that

$$\begin{aligned} (2(a-1))^2 + (2b+1)^2 &= 4\left((a-1)^2 + \left(b + \frac{1}{2}\right)^2\right) \\ &= 4\left(a^2 - 2a + 1 + b^2 + b + \frac{1}{4}\right) \\ &= 4\left((a^2 - 2a + b^2 + b) + \frac{5}{4}\right) \\ &= 4\left(5 + \frac{5}{4}\right) = 25. \end{aligned} \tag{1}$$

Since there are only two possibilities of writing 25 as the sum of the squares of two non-negative integers, namely $25 = 0^2 + 5^2$ and $25 = 3^2 + 4^2$, equation (1) gives us the solutions $(a, b) = (3, 1)$ and $(a, b) = (1, 2)$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FLORENCIO CANO VARGAS, Inca, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GILI RUSAK, student, Shaker High School, Latham, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposer. One incomplete solution was received.

M474. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia*

Let a, b and x be positive integers such that $x^2 - bx + a - 1 = 0$. Prove that $a^2 - b^2$ is not a prime number.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Assume that a^2-b^2 is a prime number. Since $a^2-b^2 = (a-b)(a+b)$, and $a, b > 0$, hence $a-b = 1$. Therefore $x^2-bx+b = 0$, so $(x^2-1)-b(x-1) = -1$ which implies that $(x-1)(x+1-b) = -1$.

Since x is a positive integer, we must have $x - 1 = 1$, $x + 1 - b = -1$ or $x = 2$, $b = 4$ which implies that $a = 5$. This yields $a^2 - b^2 = 9$, which is not prime, a contradiction, and we are done.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker High School, Latham, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and the proposer.

The proposer and most of the solvers worked on the assumption that a prime is a positive integer. If we are working with integers, then we can consider numbers like -11 to be prime as well. In this case, 11 and -11 are called **associates**, *and ± 1 are called* **units**. *Then a prime number is only divisible by a unit or one of its associates. This idea is used in number theory when extending the idea of number system and hence extending the idea of what a prime is in this number system. Only Peiró considered the case where $a^2 - b^2 < 0$ which leads to $a = 1$, $b = 2$ and $a^2 - b^2 = -3$, a prime, which contradicts the problem.*

M475. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x, that is, $\{x\} = x - \lfloor x \rfloor$. For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Show that there exist infinitely many irrational numbers x such that $x \cdot \{x\} = \lfloor x \rfloor$.

Solution by Florencio Cano Vargas, Inca, Spain.

First note that the only rational solution is $x = 0$ (this was problem M437 [2010 : 135,136; 2011 : 14]). Next note there there is no solution if $x < 0$. Indeed if $x < 0$, then $x \cdot \{x\} > x > \lfloor x \rfloor$. Also note that if $0 < x < 1$, then $\{x\} = x$ so $x \cdot \{x\} = x^2 > 0 = \lfloor x \rfloor$. Clearly, x and $\{x\}$ are either both rational or both irrational. Thus we will limit ourselves to the case $x > 1$, where x is irrational. We will solve the equation for $\{x\}$. Let $\lfloor x \rfloor = k > 0$, for some positive integer

$\boldsymbol{k}$. Then $\boldsymbol{x = k + \{x\}}$ with $\boldsymbol{0 < \{x\} < 1}$. The given equation is then

$$(\{x\} + k) \cdot \{x\} = k,$$

with solution given by

$$\{x\} = \frac{k}{2}\left[\sqrt{1 + \frac{4}{k}} - 1\right],$$

where we have discarded the "minus" solution since $\boldsymbol{0 < \{x\} < 1}$. We see that this solution falls in this range since

$$\sqrt{1 + \frac{4}{k}} - 1 > 1 - 1 = 0$$

and

$$\begin{aligned}
\frac{k}{2}\left[\sqrt{1 + \frac{4}{k}} - 1\right] &= \frac{1}{2}[\sqrt{k^2 + 4k} - k] \\
&= \frac{1}{2}[\sqrt{(k+2)^2 - 4} - k] \\
&< \frac{1}{2}[\sqrt{(k+2)^2} - k] = 1.
\end{aligned}$$

Therefore we end up with infinitely many irrational solutions for $\boldsymbol{\{x\}}$, and hence we get the infinite family of solutions

$$x = \frac{k}{2}\left[\sqrt{1 + \frac{4}{k}} + 1\right],$$

where $\boldsymbol{k}$ is any positive integer.

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker High School, Latham, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and the proposer.

THE OLYMPIAD CORNER

No. 297

R.E. Woodrow and Nicolae Strungaru

The problems from this issue come from the Indian IMO Selection Test, the Colombian Mathematical Olympiad, the Singapore Mathematical Olympiad, the Serbian Mathematical Olympiad, the Romanian National Olympiad, and the Finnish National Olympiad. Our thanks go to Adrian Tang for sharing the material with the editors.

The solutions to the problems are due to the editors by **1 November 2012**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editors thank Jean-Marc Terrier of the University of Montreal for translations of the problems.

OC41. Let P be a point in the interior of a triangle ABC. Show that

$$\frac{PA}{BC} + \frac{PB}{AC} + \frac{PC}{AB} \geq \sqrt{3}\,.$$

OC42. Find the smallest n for which $n!$ has at least 2010 different divisors.

OC43. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)\,;\ \forall x, y \in \mathbb{R}\,.$$

OC44. In a scalene triangle ABC, we denote by α and β the interior angles at A and B. The bisectors of these angles meet the opposite sides of the triangle at points D and E respectively. Prove that the acute angle between the lines D and E does not exceed $\frac{|\alpha-\beta|}{3}$.

OC45. Let $a_1, a_2, a_3, \ldots, a_{15}$ be prime numbers forming an arithmetic progression with common difference $d > 0$. If $a_1 > 15$, prove that $d > 30,000$.

OC46. Let p be a prime number, and let x, y, z be integers so that $0 < x < y < z < p$. Suppose that x^3, y^3 and z^3 have the same remainders when divided by p. Prove that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

OC47. Let a, b be two distinct odd positive integers. Let a_n be the sequence defined as $a_1 = a\,;\, a_2 = b\,;\, a_n =$ the largest odd divisor of $a_{n-1} + a_{n-2}$. Prove that there exists a natural number N so that, for all $n \geq N$ we have $a_n = \gcd(a, b)$.

OC48. The angles of a triangle ABC are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. The bisectors meet the opposite sides at A', B' and C'. Prove that $A'B'C'$ is an isosceles triangle.

OC49. Let N be a positive integer. How many non congruent triangles are there, whose vertices lie on the vertices of a regular $6N$-gon?

OC50. Let $n \geq 2$. If n divides $3^n + 4^n$, prove that 7 divides n.

. . . .

OC41. Soit P un point intérieur d'un triangle ABC. Montrer que

$$\frac{PA}{BC} + \frac{PB}{AC} + \frac{PC}{AB} \geq \sqrt{3}\,.$$

OC42. Trouver le plus petit n pour lequel $n!$ possède au moins 2010 diviseurs différents.

OC43. Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ satisfaisant

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)\,;\ \forall x, y \in \mathbb{R}\,.$$

OC44. Dans un triangle scalène ABC, notons α et β les angles intérieurs en A et B. Les bissectrices respectives de ces angles coupent les côtés opposés du triangle aux points D et E. Montrer que l'angle aigu entre les droites D et E n'excède pas $\frac{|\alpha-\beta|}{3}$.

OC45. Soit $a_1, a_2, a_3, \ldots, a_{15}$ des nombres premiers formant une progression arithmétique de raison $d > 0$. Si $a_1 > 15$, montrer que $d > 30,000$.

OC46. Soit p un nombre premier, et soit x, y, z trois entiers tels que $0 < x < y < z < p$. Supposons que x^3, y^3 et z^3 ont les mêmes restes lorsqu'on les divise par p. Montrer que $x^2 + y^2 + z^2$ est divisible par $x + y + z$.

OC47. Soit a, b deux entiers positifs impairs distincts. Soit a_n la suite définie par $a_1 = a\,;\, a_2 = b\,;\, a_n =$ le plus grand diviseur impair de $a_{n-1} + a_{n-2}$. Montrer qu'il existe un nombre naturel N tel que, pour tous les $n \geq N$, on a $a_n = \gcd(a, b)$.

OC48. Les angles d'un triangle ABC sont $\frac{\pi}{7}$, $\frac{2\pi}{7}$ et $\frac{4\pi}{7}$. Les bissectrices coupent les côtés opposés en A', B' et C'. Montrer que $A'B'C'$ est un triangle isocèle .

OC49. Soit N un entier positif. Combien y a-t-il de triangles non congruents dont les sommets sont sur les sommets d'un $6N$-gone régulier ?

OC50. Soit $n \geq 2$. Si n divise $3^n + 4^n$, montrer que 7 divise n.

First the editor apologize to Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; whose solutions were misfiled. Correct solutions were received for: Indian Team Selection Test 2007, #2 and #7; Mediterranean Mathematics Competition 2007, #1 and #4; and Bulgarian Team First Selection Test, #1. Next we turn to the Olimpiada Nacional Escolar de Matematica 2009, Level 2, given at [2010: 373].

1. Let a, b, c and d be four integer numbers whose sum is 0. Let

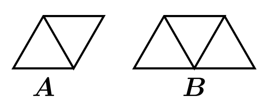
$$M = (bc - ad)(ac - bd)(ab - cd) .$$

Show that there is a whole number P such that $P^2 = M$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Ricardo's write-up.

Replacing d by $-a - b - c$ in each factor of M and factoring, we see that $bc-ad = (a+b)(a+c)$, $ac-bd = (a+b)(b+c)$, and $ab-cd = (a+c)(b+c)$. Thus $M = (bc - ad)(ac - bd)(ab - cd) = (a + b)^2(a + c)^2(b + c)^2 = [(a + b)(a + c)(b + c)]^2 = P^2$.

2. An equilateral triangle of side length 6 is divided into 36 small equilateral triangles of side length 1. The resulting chart is covered by m markers of type A and n markers of type B without doubling or leaving empty spaces. Markers of type A are formed by two equilateral triangles of side length 1 and markers of type B are formed from 3 small triangles, as shown in the figure. Determine all possible values of m.



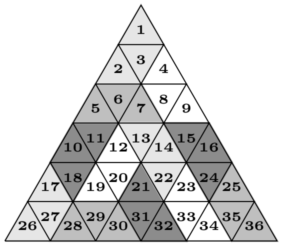
Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Since each marker of type A covers two triangles and each marker of type B covers three triangles, we have

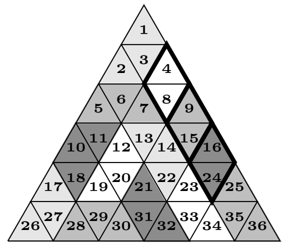
$$2m + 3n = 36,$$

for nonnegative integers m and n. This implies that $m \in \{0, 3, 6, 9, 12, 15, 18\}$. We will show that $m = 0, 3, 6, 9$ are possible, while $m = 12, 15, 18$ are impossible.

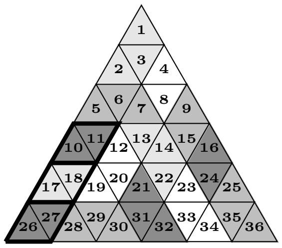
We orient the large triangle and number the smaller triangles as shown in the following diagrams.



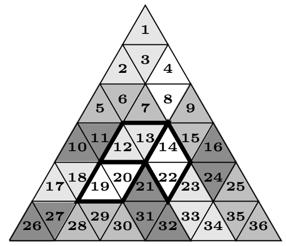
$m = 0$



$m = 3$



$m = 6$



$m = 9$

The diagrams show that $m = 0, 3, 6, 9$ are possible. For $m = 3$, we replaced the triples $(4, 8, 9)$ and $(15, 16, 24)$ covered by markers of type B in the $m = 0$ case with the pairs $(4, 8)$, $(9, 15)$, $(16, 24)$ covered by markers of type A. Similarly for $m = 6$, we make the further switch of $(11, 10, 18)$ and $(17, 27, 26)$ to $(11, 10)$, $(18, 17)$, and $(27, 26)$; and for $m = 9$, we make the additional switch of $(13, 14, 22)$ and $(12, 20, 19)$ to $(14, 22)$, $(13, 12)$, and $(20, 19)$.

We note that of the 36 small triangles 21 have a vertex uppermost, while 15 have an edge uppermost. Each marker of type A always covers one triangle of each of these types, while each marker of type B covers two of one type and one of the other.

If $m = 12$, then of the 12 triangles covered by markers of type B, nine have a vertex on top, and three have the edge on top, which is impossible since at least one third of the triangles covered by markers of type B must have an edge on top. Hence, $m \neq 12$. If $m = 15$, then all of the six triangles covered by markers of type B have a vertex on top. Hence, $m \neq 15$. Likewise, $m \neq 18$ since 18 markers of type A would require 18 triangles with an edge on top.

Thus, as claimed,

$$m \in \{0, 3, 6, 9\}.$$

3. For each positive integer n let d be the largest divisor of n with $d \leq \sqrt{n}$, and define $a_n = \frac{n}{d} - d$. Show that in the sequence $a_1, a_2, a_3 \ldots$, each nonnegative integer k appears infinitely often.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution by Curtis.

List the primes $2 = p_1 < p_2 < p_3 < \cdots$. Given a nonnegative integer k, for each positive integer j with $p_j > k$ we define $n_j(k) = p_j(p_j - k)$, and $d_j = p_j - k$. Then d_j is the largest divisor of $n_j(k)$ with $d_j \leq \sqrt{n_j(k)}$, and $a_{n_j(k)} = \frac{n_j(k)}{d_j} - d_j = k$. Hence, k appears infinitely often in the sequence $\{a_n\}$.

4. On a circle $N \geq 5$ points are marked so that the N arcs formed have the same length. A coin is placed on each point, and Ricardo and Tomás play a game with the following rules:

- They play alternately.
- Ricardo starts.
- A player may take a coin only if that coin forms an acute triangle with at least two other coins.

A player loses when he cannot take any coin during his turn.

Does either player have a winning strategy? If so, what is it?

Discussion and solution by Stan Wagon, Macalester College, St. Paul, MN, USA.

The following solution is adapted from a solution in Spanish found by Witold Jarnicki at `www.fileden.com/files/2008/5/31/1938837//CuartaFase2009.pdf`. It is by Sergio Vera (the author of the problem). We use Alice and Bob for the two players, with Alice moving first.

Bob has a winning strategy for every $N \geq 4$. When N is even, Bob can simply choose the point diametrically opposite to Alice. For if Alice chooses X there is an acute triangle Xab and neither a nor b can be $-X$ and b cannot be $-a$. But then Bob has triangle $(-X, -a, -b)$, all of which are available, by the symmetry that faced Alice. Thus Alice will always face a symmetric configuration, and therefore Bob can always move after Alice does.

Now assume $N = 2m+1$. Call a configuration C of $2k+1$ coins, *balanced* if for any point $X \in C$, the diameter through X splits the remaining $2k$ points into two equalized sets. Then the initial configuration is balanced. To be precise: the diameter through X defines two semicircles. For C to be balanced, each semicircle has the point X and exactly k other points.

The following easily proved fact is useful: Three points on a circle form an acute triangle if and only if for every diameter, the three points include one on one side of the diameter and one on the other.

Two key facts:

Fact 1: For a 5-coin balanced situation, the player to move loses.

Fact 2: For $k \geq 3$ and a $2k+1$-coin balanced situation C, any move on C can be followed by another move resulting in a $2k-1$-coin balanced situation.

These two facts suffice, since starting from N Bob can keep things balanced, and so eventually the case $N = 5$ is reached and Alice loses.

Proof of Fact 1. If the first player takes coin X, the second player takes any of the "neighbours" of X (moving around the circle). If the points are P_i in order, suppose the first player takes P_1 and second takes P_2. Then $\triangle P_2P_4P_5$ is acute, for if not, there would be a diameter with P_2, P_4, and P_5 on one side. But the diameter from P_2 splits P_4 and P_5, so any diameter with P_2, P_4, and P_5 on one side would have to be on one side of diameter from P_2, impossible.

Proof of Fact 2. If the first player takes coin X, the second player takes the coin Y that is the farthest from X (thinking in terms of angular measure from the center): Let P and Q be the neighbours of X on the two sides. Then YPQ is acute as in the proof of Fact 1, and the removal of X and Y leaves a balanced position.

Note: One can summarize Bob's strategy in all cases (except $N = 5$) by simply saying: If Alice chooses X, Bob chooses the point nearest $-X$.

Alternate solution by Stephen Morris, Newbury, England; and Stan Wagon, Macalester College, St. Paul, MN, USA.

Let Alice and Bob be the players who move first or second, respectively. We let P_1 to P_N be the positions the coins occupy at the start, in counterclockwise order. If N is even, then Bob wins by always choosing the point diametrically opposite to Alice's choice (as described in the other proof).

Assume $5 \leq N = 2m+1$. Call two positions *opposite* if they are as close as is possible to being diametrically opposite; each position is opposite two positions (that is, P_1 and either P_{m+1} or P_{m+2}). We claim that a winning strategy for Bob is to take an immediately winning move when available and otherwise take the coin closest to being opposite to Alice's last move. In what follows assume that Bob plays by this strategy. We assume the standard geometrical fact that a chord in a circle subtends an obtuse angle on its near side and an acute angle on its far side.

Case 1. If a player has a legal move, then all available moves are legal.

Proof. Let P be one of the remaining points, use the diameter from P to divide the points into two sets, and choose the points Q and R, respectively, in each set that is farthest from P. Then $\triangle PQR$ is an acute triangle. The angles at Q and R are acute because these points straddle a diameter; the angle at P is acute because if not then Q and R, and therefore all the remaining coins, would be on the same side of the diameter parallel to QR, contradicting the fact that there is a legal move.

