SKOLIAD No. 135

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **August 15, 2012**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

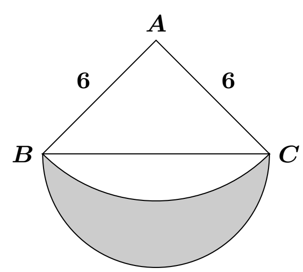
Our contest this month is the Calgary Mathematical Association 35th Junior High School Mathematics Contest, Part B, 2011. Our thanks go to Robert Woodrow and Bill Sands, both of University of Calgary, Alberta, for providing us with this contest and for permission to publish it.

L'Association mathématique de Calgary 35e Compétition Junior de Mathématique Ronde finale, partie B, 2011.

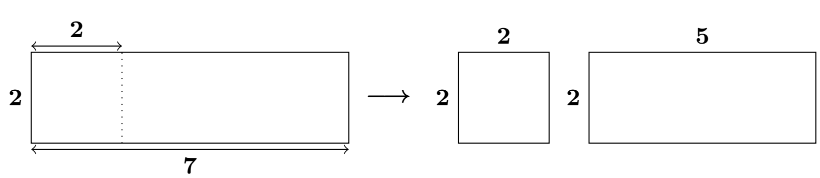
1. Ariel a acheté une certaine quantité d'abricots. **90**% du poids d'un abricot est constitué d'eau. Elle sèche les abricots jusqu'à ce que **60**% du poids d'un abricot soit constitué d'eau. **15** kg se sont ainsi évaporés. Quel était le poids initial des abricots (en kg) ?

2. Un groupe de dix amis vont ensemble au cinéma. Un autre groupe de neuf amis vont aussi au même cinéma. Quatorze des ces **19** personnes ont acheté chacune en plus une confection régulière de popcorn. Au total il s'est avéré que le coût combiné du ticket de cinéma plus les popcorn était le même pour chacun des deux groupes. Le prix du ticket de cinéma est 6\$. Trouver tous les prix possibles d'une confection régulière de popcorn.

3. Dans la figure, $|AB| = 6$, $|AC| = 6$, et $\angle BAC$ est un angle droit. On tire deux arcs de cercle passant par B et C : un arc est centré en A et l'autre est un demi-cercle de diamètre BC. Quelle est l'aire du triangle $\triangle ABC$? Quelle est la longueur de BC ? Calculer l'aire comprise entre les deux arcs de cercle, c'est-à-dire l'aire de la partie hachurée de la figure.



4. Étant donné un rectangle différent d'un carré, un *découpage carré* consiste à découper le rectangle en deux parties, dont une est un carré (l'autre étant un rectangle possiblement égal à un carré). Par exemple, un découpage carré d'un rectangle 2×7 donne un carré 2×2 et un rectangle 2×5, comme montré.

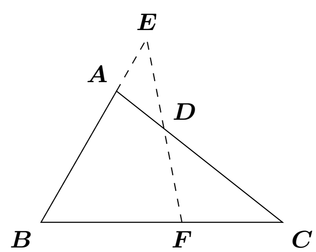


On part d'un rectangle $\mathbf{40 \times 2011}$. À chaque étape, on effectue un découpage carré de la partie rectangulaire qui n'est pas carrée. On continue jusqu'à ce que toutes les parties soient des carrés. Combien de carrés y a-t-il au total ?

5. Cinq équipes, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, et $\boldsymbol{E}$, participent à un tournoi de hockey où chaque équipe joue contre chacune autre exactement une fois. Tout match résulte en une victoire pour une équipe et une défaite pour l'autre ou bien un match nul. Le tableau comportait à l'origine tous les résultats du tournoi, mais quelques cases ont été effacées. En dépit de l'information manquante, l'issue de chaque match peut être déterminée de façon unique. Pour chacun des dix jeux, de déterminer qui a gagné ou si elle était un match nul.

équipe	victoires	défaites	nuls
$\boldsymbol{A}$	3		
$\boldsymbol{B}$	1		1
$\boldsymbol{C}$	1		
$\boldsymbol{D}$	0		
$\boldsymbol{E}$			4

6. Soit $\boldsymbol{ABC}$ un triangle dont les longueurs des côtés sont données par $|\boldsymbol{AB}| = \mathbf{5}$, $|\boldsymbol{AC}| = \mathbf{7}$, et $|\boldsymbol{BC}| = \mathbf{8}$. Le point $\boldsymbol{D}$ appartenant au côté $\boldsymbol{AC}$ est tel que $|\boldsymbol{AB}| = |\boldsymbol{CD}|$. On prolonge le côté $\boldsymbol{BA}$ au-delà de $\boldsymbol{A}$ jusqu'à un point $\boldsymbol{E}$ tel que $|\boldsymbol{AC}| = |\boldsymbol{BE}|$. Appelons $\boldsymbol{F}$ le point d'intersection de la droite $\boldsymbol{ED}$ et du côté $\boldsymbol{BC}$.



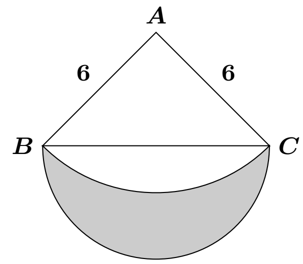
Trouver les longueurs de $\boldsymbol{AD}$, $\boldsymbol{AE}$, $\boldsymbol{BF}$, et $\boldsymbol{FC}$.

Calgary Mathematical Association
35th Junior High School Mathematics Contest
Final Round, Part B, 2011.

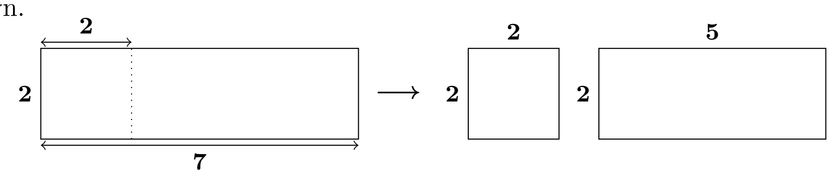
1. Ariel purchased a certain amount of apricots. $\mathbf{90}\%$ of the apricot weight was water. She dried the apricots until just $\mathbf{60}\%$ of the apricot weight was water. $\mathbf{15}$ kg of water was lost in the process. What was the original weight of the apricots (in kg)?

2. A group of ten friends all went to a movie together. Another group of nine friends also went to the same movie together. Fourteen of these nineteen people each bought a regular bag of popcorn as well. It turned out that the total cost of the movie plus popcorn for one of the two groups was the same as for the other group. A movie ticket costs \$6. Find all possible costs of a regular bag of popcorn.

3. In the diagram, $|AB| = 6$, $|AC| = 6$, and $\angle BAC$ is a right angle. Two arcs are drawn: a circular arc with centre A and passing through B and C, and a semi-circle with diameter BC. What is the area of $\triangle ABC$? What is the length of BC? Find the area between the two arcs; that is, find the area of the shaded region in the diagram.



4. Given a non-square rectangle, a *square-cut* is a cutting-up of the rectangle into two pieces, a square and a rectangle (which may or may not be a square). For example, performing a square-cut on a 2×7 rectangle yields a 2×2 square and a 2×5 rectangle, as shown.

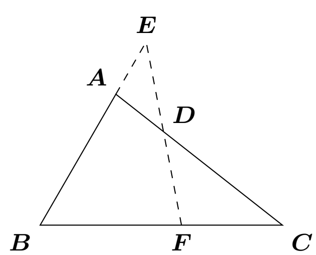


You are initially given a 40×2011 rectangle. At each stage, you make a square-cut on the non-square piece. You repeat this until all pieces are squares. How many square pieces are there at the end?

5. Five teams, A, B, C, D, and E, participate in a hockey tournament where each team plays against each other team exactly once. Each game either ends in a win for one team and a loss for the other, or ends in a tie for both teams. The table originally showed all of the results of the tournament, but some of the entries in the table have been erased. The result of each game played can be uniquely determined. For each of the ten games, determine who won or if it was a tie.

Team	Wins	Losses	Ties
A	3		
B	1		1
C	1		
D	0		
E			4

6. A triangle ABC has sides $|AB| = 5$, $|AC| = 7$, and $|BC| = 8$. Point D is on side AC such that $|AB| = |CD|$. We extend the side BA past A to a point E such that $|AC| = |BE|$. Let the line ED intersect side BC at a point F.



Find the lengths of AD, AE, BF, and FC.

Next follow solutions to the City Competition of the Croatian Mathematical Society, 2010, secondary level, grade 1, given in Skoliad 129 at [2010:481–483].

1. Let n be a positive integer and a a non-zero real number. Reduce the fraction

$$\frac{a^{3n+1}-a^4}{a^{2n+3}+a^{n+4}+a^5}\,.$$

Solution by Harris Lin, student, Killarney Secondary School, Vancouver, BC.

If $n = 1$, the expression equals $\frac{a^4-a^4}{a^5+a^5+a^5} = 0$.
If $n = 2$, the expression equals

$$\frac{a^7-a^4}{a^7+a^6+a^5} = \frac{a^4(a^3-1)}{a^5(a^2+a+1)}\,,$$

but $a^3 - 1^3 = (a-1)(a^2+a+1)$, so the expression equals

$$\frac{a^4(a-1)(a^2+a+1)}{a^5(a^2+a+1)} = \frac{a-1}{a}\,.$$

If $n = 3$, the expression equals

$$\frac{a^{10}-a^4}{a^9+a^7+a^5} = \frac{a^4(a^6-1)}{a^5(a^4+a^2+1)}\,,$$

but, again,

$$a^6-1 = (a^2)^3-1^3 = (a^2-1)((a^2)^2+(a^2)+1) = (a^2-1)(a^4+a^2+1)\,,$$

so the expression equals

$$\frac{a^4(a^2-1)(a^4+a^2+1)}{a^5(a^4+a^2+1)} = \frac{a^2-1}{a}\,.$$

Now hazard the guess that

$$\frac{a^{3n+1}-a^4}{a^{2n+3}+a^{n+4}+a^5} = \frac{a^{n-1}-1}{a}\,.$$

The equation holds if and only if

$$(a^{3n+1}-a^4)a = (a^{n-1}-1)(a^{2n+3}+a^{n+4}+a^5)\,,$$

so

$$\begin{aligned} a^{3n+2}-a^5 &= a^{n-1}a^{2n+3}+a^{n-1}a^{n+4}+a^{n-1}a^5-a^{2n+3}-a^{n+4}-a^5 \\ &= a^{3n+2}+a^{2n+3}+a^{n+4}-a^{2n+3}-a^{n+4}-a^5 \\ &= a^{3n+2}-a^5\,, \end{aligned}$$

but this is obviously true, so the guess is correct.

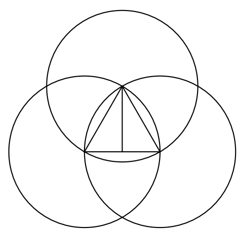
2. Find a positive integer which when multiplied by 9 gives an integer between 1100 and 1200, and when multiplied by 13 gives an integer between 1500 and 1600.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

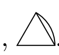
Since $122 \cdot 9 = 1098 < 1100$ and $124 \cdot 13 = 1612 > 1600$, the only integer that could satisfy the conditions is 123. Now, $123 \cdot 9 = 1107$ and $123 \cdot 13 = 1599$, so 123 does indeed satisfy the conditions.

3. Three circles, each with radius 2, are given in the plane such that the centre of each lies on the intersection of the other two. Determine the area of the intersection of the three disks bounded by those circles.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.



The three centres form an equilateral triangle with side length 2. The intersection of the three disks consists of this triangle and three congruent segments. Using the Pythagorean Theorem, the triangle has height $\sqrt{2^2 - 1^2} = \sqrt{3}$. Thus the area of the triangle is $\frac{2\sqrt{3}}{2} = \sqrt{3}$.

Each of the segments are a part of a sector in a circle of radius 2, . Since the triangle is equilateral, the sector angle is 60°, so the area of the sector is $\frac{60}{360}\pi 2^2 = \frac{2}{3}\pi$. The triangle has area $\sqrt{3}$, so each segment has area $\frac{2}{3}\pi - \sqrt{3}$.

Therefore the area of the intersection is $\sqrt{3} + 3(\frac{2}{3}\pi - \sqrt{3}) = 2\pi - 2\sqrt{3}$.

4. Consider the integer n. Let m be the integer obtained from n by removing its ones digit. If $n - m = 2010$, find n.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Introduce letters for the digits of n and m so the problem has the form

$$\begin{array}{r} abcd \\ -\ abc \\ \hline 2010 \end{array}\ .$$

Since abc is at most 999, the digit a must be 2 or 3. However, if $a = 3$, then $n - m > 3000 - 399 = 2601 > 2010$, so $a = 2$. Now the problem has the form

$$\begin{array}{r} 2bcd \\ -\ 2bc \\ \hline 2010 \end{array}\ .$$

Therefore, b must be 2 or 3. If $b = 3$, then $n - m > 2300 - 239 = 2061 > 2010$, so $b = 2$, and the problem has the form

$$\begin{array}{r} 22cd \\ -\ 22c \\ \hline 2010 \end{array}\ .$$

Again, c must be 3 or 4, but if $c = 4$, then $n - m > 2240 - 224 = 2016 > 2010$, so $c = 3$, and the problem has the form

$$\begin{array}{r} 223d \\ -\ 223 \\ \hline 2010 \end{array}\ .$$

Now d must be 3. Thus $n = 2233$.

Alternatively, let d denote the ones digit of n. Then $n = 10m + d$, so $2010 = n - m = 9m + d$. Since $2010 \div 9 \approx 233.3$, $m = 233$. Finally, $d = 2010 - 9m = 3$, and $n = 2233$ as above.

5. A bag contains a sufficient number of red, white, and blue balls. Each child in a given group takes three balls at random from the bag. What is the smallest number of children in the group that ensures that two of them have taken the same combination of balls, that is, the same number of balls of each colour?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The possible colour combinations are RRR, WWW, BBB, RRW, RWW, RRB, BBR, WWB, WBB, RWB. Since there are 10 colour combinations, 10 children could have all different combinations, but with 11 children, at least two would have the same combination. Thus the answer is 11 children.

6. If $a^2 + 2b^2 = 3c^2$, prove that

$$\left(\frac{a+b}{b+c} + \frac{b-c}{b-a}\right) \cdot \frac{a+2b+3c}{a+c}$$

is a positive integer.

Solution by Harris Lin, student, Killarney Secondary School, Vancouver, BC.

Note that

$$\frac{a+b}{b+c} \cdot \frac{a+2b+3c}{a+c} = \frac{a^2 + 2ab + 3ac + ab + 2b^2 + 3bc}{ab + bc + ac + c^2}\ .$$

Since $a^2 + 2b^2 = 3c^2$, this equals

$$\frac{3ab + 3ac + 3bc + 3c^2}{ab + bc + ac + c^2} = \frac{3(ab + ac + bc + c^2)}{ab + bc + ac + c^2} = 3\ .$$

Similarly,

$$\frac{b-c}{b-a}\cdot\frac{a+2b+3c}{a+c}=\frac{ab+2b^2+3bc-ac-2bc-3c^2}{ab+bc-a^2-ac}\,.$$

Since $a^2 = 3c^2 - 2b^2$, this equals

$$\frac{ab+bc-ac-a^2}{ab+bc-a^2-ac}=1\,.$$

Thus

$$\left(\frac{a+b}{b+c}+\frac{b-c}{b-a}\right)\cdot\frac{a+2b+3c}{a+c}=3+1=4\,.$$

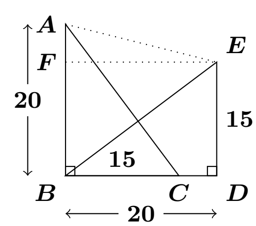
7. A right triangle, $\triangle ABC$, with legs of lengths 15 and 20 and the right angle at vertex B is congruent to a triangle, $\triangle BDE$, with the right angle at vertex D. The point C lies strictly inside the segment $\overline{BD}$, and the points A and E are on the same side of the straight line BD.

(a) Find the distance between points A and E.

(b) Find the area of the intersection of $\triangle ABC$ and $\triangle BDE$.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

(a) The triangles must sit as in the diagram on the left. Since $\angle ABC = \angle BDE = 90^\circ$, there exists a point F on AB such that $BDEF$ is a rectangle. Now $|AF| = 5$ and $|EF| = 20$, so the Pythagorean Theorem yields that $|AE|^2 = 20^2 + 5^2 = 425$. Thus $|AE| = \sqrt{425} = 5\sqrt{17}$.



(b) Now impose a coordinate system such that $A = (0,20)$, $B = (0,0)$, $C = (15,0)$, $D = (20,0)$, and $E = (20,15)$. Then the line through B and E has the equation $y = \frac{15}{20}x = \frac{3}{4}x$. Moreover, the line through A and C has the equation $y = \frac{-20}{15}x + 20 = -\frac{4}{3}x + 20$. These two lines intersect when $\frac{3}{4}x = -\frac{4}{3}x + 20$, thus $\frac{9+16}{12}x = 20 \Rightarrow \frac{25}{12}x = 20$, so $x = \frac{48}{5}$, and $y = \frac{3}{4}x = \frac{36}{5}$.

Thus the intersection between $\triangle ABC$ and $\triangle BDE$ is itself a triangle with height $\frac{36}{5}$ and base 15. Therefore the desired area is $\frac{1}{2}\cdot 15\cdot\frac{36}{5} = 54$.

8. Let p and q be different odd prime numbers. Prove that the integer $(pq+1)^4 - 1$ has at least four different prime divisors.

Solution by the editors.

Since $a^2 - b^2 = (a-b)(a+b)$,

$$\begin{aligned}(pq+1)^4 - 1 &= \Big((pq+1)^2 - 1\Big)\Big((pq+1)^2 + 1\Big) \\ &= (pq+1-1)(pq+1+1)((pq)^2 + 2pq + 1 + 1) \\ &= pq(pq+2)(p^2q^2 + 2pq + 2)\,.\end{aligned}$$

Since p and q are odd, $pq+2$ is not divisible by either p or q. Let n denote $pq+2$, and note that n is odd. Then $pqn + 2 = pq(pq+2) + 2 = p^2q^2 + 2pq + 2$, so

$$(pq+1)^4 - 1 = pqn(pqn+2)\,.$$

Note that $pqn + 2$ is not divisible by p or q since they are odd.

If n is not a power of a single prime, then n is divisible by at least two different primes, and $(pq+1)^4 - 1$ is divisible by at least four different primes.

If n is a power of a prime, say $n = r^k$ where r is a prime and k is a positive integer, then r is odd because n is odd. Therefore $pqn + 2$ is not divisible by either of p, q, or r, so $pqn + 2$ contains a prime factor different from p, q, and r. Thus $(pq+1)^4 - 1$ is divisible by at least four different primes.

*This problem was the Problem of the Month in the December 1999 issue of **Crux Mathematicorum**, [1999 : 495]. We encourage the reader to look up the solution at http://cms.math.ca/crux/*

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

We hope to receive more reader solutions to this issue's featured contest.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

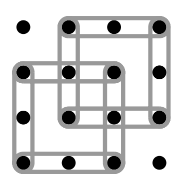
Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **1 septembre 2012**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

M501. *Proposé par l'Équipe de Mayhem.*

Un grillage **4** par **4** est formé de chevilles amovibles se situant à un centimètre l'une de l'autre, tel qu'illustré dans le schéma. Des élastiques sont attachés aux chevilles de façon à former des carrés ; deux carrés différents de taille **2** par **2** sont illustrés dans le schéma. Combien de différents carrés sont possibles ?



M502. *Proposé par l'Équipe de Mayhem.*

Lors de leur dernier match de basket, Arianne, Bernard et Claudine ont compté un total de **23** points entre eux. Chaque joueur a compté au moins **1** point ; Claudine a compté au moins **10** fois. De combien de façons les **23** points peuvent-ils être répartis, s'il faut satisfaire aux conditions ci-haut ? Par exemple, **5** points par Arianne, **3** points par Bernard et **15** points par Claudine serait une possibilité et **3** points par Arianne, **5** par Bernard et **15** par Claudine en serait une autre.

M503. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Écrire n'importe quel nombre entier et, à la suite, y adjoindre son renversement. Par exemple, si on a écrit **13**, on obtiendra **1331**. Montrer que le résultat de cette opération est toujours divisible par **11**.

M504. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

À l'intérieur d'un triangle rectangle de côtés **3**, **4** et **5**, deux cercles égaux sont tracés de façon à ce qu'ils soient tangents l'un à l'autre et à un des côtés. De plus, un des cercles est tangent à l'hypoténuse ; l'autre cercle de la paire est tangent à l'autre côté. Déterminer les rayons des cercles dans les deux cas.

