EDITORIAL

Shawn Godin

Hello again ***CRUX with MAYHEM*** readers. Is it April already? I just wanted to thank you for your patience as I ease into my role as editor of this great publication many months later than I would have liked. You may (or may not) have noticed some subtle changes already. For one thing, we have a new font that is a lot more user friendly from my end. We have also changed the package that we use to create diagrams so it is much easier to create nicer looking diagrams.

We are in the process of going through a couple of years of proposals to decide which ones we will use in the journal. The hope is that you will hear about the fate of your proposals soon after you have submitted them. It is the plan that in the next couple of months we will review the problems that have already be submitted in **2011**. After that point, as we continue to work our way through the older problems still in our banks, problem proposers will be informed of the status of their proposal in a much more timely fashion. We will also look into classifying problems so we can inform proposers which areas already have a long queue of problems waiting to appear and which areas have a shortage.

We are still behind publication schedule and will be for the remainder of **2011**, but we should be closer to our real schedule going into **2012** (so you should get the February issue before July!). We will continue to use the "old" deadlines in print, but deadlines will be extended until we are ready to work on the material.

A ***CRUX with MAYHEM*** page has been created on Facebook at

`www.facebook.com/pages/`
`Crux-Mathematicorum-with-Mathematical-Mayhem/152157028211955`.

Updates to our progress will be posted on the site. Messages informing you when certain problems have been sent to the editors for moderation will appear to give you a better idea of what the "real" deadline for the problems will be. We will also give you some hints as to what will appear in future issues.

As I mentioned in an earlier editorial we are looking to make some changes. With increases in publishing and mailing costs we are exploring alternate ways to deliver ***CRUX with MAYHEM***. Also, in an increasingly digital world, we are looking into ways that we can better interact with our readers. We hope that, sometime in the not too distant future, ***CRUX with MAYHEM*** readers can submit problem proposals and solutions online. You would then be able to check the status of your proposal to see if it has been accepted and when it is projected to appear.

We are preparing a short online survey to get some feedback on various parts of ***CRUX with MAYHEM*** and to get your input into some of the proposed changes. Check the Facebook page for details and please take a few minutes to give us your opinion.

SKOLIAD No. 132

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **December 15, 2011**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Maritime Mathematics Competition, 2010. Our thanks go to David Horrocks, University of Prince Edward Island, for providing us with this contest and for permission to publish it.

Maritime Mathematics Competition, 2010
2 hours allowed

1. The Valhalla Winter Games are held in February, and the closing ceremonies are on the last day of the month. The first Valhalla Winter Games were held in the year 750, and since that year, they have been held every five years. How many times have the closing ceremonies been held on February 29$^{\text{th}}$? Note that year Y is a leap year if exactly one of the following conditions is true:

(a) Y is divisible by 4 but Y is not divisible by 100.

(b) Y is divisible by 400.

2. A triangle with vertices $A(0,0)$, $B(3,4)$, and $C(2,c)$ has area 5. Find all possible values of the number c.

3. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers. Assume that $f(0)$, $f(1)$, and $f(2)$ are all integers.

(a) Prove that $f(2010)$ is also an integer.

(b) Decide if $f(2011)$ is an integer.

4. If x is a real number, let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x. For example, $\lfloor 7.012 \rfloor = 7$. If n is any positive integer, find a (simple) formula for

$$\left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2(n+1)}{3} \right\rfloor + \left\lfloor \frac{2(n+2)}{3} \right\rfloor .$$

5. (a) If a is a positive number, prove that

$$a + \frac{1}{a} \geq 2 .$$

(b) If a and b are both positive numbers, prove that

$$a + \frac{1}{a} + b + \frac{1}{b} + \frac{1}{ab} \geq 4.5 .$$

You may assume without proof that $f(x) = x + \frac{1}{x}$ is an increasing function for $x \geq 1$.

6. A hole in a concrete wall has the shape of a semi-circle with a radius of $\sqrt{2}$ metres. A utility company wants to place one large circular pipe or two smaller circular pipes of equal radius through the hole to supply water to Watertown. If they want to maximise the amount of water that could flow to Watertown, should they use one pipe or two pipes, and what size pipes(s) should they use?

Concours de Mathématique des Maritimes, 2010
2 heures a permis

1. Les Jeux d'hiver de Valhalla de déroulent en février, les cérémonies de clôture se tenant let dernier jour du mois. Les premier Jeux d'hiver de Valhalla se sont déroulés en l'an 750 et les Jeux prennent place depuis à tous cinq ans. Combien de fois le cérémonies de clôture ont-elle eu lieu le 29 février ? Rappelons qu'une année Y est une année bissextile si l'une des conditions suivantes est satisfaite :

(a) Y est divisible par 4 mais n'est pas divisible par 100.

(b) Y est divisible par 400.

2. Si les sommets d'un triangle d'aire 5 sont $A(0,0)$, $B(3,4)$ et $C(2,c)$, trouver toutes les valeurs possibles de c.

3. Soit $f(x) = ax^2 + bx + c$ où a, b, et c sont des nombres réel. Supposons que $f(0)$, $f(1)$, et $f(2)$ sont des entier.

(a) Montre que $f(2010)$ est aussi un entier.

(b) Déterminer si $f(2011)$ est un entier.

4. Si x est une nombre réel, dénotons par $\lfloor x \rfloor$ le plus grand entier inférieur ou égal à x. Par exemple, $\lfloor 7.012 \rfloor = 7$. Si n est un entier positif quelconque, trouver une formule (simple) pour

$$\left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2(n+1)}{3} \right\rfloor + \left\lfloor \frac{2(n+2)}{3} \right\rfloor .$$

5. (a) Si a est un nombre réel strictement positif, montre que

$$a + \frac{1}{a} \geq 2 .$$

(b) Si a et b sont des nombres réels strictement positifs, montre que

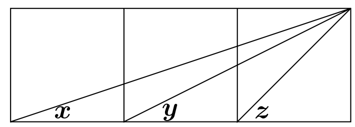
$$a + \frac{1}{a} + b + \frac{1}{b} + \frac{1}{ab} \geq 4.5 .$$

Vous pouvez supposer sans preuve que la fonction $f(x) = x + \frac{1}{x}$ est croissante pour $x \geq 1$.

6. Un mur de béton dans la ville de Watertown est percé d'un trou en forme de demi-cercle de rayon $\sqrt{2}$ mètres. La ville veut se fournir de l'eau au moyens de tuyaux passés à travers le trou. Pour maximiser le débit de l'eau, est-il préférable d'utiliser en seul grand tuyau circulaire ou deux tuyaux circulaires plus petits de même rayon, et quelle est la dimension des tuyaux à utiliser ?

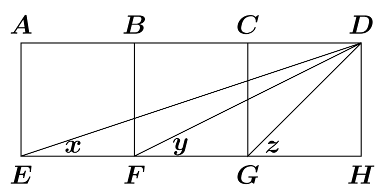
Next a comment from a reader about a problem whose solutions appeared in Skoliad 126 at [2010 : 262–263].

4. The diagram shows three squares and angles x, y, and z. Find the sum of the angles x, y, and z.

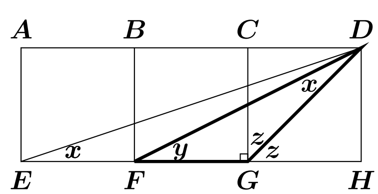


Comment by Solomon W. Golomb, University of Southern California, Los Angeles, CA, USA.

Problem **4.** on pages 262–263 of the September 2010 issue of ***CRUX with MAYHEM*** asks for the sum of the three angles x, y and z, in the figure below, consisting of three adjacent unit squares, and three diagonals. Two proofs that $x + y + z = 90°$, one trigonometric and one geometric, are given.



Years ago, an equivalent problem (same diagram, and to show that $z = x + y$) appeared in Martin Gardner's "Mathematical Games" column in *Scientific American*, and drew a great deal of reader response. However, the most elegant solution, by the late Leon Bankoff, has never been previously published. It uses no trigonometry, and needs no additional lines to be drawn. Bankoff's keen observation was that the triangles DEG and DGF are similar! (They have $\angle DGF$ is common, and the including sides are in the same ratio, namely $|DG| : |GF| = \sqrt{2} : 1 = \sqrt{2}$, and $|EG| : |GD| = 2 : \sqrt{2} = \sqrt{2}$.) Hence angle $x = \angle DEG$ in the larger triangle equals $\angle GDF = x$ in the smaller triangle.



So we have that the three angles of triangle DGF are x, y, and $90° + z$, and must sum to $180°$. Hence $x + y + z = 90°$.

Next follow solutions to the Swedish Junior High School Mathematics Contest, Final Round, 2009/2010, given in Skoliad 125 at [2010 : 259–260].

1. A 2009×2010 grid is filled with the numbers 1 and -1. For each row, calculate the product of the entries in that row. Do likewise for the columns. Show that the sum of all the row products and all the column products cannot be zero.

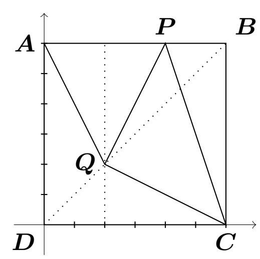
Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Each of the 2009 row products and each of the 2010 column products is either 1 or -1. For the sum of a collection of 1's and -1's to be zero, you must have equally many of each, but this is impossible since you have 4019 numbers. Thus the sum of all the row products and all the column products cannot be zero.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; and JULIA PENG, student, Campbell Collegiate, Regina, SK.

2. The square $ABCD$ has side length 6. The point P splits side AB such that $|AP| : |PB| = 2 : 1$. A point Q inside the square is chosen such that $|AQ| = |PQ| = |CQ|$. Find the area of $\triangle CPQ$.

Solution by Monica Hsieh, student, Burnaby North Secondary School, Burnaby, BC.



Impose a coordinate system so that $A = (0,6)$, $B = (6,6)$, $C = (6,0)$, and $D = (0,0)$. Since $|AP| : |PB| = 2 : 1$, $P = (4,6)$. Since $|AQ| = |PQ|$, Q is on the line given by $x = 2$. Since $|AQ| = |CQ|$, Q is on the line through B and D, $y = x$. Thus $Q = (2,2)$.

Now, the area of $\triangle CDQ$ is $\frac{6\cdot 2}{2} = 6$, the area of $\triangle BCP$ is $\frac{2\cdot 6}{2} = 6$, the area of $\triangle APQ$ is $\frac{4\cdot 4}{2} = 8$, the area of $\triangle ADQ$ is $\frac{6\cdot 2}{2} = 6$, and the area of $ABCD$ is $6^2 = 36$. Therefore the area of $\triangle CPQ$ is $36 - 6 - 6 - 8 - 6 = 10$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

3. The product of three positive integers is 140. Determine the sum of the three integers if the second integer is seven times the first one.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Say the three integers are a, $7a$, and b. Then $7a^2b = 140$. If $a = 1$, then $b = 20$, and the sum of the three positive integers is $a+7a+b = 1+7+20 = 28$. If $a = 2$, then $b = 5$, and the sum is $2 + 14 + 5 = 21$. If $a = 3$, then $b = \frac{20}{9}$, which is not an integer. If $a = 4$, then $b = \frac{5}{4}$, which is not an integer. If $a \geq 5$, then $b \leq \frac{4}{5}$, so b cannot be a positive integer. Hence, the sum is either 21 or 28.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC.

Rather than going through so many cases, it may be easier to consider that if $7a^2b = 140$*, then* $a^2b = 20 = 2^2 \cdot 5$*. The only squares that divide* $2^2 \cdot 5$ *are* 1^2 *and* 2^2*, so* $a = 1$ *or* $a = 2$*.*

4. Five points are placed at the intersections of a rectangular grid. Then the midpoint of each pair of points is marked. Prove that at least one of these midpoints lands on an intersection point of the grid.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC; Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and Monica Hsieh, student, Burnaby North Secondary School, Burnaby, BC.

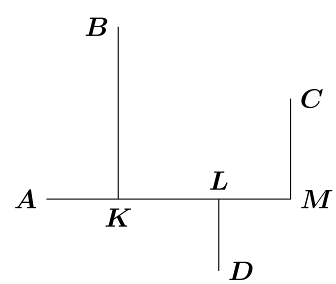
Impose a coordinate system such that $(0, 0)$ is a grid point and the size of the grid is 1. Then the five points all have integer coordinates and the midpoints are grid points if and only if they, too, have integer coordinates.

The midpoint between (x_1, y_1) and (x_2, y_2) is $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$. Therefore a midpoint is a grid point if and only if $x_1 + x_2$ and $y_1 + y_2$ are both even. Note that "even plus even is even" and "odd plus odd is even" while the sum of two numbers of opposite parity is odd.

In terms of parity of their coordinates, the grid points fall into four classes: $(\text{even}, \text{even})$, $(\text{even}, \text{odd})$, $(\text{odd}, \text{even})$, and $(\text{even}, \text{odd})$. Since you have five grid points, two must be in the same class. Say (x_1, y_1) and (x_2, y_2) are in the same class. Then x_1 and x_2 have the same parity, so $x_1 + x_2$ is even. Likewise, $y_1 + y_2$ is even, so the midpoint between (x_1, y_1) and (x_2, y_2) is a grid point.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

5. Points K and L on segment AM are placed such that $|AK| = |LM|$. Place the points B and C on one side of AM and point D on the other side of AM such that $|BK| = |KM|$, $|CM| = |KL|$, and $|DL| = |LM|$, and such that BK, CM, and DL are all perpendicular to AM. Prove that $ABCD$ is a square.



Solution by Julia Peng, student, Campbell Collegiate, Regina, SK.

Note that $|AL| = |AK| + |KL| = |LM| + |KL| = |KM| = |BK|$ and that $|AK| = |LM| = |DL|$. Since $\triangle ABK$ and $\triangle DAL$ are both right-angled, $|AB| = |AD|$ and $\triangle ABK \cong \triangle DAL$.

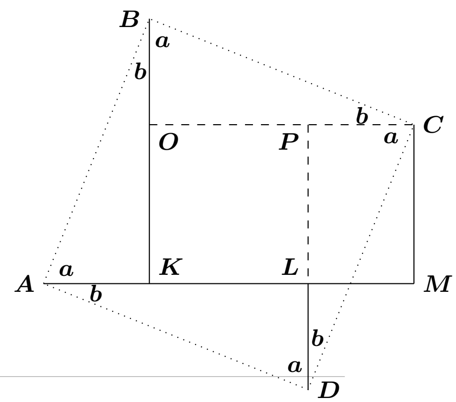
Let O be the point on BK such that $KMCO$ is a rectangle. Let P be the intersection of CO and the extension of DL. Then $OPLK$ is a rectangle. Moreover, $|KL| = |CM| = |PL|$, so $OPLK$ is a square.

Now,

$$|DP| = |DL| + |LP|$$
$$= |AK| + |KL| = |AL|$$

and

$$|PC| = |OC| - |OP|$$
$$= |KM| - |KL|$$
$$= |LM| = |DL|\,.$$



Again, both $\triangle CDP$ and $\triangle DAL$ are right-angled, so $\triangle CDP \cong \triangle DAL$.

Finally, $|OC| = |KM| = |AL|$ and $|BO| = |BK| - |OK| = |AL| - |KL| = |AK|$, so $\triangle BCO \cong \triangle ABK$ since both triangles are right-angled.

Since $\triangle ABK \cong \triangle BCO \cong CDP \cong \triangle DAL$, the sides of $ABCD$ are all equal and the angles are equal as labelled. Since the angle sum in $\triangle BAK$ is $180°$, $a + b = 90°$, so $\angle DAB = 90°$, and $ABCD$ is a square.

Several solvers showed that $ABCD$ is a rhombus.

6. Let N be a positive integer. Ragnhild writes down all the divisors of N other than 1 and N. She then notes that the largest divisor is 45 times the smallest one. Which positive integers satisfy this condition?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

The smallest possible proper divisor of N would be 2. If 2 is a divisor, then $2 \cdot 45 = 90$ would be the largest proper divisor. Since N must be the product of its largest and smallest proper divisors, $N = 2 \cdot 90 = 180$.

If 3 is the smallest proper divisor, then $3 \cdot 45 = 135$ is the largest proper divisor, and $N = 3 \cdot 135 = 405$.

If $m > 3$ and m is the smallest proper divisor, then $45m$ is the largest proper divisor, and $N = 45m^2$. But $45m^2$ is divisible by 3 since 45 is divisible by 3. Thus m is not the smallest proper divisor after all. This contradiction shows that the smallest divisor is at most 3. Hence $N = 180$ or $N = 405$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Monica Hsieh, student, Burnaby North Secondary School, Burnaby, BC.

We hope that our readers will enjoy the featured contest and share their solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The interim Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School, Orleans, ON). The Assistant Mayhem Editor is Lynn Miller (Cairine Wilson Secondary School, Orleans, ON). The other staff members are Ann Arden (Osgoode Township District High School, Osgoode, ON) and Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON).

Mayhem Problems

Please send your solutions to the problems in this edition by ***15 November 2011****. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.

M482. *Proposed by the Mayhem Staff.*

Using four sticks with lengths of 1 cm, 2 cm, 3 cm, and 5 cm, respectively, you can measure any integral length from 1 cm to 10 cm. Note that a stick may only be used once in a particular measurement, so the 1 cm, 2 cm, and 3 cm sticks could be used to measure 6 cm, but not the 3 cm stick twice.

(a) Find a set of ten stick lengths that can be used to represent any integral length from 1 cm to 100 cm.

(b) What is the fewest number of sticks that are needed to represent any integral length from 1 cm to 100 cm?

M483. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC has $\angle BAC = 90^\circ$. The feet of the perpendiculars from A to the internal bisectors of $\angle ABC$ and $\angle ACB$ are P and Q, respectively. Determine the measure of $\angle PAQ$.

M484. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Solve the equation

$$x^2 + 4\left(\frac{x}{x-2}\right)^2 = 45\,.$$

M485. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Prove that

$$\prod_{k=1}^{n} \binom{n}{k} = \frac{1}{n!}\prod_{k=1}^{n} \frac{k^k}{(n-k)!}$$

for all $n \in \mathbb{N}$.