Case 2. If there are four coins left and they do not lie on a semicircle then Bob has an immediately winning move.

Proof. The diameter through one of the points divides the other three into a pair and a singleton. The singleton wins.

Case 3. If a set of coin positions is contained in a semicircle, then there is a set of m contiguous vacancies.

Proof. The diameter through any position partitions the rest into two sets of size m. A diameter not through a position can be rotated until it just touches a position without affecting the state.

Case 4. At the start, and after each move by Bob, vacant positions can be paired so that they are opposite each other.

Proof. At the start there are no vacancies. Suppose the assertion is true, Alice chooses a coin, and Bob has a legal choice. If Bob takes an opposite coin then it remains true. Suppose Bob cannot take an opposite coin. Suppose without loss of generality, that Alice takes P_1 and Bob takes coin P_{m+r+1} (a legal choice by Case 1). Positions $m+2, \ldots, m+r$ must be vacant and be paired in the initial pairing. Since 1 is unavailable, $m+2$ is paired with 2 and, similarly, these pairings must be $(m+2, 2), (m+3, 3), \ldots, (m+r, r)$. Following Bob's move there is a new pairing that includes $(m+2, 1), (m+3, 2), \ldots, (m+r+1, r)$ and is otherwise the same.

Case 5. Alice never has an immediately winning move. This means that the given strategy wins for Bob.

Proof. Suppose she did. If Bob had been faced with 4 coins and had a legal move, then he would have won by Case 2. Therefore, Alice was facing at least 5 coins. For Alice's move to be winning, she must see a semicircle having only one coin in it. By Case 3, she is faced with a set of m contiguous positions containing one coin. No positions in this set are opposite so there are $m-1$ vacancies paired with other positions in the pairing from Case 4. This leaves at most two other positions that can contain coins. But that only allows for three coins, contradiction.

We return to the files of solutions for the October 2010 number of the *Corner* and the Sélection OIM 2006 given at [2010: 374–376].

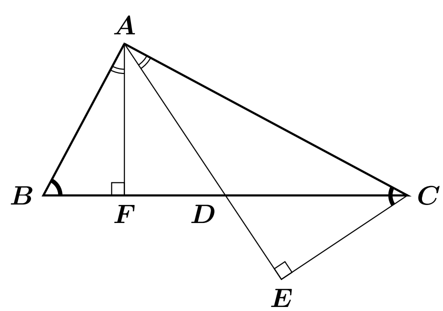
1. Dans le triangle ABC soit D le milieu du côté BC et E la projection de C sur AD. On suppose que $\angle ACE = \angle ABC$. Montrer que le triangle ABC est soit isocèle, soit rectangle.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.

We put $CA = b$, $AB = c$ and denote by F the foot of the perpendicular from A to BC.

We have $\angle BAF = \angle EAC = \angle DAC$ (since both $\angle BAF$ and $\angle EAC$ are complementary to $\angle ABC$).

Thus the two lines AF and AD are equally inclined to the arms of $\angle A$. Since AD is a median, AF is the symmedian of $\triangle ABC$ at A and so AF



divides the side BC in the ratio of the square of the sides:

$$\frac{BF}{FC} = \frac{c^2}{b^2}. \tag{1}$$

Now, $BF = c\cos B$ and $FC = b\cos C$. Hence, from equation (1), we get

$$\frac{\cos B}{\cos C} = \frac{c}{b} = \frac{\sin C}{\sin B},$$

by the law of sines, which is equivalent to

$$\sin B\cos B - \sin C\cos C = 0$$

or, equivalently,

$$\sin 2B - \sin 2C = 0$$

which we rewrite as

$$2\cos\left(\frac{2B+2C}{2}\right)\sin\left(\frac{2B-2C}{2}\right) = 0,$$

that is,

$$\cos(B+C)\sin(B-C) = 0.$$

Hence, either $\cos(B+C) = 0$ or $\sin(B-C) = 0$.

Thus, if $\cos(B+C) = 0$, then $B+C = 90^\circ$ and $\triangle ABC$ is right-angled at A. If $\sin(B-C) = 0$, then $B = C$ and $\triangle ABC$ is isosceles with $b = c$.

4. Soient $1 = d_1 < d_2 < \ldots < d_k = n$ les diviseurs positifs de n. Déterminer tous les n tels que

$$2n \;=\; d_5^2 + d_6^2 - 1\,.$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim that the only solution is $n = 272$.

Note first that $2n > d_5^2$, and $2n < 2d_6^2$, so that $d_5 < \sqrt{2n}$ and $d_6 > \sqrt{n}$. These imply that n has at most 10, but at least 6 positive divisors. If $n = p_1^{r_1}p_2^{r_2}\cdots p_\ell^{r_\ell}$, where $p_1 < p_2 < \cdots < p_\ell$ are primes, then the number of divisors of n is $(r_1+1)(r_2+1)\cdots(r_\ell+1) \le 10$. Hence, $\ell \le 3$.

Case 1. Suppose $n = p^r$, where p is prime. Then the divisors of n are $1, p, p^2, p^3, \ldots, p^r$. Thus, $5 \le r \le 9$. Hence,

$$2p^r = (p^4)^2 + (p^5)^2 - 1.$$

This implies that p divides 1, a contradiction.

Case 2. Suppose $n = p_1^{r_1}p_2^{r_2}$, where $p_1 < p_2$ are primes and $r_1, r_2 \ge 1$. If p_1 and p_2 are both odd, then $2n = d_5^2 + d_6^2 - 1$ is odd, a contradiction. Hence, $p_1 = 2$.

Letting $p = p_2$, we have $d_5 = 2^{s_1} p^{s_2}$ and $d_6 = 2^{t_1} p^{t_2}$ for $0 \le s_i, t_i \le r_i$, $i = 1, 2$. Thus the condition of the problem becomes

$$2^{r_1+1} p^{r_2} = (2^{s_1} p^{s_2})^2 + (2^{t_1} p^{t_2})^2 - 1.$$

If either $s_1 \ge 1$ and $t_1 \ge 1$ or $s_2 \ge 1$ and $t_2 \ge 1$, then a prime divides 1, a contradiction. Since $(r_1 + 1)(r_2 + 1) \le 10$, it follows that $r_1, r_2 \le 4$. In the following table, showing the possible pairs (r_1, r_2), the condition is satisfied by primes p that satisfy $d_5^2 + d_6^2 - 1 - 2n = 0$.

(r_1, r_2)	n	divisors of n in increasing order	Notes	d_5	d_6	$d_5^2 + d_6^2$ $-1 - 2n$	p
$(1, 2)$	$2p^2$	$\{1, 2, p, 2p, p^2, 2p^2\}$		p^2	$2p^2$	$5p^4 - 1 - 4p^2$	None
$(1, 3)$	$2p^3$	$\{1, 2, p, 2p, p^2, 2p^2,$ $p^3, 2p^3\}$		p^2	$2p^2$	$5p^4 - 1 - 4p^3$	None
$(1, 4)$	$2p^4$	$\{1, 2, p, 2p, p^2,$ $2p^2, p^3, 2p^3 p^4, 2p^4\}$		p^2	$2p^2$	$5p^4 - 1 - 4p^4$	None
$(2, 1)$	$4p$	$\{1, 2, 4, p, 2p, 4p\}$	$p \ge 5$	$2p$	$4p$	$20p^2 - 1 - 8p$	None
$(2, 1)$	$4p$	$\{1, 2, p, 4, 2p, 4p\}$	$p = 3$	$2p$	$4p$	$20p^2 - 1 - 8p$	None
$(2, 2)$	$4p^2$	$\{1, 2, 4, p, 2p, 4p,$ $p^2, 2p^2, 4p^2\}$	$p \ge 5$	$2p$	$4p$	$20p^2 - 1 - 8p^2$	None
$(2, 2)$	$4p^2$	$\{1, 2, p, 4, 2p, p^2,$ $4p, 2p^2, 4p^2\}$	$p = 3$	$2p$	p^2	$4p^2 + p^4$ $-1 - 8p^2$	None
$(3, 1)$	$8p$	$\{1, 2, 4, 8, p, 2p,$ $4p, 8p\}$	$p \ge 11$	p	$2p$	$5p^2 - 1 - 16p$	None
$(3, 1)$	$8p$	$\{1, 2, 4, p, 8, 2p,$ $4p, 8p\}$	$p = 5, 7$	8	$2p$	$64 + 4p^2$ $-1 - 16p$	None
$(3, 1)$	$8p$	$\{1, 2, p, 4, 2p, 8,$ $4p, 8p\}$	$p = 3$	$2p$	8	$4p^2 + 64$ $-2 - 16p$	None
$(4, 1)$	$16p$	$\{1, 2, 4, 8, 16, p,$ $2p, 4p, 8p, 16p\}$	$p \ge 17$	16	p	$256 + p^2$ $-1 - 32p$	17
$(4, 1)$	$16p$	$\{1, 2, 4, 8, p, 16,$ $2p, 4p, 8p, 16p\}$	$p = 11, 13$	p	16	$p^2 + 256$ $-1 - 32p$	17
$(4, 1)$	$16p$	$\{1, 2, 4, p, 8, 2p,$ $16, 4p, 8p, 16p\}$	$p = 5, 7$	8	$2p$	$64 + 4p^2$ $-1 - 32p$	None
$(4, 1)$	$16p$	$\{1, 2, p, 4, 2p, 8,$ $4p, 16, 8p, 16p\}$	$p = 3$	$2p$	8	$4p^2 + 64$ $-1 - 32p$	None

Hence, the only solution in this case is $p = 17$, corresponding to $n = 16 \cdot 17 = 272$. For this n, we have $d_5 = 16$ and $d_6 = 17$.

Case 3. Suppose $n = p_1^{r_1} p_2^{r_2} p_3^{r_3}$, with $p_1 < p_2 < p_3$ primes and $r_1, r_2, r_3 \ge 1$. Then $2 \cdot 2 \cdot (r_3 + 1) \le (r_1 + 1)(r_2 + 1)(r_3 + 1) \le 10$, implies that $r_3 \le \frac{3}{2}$. Hence, $r_3 = 1$. Likewise, $r_1 = r_2 = 1$, so that $n = p_1 p_2 p_3$. As in case 2, $p_1 = 2$. We have

$$4 p_2 p_3 = 2^{2\alpha_1} p_2^{2\alpha_2} p_3^{2\alpha_3} + 2^{2\beta_1} p_2^{2\beta_2} p_3^{2\beta_3} - 1,$$

where, for $i = 1, 2, 3$, $\alpha_i, \beta_i \in \{0, 1\}$, and either $\alpha_i = 0$ or $\beta_i = 0$. Also, α_1 and β_1 cannot both be 0. We test the possibilities.

$(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$	$4p_2p_3 = 2^{2\alpha_1}p_2^{2\alpha_2}p_3^{2\alpha_3} + 2^{2\beta_1}p_2^{2\beta_2}p_3^{2\beta_3} - 1$	Implication	Possible?
$(0,1,0,0,0,0)$	$4p_2p_3 = 1 + 4 - 1 = 4$	$p_2 = p_3 = 1$	No
$(0,1,0,0,0,1)$	$4p_2p_3 = 1 + 4p_3^2 - 1 = 4p_3^2$	$p_2 = p_3$	No
$(0,1,0,0,1,0)$	$4p_2p_3 = p_3^2 + 4 - 1 = p_3^2 + 3$	$p_3 = 3$	No
$(0,1,0,1,0,0)$	$4p_2p_3 = 1 + 4p_2^2 - 1 = 4p_2^2$	$p_2 = p_3$	No
$(0,1,0,1,0,1)$	$4p_2p = 1 + 4p_2^2p_3^2 - 1 = 4p_2^2p_3^2$	$p_2 = p_3 = 1$	No
$(0,1,0,1,1,0)$	$4p_2p_3 = p_3^2 + 4p_2^2 - 1$		
$(0,1,1,0,0,0)$	$4p_2p_3 = p_2^2 + 4 - 1 = p_2^2 + 3$	$p_2 = 3,\ p_3 = 1$	No
$(0,1,1,0,0,1)$	$4p_2p_3 = p_2^2 + 4p_3^2 - 1$		
$(0,1,1,0,1,0)$	$4p_2p_3 = p_2^2p_3^2 + 4 - 1 = p_2^2p_3^2 + 3$		No
$(1,0,0,0,0,0)$	$4p_2p_3 = 4 + 1 - 1 = 4$		No
$(1,0,0,0,0,1)$	$4p_2p_3 = 4 + p_3^2 - 1 = p_3^2 + 3$	$p_3 = 3$	No
$(1,0,0,0,1,0)$	$4p_2p_3 = 4p_3^2 + 1 - 1 = 4p_3^2$		No
$(1,0,0,1,0,0)$	$4p_2p_3 = 4 + p_2^2 - 1 = p_2^2 + 3$	$p_2 = 3,\ p_3 = 1$	No
$(1,0,0,1,0,1)$	$4p_2p_3 = 4 + p_2^2p_3^2 - 1 = p_2^2p_3^2 + 3$	$p_2p_3 = 1$ or 3	No
$(1,0,0,1,1,0)$	$4p_2p_3 = 4p_3^2 + p_2^2 - 1$		
$(1,0,1,0,0,0)$	$4p_2p_3 = 4p_2^2 + 1 - 2 = 4p_2^2$	$p_3 = p_2$	No
$(1,0,1,0,0,1)$	$4p_2p_3 = 4p_2^2 + p_3^2 - 1$		
$(1,0,1,0,1,0)$	$4p_2p_3 = 4p_2^2p_3^2 + 1 - 1 = 4p_2^2p_3^2$	$p_2p_3 = 1$	No

Hence there are four possibilities to consider. We note that the divisors of n, in increasing order, are

$$\{1, 2, p_2, 2p_2, p_3, 2p_3, p_2p_3, 2p_2p_3\} \text{ or } \{1, 2, p_2, p_3, 2p_2, 2p_3, p_2p_3, 2p_2p_3\}.$$

- Suppose $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = (0,1,0,1,1,0)$. Then $d_5 = p_3$ and $d_6 = 2p_2$. In neither of the above orders for the divisors of n does $d_6 = 2p_2$.
- Suppose $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = (0,1,1,0,0,1)$. Then $d_5 = p_2$ and $d_6 = 2p_3$. But p_2 is the third smallest rather than the fifth smallest divisor of n.
- Suppose $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = (1,0,0,1,1,0)$. Then $d_5 = 2p_3$ and $d_6 = p_2 < d_6$, a contradiction.
- Suppose $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = (1,0,1,0,0,1)$. Then $d_5 = 2p_2$ and $d_6 = p_3$, which is also impossible since the sixth smallest divisor of n is $2p_3$ rather than p_3.

Hence, the case of three distinct prime factors for n is impossible.

Thus, the only solution is $n = 272$.

6. Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que pour tout, $x, y \in \mathbb{R}$ on ait l'égalité suivante

$$f(f(x) - y^2) \;=\; f(x)^2 - 2f(x)y^2 + f(f(y))\,. \tag{1}$$

Solved by Michel Bataille, Rouen, France.

It is readily checked that the zero function $x \mapsto 0$ and the square function $x \mapsto x^2$ are solutions. We show that there are no other solutions. To this aim, consider f, different from the zero function, satisfying the given functional equation (1). Taking $x = y = 0$ in (1), we obtain $f(0) = 0$ and then with $y = 0$, (1) gives $f(f(x)) = f(x)^2$ for all $x \in \mathbb{R}$. It follows that (1) rewrites as

$$f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(y)^2. \tag{2}$$

With $x = 0$ and $x = y$ in turn, (2) yields

$$f(-y^2) = f(y)^2 \tag{3}$$

and

$$f(f(x) - x^2) = 2f(x)(f(x) - x^2). \tag{4}$$

Using (3), we can rewrite (2) as

$$f(f(x) - y^2) - (f(x) - y^2)^2 = f(-y^2) - (-y^2)^2 \tag{5}$$

and then, taking the images under f and using (4), we are led to

$$f(x)(f(x) - 2y^2)(f(y)^2 - y^4) = 0 \quad (x, y \in \mathbb{R}). \tag{6}$$

Now, assume that for some real number $f(a)^2 \neq a^4$. Then, $a \neq 0$ and from (3), we would have $f(x) = 0$ or $f(x) = 2a^2$ for all $x \in \mathbb{R}$. Taking $b \in \mathbb{R}$ such that $f(b) \neq 0$ (this is possible because f is not the zero function), we obtain $f(b) = 2a^2$ and from (5) with $x = b$

$$4a^4 - 4a^2y^2 = f(2a^2 - y^2) - f(y)^2$$

for all $y \in \mathbb{R}$. Since the right-hand side can take at most 3 values (namely $0, 2a^2, -2a^2$) while the left-hand side can take infinitely many values, we have reached a contradiction. We conclude that for all y, we have $f(y)^2 = y^4$. Recalling (1), $f(-y^2) = y^4$, hence $f(u) = u^2$ for all negative real numbers u. Also, since $y^8 = f(-y^2)^2 = f(f(-y^2)) = f(y^4)$, we see that $f(u) = u^2$ when u is positive and finally, f is the square function.

7. Les trois zéros réels du polynôme $P(x) = x^3 - 2x^2 - x + 1$ sont $a > b > c$. Trouver la valeur de l'expression

$$a^2b + b^2c + c^2a\,.$$

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's write-up.

Since $P(-1) = -1$, $P(0) = 1$, $P(1) = -1$, $P(2) = -1$, $P(3) = 7$, it follows that $a \in (2, 3)$, $b \in (0, 1)$, $c \in (-1, 0)$.