M505. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Démontrer que pour tout entier positif n, les nombres $A = 5n + 7$ et $B = 6n^2 + 17n + 12$ sont relativement premiers (i.e. n'ont aucun facteur commun sauf **1**).

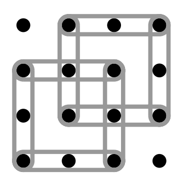
M506. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB ; et Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Nous tentons de créer un ensemble de nombres, tels que chacun est obtenu en prenant seulement ses propres chiffres et leur appliquant les quelconques opérations arithmétiques et/ou symboles qui vous sont familiers. Chaque expression doit inclure au moins un symbole/opération ; le nombre de fois qu'un chiffre apparait est le même que dans le nombre lui-même. Par exemple, $1 = \sqrt{1}$, $36 = 6 \times 3!$ et $121 = 11^2$. Toute contribution valide sera reconnue.

. . . .

M501. *Proposed by the Mayhem Staff.*

A **4** by **4** square grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form squares, two different **2** by **2** squares are shown in the diagram. How many different squares are possible?



M502. *Proposed by the Mayhem Staff.*

At their last basketball game Alice, Bob and Cindy scored a total of **23** points between them. Each player got at least **1** point, and Cindy scored at least **10**. How many different ways could the **23** points been awarded to satisfy the conditions? For example: **5** points for Alice, **3** points for Bob, **15** for Cindy; and **3** points for Alice, **5** points for Bob, **15** for Cindy; are two different possibilities.

M503. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Write any number and then follow that number by adjoining its reversal. For example, if you write **13** then you would get **1331**. Show that the resulting number is always divisible by **11**.

M504. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides 3, 4, 5, two equal circles are drawn that are tangent to one another and to one leg. One circle of the pair is tangent to the hypotenuse. The other circle of the pair is tangent to the other leg. Determine the radii of the circles in both cases.

M505. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that, for all positive integers n, the quantities $A = 5n + 7$ and $B = 6n^2 + 17n + 12$ are coprime (i.e. have no common factors other than 1).

M506. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

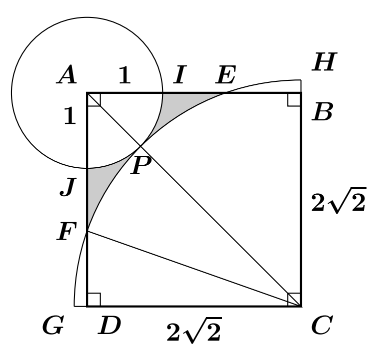
We are trying to create a set of positive integers, that each can be formed using their own digits only, along with any mathematical operations and/or symbols that are familiar to you. Each expression must include at least one symbol/operation; the number of times a digit appears is the same as in the number itself. For example, $1 = \sqrt{1}$, $36 = 6 \times 3!$ and $121 = 11^2$. All valid contributions will be acknowledged.

Mayhem Solutions

M463. *Proposed by the Mayhem Staff.*

The square $ABCD$ has side length $2\sqrt{2}$. A circle with centre A and radius 1 is drawn. A second circle with centre C is drawn so that it just touches the first circle at point P on AC. Determine the total area of the regions inside the square but outside the two circles.

Solution by Gloria Fang, student, University of Toronto Schools, Toronto, ON.



From the Pythagorean Theorem we get $AC = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = \sqrt{16} = 4$. Since CP is a radius of the larger circle, and $AP = 1$ we get $CP = CA - PA = 4 - 1 = 3$, therefore the radius of the large circle is 3.

Using $[A]$ to represent the area of figure A we get

$$
\begin{aligned}
[GDF] &= [FCG] - [FCD] \\
&= \frac{\alpha}{2\pi} \cdot \pi(3)^2 - \frac{\sqrt{3^2 - (2\sqrt{2})^2} \cdot 2\sqrt{2}}{2} \\
&= \frac{9}{2}\alpha - \sqrt{2}
\end{aligned}
$$

where $\alpha = \angle FCG = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$. Next, we have $[PCDF] = [PCG] - [GDF] = \frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}$. Thus $[JPF] = [ACD] - [APJ] - [PCDF] = 4 - \frac{\pi}{8} - \left(\frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}\right) = 4 - \sqrt{2} - \frac{5\pi}{4} + \frac{9}{2}\alpha$.

By symmetry $[IPE] = [JPE]$ thus the area of the regions inside the square but not inside a circle is $2[JPE]$, or

$$8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\alpha = 8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) \doteq 0.38 \text{ units.}$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. Four incorrect solutions were received.

M464. *Proposed by the Mayhem Staff.*

Let $\lfloor x \rfloor$ be the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Find all real numbers x for which $\lfloor \sqrt{x^2+1} - 1 \rfloor = 2$.

Solution by Florencio Cano Vargas, Inca, Spain.

The given equation is equivalent to the inequalities:

$$
\begin{aligned}
2 \le \sqrt{x^2+1} - 1 < 3 &\Leftrightarrow 3 \le \sqrt{x^2+1} < 4 \\
\Leftrightarrow 9 \le x^2 + 1 < 16 &\Leftrightarrow 8 \le x^2 < 15
\end{aligned}
$$

and then, the result is the interval $x \in (-\sqrt{15}, -2\sqrt{2}] \cup [2\sqrt{2}, \sqrt{15})$.

Also solved by GLORIA FANG, student, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. Two incorrect solutions were received.

M465. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

The integer 20114022 is divisible by 2011. Determine if there exists a positive integer that is divisible by 2011 and whose digits add to 2011.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

The answer is yes. The positive integer $\mathbf{201120112011 \cdots 201110055}$, which contains $\mathbf{500}$ groups with digits $\mathbf{2011}$ and $\mathbf{10055 = 5 \times 2011}$ as its final digits, is divisible by $\mathbf{2011}$ because a direct division would show that it is equal to $\mathbf{2011 \times 100010001 \cdots 1000100005}$, where the last number of this product contains $\mathbf{499}$ groups with digits $\mathbf{1000}$ and the digits $\mathbf{100005}$ as its final digits. Another calculation shows that the digits add to

$$(2+0+1+1)+(2+0+1+1)+\cdots+(2+0+1+1)+(1+0+0+5+5)$$
$$= \overbrace{4+4+4+\cdots+4}^{500 \text{ times}} + 11 = 2011\,.$$

Other solutions can be obtained in a similar way, for example $\mathbf{20112011 \cdots 201112066}$, made up of $\mathbf{499}$ groups with digits $\mathbf{2011}$ and $\mathbf{12066 = 3 \times 2011}$ as its final digits.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M466. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all pairs (m, n) of positive integers such that $2^m - 2 = n!$.

I. Solution by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.

The condition is equivalent to $2^{m-1} - 1 = \frac{n!}{2}$. Since $\frac{n!}{2}$ is non-zero, it must be odd, thus n can only be 1, 2 or 3. Trying each case we find the only solutions are $m = n = 2$ or $m = n = 3$.

II. Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

We have $2^m - 2 = n!$ or $2^m = 2 + n!$. If $n \ge 4$, then $2 + n! \ge 26$, thus $m > 4$. Hence, $2^m \equiv 0 \pmod 4$ and $n! + 2 \equiv 2 \pmod 4$, a contradiction.

Therefore, $n \le 3$. By direct calculation we get $(m, n) = (3, 3)$ or $(m, n) = (2, 2)$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer. One incorrect solution was received.

M467. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all real numbers x for which

$$(x - 2010)^3 + (2x - 2010)^3 + (4020 - 3x)^3 = 0\,.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

First, we observe that the given equation can be written as

$$(x - 2010)^3 + (2x - 2010)^3 = (3x - 4020)^3\,.$$

Putting $a = x - 2010$, $b = 2x - 2010$, and $c = 3x - 4020$, we have $a + b = c$ and $a^3 + b^3 = c^3$ from which immediately follows $(a + b)^3 = a^3 + b^3$ or $3ab(a + b) = 0$. To get the solutions we consider the following two cases:

(i) $a + b = 3x - 4020 = 0$ from which we obtain the solution $x = 1340$;

(ii) $ab = (x - 2010)(2x - 2010) = 0$ from which we obtain the solutions $x = 2010$ and $x = 1005$.

Since the polynomial has degree three and we have found three solutions, on account of the Fundamental Theorem of Algebra, the given equation does not have more roots and we are done.

Also solved by MIHÁLY BENCZE, Brasov, Romania; FLORENCIO CANO VARGAS, Inca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

Bencze generalized the problem by writing it as $(ax - b)^3 + (cx - d)^3 + (b + d - (a + c)x)^3 = 0$, *with* $ac(a + c) \neq 0$, *which has solutions* $x_1 = \frac{b}{a}$, $x_2 = \frac{d}{c}$ *and* $x_3 = \frac{b+d}{a+c}$. *Thus setting* $a = 1$, $b = 2010$, $c = 2$ *and* $d = 2010$ *yields the given problem and its solutions.*

M468. *Proposed by Gheorghe Ghiță, M. Eminescu National College, Buzău, Romania.*

Determine all pairs (p, q) of prime numbers for which each of $p + q$, $p + q^2$, $p + q^3$, and $p + q^4$ is a prime number.

Solution by Florencio Cano Vargas, Inca, Spain.

Let p, q be prime numbers. If we require that $p + q > 2$ has to be prime, either $p = 2$ or $q = 2$ to ensure that $p + q$ is odd. Let us consider two cases separately:

Case 1: $q = 2$

In this case, p must be an odd prime number $p \geq 3$. Clearly, $p = 3$ is a solution since $p + q = 5$, $p + q^2 = 7$, $p + q^3 = 11$ and $p + q^4 = 19$ are prime numbers. For $p > 3$ we will study the values of the expressions modulo 3. Let $p \equiv t (\text{mod } 3)$ where $t = 1$ or $t = 2$ since $p > 3$ is a prime. Then

$$p + q \equiv t + 2 (\text{mod } 3)$$
$$p + q^2 \equiv t + 4 \equiv t + 1 (\text{mod } 3)$$

and therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Case 2: $p = 2$

In this case, q must be an odd prime number $q \geq 3$. Clearly, $q = 3$ is a solution since $p + q = 5$, $p + q^2 = 11$, $p + q^3 = 29$ and $p + q^4 = 83$ are prime numbers. For $q > 3$ we will study the values of the expressions modulo 3. Let $q \equiv t (\bmod 3)$ where $t = 1$ or $t = 2$ since $q > 3$ is a prime, then

$$p + q \equiv 2 + t (\bmod 3)$$
$$p + q^2 \equiv 2 + t^2 (\bmod 3) .$$

If $t = 1$, $p + q$ is divisible by three and if $t = 2$, $p + q^2$ is divisible by three, therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Summarizing, the only pairs (p, q) of prime numbers which satisfy the condition of the problem are $(2, 3)$ and $(3, 2)$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GLORIA FANG, student, University of Toronto Schools, Toronto, ON; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

M469. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

Prove that, for all real numbers x, we have $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2-\sqrt{2}}$.

Solution by Gloria Fang, student, University of Toronto Schools, Toronto, ON.

By the AM-GM inequality we have that $\frac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x} \cdot 2^{\cos x}}$, so $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 4 \cdot 2^{\sin x} \cdot 2^{\cos x}$ thus $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2+\sin x+\cos x}$. Using well known trigonometric identities we get

$$\begin{aligned} \sin x + \cos x &= \frac{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x}{\frac{1}{\sqrt{2}}} \\ &= \frac{\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2} \sin \left(\frac{\pi}{4} + x\right) . \end{aligned}$$

Since $-1 \leq \sin\left(\frac{\pi}{4} + x\right) \leq 1$, we must have $\sqrt{2} \sin\left(\frac{\pi}{4} + x\right) \geq -\sqrt{2}$. Thus

$$\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2+\sin x+\cos x} = 2^{2+\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)} \geq 2^{2-\sqrt{2}} .$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; FLORENCIO CANO VARGAS, Inca, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

[Ed.: Note the solution could be shortened by noting that $\sin 2x = 2\sin(x)\cos(x) \leq 1 \Rightarrow (\sin(x) + \cos(x))^2 \leq 2 \Rightarrow 2 - \sqrt{2} \leq 2 + \sin(x) + \cos(x)$, *from which the result follows.]*

THE OLYMPIAD CORNER

No. 296

R.E. Woodrow and Nicolae Strungaru

The problems from this issue come from the Italian Team Selection Test, the British Mathematical Olympiad, the Macedonian Mathematical Olympiad, and the Chinese Mathematical Olympiad. Our thanks go to Adrian Tang for sharing the material with the editors.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobre 2012**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC31. Trouver toutes les paires (p, q) de nombres premiers tels que $pq \mid (5^p + 5^q)$.

OC32. Soit ABC un triangle acutangle avec $\angle B = \angle C$. Soit O le centre de son cercle circonscrit et H son orthocentre. Montrer que le centre du cercle BOH se situe sur la droite AB.

OC33. Soit n et k deux entiers tels que $n \geq k \geq 1$. On considère un cercle de n ampoules électriques, toutes éteintes. A chaque tour, on peut changer le statut d'un ensemble quelconque de k ampoules consécutives. Parmi les 2^n combinaisons possibles, combien peut-on en engendrer

(a) si k est un premier impair ?

(b) si k est un entier impair ?

(c) si k est un entier pair ?

OC34. Soit m et n deux entiers avec $4 < m < n$, et $A_1A_2 \cdots A_{2n+1}$ un $2n+1$-gone régulier. Soit $P = \{A_1, A_2, \cdots, A_{2n+1}\}$. Trouver le nombre de m-gones convexes avec exactement deux angles droits internes et dont les sommets sont tous dans P.

OC35. Trouver toutes les paires d'entiers (x, y) telles que

$$y^3 = 8x^6 + 2x^3y - y^2 .$$

OC36. Soit a, b et c les longueurs des côtés opposés respectivement aux angles $\angle A$, $\angle B$ et $\angle C$ d'un triangle obtusangle ABC. Montrer que

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc\,.$$

OC37. Trouver tous les entiers n rendant possible le coloriage de toutes les arêtes et diagonales d'un n-gone convexe avec n couleurs satisfaisant les conditions suivantes :

(i) Chacune des arêtes et diagonales est coloriée par une seule couleur ;

(ii) Pour tout ensemble de trois couleurs distinctes, il existe un triangle dont les sommets sont des sommets du n-gone et les trois arêtes sont respectivement coloriées par les trois couleurs de cet ensemble.

OC38. Soit a, b et c trois nombres réels positifs tels que $ab + bc + ca = \frac{1}{3}$. Montrer que l'inégalité suivante est satisfaite :

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}\,.$$

OC39. Étant donné un entier positif n, soit $b(n)$ le nombre d'entiers positifs dont les représentations binaires apparaissent comme blocs d'entiers consécutifs dans la représentation binaire de n. Par exemple $b(13) = 6$, puisque $13 = 1101_2$, qui contient comme blocs consécutifs les représentations binaires de $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ et $1 = 1_2$.

Montrer que si $n \leq 2500$, alors $b(n) \leq 39$, et déterminer les valeurs de n pour lesquelles on a égalité.

OC40. Soit M et N les intersections de deux cercles Γ_1 et Γ_2. Soit AB la tangente commune aux deux cercles, la plus rapprochée de M, disons $A \in \Gamma_1$ et $B \in \Gamma_2$. Soit respectivement C et D les points symétriques de A et B par rapport à M. Soit respectivement E et F les intersections du cercle circonscrit de DCM et des cercles Γ_1 et Γ_2.

Montrer que les rayons des cercles circonscrits des triangles MEF et NEF sont d'égale longueur.

. . . .

OC31. Find all pairs (p, q) of prime numbers such that $pq \mid (5^p + 5^q)$.

OC32. Let ABC be an acute-angled triangle with $\angle B = \angle C$. Let the circumcentre be O and the orthocentre be H. Prove that the centre of the circle BOH lies on the line AB.

OC33. Let n and k be integers such that $n \geq k \geq 1$. There are n light bulbs placed in a circle. They are all turned off. Each turn, you can change the state of any set of k consecutive light bulbs.

How many of the 2^n possible combinations can be reached

(a) if k is an odd prime?

(b) if k is an odd integer?

(c) if k is an even integer?

OC34. Let m, n be integers with $4 < m < n$, and $A_1A_2 \cdots A_{2n+1}$ be a regular $2n+1$-gon. Let $P = \{A_1, A_2, \cdots, A_{2n+1}\}$. Find the number of convex m-gons with exactly two acute internal angles whose vertices are all in P.

OC35. Find all pairs of integers (x, y) such that

$$y^3 = 8x^6 + 2x^3y - y^2 .$$

OC36. The obtuse-angled triangle ABC has sides of length a, b, and c opposite the angles $\angle A$, $\angle B$ and $\angle C$ respectively. Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc .$$

OC37. Find all integers n such that we can colour all the edges and diagonals of a convex n-gon by n given colours satisfying the following conditions:

(i) Every one of the edges or diagonals is coloured by only one colour;

(ii) For any three distinct colours, there exists a triangle whose vertices are vertices of the n-gon and the three edges are coloured by the three colours, respectively.

OC38. Let a, b, c be positive real numbers such that $ab + bc + ca = \frac{1}{3}$. Prove the inequality:

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c} .$$

OC39. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.

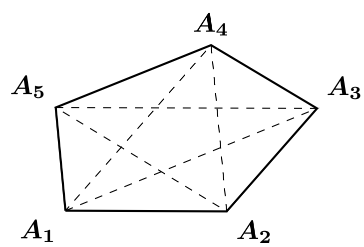
OC40. Let M and N be the intersection of two circles, Γ_1 and Γ_2. Let AB be the line tangent to both circles closer to M, say $A \in \Gamma_1$ and $B \in \Gamma_2$. Let C be the point symmetrical to A with respect to M, and D the point symmetrical to B with respect to M. Let E and F be the intersections of the circle circumscribed around DCM and the circles Γ_1 and Γ_2, respectively.

Show that the circles circumscribed around the triangles MEF and NEF have radii of the same length.

First we look at the remainder of the solutions for the 48th IMO Bulgarian Team, First Selection Test, given at [2010: 275] that we started last issue.

2. Let $A_1A_2A_3A_4A_5$ be a convex pentagon such that the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5$, $A_4A_5A_1$, $A_5A_1A_2$ have the same area. Prove that there exists a point M such that the triangles A_1MA_2, A_2MA_3, A_3MA_4, A_5MA_1 have the same area.

Solved by Titu Zvonaru, Cománeşti, Romania, shortened by the editor.



Triangles $A_5A_1A_2$ and $A_1A_2A_3$ have the same base and area, whence $A_1A_2||A_5A_3$. Similarly we deduce that each side of the given pentagon is parallel to a diagonal. If in a convex pentagon each side is parallel to a diagonal, then the ratio of a diagonal to the corresponding parallel side is the golden section $\varphi = \frac{1+\sqrt{5}}{2}$. For a simple proof of this result see the solution to problem M133 [2005: 278-279]. Recall that affine transformations preserve the ratios of segment lengths along parallel lines, so that the affine transformation that takes the first three vertices of the regular pentagon $A_1'A_2'A_3'A_4'A_5'$ to $A_1A_2A_3$ will take the vertex A_4' (which is the point on the line through A_1' parallel to $A_2'A_3'$ for which the directed segment $A_1'A_4'$ is φ times the length of the directed segment $A_2'A_3'$) to the point A_4. Similarly, A_5' is taken to A_5. Of course, the centre of gravity M' of the regular pentagon has the property that the areas of the five triangles $A_i'M'A_{i+1}'$ are equal. (In fact the triangles are congruent.) Our affine transformation takes M' to the centre of gravity M of the given pentagon. Because affine transformations preserve ratios of areas, the areas of A_iMA_{i+1} are equal, as required.

3. Prove that there are no distinct positive integers x and y such that

$$x^{2007} + y! \;=\; y^{2007} + x!\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Geupel.

We prove the more general result that for each integer $n \geq 2$ such that $n \neq 2^m - 1$ ($m \in \mathbb{N}$) the function $f : \mathbb{N} \to \mathbb{Z} : f(x) = x! - x^n$ is injective.

Firstly, we show that f is increasing for $x \geq 2n$. Indeed, for $x \geq 2n$ we have $x! \geq (1 \cdot x)(2 \cdot (x-1)) \cdots (n \cdot (x+1-n)) \geq x \cdot x \cdots x = x^n$. Taking into account that

$$\left(1+\frac{1}{x}\right)^n \leq \left(1+\frac{1}{2n}\right)^{2n} \leq e,$$

we obtain

$$f(x+1) \;=\; \left(1+\frac{1}{x}\right)^n (x! - x^n) + \left[x+1-\left(1+\frac{1}{x}\right)^n\right] x!$$
$$\geq\; f(x) + \left[x+1-\left(1+\frac{1}{2n}\right)^{2n}\right] x! \;\geq\; f(x) + (x+1-e)x! \;>\; f(x),$$

which completes the proof that f is increasing for $x \geq 2n$.