M486. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

How many distinct numbers are in the list

$$\frac{1^2-1+4}{1^2+1},\ \frac{2^2-2+4}{2^2+1},\ \frac{3^2-3+4}{3^2+1},\ \ldots,\ \frac{2011^2-2011+4}{2011^2+1}\ ?$$

M487. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let m be a positive integer. Find all real solutions to the equation

$$m + \sqrt{m + \sqrt{m + \cdots \sqrt{m + \sqrt{m + \sqrt{x}}}}} = x,$$

in which the integer m occurs n times.

. . . .

M482. *Proposé par l'Équipe de Mayhem.*

À l'aide de baguettes de longueurs **1** cm, **2** cm, **3** cm et **5** cm, on peut mesurer toute longueur entière de **1** cm à **10** cm. Noter qu'une baguette oeut être utilisée une seule fois lors d'une mesure ; par exemple, les baguettes de **1** cm, **2** cm et **3** cm peuvent être utilisées pour mesurer **6** cm, tandis qu'on ne peut pas utiliser deux baguettes de **3** cm pour mesurer **6** cm.

(a) Déterminer un ensemble de dix baguettes de longueurs entières pouvant mesurer toute longueur entière de **1** cm à **100** cm.

(b) Quel est le plus petit nombre de baguettes permettant de mesurer toute longueur entière de **1** cm à **100** cm ?

M483. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Le triangle ABC est tel que $\angle BAC = 90°$. Les pieds des perpendiculaires de A jusqu'aux bissectrices internes des angles $\angle ABC$ et $\angle ACB$ sont P et Q respectivement. Déterminer la mesure de $\angle PAQ$.

M484. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Résoudre l'équation

$$x^2 + 4\left(\frac{x}{x-2}\right)^2 = 45\,.$$

M485. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Démontrer que

$$\prod_{k=1}^{n}\binom{n}{k} = \frac{1}{n!}\prod_{k=1}^{n}\frac{k^k}{(n-k)!}$$

pour tout $n \in \mathbb{N}$.

M486. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Combien de nombres distincts y a-t-il dans la liste

$$\frac{1^2-1+4}{1^2+1},\ \frac{2^2-2+4}{2^2+1},\ \frac{3^2-3+4}{3^2+1},\ \ldots,\ \frac{2011^2-2011+4}{2011^2+1}\ ?$$

M487. *Proposé par Samuel Gómez Moreno, Université de Jaén, Jaén, Espagne.*

Soit m un entière positif. Déterminer toutes les solutions réelles à l'équation

$$m + \sqrt{m + \sqrt{m + \cdots \sqrt{m + \sqrt{m + \sqrt{x}}}}} = x,$$

dans laquelle l'entière m a lieu n fois.

Mayhem Solutions

M440. *Proposed by the Mayhem Staff.*

In trapezoid $ABCD$, AB is parallel to DC and AD is perpendicular to AB. If $AB = 20$, $BC = 5x$, $CD = x^2 + 3x$, and $DA = 3x$, determine the value of x.

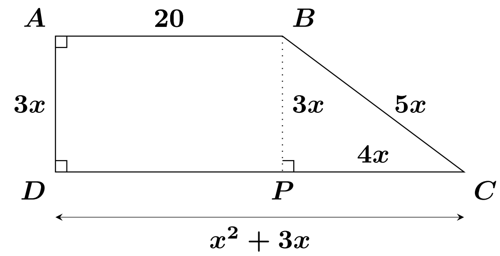
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

There are two cases to consider.

Case I: $x^2 + 3x > 20$.

Let P be the foot of the perpendicular from B to DC. Then $BP = 3x$ and, according to Pythagoras, $PC = 4x$. Therefore,

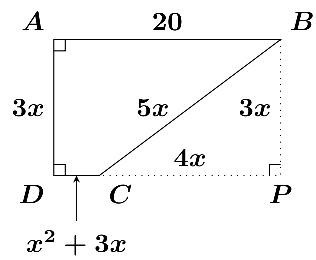
$$\begin{aligned} x^2 + 3x &= 20 + 4x, \\ x^2 - x - 20 &= 0, \\ (x-5)(x+4) &= 0, \\ x &= 5 \quad (\text{since } x > 0). \end{aligned}$$



Case II: $x^2 + 3x < 20$.

Proceeding as in Case I, we obtain

$$\begin{aligned} (x^2 + 3x) + 4x &= 20, \\ x^2 + 7x - 20 &= 0, \\ x &= \frac{-7 + \sqrt{129}}{2}. \end{aligned}$$



Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania. Eight incorrect solutions were received. Most of the incorrect solutions neglected one of the cases.

M442. *Proposed by Carl Libis, Cumberland University, Lebanon, TN, USA.*

Consider the square array

$$\begin{bmatrix} 1 & 2 & \cdots & n-1 & n \\ n+1 & n+2 & \cdots & 2n-1 & 2n \\ \vdots & \vdots & & \vdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2-1 & n^2 \end{bmatrix}$$

formed by listing the numbers 1 to n^2 in order in consecutive rows. Determine the sum of the numbers on each diagonal. How does this sum compare to the "magic constant" that would be obtained if the n^2 entries were rearranged to form a magic square?

Solución de Ricard Peiró, IES "Abastos", Valencia, Spain.

Los elementos de la diagonal principal son: $1, n+2, 2n+3, \ldots, (n-1)n+n$. La suma es:

$$\begin{aligned} D_1(n) &= 1 + (n+2) + (2n+3) + \cdots + [(n-1)n+n] \\ &= [1+2+3+\cdots+n] + n[1+2+3+\cdots+(n-1)] \\ &= \frac{n(n+1)}{2} + n\left(\frac{(n-1)n}{2}\right) \\ &= \frac{n^3+n}{2}. \end{aligned}$$

Los elementos de la diagonal secundaria son: $n, 2n-1, 3n-2, \ldots, n\cdot n-(n-1)$. La suma es:

$$\begin{aligned} D_2(n) &= n + (2n-1) + (3n-2) + \cdots + [n\cdot n-(n-1)] \\ &= n[1+2+3+\cdots+n] - [1+2+3+\cdots+(n-1)] \\ &= n\left(\frac{n(n+1)}{2}\right) - \frac{(n-1)n}{2} \\ &= \frac{n^3+n}{2}. \end{aligned}$$

La constante mágica de un cuadrado mágico $n \times n$ es:

$$\begin{aligned} M(n) &= \frac{1}{n}(1+2+3+\cdots+n^2) \\ &= \frac{1}{n}\left(\frac{n^2(n^2+1)}{2}\right) \\ &= \frac{n^3+n}{2} = D_1(n) = D_2(n). \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JACLYN CHANG, student, University of Calgary, Calgary, AB; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and ALLEN ZHU, Conestoga High School, Berwyn, PA, USA.

M445. *Proposed by the Mayhem Staff.*

The lines with equations $y = x + 1$, $y = mx - 1$, and $y = -4x + 2m$ pass through the same point. Determine all possible values for m.

Solution by Afiffah Nuur Mila Husniana, student, SMPN 8, Yogyakarta, Indonesia.

Given are three linear equations

$$y = x + 1, \tag{1}$$
$$y = mx - 1, \tag{2}$$
$$y = -4x + 2m. \tag{3}$$

From (1) and (2) we have

$$\begin{aligned} x+1 &= mx-1 \\ x(1-m) &= -2 \\ x &= \frac{-2}{1-m} \end{aligned} \tag{4}$$

From (2) and (3) we have

$$\begin{aligned} mx-1 &= -4x+2m \\ x(m+4) &= 2m+1 \\ x &= \frac{2m+1}{m+4} \end{aligned} \tag{5}$$

From (4) and (5) we have

$$\begin{aligned} \frac{-2}{1-m} &= \frac{2m+1}{m+4} \\ -2(m+4) &= (1-m)(2m+1) \\ -2m-8 &= 2m-2m^2+1-m \\ 2m^2-3m-9 &= 0 \\ (2m+3)(m-3) &= 0 \end{aligned}$$

So the possible values for m are $m=-\frac{3}{2}$ or $m=3$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; DAVINIA CERVERA GARCÍA, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; DEBRA A. OHL, student, Angelo State University, San Angelo, TX, USA; KONSTANTINOS AL. NAKOS, Agrinio, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Two incorrect solutions were submitted.

M446. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Let a, b, and c be positive digits. Suppose that b equals the product of a, b, and c, and $\underline{ac}=a+b+c$. Determine a, b, and c. (Here $\underline{ab}$ is the two-digit positive integer with tens digit a and units digit b.)

Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

We are given

$$\begin{aligned} b &= a\cdot b\cdot c \\ 10a+c &= a+b+c \end{aligned}$$

Since a, b, c are positive digits, then $1 \le a, b, c \le 9$. Since $b \neq 0$ then $b = a \cdot b \cdot c$ gives $a \cdot c = 1$; which implies that $a = 1 = c$. From $10a + c = a + b + c$, then we have $10 \cdot 1 + 1 = 1 + b + 1$; hence $b = 11 - 2 = 9$. Therefore $a = 1, b = 9, c = 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; YOUNGHUAN JUNG, The Woodlands School, Mississauga, ON; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DONGCHAN LEE, University of Toronto, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MUHAMMAD ROIHAN MUNAJIH, student, SMPN 8, Yogyakarta, Indonesia; ALEECE NALBANDIAN, California State University, Fresno, CA, USA; DEBRA A. OHL, student, Angelo State University, San Angelo, TX, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ANDRÉS PLANELLS CÁRCEL, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and INGESTI BILKIS ZULFATINAAS, student, SMPN 8, Yogyakarta, Indonesia. One incorrect solution was submitted.

M447. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let $ABCD$ be a parallelogram. The sides AB and AD are extended to points E and F (respectively) so that E, C, and F all lie on a straight line. Prove that $BE \cdot DF = AB \cdot AD$.

Solution by George Apostolopoulos, Messolonghi, Greece.

The triangles BCE and FDC are similar, so $\frac{BE}{DC} = \frac{BC}{DF}$. Since $ABCD$ is a parallelogram we know $BC = AD$ and $DC = AB$. So $\frac{BE}{AB} = \frac{AD}{DF}$, hence $BE \cdot DF = AB \cdot AD$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DONGCHAN LEE, University of Toronto, Toronto, ON; DEBRA A. OHL, student, Angelo State University, San Angelo, TX, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JORGE SEVILLA LACRUZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; LOU VANG, California State University, Fresno, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and INGESTI BILKIS ZULFATINAAS, student, SMPN 8, Yogyakarta, Indonesia. One incorrect solution was submitted.

M448. *Proposed by the Mayhem Staff.*

A polyhedron with exactly $m + n$ faces has m faces that are quadrilaterals and n faces that are triangles. Exactly four faces meet at each vertex. Prove that $n = 8$.

Solution by Dongchan Lee, University of Toronto, Toronto, ON.

The number of faces is $F = m + n$. Since there are 4 vertices and 4 edges in a quadrilateral, and 3 vertices and 3 edges in a triangle, then the total number of edges would be $E = \frac{4m+3n}{2}$. The total number of vertices will be $V = \frac{4m+3n}{4}$ since it is given that exactly four faces meet at each vertex. Using Euler's polyhedron formula, which says that the sum of the number of faces and the number of vertices is equal to the number of edges plus two,

$$F + V = E + 2,$$
$$m + n + \frac{4m + 3n}{4} = \frac{4m + 3n}{2} + 2.$$

Solving the equation, we get $n = 8$.

Also solved by ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JORGE ARMERO JIMÉNEZ, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.

M449. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $E(x) = \dfrac{4^x}{4^x + 2}$.

(a) Prove that $E(x) + E(1 - x) = 1$.

(b) Find the value of $E\left(\frac{1}{2010}\right) + E\left(\frac{2}{2010}\right) + \cdots + E\left(\frac{2008}{2010}\right) + E\left(\frac{2009}{2010}\right)$.

Solution by Afiffah Nuur Mila Husniana, student, SMPN 8, Yogyakarta, Indonesia.

(a) From the given equation $E(x) = \frac{4^x}{4^x+2}$ so,

$$\begin{aligned} E(x) + E(1 - x) &= \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2} \\ &= \frac{4^x(4^{1-x} + 2) + 4^{1-x}(4^x + 2)}{(4^x + 2)(4^{1-x} + 2)} \\ &= \frac{4 + 2(4^x) + 4 + 2(4^{1-x})}{4 + 2(4^x) + 2(4^{1-x}) + 4} \\ &= \frac{8 + 2(4^x) + 2(4^{1-x})}{8 + 2(4^x) + 2(4^{1-x})} \\ &= 1 \end{aligned}$$

[*Ed. – Note that* $E(1-x) = \dfrac{4^{1-x}}{4^{1-x} + 2} \times \dfrac{\frac{4^x}{2}}{\frac{4^x}{2}} = \dfrac{2}{2 + 4^x}$ *and the conclusion follows immediately.*]

(b) From (a) we know that $E(x) + E(1-x) = 1$, thus

$$E\left(\frac{1}{2010}\right) + E\left(\frac{2009}{2010}\right) = E\left(\frac{1}{2010}\right) + E\left(1 - \frac{1}{2010}\right) = 1$$
$$E\left(\frac{2}{2010}\right) + E\left(\frac{2008}{2010}\right) = E\left(\frac{2}{2010}\right) + E\left(1 - \frac{2}{2010}\right) = 1$$

and so on. Then we have

$$E\left(\frac{1}{2010}\right)+E\left(\frac{2}{2010}\right)+\ldots+E\left(\frac{2008}{2010}\right)+E\left(\frac{2009}{2010}\right) = 1004\times 1+E\left(\frac{1005}{2010}\right)$$

Since $E\left(\frac{1005}{2010}\right) = \frac{4^{\frac{1}{2}}}{4^{\frac{1}{2}}+2}$, then $E\left(\frac{1005}{2010}\right) = \frac{1}{2}$.

Therefore the sum is $\mathbf{1004.5}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; CHAO-PING CHEN, Henan Polytechnic University, Jiaozuo City, China and Mihály Bencze, Lajos Aprily High-school, Brasov, Romania; DIANA DOMINGUEZ, California State University, Fresno, CA, USA; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; PABLO PARDAL GARCES, Club Mathématique de l'Instituto de Ecuación Secundaria No. 1, Requena-Valencia, Spain; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DONGCHAN LEE, University of Toronto, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; KONSTANTINOS AL. NAKOS, Agrinio, Greece; CARLOS TORRES NINAHUANCA, Lima, Perú; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and INGESTI BILKIS ZULFATINAAS, student, SMPN 8, Yogyakarta, Indonesia;

M450. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Prove that if n is an odd positive integer, then $n^{n+2} + (n+2)^n$ is divisible by $2(n+1)$.

Solution by Osman Ekiz, Eskisehir, Turkey.

From the Binomial Theorem, $(n+2)^n$ can be written as:

$$\begin{aligned}(n+2)^n &= \big((n+1)+1\big)^n \\ &= \binom{n}{0}(n+1)^n + \binom{n}{1}(n+1)^{n-1} + \cdots + \binom{n}{n-1}(n+1) + 1.\end{aligned}$$

Since n is an odd number we can also write:

$$\begin{aligned}n^{n+2} &= (n+1-1)^{n+2} \\ &= \binom{n+2}{0}(n+1)^{n+2} - \binom{n+2}{1}(n+1)^{n+1} + \cdots \\ &\qquad + \binom{n+2}{n+1}(n+1) - 1.\end{aligned}$$

If we add the two expansions together, the constant terms cancel each other. Therefore, we have:

$$\begin{aligned}(n+2)^n + n^{n+2} = \Bigg[&\binom{n}{0}(n+1)^n + \binom{n}{1}(n+1)^{n-1} + \cdots \\ &+ \binom{n}{n-1}(n+1) + \binom{n+2}{0}(n+1)^{n+2} \\ &- \binom{n+2}{1}(n+1)^{n+1} + \cdots + \binom{n+2}{n+1}(n+1)\Bigg]\end{aligned}$$

Since all of the terms have a factor of $n+1$, then $(n+2)^n + n^{n+2}$ is divisible by $n+1$ and we can rewrite the expression as:

$$\begin{aligned}(n+2)^n + n^{n+2} = (n+1)\Bigg[&\binom{n}{0}(n+1)^{n-1} + \binom{n}{1}(n+1)^{n-2} + \ldots \\ &+ \binom{n}{n-1} + \binom{n+2}{0}(n+1)^{n+1} \\ &- \binom{n+2}{1}(n+1)^n + \cdots + \binom{n+2}{n+1}\Bigg]\end{aligned}$$

Now we must prove that the expression in the square brackets above is an even number. Since we know that $n+1$ is an even number, all of the terms with a factor of $n+1$ are also even. Then we are left with only two terms, $\binom{n}{n-1}$ and $\binom{n+2}{n+1}$. Since $\binom{n}{n-1} = n$ and $\binom{n+2}{n+1} = n+2$, then we have $\binom{n}{n-1} + \binom{n+2}{n+1} = 2n+2$ which is an even number.

Hence $(n+2)^n + n^{n+2}$ is divisible by $2(n+1)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ADAM GREGSON, teacher, University of Toronto Schools, Toronto, ON; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; DONGCHAN LEE, University of Toronto, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

Note that since n *is odd* $\dfrac{n^{n+2}+1}{n+1} = n^{n+1} - n^n + n^{n-1} - \cdots + 1$ *is odd. Also* $\dfrac{(n+2)^n - 1}{n+1} = (n+2)^{n-1} + (n+2)^n + \cdots + 1$ *is also odd. Thus their sum is even.*

Problem of the Month

Ian VanderBurgh

Problems involving probability can be very interesting and can lead to lots of discussion and debate. (If you don't believe me about debate, try looking up the Monty Hall Problem.) These problems also give lots of opportunity for both creative solutions and plausible incorrect solutions.

Here are two problems involving probability that have very different flavours.

Problem 1 (2011 Euclid Contest) Three different numbers are chosen at random from the set $\{1, 2, 3, 4, 5\}$. The numbers are arranged in increasing order. What is the probability that the resulting sequence is an arithmetic sequence?

This type of problem is pretty standard. We are given a set of objects and a property. We need to determine the probability that a randomly chosen object from the set has the desired property.

Often, the most direct approach is to count the total number of objects in the set and to count the number of objects in the set that have the desired property. The probability that we are after is the second number divided by the first number. Here is an illustration of this technique.

