Contributor Profiles: Arkady Alt

Arkady Alt was born in Odessa, Ukraine, by the Black Sea. His interest in mathematics began when he was in middle school. At first, his mother had to constantly demand persistence from him and fight his negligence when he was solving math problems, but since he was very interested in astronomy and physics he quickly realized that to really understand those subjects he had to know mathematics.

His professional career initially began in applied mathematics, focusing on engineering and economics, but in 1981 he switched to an educational career path and began teaching at High School No. 100 in Odessa, where he created a specialized math course that exposed students to advanced topics at the Olympiad and University level. This evolved into a special two-year program which united Mathematics and Computer Science into one course, unique in the country at the time. Many graduates from this program subsequently participated in and won national level Mathematics competitions, with some going into academia and beyond.

When the cold war ended he moved his family to Israel, where he taught children and adults ranging from high school to university and Olympiad level. Several of his students completed their high school math requirements early and enrolled in university prior to full high school graduation. Two of them won gold and silver medals at the IMO in 1995 – 1997.

At university Arkady Alt was drawn to Topology and Algebra. Soon after graduating he discovered Serge Lang's *Algebra*. Finally group theory, number theory, topology, and linear algebra were all yoked together and he was inspired to publish on those topics in conjunction with concrete problems. At his suggestion many students have read this book from cover to cover.

After his immigration to Israel he began working in the field of inequalities, where he saw amazing opportunities to apply a host of techniques. He delights in finding elegant solutions free of heavy machinery so he can share the problems with his students.

He now lives in San Jose, California, with his wife and two adult children, who were also raised to love mathematics. He still teaches and is currently working on a book on inequalities that will contain the methods he has developed over the years. He is an active problem solver and proposer in many current mathematics publications, where he often finds new problems for himself and his students.

His other interests include philosophy, history, music, and reading.

EDITORIAL

Václav Linek

This issue of ***CRUX with MAYHEM*** marks the return of Contributor Profiles, and there will be more coming up this year.

Since we are living in an electronic age and the possibilities for processing and sending information are virtually limitless, some feedback is in order for ***CRUX with MAYHEM*** contributors who are savvy in this area.

For Contributor Profiles, black and white photos of the contributor are good starting materials. If an electronic file is sent, then a high-resolution (at least 600 dpi) black and white encapsulated postscript file (EPS) with a bounding box is preferred. We can also accept colour JPG files, but these are harder to work with as the final result will be black and white.

We continue to accept all file formats, but some are easier to work with than others. Plain text (TXT) files, TEX, and LATEX files are preferred to DOCX files, as the latter leave "ghost characters" when they are copied into TEX or LATEX files. The optimal combination is a LATEX file with the PDF output. In general DVI and PS files are less preferable, since these will be converted to PDF files in any case and PS files take up a considerable amount of space. If you are sending a batch of files you may consider compressing them into a ZIP file to save space and keep them all together in your email.

Regarding email etiquette, please choose an informative subject heading that gives a good idea of what the email is about, rather than (say) hitting "reply" and generating a disconnected subject heading. For example, putting the number(s) of the ***CRUX with MAYHEM*** problem(s) you have just solved into the subject immediately alerts us as to what your email is about. If possible, please use plain Roman characters in your emails, since (for example) Greek, Cyrillic, or Chinese characters are often converted into question marks when they are received. Please type at least a few words in your email rather than leaving the body blank and letting us guess what is intended.

Choosing a good file name also helps, such as (again) putting the number(s) of the ***CRUX with MAYHEM*** problem(s) you have just solved into the file name. If you solve big batches of problems from a single issue then you may want to include the volume number, issue number, month, and year into the file name of a zipped file (for example: `CRUX_v35n4_May2009_solns.ZIP`).

We of course still love to receive your letters and exquisite handwritten mathematics on all sizes, shapes, and colours of paper!

Finally, we urge all problem solvers and proposers to start each solution or proposal on a new page, with the full name and affiliation of the contributor.

Václav Linek

SKOLIAD No. 123

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 August, 2010**. A copy of ***CRUX*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

This month's Skoliad contest is the City Competition of the Croatian Mathematical Society, 2009, Secondary Level, Grade 1. Our thanks go to Željko Hanjš, University of Zagreb, Croatia, for providing us with this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

Compétition 2009 de la Société mathématique croate
Niveau secondaire, première année

1. Réduire la fraction

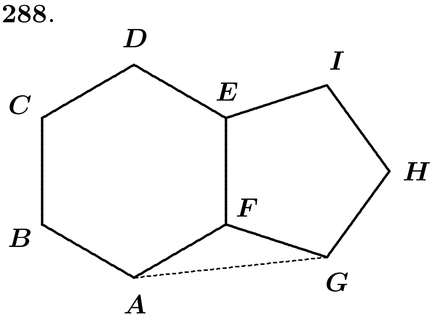
$$\frac{a^4 - 2a^3 - 2a^2 + 2a + 1}{(a+1)(a+2)}.$$

2. Si on écrit le chiffre 3 à gauche d'un entier à deux positions décimales, alors on obtient, bien sûr, un entier à trois positions décimales. Si le double de l'entier à trois positions égale 27 fois l'entier à deux positions, quel était l'entier à deux positions, au départ ?

3. Déterminer le plus grand entier n tel que $3\left(n - \frac{5}{3}\right) - 2(4n+1) > 6n+5$.

4. Déterminer le nombre de diviseurs de 288.

5. Dans la figure, $ABCDEF$ est un hexagone régulier tandis que $EFGHI$ est un pentagone régulier. Déterminer l'angle $\angle GAF$.



6. Dans un trapèze $ABCD$, l'angle à B est rectangle, et la diagonale BD est perpendiculaire au côté AD. Le côté BC est de longueur 5, tandis que la diagonale BD est de longueur 13. Déterminer la surface du trapèze $ABCD$.

7. Lors de la fête de Thérèse, le premier coup de sonnette annonça la première invitée. Ensuite, à chaque coup de sonnette, le nombre d'invitées qui se présentaient augmentait de deux. Si la sonnette a retentit n fois, combien d'invitées se sont-elles présentées à la fête ?

8. Déterminer tous les entiers positifs n tels que $n^2 - 440$ est le carré d'un entier.

City Competition of the Croatian Mathematical Society, 2009
Secondary Level, Grade 1

1. Reduce the fraction

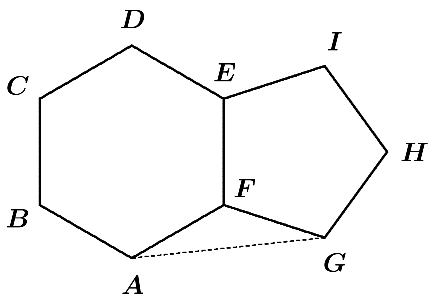
$$\frac{a^4 - 2a^3 - 2a^2 + 2a + 1}{(a+1)(a+2)}.$$

2. If you write the digit 3 on the left side of a two-digit number, you obtain, of course, a three-digit number. If twice the three-digit number equals 27 times the two-digit number, what is the original two-digit number?

3. Find the largest integer n such that $3\left(n - \frac{5}{3}\right) - 2(4n+1) > 6n + 5$.

4. Find the number of divisors of 288.

5. In the figure, $ABCDEF$ is a regular hexagon while $EFGHI$ is a regular pentagon. Determine the angle $\angle GAF$.



6. In a trapezoid $ABCD$, the angle at B is a right angle, and the diagonal BD is perpendicular to the leg AD. The length of the leg BC is 5, and the length of the diagonal BD is 13. Find the area of the trapezoid $ABCD$.

7. At Tihana's birthday party, the first guest arrived the first time the bell rang. Each time the bell rang thereafter the number of guests arriving was two more than the number that had arrived the previous time the bell rang. If the bell rang n times, how many guests attended the party?

8. Determine all positive integers n such that $n^2 - 440$ is the square of an integer.

Next follow solutions to the Calgary Mathematical Association Junior High School Mathematics Contest, Part B, 2009, that appeared previously in Skoliad 117 [2009 : 194–196].

1. Richard needs to go from his house to the park by taking a taxi. There are two taxi companies available. The first taxi company charges an initial cost of \$10.00, plus \$0.50 for each kilometre travelled. The second taxi company charges an initial cost of \$4.00, plus \$0.80 for each kilometre travelled. Richard realises that the cost to go to the park is the same regardless of which taxi company he chooses. What is the distance in km from his house to the park?

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let d denote the distance to the park (in kilometres). Then the cost (in cents) of using the first taxi company is $1000 + 50d$ while the cost of using the second company is $400 + 80d$. Since these two costs are equal, $1000 + 50d = 400 + 80d$, so $600 = 30d$, and, thus, $d = 20$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

2. A radio station runs a contest in which each winner will get to attend two Calgary Flames playoff games and to take one guest to each game. The winner does not have to take the same guest to the two games. Luckily, five school friends Alice, Bob, Carol, David, and Eva are all winners of this contest. Show how each winner can choose two others from this group to be his or her guests, so that each pair of the five friends gets to go to at least one playoff game together.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The table shows a possible choice of guests.

Winner	First guest	Second guest
Alice	Bob	Carol
Bob	Carol	David
Carol	David	Eva
David	Eva	Alice
Eva	Alice	Bob

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

3. A class was given two tests. In each test, each student was given a non-negative integer score with a maximum possible score of 10. Adrian noticed that in each test, only one student scored higher than he did and nobody got the same score as he did. But then the teacher posted the averages of the two scores for each student, and now there was more than one student with an average score higher than Adrian.

(a) Give an example (using exact scores) to show that this could happen.

(b) What is the largest possible number of students whose average score could be higher than Adrian's average score? Explain clearly why your answer is correct.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

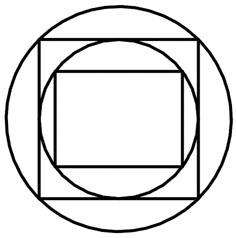
(a) The table shows a possible set of scores where two students have a higher average than Adrian.

	First Test	Second Test	Average
Adrian	7	5	6.0
Bob	6	9	7.5
Carol	10	4	7.0

(b) To get a better average than Adrian a student must score better than Adrian on at least one test. Since only one student scored better than Adrian on each of two tests, at most two students can have a higher average than Adrian.

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

4. A rectangle with dimensions 6 cm by 8 cm is drawn. A circle is drawn circumscribing this rectangle. A square is drawn circumscribing this circle. A second circle is drawn that circumscribes this square. What is the area in cm^2 of the bigger circle?



Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The diagonal of the rectangle is $\sqrt{6^2+8^2} = 10$ by the Pythagorean Theorem. But this diagonal is also a diameter of the inner circle, and the diameter of the inner circle equals the length of the side of the square. Thus the square has side length 10. By the Pythagorean Theorem again, the length of the diagonal of the square is $\sqrt{10^2+10^2} = \sqrt{200} = 10\sqrt{2}$. Since the diagonal of the square is also a diameter of the bigger circle, the radius of the bigger circle is $5\sqrt{2}$. It follows that the area of the bigger circle is $\pi\left(5\sqrt{2}\right)^2 = 50\pi$.

5. If A is a two-digit positive integer that does not contain zero as a digit, B is a three-digit positive integer, and $A\%$ of B is 400, find all possible values of A and B.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since $(A/100)B = 400$, it follows that $AB = 40000 = 2^6 \cdot 5^4$. Since A does not contain the digit zero, it must have factors of only 2 or only 5. Therefore, the possible two-digit values for A are 25, 16, 32, and 64.