We prove our initial claim by contradiction.

Assume that we have $x \leq y$ and $f(x) = f(y)$. For $k \in \mathbb{Z}$ and p prime, let $d_p(k)$ denote the greatest $\alpha \in \mathbb{Z}$ such that $p^\alpha \mid k$. Let p be any prime divisor of x. Then, $p \mid x!$, $p \mid x^n$, and $p \mid y!$. Hence, $p \mid x^n - x! + y!$, that is $p \mid y^n$ and therefore $p \mid y$. From $y! - x! = x![(x+1)(x+2)\cdots y - 1]$, we see that $d_p(x!) = d_p(y! - x!) = d_p(y^n - x^n) \geq n$.

By

$$d_p(x!) = \left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots \leq x\left(\frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \frac{x}{p-1},$$

we deduce $x \geq (p-1)d_p(x!) \geq (p-1)n$. Because $f(z)$ is increasing for $z \geq 2n$, we must have $p = 2$ and $x = 2^m$. Thus, $n \leq 2^m < 2n$. From

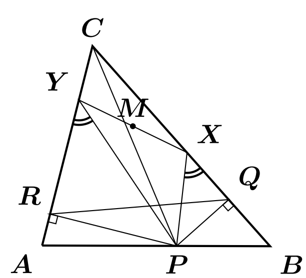
$$d_2(y! - x!) = d_2(x!) = 2^m\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^m}\right) = 2^m - 1$$

and $n \mid d_2(y^n - x^n)$, we conclude $n \mid 2^m - 1$. Consequently $n = 2^m - 1$, a contradiction.

4. Given a point P on the side AB of a triangle ABC, consider all pairs of points (X, Y), $X \in BC$, $Y \in AC$ such that $\angle PXB = \angle PYA$. Prove that the mid-points of the segments XY lie on a straight line.

Solved by Titu Zvonaru, Comăneşti, Romania.

Let Q, R be the projections of P onto BC and AC, respectively, and let M be the midpoint of XY.



We denote $m = CR$, $n = CQ$, $P = RQ$, $\alpha = \angle PXB = \angle PYA$. By Stewart's Theorem we obtain

$$\begin{aligned} & QC^2 \cdot RY - QY^2 \cdot CR + QR^2 \cdot CY = CR \cdot CY \cdot YR \\ \Leftrightarrow \quad & m \cdot QY^2 = n^2 \cdot RY + p^2(m - RY) - m(m - RY) \cdot RY \\ \Leftrightarrow \quad & m \cdot QY^2 = RY(n^2 - p^2 - m^2) + mp^2 + m \cdot RY^2 \\ \Leftrightarrow \quad & m \cdot QY^2 = RY(-2mp\cos\angle CRQ) + mp^2 + m \cdot RY^2 \end{aligned}$$

Since the quadrilateral $CRPQ$ is cyclic, we deduce

$$\cos\angle CRQ = \cos\angle CPQ = \frac{PQ}{CP} = \frac{RX\tan\alpha}{CR}\,.$$

It follows that

$$QY^2 = -2p \cdot \frac{RX \cdot RY \cdot \tan\alpha}{CP} + p^2 + RY^2 \tag{1}$$

and similarly

$$RX^2 = -2p \cdot \frac{RX \cdot RY \cdot \tan\alpha}{CP} + p^2 + QX^2\,. \tag{2}$$

By (1) and (2) we have

$$\begin{aligned} QY^2 - RY^2 &= RX^2 - QX^2 \\ \Leftrightarrow 2(RX^2 + RY^2) - XY^2 &= 2(QX^2 + QY^2) - XY^2 \\ \Leftrightarrow RM^2 &= QM^2\,, \end{aligned}$$

hence the point M lies on the perpendicular bisector of QR.

5. The real numbers a_i, b_i, $1 \le i \le n$, are such that

$$\sum_{i=1}^{n} a_i^2 \;=\; 1\,, \quad \sum_{i=1}^{n} b_i^2 \;=\; 1 \quad \text{and} \quad \sum_{i=1}^{n} a_i b_i \;=\; 0\,.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \;\le\; n\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; and Henry Ricardo, Tappan, NY, USA. We use the write-up of Apostolopoulos.

Let $x = \sum_{i=1}^{n} a_i$ and $y = \sum_{i=1}^{n} b_i$. By Cauchy-Schwarz Inequality we have

$$
\begin{aligned}
(x^2+y^2)^2 &= \left[\sum_{i=1}^{n}(a_ix+b_iy)\right]^2 \le n\sum_{i=1}^{n}(a_ix+b_iy)^2 \\
&= n\sum_{i=1}^{n}(a_i^2x^2+b_i^2y^2+2a_ib_ixy) \\
&= n\left(x^2\sum_{i=1}^{n}a_i^2+y^2\sum_{i=1}^{n}b_i^2+2xy\sum_{i=1}^{n}a_ib_i\right) \\
&= n(x^2+y^2)
\end{aligned}
$$

so $(x^2+y^2)^2 \le n(x^2+y^2)$, namely

$$x^2+y^2 \le n \Leftrightarrow \left(\sum_{i=1}^{n}a_i\right)^2+\left(\sum_{i=1}^{n}b_i\right)^2 \le n.$$

6. For a finite set S denote by $\mathcal{P}(S)$ the set of all subsets of S. The function $f:\mathcal{P}(S)\to\mathbb{R}$ is such that

$$f(X\cap Y) = \min(f(X),f(Y))$$

for any two subsets $X,Y\in\mathcal{P}(S)$. Find the largest number of distinct values that f can take.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Given

$$S=\{a_1,a_2,\dots,a_n\},$$

define, for $i=1,2,\dots,n$,

$$X_i = S\backslash\{a_i\}.$$

Set $x_i=f(X_i)$, and define $M=f(S)$. This defines f on all other subsets of S since all others can be formed from intersections of these. Moreover, these other function values are in $\{x_1,x_2,\dots,x_n,M\}$. Hence, f has at most $n+1$ distinct values. In fact, each subset of S can be obtained in a unique way (apart from order) from intersections of the X_i and S; hence, any set of choices of the x_i and of M gives an allowable f. Accordingly, f can have $n+1$ distinct values.

Next we turn to the solutions of problems of the 48$^{\text{th}}$ IMO Bulgarian Team, Second Selection Test, given at [2010; 276].

2. Find all positive integers m such that

$$\frac{2^m\alpha^m-(\alpha+\beta)^m-(\alpha-\beta)^m}{3\alpha^2+\beta^2}$$

is an integer for all integer values of α, β with $\alpha\beta \neq 0$.

Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Apostolopoulos.

Denote $x = \alpha + \beta$, $y = \alpha - \beta$. Then

$$\frac{(2\alpha)^m - (\alpha+\beta)^m - (\alpha-\beta)^m}{3\alpha^2 + \beta^2} = \frac{(x+y)^m - x^m - y^m}{x^2 + xy + y^2},$$

where $\alpha\beta \neq 0 \Leftrightarrow x^2 \neq y^2$. Now we consider $(x+y)^m - x^m - y^m$ and $x^2 + xy + y^2$ as polynomials with one variable x. Do the division and we get that $(x+y)^m - x^m - y^m = (x^2+xy+y^2)f(x,y) + g(y)x + h(y)$, where $f(x,y)$, $g(y)$, $h(y)$ are polynomials with integer coefficients. Suppose m satisfies that $x^2 + xy + y^2 \mid (x+y)^m - x^m - y^m$. Fix y, let x vary in the set of positive integers. We have $x^2 + xy + y^2 \mid g(y)x + h(y)$. But for very large x, $|x^2+xy+y^2| > |g(y)x+h(y)|$, then $g(y)x+h(y) = 0 \Rightarrow g(y) = h(y) = 0$. We deduce that for every y, $g(y) = h(y) = 0$. Thus g and h are both zero polynomials. On the other hand if g and h are zero polynomials, it is clear that $x^2+xy+y^2 \mid (x+y)^m - x^m - y^m$ for $x^2 \neq y^2$. Thus m is valid $\Leftrightarrow g \equiv h \equiv 0$.

Next let $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, then $w^2 + w + 1 = 0$, $w^3 = 1$. We claim that $g \equiv h \equiv 0 \Leftrightarrow (w+1)^m - w^m - 1 = 0$. If $g \equiv h \equiv 0$, then $(x+y)^m - x^m - y^m = (x^2+xy+y^2)f(x,y)$. This holds for all x and y. Take $x = wy$, then $x^2 + xy + y^2 = 0 \Rightarrow (x+y)^m - x^m - y^m = 0 \Rightarrow y^m[(w+1)^m - w^m - 1] = 0$, choose $y \neq 0$ then $(w+1)^m - w^m - 1 = 0$. If $(w+1)^m - w^m - 1 = 0$ we take $x = wy$, then $(x+y)^m - x^m - y^m = 0$ and $x^2 + xy + y^2 = 0$. We deduce that $g(y)wy + h(y) = 0$. Since $g(y)$ and $h(y)$ are real numbers and w is not, $g(y)y$ must be zero. Then $g(y) \equiv 0$ for all y. Finally $h(y) \equiv 0$.

Now we only need to find m such that

$$(w+1)^m - w^m - 1 = 0\,, \quad w + 1 = -w^2\,, \quad (-1)^m w^{2m} - w^m - 1 = 0\,.$$

Case 1: If $m \equiv 0 \pmod 3$, then $(-1)^m w^{2m} = (-1)^m$, $w^m = 1$, and $(-1)^m - 2 = 0$, no solution.

Case 2: If $3 \nmid m$ we have

$$(-1)^m \omega^{2m} - \omega^m - 1 = 0\,.$$

Also, since ω is a third root of unity and $3 \nmid m$, we have

$$\omega^{2m} + \omega^m + 1 = 0\,.$$

By adding these two relations we get

$$\omega^{2m}[1 + (-1)^m] = 0\,.$$

Thus

$$(-1)^m = -1 \Rightarrow 2 \nmid m\,.$$

Thus

$$3 \nmid m;\ 2 \nmid m \Rightarrow m = \pm 1 \pmod 6 .$$

Finally all valid m are $m \equiv \pm 1 \pmod 6$.

3. Find all integers $n \geq 3$ such that: for any two positive integers $m < n-1$, $r < n-1$ there exist m distinct elements of the set $\{1, 2, \dots, n-1\}$ whose sum is congruent to r modulo n.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Curtis.

We claim that the n's with the desired property are exactly the odd $n \geq 3$.

1. Suppose that k is a positive integer, $n = 2k+2$, $m = 2k$, $r = k+1$, and $S_n = \{1, 2, \dots, n-1\}$. For $j \in S_n$, let x_j denote the sum of all elements of S_n except the j^{th}. Then $x_j = (k+1)(2k+1) - j$. Thus,

$$\begin{aligned} x_j \equiv r \bmod n \quad &\Leftrightarrow \quad (k+1)(2k+1) - j \equiv (k+1) \bmod 2k+2 \\ &\Leftrightarrow \quad (k+1)(2k) \equiv j \bmod 2(k+1) \\ &\Leftrightarrow \quad 2(k+1) \text{ divides } 2k(k+1) - j \end{aligned}$$

   Since $2(k+1)$ divides $2k(k+1)$, n must divide j. But $1 \leq j \leq n-1$, a contradiction. Hence, no even n has the desired property.

2. Now suppose that k is a positive integer and $n = 2k+1$. Let $r \in S_n$.

   (a) Suppose $m = 2l+1$, where l is a positive integer less than k. Let

$$A = \begin{cases} \{1, 2, 3, \dots, r-1, r+1, \dots, l+1\} & \text{if} \quad r \leq l \\ \{1, 2, 3, \dots, l\} & \text{if} \quad l+1 \leq r \leq 2k-l \\ \{1, 2, \dots, 2k-r, 2k+2-r, \dots, l+1\} & \text{if} \quad r \geq (2k+1)-l. \end{cases}$$

   Then

$$\sum_{i \in A} [i + (n-i)] + r$$

   is a sum of m distinct elements of S_n. This sum is congruent to $r \bmod n$ since $i + (n-i) \equiv 0 \bmod n$ for each i.

   (b) Suppose $m = 2l$ and $r \geq 3$. Let

$$A = \begin{cases} \{2, 3, \dots, r-2, r, r+1, \dots, l+1\} & \text{if} \quad 3 \leq r \leq l \\ \{2, 3, \dots, l\} & \text{if} \quad l+1 \leq r \leq 2k-l \\ \{2, 3, \dots, 2k+1-r, 2k+3-r, \dots, l+1\} & \text{if} \quad r \geq (2k+1)-l. \end{cases}$$

   Then

$$1 + (r-1) + \sum_{i \in A} [i + (n-i)]$$

   is a sum of m distinct elements of S_n. The sum is congruent to $r \bmod n$.

(c) Suppose $m = 2l$ and $r = 1$. Then

$$2 + 2k + \sum_{i=3}^{l+1} [i + (n - i)] \equiv 1 \bmod n.$$

(d) Suppose $m = 2l \le 2k - 4$ and $r = 2$. Then

$$3 + 2k + \sum_{i=4}^{l+2} [i + (n - i)] \equiv 2 \bmod n.$$

(e) Suppose $m = 2l = 2k - 2 = n - 3$ and $r = 2$. Since

$$\sum_{i \in S_n} i = \frac{(n-1)n}{2} = kn \equiv 0 \bmod n,$$

$$\sum_{\substack{i \in S_n \\ i \neq 1, 2k-2}} i \equiv 0 - (2k - 1) \equiv 2 \bmod n.$$

Thus, for $n = 2k + 1 \ge 3$, and $1 \le m, r < n - 1$, there exist m distinct elements of S_n with sum congruent to $r \bmod n$.

4. Solve the system

$$\begin{cases} x^2 + yu &= (x + u)^n \\ x^2 + yz &= u^4 \end{cases}$$

where x, y and z are prime numbers and u is a positive integer.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution by Curtis.

From the second equation, $yz = (u^2 + x)(u^2 - x)$. Since y and z are prime, the only factors of the left-hand side are 1, y, z, and yz. Thus there are at most four possibilities.

1. The case $u^2 + x = 1$, $u^2 - x = yz$ is impossible since u and x are both positive.
2. If $u^2 + x = yz$ and $u^2 - x = 1$, then $x = (u + 1)(u - 1)$. Since x is prime, we must have $u - 1 = 1$ and $u + 1 = x$. Thus, $u = 2$ and $x = 3$. From $u^2 + x = yz$, we obtain $7 = yz$, which is impossible.
3. If $u^2 + x = y$ and $u^2 - x = z$, the first equation gives

   $$x^2 + u(u^2 + x) = (x + u)^n,$$

   or

   $$x^2 + u^3 + ux = (x + u)^n.$$

   If $n \ge 3$, then $(x + u)^n \ge (x + u)^3 > x^2 + u^3 + ux$. Hence any solution must have $n < 3$.

(a) If $n = 1$, then $x^2 + u^3 + ux = x + u$, which can be rewritten as $x(x-1)+u(u^2-1)+ux = 0$ which is impossible since the left-hand side is positive.

(b) If $n = 2$, then $x^2+u^3+ux = x^2+2ux+u^2$, which can be rewritten as $u[u(u-1)-x] = 0$. Since $u > 0$, we have $x = u(u-1)$. Since x is prime, $u = 2$ and $x = 2$, implying that $y = 6$ and $z = 2$.

4. If $u^2 + x = z$ and $u^2 - x = y$, the first equation gives

$$x^2 + u(u^2 - x) = (x + u)^n.$$

As before, if $n \geq 3$, the right-hand side is greater than the left-hand side.

(a) If $n = 1$, then $x^2 + u(u^2 - x) = x + u$, which can be rewritten as $x(x-1) + u(u^2 - x - 1) = 0$. But $u^4 - x^2 = yz \geq 4$ so that $u^4 \geq x^2 + 4$ and $u^2 > x$. Thus, $u^2 - x - 1 \geq 0$, and $x(x-1) + u(u^2 - x - 1) > 0$, a contradiction.

(b) If $n = 2$, then $x^2 + u(u^2 - x) = (x+u)^2$, which can be rewritten as $u[u(u-1)-3x] = 0$. Thus $u(u-1) = 3x$ and $u-1 \in \{1, 3, x, 3x\}$.

- If $u - 1 = 1$, then $u = 2$, and $3x = 2$, which is impossible.
- If $u - 1 = 3$, then $u = 4$, and $x = 4$, which is not a prime.
- If $u - 1 = x$, then $x(x+1) = 3x$, implying that $x = 2$. In this case, $u = 3$, $y = 7$, and $z = 11$.
- If $u - 1 = 3x$, then $u = 1$, so that $x = 0$, a contradiction.

In summary, the only solutions are

$$(x, y, z, u, n) \in \{(2, 6, 2, 2, 2), (2, 7, 11, 3, 2)\}.$$

And now we look at solutions to problems of the 2007 Mediterranean Mathematical Competition, given at [2010: 277].

1. Let $x \leq y \leq z$ be real numbers, such that $xy + yz + zx = 1$. Prove that $xz < \frac{1}{2}$. Is it possible to improve the value of the constant $\frac{1}{2}$?

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang and Dexter Wei, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Wang and Wei.

If $x = 0$, then $xz = 0 < \frac{1}{2}$. If $x > 0$, then $2xz = xz + xz \leq yz + zx < xy + yz + zx < 1$ so $xz < \frac{1}{2}$. Suppose $x < 0$. If $z \geq 0$, then $xz \leq 0 < \frac{1}{2}$ so it suffices to consider the case when $x \leq y \leq z < 0$.

Set $r = -z$, $s = -y$, and $t = -z$. Then r, s, and t are all positive such that $t \le s \le r$ and $ts + sr + rt = 1$. Hence, $tr < \frac{1}{2}$ by what we showed above. Then $xz < \frac{1}{2}$ follows.

Now we prove that $\frac{1}{2}$ is the best upper bound for xz by showing that for any $\varepsilon > 0$, there exists x, y, z satisfying $x \le y \le z$, $xy + yz + zx = 1$ and $xz > \frac{1}{2} - \varepsilon$.

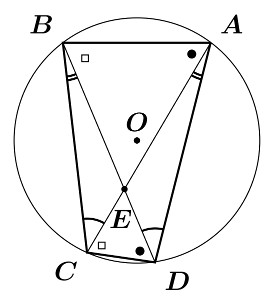
If $\varepsilon \ge \frac{1}{2}$, simply take $x = y = z = \frac{\sqrt{3}}{3}$. Hence we may assume that $0 < \varepsilon < \frac{1}{2}$.

We set $x = y = \frac{\sqrt{2\varepsilon}}{2}$ and $z = \frac{z-\varepsilon}{2\sqrt{2\varepsilon}}$. Then $y \le z$ is equivalent to $2\varepsilon < 2 - \varepsilon$ or $\varepsilon < \frac{2}{3}$ which is true.

Next, $xy + yz + zx = x^2 + 2xz = \frac{\varepsilon}{2} + \frac{2-\varepsilon}{2} = 1$. Finally, $xz > \frac{1}{2} - \varepsilon$ is equivalent, in succession, to $\frac{2-\varepsilon}{4} > \frac{1}{2} - \varepsilon$; $\frac{1}{2} - \frac{\varepsilon}{4} > \frac{1}{2} - \varepsilon$; $\frac{\varepsilon}{4} < \varepsilon$ which is clearly true and our proof is complete.

2. The quadrilateral $ABCD$ is convex and cyclic, and the diagonals AC and BD intersect at the point E. Given that $AB = 39$, $AE = 45$, $AD = 60$ and $BC = 56$, determine the length of CD.

Solved by George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We use the solution of Apostolopoulos.



The triangles BEC and AED are similar, so

$$\frac{BE}{AE} = \frac{BC}{AD} \Rightarrow BE = \frac{AE \cdot BC}{AD} = \frac{45 \cdot 56}{60} = 42\,.$$

Also, the triangles AEB and CED are similar, so $\frac{CD}{AB} = \frac{ED}{AE} = \frac{CE}{BE}$, namely $\frac{CD}{29} = \frac{ED}{45} = \frac{CE}{42} = l > 0$. Using Ptolemy's Theorem yields:

$$\begin{aligned} & AB \cdot CD + AD \cdot BC = AC \cdot BD \\ \Rightarrow\quad & 39(39l) + 60 \cdot 56 = (45 + CE)(42 + ED) \\ \Rightarrow\quad & 39^2 l + 60 \cdot 56 = (45 + 42l)(42 + 45l) \\ \Leftrightarrow\quad & 315l^2 + 378l - 245 = 0\,, \end{aligned}$$

so

$$l = \frac{7}{15} \quad \text{or} \quad l < 0$$

namely $l = \frac{7}{15}$, thus $CD = 39l = 39 \cdot \frac{7}{15} = 18.2$.