Solution to Problem 1. We consider choosing the three numbers all at once. We list the possible sets of three numbers that can be chosen:

$$\{1,2,3\} \quad \{1,2,4\} \quad \{1,2,5\} \quad \{1,3,4\} \quad \{1,3,5\}$$

$$\{1,4,5\} \quad \{2,3,4\} \quad \{2,3,5\} \quad \{2,4,5\} \quad \{3,4,5\}$$

We have listed each in increasing order because once the numbers are chosen, we arrange them in increasing order.

There are 10 sets of three numbers that can be chosen. Of these 10, the 4 sequences $1, 2, 3$ and $1, 3, 5$ and $2, 3, 4$ and $3, 4, 5$ are arithmetic sequences. Therefore, the probability that the resulting sequence is an arithmetic sequence is $\frac{4}{10}$ or $\frac{2}{5}$. □

Sometimes, a good problem solving strategy helps us think about a problem in a more straightforward way. The problem itself isn't any easier, but it might be easier to attack. Separating the problem into counting these two different sets of objects makes this easier to approach.

When this contest was marked, the most popular incorrect answer to this problem was $\frac{4}{60}$ (or its reduced form of $\frac{1}{15}$). Can you figure out what mistake might lead to this answer?

The next problem is also about probability, but seems very different at first.

Problem 2 (2011 Euclid Contest) A 75 year old person has a 50% chance of living at least another 10 years. A 75 year old person has a 20% chance of living at least another 15 years. An 80 year old person has a 25% chance of living at least another 10 years. What is the probability that an 80 year old person will live at least another 5 years?

This is a really interesting problem that has a good "real life context". The data given is close to the actual data for Canadian adults. One approach is to use the given probabilities directly.

Solution 1 to Problem 2. Suppose that the probability that a 75 year old person lives to 80 is $\boldsymbol{p}$, the probability that an 80 year old person lives to 85 is $\boldsymbol{q}$, and the probability that an 85 year old person lives to 90 is $\boldsymbol{r}$. We want to the determine the value of $\boldsymbol{q}$.

For a 75 year old person to live at least another 10 years, they must live another 5 years (to age 80) and then another 5 years (to age 85). The probability of this is equal to $\boldsymbol{pq}$. We are told in the question that this is equal to 50% or 0.5. Therefore, $\boldsymbol{pq = 0.5}$.

For a 75 year old person to live at least another 15 years, they must live another 5 years (to age 80), then another 5 years (to age 85), and then another 5 years (to age 90). The probability of this is equal to $\boldsymbol{pqr}$. We are told in the question that this is equal to 20% or 0.2. Therefore, $\boldsymbol{pqr = 0.2}$.

Similarly, since the probability that an 80 year old person will live another 10 years is 25%, then $\boldsymbol{qr = 0.25}$.

Since $\boldsymbol{pqr = 0.2}$ and $\boldsymbol{pq = 0.5}$, then $\boldsymbol{r = \dfrac{pqr}{pq} = \dfrac{0.2}{0.5} = 0.4}$.

Since $\boldsymbol{qr = 0.25}$ and $\boldsymbol{r = 0.4}$, then $\boldsymbol{q = \dfrac{qr}{r} = \dfrac{0.25}{0.4} = 0.625}$.

Therefore, the probability that an 80 year old person will live at least another 5 years is 0.625, or 62.5%. □

A second approach is actually to use the "count the objects" method that we discussed earlier. You might wonder what the set of objects is. Here is one way to do this.

Solution 2 to Problem 2. Consider a population of 100 people, each of whom is 75 years old and who behave according to the probabilities given in the question.

Each of the original 100 people has a 50% chance of living at least another 10 years, so there will be $\mathbf{50\% \times 100 = 50}$ of these people alive at age 85. Each of the original 100 people has a 20% chance of living at least another 15 years, so there will be $\mathbf{20\% \times 100 = 20}$ of these people alive at age 90.

Since there is a 25% (or $\mathbf{\frac{1}{4}}$) chance that an 80 year old person will live at least another 10 years (that is, to age 90), then there should be 4 times as many of these people alive at age 80 than at age 90. Since there are 20 people alive at age 90, then there are $\mathbf{4 \times 20 = 80}$ of the original 100 people alive at age 80.

In summary, of the initial 100 people of age 75, there are 80 alive at age 80, 50 alive at age 85, and 20 people alive at age 90. Because 50 of the 80 people alive at age 80 are still alive at age 85, then the probability that an 80 year old person will live at least 5 more years (that is, to age 85) is $\mathbf{\frac{50}{80} = \frac{5}{8}}$, or 62.5%. □

That works pretty well doesn't it? It is always fascinating to me when mathematics becomes so connected to real life. Problem 2 is related to an area of mathematics called actuarial science, which has lots of applications to things like insurance and pensions. If you are interested in the idea of applying mathematics to the financial industry, check out this field!

THE OLYMPIAD CORNER

No. 293

R.E. Woodrow and Nicolae Strugaru

In this issue we begin a transition in the Corner. Problems editor Nicolae Strungaru, from Grant MacEwan University in Edmonton, has agreed to take over from Robert Woodrow who has been the editor of the Corner since 1987. Robert's dedication to ***CRUX with MAYHEM*** over the years is greatly appreciated and he will be sorely missed. Material from Robert will continue to appear in ***CRUX with MAYHEM*** as we wrap up the solutions to the last sets of problems he published.

The format of the Corner is changing slightly. It will still consist of problems from Olympiads from around the world, but, rather than printing the contests in their entirety, each column will consist of 10 questions, in both English and French, selected from different contests. The origin of the question will be revealed when the solutions are published.

We will have the same time lines as we do with the ***CRUX*** problems. Solutions will be due six months from the issue date and will appear in the same issue number of the next volume, one year later. The first set of new Olympiad Corner problems is below, please send your solutions to Nicolae by email (preferred) at:

`crux-olympiad@cms.math.ca`

or by mail to

Nicolae Strungaru
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Grant MacEwan University
Edmonton, AB
Canada
T5J 4S2

Enjoy the new Corner!

The solutions to the problems are due to the editor by **1 January 2012**.

OC1. Find all positive integers w, x, y and z which satisfy $w! = x! + y! + z!$.

OC2. Suppose that f is a real-valued function for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real numbers x and y.

(a) Give a nonconstant polynomial that satisfies the condition.

(b) Prove that $f(x) \geq 0$ for all real x.

OC3. Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$
$$\angle ABD = 2\angle CDB$$
$$\text{and} \qquad AB = CB.$$

Prove that $AD = CD$.

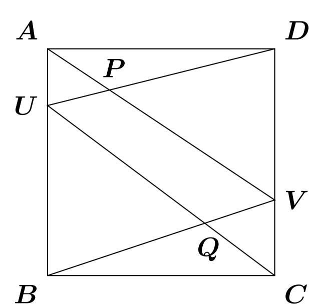
OC4. Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.

OC5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0,$$
$$a_1 + a_2 \leq 100$$
$$\text{and} \qquad a_3 + a_4 + \cdots + a_{100} \leq 100.$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

OC6. In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$.



OC7. Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

OC8. For each real number r let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r2^r x + 2^r y)$. Let F be the family of all such transformations *i.e.* $F = \{T_r : r \in \mathbb{R}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.

OC9. A deck of $2n+1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n+1$ cards are placed in a row, with the joker in the middle. For each k with $1 \leq k \leq n$, the two cards numbered k have exactly $k-1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?

OC10. The number 1987 can be written as a three digit number xyz in some base b. If $x+y+z = 1+9+8+7$, determine all possible values of x, y, z, b.

. . . .

OC1. Trouver tous les entiers positifs w, x, y et z qui satisfont $w! = x! + y! + z!$.

OC2. Supposer que f est une fonction à valeurs réelles qui satisfait

$$f(xy) + f(y-x) \geq f(y+x)$$

pour tous nombres réels x et y.

(a) Donner un polynôme non constant qui satisfait cette condition.

(b) Montrer que $f(x) \geq 0$ pour tout nombre réel x.

OC3. On considère un quadrilatère convexe $ABCD$ dans lequel

$$\begin{aligned} \angle CBD &= 2\angle ADB, \\ \angle ABD &= 2\angle CDB \\ \text{et} \qquad AB &= CB. \end{aligned}$$

Démontrer que $AD = CD$.

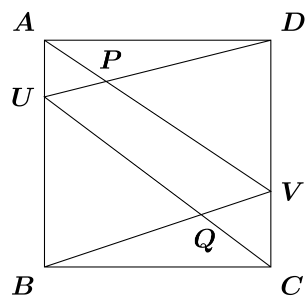
OC4. Considérer les nombres n à 70 chiffres avec la propriété que chacun des chiffres $1, 2, 3, \ldots, 7$ apparaît dix fois dans l'expansion décimale de n (et que 8, 9 et 0 n'y apparaissent pas). Montrer qu'aucun nombre de cette forme ne peut être divisé par un autre nombre de la même forme.

OC5. Supposons que les nombres réels $a_1, a_2, \ldots, a_{100}$ satisfont aux conditions suivantes

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0, \\ a_1 + a_2 \leq 100 \\ \text{et} \qquad a_3 + a_4 + \cdots + a_{100} \leq 100. \end{aligned}$$

Déterminer la valeur maximale possible de $a_1^2 + a_2^2 + \cdots + a_{100}^2$, et trouver toutes les suites possibles $a_1, a_2, \ldots, a_{100}$ pour lesquelles ce maximum est atteint.

OC6. Sur le diagramme ci-dessous, $ABCD$ est un carré sur lequel on choisit des points U et V intérieurs aux côtés AB et CD respectivement. Déterminer toutes les façons possibles de choisir U et V de telle sorte que la surface du quadrilatère $PUQV$ soit maximale.



OC7. Soit n un nombre naturel tel que $n \geq 2$. Montrer que

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

OC8. Soit F la famille des transformations $F = \{T_r : r \in \mathbb{R}\}$ où T_r transforme le point (x, y) en le point $(2^r x, r2^r x + 2^r y)$. Trouver toutes les courbes $y = f(x)$ dont le graphe est invariant pour chacune des transformations de F.

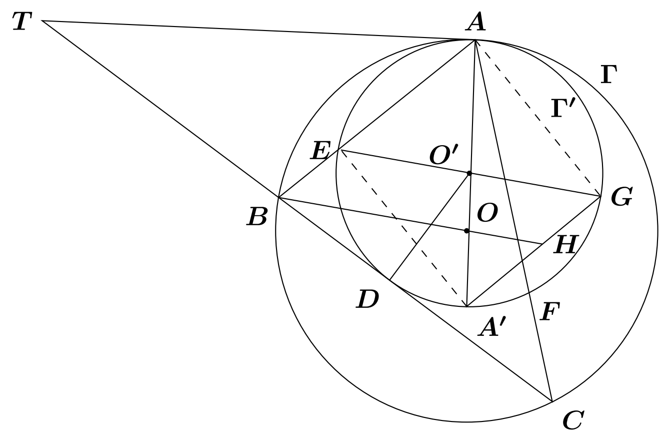
OC9. Un jeu de $2n+1$ cartes contient un joker et, pour chaque nombre entier de 1 à n inclusivement, 2 cartes marquées de ce numéro. Les $2n+1$ cartes sont alors alignées avec le joker au milieu. De plus, pour chaque nombre entier k avec $1 \leq k \leq n$, les deux cartes numérotées k ont exactement $k-1$ autres cartes entre elles. Trouver toutes les valeurs de n ne dépassant pas 10 pour lesquelles cet arrangement soit possible. Maintenant, pour quelles valeurs de n est-ce impossible ?

OC10. Le nombre 1987 s'écrit à trois chiffres, xyz, dans une certaine base b. Si $x + y + z = 1 + 9 + 8 + 7$, déterminer toutes les valeurs possible de x, y, z et b.

First we look at solutions from the files to the 20[th] Korean Mathematical Olympiad, given at [2010: 152–153].

1. Triangle ABC is acute with circumcircle Γ and circumcentre O. The circle Γ' has centre O', is tangent to O at A and to the side BC at D, and intersects the lines AB and AC again at E and F, respectively. The lines OO' and EO' intersect Γ' again at A' and G, respectively. The lines BO and $A'G$ intersect at H. Prove that $DF^2 = AF \cdot GH$.

Solved by Titu Zvonaru, Comăneşti, Romania.



Since $O'E = O'G$ and $O'A = O'A'$, the quadrilateral $AEA'G$ is a parallelogram, hence

$$A'G \| AE \qquad (1)$$

The triangles $O'AE$ and OAB are isosceles. It follows that

$$\angle O'EA = \angle EAO' = \angle BAO = \angle ABO,$$

hence

$$EO' \| BO. \qquad (2)$$

By (1) and (2) we deduce that $BHGE$ is a parallelogram; thus $HG = BE$ and we have to prove that

$$DF^2 = AF \cdot BE \qquad (3)$$

Let $a = BC$, $b = CA$, $c = AB$. If $b = c$, then D is the midpoint of BC, AD is a diameter of Γ', $BE = CF$ and $DF \perp AC$. It is easy to see that, in $\triangle ADC$ with $DF \perp AC$, the equation (3) is true.

We may assume that $b > c$, and we denote by T the intersection of the line BC with the tangent to Γ at A.

Using the power of point T with respect to Γ, we obtain

$$TA^2 = TB \cdot TC \Leftrightarrow TA^2 = TB(TB + a),$$

and applying the Law of Cosines in $\triangle ABT$ (with $\angle ABT = 180^\circ - B$), we have

$$TA^2 = TB^2 + AB^2 - 2TB\cdot AB\cdot\cos\angle ABT$$
$$\Leftrightarrow TB^2 + aTB = TB^2 + c^2 + 2c\cdot TB\cdot\cos B,$$

hence

$$TB \;=\; \frac{c^2}{a - 2c\cos B}.$$

Since $O'A = O'D$, then $TD = TA$ and, using again the Law of Cosines, we deduce that

$$TB + a \;=\; \frac{c^2}{a-2c\cos B} + a \;=\; \frac{c^2+a^2-2ac\cos B}{a-2c\cos B} \;=\; \frac{b^2}{a-2c\cos B},$$
$$BD \;=\; TD - TB \;=\; \frac{bc}{1-2c\cos B} - \frac{c^2}{a-2c\cos B} \;=\; \frac{c(b-c)}{a-2c\cos B}.$$

Denoting $\alpha = \dfrac{b-c}{a-2c\cos B}$, we have $\alpha = \dfrac{a(b-c)}{a^2-2ac\cos B} = \dfrac{a(b-c)}{b^2-c^2} = \dfrac{a}{b+c}$; it results that $BD = c\alpha$, $DC = b\alpha$.

Using the power of points B and C with respect to circle Γ', we get:

$$BE\cdot BA \;=\; BD^2,\quad CF\cdot CA \;=\; CD^2,$$

hence $BE \;=\; c\alpha^2,\ CF \;=\; b\alpha^2,\ AF = b(1-\alpha^2).$

The equality (3) is equivalent to:

$$\begin{aligned}
DF^2 = AF\cdot BE &\Leftrightarrow DC^2 + CF^2 - 2DC\cdot CF\cdot\cos C = AF\cdot BE\\
&\Leftrightarrow b^2\alpha^2 + b^2\alpha^4 - 2b^2\alpha^3\cos C = bc\alpha^2(1-\alpha^2)\\
&\Leftrightarrow b + b\alpha^2 - 2b\alpha\cos C = c(1-\alpha^2)\\
&\Leftrightarrow b - c + (b+c)\cdot\frac{a^2}{(b+c)^2} - 2b\cdot\frac{a}{b+c}\cos c = 0\\
&\Leftrightarrow b^2 - c^2 + a^2 - 2ab\cos c = 0.
\end{aligned}$$

which is true (by the Law of Cosines in $\triangle ABC$).

3. Find all triplets (x, y, z) of positive integers satisfying $1 + 4^x + 4^y = z^2$.

Solved by Michel Bataille, Rouen, France; Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of De.

Observe that z is odd and is at least 3. Let $z = 2m+1$ where m is a positive integer. Then the equation reduces to

$$4^{x-1} + 4^{y-1} \;=\; m(m+1) \qquad (1)$$

Assume that $x \geq y$ and rewrite (1) as

$$4^{y-1}(4^{x-y}+1) \;=\; m(m+1) \qquad (2)$$

Observe that $\gcd(m, m+1) = 1$. Therefore either

(a) $m = 4^{y-1}$, $m + 1 = 4^{x-y} + 1$; or

(b) $m + 1 = 4^{y-1}$, $m = 4^{x-y} + 1$.

If (a) holds then $x = 2y - 1$ and $z = 2^{2y-1} + 1$. If (b) holds then we obtain

$$2^{2y-3} - 2^{2x-2y-1} \;=\; 1 \tag{3}$$

The solution of (3) is $(x, y) = (\frac{5}{2}, 2)$ which is inadmissible because x is not an integer.

Hence the solution set in positive integers of this equation is

$$\{(2k-1, k, 2^{2k-1}+1) : k \in Z^+\} \cup \{(k, 2k-1, 2^{2k-1}+1) : k \in Z^+\}.$$

where Z^+ is the set of positive integers.

4. Find all pairs (p, q) of primes such that $p^p + q^q + 1$ is divisible by pq.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

Suppose that (p, q) is such a pair of primes. Then,

$$\left\{ \begin{array}{c} p^p + q^q + 1 \;=\; kpq, \\ \text{for some positive integer } k \\ p \text{ and } q \text{ are primes.} \end{array} \right\} \tag{1}$$

Since equation (1) is symmetric with respect to p and q; and since $p \neq q$ (by inspection, $p = q$ would imply $p \mid 1$). There is no loss of generality in assuming

$$p \;<\; q. \tag{2}$$

We distinguish between two cases: Case 1, in which p and q are both odd primes; and Case 2 wherein, $p = 2$ and q is an odd prime.

Case 1. p and q are both odd primes.

Thus, by (2) we must have

$$3 \;\leq\; p \;<\; p+2 \;\leq\; q, \quad 5 \leq\; q. \tag{3}$$

We will prove that no primes satisfying (1) and (3) exist.