Solving the equation for B yields that $B = 40000/A$, so (A, B) is one of $(25, 1600)$, $(16, 2500)$, $(32, 1250)$, and $(64, 625)$. Since B has three digits, the only solution is $(A, B) = (64, 625)$.

6. Find a rectangle with the following two properties: (i) its perimeter is an odd integer; and (ii) none of its sides is an integer.

Next, find a rectangle with the following two properties: (i) its area is an even integer; and (ii) none of its sides is an integer.

Finally, find a quadrilateral (not necessarily a rectangle) with the following three properties: (i) its perimeter is a positive integer; (ii) its area is a positive integer; and (iii) none of its sides is an integer.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The rectangle with sides $\frac{1}{4}$ and $\frac{5}{4}$ has perimeter 3. This solves the first part.

The rectangle with sides $2 + \sqrt{2}$ and $2 - \sqrt{2}$ has perimeter 8 and area

$$\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right) \;=\; 2^2 - \left(\sqrt{2}\right)^2 \;=\; 4-2 \;=\; 2\,.$$

This solves the last two parts.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The current editors of this section inherited the name *Skoliad*, and we have wondered about its origins. The best we have been able to reconstruct is this: Olympiad is a derivative of Mount Olympus; the Skoliad section features problems that are more accessible than Math Olympiad problems and is therefore named after a less exalted place, namely the town Skolos (or Scolus) as mentioned in *The Iliad*, second song, line 497.

The paragraph above prompted the proofreaders of this issue of Skoliad to inform us that the name originated with Richard K. Guy, who found Skolos as the name of a mountain on a map of Greece at the University of Calgary library.

We solicit information on the exact location of Skolos (which seems to be an open problem in Greek archaeology) as well as reader solutions to the featured contest.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 juin 2010**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M426. *Proposé par l'Équipe de Mayhem.*

Déterminer le nombre d'entiers positifs plus petits ou égaux à $1\,000\,000$ et qui soient divisibles par tous les entiers 2, 3, 4, 5, 6, 7, 8, 9 et 10.

M427. *Proposé par l'Équipe de Mayhem.*

Dans un demi-cercle de diamètre AB, on dessine un triangle équilatéral ABC. Trouver l'aire de la région comprise à l'intérieur du triangle mais à l'extérieur du demi-cercle.

M428. *Proposed by Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Trouver tous les entiers x pour lesquels on a

$$(4-x)^{4-x} + (3-x)^{3-x} + 20 \;=\; 4^x + 3^x\,.$$

M429. *Proposé par Samuel Gómez Moreno, Université de Jaén, Jaén, Espagne.*

Trouver tous les triplets (a, b, c) d'entiers positifs, avec $a^{(b^c)} = \left(a^b\right)^c$.

M430. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit p_n le n^{e} nombre premier. Montrer que $p_n > 3n$ pour tous les $n \geq 12$.

M431. *Proposé par Shailesh Shirali, École Rishi Valley, Inde.*

Dans un triangle acutangle ABC, soit D le pied de la perpendiculaire abaissée de A sur BC, E le pied de la perpendiculaire abaissée de D sur AC. Soit F un point sur le segment DE tel que $\frac{DF}{FE} = \frac{\cot C}{\cot B}$. Montrer que AF et BE sont perpendiculaires.

. . . .

M426. *Proposed by the Mayhem Staff.*

Determine the number of positive integers less than or equal to $1\,000\,000$ that are divisible by all of the integers 2, 3, 4, 5, 6, 7, 8, 9, and 10.

M427. *Proposed by the Mayhem Staff.*

A semicircle has diameter AB. Equilateral triangle ABC is drawn on the same side of AB as the semicircle. Determine the area of the region that lies inside the triangle and outside the semicircle.

M428. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all integers x for which

$$(4-x)^{4-x} + (3-x)^{3-x} + 20 \;=\; 4^x + 3^x\,.$$

M429. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Determine all triples (a, b, c) of positive integers with $a^{(b^c)} = \left(a^b\right)^c$.

M430. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let p_n be the n^{th} prime number. Prove that $p_n > 3n$ for all $n \geq 12$.

M431. *Proposed by Shailesh Shirali, Rishi Valley School, India.*

In acute triangle ABC, the foot of the perpendicular from A to BC is D, and the foot of the perpendicular from D to AC is E. Point F is located on line segment DE such that $\frac{DF}{FE} = \frac{\cot C}{\cot B}$. Prove that AF and BE are perpendicular.

Mayhem Solutions

M394. *Proposed by the Mayhem Staff.*

The numbers a, b, c, d, and e are five consecutive integers, in that order. Prove that the difference between the average of the squares of c and e and the average of the squares of a and c is equal to four times c.

Solution by all the solvers below indicated by a star.

We write the numbers a, b, c, d, and e as $n-2$, $n-1$, n, $n+1$, and $n+2$, respectively. Then

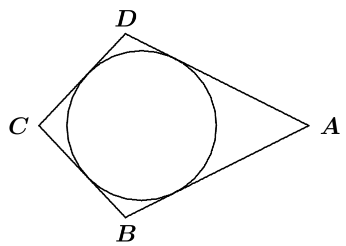
$$\begin{aligned}\frac{1}{2}\left(c^2+e^2\right)-\frac{1}{2}\left(a^2+c^2\right) &= \frac{1}{2}\left(e^2-a^2\right) \\ &= \frac{1}{2}\left((n+2)^2-(n-2)^2\right) \\ &= \frac{1}{2}(2n\cdot 4) \;=\; 4n \;=\; 4c\,,\end{aligned}$$

as required.

*Solved by *EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; *JACLYN CHANG, student, Western Canada High School, Calgary, AB; *RICHARD I. HESS, Rancho Palos Verdes, CA, USA; *HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; *RICARD PEIRÓ, IES "Abastos", Valencia, Spain; *BRUNO SALGUEIRO FANEGO, Viveiro, Spain; *JOSÉ JAIME SAN JUAN CASTELLANOS, student, Universidad technológica de la Mixteca, Oaxaca, Mexico; *JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; *GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; *OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.*

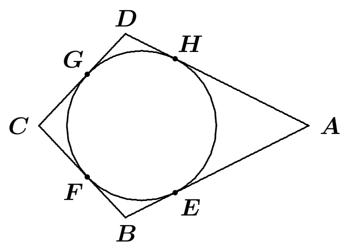
M395. *Proposed by the Mayhem Staff.*

The quadrilateral $ABCD$ is such that each of its sides is tangent to a given circle, as shown. If $AB = AD$, prove that $BC = CD$.



Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

In the diagram, $AB = AD$, and AB, BC, CD, DA are tangent to the circle at E, F, G, H, respectively.



Because of the theorem that says that the two tangents to a circle from a given exterior point have the same length, then $AE = AH$, $BE = BF$, $CF = CG$, and $DG = DH$.

Also, since $AB = AD$, then $AH + DH = AE + BE$. But $AH = AE$, so $DH = BE$. But $DG = DH$ and $BE = BF$, so $DG = BF$. Since $CF = CG$, then $BC = BF + CF = DG + CG = CD$, as required.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There was one incomplete solution submitted.

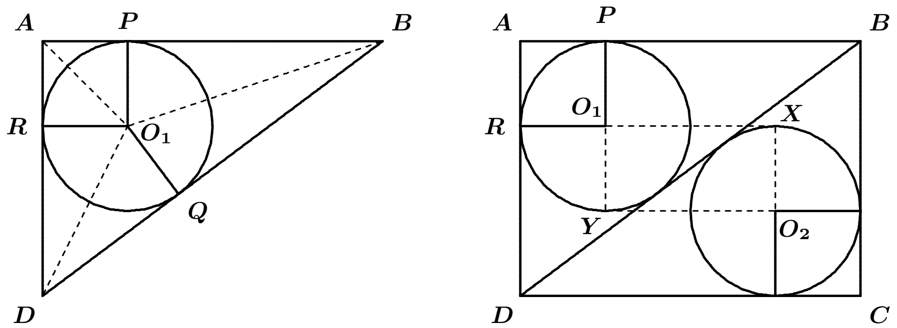
M396. *Proposed by the Mayhem Staff.*

The rectangle $ABCD$ has side lengths $AB = 8$ and $BC = 6$. Circles with centres O_1 and O_2 are inscribed in triangles ABD and BCD. Determine the distance between O_1 and O_2.

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

We know that $AD = CB$, $AB = CD$, and $BD = DB$. Hence $\triangle ABD$ is congruent to $\triangle BCD$. This means that the two incircles have equal radii.

Next, we find the radius, r, of these circles by finding the radius of the incircle of $\triangle ABD$ (see the first figure below). Connect O_1 to each of A, B, and D. Also, let points P, Q, and R on AB, BD, and DA, respectively, be the points of tangency of the circle to the sides of the triangle; connect O_1 to P, Q, and R.



Since $AD = 6$ and $AB = 8$, then by the Pythagorean Theorem, we have

$$BD = \sqrt{AD^2 + AB^2} = \sqrt{6^2 + 8^2} = 10 .$$

Also, the area of $\triangle ABD$ is $\frac{1}{2}(AD)(AB) = \frac{1}{2}(8)(6) = 24$. But this area also equals the sum of the areas of $\triangle AO_1B$, $\triangle BO_1D$, and $\triangle DO_1A$. Since O_1P, O_1Q, and O_1R are perpendicular to AB, BD, and DA, respectively, then these areas equal $\frac{1}{2}(AB)(O_1P)$, $\frac{1}{2}(BD)(O_1Q)$, and $\frac{1}{2}(DA)(O_1R)$, respectively. Therefore, $\frac{1}{2}(8r) + \frac{1}{2}(10r) + \frac{1}{2}(6r) = 24$, or $12r = 24$, and so $r = 2$.

Lastly, construct rectangle O_1XO_2Y with sides parallel to the sides of the original rectangle (see the second figure on the preceding page). Note that $O_1X = 8 - 2r$, since O_1 is r units from AD and O_2 is r units from BC. Thus, $O_1X = 4$. Similarly, $XO_2 = 6 - 2r = 2$. Therefore,

$$O_1O_2 \;=\; \sqrt{4^2 + 2^2} \;=\; \sqrt{20} \;=\; 2\sqrt{5}\,.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JACLYN CHANG, student, Western Canada High School, Calgary, AB; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were two incomplete solutions submitted.

M397. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Determine all pairs (x, y) of integers such that

$$x^4 - x + 1 \;=\; y^2\,.$$

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

We consider four cases: $x \le -1$, $x = 0$, $x = 1$, and $x \ge 2$.

If $x \le -1$, then $x < 1$, so $(x^2)^2 = x^4 < x^4 - x + 1$. Also, $x < 0$ and $2x + 1 \le -1 < 0$, so $x(2x + 1) > 0$, which yields $2x^2 > -x$ and

$$x^4 - x + 1 \;<\; x^4 + 2x^2 + 1 \;=\; \left(x^2 + 1\right)^2\,.$$

Therefore, $(x^2)^2 < x^4 - x + 1 < (x^2 + 1)^2$. Since $x^4 - x + 1$ is strictly between two consecutive perfect squares, then it cannot be a perfect square itself, so it cannot equal y^2 in this case.

If $x = 0$, then the equation becomes $y^2 = 1$, so $y = \pm 1$. This yields the solutions $(x, y) = (0, 1)$ and $(0, -1)$.

If $x = 1$, then the equation becomes $y^2 = 1$, so $y = \pm 1$. This yields the solutions $(x, y) = (1, 1)$ and $(1, -1)$.