We let $\alpha = a^2b + b^2c + c^2a$, and $\beta = ab^2 + bc^2 + ca^2$.

Since $c > -1$ and $1 > b > 0$, we have

$$c > -1 \Rightarrow bc > -b > -1,$$

hence $bc + 1 > 0$ or $b^2c + b > 0$.

It follows that, because $a > 1$,

$$\alpha > a^2b + b^2c = a^2b + b^2c + b - b > a^2b - b = b(a^2 - 1) > 0,$$

hence $\alpha > 0$.

Since a, b, c are the roots of the equation $P(x) = 0$, we have

$$\begin{aligned} a + b + c &= 2, \\ ab + bc + ca &= -1, \\ abc &= -1. \end{aligned}$$

From these, we obtain

$$-2 = (a+b+c)(ab+bc+ca) = a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc,$$

that is,

$$\alpha + \beta = 1. \tag{1}$$

Furthermore, we have

$$\begin{aligned} \alpha\beta &= (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \\ &= a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 + abc(a^3 + b^3 + c^3). \end{aligned} \tag{2}$$

Since $(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3x^2z + 3xz^2 + 6xyz$, we deduce that

$$\begin{aligned} 8 &= (a + b + c)^3 = a^3 + b^3 + c^3 + 3\alpha + 3\beta - 6 \\ &\Rightarrow a^3 + b^3 + c^3 = -3\alpha - 3\beta + 14 \end{aligned} \tag{3}$$

and

$$\begin{aligned} -1 &= (ab + bc + ca)^3 \\ &= a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^3c + 3a^3b^2c + 3ab^3c^2 \\ &\qquad + 3a^3bc^2 + 3ab^2c^3 + 3a^2bc^3 + 6a^2b^2c^2 \\ &= a^3b^3 + b^3c^3 + c^3a^3 + 3abc(\alpha + \beta) + 6a^2b^2c^2 \\ &\Rightarrow a^3b^3 + b^3c^3 + c^3a^3 = 3\alpha + 3\beta - 7. \end{aligned} \tag{4}$$

Using (3) and (4), by (2) we obtain

$$\alpha\beta = 3\alpha + 3\beta - 7 + 3 + 3\alpha + 3\beta - 14,$$

that is

$$\alpha\beta = -12. \tag{5}$$

Solving the system (1) and (5),

$$\begin{aligned} \alpha + \beta &= 1 \\ \alpha\beta &= -12, \end{aligned}$$

we obtain $(\alpha, \beta) = (4, -3)$ and $(\alpha, \beta) = (-3, 4)$.

Since $\alpha > 0$, we must have $\alpha = 4$, hence

$$a^2b + b^2c + c^2a = 4.$$

10. Soient a, b, c des nombres réels positifs avec $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Démontrer l'inégalité suivante:

$$\sqrt{ab+c} + \sqrt{bc+a} + \sqrt{ca+b} \;\geq\; \sqrt{abc} + \sqrt{a} + \sqrt{b} + \sqrt{c}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Henry Ricardo, Tappan, NY, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution by Ricardo.

We have

$$\sqrt{abc} = \sqrt{abc}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}.$$

Thus it is sufficient to prove that

$$\sum_{\text{cyclic}} \sqrt{a+bc} \geq \sum_{\text{cyclic}} \left(\sqrt{a} + \sqrt{\frac{bc}{a}}\right). \tag{1}$$

For positive numbers x, y, z, with $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, we see that $\sqrt{x+yz} \geq \sqrt{x} + \sqrt{\frac{yz}{x}}$ is equivalent to

$$x + yz \geq x + \frac{yz}{x} + 2\sqrt{yz} = x + yz\left(1 - \frac{1}{y} - \frac{1}{z}\right) + 2\sqrt{yz},$$

or $y + z \geq 2\sqrt{yz}$, which is true by the AGM inequality. Applying this result to each term of the left-hand cyclic sum in (1), our inequality is proved.

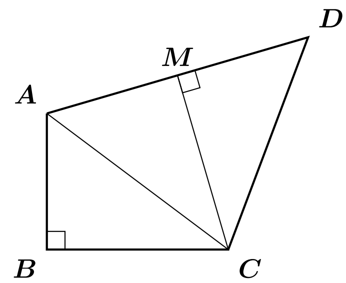
Next we turn to the Japanese Mathematical Olympiad, First Round, given at [2010 : 376–377].

1. Let $ABCD$ be a convex quadrilateral with $AB = 3$, $BC = 4$, $CD = 5$, $DA = 6$ and $\angle ABC = 90^\circ$. Find the area of $ABCD$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Zvonaru.

By the pythagorean theorem we obtain $AC = 5$. It follows that the triangle ACD is isosceles.

Let M be the midpoint of AD, hence CM is the altitude of triangle ACD. Thus we have $AM = 3$, and $AC = 5$, from which we deduce that $MC = 4$.



It follows that

$$\text{area of } ABCD = \text{ area of } ABC + \text{ area of } ACD = \frac{3 \cdot 4}{2} + \frac{6 \cdot 4}{2} = 18.$$

2. Determine the tens place of $11^{12^{13}}$ (12^{13}th power of 11, not the 13^{th} power of 11^{12}).

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up by Midttun.

From the binomial theorem we find

$$\begin{aligned} 11^{10} = (10+1)^{10} &= \sum_{i=0}^{10} \binom{10}{i} \cdot 10^i \cdot 1^{10-i} \\ &= \binom{10}{0} \cdot 10^0 + \binom{10}{1} \cdot 10^1 + \sum_{i=2}^{10} \binom{10}{i} \cdot 10^i \\ &= 1 + 100 + 100 \sum_{i=2}^{10} \binom{10}{i} \cdot 10^{i-2} = 100N + 1, \end{aligned}$$

for some positive integer N. Similarly, we find

$$11^{10n} = (11^{10})^n = (100N+1)^n = 100N_1 + 1$$

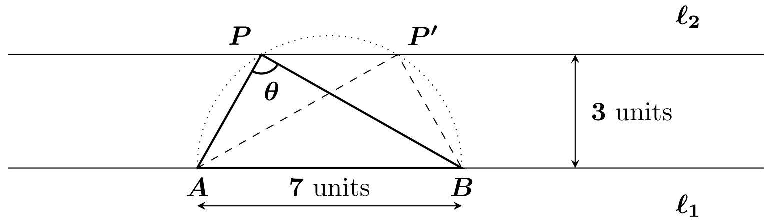
and $12^{13} = 10M + 2$. Now we have

$$(11)^{12^{13}} = 11^{10M+2} = 11^{10M} \cdot 11^2 + (100M_1 + 1)(100 + 21) = 100M_2 + 21.$$

So the tens digit is 2.

3. AB is a segment on a plane with length 7, and P is a point such that the distance between P and line AB is 3. Find the smallest possible value of $AP \times BP$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We use the solution by Zelator.



Let ℓ_1 be the line through A and B, and let ℓ_2 be a line parallel to ℓ_1 such that the distance between ℓ_1 and ℓ_2 is 3 units. Without loss of generality we can assume that P is on ℓ_2. Let θ be the measure of the $\angle APB$. The area of the triangle APB is equal to $\frac{1}{2}(AP)(PB)\sin\theta$, yet is also equal to $\frac{1}{2}(AB)(3) = \frac{1}{2}(7)(3) = \frac{21}{2}$. Thus, $\frac{1}{2}(AP)(PB)\sin\theta = \frac{21}{2}$ and hence

$$(AP)(PB) = \frac{21}{\sin\theta}. \tag{1}$$

Clearly, according to (1), the product $AP \cdot BP$ will be at minimum, when $\sin\theta$ is at maximum, that is, when $\sin\theta = 1$, so $\theta = 90^\circ$. Note that there are two such positions of the point P along the line l_2 which produce the smallest possible value of $AP \cdot BP$. These two positions, P and P', are the intersection points of the line l_2 and the semicircle with diameter $AB = 7$.

Therefore, the smallest possible value of $AP \cdot BP$ is 21.

4. The tens digit of the 4-digit integer n is nonzero. If we take the first 2 digits and the last 2 digits as two 2-digit integers, their product is a divisor of n. Determine all n with this property.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Curtis.

Write $n = 100x + y$, where $10 \le x \le 99$ and $10 \le y \le 99$. Then xy divides $100x + y$. This implies that x divides y. Write $y = tx$. The constraints on x and y imply that $1 \le t \le 9$. Note that $\frac{n}{xy} = \frac{100x+y}{xy} = \frac{100x+tx}{tx^2} = \frac{100+t}{tx}$. In particular, t must divide 100. Thus, $t \in \{1, 2, 4, 5\}$.

- If $t = 1$, then $y = x$, and $\frac{n}{xy} = \frac{101}{x}$, which is not an integer for $10 \le x \le 99$.
- If $t = 2$, then $y = 2x$, and $\frac{n}{xy} = \frac{102}{2x} = \frac{51}{x}$. Since $99 \ge y = 2x \ge 20$, we must have $x = 17$. This corresponds to $n = 1734$.
- If $t = 4$, then $y = 4x$, and $\frac{n}{xy} = \frac{104}{4x} = \frac{26}{x}$. The only suitable x is $x = 13$, corresponding to $n = 1352$.
- If $t = 5$, then $\frac{n}{xy} = \frac{105}{5x} = \frac{21}{x}$. None of the divisors of 21 is in the appropriate interval.

Hence, the only n's with the desired property are $n \in \{1352, 1734\}$.

6. We have 15 cards numbered $1, 2, \ldots, 15$. How many ways are there to choose some (at least one) cards so that all numbers on these cards are greater than or equal to the number of cards chosen?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Norvald Midttun, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

We consider the general problem for which there are n cards numbered $1, 2, \ldots, n$, respectively.

Let $f(n)$ denote the number of ways of choosing k (at least one) cards so that all numbers on these cards are at least k.

We prove that $f(n) = F_{n+2} - 1$ where F_n denotes the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Let the n cards be denoted by $c_1, c_2, \ldots, c_n$ where the number on c_i is i, $1 \leq i \leq n$. Suppose we choose only k cards, $k \geq 1$. Then clearly the given condition is satisfied if and only if all of these cards are chosen from $c_k, c_{k+1}, \ldots, c_n$. This is possible if and only if $k \leq n + 1 - k$ or $k \leq \lfloor \frac{n+1}{2} \rfloor$.

Hence the total number of possible selections is

$$f(n) = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} = \binom{n}{1} + \binom{n-1}{2} + \cdots + \binom{n+1-\lfloor \frac{n+1}{2} \rfloor}{\lfloor \frac{n+1}{2} \rfloor}.$$

Now we recall the following well known fact which can be found, for example, on p. 87 of *Basic Techniques of Combinatorial Theory* by Daniel I.A. Cohen and can be proved by induction:

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_{n+1}.$$

Replacing n by $n+1$, we then have

$$f(n) = \binom{n}{1} + \binom{n-1}{2} + \cdots = F_{n+2} - 1.$$

In particular, for the given problem, the answer is $f(15) = F_{17} - 1$. By straightforward computations, we find $F_{17} = 1597$ so $f(15) = 1596$.

7. In how many ways can 100 be written as a sum of nonnegative powers of 3?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the write-up by Curtis.

This is equivalent to finding the number of solutions in nonnegative integers to the equation

$$81x_4 + 27x_3 + 9x_2 + 3x_1 + x_0 = 100.$$

Once x_4, x_3, x_2, and x_1 are determined, x_0 is determined. We must have $x_4 \in \{0, 1\}$.

Case 1. If $x_4 = 1$, then $x_3 = 0$, and $x_2 \in \{0, 1, 2\}$.

(a) If $x_2 = 2$, then $x_1 = 0$, yielding only one solution.

(b) If $x_2 = 1$, then $x_1 \in \{0, 1, 2, 3\}$, yielding 4 solutions.

(c) If $x_2 = 0$, then $x_1 \in \{0, 1, 2, 3, 4, 5, 6\}$, yielding 7 solutions.

Case 2. If $x_4 = 0$, then $x_3 \in \{0, 1, 2, 3\}$.

(a) If $x_3 = 3$, then $x_2 \in \{0, 1, 2\}$, yielding 12 solutions, as in the total for case 1.

(b) If $x_3 = 2$, then $x_2 \in \{0, 1, 2, 3, 4, 5\}$, yielding (in reverse order) 1, 4, 7, 10, 13, and 16 solutions, for a total of 51.

(c) If $x_3 = 1$, then $x_2 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, yielding (in reverse order) 1, 4, 7, 10, 13, 16, 19, 22, and 25 solutions, for a total of 117.

(d) If $x_3 = 0$, then $x_2 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, yielding (in reverse order) 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, and 34 solutions, for a total of 210.

In case 1, we have 12 solutions; in case 2, we have 390, for a grand total of 402 solutions.

9. How many pairs of integers (a, b) satisfy $a^2b^2 = 4a^5 + b^3$?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Curtis, noting that the solution does not consider the possibility of negative values such as $(-1, 2)$ *and* $(2, -4)$.

Any odd prime dividing a divides b^3 and hence b; conversely, any odd prime dividing b divides $4a^5$ and hence a. Let p be an odd prime dividing a and b. Write $a = p^r u$ and $b = p^s v$, with $r, s \geq 1$, and $\gcd(p, u) = \gcd(p, v) = 1$. Then

$$p^{2r+2s}u^2v^2 = 4p^{5r}u^5 + p^{3s}v^3. \tag{1}$$

We claim that the largest exponent on p in equation (1) is $2r + 2s$. Indeed, the smallest two exponents on p in equation (1) must be the same, since otherwise, dividing by the smallest power of p would give two terms still divisible by p while the third term would not be.

- If $2r + 2s = 5r \leq 3s$, then $r = \frac{2}{3}s$, so that $\frac{10}{3}s \leq 3s$, a contradiction.
- If $2r + 2s = 3s \leq 5r$, then $s = 2r$, so that $6r \leq 5r$, a contradiction.

That proves the claim. Thus, $5r = 3s \leq 2r + 2s$, so that

$$p^m u^2 v^2 = 4u^5 + v^3,$$

where $m = (2r + 2s) - 3s = 2r - s = 2r - \frac{5}{3}r = \frac{1}{3}r$. Applying this argument to each of the odd primes dividing a and b, we obtain

$$\prod_{i=1}^{k} p_i^{m_i} \cdot 2^{2x+2y} = 2^{5x+2} + 2^{3y}.$$

By the same reasoning as before, the two smallest exponents on 2 must be equal.

Case 1. If $x = 0$, then

$$\prod_{i=1}^{x} p_i^{m_i} \cdot 2^{2y} = 4 + 2^{3y}.$$

If $y \geq 2$, then all of the exponents on 2 are different. Hence, $y \in \{0, 1\}$.

(a) If $y = 0$, then

$$\prod_{i=1}^{k} p_i^{m_i} = 5,$$

so that the only odd prime dividing a and b is 5. Also, $1 = m = m_1 = \frac{1}{3}r$. Therefore, $r = 3$ and $s = 5$ so that $a = 5^3$ and $b = 5^5$.

(b) If $y = 1$, then

$$\prod_{i=1}^{k} p_i^{m_i} \cdot 4 = 12$$

$$\prod_{i=1}^{k} p_i^{m_i} = 3,$$

so that the only odd prime dividing a and b is 3. We again obtain $r = 3$ and $s = 5$ so that $a = 3^3$ and $b = 2 \cdot 3^5$.

Case 2. If $x = 1$, then

$$\prod_{i=1}^{k} p_i^{m_i} \cdot 2^{2+2y} = 2^7 + 2^{3y},$$

and we must have $y = 2$. Thus,

$$\prod_{i=1}^{k} p_i^{m_i} = 3.$$

We obtain $a = 2 \cdot 3^3$ and $b = 2^2 \cdot 3^5$.

Case 3. If $x = 2$, then

$$\prod_{i=1}^{k} p_i^{m_i} \cdot 2^{4+2y} = 2^{12} + 2^{3y},$$

and $y = 4$, so that

$$\prod_{i=1}^{k} p_i^{m_i} = 2,$$

a contradiction.

Case 4. Now suppose $x > 2$.

(a) If $2x + 2y = 5x + 2 \leq 3y$, then $y = \frac{3}{2}x + 1$, so that $5x + 2 \leq \frac{9}{2}x + 3$, implying that $x \leq 2$, a contradiction.

(b) If $2x + 2y = 3y \leq 5x + 2$, then $y = 2x$, and $6x \leq 5x + 2$, again yielding the contradiction $x \leq 2$.

Hence $5x + 2 = 3y \leq 2x + 2y$. Thus, $y = \frac{5}{3}x + \frac{2}{3}$, so that $x \equiv 2 \bmod 3$. Write $x = 3z + 2$. Then $y = 5z + 4$, and

$$\begin{aligned} \prod_{i=1}^{k} p_i^{m_i} \cdot 2^{16z+12} &= 2^{15z+12} + 2^{15z+12} \\ \prod_{i=1}^{k} p_i^{m_i} \cdots 2^{z} &= 2. \end{aligned}$$

This implies that $z = 1$, and no odd primes divide a or b. Also, $x = 5$ and $y = 9$. Thus, $a = 2^5$ and $b = 2^9$.

In summary, $(a, b) \in \{(125, 3125), (27, 486), (54, 972), (32, 512)\}$.

10. A set of cards with positive integers on them is given, and the sum of these integers is 2007. For any integer $k = 1, 2, \ldots, 2006$, there is only one way to choose some of these cards so that the sum of the numbers on them is k. How many such sets of cards are there?