We make use of the concept of the order of a positive integer a modulo an odd prime r. If a and r are relatively prime, then the order of a modulo r is the smallest positive integer n such that $a^n \equiv 1 \pmod{r}$. When $a \equiv 1 \pmod{r}$, the order of a is equal to 1. Otherwise, it is some positive integer. The order of a exists, since by Fermat's Little Theorem, we know that $a^{r-1} \equiv 1 \pmod{r}$. (Thus the set of all natural numbers n such that $a^n \equiv 1 \pmod{r}$ is nonempty).

The following lemma is well-known in elementary number theory, we state it without proof.

Lemma 1. Let r be an odd prime, and a a positive integer *not* divisible by r, and let n be the order of a modulo r. Then, if m is a positive integer such that $a^m \equiv 1 \pmod{r}$, *n is a divisor of m.*

From (1) it follows that,

$$\begin{aligned} q^q &\equiv -1 \pmod p \\ \Rightarrow q^{2q} &\equiv (-1)^2 \equiv 1 \pmod p\,. \end{aligned} \tag{4}$$

Let n be the order of q modulo p. By (4) and Lemma 1, it follows that n is a divisor of $2q$, which means that $n = 1, 2, q$, or $2q$.

If $n = 1$; then $q \equiv 1 \pmod p$; $q = 1 + pl$, for some positive integer l, and going back to (1) we have

$$p^p + (1 + p \cdot l)^q + 1 \;=\; k \cdot p \cdot q \tag{5}$$

It is evident from the binomial expansion of $(1 + p \cdot l)^2$, that (5) implies $2 + \lambda p = kpq$, for some positive integer λ, which is impossible since this last equation implies $p \mid 2$; we know that $p \geq 3$.

Next, consider the case in which the order n (of q modulo p) is q or $2q$. We know from Fermat's Little Theorem that

$$q^{p-1} \;\equiv\; 1 \pmod p\,.$$

By Lemma 1, the order n ($= q$ or $2q$) must divide $p - 1$. Since $p - 1$ is even and is q odd; we see that in either case $2q$ must divide p: therefore

$$\begin{aligned} p - 1 &= 2q \cdot t, \text{ for some positive integer } t. \\ p &= 2qt + 1 \;>\; q, \end{aligned}$$

which contradicts (3).

There remains only one possibility to consider: the order n (of q modulo p) is equal to 2.

$$\begin{aligned} q^2 \equiv 1 \pmod p &\Leftrightarrow (q-1)(q+1) \equiv 0 \pmod p \\ &\Leftrightarrow q \equiv \pm 1 \pmod p \text{ (since } p \text{ is prime)}. \end{aligned} \tag{6}$$

The case $q \equiv 1 \pmod p$ has already been examined above (this was done in the case order $n = 1$). So, then suppose that $q \equiv -1 \pmod p$,

$$q \;=\; p \cdot v - 1,\; v \in \mathbb{Z}, v \geq 2 \tag{7}$$

We go back to (1) and this time we work modulo q:

$$p^p \equiv -1 \pmod q \Rightarrow p^{2p} \equiv 1 \pmod q$$

which implies by Lemma 1 that the order f of p modulo q must be a divisor of $2p$. Thus, $f = 1, 2, p$, or $2p$. Once again, by Fermat's Little Theorem, we know that the order f must divide $q - 1$ by virtue of $p^{q-1} \equiv 1 \pmod q$. Hence, $q - 1 = f \cdot u$

$$q = f \cdot u + 1, \quad \text{where } f = 2, p, \text{ or } 2p. \tag{8}$$

Note that the possibility $f = 1$ is ruled out: if $f = 1$ then $p \equiv 1 \pmod q$ which implies (since both p and q are positive and ≥ 3) that $p > q$; contrary to (3).

If $f = p$ or $2p$, then combining (7) with (8) yields $p \cdot v - f \cdot u = 2$, which implies (since $f = p$ or $2p$) that p divides 2; an impossibility since $p \geq 3$.

Finally suppose that $f = 2$. Then,

$$p^2 \equiv 1 \pmod q \Leftrightarrow (p-1)(p+1) \equiv 0 \pmod q\,;$$

and since q is a prime, we must have either $p = 1 + q \cdot w$ or $p = -1 + q \cdot w$ for some positive integer w which again contradicts the conditions in (3); for either possibility implies $p > q$ (note that in either case, $w \geq 2$). It is now clear that there are *no odd primes* p and q which satisfy (1).

Case 2. $p = 2$ and q is an odd prime.

From (1) we have, $2^2 + q^q + 1 \; = \; 2kq$;

$$5 \; = \; q \cdot (2k - q^{q-1}). \tag{9}$$

Equation (9) clearly shows that $q \mid 5$; and since q is a prime; we must have $q = 5$ and $2k - q^{q-1} = 1$ so $2k = 5^4 + 1$, thus $k = \frac{626}{2} = 313$.

Conclusion: Taking into account symmetry, there exist exactly two pairs with the problem's property: $(p, q) = (2, 5), (5, 2)$.

5. For the vertex A of $\triangle ABC$, let A' be the point of intersection of the angle bisector at A with side BC, and let ℓ_A be the distance between the feet of the perpendiculars from A' to the lines AB and C, respectively. Define ℓ_B and ℓ_C similarly, and let ℓ be the perimeter of $\triangle ABC$. Prove that

$$\frac{\ell_A \ell_B \ell_C}{\ell^3} \; \leq \; \frac{1}{64}.$$

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's solution.

We adopt the standard notations for the elements of ΔABC and denote the orthogonal projections of A' onto AB, AC by H, K, respectively. Since the line segment HK is a chord subtending $\angle BAC$ of the circle with diameter AA', we have $\ell_A = HK = AA' \sin A$. As it is well-known, the length of the bisector is given by $AA' = \dfrac{2bc\cos(A/2)}{b+c}$ so we obtain

$$\ell_A = \frac{2bc\sin(A/2)}{b+c} \cdot 2\cos^2(A/2) = \frac{2bc\sin(A/2)}{b+c} \cdot \left(1 + \frac{b^2+c^2-a^2}{2bc}\right).$$

It quickly follows that

$$\frac{\ell_A}{\ell} \; = \; \frac{2(s-a)\sin(A/2)}{b+c}.$$

With similar results for ℓ_B and ℓ_C, we finally have

$$\frac{\ell_A\ell_B\ell_C}{\ell^3} = \frac{\sin(A/2)\sin(B/2)\sin(C/2)\cdot 8(s-a)(s-b)(s-c)}{(b+c)(c+a)(a+b)}.$$

From the following known formulas:

$$rs = \sqrt{s(s-a)(s-b)(s-c)},\ \sin(A/2)\sin(B/2)\sin(C/2) = \frac{r}{4R},$$

$$abc = 4rRs, \text{ and } ab+bc+ca = s^2+r^2+4rR$$

we first deduce

$$\begin{aligned}(b+c)(c+a)(a+b) &= (a+b+c)(ab+bc+ca) - abc \\ &= 2s(s^2+r^2+4rR) - 4rRs \\ &= 2s(s^2+r^2+2rR)\end{aligned}$$

and then

$$\frac{\ell_A\ell_B\ell_C}{\ell^3} = \frac{r^3}{Rs^2+Rr^2+2rR^2} \qquad (1)$$

By AM-GM, we have

$$\frac{s}{3} = \frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)} = \sqrt[3]{r^2 s}$$

so that $s^2 \geq 27r^2$. Recalling Euler's inequality $R \geq 2r$, we obtain

$$Rs^2 + Rr^2 + 2rR^2 \geq 54r^3 + 2r^3 + 8r^3 = 64r^3$$

and from (1), $\dfrac{\ell_A\ell_B\ell_C}{\ell^3} \leq \dfrac{1}{64}$.

Next we turn to the 2006/2007 British Mathematical Olympiad, Round 1, given at [2010: 153].

1. Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

Solved by Arkady Alt, San Jose, CA, USA; Geoffrey A. Kandall, Hamden, CT, USA; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Ricardo's write-up.

Using the familiar 'difference of squares' identity repeatedly, we can write

$$3^{32} - 2^{32} = 5\prod_{k=1}^{4}(3^{2^k} + 2^{2^k}).$$

For $k \leq 2$, the factors $3^{2^k} + 2^{2^k}$ are less than 100 and it is easy to pick out 5, 13 (when $k = 1$), and 97 (when $k = 2$) as prime factors. Now we observe that $3^{32} = 9^{16} \equiv 1 \pmod{17}$ and $2^{32} = 4^{16} \equiv 1 \pmod{17}$ by Fermat's Little Theorem. Thus $3^{32} - 2^{32} \equiv 0 \pmod{17}$ and 17 is the fourth prime factor we seek.

2. In the convex quadrilateral $ABCD$, points M, N lie on the side AB such that $AM = MN = NB$, and points P, Q lie on the side CD such that $CP = PQ = QD$. Prove that

$$\text{Area of } AMCP \;=\; \text{Area of } MNPQ \;=\; \frac{1}{3}\,\text{Area of } ABCD\,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Geoffrey A. Kandall, Hamden, CT, USA; by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and by Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.

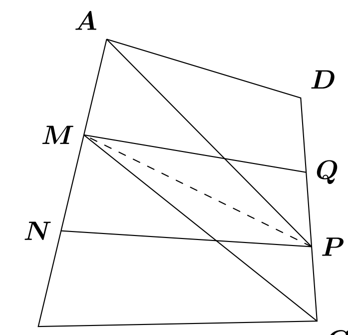


Figure 1

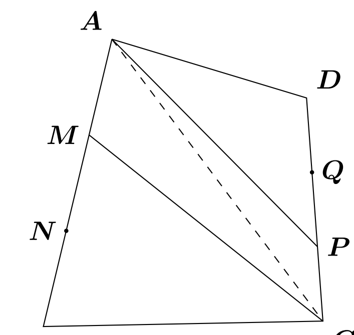


Figure 2

Since $CP = PQ$, we have (figure 1)

$$\text{Area of } \triangle CPM \;=\; \text{Area of } \triangle PQM$$

Since $AM = MN$, we have

$$\text{Area of } \triangle AMP \;=\; \text{Area of } \triangle MNP$$

Hence,

$$\text{Area of } \triangle CPM + \text{Area of } \triangle AMP \;=\; \text{Area of } \triangle PQM + \text{Area of } \triangle MNP$$

that is,

$$\text{Area of } AMCP \;=\; \text{Area of } MNPQ$$

Now, since the areas of triangles with equal altitudes are proportional to the bases of the triangles, we have (figure 2)

$$\text{Area of } \triangle AMC \;=\; \frac{1}{3}\,(\text{Area of } \triangle ABC)$$

and

$$\text{Area of } \triangle CPA = \frac{1}{3} (\text{Area of } \triangle CDA)$$

Hence,

$$\text{Area of } \triangle AMC + \text{Area of } \triangle CPA = \frac{1}{3} (\text{Area of } \triangle ABC + \text{Area of } \triangle CDA)$$

that is,

$$\text{Area of } AMCP = \frac{1}{3} (\text{Area of } ABCD)$$

and we are done.

3. The number **916238457** is an example of a nine-digit number which contains each of the digits **1** to **9** exactly once. It also has the property that the digits **1** to **5** occur in their natural order, while the digits **1** to **6** do not. How many such numbers are there?

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

$$\begin{array}{ccccccccc} \underline{x_1} & \underline{x_2} & \underline{x_3} & \underline{x_4} & \underline{x_5} & \underline{x_6} & \underline{x_7} & \underline{x_8} & \underline{x_9} \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} & \textcircled{8} & \textcircled{9} \end{array}$$

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Note that there are exactly $9! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$ nine-digit numbers with distinct *nonzero* digits.

Let S be the set of all nine-digit numbers with distinct nonzero digits and such that the digits **1** to **5** occur in their natural order, and $m = n(S) =$ cardinality of the set S.

$$\begin{array}{ccccccccc} \underline{\quad} & \underline{1} & \underline{\quad} & \underline{2} & \underline{3} & \underline{\quad} & \underline{4} & \underline{\quad} & \underline{5} \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} & \textcircled{8} & \textcircled{9} \end{array}$$

$$\begin{array}{ccccccccc} \underline{\quad} & \underline{1} & \underline{2} & \underline{\quad} & \underline{3} & \underline{4} & \underline{\quad} & \underline{5} & \underline{6} \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} & \textcircled{8} & \textcircled{9} \end{array}$$

Let S_1 be the set of all nine-digit numbers with distinct nonzero digits and such that the digits **1** to **6** occur in their natural order, and $m_1 = n(S_1) =$ cardinality of the set S_1.

Let S_2 be the set of all nine-digit numbers with distinct nonzero digits and such that the digits **1** to **5** occur in their natural order; but the (numbers) digits **1** to **6** do not occur in their natural order; and $m_2 = n(S_2) =$ cardinality of the set S_2.

Then $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Therefore,

$$\begin{aligned} n(S) &= n(S_1) + n(S_2); \\ m &= m_1 + m_2; \\ m_2 &= m - m_1. \qquad (1) \end{aligned}$$

To calculate m, observe that any five of the positions ① through ⑨ may be chosen; for any such choice, the numbers **1** to **5** are placed in their natural order on those positions. Moreover, for each such choice of five positions; there are **4!** ways to place the remaining numbers **6** to **9**, on the remaining four positions. Hence,

$$m \;=\; (4!)\cdot\binom{9}{5} \;=\; 4!\frac{9!}{4!5!} \;=\; \frac{9!}{5!} \;=\; 6\cdot 7\cdot 8\cdot 9.$$

Similarly,

$$m_1 \;=\; (3!)\binom{9}{6} \;=\; 3!\frac{9!}{3!6!} \;=\; \frac{9!}{6!} \;=\; 7\cdot 8\cdot 9.$$

Hence by (1)

$$\begin{aligned} m_2 \;=\; m - m_1 \;&=\; 6\cdot 7\cdot 8\cdot 9 - 7\cdot 8\cdot 9 \\ &=\; (7\cdot 8\cdot 9)(6-1) \\ &=\; 7\cdot 8\cdot 9\cdot 5 \;=\; 2520 \end{aligned}$$

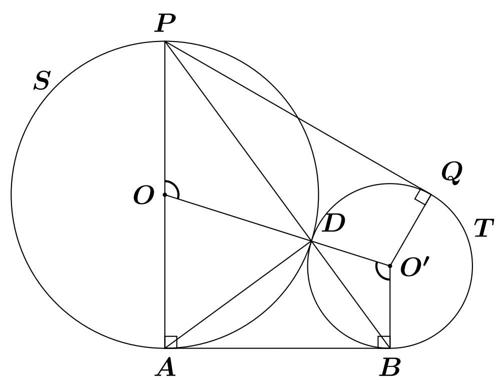
Conclusion: There are exactly **2520** such numbers.

4. Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the two solutions of Amengual Covas.

Solution 1.

Let O and O' be the centers of S and T, respectively.



Then the line OO', joining the centers of the touching circles, goes through the point of contact D.

Now, PA is perpendicular to AB, as is the radius $O'B$ to the point of contact with AB. Thus PA and $O'B$ are parallel and the alternate angles POD and $BO'D$ are equal. But triangles POD and $BO'D$ are isosceles, and since their vertical angles are equal, so are their base angles. Therefore $\angle ODP = \angle O'DB$, and D lies on PB.

Next, diameter PA subtends a right angle at D, making AD the altitude to the hypotenuse in right-triangle ADP. By a standard mean proportion, then, we have

$$PA^2 \;=\; PD\cdot PB \tag{1}$$

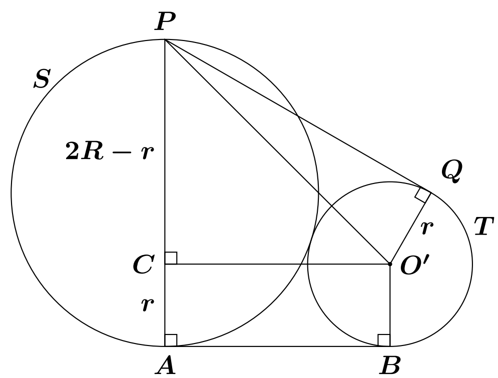
On the other hand, the power of P with respect to T is PQ^2 and also $PD \cdot PB$; hence,

$$PQ^2 = PD \cdot PB \qquad (2)$$

By (1) and (2), $PQ^2 = AP^2$. It follows that $PQ = AP$, as desired.

Solution 2.

Let R and r be the radii of circles S and T, respectively. Let O' be the center of T and denote by C the foot of the perpendicular from O' to AP. By the Pythagorean theorem, applied to right triangles PQO' and PCO',



$$\begin{aligned} PQ^2 + QO'^{\,2} &= PO'^{\,2} \\ &= PC^2 + CO'^{\,2} \end{aligned}$$

that is,

$$PQ^2 + QO'^{\,2} = PC^2 + AB^2$$

and since $AB = 2\sqrt{Rr}$ (for a proof, see e.g. *Japanese Temple Geometry Problems*, by H. Fukagawa and D. Pedoe, Canada, 1989, Example 1.1 on p. 3), $PC = 2R - r$, $QO' = r$, we have

$$PQ^2 + r^2 = (2R - r)^2 + 4Rr\,.$$

Hence

$$\begin{aligned} PQ^2 &= 4R^2 \\ &= (2R)^2 \\ &= AP^2 \end{aligned}$$

It follows that $PQ = AP$, as desired.

5. For positive real numbers a, b, c prove that

$$(a^2 + b^2)^2 \ \geq\ (a + b + c)(a + b - c)(b + c - a)(c + a - b)\,.$$

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Giulio Loddi, High School student, Cagliari, Italy; Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Loddi's solution.

We will treat a and b like constants and the right-hand side as a function of c:

$$\begin{aligned} f(c) &= (a + b + c)(a + b - c)(b + c - a)(c + a - b) \\ &= [(a + b)^2 - c^2] \cdot [c^2 - (a - b)^2] \\ &= -c^4 + c^2[(a - b)^2 + (a + b)^2] - (a - b)^2(a + b)^2 \\ &= -c^4 + 2c^2(a^2 + b^2) - (a^2 - b^2)^2 \end{aligned}$$

Let us find the maximum of $f(c)$. Differentiating with respect to c yields:

$$f'(c) = -4c^3 + 4c(a^2+b^2) = 4c[-c^2+a^2+b^2].$$

The derivative of $f(c)$ is zero when $c = 0$ or when $c^2 = a^2 + b^2$. $f(0) = -(a^2-b^2)^2$ and $f(\sqrt{a^2+b^2}) = (a^2+b^2)^2 - (a^2-b^2)^2 = 4a^2b^2$, so $f(0) < f(\sqrt{a^2+b^2})$. We can guess that there is a maximum when $c^2 = a^2+b^2$.