If $x \ge 2$, then $x > 1$ so $x^4 - x + 1 < x^4 = (x^2)^2$. Also, $x(2x - 1) > 0$ so $-x > -2x^2$, which yields

$$x^4 - x + 1 \;>\; x^4 - 2x^2 + 1 \;=\; \left(x^2 - 1\right)^2\,.$$

Therefore, $(x^2 - 1)^2 < x^4 - x + 1 < (x^2)^2$. Since $x^4 - x + 1$ is again strictly between two consecutive perfect squares, then it cannot be a perfect square itself, so it cannot equal y^2 in this case.

This covers all possible cases. Therefore, the solutions are $(0, 1)$, $(0, -1)$, $(1, 1)$, and $(1, -1)$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were three incomplete solutions submitted.

M398. *Proposed by the Mayhem Staff.*

(a) The cubic equation $w^3 - bw^2 + cw - d = 0$ has roots r, s, and t. Determine b, c, and d in terms of r, s, and t.

(b) Suppose that a is a real number. Determine all solutions to the system of equations

$$\begin{aligned} x+y+z &= a\,, \\ xy+yz+zx &= -1\,, \\ xyz &= -a\,. \end{aligned}$$

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

(a) Since r, s, and t are the roots of $w^3 - bw^2 + cw - d = 0$, then we have

$$\begin{aligned} w^3 - bw^2 + cw - d &= (w-r)(w-s)(w-t)\,; \\ w^3 - bw^2 + cw - d &= w^3 - (r+s+t)w^2 + (rs+st+rt)w - rst\,. \end{aligned}$$

Since these cubics are equal for any value of w, the corresponding coefficients are equal, so we have $b = r+s+t$, $c = rs+st+rt$, and $d = rst$.

(b) Suppose that x, y, and z are the roots of the equation

$$m^3 - (x+y+z)m^2 + (xy+yz+xz)m - xyz = 0\,.$$

From the given information, this means that x, y, and z are the roots of $m^3 - am^2 - m + a = 0$, which can be rewritten as

$$\begin{aligned} m^2(m-a) - (m-a) &= 0\,; \\ (m^2-1)(m-a) &= 0\,; \\ (m-1)(m+1)(m-a) &= 0\,. \end{aligned}$$

Therefore, the roots are $m = 1$, $m = -1$, and $m = a$.

Therefore, the possible values of x, y, and z are 1, -1, and a. In order to satisfy the given equations, x, y, and z need to take all three of these values, in some order. Therefore, the possible triples (x,y,z) are $(1,-1,a)$, $(-1,1,a)$, $(a,1,-1)$, $(a,-1,1)$, $(1,a,-1)$, and $(-1,a,1)$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incomplete solutions submitted.

Zelator noted that some of these solutions are redundant when $a = 1$ or $a = -1$.

M399. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all triples (a,b,c) of positive integers for which $\dfrac{3ab-1}{abc+1}$ is a positive integer.

Solution by Oscar Xia, student, St. George's School, Vancouver, BC.

Suppose that $\frac{3ab-1}{abc+1} = n$, where n is a positive integer. Then we have $3ab - 1 = nabc + n$, or $3ab - nabc = n + 1$, or $ab = \frac{n+1}{3-nc}$.

Since a, b, c, and n are positive integers, then $3-nc$ is a positive integer (since ab is positive) so $(n, c) = (1, 1)$, $(2, 1)$, or $(1, 2)$.

If $(n, c) = (1, 1)$, then $ab = 1$ so $(a, b) = (1, 1)$.

If $(n, c) = (2, 1)$, then $ab = 3$ so $(a, b) = (3, 1)$ or $(1, 3)$.

if $(n, c) = (1, 2)$, then $ab = 2$ so $(a, b) = (2, 1)$ or $(1, 2)$.

Therefore, the five triples are $(a, b, c) = (1, 1, 1)$, $(3, 1, 1)$, $(1, 3, 1)$, $(2, 1, 2)$, $(1, 2, 2)$. (We can check that each triple satisfies the requirements.)

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEOFFREY A. KANDALL, Hamden, CT, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incorrect solutions and one incomplete solution submitted.

M400. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that a, b, and c are positive real numbers. In addition, suppose that $a^n + b^n = c^n$ for some positive integer n with $n \geq 2$. Prove that if k is a positive integer with $1 \leq k < n$, then a^k, b^k, and c^k are the side lengths of a triangle.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain, modified by the editor.

Suppose that $a^n + b^n = c^n$ for some positive integer $n \geq 2$. Since the numbers a, b, and c are positive, then $a < c$ and $b < c$.

Suppose that k is a positive integer with $1 \leq k < n$. To show that a^k, b^k, and c^k are the side lengths of a triangle, we need to prove three inequalities, namely we need to prove that $a^k + b^k > c^k$, that $a^k + c^k > b^k$, and that $b^k + c^k > a^k$.

Since $a < c$, then $b^k + c^k > a^k$. Since $b < c$, then $a^k + c^k > b^k$. It remains to prove that $a^k + b^k > c^k$.

Since $0 < a < c$ and $0 < b < c$ and $k - n < 0$, then $a^{k-n} > c^{k-n} > 0$ and $b^{k-n} > c^{k-n} > 0$. Therefore,

$$\begin{aligned} c^k &= c^{k-n}c^n \\ &= c^{k-n}(a^n + b^n) \\ &= c^{k-n}a^n + c^{k-n}b^n \\ &< a^{k-n}a^n + b^{k-n}b^n \\ &= a^k + b^k\,, \end{aligned}$$

as required.

Therefore, a^k, b^k, and c^k are the side lengths of a triangle.

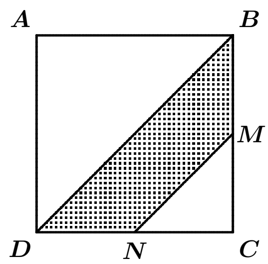
There were two incorrect solutions submitted.

Problem of the Month

Ian VanderBurgh

Sometimes seemingly unrelated problems have surprising connections.

Problem 1 (2001 Fermat Contest) In the diagram, square $ABCD$ has side length 2, with M the midpoint of BC and N the midpoint of CD. What is the area of the shaded region $BMND$?



Problem 2 (2001 Fermat Contest) A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?

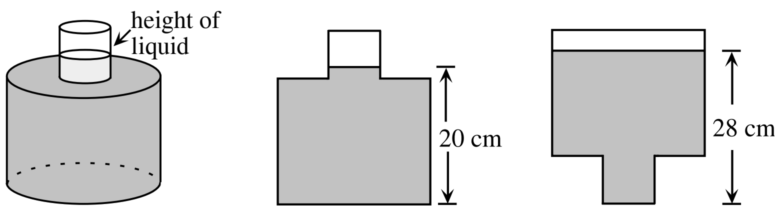


Figure A Figure B Figure C

These two problems have a more obvious connection and a less obvious connection. The superficial connection is pretty clear – they come from the same contest in the same year! But there is a more subtle connection.

Solution to Problem 1 We calculate the area of the region $BMND$ by calculating the area of the entire square $ABCD$ and subtracting the areas of triangles BAD and MCN.

Since the side length of square $ABCD$ is 2, then the area of square $ABCD$ is $2 \cdot 2 = 4$.

Triangle BAD is right-angled at A and has $BA = AD = 2$, so has area $\frac{1}{2} \cdot BA \cdot AD = \frac{1}{2} \cdot 2 \cdot 2 = 2$.

Triangle MCN is right-angled at C. Since M and N are the midpoints of BC and CD, then $MC = CN = 1$. Therefore, the area of triangle MCN is $\frac{1}{2} \cdot MC \cdot CN = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

Thus, the area of $BMND$ is $4 - 2 - \frac{1}{2} = \frac{3}{2}$. ∎

It's hard to come up with much of a connection between these problems without actually trying the second problem. Let's look at what to me seems like the most natural way of solving the second problem.

Solution 1 to Problem 2 Let the height of the larger cylinder be H cm and the height of the smaller cylinder be h cm. We want to determine the value of $H + h$. We do this by calculating the total volume of water in the bottle in two different ways, using Figure B and Figure C.

In Figure B, the water entirely fills the larger cylinder and partially fills the smaller cylinder. Put another way, we have H cm of water in the large cylinder and 20 cm of water in the bottle, so $(20 - H)$ cm of water in the small cylinder. The total volume of water in the bottle, in cubic centimetres, is thus $\pi \cdot 3^2 \cdot H + \pi \cdot 1^2 \cdot (20 - H) = 9\pi H + \pi(20 - H) = 20\pi + 8\pi H$.

Similarly, in Figure C, we have h cm of water in the small cylinder and 28 cm of water in the bottle, so we have $(28 - h)$ cm of water in the large cylinder. The total volume of water in the bottle, in cubic centimetres, is thus $\pi \cdot 1^2 \cdot h + \pi \cdot 3^2 \cdot (28 - h) = \pi h + 9\pi(28 - h) = 252\pi - 8\pi h$.

Equating these expressions for the amount of water in the bottle yields $20\pi + 8\pi H = 252\pi - 8\pi h$, hence $8\pi h + 8\pi H = 232\pi$ and $h + H = 29$.

Therefore, the total height of the bottle is 29 cm. ■

This solution was not all that difficult, except for a bit of thinking and a bit of algebra. We note that we didn't actually calculate h or H (nor could we have). This method of solving Problem 2 is the most natural in some sense, but it does not give us any hint as to the connection with the first problem.

So what is this elusive connection? In the first problem, we found the area of the shaded region by calculating the "missing area" and subtracting from the total area of the figure. We did not even consider trying to find the dimensions of the trapezoid (yes, $BMND$ is a trapezoid) and finding the area this way, since this method would be a fair bit more involved.

Can we apply this type of thinking to Problem 2? Yes, we can, by making a key observation. In Solution 1, we used the fact that the volume of water was the same in both configurations.

Solution 2 to Problem 2 Let x cm be the total height of the bottle. Then the height of the column of air in Figure B is $x - 20$ and the height of the column of air in Figure C is $x - 28$. Notice that the column of air in Figure B is a cylinder of radius 1 and in Figure C it is a cylinder of radius 3.

Since the volume of water is the same in Figure B as in Figure C, then the volume of air in each configuration is also the same. The volume of air in Figure B is $\pi \cdot 1^2 \cdot (x - 20)$ and the volume of air in Figure C is $\pi \cdot 3^2 \cdot (x - 28)$.

Therefore, $\pi(x - 20) = 9\pi(x - 28)$, or $9(28) - 20 = 9x - x$ and so $8x = 252 - 20 = 232$, whence $x = 29$. Hence, the total height of the cylinder is 29 cm. ■

This type of thinking is often really useful in mathematics. Often, we can solve a problem (or solve it more easily) by focussing on what is missing, rather than what is there.

THE OLYMPIAD CORNER

No. 284

R.E. Woodrow

We start this issue with the problems of the four Team Selection Tests for BMO 2007 and IMO 2007 of the Republic of Moldova. My thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

REPUBLIC OF MOLDOVA
Team Selection Tests for BMO 2007 and IMO 2007
First Test — 5 March 2007

1. In triangle ABC the points M, N, and P are the midpoints of the sides BC, AC, and AB, respectively. The lines AM, BN, and CP intersect the circumcircle of ABC at A_1, B_1, and C_1, respectively. Prove that the area of the triangle ABC does not exceed the sum of the areas of the triangles BA_1C, AB_1C, and AC_1B.