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

To be able to obtain a sum of 1, there must be at least one card with the number 1. One possibility is that there are 2007 cards, all of which are 1's. If there are exactly m cards with a 1, with $m < 2007$, then all of the sums from 1 through m and no others can be obtained with these m cards. Accordingly, the next highest card must be $m + 1$. Together with the m 1's, each sum less than or equal to $2m + 1$ can then be obtained. Therefore, the next card must be $2m + 2 = 2(m + 1)$. Continuing in this way, the subsequent cards must be $4(m + 1)$, $8(m + 1)$, $16(m + 1)$, and so on. The sum of the integers on the cards is then

$$m + (m + 1)(1 + 2 + 4 + \cdots + 2^n) = m + (m + 1)(2^{n+1} - 1).$$

If $m < 2007$, we must therefore have

$$\begin{aligned} -1 + (m+1)2^{n+1} &= 2007 \\ (m+1)2^{n+1} &= 2008 \\ (m+1)2^{n+1} &= 2^3 \cdot 251. \end{aligned}$$

Thus, $n \in \{0, 1, 2\}$.

- If $n = 0$, then $m = 1003$.
- If $n = 1$, then $m = 501$.
- If $n = 2$, then $m = 250$.

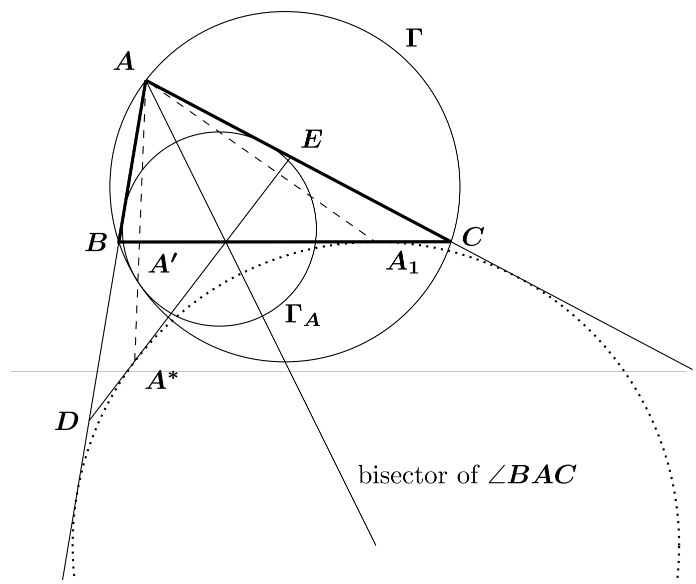
There are thus four suitable sets of cards:

- 2007 cards, all of which are 1's;
- 1003 cards which are 1's and a 1004;
- 501 cards which are 1's, a 502, and a 1004;
- 250 cards which are 1's, a 251, a 502, and a 1004.

Next we give readers' solutions to problems of the 17$^{\text{th}}$Japanese Mathematical Olympiad, Final Round, given at [2010: 378].

3. Let Γ be the circumcircle of triangle ABC. Let Γ_A be the circle tangent to AB, AC and tangent internally to Γ, and let Γ_B and Γ_C be defined similarly. Let Γ_A, Γ_B, Γ_C be tangent to Γ at A', B', C', respectively. Prove that the lines AA', BB', CC' are concurrent.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Ricardo Barroso Campos, University of Seville, Seville, Spain. We give the solution of Amengual Covas.



Let the excircle opposite $\boldsymbol{A}$ touch the side $\boldsymbol{BC}$ at $\boldsymbol{A_1}$, and let $\boldsymbol{B_1}$ and $\boldsymbol{C_1}$ be defined similarly.

Let $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{A^*}$ be the symmetric points of $\boldsymbol{C}$, $\boldsymbol{B}$ and $\boldsymbol{A_1}$ respectively with respect to the bisector of $\angle\boldsymbol{BAC}$.

Then

(a) the excircle opposite $\boldsymbol{A}$ touches $\boldsymbol{DE}$ at $\boldsymbol{A^*}$; and

(b) the lines $\boldsymbol{AA_1}$ and $\boldsymbol{AA^*}$ are isogonal at $\boldsymbol{A}$.

We invert the diagram in the circle with center at $\boldsymbol{A}$ and radius $\sqrt{\boldsymbol{AB\cdot AC}}$. The circumcircle $\boldsymbol{\Gamma}$ becomes the straight line through $\boldsymbol{D}$ and $\boldsymbol{E}$; the circle $\boldsymbol{\Gamma_A}$ inverts into the excircle opposite $\boldsymbol{A}$. Tangencies are preserved, so that $\boldsymbol{A'}$ and $\boldsymbol{A^*}$ are inverse points, implying that $\boldsymbol{A}$, $\boldsymbol{A'}$ and $\boldsymbol{A^*}$ are collinear. That is, $\boldsymbol{AA_1}$ and $\boldsymbol{AA'}$ are isogonal lines.

Analogously, $\boldsymbol{BB_1}$ and $\boldsymbol{BB'}$ are isogonal at $\boldsymbol{B}$, and $\boldsymbol{CC_1}$ and $\boldsymbol{CC'}$ are isogonal at $\boldsymbol{C}$.

Since $\boldsymbol{AA_1}$, $\boldsymbol{BB_1}$ and $\boldsymbol{CC_1}$ concur (at the Nagel point of $\triangle\boldsymbol{ABC}$), the isogonal lines $\boldsymbol{AA'}$, $\boldsymbol{BB'}$ and $\boldsymbol{CC'}$ also concur, as desired.

Next, we look at the readers' solutions to problems given in the November 2010 issue, the last issue that featured Olympiad problem sets and an invitation to submit solutions. We begin with the Croatian Mathematical Competition 2007, National Competition, given at [2010: 435–436].

3^{rd} Grade

1. Let n be a positive integer such that $n+1$ is divisible by 24.

(a) Prove that n has an even number of divisors (including 1 and n itself).

(b) Prove that the sum of all divisors of n is divisible by 24.

(Simplified from Putnam Competition 1969)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution and comment by Wang.

(a) We show that the conclusion actually holds under the weaker assumption that $n+1$ is divisible by 4. Clearly $n \neq 1$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be the prime powers decomposition of n where α_i's are natural numbers and p_i's are primes, $i = 1, 2, \dots, k$. Let $\tau(n)$ denote the number of (positive) divisors of n. It is well known that $\tau(n) = \prod_{i=1}^{k}(1+\alpha_i)$. Suppose $\tau(n)$ is odd. Then each $1+\alpha_i$ is odd so α_i must be even for all $i = 1, 2, \dots, k$. Thus, n is a perfect square. Let $n = m^2$ where m is a natural number. Clearly n is odd so m is also odd. Hence $m^2 \equiv 1 \pmod 4$. Therefore, $n+1 = m^2+1 \equiv 2 \pmod 4$ contradicting the fact that $n+1 \equiv 0 \pmod 4$.

(b) This is exactly the same as problem B1 of the 1969 Putnam Math Competition. A full solution can be found on p. 63 of *The William Lowell Putnam Math Competition, Problems and Solutions*; edited by Gerald L. Alexanderson et al. The proof given there actually establishes the stronger fact that $d + (n/d)$ is divisible by 24 for all divisors d of n. The conclusion then follows immediately from part (a).

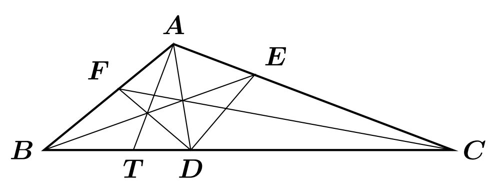
2. In the triangle ABC, with $\angle BAC = 120°$, the bisectors of the angles $\angle BAC$, $\angle ABC$ and $\angle BCA$ intersect the opposite sides in the points D, E, and F, respectively. Prove that the circle with diameter $\overline{EF}$ passes through D. (British Mathematical Olympiad 2005)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Cománeşti, Romania. We give the solutions of Zvonaru.

First Solution.

The bisector of $\angle BAD$ meets the side BD at the point T. Since

$$\begin{aligned} \angle TAC &= \angle TAD + \angle DAC \\ &= 30° + 60° = 90°, \end{aligned}$$



we deduce that AC is the external bisector of $\angle BAD$. Because BE is the bisector of $\angle ABD$, it follows that DE is an external bisector of $\angle ADB$, that is DE is a bisector of $\angle ADC$.

Similarly we obtain that DF is a bisector of $\angle ADB$; it follows that $\angle EDF = \frac{1}{2}(\angle ADC + \angle ADB) = 90°$, hence the circle with diameter EF passes through D.

Second Solution. As usual we write $a = BC$, $b = CA$, $c = AB$. By the Bisector's Theorem we obtain $DC = \frac{ab}{b+c}$.

It is known that the bisector $AD = \frac{2bc\cos\frac{A}{2}}{b+c} = \frac{bc}{b+c}$.

We have, again using the Bisector's Theorem

$$\frac{AD}{DC} = \frac{c}{a} = \frac{AE}{EC}$$

and, by the converse of the Bisector's Theorem, it follows that DE is the bisector of $\angle ADC$. Similarly, DF is the bisector of $\angle ADB$, hence $\angle EDF = 90°$.

3. In triangle ABC the distances of vertex A from the centre of the circumscribed circle and the orthocentre are equal. Determine the angle $\alpha = \angle BAC$. (USA proposal for IMO 1989)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution by Curtis.

Let $a = BC$, $b = CA$, $c = AB$, and let O and H be the circumcenter and orthocenter, respectively. Let C' be the point at which line HC intersects side AB. By the extended Law of Sines, the circumradius is given by

$$AO = \frac{b}{2\sin B}.$$

On the other hand, $AC' = \pm b\cos A$, so that $AH = \pm\frac{b\cos A}{\sin B}$. Thus,

$$\frac{b}{2\sin B} = \pm\frac{b\cos A}{\sin B}.$$

Thus, $\cos A = \pm\frac{1}{2}$, so that $\alpha = 60°, 120°$.

Next we give the solution of Geupel.

Denote the circumcentre and the orthocentre by O and H, respectively. We prove that the condition $AO = AH$ is equivalent to $\alpha = 60°, 120°$.

Let a, b, c, h, and o be the coordinates of A, B, C, H, and O in the plane of complex numbers. Without loss of generality let $o = 0$, $c = 1$, and $b = e^{i\vartheta}$ where $0 < \vartheta \leq \pi$.

We have $h = a+b+c = a+b+1$. The condition $AO = AH$ is therefore successively equivalent to

$$|a| = |h - a|,$$

$$1 = |b+1|,$$
$$1 = \left(1+e^{i\vartheta}\right)\left(1+e^{-i\vartheta}\right),$$
$$\cos\vartheta = -\frac{1}{2},$$
$$\angle BOC = 120^\circ, 240^\circ,$$

and finally,

$$\alpha = \frac{1}{2}\angle BOC = 60^\circ, 120^\circ.$$

The proof is complete.

4th Grade

3. In a $5 \times n$ table, where n is a positive integer, each 1×1 cell is painted either in red or in blue. Find the smallest possible n such that, for any painting of the table, one can always choose three rows and three columns for which the 9 cells in their intersection have all the same colour.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Curtis.

Each column has at least three reds or at least three blues. For $k = 1, 2, \ldots, n$, form the quadruple

$$(y_1(k), y_2(k), y_3(k), c(k)),$$

where the $y_i(k)$ are positive integers such that

$$1 \leq y_1(k) < y_2(k) < y_3(k) \leq 5, \tag{1}$$

And the cells $(k, y_1(k))$, $(k, y_2(k))$, $(k, y_3(k))$ each have colour $c(k)$. There are at most $\binom{5}{3} \cdot 2 = 20$ such quadruples. Thus, with $n = 41$, at least three columns must have the same quadruple. Choose three such columns with their corresponding y-values.

To see that no smaller n will work, assume $n \leq 40$, and, in each column, colour either exactly three cells red or exactly three cells blue in such a way that each of the possible quadruples described above appears at most twice. Then no selection of three rows and three columns provides an intersection with all cells of the same colour.

That completes the *Corner* for this number.

BOOK REVIEWS

Amar Sodhi

Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks
by Persi Diaconis and Ron Graham, with a foreword by Martin Gardner
Princeton University Press, 2012
ISBN 978-0-691-15164-9, 244 + xii pp., hardcover, US$29.95
Reviewed by ***S. Swaminathan****, Dalhousie University, Halifax, N. S.*

Magic tricks are fascinating to young and old alike. Many simple magic tricks that work by themselves are based on mathematical principles. Consider the following trick: the magician hands four playing cards to a spectator, turns his back and gives instructions to the spectator to perform actions on the cards in different ways such as turning a card face down, transferring cards from top to bottom either singly or collectively, and doing these actions any number of times in any order. When the spectator has finished performing the actions, he (she) is requested to perform one final set of transfers and then asked to mention how many cards are facing the opposite way from the others. The answer turns out to be only one card that faces the opposite way. The magician tells correctly what that particular card is! This amazing trick is explained fully with rich illustrations in the first chapter of the wonderful book under review.

The authors are eminent mathematicians; Ron Graham of Bell Labs and UC, San Diego is an expert on combinatorial mathematics, and Persi Diaconis is a professor of statistics at Stanford University. Both of them are skilled performers of magic; Ron is a juggler and Persi is a skilled card magician. They have been mixing entertainment with mathematics for most of their lives besides teaching, publishing papers with deep results and conjecturing new results.

The book consists of ten chapters with the following titles: Mathematics in the Air, In Cycles, Is This Stuff Actually Good for Anything?, Universal Cycles, From Gilbreath Principle to the Mandelbrot Set, Neat Shuffles, The Oldest Mathematical Entertainment?, Magic in the Book of Changes, What Goes Up must Come Down, Stars of Mathematical Magic (and Some of the Best Tricks in the Book), Going Further, On Secrets, Notes, Index.

Easy, step by step instructions are provided for each trick, explaining clearly how the effect is set up and offering tips on what to say and do during the performance. Each card trick introduces a new mathematical idea, and varying the tricks takes the readers to the very threshold of current mathematical knowledge. However, sophisticated math terminology is avoided. For example, the underlying theme of the trick mentioned in the first paragraph of this review is that a large permutation group ('group' in the mathematical sense) leaves an interesting set of invariants under the group fixed while the cards are getting mixed, which contributes to the startling final effect. Other card tricks link to mathematical secrets of combinatorics, graph theory, number theory, topology, the Riemann

Hypothesis, and even Fermat's Last Theorem. The book contains a wealth of conjuring lore, including some closely guarded secrets of legendary magicians. It discusses the mathematics of juggling, and shows how I Ching, an ancient Chinese fortune-telling book, connects to the history of probability and tricks both old and new. Stories underlying some tricks of eccentric and brilliant inventors of mathematical magic are discussed. Copious colourful illustrations and pictures are provided to illustrate the text.

Readers are sure to enjoy this brilliant book. Their interest in magic will get kindled if it is not already there. They will get introduced to little-known mathematical theorems. The book will certainly become a classic.

Unsolved Crux Problem

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux* [2010 : 545, 547]. Below is a sample of one of these unsolved problems:

154. [1976 : 110, 159, 197, 225-226; 1977 : 20-22, 108-109, 191-193] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, ON.*

Let p_n denote the n^{th} prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given p_1, p_2, $\ldots$, p_n.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \cdots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \cdots p_n^{a_{n-1}}$ in a second row. Let ℓ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

Example. Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 9 & 7 & 15 & 11 & 27 & 25 & 21 & 13 & 45 \end{array}$$

We observe that $\ell = 17 = p_7$.

RECURRING CRUX CONFIGURATIONS 3

J. Chris Fisher

Triangles whose angles satisfy $2B = C + A$

Because the angles of a triangle ABC sum to $180°$, they are in arithmetic progression if and only if the intermediate angle measures $60°$. I found nearly two dozen problems in **CRUX with MAYHEM** that deal with triangles having a $60°$ angle; there was considerable overlap, so for this month's column instead of listing those problems, I will simply list and discuss the properties that readers have discovered, and provide references to where proofs can be found. As usual, A, B, and C will represent either the vertices of a triangle or the measure of its angles, depending on the context; a, b, and c represent either the opposite sides or their lengths; $s = (a + b + c)/2$ is the semiperimeter, while H, I, O, and G are the orthocentre, incentre, circumcentre, and centroid, respectively. We shall use $2B = C + A$ and $B = 60°$ interchangeably.

The first eight properties came, in part, from Problem 724 [1982 : 78; 1983 : 92-94] (proposed by Hayo Ahlburg) and the comments found there.

Property 1. $\angle B = 60°$ if and only if $\sin(A - B) = \sin A - \sin C$.

The simple proof of only if is on page 92; for the if part note that $\sin C = \sin(A + B)$ and expand that and $\sin(A - B)$ to get an equation that reduces to $\cos B = \frac{1}{2}$.

Property 2. $\angle B = 60°$ if and only if $a^2 - b^2 = c(a - c)$.

Property 2 [1983 : 93] is just the cosine law. It forms the basis of an olympiad problem proposal of Murray Klamkin that was never used: *One of the angles of a triangle is $60°$ if and only if the square of the side opposite that angle equals the sum of the cubes of the sides divided by the perimeter*; that is,

$$\angle B = 60° \text{ if and only if } b^2 = \frac{a^3 + b^3 + c^3}{a + b + c}.$$

A more substantial use of Property 2 came in [3] to obtain a characterization of integer-sided triangles having an angle of $60°$: Let p and q be (i) relatively prime integers with (ii) one of them odd, the other even, and (iii) p not a multiple of 3; use p and q to define the integers $x = |p^2 - 3q^2|$ and $y = 2pq$. If $x > y$ we set $b = p^2 + 3q^2, a = x + y$ and $c = 2y, x - y$ (there are two values of c for each a and b because the quadratic equation of Property 2 will have two integer zeros); if $x < y$ we use the same b but set $a = 2y$ and $c = y \pm x$. Then in either case with either value of c, ABC is a triangle with $\angle B = 60°$ and side lengths $a > b > c$ that are relatively prime integers; conversely, for any such triangle there exists a pair of integers p and q that produce that triangle using the given recipe. For example, $p = 1, q = 2$ determines $x = 11, y = 4$ and triangles with sides 15, 13, 8 as well as 15, 13, 7; for $p = 2, q = 1$, the parameters are $x = 1, y = 4$, and the resulting triangles have sides 8,7, 5, and 8, 7, 3. Similar results and further references can be found in [4].