But

$$f(\sqrt{a^2+b^2}) = 4a^2b^2 \geq -c^4 + 2c^2(a^2+b^2) - (a^2-b^2)^2 = f(c)$$

when $c^4 - 2c^2(a^2+b^2) + (a^2-b^2)^2 + 4a^2b^2 \geq 0$. By computing the discriminant of this quadratic (in c^2):

$$\triangle = 4(a^2+b^2)^2 - 4(a^2-b^2)^2 - 16a^2b^2 = 16a^2b^2 - 16a^2b^2 = 0,$$

so $f(\sqrt{a^2+b^2}) - f(c) \geq 0$ for all c. Finally, $(a^2+b^2)^2 \geq 4a^2b^2$ follows from $(a^2-b^2)^2 \geq 0$ and thus

$$LHS \geq 4a^2b^2 = f(\sqrt{a^2+b^2}) \geq f(c) = RHS \qquad \forall c > 0.$$

Ed. – Note that $f(c) = -\left(c^2 - (a^2+b^2)\right)^2 + 4a^2b^2$ *which yields the same result.*

6. Let n be an integer. Show that, if $2 + 2\sqrt{1+12n^2}$ is an integer, then it is a perfect square.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Henry Ricardo, Tappan, NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bataille's write-up.

Suppose that $2 + 2\sqrt{1+12n^2}$ is a positive integer m. Then $(m-2)^2 = 4(1+12n^2)$ so that $1 + 12n^2$ must be a perfect square, say $1 + 12n^2 = a^2$ where a is a positive integer. It follows that

$$a^2 - 3(2n)^2 = 1 \tag{1}$$

and the pair $(a, 2n)$ is a solution to the Fermat equation $x^2 - 3y^2 = 1$ with $x \geq 1$ and y even. It is well-known that the solutions to this equation in nonnegative integers are the pairs (x_k, y_k) such that $x_k + y_k\sqrt{3} = (2+\sqrt{3})^k$, $k = 0, 1, 2, \ldots$. Since $x_{k+1} + \sqrt{3}y_{k+1} = (x_k + y_k\sqrt{3})(2+\sqrt{3})$ the sequences $(x_k), (y_k)$ are given by the recursion $x_{k+1} = 2x_k + 3y_k$, $y_{k+1} = x_k + 2y_k$ and $x_0 = 1$, $y_0 = 0$. Using induction, it is easy to see that y_k is even if and only if k is even. Note also that $2x_k = (2+\sqrt{3})^k + (2-\sqrt{3})^k$.

Returning to (1) and assuming that $n \geq 0$ without lost of generality, we must have $a = x_k$ and $2n = y_k$ for some even k. Setting $k = 2\ell$, we first deduce $2a = (2+\sqrt{3})^{2\ell} + (2-\sqrt{3})^{2\ell}$ and then

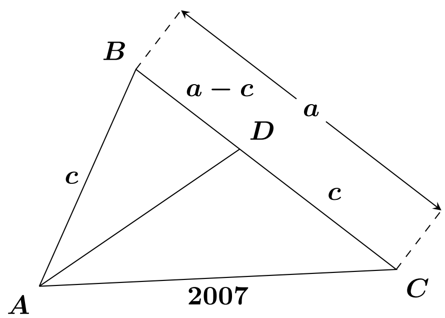
$$\begin{aligned} m = 2 + 2a &= (2+\sqrt{3})^{2\ell} + (2-\sqrt{3})^{2\ell} + 2(2+\sqrt{3})^{\ell}(2-\sqrt{3})^{\ell} \\ &= \left((2+\sqrt{3})^{\ell} + (2-\sqrt{3})^{\ell}\right)^2 = x_\ell^2, \end{aligned}$$

a perfect square.

Next up are solutions to problems of the 2006/2007 British Mathematical Olympiad, Round 2, given at [2010: 154].

1. Triangle ABC has integer-length sides, and $AC = 2007$. The internal bisector of $\angle BAC$ meets BC at D. Given that $AB = CD$, determine AB and BC.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We use Kandall's version.



Let $BC = a$, $AB = c$, so that $BD = a - c$. Note that $2007 = 3^2 \cdot 223$ (prime factorization).

Since AD is an angle bisector, we have

$$\frac{a-c}{c} = \frac{c}{2007}. \tag{1}$$

Thus, $c^2 = 3^2 \cdot 223(a - c)$. Both 3 and 223 divide c, so $c = 3 \cdot 223k$ (k a positive integer). From (1),

$$a = \frac{c^2}{2007} + c = 223(k^2 + 3k).$$

Since $a < c + 2007$ (triangle inequality), we have

$$223(k^2 + 3k) < 3 \cdot 223k + 9 \cdot 223,$$

which reduces easily to $k^2 < 9$. Thus, $k = 1$ or $k = 2$.

If $k = 1$, then $c = 3 \cdot 223$, $a = 4 \cdot 223$, so $c + a = 7 \cdot 223 < 2007$, which violates the triangle inequality.

Therefore, $k = 2$, which means that $c = 6 \cdot 223 = 1338$ and $a = 10 \cdot 223 = 2230$.

2. Show that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{m+1}{n} + \frac{n+1}{m}$$

is a positive integer.

Solved by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India.

Let $f(m, n) = \frac{m+1}{n} + \frac{n+1}{m}$. Observe that $f(1, 1) = 4$.

We claim that there are infinitely many pairs of positive integers (m, n) such that $f(m, n) = 4$.

$f(m, n) = 4$ implies $m^2 + (1 - 4n)m + n^2 + n = 0$. Viewing this as a quadratic in m and solving we get

$$m = \frac{(4n-1) \pm \sqrt{12n^2 - 12n + 1}}{2}. \tag{1}$$

Observe that $t = 12n^2 - 12n + 1$ is odd and m is an integer if and only if t is a perfect square. So let $t = p^2$, for some positive integer p, then

$$p^2 = 3q^2 - 2 \tag{2}$$

where $q = 2n - 1$. If (p, q) satisfies (2) then both p and q must be odd. Equation (2) is satisfied by $(p_1, q_1) = (1, 1)$ and if the positive integral pair (p_k, q_k) satisfies (2) then so does (p_{k+1}, q_{k+1}) where

$$\begin{aligned} p_{k+1} &= 2p_k + 3q_k \\ q_{k+1} &= p_k + 2q_k. \end{aligned}$$

Observe that $\{p_k\}$ and $\{q_k\}$ are increasing sequences and $p_k > q_k$ for $k > 1$. Now define

$$n_k = \frac{q_k + 1}{2},$$

$$m_k = \frac{(2n_k - 1) + \sqrt{12n_k^2 - 12n_k + 1}}{2} = \frac{2q_k + 1 + p_k}{2} = \frac{q_{k+1} + 1}{2}$$

for $k \geq 1$. Observe that both m_k and n_k are positive integers as q_k and q_{k+1} are odd positive integers.

The set $S = \{(m_k, n_k) : k \geq 1\}$ is an infinite set (because $\{q_k\}$ is an increasing sequence) and consists of pairs of positive integers satisfying

$$f(m, n) = 4.$$

Thus we have produced infinitely many pairs of positive integers (m, n) for which

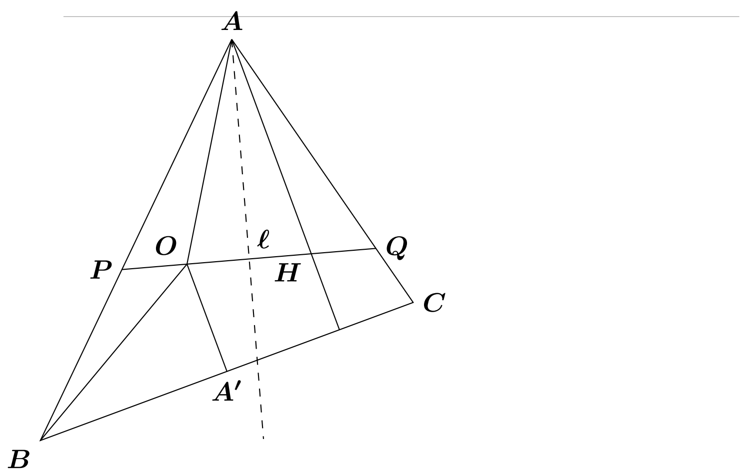
$$\frac{m+1}{n} + \frac{n+1}{m}$$

is a positive integer.

3. Let $\boldsymbol{ABC}$ be an acute-angled triangle with $\boldsymbol{AB > AC}$ and $\boldsymbol{\angle BAC = 60^\circ}$. Denote the circumcentre by $\boldsymbol{O}$ and the orthocentre by $\boldsymbol{H}$ and let $\boldsymbol{OH}$ meet $\boldsymbol{AB}$ at $\boldsymbol{P}$ and $\boldsymbol{AC}$ at $\boldsymbol{Q}$. Prove that $\boldsymbol{PO = HQ}$.

Note: The circumcentre of triangle $\boldsymbol{ABC}$ is the centre of the circle which passes through the vertices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. The orthocentre is the point of intersection of the perpendiculars from each vertex to the opposite side.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's solution.

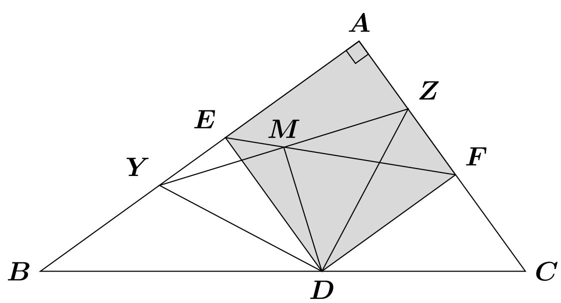


Note that from $\boldsymbol{AB > AC}$, we have $\boldsymbol{C = \angle ACB > B = \angle ABC}$, hence $\boldsymbol{2C > B + C = 180^\circ - A = 120^\circ}$ and so $\boldsymbol{C > 60^\circ}$. Also, since $\boldsymbol{\Delta ABC}$ is acute-angled, $\boldsymbol{AH = 2OA'}$ where $\boldsymbol{A'}$ is the midpoint of $\boldsymbol{BC}$, that is $\boldsymbol{AH = 2R\cos A = R}$ (denoting the circumcentre by $\boldsymbol{H}$). It follows that $\boldsymbol{\Delta OAH}$ is isosceles with $\boldsymbol{AO = AH}$. Moreover, we have $\boldsymbol{\angle HAC = 90^\circ - C}$ and $\boldsymbol{\angle PAO = \frac{1}{2}(180^\circ - \angle BOA) = \frac{1}{2}(180^\circ - 2C) = 90^\circ - C}$ as well. It follows that the angle bisectors of $\boldsymbol{\angle BAC}$ and $\boldsymbol{\angle OAH}$ are the same line $\boldsymbol{\ell}$. Now, if $\boldsymbol{\rho_\ell}$ denotes the reflection in $\boldsymbol{\ell}$, the image $\boldsymbol{\rho_\ell(OH)}$ of the line $\boldsymbol{OH}$ is $\boldsymbol{OH}$ itself (since $\boldsymbol{OH \perp \ell}$) and the image $\boldsymbol{\rho_\ell(AB)}$ is $\boldsymbol{AC}$. As a result, the image of $\boldsymbol{P}$, the intersection of $\boldsymbol{OH}$ and $\boldsymbol{AB}$ is $\boldsymbol{Q}$, the intersection of $\boldsymbol{OH}$ and $\boldsymbol{AC}$. Finally, $\boldsymbol{\rho_\ell(P) = Q,\ \rho_\ell(O) = H}$ and so $\boldsymbol{PO = QH}$.

Next we move to the May 2010 number of the *Corner* and solutions from our readers to problems of the XV Olympíada Matemática Rioplatense, Nivel 2, given at [2010; 214].

1. Let ABC be a right triangle with right angle at A. Consider all the isosceles triangles XYZ with right angle at X, where X lies on the segment BC, Y lies on AB, and Z is on the segment AC. Determine the locus of the medians of the hypotenuses YZ of such triangles XYZ.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Geupel.



Let $AEDF$ be the square where D, E, and F are on the segments BC, AB, and AC, respectively. We prove that $D = X$ and that the locus of the midpoints M of segments YZ is the line segment EF. The end point E and accordingly F is included if and only if $AB \geq AC$ and $AB \leq AC$, respectively.

Consider $\triangle XYZ$ as described in the problem. By $\angle YAZ = \angle YXZ = 90^\circ$, the quadrilateral $AYXZ$ is cyclic. From the condition $XY = XZ$, we see that $\angle XAY = \angle XAZ = 45^\circ$; hence $X = D$. By $\angle DMY = \angle DEY = 90^\circ$, the quadrilateral $DMEY$ is cyclic. Thus, $\angle DEM = \angle DYM = 45^\circ$. Consequently, M is on EF.

Vice versa, let M lie on EF. The cases $M = E$ and $M = F$ are possible if and only if $AB \geq AC$ and $AB \leq AC$, respectively. Let us suppose that $M \neq E, F$. The perpendicular to DM through M cuts AB and AC at Y and Z, respectively. By $\angle DMY = \angle DEY = 90^\circ$, the quadrilateral $DMEY$ is cyclic; hence $\angle DYM = \angle DEM = 45^\circ$. Similarly $\angle DZM = 45^\circ$. Consequently, XYZ is an isosceles right triangle, which completes the proof.

3. A finite number of (possibly overlapping) intervals on a line are given. If the rightmost $1/3$ of each interval is deleted, an interval of length 31 remains. If the leftmost $1/3$ of each interval is deleted, an interval of length 23 remains. Let M and m be the maximum and minimum of the lengths of an interval in the collection, respectively. How small can $M - m$ be?

Solved by Oliver Geupel, Brühl, NRW, Germany.

The solution is 24.

Consider the intervals $[0,\ 33]$, $[19,\ 28]$, and $[25,\ 34]$. If the rightmost $1/3$ of each interval is deleted, then the union of the resulting intervals is the interval $[0,\ 31]$ with length 31. If the leftmost $1/3$ of each interval is deleted, then the union of the resulting intervals is $[11,\ 34]$ with length 23. We have $M = 33$, $m = 9$; therefore $M - m = 24$.

We prove that generally $M - m \geq 24$.

Let $[a, b]$ be the minimal closed interval that contains all the given intervals.

If the rightmost $1/3$ of each interval is deleted, then an interval $[a,\ r]$ of length 31 remains. Thus,

$$r - a = 31.$$

At least one of the intervals with right end point in the interval $[r,\ b]$ will be reduced by a segment not greater than $b - r$. The length of such an interval is not greater than $3(b - r)$, which implies that

$$m \leq 3(b - r).$$

If the leftmost $1/3$ of each interval is deleted, then an interval $[\ell, b]$ of length 23 remains. Thus,

$$b - \ell = 23.$$

The initial collection of intervals contains an interval with left bound a. Its left bound after the deletion of the left $1/3$ is not less than ℓ. Hence, its length is not less than $3(\ell - a)$, which implies that

$$M \geq 3(\ell - a).$$

We conclude

$$M - m \geq 3(\ell - a) - 3(b - r) = 3\left[(r - a) - (b - \ell)\right] = 3(31 - 23) = 24,$$

which completes the proof.

BOOK REVIEWS

Amar Sodhi

Pythagoras' Revenge: A Mathematical Mystery
by Arturo Sangalli
Princeton University Press, 2009
ISBN: 978-0-6910-4955-7, cloth, 188 + xviii pp. US$24.95
Reviewed by ***Mark Taylor****, Halifax, N.S.*

Dr. Jule (formerly Jules) Davidson teaches group theory and non-Euclidean geometry at Indiana State University. He has reached the age of 34 and is becoming increasingly bored with the routine of academic life. On top of that he has "*all but given up hope of becoming a famous mathematician*". Presumably in an effort to complete the work for his Fields Medal before time runs out, Jule spends his evenings visiting `canyousolveit.com`, a website devoted to math puzzles. His attention is caught by the following problem:

A group of twelve baseball players put their caps in a bag. After the caps are well shuffled, each player picks one at random. (1) Calculate the probability that none of the players will pick up his own hat; (2) What is this probability if there are infinitely many players in the group?

In the preface to the book, Sangalli states that one of his aims is "*to reach those who usually shun mathematics*". The question above certainly has the potential to fulfill this aim. The problem can be explained in such a way that readers who may not understand the mathematical meaning of "probability" or "infinitely many" can come to believe they understand what the question asks. Unfortunately the calculation of probabilities can be difficult to understand even when the problem is simple to pose – the Monty Hall and the Birthday problems are two good examples.

Jule's answers are **0.3679** and $\mathbf{1/e}$ for parts (1) and (2) respectively. A hint to the solution is provided in appendix 1. It uses the Inclusion-Exclusion Principle. This may have the effect of damaging some readers for life or at least cause them to shun the appendices; the latter being unfortunate because the appendices also include Infinitely Many Primes, A Simple Visual Proof of the Pythagorean Theorem and some paragraphs on Perfect, Triangular and Square Numbers all of which should be accessible to anyone with a Grade 10 education.

Jule's solution to the hat problem results in an invitation to compete for the "*opportunity to help solve a 2,500 - year - old enigma*". Indiana Jule is quick to seize the opportunity and, after passing a number of tests, joins an esoteric group seeking the reincarnation of Pythagoras. The genesis of the Pythagorean cult and its beliefs are explained in Chapter 8 where we also learn Pythagoras prophesied his own reincarnation, and left instructions in a secret document which was to be guarded through the generations until he reappeared on earth. When questioned how the custodian of the secret document would recognize the reincarnation, Pythagoras states, "*He will be an extraordinary gifted man, eminently versed in the secrets of Number, of whom many wonderful things will be persistently related*".