2. Let p be a prime number, $p \neq 2$, and m_1, m_2, $\dots$, m_p consecutive positive integers, and σ a permutation of the set $A = \{1, 2, \dots, p\}$. Prove that A contains two distinct numbers k, l such that $p \mid (m_k m_{\sigma(k)} - m_l m_{\sigma(l)})$.

3. Inside the triangle ABC there is a point T such that

$$\angle ATB = \angle BTC = \angle CTA = 120^\circ .$$

Prove that the Euler lines of the triangles ATB, BTC, ATC are concurrent.

4. Let $P = A_1A_2 \dots A_n$ be a convex polygon. For any point M in the interior, let B_i be the point where A_iM intersects the perimeter. We say that P is *balanced* if for some such M the points B_1, B_2, $\dots$, B_n are interior to distinct sides of P. Determine all n for which there exists a *balanced* polygon with n sides.

Second Test — 23 March 2007

5. Determine the smallest positive integers m and k such that

(a) there exist $2m + 1$ consecutive positive integers whose cubes sum to a perfect cube;

(b) there exist $2k + 1$ consecutive positive integers whose squares sum to a perfect square.

6. Let I be the incentre of triangle ABC and let R be its circumradius. Prove that $AI + BI + CI \leq 3R$.

7. Let U, V be two points inside the angle BAC such that $\angle BAU = \angle CAV$. Let X_1, X_2 be the projections of U onto the angle sides AC, AB; and let Y_1, Y_2 be the projections of V onto the angle sides AC, AB. Let the lines X_2Y_1 and X_1Y_2 interesect at W. Prove that U, V, W are collinear.

8. The convex hull of five points in the plane has area S. Prove that three of these points form a triangle of area not greater than $\left(\frac{5-\sqrt{5}}{10}\right) S$.

Third Test — 24 March 2007

9. Let a_1, a_2, $\ldots$, a_n ($n \geq 2$) be real numbers in the interval $[0, 1]$. Let $S = a_1^3 + a_2^3 + \cdots + a_n^3$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2n + 1 + S - a_i^3} \leq \frac{1}{3}.$$

10. Find all polynomials f with integer coefficients, such that $f(p)$ is a prime for every prime p.

11. Let ABC be a triangle with $a = BC$, $b = AC$, $c = AB$, inradius r, and circumradius R. Let r_A, r_B, and r_C be the radii of the excircles of the triangle ABC. Prove that

$$a^2\left(\frac{2}{r_A} - \frac{r}{r_Br_C}\right) + b^2\left(\frac{2}{r_B} - \frac{r}{r_Cr_A}\right) + c^2\left(\frac{2}{r_C} - \frac{r}{r_Br_A}\right) = 4(R + 3r).$$

12. Consider n distinct points in the plane, $n \geq 3$, arranged such that the number $r(n)$ of segments of length l is maximized. Prove that $r(n) \leq \frac{n^2}{3}$.

Fourth Test — 25 March 2007

13. Prove that the plane cannot be covered by the inner regions of finitely many parabolas.

14. Let b_1, b_2, $\ldots$, b_n ($n \geq 1$) be nonnegative real numbers at least one of which is positive. Prove that $P(X) = X^n - b_1X^{n-1} - \cdots - b_{n-1}X - b_n$ has a single positive root p, which is simple, and that the absolute value of each root of $P(X)$ is not greater than p.

15. A circle is tangent to the sides AB and AC of the triangle ABC and to its circumcircle at P, Q, and R respectively. Prove that if $PQ \cap AR = \{S\}$, then $\angle SBA = \angle SCA$.

16. Prove that there are infinitely many primes p for which there exists a positive integer n such that p divides $n! + 1$ and n does not divide $p - 1$.

Next we give selected problems of the Thai Mathematical Olympiad Examinations 2006. Thanks again go to Bill Sands, Canadian team Leader to the IMO in Vietnam, for collecting them for us.

THAI MATHEMATICAL OLYMPIAD EXAMINATIONS 2006
Selected Problems

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$f(x^2 + x + 3) + 2f(x^2 - 3x + 5) \;=\; 6x^2 - 10x + 17$$

for all real x. Find $f(85)$.

2. Evaluate

$$\sum_{k=84}^{8000} \binom{k}{84}\binom{8084-k}{84}.$$

3. Find all integers n such that $n^2 + 59n + 881$ is a perfect square.

4. Find the least positive integer n such that

$$\sqrt{3}z^{n+1} - z^n - 1 \;=\; 0$$

has a complex root z with $|z| = 1$.

5. Let p_k denote the k^{th} prime number. Find the remainder when

$$\sum_{k=2}^{2550} p_k^{p_k^4 - 1}$$

is divided by 2550.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{i=1}^{2005} f(x_i + x_{i+1}) + f\left(\sum_{i=1}^{2006} x_i\right) \;\le\; \sum_{i=1}^{2006} f(2x_i)$$

for all real numbers $x_1, x_2, \ldots, x_{2006}$.

7. A triangle has perimeter $2s$, inradius r, and the distances from its incentre to the vertices are s_a, s_b, and s_c. Prove that

$$\frac{3}{4} + \frac{r}{s_a} + \frac{r}{s_b} + \frac{r}{s_c} \;\le\; \frac{s^2}{12r^2}.$$

8. Let $\mathbb{N}$ be the set of positive integers. Is there a bijection $f : \mathbb{N} \to \mathbb{N}$ with the three properties below?

(a) $f(n + 2006) = f(n) + 2006$ for all $n \in \mathbb{N}$;

(b) $f(f(n)) = n + 2$ for $n = 1, 2, 3, \ldots, 2004$;

(c) $f(2549) > 2550$.

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9. Find all prime numbers p such that $\dfrac{2^{p-1} - 1}{p}$ is a perfect square.

10. In a school yard 229 boys and 271 girls are divided into 10 groups of 50 students each, with the students in each group numbered from 1 to 50. Four students are selected from two groups such that two pairs of students have identical numbers and the number of girls is odd. Show that the number of ways to select four students in this manner is odd.

11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + \cos(2007y)) = f(x) + 2007\cos(f(y))$$

for all real numbers x and y.

Next we give the problems of the Turkish Mathematical Olympiad 2006. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for providing these for our use.

14th TURKISH MATHEMATICAL OLYMPIAD 2006
December 16–17, 2006

1. Let E and F be points on the side CD of a convex quadrilateral $ABCD$ satisfying $0 < DE = FC < CD$. The circumcircles of triangles ADE and ACF intersect at $K \neq A$, and the circumcircles of triangles BDE and BCF intersect at $L \neq B$. Prove that A, B, K, and L are concyclic.

2. Find the largest real number t such that, in any school with 2006 students and 14 teachers where every student is acquainted with at least one teacher, a student and a teacher can be found such that they are acquainted and the ratio of the number of students who are acquainted with the teacher to the number of teachers who are acquainted with the student is at least t.

3. Find all positive integers n for which every coefficient of the polynomial

$$P_n(x) = (x^2 + x + 1)^n - (x^2 + x)^n - (x^2 + 1)^n - (x + 1)^n + x^{2n} + x^n + 1$$

is divisible by 7.

4. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be positive real numbers satisfying the relation $t = a_1 + a_1 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2$. Prove that

$$\sum_{i \neq j} \frac{a_i}{a_j} \geq \frac{(n-1)^2 t}{t-1} .$$

5. Let A_1, B_1, C_1 be the feet of the altitudes from vertices A, B, C in acute triangle ABC, respectively, and let O_A, O_B, O_C be the incentres of the triangles AB_1C_1, BC_1A_1, CA_1B_1, respectively. Let T_A, T_B, T_C be the points of tangency of the incircle of ABC to the sides BC, CA, AB, respectively. Show $T_AO_CT_BO_AT_CO_B$ is a *regular* hexagon.

6. Prove that there exists no triangle whose side lengths, area, and angles (measured in degrees) are all rational numbers.

As a final set for this number we give the Turkish Team Selection Test for the IMO 2007. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for us.

TURKISH TEAM SELECTION TEST FOR IMO 2007
March 24–25, 2007

1. An airline company is planning to run two-way flights between some of the six cities A, B, C, D, E, and F. Determine the number of ways these flights can be arranged so that it is possible to travel between any two of these six cities using only the flights of this company.

2. Let A and B be distinct points on a circle Γ. For a variable point P on Γ distinct from A and B, find the locus of the point M such that PM is the opposite ray to the angle bisector of $\angle APB$ and $MP = AP + PB$.

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ca + 2b^2 + 2b} \geq \frac{1}{ab + bc + ca} .$$

4. The acute triangle ABC is similar to the triangle $A_1B_1C_1$ whose vertices B_1, C_1, A_1 lie on the rays AC, BA, CB, respectively. Prove that the orthocentre of $A_1B_1C_1$ is the circumcentre of ABC.

5. Determine all positive odd integers n for which there exist odd integers $x_1, x_2, \ldots, x_n$ such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = n^4 .$$

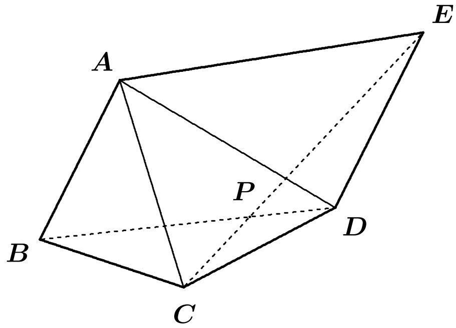
6. In how many ways can the numbers 1 and -1 be assigned to the unit squares of a 2007×2007 chessboard so that the absolute value of the sum of the numbers in any square made up from the unit squares of the chessboard does not exceed 1?

Next we return to solutions from our readers to problems proposed but not used at the 47th International Mathematical Olympiad 2006 in Slovenia given at [2008: 461–464].

G2. Let $ABCDE$ be a convex pentagon such that

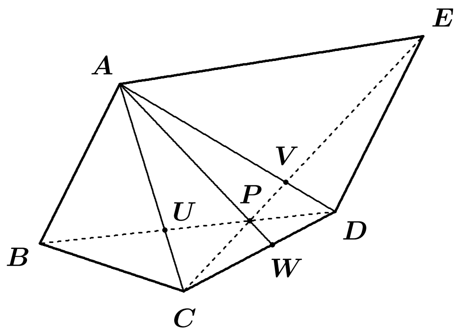
$$\angle BAC = \angle CAD = \angle DAE\,;$$
$$\angle ABC = \angle ACD = \angle ADE\,.$$

The diagonals BD and CE intersect at P. Prove that the line AP bisects the side CD.



Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's version.

Let $\mathcal{S}$ denote the direct similarity with centre A transforming B into C. From the hypotheses, we have $\mathcal{S}(C) = D$ and $\mathcal{S}(D) = E$. Let U be the point of intersection of the line segments AC and BD. Since $\mathcal{S}(AC) = AD$ and $\mathcal{S}(BD) = CE$, the image V of U under $\mathcal{S}$ is the point of intersection of AD and CE. It follows that



$$\frac{UA}{UC} = \frac{VA}{VD}\,. \tag{1}$$

Now, if AP meets CD at W, we have from Ceva's theorem

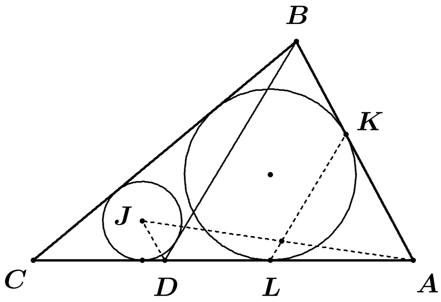
$$\frac{AU}{UC} \cdot \frac{DV}{VA} \cdot \frac{CW}{WD} = 1$$

and using (1), it follows that $\dfrac{CW}{WD} = 1$. This means that W is the midpoint of CD, so the proof is complete.