3. In the triangle ABC, the angle $\alpha = \angle A$ and the side $a = |BC|$ are given. It is known that $a = \sqrt{rR}$, where r is the inradius and R is the circumradius of ABC. Determine all such triangles, that is, compute the sides b and c of all such triangles.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Amengual Covas.

Let $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$.

From $a = \sqrt{rR}$, we obtain

$$r = \frac{a^2}{R}. \tag{1}$$

Note that the area of $\triangle ABC$ may be expressed as $\frac{abc}{4R}$, and also as rs. Equating those we get $abc = 4sRr = 4sa^2$. Thus

$$bc = 4as. \tag{2}$$

Then $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc} = \frac{s(s-a)}{4as} = \frac{s-a}{4a}$. Since $\cos\alpha = 2\cos^2\frac{A}{2} - 1$, this is equivalent to $\cos\alpha = \frac{s-3a}{2a}$ or $s = a(3+2\cos\alpha)$, so that (2) becomes

$$bc = 4a^2(3+2\cos\alpha). \tag{3}$$

We also have

$$\begin{aligned} b+c &= 2s-a \\ &= 2a(3+2\cos\alpha)-a \\ &= a(5+4\cos\alpha). \end{aligned} \tag{4}$$

We solve the system (3) and (4) by considering b and c as roots of a quadratic equation with coefficients determined by the product (3) and the sum (4) of the roots:

$$x^2 - a(5+4\cos\alpha)x + 4a^2(3+2\cos\alpha) = 0. \tag{5}$$

The requirement that b and c be sides of $\triangle ABC$ forces the discriminant of (5)

$$a^2[(5+4\cos\alpha)^2 - 16(3+2\cos\alpha)] = a^2[(4\cos\alpha+1)^2 - 24]$$

to be non-negative. This is true only if $\cos\alpha \geq \frac{\sqrt{24}-1}{4}$, that is, only if $\alpha \leq 12^\circ 54' 15''$. If this holds, the roots of (5)

$$x = \frac{1}{2}a[5+4\cos\alpha \pm \sqrt{(4\cos\alpha+1)^2-24}]$$

yield the length of the sides CA and AB of $\triangle ABC$.

4. Let $x > 1$ be a noninteger number. Prove that

$$\left(\frac{x+\{x\}}{[x]} - \frac{[x]}{x+\{x\}}\right) + \left(\frac{x+[x]}{\{x\}} - \frac{\{x\}}{x+[x]}\right) > \frac{9}{2},$$

where $[x]$ and $\{x\}$ represents the integer and the fractional part of x.

Solved by Mohammed Aassila, Strasbourg, France; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the improved results of Bataille.

Let $L = \left(\frac{x+\{x\}}{\lfloor x\rfloor} - \frac{\lfloor x\rfloor}{x+\{x\}}\right) + \left(\frac{x+\lfloor x\rfloor}{\{x\}} - \frac{\{x\}}{x+\lfloor x\rfloor}\right)$. We show that $L > \frac{16}{3}$, slightly improving the proposed inequality. Using $\lfloor x\rfloor + \{x\} = x$, we calculate

$$\frac{x+\{x\}}{\lfloor x\rfloor} + \frac{x+\lfloor x\rfloor}{\{x\}} = \frac{x^2+\{x\}^2+\lfloor x\rfloor^2}{\{x\}\lfloor x\rfloor}$$

and

$$\frac{\lfloor x\rfloor}{x+\{x\}} + \frac{\{x\}}{x+\lfloor x\rfloor} = \frac{x^2+\{x\}^2+\lfloor x\rfloor^2}{(x+\{x\})(x+\lfloor x\rfloor)} = \frac{x^2+\{x\}^2+\lfloor x\rfloor^2}{2x^2+\{x\}\lfloor x\rfloor}.$$

It follows that

$$L = \frac{2x^2(x^2+\{x\}^2+\lfloor x\rfloor^2)}{\{x\}\lfloor x\rfloor(2x^2+\{x\}\lfloor x\rfloor)}.$$

Now, recalling that $2(a^2+b^2) > (a+b)^2$ for distinct positive real numbers a, b, we see that

$$2x^2(x^2+\{x\}^2+\lfloor x\rfloor^2) > x^2(2x^2+(\{x\}+\lfloor x\rfloor)^2) = 3x^4$$

(note that $\{x\} \in [0,1)$ and $\lfloor x\rfloor \geq 1$ so that $\{x\} \neq \lfloor x\rfloor$). In addition, from AM-GM,

$$\{x\}\lfloor x\rfloor(2x^2+\{x\}\lfloor x\rfloor) < \frac{(\{x\}+\lfloor x\rfloor)^2}{4} \cdot \left(2x^2 + \frac{(\{x\}+\lfloor x\rfloor)^2}{4}\right) = \frac{9x^4}{16}$$

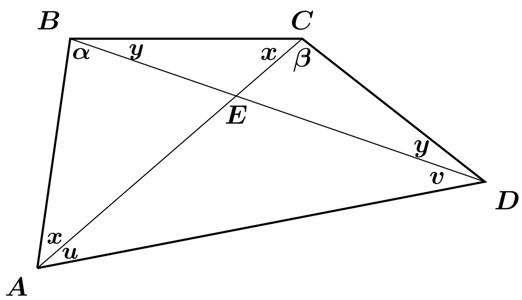
and we finally deduce

$$L > (3x^4) \cdot \frac{16}{9x^4} = \frac{16}{3}.$$

We return to the files of solutions from our readers and the 24th Balkan Mathematical Olympiad 2007 given at [2010: 277–278].

1. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$ and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^\circ$.

Solved by Mohammed Aassila, Strasbourg, France; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.



Let $\angle DAE = u$, $\angle EDA = v$, $\angle ABD = \alpha$ and $\angle ACD = \beta$. Letting the base angles in isosceles triangles ABC and BCD be x and y, respectively, we have $u + v = x + y$ in $\triangle AED$, $\triangle BEC$ and $x + \alpha = y + \beta$ in $\triangle ABE$, $\triangle CDE$, respectively, because of the vertically opposite angles at E. Therefore,

$$\begin{aligned} \angle BAD + \angle ADC &= (x+u)+(v+y) \\ &= (u+v)+(x+y) \qquad (1) \\ &= 2(x+y) \end{aligned}$$

We must have $\alpha \neq \beta$. Suppose $\alpha = \beta$: Then the condition $x + \alpha = y + \beta$ implies $x = y$ and we would have $\angle ABC = \angle BCD$, making $\triangle ABC$ and $\triangle BCD$ congruent which contradicts the assumption $AC \neq BD$.

Now, by the law of sines,

$$\begin{aligned} \frac{AE}{\sin\alpha} &= \frac{AB}{\sin\angle BEA} \\ &= \frac{CD}{\sin\angle DEC} \quad (\angle BEA \text{ and } \angle DEC \text{ are vertically opposite angles}) \\ &= \frac{DE}{\sin\beta}. \end{aligned}$$

so

$$\begin{aligned} AE = DE &\Leftrightarrow \sin\alpha = \sin\beta \\ &\Leftrightarrow \alpha + \beta = 180^\circ \quad (\text{since } a \neq \beta) \\ &\Leftrightarrow \angle BAD + \angle ADC + x + y = 180^\circ \\ &\qquad (\text{since the angles of } ABCD \text{ add up to } 360^\circ) \\ &\Leftrightarrow \angle BAD + \angle ADC + \frac{1}{2}(\angle BAD + \angle ADC) = 180^\circ \quad \text{by (1)} \\ &\Leftrightarrow \angle BAD + \angle ADC = 120^\circ \end{aligned}$$

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(f(x)+y) = f(f(x)-y) + 4f(x)y.$$

Solved by Mohammed Aassila, Strasbourg, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give the solution by Curtis.

We claim that the only solutions are the 0 function and functions $f(x) = x^2 + b$, where b is a real number. It is readily verified that these satisfy the functional equation, so we assume that f not identically 0 satisfies the functional equation. Choose $\bar{x} \in \mathbb{R}$ such that $f(\bar{x}) \neq 0$. Let $z \in \mathbb{R}$, and set $\bar{y} = \dfrac{z}{8f(\bar{x})}$. Then

$$f\left(f(\bar{x}) + \frac{z}{8f(\bar{x})}\right) - f\left(f(\bar{x}) - \frac{z}{8f(\bar{x})}\right) = \frac{z}{2}.$$

Let $x_1 = f(\bar{x}) + \bar{y}$ and $x_2 = f(\bar{x}) - \bar{y}$. Then

$$z = 2[f(x_1) - f(x_2)].$$

With

$$v = f(x_1) - 2f(x_2),$$

we have

$$z = f(x_1) + v,$$

so that

$$\begin{aligned} f(z) = f(f(x_1) + v) &= f(f(x_1) - v) + 4f(x_1)v \\ &= f(2f(x_2)) + 4f(x_1)v. \end{aligned}$$

Let $b = f(0)$, and in the functional equation, let $x = x_2$ and $y = f(x_2)$. Then

$$f(2f(x_2)) = b + 4[f(x_2)]^2.$$

Thus,

$$\begin{aligned} f(z) = b + 4[f(x_2)]^2 + 4f(x_1)v &= b + 4[f(x_2)]^2 + 4f(x_1)[f(x_1) - 2f(x_2)] \\ &= b + 4[f(x_1) - f(x_2)]^2 \\ &= b + z^2. \end{aligned}$$

3. Find all positive integers n such that there is a permutation σ of the set $\{1, 2, \ldots, n\}$ such that the number below is a rational number

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}$$

Ed.: A permutation of the set $\{1, 2, \ldots, n\}$ is a one-to-one function of this set to itself.

Solved by Mohammed Aassila, Strasbourg, France; and George Apostolopoulos, Messolonghi, Greece. We give the version of Apostolopoulos.

Suppose that for some $n \in \mathbb{N}^*$ we have

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} = v_1 \in \mathbb{Q}$$

then $\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}$ is a rational number. Similar, for each $k \in \{1, 2, \dots, n\}$ the number $v_k = \sqrt{\sigma(k) + \sqrt{\sigma(k+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}$ is a rational number. Define $\alpha_k = \sqrt{n + \sqrt{n + \sqrt{\cdots + \sqrt{n}}}}$, for each $k \in \mathbb{N}^*$. By an easy induction, we can prove that $\alpha_k < \sqrt{n} + 1$, for each $k \in \mathbb{N}^*$, as a result we will have $v_1 < \alpha_n < \sqrt{n} + 1$. If $l > 0$ is such that $l^2 \le n < (l+1)^2$ then for some $i \in \{1, 2, \dots, n\}$ we have $\sigma(i) = l^2$.

Case 1. $i \neq n$.

Then we have

$$l < \sqrt{l^2 + \sqrt{\sigma(i+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} < \sqrt{n} + 1 < l + 2$$
$$\Rightarrow \sqrt{l^2 + \sqrt{\sigma(i+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} = l + 1$$
$$\Rightarrow 2l + 1 = \sqrt{\sigma(i+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}} < \sqrt{n} + 1 < l + 2 \Rightarrow l < 1\,,$$

a contradiction.

Case 2. $i = n$.

If $l > 1$, then $(l^2 - 1) \in \{\sigma(1), \sigma(2), \dots, \sigma(n-1)\}$. Suppose that $j < n$ is such that $\sigma(j) = l^2 - 1$. Similarly to Case 1, we get

$$l < \sqrt{l^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{\cdots + \sqrt{l^2}}}} < \sqrt{n} + 1 < l + 2$$
$$\Rightarrow \sqrt{l^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{\cdots + \sqrt{l^2}}}} = l + 1$$
$$\Rightarrow 2l + 2 = \sqrt{\sigma(j+1) + \sqrt{\cdots + \sqrt{l^2}}} < \sqrt{n} + 1 < l + 2\,,$$

a contradiction.

If $l = 1$, then $n \in \{1, 2, 3\}$. We conclude that for $n = 1$ and for $n = 3$ there are permutations that satisfy the terminus relation. For $n = 1$, we have $\sqrt{1} = 1$ and for $n = 3$, we have $\sqrt{2 + \sqrt{3 + \sqrt{1}}} = 2$. But for $n = 2$ no such permutation exists.

So $n = 1$, $n = 3$.

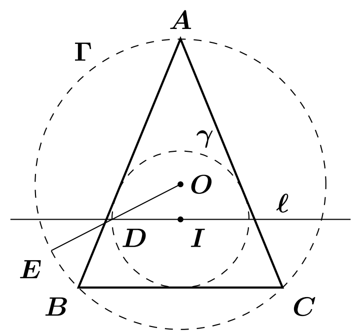
Next we turn to the Indian Team Selection Test 2007 given at [2010: 278–279].

1. Let ABC be a triangle with $AB = AC$, and let Γ be its circumcircle. The incircle γ of ABC moves (slides) on BC in the direction of B. Prove that when γ touches Γ internally, it also touches the altitude through A.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Smeenk.

We denote $\Gamma = \Gamma(O, R)$, $\quad \gamma = \gamma(I, r)$.

Line ℓ passes through I and is parallel to BC. D lies on ℓ, and $ID = r$. OD intersects Γ at E.



It suffices to show that $DE = r$, and so $OD = R - r$. $OI = R\cos\alpha - r$, $ID = r$, and we are to show

$$(R-r)^2 = (R\cos\alpha - r)^2 + r^2 \qquad (1)$$

As $\beta = \gamma$ we have

$$\begin{aligned} r &= R(\cos\alpha + \cos\beta + \cos\gamma - 1) \\ &= R(-\cos 2\beta + 2\cos\beta - 1) \\ &= 2R\cos\beta(1 - \cos\beta). \end{aligned} \qquad (2)$$

Substituting (2) in (1) we see that it holds and we are done.

2. Consider the quadratic polynomial $p(x) = x^2 + ax + b$, where a, b are in the interval $[-2, 2]$. Determine the range of the real roots of $p(x) = 0$ as a and b vary over $[-2, 2]$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The polynomial $p(x)$ has real roots if and only if its discriminant $D = a^2 - 4b$ is nonnegative. Set

$$x_+ = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad x_- = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

Since $-2 \le b \le \frac{1}{4}a^2$, we have $0 \le a^2 - 4b \le a^2 + 8$. Hence,

$$-\frac{a}{2} \le x_+ \le \frac{-a + \sqrt{a^2 + 8}}{2} \quad \text{and} \quad \frac{-a - \sqrt{a^2 + 8}}{2} \le x_- \le \frac{-a}{2}.$$

The function $f(t) = \frac{1}{2}(-t + \sqrt{t^2 + 8})$ is continuous and decreasing on $[-2, 2]$; hence

$$-1 \le x_+ \le 1 + \sqrt{3}.$$

The function $g(t) = \frac{1}{2}(-t - \sqrt{t^2 + 8})$ is also continuous and decreasing on $[-2, 2]$; hence,

$$-1 - \sqrt{3} \le x_- \le 1.$$

Thus the range of the real roots of $p(x)$ is $[-1 - \sqrt{3}, 1 + \sqrt{3}]$.

3. Let triangle ABC have side lengths a, b, c; circumradius R, and internal angle bisector lengths w_a, w_b, w_c. Prove that

$$\frac{b^2+c^2}{w_a}+\frac{c^2+a^2}{w_b}+\frac{a^2+b^2}{w_c} \;>\; 4R\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Apostolopoulos.

It is well-known that $w_a=\dfrac{2bc\cos\dfrac{A}{2}}{b+c}$, and thus we have

$$\begin{aligned}\sum_{\text{cyclic}}\frac{b^2+c^2}{w_a}>4R \quad&\Leftrightarrow\quad \sum_{\text{cyclic}}\frac{(b^2+c^2)(b+c)}{2bc\cos\frac{A}{2}}>4R\\ &\Leftrightarrow\quad \sum_{\text{cyclic}}\frac{(b^2+c^2)(b+c)}{4Rbc\cos\frac{A}{2}}>2\\ &\Leftrightarrow\quad \sum_{\text{cyclic}}\frac{(b^2+c^2)(b+c)\sin\frac{A}{2}}{2abc}>1.\end{aligned}$$

Note that $b^2+c^2\geq 2bc$, $b+c>a$. It suffices to prove that $\sum\limits_{\text{cyclic}}\sin\dfrac{A}{2}>1$. Let r be the inradius of the triangle. Then $\sum\limits_{\text{cyclic}}\cos A+1+\dfrac{r}{R}>1$. This inequality holds for all triangles. Note that $\dfrac{\pi-A}{2}$, $\dfrac{\pi-B}{2}$, $\dfrac{\pi-C}{2}$ are also the three angles of a triangle thus $\sum\limits_{\text{cyclic}}\cos\dfrac{\pi-A}{2}>1$. That is $\sum\limits_{\text{cyclic}}\sin\dfrac{A}{2}>1$.

5. Show that in a non-equilateral triangle, the following are equivalent:

(a) The angles of the triangle are in arithmetic progression;

(b) The common tangent to the nine-point circle and the in-circle is parallel to the Euler line.

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

We show that both (a) and (b) are equivalent to: (c) one of the angles of the triangle is 60°.

Let ABC be the given non-equilateral triangle with $BC=a, CA=b, AB=c$.

If (c) holds, then the angles of ΔABC are of the form 60°, $60^\circ+\alpha$, $60^\circ-\alpha$, hence (a) holds. Conversely, if, say, $B=\frac{A+C}{2}$, then $B=\frac{180^\circ-B}{2}$, hence $B=60^\circ$. Thus, (a) $\Longleftrightarrow$ (c).

Now, let $O, H, I,$ and N denote the circumcentre, the orthocentre, the incentre, and the centre of the nine-point circle, respectively. As usual, let R and s be the circumradius and the semi-perimeter of ΔABC. It is well-known that the incircle is internally tangent to the nine-point circle. It follows that (b) is equivalent to $OH \perp IN$. Since $\overrightarrow{NI} = \overrightarrow{OI} - \overrightarrow{ON} = \overrightarrow{OI} - \frac{1}{2}\overrightarrow{OH}$, the condition $OH \perp IN$ is equivalent to $\overrightarrow{OH} \cdot \overrightarrow{OI} = \frac{OH^2}{2}$. Using $2s\overrightarrow{OI} = a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}$ and $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, a simple computation shows that $\overrightarrow{OH} \cdot \overrightarrow{OI} = \frac{OH^2}{2}$ is itself equivalent to

$$s = -a\cos 2A - b\cos 2B - c\cos 2C \qquad (1)$$

(note that for example $\overrightarrow{OA} \cdot \overrightarrow{OB} = R^2 \cos 2C$, C being acute or not).

We successively rewrite (1) as

$$a(1 + 2\cos 2A) + b(1 + 2\cos 2B) + c(1 + 2\cos 2C) = 0$$
$$\sin A + 2\sin A\cos 2A + \sin B + 2\sin B\cos 2B + \sin C + 2\sin C\cos 2C = 0$$
$$\sin 3A + \sin 3B + \sin 3C = 0$$

(the latter because $2\sin A\cos 2A = \sin 3A - \sin A$, etc.).

As a result, (b) is equivalent to $\sin 3A + \sin 3B + \sin 3C = 0$, or, with the help of the familiar trig formulas, to $4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2} = 0$. Finally, (b) is equivalent to $90^\circ = \frac{3A}{2}$ or $\frac{3B}{2}$ or $\frac{3C}{2}$ and (b) $\iff$ (c).

7. Let a, b, c be nonnegative real numbers such that $a+b \le c+1$, $b+c \le a+1$ and $c+a \le b+1$. Prove that

$$a^2 + b^2 + c^2 \le 2abc + 1\,.$$

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Titu Zvonaru, Comănești, Romania. We use the solution by Alt.

First note that $a, b, c \le 1$. Indeed,

$$c + a \le b + 1 \Rightarrow c + 2a \le a + b + 1 \le c + 1 + 1 \Rightarrow 2a \le 2 \Rightarrow a \le 1\,,$$

and, similarly, $b, c \le 1$.