Chapter 9 introduces Norton Thorp who at the age of 9 months spoke in complete sentences. When Norton was barely five an incredible thing happened to him. His guardian aunt, Therese, put him to bed and then sat down with her erstwhile lover Morris to a meal that consisted of "*an assortment of dips that included grilled eggplant and lemon puree, a spread made from feta cheese spiced with chili pepper and garlic, and meat cooked in tomato and red wine sauce. An entre of burghul and potato cakes with lamb and apricot filling was followed by the main course: swordfish baked in a lemon and paprika sauce and served on a bed of pilaf rice*".

Just before dessert was to be served, the couple heard a piano playing. The young Norton, who had never had a piano lesson in his life, was playing the third movement from Mozart's piano sonata in A major, K 331, with all the skill of a concert pianist!

Ten years after the piano incident, Thorp is subject to another supernatural incident. This time he writes an excerpt from The Odyssey in an ancient Greek script.

The search for the reincarnation of Pythagoras is paralleled in the book by a search for a copy of a manuscript by Pythagoras (no such manuscript was believed to exist). Of course, the latter search, driven by Professor Elmer Galway of Oriel College, Oxford, results in the unearthing of Pythagoras' secret document. The discovery takes place in Rome and the chapter heading is the old proverb - to think that if the manuscript had been left in Greece the heading might have read "All Roams lead to Rhodes".

I should mention that Jule has a twin sister, Johanna Davidson, who has a Ph.D. in computer science. Johanna's purpose in the book seems to be to link Jule to Norton Thorp, and to facilitate a discourse on randomness. Unfortunately she also affords Sangalli the opportunity to display his inadequacies as a writer. I shudder, or perhaps cringe is a better word, to recall his description of Johanna. I can only assume that the material on random numbers had numbed the editor's brain to such an extent that the paragraphs immediately following failed to register.

You may detect a certain lack of enthusiasm for the book on my part. Such an observation is correct. However, I think Sangalli has produced the basis for what could be a commercially successful screenplay. His ingenious twists would transfer to film without difficulty and the pace could make more acceptable the irrationalities within the plot because there would be little time for reflection.

Of course, the mathematics would have to be toned down; perhaps reduced to the standard esoteric scribblings, with plenty of subscripts, superscripts, multiple integrals, Greek letters and tensor products.

Yes, my advice would be to wait for the film.

PROBLEMS

*Solutions to problems in this issue should arrive no later than **1 November 2011**.*
An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3626. *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Let x, y, and z be positive real numbers such that $x^2+y^2+z^2=3$. Prove that

$$\frac{1+x^2}{z+2}+\frac{1+y^2}{x+2}+\frac{1+z^2}{y+2} \;\geq\; 2\,.$$

3627. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all quadruples a, b, c, d of positive real numbers that are solutions to the system of equations

$$a+b+c+d \;=\; 4\,,$$
$$\left(\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}\right)(1+3abcd) \;=\; 16\,.$$

3628. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c and r be the edge-lengths and the inradius of a triangle ABC. Find the minimum value of the expression

$$E=\left(\frac{a^2b^2}{a+b-c}+\frac{b^2c^2}{b+c-a}+\frac{c^2a^2}{c+a-b}\right)r^{-3}.$$

3629. *Proposed by Michel Bataille, Rouen, France.*

Find the greatest positive integer m such that 2^m divides

$$2011^{\left(2013^{2016}-1\right)}-1.$$

3630. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c \;=\; 3$. Prove that

$$\frac{ab(b+c)}{2+c}+\frac{bc(c+a)}{2+a}+\frac{ca(a+b)}{2+b} \;\leq\; 2\,.$$

3631. *Proposed by Michel Bataille, Rouen, France.*

Let $\{x_n\}$ be the sequence satisfying $x_0 = 1$, $x_1 = 2011$, and $x_{n+2} = 2012x_{n+1} - x_n$ for all nonnegative integer n. Prove that

$$\frac{(2010 + x_n^2 + x_{n+1}^2)(2010 + x_{n+2}^2 + x_{n+3}^2)}{(2010 + x_{n+1}^2)(2010 + x_{n+2}^2)}$$

is independent of n.

3632. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let k be a real number such that $0 \le k \le 56$. Prove that the equation below has exactly two real solutions:

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; k(x^2 - 7x) + 720\,.$$

3633. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $g_1(x) = x$ and for natural numbers $n > 1$ define $g_n(x) = x^{g_{n-1}(x)}$. Let $f : (0,1) \to \mathbb{R}$ be the function defined by $f(x) = g_n(x)$, where $n = \left\lfloor \frac{1}{x} \right\rfloor$. For example, $f\left(\frac{1}{3}\right) = \frac{1}{3}^{\frac{1}{3}^{\frac{1}{3}}}$. Here $\lfloor a \rfloor$ denotes the floor of a. Determine $\lim\limits_{x \to 0^+} f(x)$ or prove it does not exist.

3634. *Proposed by Michel Bataille, Rouen, France.*

ABC is an isosceles triangle with $AB = AC$. Points X, Y and Z are on rays $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{AC}$ respectively with $AZ > AC$ and $AX = BY = CZ$.

(a) Show that the orthogonal projection of X onto BC is the midpoint of YZ.

(b) If BZ and YC intersect in W, show that the triangles CYA and CWZ have the same area.

3635. *Proposed by Mehmet Sahin, Ankora, Turkey.*

Let ABC be an acute-angled triangle with circumradius R, inradius r, semiperimeter s, and with points $A' \in BC, B' \in CA$, and $C' \in AB$ arranged so that

$$\angle ACC' = \angle CBB' = \angle BAA' = 90^\circ.$$

Prove that:

(a) $|BC'||CA'||AB'| = abc$;

(b) $\dfrac{|AA'|}{|BC'|}\dfrac{|BB'|}{|CA'|}\dfrac{|CC'|}{|AB'|} = \tan A \tan B \tan C$;

(c) $\dfrac{Area(A'B'C')}{Area(ABC)} = \dfrac{4R^2}{s^2-(2R+r)^2} - 1$.

3636. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a+b+c+d=2$. Prove that

$$ab(a^2+b^2+c^2)+bc(b^2+c^2+d^2)+cd(c^2+d^2+a^2)+da(d^2+a^2+b^2) \;\leq\; 2\,.$$

3637. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let x be a real number with $|x| < 1$. Determine

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\ln(1-x)+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}\right)\,.$$

. . . .

3626. *Proposé par Thanos Magkos, 3[ième] -Collège de Kozanie, Kozani, Grèce.*

Soit x, y et z trois nombres réels positifs tels que $x^2+y^2+z^2=3$. Montrer que

$$\frac{1+x^2}{z+2}+\frac{1+y^2}{x+2}+\frac{1+z^2}{y+2} \;\geq\; 2\,.$$

3627. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Trouver tous les quadruplets a, b, c, d de nombres réels positifs qui sont solutions du système d'équations

$$a+b+c+d \;=\; 4\,,$$
$$\left(\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}\right)(1+3abcd) \;=\; 16\,.$$

3628. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c les longueurs des côtés d'un triangle ABC et r le rayon de son cercle inscrit. Trouver la valeur minimale de l'expression

$$E = \left(\frac{a^2b^2}{a+b-c} + \frac{b^2c^2}{b+c-a} + \frac{c^2a^2}{c+a-b}\right) r^{-3}.$$

3629. *Proposé par Michel Bataille, Rouen, France.*

Trouver le plus grand entier positif m tel que 2^m divise

$$2011^{\left(2013^{2016}-1\right)} - 1.$$

3630. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b et c trois nombres réels non négatifs tels que $a + b + c = 3$. Montrer que

$$\frac{ab(b+c)}{2+c} + \frac{bc(c+a)}{2+a} + \frac{ca(a+b)}{2+b} \leq 2\,.$$

3631. *Proposé par Michel Bataille, Rouen, France.*

Soit $\{x_n\}$ une suite satisfaisant $x_0 = 1$, $x_1 = 2011$ et, pour tout entier non négatif n, $x_{n+2} = 2012x_{n+1} - x_n$. Montrer que

$$\frac{(2010 + x_n^2 + x_{n+1}^2)(2010 + x_{n+2}^2 + x_{n+3}^2)}{(2010 + x_{n+1}^2)(2010 + x_{n+2}^2)}$$

est indépendant de n.

3632. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit k un nombre réel tel que $0 \leq k \leq 56$. Montrer que l'équation ci-dessous possède exactement deux solutions réelles :

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = k(x^2 - 7x) + 720\,.$$

3633. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $g_1(x) = x$ et, pour les nombres naturels $n > 1$, on définit $g_n(x) = x^{g_{n-1}(x)}$. Soit $f : (0,1) \to \mathbb{R}$ la fonction définie par $f(x) = g_n(x)$, où $n = \left\lfloor \frac{1}{x} \right\rfloor$. Par exemple, $f\left(\frac{1}{3}\right) = \frac{1}{3}^{\frac{1}{3}^{\frac{1}{3}}}$. Ici, $\lfloor a \rfloor$ dénote la partie entière de a. Trouver la limite $\lim\limits_{x \to 0^+} f(x)$ ou montrer qu'elle n'existe pas.

3634. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle isocèle avec $AB = AC$. On choisit respectivement trois points X, Y et Z sur les rayons $\overrightarrow{AC}$, $\overrightarrow{BA}$ et $\overrightarrow{AC}$ avec $AZ > AC$ et $AX = BY = CZ$.

(a) Montrer que la projection orthogonale de X sur BC est le point milieu de YZ.

(b) Si BZ et YC se coupent en W, montrer que les triangles CYA et CWZ ont la même aire.

3635. *Proposé par Mehmet Sahin, Ankora, Turkey.*

Soit ABC un triangle acutangle, r le rayon de son cercle inscrit, R le rayon de son cercle circonscrit, s son demi-périmètre. Soit de plus les points $A' \in BC, B' \in CA$ et $C' \in AB$ arrangés de telle sorte que

$$\angle ACC' = \angle CBB' = \angle BAA' = 90^\circ.$$

Montrer que :

(a) $|BC'||CA'||AB'| = abc$;

(b) $\dfrac{|AA'|}{|BC'|}\dfrac{|BB'|}{|CA'|}\dfrac{|CC'|}{|AB'|} = \tan A \tan B \tan C$;

(c) $\dfrac{Aire(A'B'C')}{Aire(ABC)} = \dfrac{4R^2}{s^2 - (2R + r)^2} - 1$.

3636. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Soit a, b, c et d des nombres réels non négatifs tels que $a + b + c + d = 2$. Montrer que

$$ab(a^2+b^2+c^2)+bc(b^2+c^2+d^2)+cd(c^2+d^2+a^2)+da(d^2+a^2+b^2) \leq 2 .$$

3637. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit x un nombre réel avec $|x| < 1$. Déterminer

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \left(\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} \right) .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3521. [2010 : 108, 111] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let x_1, x_2, $\dots$, x_n be real numbers in the interval $[e,\infty)$ and for each index k let $e_k = \dfrac{x_1+x_2+\cdots+x_k}{x_k}$. Prove that

$$x_1^{e_1} + x_2^{e_2} + \cdots + x_n^{e_n} \geq nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n\,.$$

Solution by Richard Eden, student, Purdue University, West Lafayette, IN, USA.

It is easy to show that $f(x) = x^{1/x}$ is decreasing on $[e,\infty)$. [*Ed.:* $f(x) = \exp(\ln(x)/x)$ *and* $d^2/dx^2 \ln(x)/x = (1-\ln(x))/x^2 \leq 0$ *for* $x \in [e,\infty)$, *so* $\ln(x)/x$ *decreases on the given interval and so does* $f(x)$.]

For each index k,

$$x_k^{1/x_k} \geq (x_1+x_2+\cdots+x_k)^{1/(x_1+x_2+\cdots+x_k)}\;,$$

so

$$x_k^{e_k} = x_k^{(x_1+x_2+\cdots+x_k)/x_k} \geq x_1+x_2+\cdots+x_k\,.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n} x_k^{e_k} &\geq (x_1) + (x_1+x_2) + \cdots + (x_1+x_2+\cdots+x_n) \\ &= nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n\,.\end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Most solvers used a similar approach to that of the featured solution, except Curtis, who used Bernoulli's inequality.

Geupel noted that problem 3521 is equivalent to Mathematics Magazine *problem 1794 by the same proposer.*

3522. [2010 : 108, 111] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

If a, b, c, and d are positive real numbers satisfying $abcd = 1$, prove that

$$\left(1+\frac{a}{b}\right)^{cd}\left(1+\frac{b}{c}\right)^{da}\left(1+\frac{c}{d}\right)^{ab}\left(1+\frac{d}{a}\right)^{bc} \geq 2^{\left(\frac{16}{a^2+b^2+c^2+d^2}\right)}.$$

Solution by Michel Bataille, Rouen, France.

The inequality is equivalent to

$$cd\ln\left(1+\frac{a}{b}\right)+da\ln\left(1+\frac{b}{c}\right)+ab\ln\left(1+\frac{c}{d}\right)+bc\ln\left(1+\frac{d}{a}\right) \geq \frac{16}{a^2+b^2+c^2+d^2}\cdot\ln 2,$$

or, taking $abcd = 1$ into account and setting $f(x) = \sqrt{x\ln(1+x)}$,

$$(a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}f\left(\frac{a}{b}\right)^2+\frac{1}{b^2}f\left(\frac{b}{c}\right)^2+\frac{1}{c^2}f\left(\frac{c}{d}\right)^2+\frac{1}{d^2}f\left(\frac{d}{a}\right)^2\right) \geq 16\ln 2.$$

By the Cauchy-Schwarz Inequality, the left side L satisfies $L \geq M^2$, where

$$M = f(a/b)+f(b/c)+f(c/d)+f(d/a),$$

so it suffices to show that $M \geq 4\sqrt{\ln 2}$.

Now, let $x_1 = \ln(a/b)$, $x_2 = \ln(b/c)$, $x_3 = \ln(c/d)$, $x_4 = \ln(d/a)$, and for real x, let $g(x) = f(e^x) = e^{x/2}(\ln(1+e^x))^{1/2}$. Then

$$g''(x) = \frac{e^{x/2}\left(\ln(1+e^x)\right)^{-3/2}}{4}\cdot\left[\ln(1+e^x)^2 + \frac{2e^x\ln(1+e^x)}{(e^x+1)^2}+\frac{e^x}{e^x+1}\left(2\ln(1+e^x)-\frac{e^x}{e^x+1}\right)\right].$$

So $g''(x) > 0$ certainly holds, since $2\ln(1+e^x) - \frac{e^x}{e^x+1} > \frac{e^x}{e^x+1}$ (as it follows from $\ln(1+u) > \frac{u}{1+u}$ for positive u).

Thus, g is convex on $\mathbb{R}$, and from Jensen's inequality we obtain

$$g(x_1)+g(x_2)+g(x_3)+g(x_4) \geq 4g\left(\frac{x_1+x_2+x_3+x_4}{4}\right).$$

Since $x_1+x_2+x_3+x_4 = 0$, we have the desired result

$$M \geq 4\cdot e^0\cdot\left(\ln(1+e^0)\right)^{1/2} = 4\sqrt{\ln 2}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer. Two incorrect solutions were submitted.

Each of the two incorrect submissions used Bernoulli's inequality, but overlooked the fact that $(1+x)^r < 1+rx$ *for* $x > 0$ *and* $0 < r < 1$.

3527. [2010 : 171, 173] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sum_{\text{cyclic}}\left(a^2b+\frac{3}{2}\right)\left(b^2c+\frac{3}{2}\right)\ \le\ \frac{75}{4}\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece.

First we will prove that if a, b, c are nonnegative real numbers satisfying $a+b+c=3$, then

$$ab^2+bc^2+ca^2\ \le\ 4-abc\,. \tag{1}$$

We will use the Rearrangement Inequality to prove this. Let (x,y,z) be a permutation of (a,b,c) such that $x\ge y\ge z$. Since $xy\ge xz\ge yz$, we have

$$\begin{aligned}ab^2+bc^2+ca^2 &= b\cdot ab+c\cdot bc+a\cdot ac\\ &\le x\cdot xy+y\cdot xz+z\cdot yz\\ &= y(x+z)^2-xyz\ =\ y(x+z)^2-abc\,.\end{aligned}$$

It suffices to show $y(x+z)^2\le 4$, which follows from the AM-GM Inequality:

$$\begin{aligned}2y(x+z)^2 &= 2y(x+z)(x+z)\\ &\le \left(\frac{2y+(x+z)+(x+z)}{3}\right)^3\ =\ \left(\frac{2(x+y+z)}{3}\right)^3\ =\ 8\,;\end{aligned}$$

and (1) is established.

Now

$$\sum_{\text{cyclic}}\left(a^2b+\frac{3}{2}\right)\left(b^2c+\frac{3}{2}\right)\ \le\ \frac{75}{4}\iff rA+3B-12\ \le\ 0\,,$$

where $r=abc$, $A=ab^2+bc^2+ca^2$, and $B=a^2b+b^2c+c^2a$.

By the AM–GM Inequality we have $r\le 1$. Also, by Schur's inequality,

$$(a+b+c)^3\ \ge\ 3abc+4ab(a+b)+4bc(b+c)+4ca(c+a)\,,$$

from which we obtain $A+B\le\dfrac{27-3r}{4}$. From (1) we have $A\le 4-r$.

Finally, we have

$$\begin{aligned}rA+3B-12 &= (r-3)A+3(A+B)-12\\ &\le (r-3)(4-r)+3\left(\frac{27-3r}{4}\right)-12\\ &= \frac{-4r^2+19r-15}{4}\ =\ -(r-1)\left(r-\frac{15}{4}\right)\ \le\ 0\,,\end{aligned}$$

which holds because $0 \leq r \leq 1$ and $r - \frac{15}{4} < 0$.

Equality holds if and only if $(a, b, c) = (1, 1, 1)$ or (a, b, c) is a permutation of $(0, 1, 2)$.

Also solved by MARIAN DINCĂ, Bucharest, Romania; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3528. [2010 : 171, 173] *Proposed by Hiroshi Kinoshita and Katsuhiro Yokota, Tokyo, Japan.*

The incircle of triangle ABC touches the sides BC, AC, AB at the points A', B', C', respectively. Let ρ, r_a, r_b, r_c denote the inradii of the triangles $A'B'C'$, $AB'C'$, $BC'A'$, $CA'B'$, respectively, and let r be the inradius of the triangle ABC. Prove that

$$r = \frac{1}{2}(\rho + r_a + r_b + r_c)\,.$$

Similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposers.