G3. A point D is chosen on the side AC of a triangle ABC with

$$\angle ACB < \angle BAC < 90^\circ$$

in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.



Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel's write-up.

It is a basic fact that, if the incircle of $\triangle PQR$ is tangent to the side PQ at the point T, then $2PT = PQ + PR - QR$. (See, for example, H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, The Mathematical Association of America, 1967, Theorem 1.4.1, page 11.)

Let L' be the point on the side AC such that $JL' \| LK$. Denote the point of intersection of AJ and KL by M. Let the incircle of $\triangle BCD$ meet CD at point T. Because J is on the internal bisector of $\angle BDC$, we have that $\angle JDL' = 90^\circ - \frac{1}{2}\angle ADB = \angle DL'J$; hence $DL' = 2DT$. Using the basic fact above, we obtain

$$DL' = 2DT = BD + CD - BC = AB + CD - BC$$

and

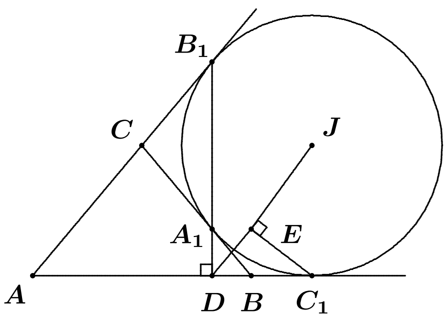
$$2AL = AB + AC - BC\,.$$

Consequently,

$$\begin{aligned} AL' &= AC - CD + DL' \\ &= AC - CD + (AB + CD - BC) = 2AL\,. \end{aligned}$$

We conclude that $\dfrac{AJ}{AM} = \dfrac{AL'}{AL} = 2$, which completes the proof.

G4. In triangle ABC, let J be the centre of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.



Solution by Titu Zvonaru, Comănești, Romania.

As usual we write $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. It is known that the following hold: $BA_1 = BC_1 = s - c$, $AB_1 = AC_1 = s$, and $CA_1 = CB_1 = s - b$. Since $B_1D \perp AB$, we have

$$AD = s\cos A\,, \qquad DB = (s-c)\cos B\,.$$

By Menelaus' theorem we obtain

$$\begin{aligned}\frac{B_1C}{B_1A}\cdot\frac{DA}{DB}\cdot\frac{A_1B}{A_1C} = 1 &\iff \frac{s-b}{s}\cdot\frac{s\cos A}{(s-c)\cos B}\cdot\frac{s-c}{s-b} = 1\\ &\iff \cos A = \cos B\,,\end{aligned}$$

hence the given triangle ABC is isosceles with $\angle A = \angle B$.

We denote by h the altitude from C to the line AB, and by $[ABC]$ the area of $\triangle ABC$. We now have the following calculations and deductions:

$$\begin{aligned}
JC_1 &= \frac{[ABC]}{s-a} = \frac{ch}{c} = h\,; \qquad h^2 = a^2 - \frac{c^2}{4}\,.\\
DC_1 &= DB + BC_1 = (s-c)\cos B + (s-c)\\
&= (s-c)\left(1+\frac{c}{2a}\right) = \left(a-\frac{c}{2}\right)\left(a+\frac{c}{2}\right)\cdot\frac{1}{a} = \frac{h^2}{a}\,.\\
DJ^2 &= h^2 + \frac{h^4}{a^2} \implies DJ = \frac{h\alpha}{a}\,, \qquad \text{where } \alpha = \sqrt{a^2+h^2}\,.\\
DE &= \frac{DC_1^2}{DJ} = \frac{h^3}{a\alpha}\,; \qquad EJ = \frac{JC_1^2}{DJ} = \frac{ah}{\alpha}\,;\\
DE\cdot EJ\cdot DJ &= \frac{h^3}{a\alpha}\cdot\frac{ah}{\alpha}\cdot\frac{h\alpha}{a} = \frac{h^5}{a\alpha}\,.
\end{aligned}$$

By Stewart's theorem we obtain the following four deductions:

$$\begin{aligned}
&A_1D^2\cdot EJ - A_1E^2\cdot DJ + A_1J^2\cdot DE = EJ\cdot DJ\cdot DE\\
&\implies A_1E^2\cdot\frac{h\alpha}{a} = (s-c)^2\cdot\frac{h^2}{a^2}\cdot\frac{ah}{\alpha} + h^2\cdot\frac{h^3}{a\alpha} - \frac{h^5}{a\alpha}\\
&\implies A_1E^2 = \frac{(s-c)^2\cdot h^2}{a^2+h^2} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
&BD^2\cdot EJ - BE^2\cdot EJ + BJ^2\cdot DE = EJ\cdot DJ\cdot DE\\
&\implies BE^2\cdot\frac{h\alpha}{a} = (s-c)^2\cdot\frac{c^2}{4a^2}\cdot\frac{ah}{\alpha} + [(s-c)^2+h^2]\cdot\frac{h^3}{a\alpha} - \frac{h^5}{a\alpha}\\
&\implies BE^2 = \frac{(s-c)^2\cdot a^2}{a^2+h^2} \qquad (2)
\end{aligned}$$

$$\begin{aligned} AD^2 \cdot EJ - AE^2 \cdot DJ + AJ^2 \cdot DE &= DE \cdot EJ \cdot DJ \\ \implies AE^2 \cdot \frac{h\alpha}{a} &= s^2 \cdot \frac{c^2}{4a^2} \cdot \frac{ah}{\alpha} + (s^2+h^2) \cdot \frac{h^3}{a\alpha} - \frac{h^5}{a\alpha} \\ \implies AE^2 &= \frac{s^2a^2}{a^2+h^2} \end{aligned} \tag{3}$$

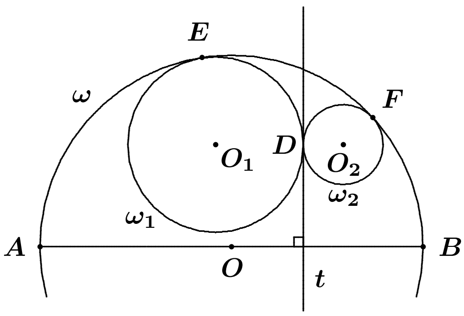
$$\begin{aligned} B_1D^2 \cdot EJ - B_1E^2 \cdot DJ + B_1J^2 \cdot DE &= DE \cdot EJ \cdot DJ \\ \implies B_1E^2 \cdot \frac{h\alpha}{a} &= s^2 \cdot \frac{h^2}{a^2} \cdot \frac{ah}{\alpha} + h^2 \cdot \frac{h^3}{a\alpha} - \frac{h^5}{a\alpha} \\ \implies B_1E^2 &= \frac{s^2h^2}{a^2+h^2}\,. \end{aligned} \tag{4}$$

It follows from (1)-(4) that

$$A_1E^2 + BE^2 = BA_1^2 \qquad \text{and} \qquad AE^2 + B_1E^2 = AB_1^2\,,$$

so by the converse of the Pythagorean Theorem, $\angle BEA_1 = \angle AEB_1 = 90^\circ$.

G5. Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, and O_1 are on the same side of t. Prove that the lines AO_1, BO_2, EF, and t are concurrent.



Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We first give the solution of Geupel.

Let O be the centre of ω. The triangles O_1ED and OEB are isosceles with $O_1E \| OE$ and $O_1D \| OB$; hence $ED \| EB$; which means that points B, D, and E are collinear. Similarly, A, D, and F are collinear. Denote by G the intersection of AO_1 with BO_2. We apply Pappus' theorem to the collinear points A, B, O, and the collinear points O_2, O_1, D, thus obtaining that the points $E = BD \cap OO_1$, $F = AD \cap OO_2$, and $G = AO_1 \cap BO_2$ are collinear.

It therefore remains to prove that G lies on the line t.

The line O_1D intersects ω_1 and ω_2 again at points H and I, respectively. The homothety with centre E that maps O_1 to O also maps ω_1 to ω and thus H to A. Therefore, H lies on the line AE. Similarly, I lies on BF. Denote by C the intersection of AE and BF. Since $AF \perp BC$ and

$BE \perp AC$ it follows that $D = AF \cap BE$ is the orthocentre of $\triangle ABC$ and t is the third altitude of $\triangle ABC$, which shows that C lies on t.

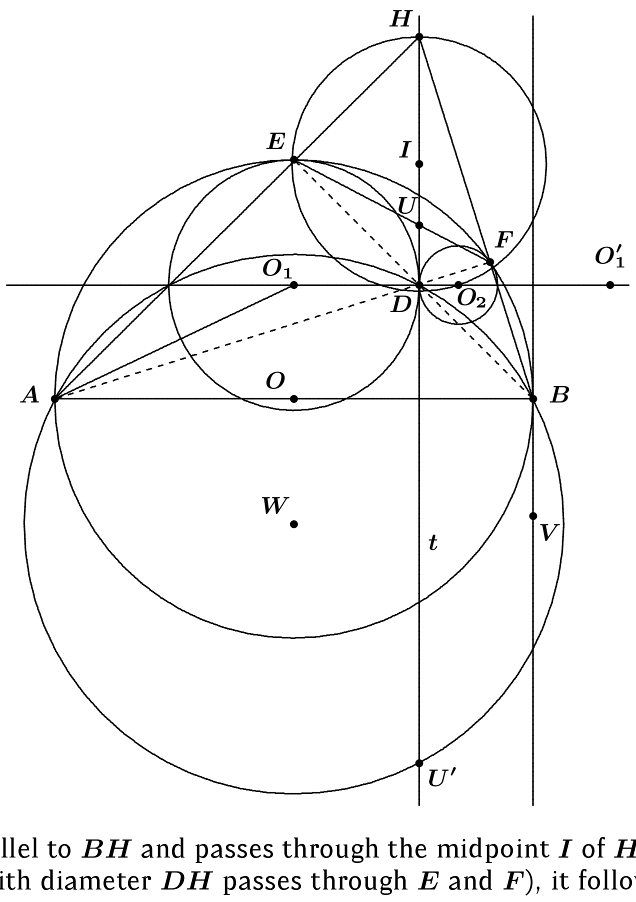
Let J denote the intersection of AB and t. Because the triangles HIC and ABC are homothetic,

$$\frac{DO_1}{DO_2} = \frac{DH}{DI} = \frac{JA}{JB}.$$

Consequently, the point G at the intersection of the lines AO_1 and BO_2 also lies on the line t, which completes the proof.

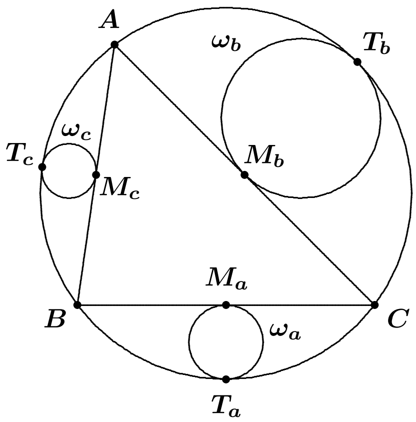
Next we give the solution of Bataille.