Let $x := 1 - a$, $y := 1 - b$, $z := 1 - c$. Then $x, y, z \in [0, 1]$, $a = 1 - x$, $b = 1 - y$, $c = 1 - z$

$$\begin{cases} a + b \le c + 1 \\ b + c \le a + 1 \\ c + a \le b + 1 \end{cases} \iff \begin{cases} z \le x + y \\ x \le y + z \\ y \le z + x \end{cases} \iff |x - y| \le z \le x + y,$$

and the original inequality becomes

$$\begin{aligned}
&(1-x)^2+(1-y)^2+(1-z)^2 \le 2(1-x)(1-y)(1-z)+1 \\
\iff\ & x^2+y^2+z^2 \le 2(xy+yz+zx)-2xyz \\
\iff\ & x^2+y^2-2xy \le 2z(x+y-xy)-z^2 \\
\iff\ & (x-y)^2 \le 2z(x+y-xy)-z^2 . \qquad (1)
\end{aligned}$$

Let $f(z) = 2z(x+y-xy)-z^2$. Since $z \in [|x-y|, x+y]$ then $\min\limits_z f(z) = \min\{f(|x-y|), f(x+y)\}$, and, therefore,

$$\begin{aligned}
(1) \iff\ & (x-y)^2 \le \min_z \left(2z(x+y-xy)-z^2\right) \\
\iff\ & (x-y)^2 \le \min\{f(|x-y|), f(x+y)\} \qquad (2) \\
\iff\ & \begin{cases} (x-y)^2 \le f(|x-y|) \\ (x-y)^2 \le f(x+y) \end{cases} \\
\iff\ & \begin{cases} (x-y)^2 \le 2|x-y|(x+y-xy)-(x-y)^2 \\ (x-y)^2 \le 2(x+y)(x+y-xy)-(x-y)^2 \end{cases} \\
\iff\ & \begin{cases} (x-y)^2 \le |x-y|(x+y-xy) \\ x^2+y^2 \le (x+y)(x+y-xy) \end{cases}
\end{aligned}$$

We have

$$\begin{aligned}
x^2+y^2 \le (x+y)(x+y-xy) \iff\ & (x+y)xy \le 2xy \\
\iff\ & 0 \le xy(2-x-y)
\end{aligned}$$

and

$$(x-y)^2 \le |x-y|(x+y-xy) \iff 0 \le |x-y|(x+y-xy-|x-y|) .$$

Since $x, y \in [0,1]$ then the inequality $0 \le xy(2-x-y)$ obviously holds and

$$\begin{aligned}
|x-y| \le x+y-xy \iff\ & xy-x-y \le x-y \le x+y-xy \\
\iff\ & \begin{cases} xy-x \le x \\ -y \le y-xy \end{cases} \iff \begin{cases} 0 \le x(2-y) \\ 0 \le y(2-x) \end{cases} .
\end{aligned}$$

8. Given a finite string S of symbols a and b, we write $\triangle(S)$ for the number of a's in S minus the number of b's. (For example, $\triangle(abbabba) = -1$.) We call a string S balanced if every substring (of consecutive symbols) T of S has the property $-1 \leq \triangle(T) \leq 2$. (Thus $abbabba$ is not balanced, as it contains the substring $bbabb$ and $\triangle(bbabb) = -3$). Find, with proof, the number of balanced strings of length n.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give Geupel's solution.

We prove that the number of balanced strings of length n is $n + 1$. If a string S is balanced, then it does not contain the string bb as a substring, because $\triangle(bb) = -2$. Moreover, S contains at most one substring of the form aa, and the remaining part of S must alternate between the symbols a and b. Conversely, any string with not more than one occurrence of the substring aa and alternating symbols in the remaining parts of the string is balanced. We have two such strings with no substring aa and $n - 1$ strings with exactly one substring aa. This completes the proof.

9. Define the functions f, g, h on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ as follows:

$$\begin{aligned} f(x,y,z) &= (3x+2y+2z, 2x+2y+z, 2x+y+2z)\,, \\ g(x,y,z) &= (3x+2y-2z, 2x+2y-z, 2x+y-2z)\,, \\ h(x,y,z) &= (3x-2y+2z, 2x-y+2z, 2x-2y+z)\,. \end{aligned}$$

Given a primitive Pythagorean triplet (x, y, z), with $x > y > z$, prove that (x, y, z) can be uniquely obtained by repeated application of f, g, h to the triple $(5, 4, 3)$. For example: $(697, 528, 455) = f \circ h \circ g \circ h(5, 4, 3)$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Representing f, g, and h by the matrices F, G, and H given by

$$F = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & -1 \\ 2 & 1 & -2 \end{bmatrix}, \quad H = \begin{bmatrix} 3 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix},$$

their inverses are given by

$$F^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 2 & 1 \\ 2 & -1 & -2 \end{bmatrix},$$

$$H^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix}.$$

Let (x, y, z) be a primitive Pythagorean triple with $x > y > z$. Then there exist relatively prime opposite-parity integers s, t with $s > t$ such that $x = s^2 + t^2$ and either $y = 2st$ and $z = s^2 - t^2$ or $y = s^2 - t^2$ and $z = 2st$.

Case 1. Suppose first that $y = 2st$ and $z = s^2 - t^2$. Then $2st > s^2 - t^2$, so that $t^2 > s^2 - 2st = s(s - 2t) > t(s - 2t)$. This implies that $3t > s$. We have

- $F^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = F^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ 2st \\ s^2 - t^2 \end{bmatrix}\right) = \begin{bmatrix} t^2 + (s - 2t)^2 \\ t^2 - (s - 2t)^2 \\ 2t(s - 2t) \end{bmatrix}$ $(3t > s > 2t)$;
- $G^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = G^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ 2st \\ s^2 - t^2 \end{bmatrix}\right) = \begin{bmatrix} t^2 + (2t - s)^2 \\ t^2 - (2t - s)^2 \\ 2t(2t - s) \end{bmatrix}$ $(2t > s > t)$;
- $H^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = H^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ 2st \\ s^2 - t^2 \end{bmatrix}\right) = \begin{bmatrix} t^2 + (2t - s)^2 \\ 2t(2t - s) \\ t^2 - (2t - s)^2 \end{bmatrix}$ $(2t > s > t)$.

Since s and t are relatively prime, $s \neq 2t$ unless $s = 2$ and $t = 1$; if $s > 2t$, then the first right-hand side is a primitive Pythagorean triple closer to the origin in $Z \times Z \times Z$ than the original; if $s < 2t$, the second and third right-hand sides are primitive Pythagorean triples closer to the origin than the original. We note that with $[x'\ y'\ z']^T = F^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $t^2 - (s - 2t)^2 > 2t(s - 2t)$, which is equivalent to $y > z$. Likewise, with $[x'\ y'\ z']^T = G^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $t^2 - (2t - s)^2 > 2t(2t - s)$, which is equivalent to $s^2 - 6st + 7t^2 < 0$. This inequality holds if and only if $(3 - \sqrt{2})t < s < (3 + \sqrt{2})t$. This also implies that with $[x'\ y'\ z']^T = H^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $s < (3 - \sqrt{2})t$. Hence, when y is even and z is odd, we find the previous Pythagorean triple by applying F^{-1} if $3t > s > 2t$, G^{-1} if $2t > s > (3 - \sqrt{2})t$, and H^{-1} if $(3 - 2\sqrt{2})t > s > t$.

Case 2. Similarly, if $y = s^2 - t^2$ and $z = 2st$, then $s^2 - t^2 > 2st$, implying that $s(s - 2t) > t^2 > 0$, so that $s > 2t$. We have

- $F^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = F^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ s^2 - t^2 \\ 2st \end{bmatrix}\right) = \begin{bmatrix} t^2 + (s - 2t)^2 \\ 2t(s - 2t) \\ t^2 - (s - 2t)^2 \end{bmatrix}$ $(3t > s > 2t)$;
- $G^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = G^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ s^2 - t^2 \\ 2st \end{bmatrix}\right) = \begin{bmatrix} (s - 2t)^2 + t^2 \\ 2(s - 2t)t \\ (s - 2t)^2 - t^2 \end{bmatrix}$ $(s > 3t)$;
- $H^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = H^{-1}\left(\begin{bmatrix} s^2 + t^2 \\ s^2 - t^2 \\ 2st \end{bmatrix}\right) = \begin{bmatrix} (s - 2t)^2 + t^2 \\ (s - 2t)^2 - t^2 \\ 2(s - 2t)t \end{bmatrix}$ $(s > 3t)$.

Suppose that $(s,t) \neq (2,1)$. If $s < 3t$, then the first right-hand side is a primitive Pythagorean triple closer to the origin than the original triple. If $s > 3t$, the second and third right-hand sides are primitive Pythagorean triples closer to the origin than the original triple. With $[x'\ y'\ z']^T = F^{-1}([x\ y\ z]^T)$, $y' > z'$ if and only if $2t(s-2t) > t^2 - (s-2t)^2$, which is equivalent to $y > z$. Similarly, $y' > z'$ if and only if $s < (3+\sqrt{2})t$ when $[x'\ y'\ z']^T = G^{-1}([x\ y\ z]^T)$, and $y' > z'$ if and only if $s > (3+\sqrt{2})t$, when $[z'\ y'\ z']^T = H^{-1}([x\ y\ z]^T)$. Hence, if y is odd and z is even, we find the previous triple by applying F^{-1} if $3t > s > 2t$; G^{-1} if $(3+\sqrt{2})t > s > 3t$, and H^{-1} if $s > (3+\sqrt{2})t$.

Thus, repeatedly applying F^{-1}, G^{-1}, and H^{-1} to a primitive Pythagorean triple yields a sequence of primitive Pythagorean triples, or equivalently a sequence of lattice points in the (s,t)-plane successively closer to the origin, terminating when $s = 2$, $t = 1$, which corresponds to the triple $(5,4,3)$. At each step, there is a unique choice of F^{-1}, G^{-1}, or H^{-1} for which $x' > y' > z'$. Reversing the process yields a unique path to (x,y,z) from $(5,4,3)$.

11. Find all pairs of integers (x,y) such that $y^2 = x^3 - p^2x$, where p is a prime such that $p \equiv 3 \pmod 4$.

Solution based on an approach of George Apostolopoulos, Messolonghi, Greece, modified by the editor.

The equation can be rewritten as $y^2 = (x-p)(x+p)x$. There are two cases.

Case 1. $p \nmid y$, Then $(p,(x-p)x(x+p)) = 1$. When x is even, then $x-p$, $x+p$ and x are pairwise relatively prime and so must all be squares. Since x is a multiple of 4 and $p \equiv 3 \pmod 4$, it follows that $x+p \equiv 3 \pmod 4$, and so cannot be square. Thus there are no solutions when x is even.

When x is odd, then $(x-p, x+p) = 2$ so that $\frac{x-p}{2}$, $\frac{x+p}{2}$ and x are pairwise relatively prime. Their product is the integer $\left(\frac{y}{2}\right)^2$, so they are all squares. Let $x = r^2$, $x-p = 2s^2$, $x+p = 2t^2$. Then $t^2 - s^2 = p$, $t = \frac{p+1}{2}$ and $s = \frac{p-1}{2}$. Thus

$$r^2 = (x-p) + p = \frac{(p-1)^2}{2} + p = \frac{p^2+1}{2},$$

whereupon $p^2 - 2r^2 = -1$. The positive solutions of this pellian equation are given by $p_n + q_n\sqrt{2} = (1+\sqrt{2})(3+2\sqrt{2})^n$ for n a nonnegative integer. Since $p_{n+1} = 6p_n - p_{n-1}$, it can be verified that $p_n \equiv (-1)^n \pmod 8$. The first few solutions are $(1,1)$, $(7,5)$, $(41,29)$, $(239,169)$. The equation is solvable if and only if n is odd and $p = p_n$ is a prime. For example, we obtain $(p,x,y) = (7, 25, 2 \times 3 \times 4 \times 5) = (7, 25, 120)$ and $(p,x,y) = (239, 169^2, 2 \times 119 \times 120 \times 169) = (239, 28561, 4826640)$.

Case 2. $p \mid y$. If $y = 0$, then $(x,y) = (p,0), (-p,0), (0,0)$ are solutions. We show that there are no other solutions.

Suppose that $y \neq 0$. Since $p \mid y$, then $(x-p, x+p, x) = p$ and $p^2 \mid y$. Dividing both sides of the equation by p^3, we have that $pb^2 = (a-1)a(a+1)$, where $x = pa$, $y = p^2b$. Since $b \neq 0$, $a - 1 > 0$.

The prime p divides exactly one of $a-1$, $a+1$, a. Since any pair of these three integers is co-prime, two of them must be squares that differ by either 1 or 2. But this is possible only when $a=0$ and $a=1$, both of which are excluded. The result follows.

Comment by the editor. The condition that $p \equiv 3 \pmod 4$ seems artificial. When p is any odd prime and does not divide y, we deduce as above that $x-p$, x and $x+p$ are all square. The only pair of squares that differ by p are $\left(\frac{p-1}{2}\right)^2$ and $\left(\frac{p+1}{2}\right)^2$. Thus there is no integer x for which all of $x-p$, x and $x+p$ are square and so there are no solutions when x is even and y is not a multiple of p.

When x is odd, we can pursue the foregoing argument to find solutions when $p \equiv 1 \pmod 8$, such as $(p,x,y) = (41, 29^2, 2\times 20\times 21\times 29) = (41, 841, 24360)$.

The reader is invited to examine the case that $p=2$.

12. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the equation

$$f(x+y)+f(x)f(y) \;=\; (1+y)f(x)+(1+x)f(y)+f(xy)\,,$$

for all $x,y\in\mathbb{R}$.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Substituting $y=1$ into

$$f(x+y)+f(x)f(y) = (1+y)f(x)+(1+x)f(y)+f(xy) \qquad (1)$$

gives

$$f(x+1) = 3f(x)+f(1)[1+x-f(x)]. \qquad (2)$$

Substituting $y=-1$ into (1) gives

$$f(x-1) = f(-x)+f(-1)[1+x-f(x)]. \qquad (3)$$

Substituting $y=0$ into (1) gives

$$f(0)[f(x)-x-2] = 0. \qquad (4)$$

Hence, either $f(0)=0$ or $f(x)=x+2$ for all real x. Substituting this last possibility into (1) yields the contradiction $2xy=0$ for all $x,y\in\mathbb{R}$. Thus

$$f(0)=0.$$

Setting $x=1$ and $y=-1$ in (1) gives $f(1)f(-1) = 2f(-1)+f(-1)$, so that $f(-1)[3-f(1)]=0$. Hence, either

$$f(1)=3 \quad \text{or} \quad f(-1)=0.$$

We note for future reference that substituting, respectively, $y=x$ and $y=-x$ in (1) gives

$$f(2x)+[f(x)]^2 = 2(1+x)f(x)+f(x^2) \qquad (5)$$

and

$$f(x)f(-x) = (1-x)f(x) + (1+x)f(-x) + f(-x^2). \qquad (6)$$

Case 1. If $f(1) = 3$, then setting $y = 1$ gives $f(x+1) + 3f(x) = 2f(x) + 3(1+x) + f(x)$, which is equivalent to $f(x+1) = 3(1+x)$. Thus, for all real x,

$$f(x) = 3x.$$

Case 2. Now suppose $f(-1) = 0$ and $f(1) \neq 3$. From (3), we have

$$f(x-1) = f(-x),$$

so that, by replacing x with $x+1$,

$$f(x) = f(-x-1). \qquad (7)$$

Set $y = -x-1$ in (1). Then

$$[f(x)]^2 = -xf(x) + (1+x)f(x) + f(-x-x^2).$$

Hence,

$$f(-x-x^2) = [f(x)]^2 - f(x).$$

In particular, if $x = 1$, we have $f(-2) = [f(1)]^2 - f(1)$. But from (7), we have $f(-2) = f(1)$. Thus, $f(1) = [f(1)]^2 - f(1)$, so that $f(1)[f(1)-2] = 0$. Therefore, either

$$f(1) = 0 \quad \text{or} \quad f(1) = 2.$$

Subcase (a). Suppose $f(1) = 0$. Equations (2) and (3) give

$$f(x+1) = 3f(x) \qquad (8)$$

and

$$f(x-1) = f(-x), \qquad (9)$$

respectively. Applying (8) with $x = 1$ yields

$$f(2) = 0. \qquad (10)$$

Thus, substituting $y = 2$ in (1) gives

$$f(x+2) = 3f(x) + f(2x). \qquad (11)$$

On the other hand, applying (8) twice yields

$$f(x+2) = 9f(x). \qquad (12)$$

From (11) and (12), we have

$$f(2x) = 6f(x). \qquad (13)$$

Replacing x with $x-1$ in (8) gives $f(x)=3f(-x)$; combining with (9) gives

$$f(-x)=\frac{1}{3}f(x). \tag{14}$$

Applying (14) and then (5) to (6) gives

$$\begin{aligned}\frac{1}{3}[f(x)]^2 &= (1-x)f(x)+\frac{1}{3}(1+x)f(x)+\frac{1}{3}f(x^2)\\ \frac{1}{3}[f(x)]^2 &= (1-x)f(x)+\frac{1}{3}(1+x)f(x)\\ &\qquad +\frac{1}{3}\{f(2x)+[f(x)]^2-2(1+x)f(x)\}.\end{aligned}$$

Simplifying gives

$$f(2x)=2(2x-1)f(x). \tag{15}$$

From (13) and (15), we obtain

$$\begin{aligned}6f(x)&=2(2x-1)f(x)\\ 4(2-x)f(x)&=0.\end{aligned}$$

Since $f(2)=0$, this last equation implies that $f(x)$ is identically 0.

Subcase (b). Now suppose that $f(1)=2$. From (2) and (3),

$$f(x+1)=f(x)+2(1+x) \tag{16}$$

and

$$f(x-1)=f(-x). \tag{17}$$

Replacing x with $x-1$ in (16) and combining with (17) gives

$$f(-x)=f(x)-2x. \tag{18}$$

Applying (18) and then (5) to (6) gives

$$\begin{aligned}f(x)[f(x)-2x] &= (1-x)f(x)+(1+x)[f(x)-2x]+f(x^2)-2x^2\\ f(x)[f(x)-2x] &= (1-x)f(x)+(1+x)[f(x)-2x]+f(2x)\\ &\qquad +[f(x)]^2-2(1+x)f(x)-2x^2\\ [f(x)]^2-2xf(x) &= 2f(x)-2x(1+x)+f(2x)+[f(x)]^2\\ &\qquad -2(1+x)f(x)-2x^2\\ 0 &= -2x(1+x)+f(2x)-2x^2\\ f(2x) &= 4x^2+2x\\ f(2x) &= (2x)^2+(2x)\end{aligned}$$

for all real x, so that

$$f(x)=x^2+x.$$

In summary, there are three possibilities, each of which is readily verified to satisfy (1).

- If $f(1) = 3$, then $f(x) = 3x$.
- If $f(-1) = 0$ and $f(1) = 0$, then $f(x) \equiv 0$.
- If $f(-1) = 0$ and $f(1) = 2$, then $f(x) = x^2 + x$.

Next we turn to the files of readers' solutions to problems given in the October 2010 number of the *Corner* and the Olimpiada Nacional Escolar de Matematica 2009, Level 1, given at [2010: 372–373].

1. If P, E, R and U represent digits different from 0 and pairwise different such that $\overline{PER} + \overline{PRU} + \overline{PUE} + 2009 = \overline{PERU}$, find all the values that $P + E + R + U$ can take.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the version of Zvonaru.

The given condition is equivalent to

$$\begin{aligned} 100P + 10E + R + 100P + 10R + U + 100P + 10U + E + 2009 & \\ = \ & 1000P + 100E + 10R + U; \\ 700P + 89E \ = \ & 2009 + \overline{UR}. \end{aligned}$$

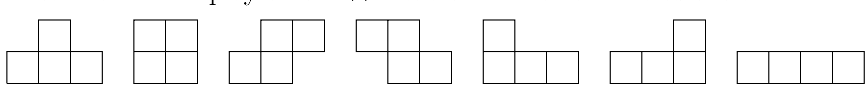
Since $2009 + \overline{UR} \leq 2009 + 98$ and $89E \geq 89$, we deduce that $P \leq 2$.

If $P = 1$, then we have $89E = 1309 + \overline{UR}$. We do not obtain a solution because $89E \leq 89 \cdot 9 = 801$.

If $P = 2$, then we obtain $89E = 609 + \overline{UR}$. Since $609 + \overline{UR} \geq 609 + 12 = 621$ and $609 + \overline{UR} \leq 609 + 98 = 707$, we deduce that $\frac{621}{89} \leq E \leq \frac{707}{89}$, hence $E = 7$.

It results that $\overline{PERU} = 2741$ and $P + E + R + U = 14$.

3. Andrés and Bertha play on a 4×4 table with tetrominos as shown.



Andrés begins the game placing 4 tetrominos of the same shape on the table without overlaps and leaving no empty space. Then Bertha must write on each square of the table one of the numbers 1, 2, 3 or 4 in such a way that each row and column has no two numbers repeated. Bertha wins if each tetromino on the chart covers 4 different numbers.

(a) Show that Bertha can always win the game.

(b) Andrés fills the table with 4 tetrominos where at least 2 are different. Is it true that in this situation, playing with the same rules, Bertha can always win?

Solved by Oliver Geupel, Brühl, NRW, Germany; and Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany (joint work).