The desired equation is a consequence of two known theorems. Problem 1.1.4, page 3 of H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, says that *if the tangents at points B' and C' of a circle Γ intersect in a point A, then the incentre of $\Delta AB'C'$ lies on Γ*. Since the simple proof is not given there, we shall prove it here. Let the line joining A to the centre of Γ meet the circle at P; we shall show that P is the incentre of $\Delta AB'C'$. Since AC' is tangent to Γ at C', we have $\angle PC'A = \angle PA'C'$ for any point A' on the arc $B'C'$ opposite P. But P is the midpoint of arc $B'C'$ so that $\angle PA'C' = \angle PA'B' = \angle PC'B'$, whence $C'P$ bisects $\angle AC'B'$. Since P also lies on the bisector of $\angle B'AC'$, it must be the incentre of $\Delta AB'C'$, as claimed.

To set up the second theorem, let I_a, I_b, I_c, and I be the incentres of the triangles $AB'C'$, $A'BC'$, $A'B'C$, and ABC, respectively. We have seen that I_a, I_b, and I_c lie on the circumcircle of $\Delta A'B'C'$, which we again call Γ; note that Γ has centre I and radius r. If d_a, d_b, and d_c are the distances from I to the sides $B'C', C'A'$, and $A'B'$, respectively, then $r = d_a + r_a = d_b + r_b = d_c + r_c$. Carnot's theorem applied to $\Delta A'B'C'$ with its circumradius r and inradius ρ says that $d_a + d_b + d_c = r + \rho$. (See, for example, Nathan Altshiller Court, *College Geometry*, page 83.) It follows that

$$3r = (d_a + r_a) + (d_b + r_b) + (d_c + r_c) = r + \rho + r_a + r_b + r_c,$$

which is equivalent to the desired result.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece;

MICHEL BATAILLE, Rouen, France; MIHAELA BLANARIU, Columbia College Chicago, Chicago, IL, USA; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposers (a second solution).

Geupel referred to the result as "well known", and provided the reference

http://www.artofproblemsolving.com/Forum/viewtopic.php?t=275874.

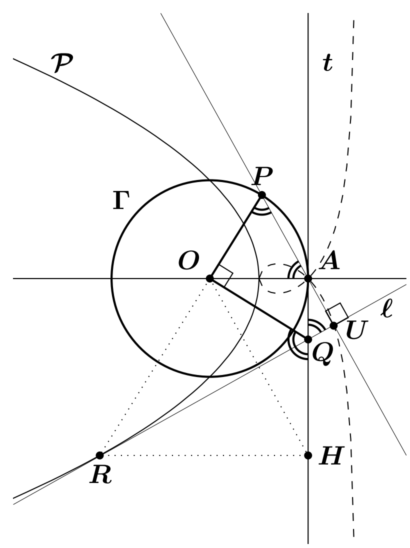
The solution on that web page is essentially the same as our featured solution. Most of the other submissions were based on formulas equivalent to Carnot's theorem which, applied to ΔABC, *become*

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1 = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

3529. [2010 : 171, 174] *Proposed by Michel Bataille, Rouen, France.*

Let A be a point on a circle Γ with centre O and t be the tangent to Γ at A. Triangle POQ is such that P is on Γ, Q is on t, and $\angle POQ = 90°$. Find the envelope of the perpendicular to AP through Q as $\triangle POQ$ varies.

I. *Solution by the proposer.*



We shall see that the envelope is the parabola, minus its vertex, with focus O and directrix t. Point P can be any point of Γ except for A and its diametrically opposed point. We denote by ℓ the perpendicular to AP through Q, and by U the point of intersection of ℓ with the line AP. Since $OP = OA$, $\angle OPA \neq 90°$ and so lines OP and ℓ must intersect, say at R. Lastly, let H be the projection of R onto t. From the figure we see that $\angle OPA = \angle OAP = \angle AQU = \angle RQH$, while $\angle OQR = 90° - \angle QRP = \angle RPU = \angle OPA$; it follows that $\angle OQR = \angle RQH$. Thus, the right triangles ROQ and RHQ are congruent; so $RO = RH$, whence, R must lie on the parabola $\mathcal{P}$ with focus O, directrix t. As the perpendicular bisector of OH, ℓ is the tangent to $\mathcal{P}$ at R. Conversely, let ℓ be the tangent to $\mathcal{P}$ at a point R of $\mathcal{P}$ distinct from its vertex, and suppose that it meets t at Q. Let the perpendicular to ℓ through A intersect Γ again at P and ℓ at U. We show that $\angle QOP = 90°$. As above, let H be the projection of R onto t. Since, using directed angles here, $\angle HOQ = \angle QHO = \angle QAU$, we have $\angle OPU = \angle OPA = \angle PAO = \angle OQR$, hence $180° = \angle UQO + \angle OQR = \angle UQO + \angle OPU$. Thus, P, O, Q, U are concyclic and the claim follows since $\angle PUQ = 90°$.

Comment. Note that this problem offers an alternative construction for the points and tangents of a parabola using the circle centred at the focus and tangent to the directrix.

II. *Solution by Oliver Geupel, Brühl, NRW, Germany.*

Consider Cartesian coordinates (x, y) with $O = (0, 0)$ and $A = (1, 0)$, and let $P = (\cos\varphi, \sin\varphi)$. Then for $\varphi \neq 0, \pi$, we have $Q = (1, -\cot\varphi)$, and the perpendicular to AP has slope $\tan\frac{\varphi}{2}$.

Firstly, take $0 < \varphi < \pi$. The family of perpendiculars to AP through Q as φ varies between 0 and π is given implicitly by $U(x, y, \varphi) = 0$, where

$$U(x, y, \varphi) = (x - 1)\tan\frac{\varphi}{2} - y - \cot\varphi.$$

According to H. v. Mangoldt and K. Knopp, *Einführung in die Höhere Mathematik*, Vol. 2, 10th ed. (S. Hirzel Verlag Leipzig, 1957) Paragraph 176, for the existence of an envelope one must check that the partial derivatives $U_x, U_y, U_\varphi, U_{\varphi x}, U_{\varphi y}$, and $U_{\varphi\varphi}$ are continuous on the domain of definition (which is easily verified here); moreover, $-U_{\varphi\varphi} = \frac{\cos(\varphi/2)}{\sin^2\varphi\sin(\varphi/2)} \neq 0$ and $U_x U_{\varphi y} - U_y U_{\varphi x} = \frac{1}{2\cos^2\varphi} \neq 0$, as required (where we inserted the values of x and y that we obtain below into the second derivatives). The theorem implies that the equation of the envelope in terms of x and y can be found by solving simultaneously the pair of equations

$$U_\varphi = 0, \quad U = 0.$$

From

$$0 = U_\varphi = (x - 1) \cdot \frac{1}{2\cos^2\frac{\varphi}{2}} + \frac{1}{\sin^2\varphi}\ ,$$

we solve for x:

$$x - 1 = -\frac{2\cos^2\frac{\varphi}{2}}{\sin^2\varphi},$$

so that

$$x = \frac{1}{2}\left(1 - \cot^2\frac{\varphi}{2}\right).$$

Plugging this into the equation $U = 0$ yields

$$y = -\cot\varphi - \frac{1}{\sin\varphi} = -\cot\frac{\varphi}{2}.$$

Combining these results, we obtain $x = \frac{1}{2}(1 - y^2)$, or

$$y^2 = -2x + 1 \qquad (y < 0),$$

which is the lower branch of the parabola with focus O and directrix t.

For $-\pi < \varphi < 0$, we obtain the upper branch of the same parabola. The desired envelope is thus the parabola with the exception of its vertex.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern

State University, Joplin, MO, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ALBERT STADLER, Herrliberg, Switzerland; and PETER Y. WOO, Biola University, La Mirada, CA, USA. Two submissions dealt instead with a related problem.

All the correct solutions used the same argument as in our solution II, except for solution I. Geupel's version stood out, however, because he provided the required details (and, consequently, was the only person other than the proposer to explicitly exclude the parabola's vertex from the envelope). The two submissions that went astray determined the locus of the point U *(in the notation of solution I) rather than the envelope of the lines* QU*. It turns out that the locus is a right strophoid—the pedal curve of a parabola with respect to the point of intersection of its axis and directrix (that is, the locus of the points where a tangent to the parabola meets the perpendicular dropped to it from the pedal point* A*). It is the curve in the accompanying figure with a vertical asymptote* $x = \frac{3}{2}$ *and a loop that is tangent to the parabola at its vertex* $(\frac{1}{2}, 0)$*. In terms of the coordinatization of solution II, the locus satisfies*

$$y^2 = (x-1)^2 \left(\frac{1+2(x-1)}{1-2(x-1)}\right).$$

3530. [2010 : 171, 174] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \to \mathbb{R}$ be an integrable function which is continuous at 1. Let k be a fixed positive integer, and let

$$a_n \;=\; \int_0^1 \frac{f(x)}{(1+x^n)(1+x^{n+k})}\, dx\,.$$

Find $L = \lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} n(L - a_n)$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We shall show that

$$L = \int_0^1 f(x)dx \tag{1}$$

and

$$\lim_{n\to\infty} n(L - a_n) = \left(\frac{1}{2} + \ln 2\right) f(1)\,. \tag{2}$$

The sequence $f_n(x) = \frac{f(x)}{(1+x^n)(1+x^{n+k})}$ of integrable functions converges to $f(x)$ almost everywhere in $[0,1]$ as $n \to \infty$. Moreover, $|f_n(x)| \le |f(x)|$ for all n. By Lebesgue's Dominated Convergence Theorem, we obtain

$$L = \lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 f(x)dx\,,$$

thus proving the limit (1).

We will first prove the limit (2) for $f \in C^1$ and subsequently for general integrable functions f. Let us assume that $f \in C^1$ and make use of the following lemma [1].

Lemma If $g : [0,1] \to \mathbb{R}$ is a continuous function such that $\lim_{n\to 0+} \frac{g(x)}{x}$ exists and is finite, then for any function $f : [0,1] \to \mathbb{R}$ of class C^1,

$$\lim_{n\to\infty} n\int_0^1 f(x)g(x^n)dx = f(1)\int_0^1 \frac{g(x)}{x}dx\,. \qquad \square$$

We have

$$\begin{aligned} n(L-a_n) &= n\int_0^1 f(x)\left(1-\frac{1}{(1+x^n)(1+x^{n+k})}\right)dx \\ &= n\int_0^1 f(x)\frac{x^n(2+x^n)}{(1+x^n)^2}dx \\ &\quad - n\int_0^1 f(x)\frac{x^n(1-x^k)}{(1+x^n)^2(1+x^{n+k})}dx\,. \end{aligned}$$

By applying the lemma with $g(x) = \frac{x(2+x)}{(1+x)^2}$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} n\int_0^1 f(x)\frac{x^n(2+x^n)}{(1+x^n)^2}dx = f(1)\int_0^1 \frac{g(x)}{x}dx \\ &= f(1)\left[\ln(1+x) - \frac{1}{1+x}\right]_0^1 = \left(\frac{1}{2}+\ln 2\right)f(1)\,. \end{aligned}$$

Moreover, f is bounded, and for $0 \le x < 1$, it holds that $\lim_{n\to\infty} nx^n = 0$. Hence, for each $x \in [0,1]$,

$$\lim_{n\to\infty} nf(x)\frac{x^n(1-x^k)}{(1+x^n)^2(1+x^{n+k})} = 0\,.$$

By Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} n\int_0^1 f(x)\frac{x^n(1-x^k)}{(1+x^n)^2(1+x^{n+k})} = 0\,.$$

This proves (2) for $f \in C^1$.

Finally, let us drop the hypothesis on f and assume that f is an integrable function which is continuous at 1. By the continuity, for each fixed $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - f(1)| < \epsilon$ whenever $1-\delta \le x < 1$. We have

$$\begin{aligned} n(L-a_n) &= n\int_0^1 f(x)\left(1-\frac{1}{(1+x^n)(1+x^{n+k})}\right)dx \\ &= n\int_0^{1-\delta} f(x)\frac{x^n(1+x^k+x^{n+k})}{(1+x^n)(1+x^{n+k})}dx \\ &\quad + n\int_{1-\delta}^1 f(x)\frac{x^n(1+x^k+x^{n+k})}{(1+x^n)(1+x^{n+k})}dx\,. \end{aligned}$$

By Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty} n\int_0^{1-\delta} f(x)\frac{x^n(1+x^k+x^{n+k})}{(1+x^n)(1+x^{n+k})}dx = 0\,.$$

since the integrand is bounded and pointwise convergent to 0. Hence,

$$\lim_{n\to\infty} n(L-a_n) = \lim_{n\to\infty} n\int_{1-\delta}^{1} f(x)\frac{x^n(1+x^k+x^{n+k})}{(1+x^n)(1+x^{n+k})}dx\,.$$

We have that $f(1)-\epsilon < f(x) < f(1)+\epsilon$ for $1-\delta \le x < 1$. Since the relation (2) holds for the constant functions $f_1(x) = f(1)-\epsilon$ and $f_2(x) = f(1)+\epsilon$, we conclude that

$$\left(\frac{1}{2}+\ln 2\right)(f(1)-\epsilon) \le \lim_{n\to\infty} n(L-a_n) \le \left(\frac{1}{2}+\ln 2\right)(f(1)+\epsilon)\,.$$

This holds for each $\epsilon > 0$. Consequently,

$$\lim_{n\to\infty} n(L-a_n) = \left(\frac{1}{2}+\ln 2\right)f(1)\,.$$

Also solved by George Apostolopoulos, Messolonghi, Greece (part 1 only); Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (part 1 only); Albert Stadler, Herrliberg, Switzerland; and the proposer. Two incorrect solutions to part 2 were received.

References [1] T.-L. T. Rădulescu, V. D. Rădulescu, T. Andreescu, *Problems in Real Analysis*, Springer, 2009, § 9.5.12., page 348.

3531. [2010 : 172, 174] *Proposed by K.S Bhanu, Institute of Science, Nagpur, India, and M.N. Deshpande, Nagpur, India.*

Let a, b be positive integers. On the real line, A stands at $-a$ and B stands at b. A fair coin is tossed, and if it shows heads then A moves one unit to the right, while if it shows tails then B moves one unit to the left. The process stops when A or B reaches the origin.

Let $P_A(a,b)$ be the probability that A reaches the origin before B, and define $P_B(a,b)$ similarly. Prove that

$$E(a,b) \;=\; 2aP_A(a+1,b) + 2bP_B(a,b+1)\,,$$

where $E(a,b)$ is the expected number of tosses before the process terminates.

Solution by the Joel Schlosberg, Bayside, NY, USA.

Consider a specfic a, b. For $0 \le k \le b-1$ let A_k be the event that A reaches the origin before B after k tosses show tails; for $0 \le k \le a-1$, let B_k be the event that B reaches the origin before A after k tosses show heads. Then $P_A(a,b) = \sum_{k=0}^{b-1} P(A_k)$ and $P_B(a,b) = \sum_{k=0}^{a-1} P(B_k)$.

Since A_k occurs if and only if there are $a+k$ tosses, of which the first $a+k-1$ contain $a-1$ heads and k tails and the $(a+k)^{\text{th}}$ toss is heads,

$$P(A_k) = \binom{a+k-1}{k}\left(\frac{1}{2}\right)^{a+k-1}\cdot\frac{1}{2} = \frac{1}{2^{a+k}}\binom{a+k-1}{k}$$

$$P_A(a,b) = \sum_{k=0}^{b-1} P(A_k) = \frac{1}{2^a}\sum_{k=0}^{b-1}\frac{1}{2^k}\binom{a+k-1}{k}$$

Similarly,

$$P(B_k) = \frac{1}{2^{b+k}}\binom{b+k-1}{k}$$

$$P_B(a,b) = \sum_{k=0}^{a-1} P(B_k) = \frac{1}{2^b}\sum_{k=0}^{a-1}\frac{1}{2^k}\binom{b+k-1}{k}$$

Furthermore,

$$\begin{aligned}
E(a,b) &= \sum_{k=0}^{b-1}(a+k)P(A_k) + \sum_{k=0}^{a-1}(b+k)P(B_k)\\
&= \sum_{k=0}^{b-1}\frac{a}{2^{a+k}}\binom{a+k}{k} + \sum_{k=0}^{a-1}\frac{b}{2^{b+k}}\binom{b+k}{k}\\
&= \frac{a}{2^a}\sum_{k=0}^{b-1}\frac{1}{2^k}\binom{a+k}{k} + \frac{b}{2^b}\sum_{k=0}^{a-1}\frac{1}{2^k}\binom{b+k}{k}\\
&= \frac{2a}{2^{a+1}}\sum_{k=0}^{b-1}\frac{1}{2^k}\binom{(a+1)+k-1}{k} + \frac{2b}{2^{b+1}}\sum_{k=0}^{a-1}\frac{1}{2^k}\binom{(b+1)+k-1}{k}\\
&= 2aP_A(a+1,b) + 2bP_B(a,b+1)
\end{aligned}$$

Also solved by George Apostolopoulos, Messolonghi, Greece; Victor Arnaiz and Pedro A. Castillejo, students, Universidad Complutense de Madrid, Madrid, Spain; Keith Ekblaw, Walla Walla, WA, USA; Oliver Geupel, Brühl, NRW, Germany; Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA; Albert Stadler, Herrliberg, Switzerland; and the proposer.

3532. Correction. [2010 : 172, 174, 239] *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC have circumradius R, inradius r, and let δ_a, δ_b, δ_c be the distances from the centroid to the sides BC, CA, AB, respectively. Prove that

$$\sqrt{r} \le \frac{\sqrt{\delta_a}+\sqrt{\delta_b}+\sqrt{\delta_c}}{3} \le \sqrt{\frac{R}{2}}\,.$$

Solution by Richard Eden, student, Purdue University, West Lafayette, IN, USA.

Let $\triangle ABC$ have centroid P, area F, and semiperimeter s. Since the area of $\triangle BPC$ is one-third of F, then $\frac{a\delta_a}{2} = \frac{F}{3}$, so $\delta_a = \frac{2F}{3a}$. Similarly, $\delta_b = \frac{2F}{3b}$ and $\delta_c = \frac{2F}{3c}$.