Since AB is a diameter of ω, we have $AF \perp FB$ and $AE \perp EB$. It follows that the point H of intersection of AE and BF is the orthocentre of $\triangle ADB$. This said, we shall make use of the inversion $\mathbf{I}$ with pole D such that $\mathbf{I}(\omega) = \omega$. We clearly have $\mathbf{I}(E) = B$ and $\mathbf{I}(F) = A$ so that the line EF is transformed into the circumcircle Γ of $\triangle ADB$. We denote by U the point of intersection of t and EF, W the centre of Γ, O the midpoint of AB, and $U' = \mathbf{I}(U)$ Note that U' is on t and Γ with U, U' on either side of D on t and that $\overrightarrow{DH} = 2\overrightarrow{WO}$.



Let t_B be the tangent to ω at B and O_1' be the reflection of D in t_B. From $\mathbf{I}(O_1) = O_1'$ (because $\mathbf{I}(\omega_1) = t_B$), we deduce that the image of the line AO_1 under $\mathbf{I}$ is the circle (DFO_1'). Let V be the centre of this circle. Clearly, V is on t_B and also on the perpendicular bisector of DF. Since the latter is parallel to BH and passes through the midpoint I of HD (note that the circle with diameter DH passes through E and F), it follows that $HIVB$ is a parallelogram. Thus $\overrightarrow{BV} = \overrightarrow{HI} = \overrightarrow{OW}$ and V is on the perpendicular to DU' through W, that is, on the perpendicular bisector of DU'. As a result, the circle (DFO_1') passes through U' and its inverse, and the line AO_1 passes through U. Similarly, the line BO_2 passes through U and we are done.

G6. In a triangle ABC, let M_a, M_b, M_c be the respective mid-points of the sides BC, CA, AB and let T_a, T_b, T_c be the mid-points of the arcs BC, CA, AB of the circumcircle of ABC not containing A, B, C, respectively. For each $i \in \{a,b,c\}$, let ω_i be the circle with diameter M_iT_i. Let p_i be the common external tangent to ω_j, ω_k such that $\{i,j,k\} = \{a,b,c\}$ and such that ω_i lies on one side of p_i while ω_j, ω_k lie on the other side. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.



Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $a = BC$, $b = CA$, $c = AB$, and $2s = a + b + c$. Let $F = [ABC]$ be the area of $\triangle ABC$ with circumradius R, circumcentre O, and incentre I.

Lemma. Lines p_a and M_bM_c are parallel, and their distance is $\dfrac{F(s-a)}{2sa}$.

Proof. For each $i \in \{a, b, c\}$, let O_i, r_i be the centre and the radius of ω_i. The perpendicular from O_b to BC meets M_bM_c and ω_b at N_b and P_b, respectively, with N_c, P_c defined similarly. Then $r_b = O_bT_b = \dfrac{R - OM_b}{2} = \dfrac{R(1-\cos B)}{2}$ and $O_bN_b = O_bM_b \cos C = r_b \cos C$; hence

$$\begin{aligned} N_bP_b &= r_b - O_bN_b = \frac{R(1-\cos B)(1-\cos C)}{2} \\ &= \frac{abc}{2F} \cdot \frac{(s-a)(s-c)}{ac} \cdot \frac{(s-a)(s-b)}{ab} = \frac{F(s-a)}{2sa}. \end{aligned}$$

Similarly, $N_cP_c = \dfrac{F(s-a)}{2sa}$. We conclude that $p_a = P_bP_c$. ∎

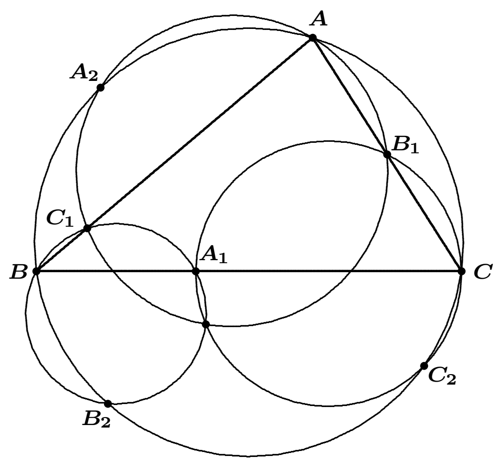
Corollary. The triangles $\triangle(p_a, p_b, p_c)$ and $M_aM_bM_c$ are homothetic with centre I and ratio 2. Thus, $\triangle(p_a, p_b, p_c) \sim \triangle ABC$ with ratio 4.

Proof. Denoting by $\overrightarrow{P}$ the position vector of point P, we have

$$\overrightarrow{I} = \frac{a\overrightarrow{A} + b\overrightarrow{B} + c\overrightarrow{C}}{a+b+c} = \frac{(s-a)\overrightarrow{M}_a + (s-b)\overrightarrow{M}_b + (s-c)\overrightarrow{M}_c}{s}.$$

Thus, $d(I, M_bM_c) = \frac{2[IM_bM_c]}{M_bM_c} = \frac{4(s-a)[M_aM_bM_c]}{sa} = 2d(p_a, M_bM_c)$, and by entirely similar calculations we have $d(I, M_cM_a) = 2d(p_b, M_cM_a)$ and $d(I, M_aM_b) = 2d(p_c, M_aM_b)$, where d denotes the Euclidean distance.

G8. Points A_1, B_1, C_1 are on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively (that is, $A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.



Solution by Oliver Geupel, Brühl, NRW, Germany.

Let us consider the situation in the plane of complex numbers, and let a, b, c, $\ldots$ denote the coordinates of the points A, B, C, $\ldots$. It is well-known (see Titu Andreescu, Dorin Andrica, *Complex numbers from A to... Z*, Birkhäuser, Boston, 2006, page 68) that triangles PQR and STU are similar (with the same orientation) if and only if $\frac{p-r}{q-r} = \frac{s-u}{t-u}$.

First we recognize inscribed angles on the circumcircles of $\triangle ABC$ and $\triangle AB_1C_1$. We have

$$\angle A_2BC_1 = \angle A_2BA = \angle A_2CA = \angle A_2CB_1$$

and

$$\begin{aligned}\angle A_2C_1B &= 180^\circ - \angle A_2C_1A \\ &= 180^\circ - \angle A_2B_1A = \angle A_2B_1C\,.\end{aligned}$$

Therefore, the (likewise) oriented triangles A_2BC_1 and A_2CB_1 are similar, which implies that

$$\frac{a_2 - c_1}{b - c_1} = \frac{a_2 - b_1}{c - b_1}\,.$$

We obtain $a_2 = \dfrac{bb_1 - cc_1}{b + b_1 - c - c_1}$. Similarly,

$$b_2 \;=\; \frac{cc_a a_1}{c + c_1 - a - a_1}\,, \qquad \text{and} \qquad c_2 \;=\; \frac{aa_1 - bb_1}{a + a_1 - b - b_1}\,.$$

Since A_3 is symmetric to A_1 with respect to the midpoint of BC, we have $a_3 - c = b - a_1$, and hence $a_3 = b + c - a_1$. Similarly $b_3 = c + a - b_1$ and $c_3 = a + b - c_1$.

By the characterization given above, it suffices to prove that

$$(a_2 - c_2)(b_3 - c_3) \;=\; (b_2 - c_2)(a_3 - c_3)\,.$$

But this can easily be verified by employing the relations above and clearing denominators. This completes the proof.

N1. Given $x \in (0,1)$ let $y \in (0,1)$ be the number whose n^{th} digit after the decimal point is the $(2^n)^{\text{th}}$ digit after the decimal point of x. Prove that if x is a rational number, then y is a rational number.

Solution by Oliver Geupel, Brühl, NRW, Germany.

For each positive integer n, let x_n be the n^{th} digit of x after the decimal point. Because x is rational, there exists a positive integer N such that the sequence $(x_n)_{n \geq N}$ is periodic with some period q. By the Pigeonhole Principle, there exist positive integers m and n such that $2^m > 2^n \geq N$ and $2^m \equiv 2^n \pmod{q}$. Then the sequence of the least nonnegative remainders $(2^k \bmod q)_{k \geq n}$ is periodic with a period not greater than $m - n$. Consequently, the sequence $(y_k)_{k \geq n} = (x_{2^k})_{k \geq n}$ is periodic; hence the decimal expansion of the number y is eventually periodic and therefore y is rational.

N2. For each positive integer n let

$$f(n) \;=\; \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right)\,,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(a) Prove that $f(n+1) > f(n)$ for infinitely many n.

(b) Prove that $f(n+1) < f(n)$ for infinitely many n.

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let $d(n)$ denote the number of positive divisors of the integer n. Observe that for each positive integer k, there are exactly $\left\lfloor\frac{n}{k}\right\rfloor$ numbers in the set $\{1, 2, \ldots, n\}$ which are divisible by k. Therefore,

$$\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor \;=\; d(1) + d(2) + \cdots + d(n)\,.$$

This implies that $f(n)$ is the arithmetic mean of the numbers of divisors of the first n positive integers.

Because the function d is unbounded, there are infinitely many numbers n, such that $d(n+1) > \max\{d(k) : 1 \le k \le n\}$. By rewriting this as

$$\begin{aligned} f(n+1) &= \frac{d(1)+d(2)+\cdots+d(n+1)}{n+1} \\ &> \frac{d(1)+d(2)+\cdots+d(n)}{n} = f(n) \end{aligned}$$

we have completed part (a).

For part (b), on the one hand since $f(6) = \frac{7}{3} > 2$ and $d(k) \ge 2$ for each $k > 1$, it follows that $f(n) > 2$ whenever $n > 5$. On the other hand, for each prime number $n+1$ we have $d(n+1) = 2$. Consequently, for any prime number $n+1 \ge 7$ we have $f(n+1) < f(n)$, and we have completed part (b).

N4. Let a and b be relatively prime integers with $1 < b < a$. Define the *weight* of an integer c, denoted by $w(c)$, to be the minimum possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by = c .$$

An integer c is called a *local champion* if $w(c) \ge \max\{w(c \pm a), w(c \pm b)\}$. Find all local champions and determine their number.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove a series of lemmas.

Lemma 1. Let x, y, and c be integers such that $y \ge 0$ and

$$ax + by = c . \tag{1}$$

Then the condition

$$w(c) = |x| + |y| \tag{2}$$

is equivalent to

$$|x+b| + |y-a| \ge |x| + y . \tag{3}$$

Proof: If (2) holds, then (3) follows immediately from the definition of the weight. Conversely, assume (3). Note that $y < a$. We are to show that for each integer n,

$$|x+nb| + |y-na| \ge |x| + y . \tag{4}$$

If $n > 0$, then (4) follows from

$$\begin{aligned} &|x+nb| + |y-na| \\ &\quad = |x+nb| + na - y \ge |x+nb| + (n-1)b + a - y \\ &\quad = |x+nb| + |(1-n)b| + |y-a| \ge |x+b| + |y-a| \ge |x| + y . \end{aligned}$$

If $n \leq 0$, then we derive (4) as follows:

$$\begin{aligned} |x+nb|+|y-na| &= |x+nb|-na+y \\ &\geq |x+nb|+|-nb|+y \ \geq\ |x|+y\,. \end{aligned} \qquad \blacksquare$$

Lemma 2. Let d and x be integers such that $-b+1 \leq d \leq b$. Then the conditions

$$|x+b| \ \geq\ |x|+d \tag{5}$$

and

$$2x+b \ \geq\ d \tag{6}$$

are equivalent. Also the conditions $|x+b| \leq |x|+d$ and $2x+b \leq d$ are equivalent.