Property 3. If $\angle B = 60°$ then the points A, C, O, I, H lie on a circle that also contains the excentre I_b opposite vertex B; its radius equals the circumradius of ΔABC and its centre O' is where the angle bisector BI again meets the circumcircle.

The solution to Problem M1046 from the 1987 U.S.S.R. journal *Kvant* [1988 : 165; 1990 : 103] follows easily: *If* $\angle B = 60°$ *then one of the bisectors of the angle between the altitudes from* A *and* C *passes through* O. One should take care with the converse of Property 3, which is the topic of problem 1521 [1990 : 74; 1991 : 126-127] (proposed by J.T. Groenman). If either A, I, O, C or A, H, I, C are concyclic, then it follows that $\angle B = 60°$ and all five points are concyclic. Such is not the case, however, with A, H, O, C concyclic because then $\angle B = 120°$ is also possible (which can be seen by changing the roles of the points B with H and O with O'). This property is clearly equivalent to the following result, which is the subject of Problem 998 [1984 : 319; 1986 : 65] (proposed by Andrew P. Guinand):

If one angle of a triangle is either $60°$ or $120°$, then the image of the orthocentre under inversion with respect to the circumcircle lies on the side (possibly extended) opposite that angle.

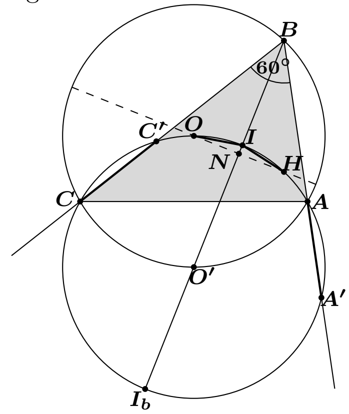


Figure 1: The angles of ΔABC satisfy $2B = C + A$.

Property 4. The circle of Property 3, with radius R and centre O', intersects the lines BA and BC at points A' and C' for which $AA' = CC' = |c - a|$. (Proof is on page [1983 : 93].)

Property 5. In any triangle, if N is the centre of its nine-point circle (and, therefore, the midpoint of OH), and P is the projection of the incentre I onto the Euler line $OGNH$, then P lies between G and H; furthermore, $P = N$ if and only if one angle of the triangle has measure $60°$.

This is Problem 260 [1977 : 155; 1978 : 58-60] (Proposed by W.J. Blundon). A variant of this property became Problem 5 on the 2007 Indian Team Selection

Test [2010 : 278; 2011 : 370-371]: *For triangles that are not equilateral, the common tangent to the incircle and the nine-point circle is parallel to the Euler line if and only if the angles of the triangle are in arithmetic progression.* This, of course, is because in all triangles that common tangent is perpendicular to IN.

Property 6. If $\angle B = 60°$ then N lies on the bisector of that angle; conversely, if the nine-point centre of a triangle lies on the interior bisector of $\angle CBA$, then the vertex B lies on the perpendicular bisector of AC or $\angle B = 60°$.

This is Problem 2855 [2003 : 316; 2004 : 308-309] (Proposed by Antreas P. Hatzipolakis and Paul Yiu); the claim that BN bisects $\angle B$ when $\angle B = 60°$ is also proved as a part of Problem 724 [1983 : 94].

Property 7. If B is the intermediate angle of ΔABC, then $\angle B = 60°$ if and only if $OI = IH$, if and only if $OI_b = I_bH$.

The equivalence of $\angle B = 60°$ and $OI = IH$ is proved as part of Problem 260 (Property 5 above). It was proved yet again as part of Problem 1521 (see Property 3 above). This result also appeared as Problem 739 [1982 : 107; 1983 : 153-154, 210-211] (proposed by G.C. Giri), where there is another proof and references to textbooks where it appears as an exercise. There is also a reference to a stronger result [2]:
If the angles of triangle ABC are labeled so that $A \leq B \leq C$ then

$$\begin{aligned}
\angle B > 60° &\Rightarrow 0 < \frac{HI}{IO} < 1, \\
\angle B = 60° &\Rightarrow HI = IO, \\
\angle B < 60° &\Rightarrow 1 < \frac{HI}{IO} < 2.
\end{aligned}$$

The proof of the result for $OI_b = I_bH$ can be found in [1].

Property 8. $\angle B = 60°$ if and only if $s = \sqrt{3}(R + r)$.

The proof is another part of the solution to Problem 260 [1978 : 58-60].

Property 9. $\angle B = 60°$ if and only if the bisector BO' of $\angle B$ is perpendicular to the Euler line OH; when these properties hold, then N is the common midpoint of BO' and OH. (Recall that O' was defined in Property 3 to be where the angle bisector again intersects the circumcircle. Note that in any triangle, BH and OO' are both perpendicular to AC.)

The claims follow from the proof in Problem 1521 (see Property 3). Three more proofs can be found in [5]. An immediate consequence is Problem 3 of Round 2 of the 2006-2007 British Mathematical Olympiad [2010: 154; 2011: 165]:
If the Euler line OH meets BA at P and BC at Q, then $\angle B = 60°$ implies that $OQ = HP$.

See [2011: 165] for an independent proof (although the problem and proof found there were unnecessarily restricted to acute-angled triangles). Another immediate consequence of Problem 1521 is Problem 1673 [1991 : 237; 1992 : 218-219] (proposed by D.J. Smeenk):

Given ΔABC *let* P *be an arbitrary point of the line* BA *and* Q *be on* BC*, neither point coinciding with a vertex. If* $\angle B = 60°$ *then the Euler lines of* ΔABC *and* ΔPBQ *are parallel; moreover, if the two Euler lines coincide then the circumcircle* PQR *contains* O'*.*

Jordi Dou added a proof that the sides PQ of those triangles whose Euler lines coincide with that of the given triangle are tangent to the parabola with focus O' and directrix OH.

Property 10. $\angle B = 60°$ or $120°$ if and only if $BH = BO$.

This is case (i) of Problem 1518 [1990 : 44; 1991 : 122] (proposed by K.R.S. Sastry). Compare Property 9; Problem 1232(b) in [5] says that $BO' || OH$ if and only if $\angle B = 120°$.

Property 11. $\angle B = 60°$ or $120°$ if and only if its internal bisector divides an altitude in the ratio $1 : 2$.

This is Problem 2526 [2000 : 177; 2001 : 271-273] (proposed by K.R.S. Sastry).

We devote the remainder of this compilation to properties that were discovered over the past 20 years by Toshio Seimiya. What a pity that we failed to invite him to write this article for us! For most of these properties we denote by D and E the points where the interior bisectors of angles A and C meet the opposite sides. Properties 12 through 16 are quite closely related.

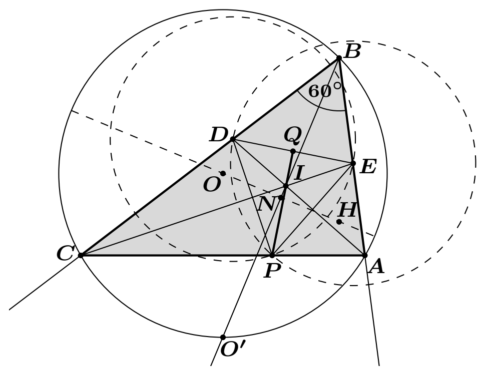


Figure 2: Seimiya's properties 12, 13, 14, and 15.

Property 12. If $\angle B = 60°$ then the points D and E are two vertices of an equilateral triangle whose third vertex lies on AC and whose incentre is I.

In other words, the bisectors of angles A and C meet the opposite sides at the centres of two circles with common radius DE that intersect on AC. This is Seimiya's counterexample to the incorrect claim made by the proposer of Problem 1446(c) [1989 : 148; 1990 : 217-219] (namely, that the existence of this inscribed equilateral triangle implied that the original triangle ABC was necessarily equilateral).

Property 13. Define P to be the point where the line perpendicular to DE

meets AC, and Q to be where it meets DE. Then $IP = 2IQ$ if and only if $\angle B = 60°$. (Problem 2011 [1995 : 52; 1996 : 80])

Property 14. Define P to be the point where the bisector of $\angle AIC$ meets AC, and Q to be where it meets DE. Then $IP = 2IQ$ if and only if $\angle B = 60°$. (Problem 2939 [2004 : 229, 232; 2005 : 243-244])

Property 15. $\angle ADE = 30°$ if and only if $\angle B = 60°$ or $\angle C = 120°$. (Problem 2263 [1997 : 364; 1998 : 432-433])

Property 16. Call F the point where DE intersects AC. If ΔABC has $BC > BA$ and $\angle DFC = \frac{1}{2}(\angle DAC - \angle ECA)$, then $\angle B = 60°$. (Problem 2314 [1998 : 107; 1999 : 117])

Property 17. Given ΔABC define P and Q to be points on the same side of AC as B, with P the point on BC for which $PC = BA$, and Q the point on BA for which $QA = BC$. Then $\angle B = 60°$ if and only if O lies on PQ. (Problem 1692 [1991 : 301; 1992 : 284-285])

Property 18. An acute-angled triangle ABC is given, and equilateral triangles ABP and BCQ are drawn outwardly on the sides AB and BC. Suppose that AQ and CP meet BC and AB at R and T, respectively, and that AQ and CP intersect at S. If the area of the quadrilateral $BRST$ is equal to the area of the triangle ASC, then $\angle B = 60°$. (Problem 2304 [1998 : 46; 1999 : 56-57])

In addition to his many problems, Semiya also wrote an article for **CRUX with MAYHEM** entitled "On Some Examples of Geometric Fallacies" [**29**:6 (October 2003) 393-396]. He began with a theorem and proposed two attempted converses, both of which came with very convincing arguments; he then pointed out the subtle but critical errors in the arguments, and provided counterexamples to show that those converses were indeed false. It is the theorem that is relevant here:

Theorem. Let ABC be a triangle with $\angle B = 60°$. Let D be the point on BC produced beyond C such that $CD = CA$, and let E be the point on BA produced beyond A such that $CA = AE$. Then $\angle DCA = 2\angle AED$, $\angle CAE = 2\angle EDC$, and $\angle EDA = 30°$.

References

[1] M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*, Dover, 1968, p.185.

[2] Anders Bager, Solution to Problem E2282 (proposed by W.J. Blundon), *American Mathematical Monthly*, **47**:79 (1972) 397-398.

[3] Bob Burn, Triangles with a $60°$ angle and Sides of Integer Length. *Mathematical Gazette*, **87**:508 (Mar. 2003) 148-153.

[4] Russell A. Gordon, Properties of Eisenstein Triples *Mathematics Magazine*, **85**:1 (Feb. 2012) 12-25.

[5] J.T. Groenman and D.J. Smeenk, Problem 1232(a), *Mathematics Magazine*, **60**:1 (Feb. 1987) 43-45. (Appeared in Feb. 1986.)

PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 November 2012****.*
An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3676. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle with semiperimeter s, inradius r and circumradius R. Let r' and R' be the inradius and circumradius of a triangle with sides $\sqrt{a(s-a)}$, $\sqrt{b(s-b)}$, and $\sqrt{c(s-c)}$. Prove that

$$Rr' \;\geq\; R'r\,.$$

3677. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n-1}(-1)^k \sin^n(k\pi/n) = \frac{(1+(-1)^n)n}{2^n}\cdot\cos\frac{n\pi}{2}\,.$$

3678. *Proposed by Michel Bataille, Rouen, France.*

Let Γ_1, Γ_2 be two intersecting circles and U one of their common points. Show that there exists infinitely many pairs of lines passing through U and meeting Γ_1 and Γ_2 in four concyclic points. Give a construction of such pairs.

3679. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$(a^2b+c)(b^2c+a)(c^2a+b) \;\leq\; 4(ab+bc+ca-abc)\,.$$

3680. *Proposed by Michel Bataille, Rouen, France.*

In a system of axes (Ox, Oy, Oz), let $U(1,1,1)$, $S(a,b,c)$ and $H(h_a,h_b,h_c)$ where a,b,c are the sides of a triangle ABC and h_a,h_b,h_c are the corresponding altitudes. Given that the lines OU and SH intersect at M such that $|HM| = \frac{1}{3}|HS|$, find the angles of ΔABC.

3681. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let D, E, and F be the points where the incircle of ΔABC touches the sides. Let Z be the Gergonne point (where AD, BE, and CF concur), and let M be the midpoint of BC. Define T to be the tangency point of the incircle with the circle through B and C that is tangent to it, and let the common tangent line at that point intersect AC at S. Prove that AB, SZ, and ME are concurrent.

3682. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 1$. Prove that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-cd}+\frac{1}{1-da}+\frac{1}{1-bd}+\frac{1}{1-ac} \leq 8\,.$$

3683. *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 2$ and z a complex number with $|z| \leq 1$. Prove that

$$\sum_{k=1}^{n} kz^{n-k} \neq 0\,.$$

3684. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given two circles that are internally tangent at T, let the chord BC of the outer circle be tangent to the inner circle at D. Let the second tangents from B and C touch the inner circle at F and E respectively, and define $J = EF \cap DT$ and $Z = BE \cap CF$. Prove that

(a) JZ intersects BC at its midpoint, and

(b) TD bisects $\angle BTC$.

Comment. This result allows for a solution to a special case of the Problem of Apollonius: Construct a circle through two given points that is tangent to a given circle which, itself, is tangent to the line joining the given points.

3685. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \to (0,\infty)$ be a bounded function which is continuous at 0. Find the value of

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f\left(\frac{1}{1}\right)}+\sqrt[n]{f\left(\frac{1}{2}\right)}+\cdots+\sqrt[n]{f\left(\frac{1}{n}\right)}}{n}\right)^{n}.$$

3686. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be real numbers such that $abc = 1$. Show that

$$\left(a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c}\right)^2 \;\leq\; 2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right).$$

3687. *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\left(k + 1 - \frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

where k^n is taken to be 1 for $k = n = 0$ and $S(n,k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1.$$

. . . .

3676. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c les côtés d'un triangle de demi-périmètre s et dont les cercles inscrit et circonscrit sont de rayons respectifs r et R. Soit d'autre part les rayons r' et R' des cercles inscrit et circonscrit d'un triangle sont les côtés sont $\sqrt{a(s-a)}$, $\sqrt{b(s-b)}$et $\sqrt{c(s-c)}$. Montrer que

$$Rr' \;\geq\; R'r.$$

3677. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit n un entier positif. Montrer que

$$\sum_{k=1}^{n-1} (-1)^k \sin^n(k\pi/n) = \frac{(1 + (-1)^n)n}{2^n} \cdot \cos\frac{n\pi}{2}.$$

3678. *Proposé par Michel Bataille, Rouen, France.*

Soit U un des deux points d'intersection de deux cercles Γ_1 et Γ_2. Montrer qu'il existe une infinité de paires de droites passant par U et coupant Γ_1 et Γ_2 en quatre points cocycliques. Donner une construction de telles paires.

3679. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a + b + c = 3$. Montrer que

$$(a^2b + c)(b^2c + a)(c^2a + b) \;\leq\; 4(ab + bc + ca - abc).$$

3680. *Proposé par Michel Bataille, Rouen, France.*

Dans un système d'axes (Ox, Oy, Oz), soit $U(1,1,1)$, $S(a,b,c)$ et $H(h_a, h_b, h_c)$ où a, b, c sont les côtés d'un triangle ABC et h_a, h_b, h_c en sont les hauteurs correspondantes. En supposant que les droites OU et SH se coupent en M de telle sorte que $|HM| = \frac{1}{3}|HS|$, trouver les angles de ΔABC.

3681. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

Soit D, E et F les points de contact du cercle inscrit du triangle ABC avec ses côtés. Soit Z le point de Gergonne (intersection de AD, BE et CF), et soit M le point milieu de BC. Notons T le point de tangence du cercle inscrit avec le cercle qui lui est tangent et qui passe par B et C, et soit S le point d'intersection de la tangente commune en T avec AC. Montrer que AB, SZ et ME sont concourantes.

3682. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit a, b, c et d quatre nombres réels non négatifs tels que $a^2+b^2+c^2+d^2 = 1$. Montrer que

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-cd} + \frac{1}{1-da} + \frac{1}{1-bd} + \frac{1}{1-ac} \le 8\,.$$

3683. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier avec $n \ge 2$ et z un nombre complexe tel que $|z| \le 1$. Montrer que

$$\sum_{k=1}^{n} k z^{n-k} \neq 0\,.$$

3684. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

On donne deux cercles intérieurement tangents en T et une corde BC du cercle extérieur tangente au cercle intérieur en D. Soit respectivement F et E les points de contact des secondes tangentes issues de B et C avec le cercle intérieur, et soit $J = EF \cap DT$ et $Z = BE \cap CF$. Montrer que

(a) JZ coupe BC en son point milieu, et

(b) TD est la bissectrice de $\angle BTC$.

Commentaire. Ce résultat permet de résoudre un cas spécial du problème d'Apollonius : Construire un cercle passant par deux points donnés qui soit tangent à un cercle donné qui, lui, est tangent à la droite joignant les deux points.