Using the Harmonic Mean – Root Mean Square Inequality, and the fact that $F = rs$, we have

$$\frac{3}{\sqrt{\frac{2F}{3a}} + \sqrt{\frac{2F}{3b}} + \sqrt{\frac{2F}{3c}}} \;\le\; \sqrt{\frac{\frac{3a}{2F} + \frac{3b}{2F} + \frac{3c}{2F}}{3}}\,,$$

or

$$\frac{3}{\sqrt{\delta_a} + \sqrt{\delta_b} + \sqrt{\delta_c}} \;\le\; \sqrt{\frac{s}{F}} \;=\; \sqrt{\frac{1}{r}}\,,$$

which is the desired first inequality.

To prove the second inequality, we write

$$\begin{aligned} \delta_a &= \frac{2F}{3a} = \frac{bc\sin A}{3a} \\ &= \frac{(2R\sin B)(2R\sin C)\sin A}{3\cdot 2R\sin A} = \frac{2R\sin B\sin C}{3}\,, \end{aligned}$$

so that

$$\sqrt{\delta_a} \;=\; \sqrt{\frac{2R}{3}}\sqrt{\sin B\sin C} \;\le\; \sqrt{\frac{2R}{3}}\left(\frac{\sin B + \sin C}{2}\right)$$

with similar statements for $\sqrt{\delta_b}$ and $\sqrt{\delta_c}$. Adding these together yields

$$\sqrt{\delta_a} + \sqrt{\delta_b} + \sqrt{\delta_c} \;\le\; \sqrt{\frac{2R}{3}}\,(\sin A + \sin B + \sin C)\;.$$

The sine function is concave on $(0,\pi)$, so by Jensen's Inequality we have

$$\frac{\sin A + \sin B + \sin C}{3} \;\le\; \sin\left(\frac{A+B+C}{3}\right) \;=\; \sin\frac{\pi}{3} \;=\; \frac{\sqrt{3}}{2}\,.$$

Therefore, $\sqrt{\delta_a} + \sqrt{\delta_b} + \sqrt{\delta_c} \le \sqrt{\frac{2R}{3}} \cdot \frac{3\sqrt{3}}{2} = 3\sqrt{\frac{R}{2}}$, from which the second inequality follows immediately.

In both inequalities, equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MIHAELA BLANARIU, Columbia College Chicago, Chicago, IL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; MARIAN DINCĂ, Bucharest, *Romania; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

It is worth noting that the proposed inequality implies Euler's inequality, $2r \le R$.

3533. [2010 : 172, 174] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Let m and n be positive real numbers satisfying $6m \le 5n$. Prove that

$$\frac{ma+nbc}{b+c} + \frac{mb+nca}{c+a} + \frac{mc+nab}{a+b} \le \frac{3m+n}{2}.$$

Solution to a corrected version of the problem by the proposer.

We prove instead that $\dfrac{ma+nbc}{b+c} + \dfrac{mb+nca}{c+a} + \dfrac{mc+nab}{a+b} \ge \dfrac{3m+n}{2}$ under the assumption that a, b, c, m, n are all positive real numbers with $a+b+c=1$ and $6m \ge 5n$.

By the AM–GM inequality , we have

$$\frac{ma+nbc}{b+c} + \frac{9(ma+nbc)(b+c)}{4} \ge 3(ma+nbc).$$

Adding the above inequality to its two cyclic variants and using Schur's inequality $abc \ge \frac{1}{9}[4(ab+bc+ca)-1]$, we obtain

$$\begin{aligned}
&\frac{ma+nbc}{b+c} + \frac{mb+nca}{c+a} + \frac{mc+nab}{a+b} \\
&\ge 3m + 3n(ab+bc+ca) \\
&\quad - \left[\frac{9(ma+nbc)(1-a)}{4} + \frac{9(mb+nca)(1-b)}{4} + \frac{9(mc+nab)(1-c)}{4}\right] \\
&= \frac{3}{4}m + \frac{1}{4}\left[3n(ab+bc+ca) + 9m(a^2+b^2+c^2) + 27nabc\right] \\
&= \frac{3}{4}m + \frac{1}{4}\left[3n(ab+bc+ca) + 9m(1-2ab-2bc-2ca) + 27nabc\right] \\
&\ge 3m - \frac{3}{4}n + \frac{1}{4}(15n-18m)(ab+bc+ca) \\
&\ge 3m - \frac{3}{4}n + \frac{1}{12}(15n-18m)(a+b+c)^2 = \frac{3m+n}{2}.
\end{aligned}$$

Counterexamples to the original problem given by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO

PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Geupel and Perfetti gave the counterexample $a = b = \frac{1}{4}$, $c = \frac{1}{2}$, $m = 1$, $n = 2$ *to the original problem.*

The proposer's original wording of the problem was almost identical to the corrected version of the problem stated above, except *that* m, n *were separately given as* negative *in the proposer's version.*

3534. [2010 : 172, 175] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, where $n \geq 2$, and let $\alpha \geq 1$. Prove that

$$(n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha}\right)\left(\sum_{k=1}^{n} x_k\right)^{\alpha}$$
$$\geq\ 2^{\alpha}\left(\sum_{1\leq i<j\leq n} x_i x_k\right)^{\alpha} + (n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha+1}\right)\left(\sum_{k=1}^{n} x_k\right)^{\alpha-1}.$$

Solution by Michel Bataille, Rouen, France.

By homogeneity, we may suppose that $\sum\limits_{k=1}^{n} x_k = 1$. The inequality to be proved becomes

$$(n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha}\right) \geq \left(2\sum_{1\leq i<j\leq n} x_i x_j\right)^{\alpha} + (n-1)^{\alpha-1}\left(\sum_{k=1}^{n} x_k^{\alpha+1}\right),$$

that is,

$$(n-1)^{\alpha-1}\left(\sum_{k=1}^{n}(1-x_k)x_k^{\alpha}\right) \geq \left(\sum_{i\neq j} x_i x_j\right)^{\alpha}. \qquad (1)$$

On the one hand,

$$\sum_{i\neq j} x_i x_j = \sum_{i=1}^{n} x_i\left(\sum_{1\leq j\leq n, j\neq i} x_j\right) = \sum_{i=1}^{n} x_i(1-x_i)$$

and on the other hand, since $\sum\limits_{i=1}^{n}(1-x_i) = n-1$ and $x \mapsto x^{\alpha}$ is convex, Jensen's inequality yields

$$(n-1)^{\alpha-1}\left(\sum_{k=1}^{n}(1-x_k)x_k^{\alpha}\right) \geq (n-1)^{\alpha-1}\cdot(n-1)\left(\sum_{k=1}^{n}\frac{(1-x_k)x_k}{n-1}\right)^{\alpha}$$
$$= \left(\sum_{i=1}^{n} x_i(1-x_i)\right)^{\alpha}.$$

The inequality (1) immediately follows.

Also solved by RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3535. [2010 : 172, 175] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, and c be positive real numbers and let $\alpha \geq 0$. Prove that

$$\left(\frac{a^2+bc}{b+c}\right)^{\alpha} + \left(\frac{b^2+ca}{c+a}\right)^{\alpha} + \left(\frac{c^2+ab}{a+b}\right)^{\alpha} \geq 3^{1-\alpha}(a+b+c)^{\alpha}\,.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

We will prove that the statement is false for $0 < \alpha \leq 0.13432$, however it hold true for $\alpha \geq 0.5$. The open question is what is the smallest positive α for which the inequality holds true.

We will prove first the following result: If the inequality holds for some $\alpha = A$ and p is any real number so that $p > 1$ then the inequality also holds for $\alpha = Ap$:

Indeed, by the Hölder inequality

$$\left[\left(\frac{a^2+bc}{b+c}\right)^{Ap} + \left(\frac{b^2+ca}{c+a}\right)^{Ap} + \left(\frac{c^2+ab}{a+b}\right)^{Ap}\right]^{\frac{1}{p}} 3^{1-\frac{1}{p}}$$
$$\geq \left(\frac{a^2+bc}{b+c}\right)^{A} + \left(\frac{b^2+ca}{c+a}\right)^{A} + \left(\frac{c^2+ab}{a+b}\right)^{A}\,.$$

Thus, since the inequality holds for $\alpha = A$, we get

$$\left[\left(\frac{a^2+bc}{b+c}\right)^{Ap} + \left(\frac{b^2+ca}{c+a}\right)^{Ap} + \left(\frac{c^2+ab}{a+b}\right)^{Ap}\right]^{\frac{1}{p}} 3^{1-\frac{1}{p}}$$
$$\geq 3^{1-A}(a+b+c)^{A}\,,$$

or equivalently

$$\left(\frac{a^2+bc}{b+c}\right)^{Ap} + \left(\frac{b^2+ca}{c+a}\right)^{Ap} + \left(\frac{c^2+ab}{a+b}\right)^{Ap} \geq 3^{1-Ap}(a+b+c)^{Ap}\,,$$

which completes our claim.

Problem **3437** [2009 : 174, 176; 2010 : 190-191], showed that the statement holds true for $\alpha = 0.5$, thus by our argument it holds true for $\alpha \geq 0.5$.

An easy computation shows that the statement fails for $a = 1.6\cdot 10^{-6}$; $b = 0.73$; $c = 0.049$ and $\alpha = 0.13432$. Thus, again by the above argument, the inequality cannot hold for $0 < \alpha \leq 0.13432$.

The question of finding the smallest positive α for which the inequality holds true is still open.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. There was also one incorrect solution.

The original problem was an "open problem", the star was missing. The question of finding the smallest α is still open.

All solvers showed that the inequality holds if α is not small (≥ 0.5 for Stadler, Apostolopoulos and Geupel, ≥ 1 for Perfetti) and fails for α small enough. Geupel and Apostolopoulos proved that there exists a small α for which there is a counterexample and Perfetti found that $\alpha = 0.1$ works.

3536. [2010 : 173, 175] *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Find all positive integers n and k such that the equation $\{x^{2n}\} = \{x\}$ has 2010 roots inside the interval $[k, k+1)$, where $\lfloor x\rfloor$ is the greatest integer not exceeding x and $\{x\} = x - \lfloor x\rfloor$.

Solution by George Apostolopoulos, Messolonghi, Greece, modified by the editor.

Observe that $\{x^{2n}\} = \{x\}$ if and only if $f(x) = x^{2n} - x$ is an integer.

Since $x \in [k, k+1)$ and $k \geq 1$, we see that $f(x) = x(x^{2n-1} - 1)$ is an increasing function of x on this interval, as it is the product of two nonnegative and increasing functions.

Therefore, since f is continuous, the number of roots of the original equation is the number of integers in the interval $[f(k), f(k+1)) = [k^{2n} - k, (k+1)^{2n} - k - 1)$, which is $(k+1)^{2n} - k^{2n} - 1$.

Thus, we seek all positive integers k, n such that $(k+1)^{2n} - k^{2n} - 1 = 2010$, or $(k+1)^{2n} - k^{2n} = 2011$. Now, 2011 is a prime number, so by factoring a difference of squares we deduce that $(k+1)^n - k^n = 1$ and $(k+1)^n + k^n = 2011$, and hence $k^n = 1005 = 3\cdot 5\cdot 67$.

Finally, since 1005 contains primes which only divide it to the first power, we see that $(k, n) = (1005, 1)$ is the only solution.

Also solved by VICTOR ARNAIZ and PEDRO A. CASTILLEJO, students, Universidad Complutense de Madrid, Madrid, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3537. [2010 : 173, 175] *Proposed by Marian Marinescu, Monbonnot, France.*

Let $f : [0,1] \to \mathbb{R}$ be continuous, and let $g : [0,1] \to \mathbb{R}$ be monotonic and differentiable with $g(0) = 0$. Prove that there is a number $0 < a < 1$ such that

$$\int_0^a f(x)g(x)\,dx \;=\; \left(\int_0^1 f(x)\,dx\right)\left(\int_0^a g(x)\,dx\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland.

Let $h(x) = f(x) - \int_0^1 f(t)dt$. Then h is continuous on $[0,1]$ and $\int_0^1 h(x)dx = 0$.

It suffices to prove that there exists an $\alpha \in (0,1)$ such that

$$\int_0^\alpha h(x)g(x)dx = 0\,.$$

By eventually replacing g with $-g$, without loss of generality we can assume g is non-decreasing.

Let $F(x) := \int_0^x h(t)dt$ and $H(x) = \int_0^x h(t)g(t)dt$. Then F, H are continuously differentiable on $[0,1]$.

We need to prove that $H(\alpha) = 0$ for some $0 < \alpha < 1$.

Lets note that if F is constant then $F \equiv 0$, and hence $h \equiv 0$, which solves the problem. So we can assume that F is not constant. Also, if g is constant, then $g \equiv 0$, and the problem is also trivial. Thus we can also assume that g is not constant.

Integration by parts yields

$$\begin{aligned} H(x) = F(t)g(t)|_0^x - \int_0^x F(t)g'(t)dt = F(x)g(x) - \int_0^x F(t)g'(t)dt \\ = \int_0^x F(x)g'(t) - F(t)g'(t)dt = \int_0^x [F(x) - F(t)]g'(t)dt \end{aligned} \tag{1}$$

Since F is continuous on $[0,1]$ it attains an absolute max and and absolute min. Also, since $F(0) = F(1) = 0$, and F is not constant, at most one of them can occur at an endpoint of the interval.

We break now the problem in two cases:

First case: F attains one of the extremum only at the end points of the interval $[0,1]$.

By eventually replacing h by $-h$, we can assume in this case without loss of generality that F attains its absolute minimum only at 0 and 1. This means that $F(x) > 0$ for all $x \in (0,1)$.

Let a be so that $F(a)$ is the absolute max of F on $[0,1]$. Since F is not constant, we get that $F(a) > 0$ and $a \in (0,1)$.

Since g is non-constant and non-decreasing, and $F(t) > 0$ for all $t \in (0,1)$ we get

$$H(1) = \int_0^1 [F(1) - F(t)]g'(t)dt = -\int_0^1 F(t)g'(t)dt < 0\,,$$

and

$$H(a) = \int_0^a [F(a) - F(t)]g'(t)dt \geq 0\,.$$

If $H(a) = 0$ we are done, while if $H(a) > 0$, by the Intermediate Value Theorem we have $H(c) = 0$ for some $c \in (a, 1)$.

Second case: F attains both the absolute maximum and the absolute minimum inside $(0, 1)$.

Let $a, b \in (0, 1)$ so that $F(a), F(b)$ are the absolute maxima and minima of F.

Then, since $g'(t) \geq 0$ we have $H(a) \geq 0$ and $H(b) \leq 0$.

Then, by the Intermediate Value Theorem, $H(c) = 0$ for some c in the closed interval defined by a and b. Since $a, b \in (0, 1)$, we get that $c \in (0, 1)$, which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer .

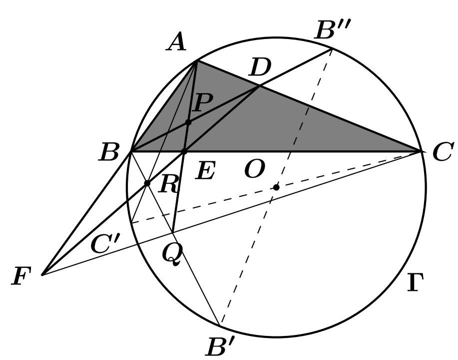
3538. [2010 : 173, 175] *Proposed by Victor Oxman, Western Galilee College, Israel.*

In the plane you are given a triangle ABC with its internal angle bisector BD, a point E on the side BC such that ED is the bisector of angle AEC, and the circumcircle of the triangle ABC (but not its centre). Construct the centre of that circle using only a straightedge.

[*Ed.: The Poncelet–Steiner Theorem says that given a circle with its centre, we can carry out all the ruler-compass constructions in the plane of that circle by straightedge only. See Crux problems 2694, 2695, and 2696* [*2002 : 553-557*].]

Combination of solutions by Michel Bataille, Rouen, France and Peter Y. Woo, Biola University, La Mirada, CA, USA.

The construction is based on classical properties of complete quadrangles. As shown in the figure, we define F to be the intersection of the given lines AB and ED, and consider the quadrangle $BFCD$. The diagonal points $A = FB \cap CD$ and $E = BC \cap FD$ determine a diagonal AE which meets the other two sides in points P on BD and Q on FC that are separated harmonically by the two diagonal points. Thus, the lines BP $(= BD)$ and BQ are separated harmonically by BE $(= BC)$ and BA. Because BP



bisects $\angle EBA$, it follows that $BQ \perp BP$. Extending BQ and BP to the second points B' and B'' where these lines meet the given circumcircle, call it Γ, $B'B''$ must be a diameter of Γ.

Apply the same argument to the quadrangle $BQCA$: $F = AB \cap QC$ and $E = BC \cap QA$ are diagonal points which separate harmonically the points R, where FE meets the side BQ, and D, where it meets AC. Thus AF and AE separate harmonically AR and AD. Because BD and ED bisect angles at B and E in triangle ABE (internally or externally depending on where E lies on the line BC), D must be a tritangent centre, whence AD is a bisector of $\angle A$ and, therefore, $AD \perp AR$. Extending AD and AR to the points C and C' where they meet Γ, we see that CC' is another diameter of Γ. The centre of Γ is then the point where CC' intersects $B'B''$. Note that to construct the centre we had to draw only the five new lines: $FC, BQ, B'B'', AR$, and CC' (and to extend some of the given segments). The construction works as long as E is a point on the line BC distinct from B and C (not just restricted to the segment BC as required by the statement of the problem).

Also solved by the proposer.

Bataille contributed the idea of using harmonic conjugates, while Woo contributed the observation that AD is a bisector of $\angle A$ in ΔABE. From that bisector property we see that E can be located as the point where the line BC meets the reflection of the line AB in AD. This implies that $\angle BAC + \angle EAC = 180^\circ$ so that, as observed by the proposer, E is interior or exterior to the segment BC according as $\angle BAC$ is obtuse or acute. When that angle is 90°, then $E = B$ and our construction for the centre of Γ fails; nevertheless, the centre is still easily constructed by straightedge using a somewhat modified procedure.

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