Let $\boldsymbol{S_1, \ldots, S_7}$ denote the given shapes from left to right. The table can not be covered with 4 pieces of shape $\boldsymbol{S_3}$ or 4 pieces of shape $\boldsymbol{S_4}$. The possible coverings with 4 pieces of the same shape $\boldsymbol{S_1}$, $\boldsymbol{S_2}$, $\boldsymbol{S_5}$, $\boldsymbol{S_6}$, $\boldsymbol{S_7}$, respectively, are each similar to one of the assemblies shown in Figure 1 below. It is demonstrated in Figure 1 how Bertha can win the game (a) in each case.

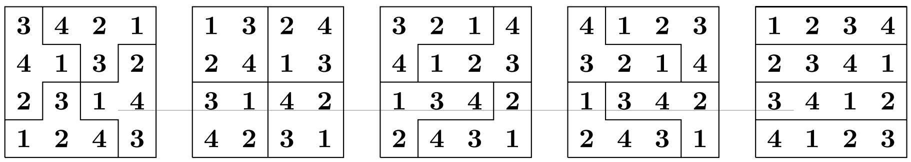


Figure 1

Andrés can win the game (b) if he puts the covering in Figure 2. For the proof, we use coordinates as in Figure 2. For example, $(\boldsymbol{A}, \boldsymbol{d})$ denotes the lower-left cell. With no loss of generality assume that Bertha writes the numbers in the upper-left piece according to Figure 2. In the lower-left piece, the number **1** can not occur in column $\boldsymbol{A}$. Hence $(\boldsymbol{B}, \boldsymbol{c}) = \mathbf{1}$. Then $(\boldsymbol{A}, \boldsymbol{b})$ in the same piece cannot be **1**. Also $(\boldsymbol{A}, \boldsymbol{b})$ cannot be **3** or **4**, because both numbers occur in the same row. Thus $(\boldsymbol{A}, \boldsymbol{b}) = \mathbf{2}$. Also, $(\boldsymbol{D}, \boldsymbol{a}) = \mathbf{4}$ and $(\boldsymbol{C}, \boldsymbol{a}) = \mathbf{3}$. Now, the occurrence of **2** in the upper-right tetromino must be $(\boldsymbol{D}, \boldsymbol{c})$, with the consequence that there is no possible entry for $(\boldsymbol{C}, \boldsymbol{c})$. Therefore, Bertha loses the game (b).

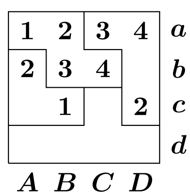


Figure 2

To complete this number of the *Corner* we look at solutions to the Olimpiada Nacional Escolar de Matematica 2009, Level 3 given at [2010 : 373–374].

1. For each positive integer $\boldsymbol{N}$ let $\boldsymbol{c(n)}$ be the number of decimal digits of $\boldsymbol{N}$. Let $\boldsymbol{A}$ be a set of positive integers such that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are two distinct elements of $\boldsymbol{A}$, then $\boldsymbol{c(a+b)+2 > c(a)+c(b)}$. Find the largest number of elements that $\boldsymbol{A}$ can have.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the presentation by Curtis.

Note that $c(n)$ is a nondecreasing function, and that for each positive integer n, $10^{c(n)-1} \le n \le 10^{c(n)} - 1$. Hence, if $a \le b$, then

$$c(a+b) \le c(2b) \le c[2 \cdot (10^{c(b)} - 1)] \le c(10^{c(b)+1} - 1) = c(b) + 1.$$

Thus,

$$c(a+b) \le 1 + \max\{c(a), c(b)\}.$$

If $c(b) \ge c(a)$, then $3 + c(b) \ge c(a+b) + 2 > c(a) + c(b)$, so that $c(a) \in \{1,2\}$. Similarly, if $c(a) > c(b)$, then $c(b) \in \{1,2\}$. If $c(b) = c(a)$, then $c(b) = c(a) \in \{1,2\}$.

Write the elements of A as

$$1 \le x_1 < x_2 < x_3 < \cdots.$$

If $i < j$, then $x_i < x_j$, implying that $c(x_i) \le c(x_j)$, so that $c(x_i) \in 1, 2$. Hence, A has at most one element greater than or equal to 100, so A is finite, and $\#(A) \le 100$. Write $A = \{x_i\}_{i=1}^{N}$. We have $N \le 100$. Suppose that $i < j$ and $c(x_i) = c(x_j) = 2$. Then $c(x_i + x_j) > 2$. In particular, the smallest two 2-digit numbers in A have sum at least 100. The table shows the maximum number of 2-digit numbers for various possible values of the smallest 2-digit number in A.

x_i	$\min x_j$	Max. num. of 2-digit num.
10	90	11
20	80	21
30	70	31
40	60	41
49	51	50
50	51	50
51	52	49

Thus, A can have at most nine 1-digit numbers, at most fifty 2-digit numbers, and at most one number with more than two digits, so $N \le 60$. In fact, A can have 60 numbers. One possible such set is

$$A = \{1, 2, 3, \ldots, 9, 50, 51, 52, \ldots, 99, 100\}.$$

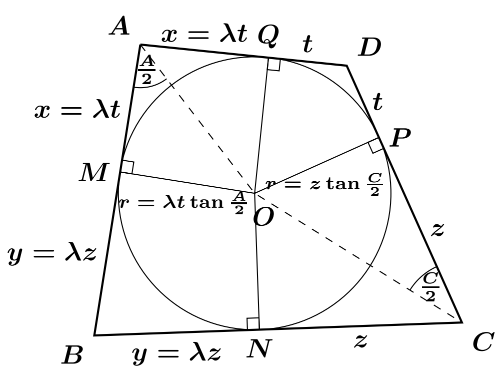
It is readily verified that any distinct pair $a, b \in A$ satisfies $c(a+b) + 2 > c(a) + c(b)$.

2. In a quadrilateral $ABCD$, a circle is drawn that is tangent to the sides AB, BC, CD and DA at the points M, N, P and Q respectively. Prove that if

$$(AM)(CP) = (BN)(DQ),$$

then $ABCD$ can be inscribed in a circle.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.



We put $AM = x$, $BN = y$, $CP = z$ and $DQ = t$. Then we have $xz = yt$, that is, $\frac{x}{t} = \frac{y}{z} = \lambda$, say, where λ is a real number. Thus, $x = \lambda t$ and $y = \lambda z$.

The tangents AM and AQ have the same length and similarly at the other vertices. Therefore

$$\begin{aligned} AB &= AM + MB = x + y = \lambda(t + z) \\ BC &= BN + NC = y + z = (\lambda + 1)z \\ CD &= CP + PD = z + t \\ DA &= DQ + QA = t + x = (\lambda + 1)t. \end{aligned}$$

By the law of cosines, applied to triangles ABD and BCD, we have

$$BD^2 = [(\lambda + 1)t]^2 + [\lambda(t + z)]^2 - 2\lambda(\lambda + 1)t(t + z)\cos A$$
$$BD^2 = [(\lambda + 1)z]^2 + (z + t)^2 - 2(\lambda + 1)z(z + t)\cos C$$

or

$$(\lambda + 1)^2(t^2 - z^2) + (t + z)^2(\lambda^2 - 1) = 2(\lambda + 1)(z + t)(\lambda t \cos A - z \cos C)$$

that is

$$(\lambda+1)(z+t)[(\lambda+1)(t-z)+(\lambda-1)(t+z)] = 2(\lambda+1)(z+t)(\lambda t \cos A - z \cos C)$$

which simplifies to $\lambda t - z = \lambda t \cos A - z \cos C$ or $\lambda t(1 - \cos A) = z(1 - \cos C)$, which is equivalent to

$$2\lambda t \sin^2 \frac{A}{2} = 2z \sin^2 \frac{C}{2} \,. \tag{1}$$

We also have

$$\lambda t \tan \frac{A}{2} = z \tan \frac{C}{2} (= r) \tag{2}$$

where r is the radius of the inscribed circle in quadrilateral $ABCD$. From (1) and (2) we obtain, by division,

$$\begin{aligned} &\frac{2\lambda t \sin^2 \frac{A}{2}}{\lambda t \tan \frac{A}{2}} = \frac{2z \sin^2 \frac{C}{2}}{z \tan \frac{C}{2}} \Rightarrow \frac{2 \sin^2 \frac{A}{2}}{\tan \frac{A}{2}} = \frac{\sin^2 \frac{C}{2}}{\tan \frac{C}{2}} \\ &\Rightarrow 2 \sin \frac{A}{2} \cos \frac{A}{2} = 2 \sin \frac{C}{2} \cos \frac{C}{2} \end{aligned}$$

so

$$\sin A = \sin C \,. \tag{3}$$

Similarly, we find

$$\sin B = \sin D \,. \tag{4}$$

From (3), either $A = C$ or $A + C = 180°$. From (4), either $B = D$ or $B + D = 180°$.

Case (a). $A + D = 180°$ or $B + D = 180°$. In this case $ABCD$ is cyclic since opposite angles are supplementary.

Case (b). $A = C$ and $B = D$. Denote the center of the incircle of $ABCD$ by O. Right-angled triangles AMO, CPO and BNO, DQO are congruent with $x = z$ and $y = t$. Since $xz = yt$, we obtain $x = y = z = t$ making OM, ON, OP, OQ the perpendicular bisectors of segments AB, BC, CA, AD, respectively. Thus, $OA = OB = OC = OD$ making $ABCD$ cyclic.

3. (a) There are $\mathbf{8}$ points placed on a circle. We say that Juliana performs "operation T" if she chooses $\mathbf{3}$ such points and paints the sides of the triangle they determine in such a way that each painted triangle has at most one vertex in common with a previously painted triangle.

What is the greatest number of operations T that Juliana can make?

(b) If in part (a), if you have $\mathbf{7}$ points instead of $\mathbf{8}$ points, then what is the greatest number of operations T Juliana can make?

Solved by Oliver Geupel, Brühl, NRW, Germany.

We interpret the problem so that each painted triangle must have at most one vertex in common with *every* previously painted triangle. We claim that the greatest number of operations T is 8 in part (a) and 7 in part (b).

We start with the proof of part (a) and denote the points by the numbers $\mathbf{1, 2, \ldots, 8}$. A possible sequence of triangles of length 8 is

$$\mathbf{123,\ 456,\ 167,\ 148,\ 268,\ 347,\ 358,\ 257.}$$

On the other hand, consider a sequence of operations T. By assumption, each painted edge belongs to only one painted triangle. Therefore, each point from $\{\mathbf{1, 2, \ldots, 8}\}$ is adjacent with an even number of edges of painted triangles, hence, with not more than 6 edges. Then, the total number of painted edges is not greater than $\mathbf{8 \cdot 6/2 = 24}$. Consequently, we have not more than $\mathbf{24/3 = 8}$ triangles, which completes the proof of part (a).

It remains to prove part (b). Denote the points by the numbers $\mathbf{1, 2, \ldots, 7}$. A possible sequence of length 7 of triangles is

$$\mathbf{123,\ 345,\ 367,\ 146,\ 157,\ 247,\ 256.}$$

On the other hand, we have not more than $\binom{\mathbf{7}}{\mathbf{2}} = \mathbf{21}$ edges, hence not more than $\mathbf{21/3 = 7}$ steps, which completes the proof of part (b).

That completes the *Corner* for this issue.

BOOK REVIEWS

Amar Sodhi

Charming Proofs : A Journey Into Elegant Mathematics
by Claudi Alsina and Roger B. Nelsen
The Mathematical Association of America, 2010
ISBN: 978-0-88385-348-1, Hardcover, 295 + xxiv pages, US$59.95
Reviewed by ***R. P. Gallant****, Grenfell Campus, Memorial University of Newfoundland, Corner Brook, NL*

Alsina and Nelsen have set out to collect some beautiful proofs in elementary mathematics. Although beauty is in the eye of the beholder, I think many will agree they have succeeded.

The book includes chapters individually devoted to results on polygons, triangles, equilateral triangles, quadrilaterals, squares, curves, and results from three-dimensional geometry. Other chapters include 'Adventures in Tiling and Coloring', 'Distinguished Numbers', 'Points in the Plane', 'A Garden of Integers', and a final chapter containing assorted results. Each chapter closes with a selection of 10–15 relevant problems for the reader to attempt. Solutions to these challenges are provided at the end of the book.

The selection of content is ripe for supporting visuals, and indeed "*Charming Proofs*" is distinguished by its numerous (over 300(!)) diagrams. The authors have written several other books devoted to the use of diagrams in mathematics, and that experience shows in this book.

This book is part of the Dolciani Mathematical Expositions series and as such is intended to be sufficiently elementary for undergraduate and even some high school students. "Charming Proofs" hardly uses calculus, and even then only in a handful of places. The book should be fully accessible to serious mathematics undergraduates, and much will be accessible to talented high school students.

I must mention an error. In a short discussion about types of proofs, the authors confuse the converse for the contrapositive (page xxii). You may feel the same pang of concern I did upon finding this doozy so early in the book, but I assure you that the very few mistakes I found (like the one on page 108 involving the perimeter of the Varignon parallelogram, as another example) are essentially typos and of no consequence if one is paying attention.

Certainly anyone interested in elegant proofs should consider this book. Also, anyone interested in mathematical competitions should find this a useful problem book, though in this case be mindful of the elementary nature and geometric emphasis of the book. The book contains results both familiar and less familiar, and should be attractive to inexperienced and experienced readers of both classes.

In summary, "Charming Proofs" is a wonderful collection of elementary proofs and related problems and should find its way onto the bookshelves of many.

RECURRING CRUX CONFIGURATIONS 2

J. Chris Fisher

Triangles for which $2b = c + a$

This month we explore triangles ABC whose sides a, b, c are in arithmetic progression; we shall see that with the triangle labeled so that b is the intermediate side, having the sides in arithmetic progression is equivalent to requiring $\angle BIO = 90°$, as well as IG parallel to AC, and many other noteworthy properties (where I, O, and G are the incentre, circumcentre, and centroid, respectively). As in last month's column, I will supply statements, references, and occasional hints, leaving the proofs as exercises.

Problem 268 [1977 : 190; 1978 : 78-79] (Proposed by Gali Salvatore = Léo Sauvé). Show that in ΔABC with $a \geq b \geq c$, the sides are in arithmetic progression if and only if

$$2\cot\frac{B}{2} = 3\left(\tan\frac{C}{2} + \tan\frac{A}{2}\right).$$

The featured solution made use of the half-angle identities

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{s-b}{s} \quad \text{and} \quad \sum_{cyclic}\left(\tan\frac{A}{2}\tan\frac{B}{2}\right) = 1,$$

where s is the semiperimeter $(a+b+c)/2$. One solver, Charles W. Trigg, added the comment that in triangles where $c + a = 2b$, seven other relationships "follow easily":

1. $\cos C + \cos A = 4\sin^2\frac{B}{2}$.
2. $a\cos C - c\cos A = 2(a - c)$.
3. $ca = 6Rr$ (where R and r are the circumradius and inradius).
4. $\cos A = \frac{4c-3b}{2c}$.
5. $B < 60°$ except when the triangle is equilateral.
6. $GI || BC$.
7. In the special case where $A = C + 90°$, then a, b, c are in the ratio $(\sqrt{7}+1) : \sqrt{7} : (\sqrt{7}-1)$.

The editor Sauvé remarked that copies of Trigg's proofs were available from him on request; sadly both he and Trigg died long ago, so today it would probably be faster for the reader to discover the proofs for himself. Number 6, however, has appeared in this journal several times, twice as a "Klamkin Quickie" [1996 : 61] and [2001 : 79]. An even quicker proof appeared as D.L. MacKay's solution to Problem E411 in [1]; namely, *Prove that if the sides of a triangle form an* *arithmetic progression the line joining the centroid to the incentre is parallel to one side*: We have $b - a = c - b$ if and only if $s = 3b/2$, and

$$r = \frac{\Delta}{s} = \frac{2\Delta}{3b} = \frac{h}{3},$$

where Δ is the area of ΔABC and h is the altitude from B to CA. Thus the incentre and centroid are equidistant from the side BC. This proof (expanded somewhat) was reproduced as Problem 82 in [5]. A refined version of the same problem had appeared a couple years earlier in [2]. For that version, recall that the Nagel point (the common intersection point of the lines joining a vertex to the point where the opposite excircle touches a side) lies on the line GI. It turns out that in a triangle whose sides are in arithmetic progression, the common difference $|a - b| = |b - c|$ equals the distance from I to the Nagel point. The *MathWorld* web page has proposed calling GI the *Nagel line*, but the name seems not to have caught on yet.

Another solver of Problem 268, Leon Bankoff, submitted two striking results (which he saw—so he said— "out of the cornea of his eye"):

i. If $c + a = 2b$, then $\cot \frac{C}{2}k + \cot \frac{A}{2} = 2\cot \frac{B}{2}$;
ii. If $c^2 + a^2 = 2b^2$, then $\cot C + \cot A = 2\cot B$.

The first follows from Problem 268, while the second is Property 3 from our previous column on root-mean-square triangles.

Problem 2870 [2003 : 399; 2004 : 382-383] (Proposed by Toshio Seimiya). Given triangle ABC with incentre I, circumcentre O, and centroid G, suppose that $\angle AIO = 90°$. Prove that $IG||BC$.

The featured solution also proved the converse (for scalene triangles); this result combined with Trigg's Property 6 (above) implies that for scalene triangles,

> $\angle BIO \leq 90°$ if and only if $2b \leq c + a$, with equality holding only simultaneously.

This version is Problem 1506 in [7], where there are two published solutions; it also appeared as problem 2 on the Second Hong Kong Mathematical Olympiad 1999, with a solution in **CRUX with MAYHEM** [2005 : 520-521]. Amengual Covas added three further references, namely [3], [4], [6] which, he said, "provide other relationships."

Problem 3197 [2006 : 516, 518; 2007 : 501-502] (Proposed by Paul Deiermann). If AB is a fixed line segment, find the triangle ABC which has maximum area among those which satisfy $\angle AIO = \pi/2$. What is this maximum area?

Michel Bataille's featured solution plugged $2a = b + c$ into Heron's formula for the area and found that the maximum area is achieved for the triangle with sides

$$c = 1, a = \frac{3 + \sqrt{3}}{3}, \text{ and } b = 2a - 1.$$

Finally, triangles with sides in arithmetic progression are mentioned in a footnote to a problem dealing with lines through vertex B that are perpendicular to IO.

Problem 2246 (reworded) [1997 : 244; 1998 : 318-319] (Proposed by D.J. Smeenk). Given the nonequilateral triangle ABC, suppose that the line through B that is perpendicular to OI intersects the bisector of $\angle BAC$ at P, and that the line through P parallel to AC intersects BC at M. Show that I, G, and M are collinear.

The problem breaks down, of course, should ΔABC be equilateral (because then I and O would coincide). Note that $BI \perp IO$ implies that $P = I$, whence the conclusion that G is on the line IM gives yet another proof (although convoluted) that $BI \perp IO$ implies $GI || AC$. A comment attached to the problem pointed out that for M to be defined, IG and BC could not be parallel, which thus forbids $AI \perp IO$, whence the condition $2a \neq b + c$ should have been added to the statement of the problem.

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[3] F. G.-M., *Exercices de géométrie*, 7th ed., pp. 464-465.

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[5] Ross Honsberger, *Mathematical Morsels.* Dolciani Mathematical Expositions No. 3, 1978, pages 209-210.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobre 2012**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3651. *Correction. Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a + b + c = 3$. Montrer que

$$a^2b + b^2c + c^2a + abc + 4abc(3 - ab - bc - ca) \leq 5\,.$$

3664. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres non négatifs tels que $a + b + c = 3$ Montrer que

$$|(1 - a^2b)(1 - b^2c)(1 - c^2a)| \leq 3|1 - abc|\,.$$

3665. *Proposé par Nguyen Thanh Binh, Hanoi, Vietnam.*

Dans une quadrilatère cyclique $ABCD$, soit M le point d'intersection des diagonales AC et BD, et soit Q le point d'intersection de la droite passant par M et le point milieu de BC. Montrer que MQ est perpendiculaire à AD si et seulement si les côtés AD et BC sont parallèles (en quel cas $ABCD$ est un trapèzoïde isocèle) ou les diagonales sont perpendiculaires (et alors on a la configuration de Brahmagupta).

3666. *Proposé par Michel Bataille, Rouen, France.*

Soit $\mathcal{R}$ l'ensemble des paires d'entiers relativement premiers et

$$\mathcal{S} = \{(a, b) \in \mathcal{R} : 5a + 42b \equiv 0 \pmod{1789}\}\,.$$

Trouver une bijection explicit entre $\mathcal{S}$ et $\mathcal{R} - S$.