3685. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $f : [0,1] \to (0,\infty)$ une fonction bornée, continue en 0. Trouver la valeur de

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f\left(\frac{1}{1}\right)}+\sqrt[n]{f\left(\frac{1}{2}\right)}+\cdots+\sqrt[n]{f\left(\frac{1}{n}\right)}}{n}\right)^n .$$

3686. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c trois nombres réels tels que $abc = 1$. Montrer que

$$\left(a-\frac{1}{a}+b-\frac{1}{b}+c-\frac{1}{c}\right)^2 \leq 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) .$$

3687. *Proposé par Albert Stadler, Herrliberg, Suisse.*

Soit n un entier non négatif. Montrer que

$$\sum_{k=0}^{\infty}\frac{k^n}{k!}\left(k+1-\frac{1}{k!}\int_1^{\infty}e^{-t}t^{k+1}dt\right)=\sum_{k=0}^{n}\frac{S(n,k)}{k+2},$$

où l'on pose $k^n = 1$ pour $k = n = 0$ et où $S(n,k)$ sont les nombres de Striling du second ordre, définis par la récursion

$$S(n,m)=S(n-1,m-1)+mS(n-1,m), S(n,0)=\delta_{0,n}, S(n,n)=1 .$$

SOLUTIONS

A filing error took place and as a result a few solvers were not acknowledged in the past few issues. The editors would like to recognize the following correct solutions: George Apostolopoulos, Messolonghi, Greece (**3754**); Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (**3566**, **3570**, **3572**, **3574**); Michel Bataille, Rouen, France (**3566**, **3570**, **3572**); John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA (**3558**); Oliver Geupel, Brühl, NRW, Germany (**3564**); Dragoljub Milošević, Gornji Milanovac, Serbia (**3457**); Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (**3564**); Henry Ricardo, Tappan, NY, USA (**3558**); Joel Schlosberg, Bayside, NY, USA (**3556**, **3563**, **3566**); Digby Smith, Mount Royal University, Calgary, AB (**3571**); and Titu Zvonaru, Cománeşti, Romania (**3572**). If any other errors or omissions occur, please send an email to `crux-editors@cms.math.ca`.

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3568. [2010 : 396, 398] *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer and let a_k be the coefficient of z^k in the McLaurin expansion of $(z-1)^n\ln(1-z)$. Prove that

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad \text{and} \quad a_k = \frac{-1}{(n+1)\binom{k}{n+1}}\,,\ \ k>n\,.$$

I. Solution by George Apostolopoulos, Messolonghi, Greece.

It is well known that $(z-1)^n = \displaystyle\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}z^i$ and $\ln(1-z) = -\displaystyle\sum_{i=1}^{\infty}\frac{z^i}{i}$, thus

$$(z-1)^n\ln(1-z) = \left(\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}z^i\right)\left(-\sum_{i=1}^{\infty}\frac{z^i}{i}\right). \qquad (1)$$

From (1) we deduce that

$$a_n = \sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i}\left(-\frac{1}{n-i}\right) = \sum_{j=1}^{n}(-1)^{j+1}\frac{\binom{n}{j}}{j}.$$

Let $g(n) = a_n = \sum_{i=1}^{n}(-1)^{i+1}\frac{\binom{n}{i}}{i}$. Recall that $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$ so

$$g(n) = \sum_{i=1}^{n}(-1)^{i+1}\frac{\binom{n-1}{i}}{i} + \sum_{i=1}^{n}(-1)^{i+1}\frac{\binom{n-1}{i-1}}{i}. \tag{2}$$

Now, since $\binom{n}{i} = 0$ for $i > n$ or $i < 0$, then

$$\sum_{i=1}^{n}(-1)^{i+1}\frac{\binom{n-1}{i}}{i} = \sum_{i=1}^{n-1}(-1)^{i+1}\frac{\binom{n-1}{i}}{i} = g(n-1). \tag{3}$$

It is easy to verify that $\binom{n-1}{i-1}\cdot\frac{1}{i} = \binom{n}{i}\cdot\frac{1}{n}$, hence

$$\begin{aligned}\sum_{i=1}^{n}(-1)^{i+1}\frac{\binom{n-1}{i-1}}{i} &= \sum_{i=1}^{n}(-1)^{i+1}\frac{\binom{n}{i}}{n} = -\frac{1}{n}\sum_{i=1}^{n}(-1)^{i}\binom{n}{i}\\ &= -\frac{1}{n}\left[(1-1)^n - (-1)^0\binom{n}{0}\right] = \frac{1}{n}.\end{aligned} \tag{4}$$

Thus (2), (3) and (4) yield $g(n) = g(n-1) + \frac{1}{n}$. Since $g(1) = (-1)^2\frac{\binom{1}{1}}{1} = 1$, we deduce that $g(n) = \sum_{i=1}^{n}\frac{1}{i}$, that is, $a_n = \sum_{i=1}^{n}\frac{1}{i}$.

From (1) we deduce that for $k > n$

$$a_k = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\left(-\frac{1}{k-i}\right).$$

Let $f(n,k) = \sum_{i=0}^{n}(-1)^{n-i}\frac{\binom{n}{i}}{k-i}$, then

$$\begin{aligned}f(n,k) &= \sum_{i=0}^{n}(-1)^{n-i}\frac{\binom{n-1}{i}}{k-i} + \sum_{i=0}^{n}(-1)^{n-i}\frac{\binom{n-1}{i-1}}{k-i}\\ &= -\sum_{i=0}^{n-1}(-1)^{n-1-i}\frac{\binom{n-1}{i}}{k-i} + \sum_{j=0}^{n-1}(-1)^{n-1-j}\frac{\binom{n-1}{j}}{k-1-j}\\ &= f(n-1,k-1) - f(n-1,k).\end{aligned}$$

Now $f(0,k) = \frac{1}{k} = \frac{1}{(0+1)\binom{k}{0+1}}$ for all positive integer values of k, and if $f(n-1,k) = \frac{1}{n\binom{k}{n}}$ holds for all $k > n-1$, then $f(n,k) = f(n-1,k-1)$ $-f(n-1,k)$ for all $k-1 > n-1$, i.e. $k > n$. Thus

$$f(n,k) = \frac{1}{n\binom{k-1}{n}} - \frac{1}{n\binom{k}{n}} = \frac{1}{(n+1)\binom{k}{n+1}}$$

for $k > n$ so $a_k = \dfrac{-1}{(n+1)\binom{k}{n+1}}$ for $k > n$.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

Considering $(z-1)^n \ln(1-z) = \left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}z^j\right)\left(-\sum_{j=1}^{\infty}\frac{1}{j}z^j\right)$, we have to prove that

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j}\binom{n}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{1}$$

and

$$\sum_{j=0}^{n}\frac{(-1)^j}{k-n+j}\binom{n}{j} = \frac{1}{(n+1)\binom{k}{n+1}}\,, \quad k > n. \tag{2}$$

For a proof of identity (1) we refer to [1]. We only prove (2).

We have $x^{k-n-1}\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j = x^{k-n-1}(1-x)^n$; hence

$$\sum_{j=0}^{n}\frac{(-1)^j}{k-n+j}\binom{n}{j} = \int_0^1 x^{k-n-1}(1-x)^n dx.$$

Integration by parts for nonnegative m, n yields

$$\begin{aligned}\int_0^1 x^m(1-x)^n dx &= \left[\frac{(1-x)^n x^{m+1}}{m+1}\right]_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx \\ &= \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx.\end{aligned}$$

By repeated application of this formula, we obtain

$$\begin{aligned}\int_0^1 x^{k-n-1}(1-x)^n dx &= \frac{n(n-1)\cdots 1}{(k-n)(k-n+1)\cdots(k-1)}\int_0^1 x^{k-1}dx \\ &= \frac{n(n-1)\cdots 1}{(k-n)(k-n+1)\cdots(k-1)}\cdot\frac{1}{k} \\ &= \frac{1}{(n+1)\binom{k}{n+1}},\end{aligned}$$

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

References

[1] Loren C. Larson, *Problem solving through problems*, Springer, second printing, 1990, example 5.1.4, p.160.

3573. [2010 : 397, 399] *Proposed by A.A. Dzhumadil'daeva, Almaty Republic Physics and Mathematics School, Almaty, Kazakhstan.*

Let $(2n+1)!! = 1\cdot 3\cdots(2n+1)$ be the double factorial, so (for example) $7!! = 105$. Make the convention that $0!! = (-1)!! = 1$. Prove that for any nonnegative integer n,

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{n}{i,j,k}(2i-1)!!(2j-1)!!(2k-1)!! \;=\; (2n+1)!!\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

From the identity

$$(2m-1)!! = 1\cdot 3\cdots(2m-1) = \frac{(2m)!}{2^m m!} = \frac{m!}{2^m}\binom{2m}{m}$$

we deduce the formal power series

$$\begin{aligned}
(1-4z)^{-\frac12} &= \sum_m \binom{-\frac12}{m}(-4)^m z^m\\
&= \sum_m \frac{(-4)^m}{m!}\left(-\frac12\right)\left(-\frac32\right)\cdots\left(-\frac{2m-1}{2}\right)z^m\\
&= \sum_m \frac{2^m(2m-1)!!}{m!}z^m = \sum_m \binom{2m}{m}z^m.
\end{aligned}$$

Hence, the number $\displaystyle\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}}\binom{2i}{i}\binom{2j}{j}\binom{2k}{k}$ is the coefficient of z^n in the series

$$\begin{aligned}
(1-4z)^{-\frac32} &= \sum_n \binom{-\frac32}{n}(-4)^n z^n\\
&= \sum_n \frac{(-4)^n}{n!}\left(-\frac32\right)\left(-\frac52\right)\cdots\left(-\frac{2n+1}{2}\right)z^n\\
&= \sum_n \frac{2^n(2n+1)!!}{n!}z^n = \sum_n \frac{n+1}{2}\binom{2n+2}{n+1}z^n.
\end{aligned}$$

We conclude

$$\begin{aligned}
&\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{n}{i,j,k}(2i-1)!!(2j-1)!!(2k-1)!!\\
&\qquad = \sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{n}{i,j,k}\frac{n!}{i!j!k!}\cdot\frac{i!}{2^i}\binom{2i}{i}\cdot\frac{j!}{2^j}\binom{2j}{j}\cdot\frac{k!}{2^k}\binom{2k}{k}\\
&\qquad = \frac{n!}{2^n}\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{2i}{i}\binom{2j}{j}\binom{2k}{k}\\
&\qquad = \frac{(n+1)!}{2^{n+1}}\binom{2n+2}{n+1} = (2n+1)!!
\end{aligned}$$

which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3576. [2010 : 459, 461] *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with interior points D, E, F such that $\angle FAB = \angle EAC$, $\angle FBA = \angle DBC$, $\angle DCB = \angle ECA$, $AF = AE$, $BF = BD$, and $CD = CE$. If R is the circumradius of ABC, r is the circumradius of EDF, and s is the semiperimeter of ABC, prove that the area of triangle EDF is $\dfrac{sr^2}{2R}$.

Solution by John G. Heuver, Grande Prairie, AB.

Let I_a, I_b, and I_c be the excentres of ΔABC. The definition of points E and F implies that the lines AE and AF are symmetric by reflection in the internal bisector AI_a of $\angle BAC$; that is, AI_a is the perpendicular bisector of EF. But AI_a is also perpendicular to I_bI_c (which is the external bisector of $\angle BAC$), whence $EF||I_bI_c$. Analogous statements hold for FD and DE. We deduce first that the sides of ΔDEF are parallel to the corresponding sides of $\Delta I_aI_bI_c$, so that the two triangles are similar, and second that I is the circumcentre of ΔDEF. Furthermore, $\angle EDF = \angle I_bI_aI_c = \frac{1}{2}(\angle B + \angle C)$, and, because I is the circumcentre of ΔDEF, $\angle EIF = 2\angle EDF = \angle B + \angle C$. Because the circumradius of ΔDEF is given to be r, similar reasoning for the angles at E

and F allows us to deduce that

$$\begin{aligned}
\text{Area}(DEF) &= \text{Area}(EIF) + \text{Area}(FID) + \text{Area}(DIE) \\
&= \frac{1}{2}r^2\left(\sin(B+C) + \sin(C+A) + \sin(A+B)\right) \\
&= \frac{1}{2}r^2(\sin A + \sin B + \sin C) = \frac{1}{2}r^2\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right) \\
&= \frac{r^2}{2}\left(\frac{s}{R}\right).
\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; JOEL SCHLOSBERG, Bayside, NY, USA; MIHAÏ STOËNESCU, Bischwiller, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Our featured solution proves only that if ΔDEF were to exist, its area would equal the predicted value. Bataille and Schlosberg both addressed the question of existence: our featured solution makes clear that reflection in AI_a takes E to F, in BI_b takes F to D, and in CI_c takes D back to E. Because the product of three reflections is an opposite isometry, while I and E are points that are fixed by this product, this isometry must be a reflection in IE. Define ℓ to be the line that makes a directed angle with CI_c equal to the directed angle from AI_a to BI_b. We conclude that it is both necessary and sufficient that E be a point of ℓ inside ΔABC different from I, for which its reflections in AI_a and CI_c, namely F and D, also lie inside that triangle.

3577. [2010 : 459,461] *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let H be the orthocentre of the acute triangle ABC with A' on the ray HA and such that $A'A = BC$. Define B', C' similarly. Prove that

$$\text{Area}(A'B'C') \ = \ 4\text{Area}(ABC) \ + \ \frac{a^2+b^2+c^2}{2}.$$

Solution by Joel Schlosberg, Bayside, NY, USA.

Editor's comment. The statement of the problem is somewhat flawed. What Schlosberg proves here is the theorem,

> Let H be the orthocentre of the arbitrary triangle ABC; define A' to be the unique point satisfying $AA' = BC$ that lies on the half-line which starts at A and extends along the line HA in the direction that misses the line BC. Define B', C' similarly. Then $\text{Area}(A'B'C') = 4\text{Area}(ABC) + \frac{a^2+b^2+c^2}{2}$.

We have modified Schlosberg's proof to make use of directed angles. Recall that $\angle XYZ$ as a *directed angle* denotes that angle (whose measure ranges from $0°$ to $180°$) through which the line XY must be rotated about Y in the positive direction in order to coincide with YZ.

Because corresponding sides of $\angle BHC$ and $\angle CAB$ are perpendicular, while $\angle ABH$ and $\angle HCA$ are complements of $\angle CAB$ in right triangles, we have

$$\angle BHC = \angle CAB = 90^\circ - \angle ABH = 90^\circ - \angle HCA.$$

It follows that

$$\begin{aligned}
\text{Area}(HB'C') &= \frac{1}{2}HB' \cdot HC' \sin\angle BHC \\
&= \frac{1}{2}(HB+b)(HC+c)\sin\angle BHC \\
&= \frac{1}{2}HB\cdot HC\sin\angle BHC + \frac{1}{2}cHB\cos\angle ABH \\
&\qquad + \frac{1}{2}bHC\cos\angle HCA + \frac{1}{2}bc\sin\angle CAB \\
&= \text{Area}(HBC) + \frac{1}{2}\overrightarrow{BA}\cdot\overrightarrow{BH} + \frac{1}{2}\overrightarrow{CA}\cdot\overrightarrow{CH} + \text{Area}(ABC).
\end{aligned}$$

Similarly,

$$\text{Area}(HC'A') = \text{Area}(HCA) + \frac{1}{2}\overrightarrow{CB}\cdot\overrightarrow{CH} + \frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{AH} + \text{Area}(ABC),$$

and

$$\text{Area}(HA'B') = \text{Area}(HAB) + \frac{1}{2}\overrightarrow{AC}\cdot\overrightarrow{AH} + \frac{1}{2}\overrightarrow{BC}\cdot\overrightarrow{BH} + \text{Area}(ABC).$$

Consequently,

$$\begin{aligned}
\text{Area}(A'B'C') &= \text{Area}(HB'C') + \text{Area}(HC'A') + \text{Area}(HA'B') \\
&= \text{Area}(HBC) + \text{Area}(HCA) + \text{Area}(HAB) + 3\text{Area}(ABC) \\
&\qquad + \frac{1}{2}\overrightarrow{BC}\cdot(\overrightarrow{BH}+\overrightarrow{HC}) + \frac{1}{2}\overrightarrow{CA}\cdot(\overrightarrow{CH}+\overrightarrow{HA}) \\
&\qquad + \frac{1}{2}\overrightarrow{AB}\cdot(\overrightarrow{AH}+\overrightarrow{HB}) \\
&= 4\text{Area}(ABC) + \frac{1}{2}\overrightarrow{BC}\cdot\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA}\cdot\overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{AB} \\
&= 4\text{Area}(ABC) + \frac{a^2+b^2+c^2}{2}.
\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; JOHN G. HEUVER, Grande Prairie, AB; ALBERT STADLER, Herrliberg, Switzerland; MIHAÏ STOËNESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

The proposer's submitted problem applied to all triangles, but since its statement was somewhat ambiguous the editor restricted it to acute triangles. Unfortunately, even the published statement of 3577 is ambiguous. We hope we got it right this time.

3578. [2010 : 459,462] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $a > 0$ and $b > 1$ be real numbers and let $f : [0,1] \to \mathbb{R}$ be a continuous function. Find

$$\lim_{n\to\infty} n^{a/b} \int_0^1 \frac{f(x)}{1+n^a x^b}\,dx\,.$$

Solution by Mohammed Aassila, Strasbourg, France (expanded slightly by the editor).

We show that the required limit is $\dfrac{\pi f(0)}{b\sin\left(\frac{\pi}{b}\right)}$.

Let L denote the given limit. Using the substitution $y = n^{\frac{a}{b}}x$, we have

$$n^{\frac{a}{b}} \int_0^1 \frac{f(x)}{1+n^a x^b}dx = \int_0^1 \frac{f(x)}{1+\left(n^{\frac{a}{b}}x\right)^b}\cdot n^{\frac{a}{b}}dx = \int_0^{n^{\frac{a}{b}}} \frac{f\left(n^{-\frac{a}{b}}y\right)}{1+y^b}dy.$$

Hence, $L = f(0)\int_0^\infty \frac{1}{1+y^b}dy$.