Proof: If $x \leq -b$, then both (5) and (6) are false. If $-b < x < 0$, then $|x+b| = x+b$ and $|x| = -x$; hence (5) is equivalent to $x+b \geq -x+d$; that is to (6). Finally, if $x \geq 0$, then both (5) and (6) are satisfied. The proof of the second claim is similar. ■

Lemma 3. Let x, y, and c be integers satisfying (1). Then, (2) is satisfied if and only if one of the following holds:

(a) $-\left(\frac{a+b}{2}\right) \leq y < -\left(\frac{a-b}{2}\right)$ and $x \leq y + \frac{a+b}{2}$;

(b) $-\left(\frac{a-b}{2}\right) \leq y \leq \frac{a-b}{2}$;

(c) $\frac{a-b}{2} < y \leq \frac{a+b}{2}$ and $y \leq x + \frac{a+b}{2}$.

Proof: We can assume without loss of generality that $y > 0$, since we can switch from x, y, c to $-x$, $-y$, $-c$, whenever $y < 0$. Now, (2) is equivalent to (3) by Lemma 1. Again $y < a$ holds; hence (3) is equivalent to

$$|x+b| \ \geq\ |x|+2y-a\,. \tag{7}$$

If $y \leq \frac{a-b}{2}$, then $2y-a \leq -b$ and (7) is true. Next, if $\frac{a-b}{2} < y \leq \frac{a+b}{2}$, then by Lemma 2 inequality (7) is equivalent to $2x+b \geq 2y-a$. Finally, for $y > \frac{a+b}{2}$ no solution exists. ■

Lemma 4. Let x, y, c be integers satisfying (1), (2). Let $\frac{a-b}{2} < y \leq \frac{a+b}{2}$. Then c is a local champion if and only if $x < 0$ and

$$x + \left\lfloor \frac{a+b}{2} \right\rfloor \ =\ y\,. \tag{8}$$

Proof: If c is a local champion, then by Lemma 3

$$y \ \leq\ x + \frac{a+b}{2} \ <\ x+1+\frac{a+b}{2}\,.$$

Again by Lemma 3, we obtain $w(c+a) = |x+1| + y$. Because c is a local champion, it follows that $|x+1| \leq |x|$; hence $x < 0$. If $y \leq x - 1 + \frac{a+b}{2}$, then we obtain from Lemma 3 that $w(c-a) = |x-1| + y = w(c) + 1$, which contradicts the hypothesis that c is a local champion. Therefore,

$$x + \frac{a+b}{2} - 1 \; < \; y \; \leq \; x + \frac{a+b}{2}\,,$$

that is (8).

Conversely, it remains to prove that c is indeed a local champion. We show that $w(c \pm a) \leq w(c)$ and $w(c \pm b) \leq w(c)$.

First, $w(c+a) = |x+1| + y \leq |x| + y = w(c)$ holds.

Second, we have $c - a = a(x+b-1) + b(y-a)$, as well as

$$-\left(\frac{a+b}{2}\right) \; < \; y - a \; < \; -\left(\frac{a-b}{2}\right)$$

and $x + b - 1 \; \leq \; y - a + \frac{a+b}{2}$. By Lemma 3, it follows $w(c-a) = |x+b-1| + a - y$. From $x + \frac{a+b-2}{2} \leq y$, we obtain $2x + b - 1 \leq 2y - a$. The second part of Lemma 2 leads to $|s+b-1| \leq |x| + 2y - a$; thus

$$w(c-a) \; \leq \; |x+b-1| + a - y \; \leq \; |x| + y \; = \; w(c)\,.$$

Third, $c + b = a(x+b) + b(y+1-a)$; hence

$$w(c+b) \; \leq \; b + a - y - 1 \; = \; a + b - 1 - \left\lfloor \frac{a+b}{2} \right\rfloor \; \leq \; \left\lfloor \frac{a+b}{2} \right\rfloor \; = \; w(c)\,.$$

Fourth, $c - b = ax + b(y-1)$; thus

$$w(c-b) \; \leq \; |x| + y - 1 \; = \; w(c) - 1\,. \qquad \blacksquare$$

Lemma 5. Let x, y, and c be integers satisfying inequalities (1) and (2). If $-\left(\frac{a+b}{2}\right) \leq y < \frac{a-b}{2}$, then c is a local champion if and only if $x > 0$ and $x = y + \left\lfloor \frac{a+b}{2} \right\rfloor$. Moreover, there are no local champions c with $|y| \leq \frac{a-b}{2}$.

Proof: The first part follows from Lemma 4 by replacing x, y by $-x$, $-y$. Let $|y| \leq \frac{a-b}{2}$. We conclude by Lemma 3 that $w(c-a) = |x-1| + |y|$ and $w(c+a) = |x+1| + |y|$. Clearly, one of the numbers $|x-1|$ and $|x+1|$ is greater than $|x|$, which completes the proof of the second part. ■

Corollary 6. There are $b-1$ local champions if ab is odd and $2(b-1)$ local champions if ab is even.

Proof: We can describe the local champions explicitly. If ab is odd, then by Lemma 4 and Lemma 5 the set of local champions is

$$\left\{ \frac{(a+b)(2-b)}{2} + n(a+b) : 0 \leq n \leq b-2 \right\}\,.$$

If ab is even, then Lemma 4 and Lemma 5 yield the disjoint sets

$$M = \left\{ \frac{2a - ab + b - b^2}{2} + n(a+b) : 0 \le n \le b-2 \right\},$$
$$N = \left\{ \frac{2a - ab + b - b^2}{2} + n(a+b) + b : 0 \le n \le b-2 \right\},$$

respectively. The set M is an arithmetic progression with difference $a + b$, and the set N is the same progression with offset b. ∎

Next we turn to solutions from our readers to problems of the Swedish Mathematical Contest 2005/2006 given at [2008 : 464–465].

1. Find all solutions in integers x and y of the equation

$$(x + y^2)(x^2 + y) = (x + y)^3 .$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Amengual Covas.

The equation is equivalent to $xy + x^2y^2 = 3x^2y + 3xy^2$, which can be rewritten as $xy(1 + xy - 3x - 3y) = 0$. We observe that $(x, y) = (0, m)$, $(m, 0)$, where m is an integer, are solutions.

Now suppose that $xy \neq 0$. Then $1 + xy - 3x - 3y = 0$. Since no pairs (x, y) with $x = 3$ can satisfy this last equation, we rewrite it in the form $y = \dfrac{3x-1}{x-3}$, or $y = 3 + \dfrac{8}{x-3}$.

It follows that y is an integer if and only if $x - 3$ divides 8. Thus, $x - 3 \in \{\pm1, \pm2, \pm4, \pm8\}$, so that $x \in \{-5, -1, 1, 2, 4, 5, 7, 11\}$.

We conclude that the complete solution set for (x, y) is

$$\{(0, m), (m, 0) : m \in \mathbb{Z}\} \bigcup \{(2, -5), (-5, 2), (-1, 1), (1, -1), (4, 11), (11, 4), (5, 7), (7, 5)\} .$$

2. A queue in front of a counter consists of 12 persons. The counter is then closed because of a technical problem and the 12 people are redirected to another one. In how many different ways can the new queue be formed if each person maintains the same position as before, or is one step closer to the front, or is one step farther from the front?

Solution by Oliver Geupel, Brühl, NRW, Germany.

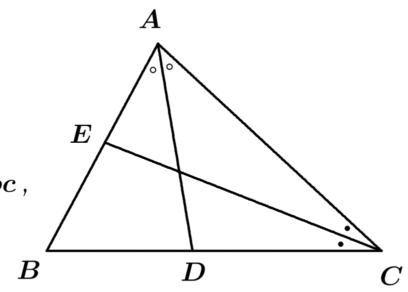
Let Q_n be the number of distinct permutations π of $\{1, 2, \dots, n\}$ such that for $1 \le k \le n$ we have $k - 1 \le \pi(k) \le k + 1$. For $n \ge 3$, if π has the desired property, then we have $\pi(n) \in \{n - 1, n\}$. First consider the case $\pi(n) = n - 1$. Then $\pi(n-1) = n$, and the numbers $1, 2, \ldots, n-2$ can be arranged in Q_{n-2} ways. Second, if $\pi(n) = n$, then the numbers $1, 2, \ldots, n-1$ can be arranged in Q_{n-1} ways. Hence, for $n \geq 3$, we have the recursion $Q_n = Q_{n-2} + Q_{n-1}$ with $Q_1 = 1$ and $Q_2 = 2$.

Therefore, Q_n is the $(n+1)^{\text{st}}$ Fibonacci number F_{n+1} (specifically, the solution of our exercise is $Q_{12} = F_{13} = 233$).

3. In the triangle ABC the angle bisector from A intersects the side BC in the point D and the angle bisector from C intersects the side AB in the point E. The angle at B is greater than $60°$. Prove that $AE + CD < AC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give the version of Apostolopoulos.

We have $\cos B < \frac{1}{2}$. By the Law of Cosines, $b^2 = a^2 + c^2 - 2ac \cdot \cos B$, so $\cos B = \dfrac{a^2+c^2-b^2}{2ac} < \frac{1}{2}$. Then



$$\begin{aligned} a^2 + c^2 &< ac + b^2\,, \\ bc + a^2 + c^2 + ab &< ab + ac + b^2 + bc\,, \\ c(b+c) + a(a+b) &< (a+b)(b+c)\,, \\ \frac{bc}{a+b} + \frac{ab}{b+c} &< b\,. \end{aligned}$$

From the Bisector Theorem, we have $AE = \dfrac{bc}{a+b}$ and $CD = \dfrac{ab}{b+c}$, hence the conclusion $AE + CD < AC$.

Comment by Zvonaru. This problem is part (c) of Problem 3 of the contest Trentième Olympiad Mathématique Belge, [2008 : 80].

4. The polynomial $f(x)$ is of degree four. The zeroes of f are real and form an arithmetic progression, that is, the zeroes are a, $a+d$, $a+2d$, and $a+3d$ where a and d are real numbers. Prove that the three zeroes of $f'(x)$ also form an arithmetic progression.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's version.

First, note that the zeroes of $f'(x)$ are also real numbers. This is obvious if $d = 0$ and follows from Rolle's theorem otherwise.

Let $x_k = a + kd$, $(k = 0, 1, 2, 3)$ and set $\lambda = \dfrac{x_0 + x_3}{2} = \dfrac{x_1 + x_2}{2}$. Then,

$$f(x) \;=\; m\prod_{k=0}^{3}(x - x_k) \;=\; m\left(x^2 - 2\lambda x + p\right)\left(x^2 - 2\lambda x + q\right)\,,$$

where m is a nonzero real number, $p = x_0x_3$, and $q = x_1x_2$. It readily follows that

$$f'(x) \;=\; 4m(x-\lambda)\left(x^2 - 2\lambda x + \frac{p+q}{2}\right) .$$

The zeroes of $f'(x)$ are λ and real numbers y_1, y_2 such that $y_1 + y_2 = 2\lambda$. It follows that y_1, λ, y_2 is an arithmetic progression, and the proof is complete.

5. Each square on a 2005×2005 chessboard is painted either black or white. This is done in such a way that each 2×2 "sub-chessboard" contains an odd number of black squares. Show that the number of black squares among the four corner squares is even. In how many different ways can the chessboard be painted so that the above condition is satisfied?

Solution by Oliver Geupel, Brühl, NRW, Germany.