3667. *Proposé par Joe Howard, Portales, NM, É-U.*

On suppose que $b_i > 0$ pour $i = 1, 2, \ldots, n$; $n \geq 3$; et $\prod_{i=1}^{n} b_i = 1$. Montrer que

$$\sum_{i=1}^{n} b_i^{n-2} \geq \sum_{i=1}^{n} b_i^{\frac{n-1}{2}} .$$

3668. *Proposé par Neven Jurič, Zagreb, Croatie.*

On suppose que p, q et r sont trois nombres premiers distincts. Combien de solutions en entiers positifs l'équation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ (où $y = pqr$) possède-t-elle ?

3669. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle inscrit dans un cercle Γ et soit r un nombre réel ($r \neq 0, 1$). On définit les points D, E, F par $\overrightarrow{BD} = r\overrightarrow{BC}$, $\overrightarrow{CE} = r\overrightarrow{CA}$, $\overrightarrow{AF} = r\overrightarrow{AB}$. Le cercle Γ coupe encore DA en D_1 et la parallèle à BC par D_1 en B'. On construit les points C' et A' de manière analogue. Pour quels r les droites AA', BB', CC' sont-elles concourantes ? Quel est alors leur point d'intersection ?

3670. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $n \geq 2$ un entier. Calculer

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{x + y + z}.$$

3671★. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit M un point à l'intérieur d'un tétrahèdre $ABCD$. Est-il vrai, oui ou non, que

$$\frac{[BCD]}{AM^2} = \frac{[ACD]}{BM^2} = \frac{[ABD]}{CM^2} = \frac{[ABC]}{DM^2} = \frac{2}{\sqrt{3}},$$

si et seulement si le tétrahèdre est régulier et que M est son centre de gravité. On note ici l'aire de T par $[T]$.

3672. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Soit x et y deux nombres réels tels que $x^2 + y^2 = 1$ Montrer que

$$\frac{1}{1 + x^2} + \frac{1}{1 + y^2} + \frac{1}{1 + xy} \geq \frac{3}{1 + \left(\frac{x+y}{2}\right)^2}$$

Trouver quand cette inégalité est valable.

3673. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Calculer le produit

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)^{(-1)^{n-1}} .$$

3674★. *Proposé par Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, Chine.*

Soit I le centre de la sphère inscrite dans un tétrahèdre $ABCD$ et soit A_1, B_1, C_1, D_1 les points respectivement symétriques de I par rapport aux plans BCD, ACD, ABD, ABC. Les quatre droites AA_1, BB_1, CC_1, DD_1 sont-elles forcément concourantes ?

3675. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c les côtés d'un triangle et s son demi-périmètre. Soit respectivement r et R les rayons de ses cercles inscrit et circonscrit. Montrer que

$$6 \;\leq\; \sum_{\text{cyclique}} \frac{b(s-b)+c(s-c)}{a(s-a)} \;\leq\; \frac{3R}{r} .$$

. . . .

3651. *Correction. Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c = 3$. Prove that

$$a^2b+b^2c+c^2a+abc+4abc(3-ab-bc-ca) \;\leq\; 5 .$$

3664. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$|(1-a^2b)(1-b^2c)(1-c^2a)| \;\leq\; 3|1-abc| .$$

3665. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the diagonals AC and BD of the cyclic quadrilateral $ABCD$ intersect at M, and let the line joining M to the midpoint of BC meet AD at Q. Prove that MQ is perpendicular to AD if and only if the sides AD and BC are parallel (in which case $ABCD$ is an isosceles trapezoid), or the diagonals are perpendicular (and we have Brahmagupta's configuration).

3666. *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{R}$ denote the set of all pairs of relatively prime integers and

$$\mathcal{S} = \{(a, b) \in \mathcal{R} : 5a + 42b \equiv 0 \pmod{1789}\}.$$

Find an explicit bijection between $\mathcal{S}$ and $\mathcal{R} - \mathcal{S}$.

3667. *Proposed by Joe Howard, Portales, NM, USA.*

Suppose $b_i > 0$ for $i = 1, 2, \ldots, n$; $n \geq 3$; and $\prod_{i=1}^{n} b_i = 1$. Prove

$$\sum_{i=1}^{n} b_i^{n-2} \geq \sum_{i=1}^{n} b_i^{\frac{n-1}{2}}.$$

3668. *Proposed by Neven Jurič, Zagreb, Croatia.*

Suppose p, q and r are distinct prime numbers. How many positive integer solutions has the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ where $y = pqr$?

3669. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle inscribed in a circle Γ and let r be a real number ($r \neq 0, 1$). Points D, E, F are defined by $\overrightarrow{BD} = r\overrightarrow{BC}$, $\overrightarrow{CE} = r\overrightarrow{CA}$, $\overrightarrow{AF} = r\overrightarrow{AB}$. The circle Γ meets again DA at D_1 and the parallel to BC through D_1 at B'. Points C' and A' are constructed in a similar way. For which r are AA', BB', CC' concurrent lines? What is then their point of concurrency?

3670. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $n \geq 2$ be an integer. Calculate

$$\int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{x + y + z}.$$

3671★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a tetrahedron and let M be a point in its interior. Prove or disprove that

$$\frac{[BCD]}{AM^2} = \frac{[ACD]}{BM^2} = \frac{[ABD]}{CM^2} = \frac{[ABC]}{DM^2} = \frac{2}{\sqrt{3}},$$

if and only if the tetrahedron is regular and M is its centroid. Here $[T]$ denotes the area of T.

3672. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2 + y^2 = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy} \ \geq \ \frac{3}{1+\left(\frac{x+y}{2}\right)^2}$$

When does this inequality occur?

3673. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the product

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)^{(-1)^{n-1}} .$$

3674★. *Proposed by Zhang Yun, High School attached to Xi$'$ An Jiao Tong University, Xi$'$ An City, Shan Xi, China.*

Let I denote the centre of the inscribed sphere of a tetrahedron $ABCD$ and let A_1, B_1, C_1, D_1 denote their symmetric points of point I about planes BCD, ACD, ABD, ABC respectively. Must the four lines AA_1, BB_1, CC_1, DD_1 be concurrent?

3675. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle and let s be its semiperimeter. Let r and R denote its inradius and circumradius respectively. Prove that

$$6 \ \leq \ \sum_{\text{cyclic}} \frac{b(s-b)+c(s-c)}{a(s-a)} \ \leq \ \frac{3R}{r} .$$

SOLUTIONS

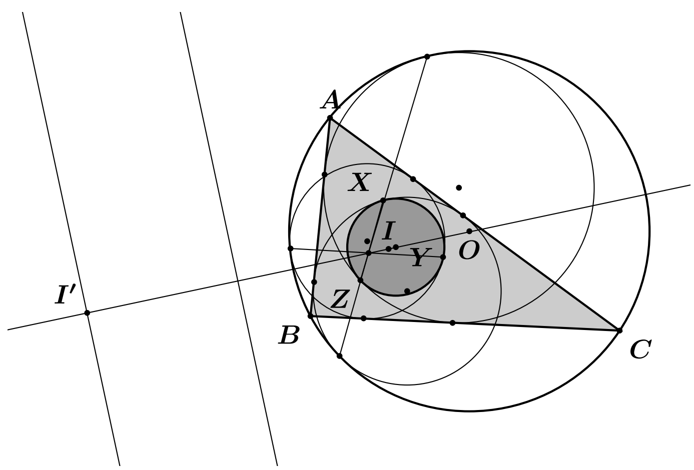
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3542★. [2010 : 240, 242; 2011 : 244] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

The mixtilinear incircles of a triangle ABC are the three circles each tangent to two sides and to the circumcircle internally. Let Γ be the circle tangent to each of these three circles internally. Prove that Γ is orthogonal to the circle passing through the incentre and the isodynamic points of the triangle ABC.

[*Ed.: Let* Γ_A *be the circle passing through* A *and the intersection points of the internal and external angle bisectors at* A *with the line* BC*. The isodynamic points are the two points that* Γ_A, Γ_B, *and* Γ_C *have in common.*]

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.



1. The circle Γ has barycentric equation

$$(a+b+c)^2(a^2+b^2+c^2-2ab-2bc-2ca)(a^2yz+b^2zx+c^2xy)$$
$$+\,8abc(x+y+z)\left(\sum_{\text{cyclic}} bc(b+c-a)x\right)=0.$$

Proof. The mixtilinear incircles are defined by the equations

$$\begin{aligned}(a+b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)f_a(x,y,z) &= 0,\\(a+b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)f_b(x,y,z) &= 0,\\(a+b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)f_c(x,y,z) &= 0,\end{aligned}$$

where

$$\begin{aligned}f_a(x,y,z) &= 4b^2c^2x+c^2(c+a-b)^2y+b^2(a+b-c)^2z,\\f_b(x,y,z) &= c^2(b+c-a)^2x+4c^2a^2y+a^2(a+b-c)^2z,\\f_c(x,y,z) &= b^2(b+c-a)^2x+a^2(c+a-b)^2y+4a^2b^2z.\end{aligned}$$

Their radical center is the point defined by

$$f_a(x,y,z)=f_b(x,y,z)=f_c(x,y,z).$$

Solving these equations we obtain the radical center in homogeneous barycentric coordinates

$$(a^2(b^2+c^2-a^2-4bc):b^2(c^2+a^2-b^2-4ca):c^2(a^2+b^2-c^2-4ab)).$$

This point divides OI in the ratio $2R:-r$. The lines joining this radical center to the points of tangency with the circumcircle intersect the respective mixtilinear incircles again at the points

$$\begin{aligned}X &= (a(a^2+2a(b+c)-3(b-c)^2):\ 2b^2(c+a-b):\ 2c^2(a+b-c)),\\Y &= (2a^2(b+c-a):\ b(b^2+2b(c+a)-3(c-a)^2):\ 2c^2(a+b-c)),\\Z &= (2a^2(b+c-a):\ 2b^2(c+a-b):\ c(c^2+2c(a+b)-3(a-b)^2)).\end{aligned}$$

Γ is the circle containing these three points.

Note: The center of Γ is the point

$$(a^2(b^2+c^2-a^2+8bc):b^2(c^2+a^2-b^2+8ca):c^2(a^2+b^2-c^2+8ab)),$$

which divides OI in the ratio $4R:r$.

2. The line $\sum_{\text{cyclic}} bc(b+c-a)x=0$ is the perpendicular bisector of II', where I' is the inversive image of the incenter I in the circumcircle.

   Proof. The polar of I in the circumcircle is the line $\sum_{\text{cyclic}} bc(b+c)x=0$. Replacing (x,y,z) by $2(a+b+c)(x,y,z)-(x+y+z)(a,b,c)$, we obtain the image of this polar under the homothety $\mathsf{h}\left(I,\frac{1}{2}\right)$. This gives the line in question. Since the pedal of I on its polar is the inversive image I', the line is the perpendicular bisector of II'.

3. One obtains the equation of the pencil of circles generated by a circle and a line by setting equal to zero a linear combination of the circle's formula and the product of the line's formula times that of the line at infinity: $x+y+z$. (Circles are conics that pass through the conjugate imaginary points on the line at infinity; all circles will contain those two points while the pencil generated by a circle and line will consist of all circles through the common pair of points of that circle and line—points which are possibly imaginary or coincident.)

From the equation of Γ in (1), we conclude that it is a member in the pencil of circles generated by the circumcircle $a^2yz + b^2zx + c^2xy = 0$ and the perpendicular bisector of II' with equation computed in (2).

Now, any circle through I and I' is orthogonal to the circumcircle.

Since the isodynamic points are inverse in the circumcircle, the circle through I, I' and one of them must also contain the other. In other words, the circle $\mathcal{C}$ through I and the isodynamic points contains I', and is orthogonal to every circle in the pencil generated by the circumcircle and the perpendicular bisector of II'.

Since Γ is a member of this pencil, it is orthogonal to circle $\mathcal{C}$.

No other solutions were received.

3556. [2010 : 315, 317] *Proposed by Arkady Alt, San Jose, CA, USA.*

For any acute triangle with side lengths a, b, and c, prove that

$$(a+b+c)\min\{a,b,c\} \ \le\ 2ab+2bc+2ca-a^2-b^2-c^2\,.$$

I. Solution by Edmund Swylan, Riga, Latvia.

Note that $a+b+c = (-a+b+c)+(a-b+c)+(a+b-c),$ where each of the three terms on the right are positive. Now,

$$\begin{aligned} -a^2+ab+ca &= (-a+b+c)a,\\ ab-b^2+bc &= (a-b+c)b,\\ ca+bc-c^2 &= (a+b-c)c. \end{aligned}$$

The result follows, for any triangle, by adding these equations, replacing the final factors on the right by $\min\{a,b,c\}$, and using the first identity above.

II. Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Taking $a \le b \le c$, the required inequality may be successively rewritten:

$$\begin{aligned} ab+2bc+ca-2a^2-b^2-c^2 &\ge 0,\\ 2a(b-a)+(c-b)(a+b-c) &\ge 0. \end{aligned}$$

By the triangle inequality, both terms are nonnegative, proving the claim.

III. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy

We introduce the well-known change of variables $a = x + z$, $b = x + y$, $c = y + z$, or $x = \frac{a+b-c}{2} \geq 0$, $y = \frac{b+c-a}{2} \geq 0$ and $z = \frac{a+c-b}{2} \geq 0$. Taking $a \leq b \leq c$ gives $y \geq x, z$ and the required inequality becomes

$$2(x + y + z)(x + z) \leq 4(xy + yz + xz),$$

which is equivalent to $(xy+zy) \geq x^2+z^2$, which holds because of the inequalities between x, y, z. The result follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Most solvers observed that the result holds regardless of whether the triangle is acute. Arslanagić, Geupel and Malikić also noted the case of equality, $a = b = c$, but all appear to have missed two separate (degenerate) cases of equality, namely (i) $c = 2a = 2b$ and (ii) $a = 0$, $b = c$.

3563. [2010 : 316, 318] *Proposed by Mikhail Kochetov and Sergey Sadov, Memorial University of Newfoundland, St. John's, NL.*

A square $n \times n$ array of lamps is controlled by an $n \times n$ switchboard. Flipping a switch in position (i, j) changes the state of all lamps in row i and in column j.

(a) Prove that for even n it is possible to turn off all the lamps no matter what the initial state of the array is. Demonstrate how to do it with the minimum number of switches.

(b) Prove that for odd n it is possible to turn off all the lamps if and only if the initial state of the array has the following property: either the number of ON lamps in every row and every column is odd, or the number of ON lamps in every row and every column is even. If this property holds, provide an algorithm to turn off all the lamps.

Solution by Steffen Weber, student, Martin-Luther-Universität, Halle, Germany.

The order in which switches are flipped does not affect the final state, and flipping a switch twice has no effect. A series of flips, avoiding this redundancy, may be coded as an $n \times n$ $(0, 1)$-matrix in which "1" represents a flip in the corresponding position. It follows that there are at most 2^{n^2} distinct transformations.

(a) If n is even, flipping all switches which are in row i or column j changes only the state of lamp (i, j). Doing this once for every ON lamp turns OFF all lamps. It follows that there are at least 2^{n^2} distinct transformations, and so *exactly* 2^{n^2} distinct transformations, in one-to-one correspondence with the coded matrices described above, which give the unique minimum series of flips attaining each transformation.

Eliminating redundant flips from the above series of flips to turn all lamps OFF we have that the unique minimum series of flips is attained by flipping each switch (i, j) if and only if there are initially an odd number of ON lamps among the $2n - 1$ positions in row i and column j.

(b) If n is odd, flipping any switch changes the parity of the number of ON lamps in every row and column. If all lamps are OFF this parity is 0 for every row and column, so as required, a solution is possible only if, in the initial state, the number of ON lamps in every row and column has the same parity.

If this condition is met, follow the procedure described in (a) for n even to turn OFF all lamps in the initial $(n - 1) \times (n - 1)$ block. Because the parity property is preserved, either every lamp in the n^{th} row and column is OFF (and we're done), or all lamps in those positions are ON; in the latter case, flipping switch (n, n) turns off these remaining lamps.

Also solved by SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; OLIVER GEUPEL, Brühl, NRW, Germany; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Some solvers observed that the solution described for odd n is also optimum. Some solvers explicitly derived relationships between the state matrix and the (coded) transformation, using arithmetic modulo 2 or representing both states and transformations (acting additively) as vectors in the space of $n \times n$ matrices over $\mathbb{Z}_2$.

The proposer notes that this problem belongs to a popular class of "switchboard problems" also known as "all ones" and "lights out" (the name of a commercial game). Problems of such type are found in many math competitions and popular math journals. The problem was inspired by [6]. However, optimality was not shown for even n and the solution for odd n was inelegant. Some other references are given below.

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3564. [2010 : 396, 398] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, d be positive real numbers. Prove that

$$a^3 + b^3 + c^3 + d^3 + \frac{32abcd}{a+b+c+d} \;\geq\; 3(abc + bcd + cda + dab)\,.$$

Solution by Joe Howard, Portales, NM, USA.

This is Problem 8 of [1], left as an exercise. We will follow the solution to Problem 3, from the same article.

Without loss of generality we can assume that $d = \min\{a, b, c, d\} = 1$. Then, the inequality is:

$$a^3 + b^3 + c^3 + 1 + \frac{32abc}{a+b+c+1} \geq 3(abc + ab + ac + bc)\,,$$

or equivalently

$$(a+b+c+1)^4 + 32abc \geq 3(a+b+c+1)^2(ab+bc+ca+a+b+c)\,.$$

Let $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$. Then $p \geq 3$ and $q \geq 3$. We need to prove

$$(p+1)^4 + 32r \geq 3(p+1)^2(p+q)\,.$$

The inequality $3q \leq p^2$ is equivalent to $ab+ac+bc \leq a^2+b^2+c^2$, which is easy to prove. Thus, we can find some $t \geq 0$ so that

$$q = \frac{p^2 - t^2}{3}\,.$$

Now, by Theorem 1 in [1] we get

$$r \geq \frac{p^3 - 3pt^2 - 2t^3}{27}\,.$$

To complete the proof it suffices to show that

$$(p+1)^4 - 3p(p+1)^2 - 3(p+1)^2\frac{p^2-t^2}{3} + \frac{32}{27}(p^3 - 3pt^2 - 2t^3) \geq 0\,. \quad (1)$$

This simplifies to

$$(5p+3)(p-3)^2 + t^2(27 + 27p^2 - 42p - 64t) \geq 0\,.$$

Using $p \geq 3$ we get $14p^2 \geq 42p$, while from $\frac{p^2-t^2}{3} = q \geq 3$ we get

$$p^2 \geq t^2 + 9 \geq 6t\,.$$

Thus

$$27p^2 = 14p^2 + 13p^2 \geq 42p + 78t \geq 42p + 64t\,,$$

which proves (1), and hence completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer . There was one incorrect solution.

References

[1] P. V. Thuận, Lê Vĩ, *A useful inequality revisited*, Crux, Vol 35(3), 2009, pp. 164-171

3565. [2010 : 396, 398] *Proposed by Max Diaz, student, San Juan Bosco High School, Huancayo, Junin, Peru.*

Find all positive integers n such that $\sigma(\tau(n)) = n$, where $\tau(m)$ and $\sigma(m)$ are, respectively, the number of positive divisors of the integer m and the sum of all the positive divisors of the integer m.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The solutions are 1, 3, 4, and 12.

Note that if m is a positive integer and $m = ab$ where $1 \le a \le b \le m$, then $a \le \sqrt{m}$. Thus, for each $d = 1, 2, \ldots, \lfloor\sqrt{m}\rfloor$, there is at most one positive integer d' such that $dd' = m$. Hence, we have

$$\tau(m) \le 2\lfloor\sqrt{m}\rfloor \le 2\sqrt{m}. \qquad (1)$$

Furthermore, if d is a positive divisor of m with $d < m$, then clearly $d \le \dfrac{m}{2}$, so $d \le \left\lfloor\dfrac{m}{2}\right\rfloor$.

Hence,

$$\sigma(m) \le m + \sum_{k=1}^{\lfloor\frac{m}{2}\rfloor} k \le m + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right) = \frac{1}{8}(m^2 + 10m). \qquad (2)$$

Now, if n is a solution to the given problem, then by (1), (2) and the fact that $\frac{1}{8}(m^2 + 10m)$ is an increasing function, we have

$$\begin{aligned} n = \sigma(\tau(n)) &\le \max_k \left\{\sigma(k) | 1 \le k \le 2\sqrt{n}\right\} \\ &\le \frac{1}{8}\left((2\sqrt{n})^2 + 10(2\sqrt{2})\right) = \frac{1}{2}(n + 5\sqrt{n}), \end{aligned}$$

from which we easily deduce that $n \le 25$.