Let $u = y^b$ so $y = u^{\frac{1}{b}}$ and $dy = \frac{1}{b}u^{\frac{1}{b}-1}du$. Then we have

$$L = \frac{f(0)}{b}\int_0^\infty \frac{u^{\frac{1}{b}}}{u(1+u)}du. \qquad (1)$$

Let $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, $x > 0$ and $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, $x > 0$, $y > 0$ denote the Gamma function and the Beta function, respectively. The following formulae are well known [1]:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad \text{and} \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \qquad (2)$$

Let $u = \dfrac{t}{1-t}$. Then $du = \dfrac{1}{1-t^2}dt$ and $u(1+u) = \dfrac{t}{(1-t)^2}$. Hence, from (1), (2) and the obvious fact that $\Gamma(1) = 1$, we have

$$\begin{aligned} L &= \frac{f(0)}{b}\int_0^1 t^{\frac{1}{b}-1}(1-t)^{-\frac{1}{b}}dt = \frac{f(0)}{b}B\left(\frac{1}{b}, 1-\frac{1}{b}\right) \\ &= \frac{f(0)}{b}\cdot\frac{\Gamma\left(\frac{1}{b}\right)\Gamma\left(1-\frac{1}{b}\right)}{\Gamma(1)} = \frac{f(0)}{b}\cdot\frac{\pi}{\sin\left(\frac{\pi}{b}\right)} \end{aligned}$$

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

There were three incomplete solutions all of which gave $f(0)\int_0^\infty \frac{dt}{1+t^b}$ *or its equivalent form, as the final answer.*

Stadler remarked that it suffices to assume that $f(x)$ *is a bounded integrable function which is continuous from the right at* 0.

References

[1] E.T. Whittaker and G.N.Watson, *A course of Mathematical Analysis*, Cambridge University Press, 1961.

3579. [2010 : 459,462] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\alpha = \dfrac{\pi}{13}$ and

$$\begin{aligned} x_1 &= \tan(4\alpha) + 4\sin(\alpha) = -\tan(\alpha) + 4\sin(3\alpha)\,, \\ x_2 &= \tan(\alpha) + 4\sin(\alpha) = -\tan(4\alpha) + 4\sin(3\alpha)\,, \\ x_3 &= \tan(6\alpha) - 4\sin(6\alpha) = \tan(2\alpha) + 4\sin(5\alpha)\,. \end{aligned}$$

Prove that the length x_1 can be constructed with compass and straightedge and determine whether or not the same is true for x_2 and x_3.

Solution by Stan Wagon, Macalester College, St. Paul, MN, USA.

Applying trigonometric expansion with the help of *Mathematica*, we find that $x_1 = \sqrt{13 - 2\sqrt{13}}$ which is clearly constructible since it contains only square roots. [*Ed.: The proposer remarked that* $x_1 = \sqrt{13 - 2\sqrt{13}}$ *can be proved from the solution to problem # 3305.*]

On the other hand, using trigonometric expansion and an algorithm to deduce the minimal polynomial satisfied by an algebraic number, we learn that each of x_2 and x_3 is a root of the irreducible polynomial $x^{12} - 78x^{10} + 1963x^8 - 20020x^6 + 81991x^4 - 138398x^2 + 81133$. Since it is a classical result that an algebraic number is constructible if and only if the degree of its minimal polynomial is a power of 2, we conclude that x_2 and x_3 are not constructible.

The proposer gave a partial answer by showing that x_1 is constructible. No other solutions were received.

3580. [2010: 460, 462] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k > 0$ and $m \geq 0$ be real numbers, and let $\{a\} = a - \lfloor a \rfloor$ denote the fractional part of a. Calculate

$$\int_0^1 \left\{ \frac{1}{x^k} - \frac{1}{(1-x)^k} \right\} x^m (1-x)^m \, dx\,.$$

Solution by Michel Bataille, Rouen, France.

Let I denote the integral. The substitution $u = 1 - x$ yields

$$I = \int_1^0 \left\{ \frac{1}{(1-u)^k} - \frac{1}{u^k} \right\} u^m (1-u)^m (-du)$$
$$= \int_0^1 \left\{ \frac{1}{(1-x)^k} - \frac{1}{x^k} \right\} x^m (1-x)^m dx \,.$$

Thus

$$2I = \int_0^1 \left[\{\phi(x)\} + \{-\phi(x)\} \right] x^m (1-x)^m dx \,,$$

where $\phi(x) := \frac{1}{(1-x)^k} - \frac{1}{x^k}$.

Since $\phi(x)$ is continuous, strictly increasing on $(0,1)$, and $\lim_{x \to 0^+} \phi(x) = -\infty$; $\lim_{x \to 1^-} \phi(x) = +\infty$, it follows that ϕ is a bijection from $(0,1)$ onto $\mathbb{R}$. Thus, for each integer $p \in \mathbb{Z}$, the equation $\phi(x) = p$ has exactly one solution $a_p \in (0,1)$.

It is easy to see that

$$\{a\} + \{-a\} = \begin{cases} 0 & \text{if } a \in \mathbb{Z} \\ 1 & \text{if } a \notin \mathbb{Z} \end{cases}$$

Thus

$$\left[\{\phi(x)\} + \{-\phi(x)\} \right] x^m (1-x)^m = x^m (1-x)^m$$

outside the countable set $\{a_p | p \in \mathbb{Z}\}$.

Hence

$$I = \frac{1}{2} \int_0^1 \left[\{\phi(x)\} + \{-\phi(x)\} \right] x^m (1-x)^m dx$$
$$= \frac{1}{2} \int_0^1 x^m (1-x)^m dx = \frac{(\Gamma(m+1))^2}{2\Gamma(2m+2)} = \frac{(m!)^2}{2(2m+1)!} \,.$$

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3582. [2010 : 460, 462] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let Γ_1, Γ_2 be circles of radius r with centres A, B (respectively), let $\{C,\ D\} = \Gamma_1 \cap \Gamma_2$, and suppose that $\angle BCA = 90^\circ$. A line through C intersects Γ_1 and Γ_2 again at E and F, respectively. The circle Γ with centre O and radius R passes through points E and F. A second line passes through C, is perpendicular to the segment EF, and intersects the circle Γ in G and H. Prove that $CH^2 + CG^2 = 4(R^2 - r^2)$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $\boldsymbol{M}$ be the midpoint of $\boldsymbol{EC}$, $\boldsymbol{N}$ the midpoint of $\boldsymbol{CF}$, $\boldsymbol{a = AM}$, and $\boldsymbol{b = BN}$. Because corresponding sides of $\boldsymbol{\Delta AMC}$ and $\boldsymbol{\Delta CNB}$ are perpendicular, while $\boldsymbol{BC = AC = r}$, the triangles are congruent. In particular, $\boldsymbol{a = CN = NF}$ and $\boldsymbol{b = MC = EM}$, so that $\boldsymbol{EF = 2(b + a)}$ and

$$\boldsymbol{r^2 = a^2 + b^2.}$$

Furthermore, if $\boldsymbol{P}$ is the midpoint of $\boldsymbol{EF}$, then $\boldsymbol{EP = b + a}$ so that $\boldsymbol{MP = a}$ and $\boldsymbol{PC = b - a}$.

Next, let $\boldsymbol{Q}$ be the midpoint of $\boldsymbol{GH}$ and $\boldsymbol{h = PO}$ ($\boldsymbol{= CQ}$). Then $\boldsymbol{OQ = PC = b - a}$, whence

$$\boldsymbol{QH^2 = R^2 - (b - a)^2.}$$

Moreover,

$$\boldsymbol{CH^2 + CG^2 = (h + QH)^2 + (h - QH)^2 = 2\left(QH^2 + h^2\right).}$$

But $\boldsymbol{R^2 = OE^2 = h^2 + (b + a)^2}$, hence

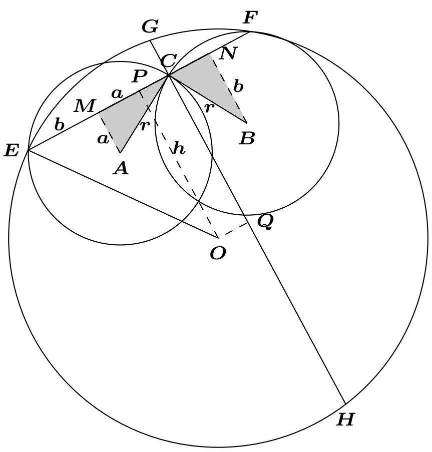
$$\boldsymbol{h^2 = R^2 - (b + a)^2.}$$

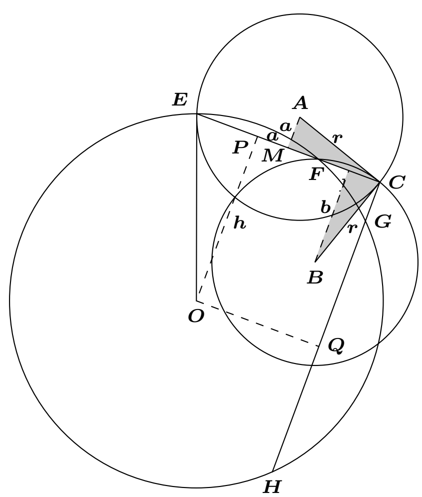
Assembling the pieces, we conclude that

$$\begin{aligned}\boldsymbol{CH^2 + CG^2} &\boldsymbol{= 2\left[\left(R^2 - (b - a)^2\right) + \left(R^2 - (b + a)^2\right)\right]}\\ &\boldsymbol{= 2\left[2R^2 - 2\left(b^2 + a^2\right)\right] = 4R^2 - 4r^2,}\end{aligned}$$

as claimed.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.





Only Bataille observed that there arise two cases: $\boldsymbol{C}$ can lie between $\boldsymbol{E}$ and $\boldsymbol{F}$ (as in the first diagram), or not between them (as in the second diagram). If we use directed line segments *along the lines EF and GH (so that $XY = -YX$ for points X and Y both on one of these lines or on a line parallel to one of them), then Woo's notation has been modified by the editor so that his argument deals with both cases simultaneously. Otherwise, one must observe that when C is not between E and F (and therefore not between G and H) and the diagram is labeled so that $b \geq a$, then $PE = b - a$ and $PC = b + a$, and the featured argument goes through without difficulty.*

3583. [2010: 460, 463] *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let α and β be nonnegative real numbers and define

$$\begin{aligned} a_n &= \big(n + \ln(n+1)\big) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k+1) + \ln(k+1)}\,, \\ p_n &= \big(\alpha + n + 1 + \ln(n+1)\big) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k+1) + \ln(k+1)}\,. \end{aligned}$$

Find those nonnegative real numbers α and β for which $\sum\limits_{n=1}^{\infty} a_n$ converges, and determine the relation between α and β that ensures that

$$\sum_{n=1}^{\infty} \left(a_n - p_n \ln\left(1 + \frac{1}{n+1}\right)\right) = (\alpha+1)(\alpha + 2 + \ln 2) - \frac{(\alpha+1)^2}{2}\,.$$

Solution by the proposer.

We claim that the series converges if and only if $\beta > \alpha + 1$.

Let's start by observing that

$$\frac{\alpha + n + 1 + \ln(n+1)}{\beta + n + 2 + \ln(n+2)} = 1 + \frac{\alpha - \beta - 1 + \ln\left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)}$$

and

$$\frac{n + 1 + \ln(n+2)}{n + \ln(n+1)} = 1 + \frac{1 + \ln\left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}.$$

Hence

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\alpha + n + 1 + \ln(n+1)}{\beta + n + 2 + \ln(n+2)} \frac{n + 1 + \ln(n+2)}{n + \ln(n+1)} \\ &= \left(1 + \frac{\alpha - \beta - 1 + \ln\left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)}\right) \cdot \left(1 + \frac{1 + \ln\left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}\right) \\ &= 1 + \frac{\alpha - \beta - 1 + \ln\left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} + \frac{1 + \ln\left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)} \\ &\quad + \frac{\alpha - \beta - 1 + \ln\left(1 - \frac{1}{n+2}\right)}{\beta + 2 + n + \ln(n+2)} \cdot \frac{1 + \ln\left(1 + \frac{1}{n+1}\right)}{n + \ln(n+1)}\,. \end{aligned}$$

In particular

$$
\begin{aligned}
&\frac{a_{n+1}}{a_n} - \left(1 + \frac{\alpha-\beta}{n}\right) \\
&= \frac{\alpha-\beta-1+\ln\left(1-\frac{1}{n+2}\right)}{\beta+2+n+\ln(n+2)} - \frac{\alpha-\beta-1}{n} + \frac{1+\ln\left(1+\frac{1}{n+1}\right)}{n+\ln(n+1)} - \frac{1}{n} \\
&\quad + \frac{\alpha-\beta-1+\ln\left(1-\frac{1}{n+2}\right)}{\beta+2+n+\ln(n+2)} \cdot \frac{1+\ln\left(1+\frac{1}{n+1}\right)}{n+\ln(n+1)} \\
&= (\alpha-\beta-1)\left(\frac{1}{\beta+2+n+\ln(n+2)} - \frac{1}{n}\right) + \left(\frac{1}{n+\ln(n+1)} - \frac{1}{n}\right) \\
&\quad + \frac{\alpha-\beta-1+\ln\left(1-\frac{1}{n+2}\right)}{\beta+2+n+\ln(n+2)} \cdot \frac{1+\ln\left(1+\frac{1}{n+1}\right)}{n+\ln(n+1)} \\
&\quad + \frac{\ln\left(1-\frac{1}{n+2}\right)}{\beta+2+n+\ln(n+2)} + \frac{\ln\left(1+\frac{1}{n+1}\right)}{n+\ln(n+1)} \\
&= (\alpha-\beta-1)\frac{-\beta-2-\ln(n+2)}{n^2+\beta n+2n+n\ln(n+2))} - \frac{\ln(n+1)}{n^2+n\ln(n+1)} \\
&\quad + \frac{\alpha-\beta-1+\ln\left(1-\frac{1}{n+2}\right)}{\beta+2+n+\ln(n+2)} \cdot \frac{1+\ln\left(1+\frac{1}{n+1}\right)}{n+\ln(n+1)} \\
&\quad + \frac{\ln\left(1-\frac{1}{n+2}\right)}{\beta+2+n+\ln(n+2)} + \frac{\ln\left(1+\frac{1}{n+1}\right)}{n+\ln(n+1)} \\
&= O\left(\frac{\ln(n)}{n^2}\right)
\end{aligned}
$$

Thus, we get

$$
\frac{a_{n+1}}{a_n} = 1 + \frac{\alpha-\beta}{n} + O\left(\frac{\ln(n)}{n^2}\right). \tag{1}
$$

Convergence for $\beta > \alpha + 1$.

For all $n \geq 2$ set $b_n = \frac{1}{n\ln^2(n)}$. Then

$$
\begin{aligned}
\frac{b_{n+1}}{b_n} &= \frac{n\ln^2(n)}{(n+1)\ln^2(n+1)} = 1 - \frac{(n+1)\ln^2(n+1) - n\ln^2(n)}{(n+1)\ln^2(n+1)} \\
&= 1 - \frac{(n+1)\ln^2(n+1) - n\ln^2(n+1)}{(n+1)\ln^2(n+1)} - \frac{n\ln^2(n+1) - n\ln^2(n)}{(n+1)\ln^2(n+1)} \\
&= 1 - \frac{1}{(n+1)} - \frac{(n\ln(n+1) - n\ln(n))\,(\ln(n+1)+\ln(n))}{(n+1)\ln^2(n+1)} \\
&= 1 - \frac{1}{(n+1)} - \frac{\ln\left(1+\frac{1}{n}\right)^n(\ln(n+1)+\ln(n))}{(n+1)\ln^2(n+1)},
\end{aligned}
$$

so

$$\begin{aligned}
\frac{b_{n+1}}{b_n} &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} - \frac{\ln\left(1+\frac{1}{n}\right)^n (\ln(n+1)+\ln(n))}{(n+1)\ln^2(n+1)} \\
&= 1 - \frac{1}{n} + \frac{1}{n(n+1)} - \frac{2}{(n+1)\ln(n+1)} \\
&\quad - \frac{\ln\left(1+\frac{1}{n}\right)^n (\ln(n+1)+\ln(n)) - 2\ln(n+1)}{(n+1)\ln^2(n+1)} \\
&= 1 - \frac{1}{n} + \frac{1}{n(n+1)} - \frac{\ln(n)+\ln(n+1)}{(n+1)\ln(n+1)} \\
&\quad - \frac{\ln \frac{(1+\frac{1}{n})^n}{e} (\ln(n+1)+\ln(n))}{(n+1)\ln^2(n+1)} \\
&= 1 - \frac{1}{n} - \frac{2}{(n+1)\ln(n+1)} + O\left(\frac{\ln(n)}{n^2}\right) .
\end{aligned} \tag{2}$$

Since $\beta > \alpha + 1$, by combining (1) and (2) we get that there exists some N so that for all $n > N$ we have

$$0 < \frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} .$$

Hence, for all $n > N$ we have

$$\frac{a_{n+1}}{b_{n+1}} < \frac{a_n}{b_n} .$$

It follows that the sequence $\frac{a_n}{b_n}$ is eventually decreasing, and hence bounded from above. Hence, there exists an M so that, for all n we have

$$0 < a_n < Mb_n .$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ is convergent by the integral test, by the Comparison Test we get that $\sum_n a_n$ is convergent.

Divergence for $\beta \leq \alpha + 1$.

For all $n \geq 2$ set $c_n = \frac{1}{n\ln^2(n)}$. Exactly as in (2) one can show that

$$\frac{c_{n+1}}{c_n} = 1 - \frac{1}{n} - \frac{1}{(n+1)\ln(n+1)} + O\left(\frac{\ln(n)}{n^2}\right) .$$

Thus, there exists some N so that, for all $n > N$ we have

$$\frac{a_{n+1}}{a_n} \geq \frac{c_{n+1}}{c_n} > 0 .$$

Hence, for all $n > N$ we have

$$\frac{a_{n+1}}{c_{n+1}} > \frac{a_n}{c_n} > 0\,.$$

It follows that the sequence $\frac{a_n}{c_n}$ is eventually increasing. Hence, there exists an m so that, for all n we have

$$a_n > mc_n\,.$$

Moreover, m can be chosen as the smallest term of $\frac{a_n}{c_n}$, and hence it can be chosen so that $m > 0$.

Since the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent, by the Comparison Test we get that $\sum_n a_n$ is also divergent.

This completes the first part of the problem.

For the second part, let's observe first that

$$a_n = (\alpha + n + 1 + \ln(n+1)) \prod_{k=1}^{n} \frac{\alpha + k + \ln(k)}{\beta + k + 1 + \ln(k+1)} - (\alpha + 1) \prod_{k=1}^{n} \frac{\alpha + k + \ln(k)}{\beta + k + 1 + \ln(k+1)}.$$

Define

$$q_n := \prod_{k=1}^{n} \frac{\alpha + k + \ln(k)}{\beta + k + 1 + \ln(k+1)}\,.$$

Then

$$a_n = p_n - (\alpha + 1)q_n\,.$$

Since by the definition of q_n we have $0 < q_n \leq a_n$, it follows that $\sum_n p_n$ and $\sum_n q_n$ are absolutely convergent.

Let's note that

$$\frac{q_{n+1}}{q_n} = \frac{\alpha + n + 1 + \ln(n+1)}{\beta + n + 2 + \ln(n+2)}. \tag{3}$$

In particular

$$\beta(q_{n+1} - q_n) + (\beta - \alpha)q_n + (n+2)q_{n+1} - q_n(n+1) = q_n \ln(n+1) - q_{n+1} \ln(n+2)\,. \tag{4}$$

Since $\sum a_n$ is convergent, we are in the case $\beta > \alpha + 1$. Thus

$$\frac{q_{n+1}}{q_n} < \frac{\beta + n + \ln(n+1)}{\beta + n + 2 + \ln(n+2)} < 1\,,$$

and hence q_n is decreasing.

We will employ the following well known lemma:

Lemma: Let q_n be monotonic and positive. If $\sum q_n$ is convergent, then

$$\lim_n nq_n = 0\,.$$

Let's denote $\sum_n q_n = U$. By summing (4) from 1 to N we get

$$\beta(q_{N+1} - q_1) + (\beta - \alpha)\sum_{n=1}^{N} q_n + (N+2)q_{N+1} - 2q_1(n+1)$$
$$= q_1 \ln(2) - q_{N+1}\ln(N+2)\,.$$

Using the above Lemma and letting $N \to \infty$ we get:

$$-\beta q_1 + (\beta - \alpha)U - 2q_1 = q_1 \ln(2)\,.$$

Since $q_1 = \dfrac{\alpha+1}{\beta+2+\ln 2}$ we get

$$U = \frac{(\beta+2+\ln(2))q_1}{\beta-\alpha} = \frac{\alpha+1}{\beta-\alpha}\,.$$

Now, let's observe that

$$\frac{p_{n+1}}{p_n} = \frac{\alpha+n+2+\ln(n+2)}{\beta+n+2+\ln(n+2)}\,,$$

and hence

$$\beta(p_{n+1} - p_n) + (\beta-\alpha)p_n + [(n+2)p_{n+1} - (n+1)p_n] - p_n$$
$$= [p_n \ln(n+1) - p_{n+1}\ln(n+2)] + [p_n \ln(n+2) - p_n \ln(n+1)]$$

By summing we get

$$\beta(p_{N+1} - p_1) + (\beta-\alpha)\sum_{n=1}^{N} p_n + [(N+2)p_{N+1} - 2p_1] - \sum_{n=1}^{N} p_n$$
$$= [p_1 \ln(2) - p_{N+1}\ln(N+2)] + \left[\sum_{n=1}^{N} p_n \ln\left(1 + \frac{1}{n+1}\right)\right]$$

Again using the Lemma, and letting $N \to \infty$ we get:

$$-\beta p_1 - 2p_1 + (\beta-\alpha)\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} p_n = p_1 \ln(2) + \sum_{n=1}^{N} p_n \ln\left(1 + \frac{1}{n+1}\right)\,.$$

Since

$$p_1 = (\alpha+2+\ln(2))\frac{\alpha+1}{\beta+2+\ln(2)}\,,$$

we get

$$(\beta-\alpha)\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{N} p_n \ln\left(1+\frac{1}{n+1}\right) = (\alpha+2+\ln(2))(\alpha+1)\,,$$

and hence

$$\sum_{n=1}^{\infty} p_n = \frac{1}{\beta-\alpha-1}\sum_{n=1}^{N} p_n \ln\left(1+\frac{1}{n+1}\right) + \frac{(\alpha+2+\ln(2))(\alpha+1)}{\beta-\alpha-1}.$$

This shows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(a_n - p_n \ln\left(1+\frac{1}{n+1}\right)\right)\\
&= \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty}(\alpha+1)q_n - \sum_{n=1}^{\infty} p_n \ln\left(1+\frac{1}{n+1}\right)\\
&= \frac{(\alpha+2+\ln(2))(\alpha+1)}{\beta-\alpha-1} + \frac{\beta-\alpha-2}{\beta-\alpha-1}\sum_{n=1}^{\infty} p_n \ln\left(1+\frac{1}{n+1}\right) - \frac{(\alpha+1)^2}{\beta-\alpha}.
\end{aligned}$$

Thus, if $\beta = \alpha + 2$, the second formula holds.

No other solution was received.

3584. [2010: 460, 463] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let ABC be a triangle with inradius r, side lengths a, b, c and medians m_a, m_b, m_c. Prove that $\frac{c}{m_a^2 m_b^2} + \frac{a}{m_b^2 m_c^2} + \frac{b}{m_c^2 m_a^2} \le \frac{3\sqrt{3}}{27r^3}$.

Solution by Michel Bataille, Rouen, France.

We first observe that

$$\begin{aligned}
m_a^2 &= \frac{2b^2+2c^2-a^2}{4} \ge \frac{b^2+2bc+c^2-a^2}{4}\\
&= \frac{(b+c)^2-a^2}{4} = p(p-a)\,,
\end{aligned}$$

where $p = \frac{a+b+c}{2}$ is the semiperimeter of the triangle ABC. Similarly we get

$$m_b^2 \ge p(p-b)\,;\; m_c^2 \ge p(p-c)\,.$$

Thus, we get

$$\frac{c}{m_a^2m_b^2}+\frac{a}{m_b^2m_c^2}+\frac{b}{m_a^2m_c^2}\le\frac{c}{p^2(p-a)(p-b)}+\frac{a}{p^2(p-b)(p-c)}+\frac{b}{p^2(p-a)(p-c)}=\frac{c(p-c)+a(p-a)+b(p-b)}{p^2(p-a)(p-b)(p-c)}$$

Let S denote the area of the triangle ABC. Since $p(p-a)(p-b)(p-c)=S^2=r^2p^2$ and

$$a(p-a)+b(p-b)+c(p-c)=\frac{2[abc+(p-a)(p-b)(p-c)]}{p},$$

to complete the proof we need to show that

$$abc+(p-a)(p-b)(p-c)\le\frac{\sqrt{3}p^4}{27r} \qquad (1)$$

By applying the AM-GM inequality to both terms on the left side we get

$$abc+(p-a)(p-b)(p-c)\le\left(\frac{a+b+c}{3}\right)^3+\left(\frac{(p-a)+(p-b)+(p-c)}{3}\right)^3=\frac{8p^3}{27}+\frac{p^3}{27}=\frac{p^3}{3}$$

Also

$$r^2p^2=p(p-a)(p-b)(p-c)\le p\left(\frac{(p-a)(p-b)(p-c)}{3}\right)^3\le\frac{p^4}{27},$$

and hence

$$3\sqrt{3}r\le p\,.$$

Thus

$$abc+(p-a)(p-b)(p-c)\le\frac{p^3}{3}=\frac{p^4}{3p}\le\frac{p^4}{9\sqrt{3}r},$$

which proves (1).

Also solved by ARKADY ALT, San Jose, CA, USA; KEE-WAI LAU, Hong Kong, China; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; and the proposer.

3585. [2011 : 461,463] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $T_n(x)$ be the Chebyshev polynomial of the first kind defined by the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Find all positive integers n such that

$$T_n(x) \;\leq\; (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}\,, \quad x \in [1,\infty)\,.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

The given recurrence defining $T_n(x)$ implies that it is a polynomial of degree n, whose leading coefficient is 2^{n-1} for all $n \geq 1$. If the required inequality is to hold for all $x \geq 1$ at a particular positive integer n, then necessarily the leading coefficient of $T_n(x)$ must be at most $2^{n-2}+1$. Thus $2^{n-1} \leq 2^{n-2}+1$, which implies that $2^{n-2} \leq 1$, which is true only when $n = 1$ or $n = 2$.

With $n = 1$, the inequality demands that $x \leq \frac{3}{2}x - \frac{1}{2}$, which clearly holds for all $x \geq 1$. However, with $n = 2$, the inequality demands that $2x^2 - 1 \leq 2x^2 - x$, which clearly fails for all $x > 1$. Thus $n = 1$ is the only positive integer with the required property.

Also solved by MICHEL BATAILLE, Rouen, France.

3586. [2010 : 461,463] *Proposed by Shai Covo, Kiryat-Ono, Israel.*

For each positive integer n, a_n is the number of positive divisors of n of the form $4m+1$ minus the number of positive divisors of n of the form $4m+3$ (so $a_4 = 1$, $a_5 = 2$, and $a_6 = 0$). Evaluate the sum $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{a_n}{n}$.

Solution by the proposer.

We show that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a_n}{n} = \frac{\pi \ln 2}{4}. \tag{1}$$

To prove (1) we first establish the following lemma:

Lemma: Let $x_k = (-1)^{k+1}k$ and $y_k = (-1)^{k+1}(2k-1)$, for each integer $k \geq 1$. Then

$$\lim_{n\to\infty}\sum_{|x_iy_j|\leq n}\frac{1}{x_i}\frac{1}{y_j} = \left(\sum_{k=1}^{\infty}\frac{1}{x_k}\right)\left(\sum_{k=1}^{\infty}\frac{1}{y_k}\right) = \frac{\pi \ln 2}{4}.$$

Proof: For any fixed n let $m = \lfloor\sqrt{n}\rfloor$. Then

$$\sum_{|x_iy_j|\leq n}\frac{1}{x_i}\frac{1}{y_j} = \sum_{i=1}^{n}\left(\frac{1}{x_i}\sum_{|x_iy_j|\leq n}\frac{1}{y_j}\right) = \sum_{i=1}^{m}b_i + \sum_{i=m+1}^{n}b_i$$

where $b_i = \dfrac{1}{x_i}\displaystyle\sum_{|x_iy_j|\leq n}\frac{1}{y_j}$.

For each fixed $i = 1, 2, \cdots, m$, let

$$\sum_{|x_i y_j| \le n} \frac{1}{y_j} = \frac{\pi}{4} + E_i.$$

Using the fact that

$$\left| \frac{\pi}{4} - \sum_{j=1}^{k} (-1)^{j+1} \frac{1}{2j-1} \right| \le \frac{1}{2k+1}$$

we have

$$|E_i| \le \frac{1}{2k+1} \quad \text{and} \quad (2k+1)i > n.$$

Thus,

$$\left| \sum_{i=1}^{m} b_i - \frac{\pi}{4} \sum_{i=1}^{m} \frac{1}{x_i} \right| \le \sum_{i=1}^{m} \left| \frac{E_i}{x_i} \right| < \frac{m}{n}.$$

It follows that

$$\lim_{n\to\infty} \sum_{i=1}^{m} b_i = \frac{\pi}{4} \sum_{i=1}^{\infty} \frac{1}{x_i} = \frac{\pi}{4} \ln 2.$$

Hence, it suffices to show that

$$\lim_{n\to\infty} \sum_{i=m+1}^{n} b_i = 0. \tag{2}$$

We consider the sums $|b_{m+1} + b_{m+2}|$, $|b_{m+3} + b_{m+4}|$, $\cdots$. (Note that $|b_n| = \frac{1}{n} \to 0$ as $n \to \infty$.)

Now define

$$w(i) = \max_{|x_i y_j| \le n} |y_j|$$

and consider an arbitrary but fixed $i \in \{m+1, m+2, \cdots\}$ where $i \le n-1$. If $w(i) = w(i+1)$, then

$$|b_i + b_{i+1}| = \left| \sum_{|x_i y_j| \le n} \frac{1}{y_j} \right| \left| \frac{1}{i} - \frac{1}{i+1} \right| \le \frac{1}{i(i+1)}. \tag{3}$$

If $w(i) \ne w(i+1)$ then $w(i+1) \le w(i) - 2$ since clearly $w(i+1) \le w(i)$ and both are odd integers. We now show that

$$w(i+1) \ge w(i) - 2. \tag{4}$$

Note first that

$$\begin{aligned} iw(i) \le n \Rightarrow (m+1)w(i) \le n &\Rightarrow \left(\lfloor \sqrt{n} \rfloor + 1\right) w(i) \le n \\ &\Rightarrow \sqrt{n}w(i) < n \Rightarrow w(i) < \sqrt{n} \\ &\Rightarrow w(i) \le \lfloor \sqrt{n} \rfloor = m \Rightarrow w(i) \le i - 1 < 2(i+1) \end{aligned}$$

so $(i+1)(w(i)-2) = iw(i) + w(i) - 2(i+1) < n$.

Returning now to the proof of (4) which clearly holds if $w(i) = 1$ since $w(i+1) \geq 1$. If $w(i) \geq 3$, then $w(i) - 2 > 0$ and is odd. Since $w(i) - 2$ is of the form $|y_j|$ for some j and since $w(i+1) = \max_{|x_{i+1}y_j| \leq n} |y_j|$, we have $(i+1)w(i+1) \geq (i+1)|y_j| = (i+1)(w(i)-2)$ from which $w(i+1) \geq w(i)-2$ follows, establishing (4).

Therefore, $w(i+1) = w(i) - 2$. Using (3) we obtain

$$\begin{aligned} |b_i + b_{i+1}| &\leq \frac{1}{i(i+1)} + \frac{1}{|x_i w(i)|} = \frac{1}{i(i+1)} + \frac{1}{iw(i)} \\ &= \frac{1}{i(i+1)} + \frac{1}{(i+1)w(i)}\left(1 + \frac{1}{i}\right) \end{aligned} \tag{5}$$

If $(i+1)w(i) \leq n$, then $|x_{i+1}|w(i) \leq n$ would imply that $|x_{i+1}||y_j| \leq n$ for all y_j such that $|x_i||y_j| \leq n$ so $w(i) < w(i+1)$, a contradiction.

Hence,

$$(i+1)w(i) > n. \tag{6}$$

From (5) and (6) we obtain

$$|b_i + b_{i+1}| < \frac{1}{i(i+1)} + \frac{2}{n}. \tag{7}$$

However, noting that $w(m+1) \leq m$ so the number of indices i for which $w(i+1) = w(i) - 2$ is bounded above by m.

Since both $\dfrac{1}{(m+1)(m+2)} + \dfrac{1}{(m+3)(m+4)} + \cdots$ and $\dfrac{m}{n}$ tend to 0 as $n \to \infty$ we conclude that $\lim\limits_{n\to\infty} \sum\limits_{i=m+1}^{n} b_i = 0$ establishing (2) and completing the proof.

Also solved by ALBERT STADLER, Herrliberg, Switzerland; who gave a 3-page proof based on the Dirichlet L-function associated with the non-trivial character $(\mathbf{mod}\ 4)$*, and analytic continuation of some complex function defined by Dirichlet series.*

3587★. [2010 : 461,463] *Proposed by Ignotus, Colegio Manablanca, Facatativá, Colombia.*

Define the *prime graph* of a set of positive integers as the graph obtained by letting the numbers be the vertices, two of which are joined by an edge if and only if their sum is prime.

(a) Prove that given any tree T on n vertices, there is a set of positive integers whose prime graph is isomorphic to T.

(b) For each positive integer n, determine $t(n)$, the smallest number such that for any tree T on n vertices, there is a set of n positive integers each not greater than $t(n)$ whose prime graph is isomorphic to T.

No solutions have been received so this problem remains open.

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Readers are invited to send their solutions to any of the problems published in ***Crux Mathematicorum with Mathematical Mayhem***. We also welcome your problem proposals and articles, as well as comments and suggestions. For the problems in *Skoliad* or *Mathematical Mayhem*, we are looking especially for solutions from pre-university students.

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Correspondence may be sent by email or regular mail (but not by FAX). Material sent by regular mail should be typed or neatly hand-written on standard $8\frac{1}{2}'' \times 11''$ or A4 paper. For problem proposals and solutions, please sign each page, and use a new page when beginning a new problem. For email submissions, it would be appreciated if LaTeX could be used, but MSWord, WordPerfect, PDF files, and PostScript are also acceptable. Graphics files should be in epic format or encapsulated PostScript.

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Acknowledgements

The support of the Ottawa Carleton District School Board is gratefully acknowledged.

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