Direct checking for $n = 1, 2, \ldots, 25$ then reveal that only 1, 3, 4 and 12 satisfy the given condition and our proof is complete.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HENRY RICARDO, Tappan, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3566. [2010 : 396, 398] *Proposed by an unknown proposer.*

Given points A and C on a circle with centre O, choose B on the shorter arc AC. Let ℓ be the line tangent to the circle at B, and let P and Q be the points where ℓ intersects the bisectors of $\angle AOB$ and $\angle BOC$, respectively. Prove that if $E = AC \cap OQ$, then PE is perpendicular to OQ.

Similar solutions by Václav Konečný, Big Rapids, MI, USA; and by Kee-Wai Lau, Hong Kong, China.

We assume that the circle has unit radius, and set $\alpha = \angle AOP = \angle POB$ and $\gamma = \angle BOQ = \angle QOC$. In triangle OEC we have $\angle EOC = \angle QOC = \gamma$ and

$$\angle OCE = \angle OCA = \angle OAC = \frac{180^\circ - 2(\alpha+\gamma)}{2} = 90^\circ - (\alpha+\gamma),$$

whence $\angle OEC = 90^\circ + \alpha$. By the sine law,

$$OE = \frac{\cos(\alpha+\gamma)}{\cos\alpha}.$$

Moreover, from the right triangle OPB,

$$OP = \frac{1}{\cos\alpha};$$

thus $\frac{OE}{OP} = \cos(\alpha+\gamma)$. But in triangle POE, $\angle POE = \alpha+\gamma$; therefore ΔPOE is a right triangle with hypotenuse OP and right angle at E. That is, PE is perpendicular to OQ, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece(2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; DAG JONSSON, Uppsala, Sweden; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; and PETER Y. WOO, Biola University, La Mirada, CA, USA(2 solutions).

All the submitted solutions were short and neat. Other nice methods came down to showing that PE is an altitude of triangle OPQ, or that the quadrilateral $OAPE$ is cyclic with a right angle at A.

3567. [2010 : 396, 398] *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Prove that

$$\int_0^\infty \frac{e^{-x}\left(1-e^{-2x}\right)\left(1-e^{-4x}\right)\left(1-e^{-6x}\right)}{x(1-e^{-14x})}\,dx \;=\; \ln 2\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let I denote the given integral.

With the substitution $u = e^{-x}$, we easily obtain

$$I = -\int_0^1 \frac{(1-u^2)(1-u^4)(1-u^6)}{(1-u^{14})\ln u}du.$$

We apply the following known result [1]:

$$\int_0^1 \frac{(1-u^p)(1-u^q)(1-u^r)u^{s-1}}{(1-u^t)\ln u}du = \ln\left\{\frac{\Gamma\left(\frac{p+s}{t}\right)\Gamma\left(\frac{q+s}{t}\right)\Gamma\left(\frac{r+s}{t}\right)\Gamma\left(\frac{p+q+r+s}{t}\right)}{\Gamma\left(\frac{p+q+s}{t}\right)\Gamma\left(\frac{q+r+s}{t}\right)\Gamma\left(\frac{p+r+s}{t}\right)\Gamma\left(\frac{s}{t}\right)}\right\},$$

where Γ denotes the gamma function and p, q, r, s, t are all positive. With $p = 2$, $q = 4$, $r = 6$, $s = 1$, and $t = 14$ we then obtain

$$I = \ln\left\{\frac{\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right)}{\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{5}{14}\right)\Gamma\left(\frac{13}{14}\right)}\right\}.$$

Since it is known [2] that

$$\frac{\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right)}{\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{5}{14}\right)\Gamma\left(\frac{13}{14}\right)} = 2,$$

the result follows.

Also solved by MOHAMMED AASSILA, Strasbourg, France; MICHEL BATAILLE, Rouen, France; and the proposer.

References

[1] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 5th edition, 1996; formula #32.

[2] The American Mathematical Monthly, Problem 11426: *Gamma Products*, v. 116(2009), p.365; solution in v. 117(2010), p. 842.

3569★. [2010 : 397, 399] *Proposed by Jian Liu, East China Jiaotang University, Nanchang City, China.*

Let the point P lie inside the triangle ABC and let the point Q lie outside the triangle. Let w_1, w_2, w_3 denote the lengths of the angle bisectors of $\angle BPC$, $\angle CPA$, $\angle APB$, respectively. Does the inequality

$$PA \cdot QA + PB \cdot QB + PC \cdot QC \ \geq\ 4(w_1w_2 + w_2w_3 + w_3w_1)$$

hold? [*At* `http://www.emis.de/journals/JIPAM/article1162.html?sid=1162` *the proposer's inequality is proved when* Q *lies* inside *the triangle.*]

Comment by Oliver Geupel, Brühl, NRW, Germany.

The cited article does not contain the promised inequality. Its main result is the weaker inequality

$$PA \cdot QA + PB \cdot QB + PC \cdot QC \geq 4(r_1 r_2 + r_2 r_3 + r_3 r_1), \qquad (1)$$

where r_i denotes the distances from P to the sides of the triangle, and both P and Q are interior points. We prove the following

Lemma. For each point Q outside the triangle there exists a point Q' on the boundary of the triangle such that $QA > QA'$, $QB > QB'$, and $QC > QC'$.

Proof. Drawing rays from the vertices of the triangle orthogonal to the sides partitions the plane outside the triangle into six regions: Three regions S_A, S_B, S_C outwardly on the sides and three regions T_A, T_B, T_C outwardly on the vertices. If Q lies in an "S" region, define Q' to be the orthogonal projection of Q onto the adjacent side of the triangle. If Q lies in an "T" region define Q' to be the adjacent vertex.

Editor's comment. Geupel's lemma implies that inequality (1) holds for all points Q in the plane. The status of the required result for angle bisectors, however, remains in doubt until somebody produces either a correct reference or a valid proof. Of course, Geupel's lemma shows that if the desired inequality holds when Q is an interior point, then it holds for arbitrary Q.

No solutions were received.

3570. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r, r_a, r_b, r_c, and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that

$$\frac{r_a^2}{a^2 + r_a^2} + \frac{r_b^2}{b^2 + r_b^2} + \frac{r_c^2}{c^2 + r_c^2} \geq \frac{4R + r}{4R - r}\,.$$

Solution by Kee-Wai Lau, Hong Kong, China.

Applying the Cauchy-Schwarz inequality to vectors

$$\left(\frac{r_a}{\sqrt{a^2 + r_a^2}}, \frac{r_b}{\sqrt{b^2 + r_b^2}}, \frac{r_c}{\sqrt{c^2 + r_c^2}}\right) \text{ and } \left(\sqrt{a^2 + r_a^2}, \sqrt{b^2 + r_b^2}, \sqrt{c^2 + r_c^2}\right)\,,$$

we have

$$\frac{r_a^2}{a^2 + r_a^2} + \frac{r_b^2}{b^2 + r_b^2} + \frac{r_c^2}{c^2 + r_c^2} \geq \frac{(r_a + r_b + r_c)^2}{\left(a^2 + r_a^2\right) + \left(b^2 + r_b^2\right) + \left(c^2 + r_c^2\right)}\,. \qquad (1)$$

Let s be the semiperimeter. We need the following known results.

$$\sum_{\text{cyclic}} r_a = 4R + r, \tag{2}$$

$$\sum_{\text{cyclic}} r_a^2 = (4R + r)^2 - 2s^2, \tag{3}$$

$$\sum_{\text{cyclic}} a^2 = 2\left(s^2 - r^2 - 4Rr\right). \tag{4}$$

Formulas (2) and (3) appear on p.61 (items 99, 103) and formula (4) on p.52 (item 5) [1]. With these formulas, we can get

$$\begin{aligned}\frac{(r_a + r_b + r_c)^2}{\left(a^2 + r_a^2\right) + \left(b^2 + r_b^2\right) + \left(c^2 + r_c^2\right)} &= \frac{(4R + r)^2}{(4R + r)^2 - 2s^2 + 2\left(s^2 - r^2 - 4Rr\right)} \\ &= \frac{(4R + r)^2}{(4R + r)(4R - r)} = \frac{4R + r}{4R - r},\end{aligned}$$

and this with (1) completes the solution.

Also solved by JOE HOWARD, Portales, NM, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Geupel and the proposer note that the identity $a^2 + b^2 + c^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - r^2$, *used in the last step of the featured solution, is interesting on its own.*

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

3571. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $n \geq 1$ be an integer. Among all increasing arithmetic progressions x_1, x_2, ..., x_n such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find the progression with the greatest common difference d.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Since the case $n = 1$ is degenerate, let us assume $n > 1$. An arithmetic progression $x_1, x_2, \ldots, x_n$ with common difference $d > 0$ has the property that $1 = x_1^2 + x_2^2 + \cdots + x_n^2$ if and only if

$$1 = \sum_{k=0}^{n-1} (x_1 + kd)^2 = nx_1^2 + 2d\left(\sum_{k=0}^{n-1} k\right)x_1 + d^2\left(\sum_{k=0}^{n-1} k^2\right).$$

Equivalently, x_1 is a real root of the following quadratic in x

$$1 = nx^2 + (n-1)nd \cdot x + \frac{1}{6}(n-1)n(2n-1)d^2,$$

which will happen if and only if $n > 1$ and $d > 0$ are such that its discriminant

$$(n-1)^2 n^2 d^2 - \frac{2}{3}(n-1)n^2(2n-1)d^2 + 4n$$

is non-negative and this is equivalent to saying $(n-1)n(n+1)d^2 \leq 12$. Consequently the greatest common difference is

$$d = \sqrt{\frac{12}{(n-1)n(n+1)}}\,,$$

and the first term of the progression is the solution of the quadratic above with this value of d, namely

$$x_1 = -\sqrt{\frac{3(n-1)}{n(n+1)}}.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incomplete solution was received.

3572. [2010 : 397, 399] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\left(\sum_{\text{cyclic}} \frac{ab}{c+ab}\right) + \frac{1}{4}\prod_{\text{cyclic}}\left(\frac{a+\sqrt{ab}}{a+b}\right) \geq 1\,.$$

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Albert Stadler, Herrliberg, Switzerland.

Note first that

$$\sum_{\text{cyclic}} \frac{ab}{c+ab} = \sum_{\text{cyclic}} \frac{ab}{c(a+b+c)+ab} = \sum_{\text{cyclic}} \frac{ab}{(c+a)(c+b)}$$
$$= \frac{1}{(a+b)(b+c)(c+a)}\sum_{\text{cyclic}} ab(a+b)\,.$$

Hence the given inequality is equivalent to

$$4\sum_{\text{cyclic}} ab(a+b) + \prod_{\text{cyclic}}(a+\sqrt{ab}) \geq 4(a+b)(b+c)(c+a),$$

or

$$\prod_{\text{cyclic}}(a+\sqrt{ab}) \geq 8abc\,.$$

By the AM-GM Inequality, we have

$$\prod_{\text{cyclic}}(a+\sqrt{ab}) = \sqrt{abc}\prod_{\text{cyclic}}(\sqrt{a}+\sqrt{b})$$
$$\geq 8\sqrt{abc}\sqrt{\sqrt{a}\sqrt{b}\sqrt{b}\sqrt{c}\sqrt{c}\sqrt{a}} = 8abc,$$

so our proof is complete. Clearly, equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

3574. [2010 : 398, 400, 548, 550] *Proposed by Michel Bataille, Rouen, France.*

Let x, y, and z be real numbers such that $x+y+z=0$. Prove that

$$\sum_{\text{cyclic}}\cosh x \;\leq\; \sum_{\text{cyclic}}\cosh^2\left(\frac{x-y}{2}\right) \;\leq\; 1+2\sum_{\text{cyclic}}\cosh x\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $a := e^x, b = e^y, c = e^z$. Then $a, b, c > 0$ and $abc = e^{x+y+z} = 1$. Let $s := a+b+c, p := ab+ac+bc$. Then

$$\sum_{cyc}\cosh(x) = \frac{1}{2}\sum_{cyc}(a+bc) = \frac{s+p}{2}\,.$$

Let's observe that

$$\cosh\left(\frac{x-y}{2}\right) = \frac{e^{\frac{x-y}{2}}+e^{\frac{y-x}{2}}}{2} = \frac{1}{2}\left(\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}}\right)$$
$$= \frac{a+b}{2\sqrt{ab}} = \frac{(a+b)\sqrt{c}}{2}\,. \qquad (1)$$

Thus

$$\sum_{cyc}\cosh^2\left(\frac{x-y}{2}\right) = \frac{1}{4}\sum_{cyc}(a+b)^2c = \frac{1}{4}\sum_{cyc}a^2c+b^2c+2 = \frac{3+sp}{4}\,.$$

Also

$$\prod_{cyc}\cosh(x) = \prod_{cyc}\frac{a+bc}{2} = \prod_{cyc}\frac{a^2+1}{2a}$$
$$= \frac{1}{8}\prod_{cyc}(a^2+1) = \frac{2+p^2+s^2-2p-2s}{8}\,.$$

Thus, the inequality to prove becomes

$$\frac{1}{2}(s+p) \leq \frac{3+sp}{4} \leq 1 + \frac{2+p^2+s^2-2p-2s}{4}\,, \tag{2}$$

or equivalently

$$2(s+p) \leq 3+sp \leq 6+p^2+s^2-2(p+s)\,. \tag{3}$$

Observing that $p \geq 3\sqrt[3]{a^2b^2c^2} = 3$ and $s \geq 3\sqrt[3]{abc} = 3$ we obtain

$$sp+3-2(s+p) = (s-3)(p-3)+(s-3)+(p-3) \geq 0\,,$$

which proves the left hand side of (3).

To prove the RHS of (3) we note that

$$\begin{aligned} 6+&p^2+s^2-2(p+s)-(3+sp) \\ &= 3+p^2+s^2-2p-2s-sp = (p-s)^2+sp+3-2(s+p) \\ &= (p-s)^2+(s-3)(p-3)+(s-3)+(p-3) \geq 0\,. \end{aligned} \tag{4}$$

This proves the RHS of (3), and thus completes the proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

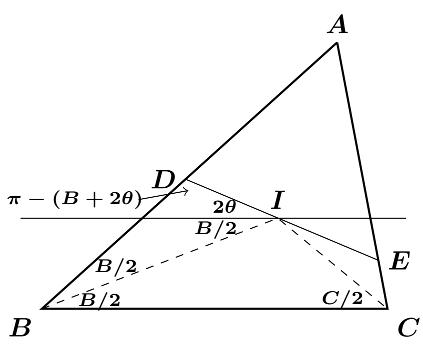
3575. [2010 : 398, 400] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with incentre I. Characterize the lines through I intersecting the sides AB and AC at D and E, respectively, such that $DE = DB + EC$ and determine how many such lines there are in terms of $\angle B$ and $\angle C$.

Solution by Joel Schlosberg, Bayside, NY, USA.

For each line DE through I intersecting sides AB and AC, $\angle BID$ and $\angle CIE$ are external to ΔBIC; moreover, because $\angle BID + \angle CIE = 180^\circ - \angle BIC = \frac{B+C}{2}$, each position of DE corresponds to a unique angle $\theta \in [-\frac{B}{4},\ \frac{C}{4}] \subset [-45^\circ,\ 45^\circ]$ such that

$$\angle BID = \frac{B}{2} + 2\theta \quad \text{and} \quad \angle CIE = \frac{C}{2} - 2\theta.$$



Because $DE = DI + IE$, the requirement that $DE = DB + EC$ is equivalent to

$$\left(\frac{DI}{BI} - \frac{DB}{BI}\right)\frac{BI}{CI} = \frac{EC}{CI} - \frac{IE}{CI}. \tag{1}$$

By the Law of Sines applied to triangles $DBI,$ BCI, and CEI, equation (1) is equivalent in turn to

$$\begin{aligned} \frac{\sin\frac{B}{2} - \sin\left(\frac{B}{2}+2\theta\right)}{\sin(B+2\theta)}\cdot\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}} &= \frac{\sin\left(\frac{C}{2}-2\theta\right) - \sin\frac{C}{2}}{\sin(C-2\theta)} \\ \frac{2\sin(-\theta)\cos\left(\frac{B}{2}+\theta\right)\sin\frac{C}{2}}{2\sin\left(\frac{B}{2}+\theta\right)\cos\left(\frac{B}{2}+\theta\right)\sin\frac{B}{2}} &= \frac{2\sin(-\theta)\cos\left(\frac{C}{2}-\theta\right)}{2\sin\left(\frac{C}{2}-\theta\right)\cos\left(\frac{C}{2}-\theta\right)}. \end{aligned}$$

Clearly $\theta = 0$ satisfies the condition; if $\theta \neq 0$ then $\sin(-\theta)$ is nonzero and can be canceled, yielding

$$\begin{aligned} \sin\left(\frac{B}{2}+\theta\right)\sin\frac{B}{2} &= \sin\left(\frac{C}{2}-\theta\right)\sin\frac{C}{2} \\ \sin^2\frac{B}{2}\cos\theta + \sin\frac{B}{2}\cos\frac{B}{2}\sin\theta &= \sin^2\frac{C}{2}\cos\theta - \sin\frac{C}{2}\cos\frac{C}{2}\sin\theta, \end{aligned}$$

whence,

$$\begin{aligned} \tan\theta &= \frac{\sin^2\frac{C}{2} - \sin^2\frac{B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\cos\frac{B}{2}} \\ &= \frac{-\left(1-2\sin^2\frac{C}{2}\right) + \left(1-2\sin^2\frac{B}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2} + 2\sin\frac{B}{2}\cos\frac{B}{2}} = \frac{-\cos C + \cos B}{\sin C + \sin B} \\ &= \frac{2\sin\frac{C+B}{2}\sin\frac{C-B}{2}}{2\sin\frac{C+B}{2}\cos\frac{C-B}{2}} = \tan\frac{C-B}{2}. \end{aligned} \tag{2}$$

Since the tangent function is strictly increasing and bijective in the domain $(-90°,\ 90°)$, which contains both the interval $\left[-\frac{B}{4},\ \frac{C}{4}\right]$ and the angle $\frac{C-B}{2}$, equation (2) is equivalent to $\theta = \frac{C-B}{2}$ for $\theta \neq 0$ and $\theta \in \left[-\frac{B}{4},\ \frac{C}{4}\right]$. These conditions imply that

$$\angle B \neq \angle C, \quad \frac{\angle B}{\angle C} \in \left[\frac{1}{2}, 2\right], \quad \text{and} \quad \angle BID = \frac{B}{2} + 2\frac{C-B}{2} = C - \frac{B}{2}.$$

Our conclusion:

- If $\angle B = \angle C$ or $\frac{\angle B}{\angle C} \notin \left[\frac{1}{2}, 2\right]$, then there is a single line with the desired property; it satisfies $\angle BID = \frac{B}{2}$ and is therefore the parallel to BC through I.

- If $\angle B \neq \angle C$ and $\frac{\angle B}{\angle C} \in \left[\frac{1}{2}, 2\right]$, then there are two lines with the desired property; they satisfy $\angle BID = \frac{B}{2}$ and $\angle BID = C - \frac{B}{2}$.

Also solved by the proposer.

We also received one incorrect and one incomplete solution. The incomplete solution provided an appealing intuitive approach, as follows: One obvious line that satisfies the required equation is the line DE through I that is parallel to BC: in this case, because BI bisects $\angle B$, $\angle DIB = \angle CBI = \angle IBD$, whence $DB = DI$; similarly $EC = IE$, so that $DE = DB + EC$, as desired. Another solution is obtained by reflecting this DE (namely the parallel to BC through I) in the angle bisector AI to obtain $E'D'$, with E' the image of D on AC and D' the image of E on AB. Here

$$\begin{aligned} D'B + E'C &= AB + AC - (AD' + AE') = AB + AC - (AE + AD) \\ &= DB + EC = DE = D'E'. \end{aligned}$$

This second solution will not exist, however, if either D' or E' fall outside ΔABC. E' falls outside when C lies between E' and A; that is, if $\angle AE'I < \angle ACI$ or, in terms of $\angle B = \angle ADE = \angle AE'D' = \angle AE'I$, if $\angle B < \frac{\angle C}{2}$. Similarly D' falls outside if $\angle C < \frac{\angle B}{2}$. The two lines coincide when $\angle B = \angle C$. In all other cases there will be at least two solutions. It remains to show that there cannot be more than two solutions. For this our correspondent makes claims about the monotonicity of the lengths of the relevant segments, first as the point E moves along AC from its position where $EI||BC$ to the foot of the angle bisector BI, and then as it moves along AC to the vertex C; he claimed that these claims are "easy to prove." I suppose it should be comforting to know that the proof is easy; however, this editor has seen many very easy proofs that he was not clever enough to devise for himself, many of which were not only easy, but correct. Please, if the proof is easy but not routine, then supply the proof.