Consider more generally integers $m \ge 2$, $n \ge 2$, and an $m \times n$ board represented by a matrix

$$\begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \cdots & \cdots & \cdots \\ s_{m1} & \cdots & s_{mn} \end{pmatrix}$$

where

$$s_{ij} \;=\; \begin{cases} 1 & \text{if square } (i,j) \text{ is painted black,} \\ 0 & \text{if square } (i,j) \text{ is painted white.} \end{cases}$$

Call the board *odd* if for each 2×2 sub-board

$$B_{ij} \;=\; \begin{pmatrix} s_{ij} & s_{i,j+1} \\ s_{i+1,j} & s_{i+1,j+1} \end{pmatrix}$$

the sum of its entries $b_{ij} = s_{ij} + s_{i,j+1} + s_{i+1,j} + s_{i+1,j+1}$ is odd, where $1 \le i \le m-1$ and $1 \le j \le n-1$.

Proposition 1. For each odd $m \times n$ board, if $2 \mid (m-1)(n-1)$, then the sum of its corner entries is even.

Proof: By hypothesis, the number

$$\begin{aligned} \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} b_{ij} \;&=\; 4\sum_{i=2}^{m-1}\sum_{j=2}^{n-1} s_{ij} \;+\; 2\sum_{i=2}^{m-1}(s_{i1}+s_{in}) \\ &\quad + 2\sum_{j=2}^{n-1}(s_{1j}+s_{mj}) \;+\; (s_{11}+s_{1n}+s_{m1}+s_{mn}) \end{aligned}$$

is even. Hence, the sum $s_{11} + s_{1n} + s_{m1} + s_{mn}$ is also even. ■

Proposition 2. For $m \ge 2$ and $n \ge 2$, the number of odd $m \times n$ boards is 2^{m+n-1}.

Proof: By induction on n.

For $n = 2$, there are eight ways to paint the topmost 2×2 sub-board. The rows $3, 4, \ldots, m$ can be successively painted, where we have two choices for each row. We obtain $8 \cdot 2^{m-2} = 2^{m+1}$ different painted boards.

In an $m \times n$ board the leftmost $n - 1$ columns form an $m \times (n - 1)$ board:

$$\begin{pmatrix} s_{11} & \cdots & s_{1,n-1} & s_{1n} \\ \vdots & & \vdots & \vdots \\ s_{m1} & \cdots & s_{m,n-1} & s_{mn} \end{pmatrix},$$

which can be painted in 2^{m+n-2} ways by the induction hypothesis. We now have two choices for s_{1n} and s_{2n}. Each further entry s_{in} in the last column is uniquely determined by the elements $s_{i-1,n-1}$, $s_{i-1,n}$, and $s_{i,n-1}$. We therefore obtain $2^{m+n-2} \cdot 2 = 2^{m+n-1}$ painted boards. ∎

Remark: For $m = n = 2005$ we obtain 2^{4009} painted boards.

That brings us to the start of the problems given in numbers of the *Corner* for 2009, and solutions from our readers to problems of the German Mathematical Olympiad, Final Round, Grades 12–13, Munich, April 29–May 2, 2006 given at [2009 : 21].

1. Determine all positive integers n for which the number

$$z_n = \underbrace{101\cdots101}_{2n+1 \text{ digits}}$$

is a prime.

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; Alex Song, student, Elizabeth Ziegler Public School, Waterloo, ON; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the write-up of Babbit, modified by the editor.

We have

$$\begin{aligned} z_n &= \sum_{i=0}^{n} 100^i = \frac{100^{n+1} - 1}{100 - 1} \\ &= \frac{10^{2n+2} - 1}{99} = \frac{\left(10^{n+1} + 1\right)\left(10^{n+1} - 1\right)}{99}. \end{aligned}$$

It is obvious that $9 \mid (10^{n+1} - 1)$ for all positive integers n. Note that 99 divides $(10^{n+1} + 1)(10^{n+1} - 1)$ because z_n is an integer, and 11 divides 99, hence 11 divides $(10^{n+1} + 1)(10^{n+1} - 1)$.

When 11 does not divide $10^{n+1} - 1$, then it divides $10^{n+1} + 1$, which is always greater than 11. Also, 9 divides $10^{n+1} - 1$, which is always greater than 9. Therefore, z_n is not prime in this case.

When 11 does divide $10^{n+1} - 1$, then we have that $10^{n+1} - 1$ is a multiple of 99. Since $10^{n+1} + 1$ is always greater than 1, z_n can only be prime when $10^{n+1} - 1 = 99$, or $n = 1$.

Therefore, z_n is not prime for $n > 1$, and $z_1 = 101$ is prime.

5. Let x be a nonzero real number satisfying the equation $ax^2 + bx + c = 0$. Furthermore, let a, b, and c be integers satisfying $|a| + |b| + |c| > 1$. Prove that

$$|x| \ \geq \ \frac{1}{|a| + |b| + |c| - 1}\,.$$

Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille.

Since a, b, and c are integers, $|a| + |b| + |c| \geq 2$. We rewrite the inequality as

$$\big(|a| + |b| + |c|\big)|x| \ \geq \ 1 + |x|\,, \qquad (1)$$

which certainly holds if $|x| \geq 1$, since then

$$(|a| + |b| + |c|)|x| \ \geq \ 2|x| \ \geq \ 1 + |x|\,.$$

Therefore, we suppose that $0 < |x| < 1$ in what follows. Now, $|x| > |x|^2$, so we have $|a| \cdot |x| + |b| \cdot |x| \geq |a| \cdot |x|^2 + |b| \cdot |x| \geq |ax^2 + bx| = |-c| = |c|$ and it follows that

$$(|a| + |b| + |c|)|x| \ \geq \ |c|(1 + |x|)\,.$$

Thus, (1) certainly holds if $|c| \geq 1$, that is, if $c \neq 0$.

Finally, if $c = 0$, then $ax + b = 0$ (since x is nonzero) and $b \neq 0$ since otherwise $a = 0$ as well and $|a| + |b| + |c| > 1$ would not be true. The left-hand side of (1) then becomes $(|a| + |b|)|x|$ and

$$\begin{aligned}(|a| + |b|)|x| &= |ax| + |b| \cdot |x| \\ &= |-b| + |b| \cdot |x| \\ &= |b|(1 + |x|) \ \geq \ 1 + |x|\,,\end{aligned}$$

since b is a nonzero integer. Thus, (1) continues to hold in that case and the proof is complete.

6. Let a circle through B and C of a triangle ABC intersect AB and AC in Y and Z, respectively. Let P be the intersection of BZ and CY, and let X be the intersection of AP and BC. Let M be the point that is distinct from X and on the intersection of the circumcircle of the triangle XYZ with BC. Prove that M is the midpoint of BC.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We reduce the geometric problem to an algebraic one, which can be solved by direct computation. Let a, b, c, x, y, z be the coordinates of A, B, C, X, Y, Z in the complex plane, respectively. Without loss of generality let $b = -1$ and $c = 1$. It suffices to prove that 0, x, y, z are concyclic. It is well known that distinct noncollinear points p_1, p_2, p_3, p_4 are concyclic if $\dfrac{(p_3 - p_2)(p_1 - p_4)}{p_1 - p_2)(p_3 - p_4)}$ is a real number (see Titu Andreescu, Dorin Andrica, *Complex numbers from A to… Z*, Birkhäuser, Boston, 2006, page 68). Therefore, it suffices to show that

$$\frac{(y-x)z}{x(y-z)} \in \mathbb{R}\,. \tag{1}$$

Let $AY = \lambda \cdot AB$; that is $y = a - \lambda(a+1)$, where $\lambda \in \mathbb{R}$. The fact that B, C, Y, Z are concyclic implies that $\triangle ABC \sim \triangle AZY$, hence $AZ = \dfrac{AB^2}{AC^2} \cdot \dfrac{AY}{AB} \cdot AC$; that is, $z = a - \dfrac{|a+1|^2}{|a-1|^2}\lambda(a-1)$. By Ceva's theorem we see that $\overrightarrow{BX} = \dfrac{\overrightarrow{BY}}{\overrightarrow{AY}} \cdot \dfrac{\overrightarrow{AZ}}{\overrightarrow{CZ}} \cdot \overrightarrow{XC}$, that is $x + 1 = \dfrac{(y+1)(z-a)}{(y-a)(z-1)} \cdot (1-x)$, and thus

$$x \;=\; \frac{|a+1|^2 - |a-1|^2}{|a+1|^2 + |a-1|^2 - 2\lambda|a+1|^2}\,.$$

Substituting these expressions into the fraction in (1) and clearing real factors yields the expression

$$\begin{aligned}\Big(2\lambda^2|a+1|^2(a+1) - \lambda(|a+1|^2(3a+1) + |a-1|^2(a+1))\\ +\ |a+1|^2(a-1) + |a-1|^2(a+1)\Big)\\ \cdot \Big(\lambda|a+1|^2(a-1) - |a-1|^2 a\Big) \cdot \left(\frac{\overline{a}-1}{a+1}\right)\,.\end{aligned}$$

We rewrite this last expression as $f_3(a)\lambda^3 + f_2(a)\lambda^2 + f_1(a)\lambda + f_0(a)$, where

$$\begin{aligned} f_3(a) &= 2|a+1|^4|a-1|^2\,,\\ f_2(a) &= |a+1|^2|a-1|^2(|a-1|^2 - 5|a|^2 - a - \overline{a} - 1)\,,\\ f_1(a) &= |a-1|^2(5|a|^2 - 2|a| - a^2 - \overline{a}^2 - 2a - 2\overline{a} - 1 + |a+1|^2|a-1|^2)\,,\\ f_0(a) &= -2|a|^2|a+1|^2|a-1|^2\,.\end{aligned}$$

Each of the four coefficients above is a real number, and so is λ. This completes the proof of the relation (1).

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

The Mathematics of the Heavens and the Earth: The Early History of Trigonometry
By Glen Van Brummelen, Princeton University Press, 2009
ISBN13: 978-0-691-12973-0, hardcover, 329+xvii pages, US$39.50
Reviewed by ***Menolly Lysne****, Simon Fraser University, Burnaby, BC*

In his book *The Mathematics of the Heavens and the Earth*, Glen Van Brummelen takes on the task of giving a historical treatment of trigonometry from its earliest roots until the beginnings of its modern development. He takes the reader on a voyage through time and space, moving from Ancient Egypt, Babylon and Greece, to Alexandrian Greece through medieval India and Islam, ending in 1550 in Renaissance Europe. Van Brummelen begins by asking the question "What is trigonometry?" His answer is twofold, involving finding lengths given angles and finding angles given lengths. With this definition in mind, the author begins with the early societies that could do one or the other of these such as the Ancient Egyptians, who lacked angular measure but were able to make computations for architectural purposes, and the Babylonians, who had angular measure. Then there were societies that could do both, such as the Alexandrian Greeks who produced chord tables so that a person could find the chord for a given angle. This allowed the Greeks to determine shadow lengths, hence permitting them to move from angles to lengths. This journey through the development of trigonometry is bound together by an investigation into the transmission of knowledge from one region to another and the controversy that surrounds this transmission. Such controversy is in India where there are disputes as to whether there was a transmission of knowledge from Greece or whether the knowledge was developed in India. After each text, Van Brummelen provides an explanation, which allows the sometimes elusive meaning to become apparent.

This book would be a wonderful addition to the library of any student, especially the graduate student researching the early history of mathematics. Van Brummelen provides an excellent bibliography and also points out the many controversies that took place over the interpretation of events. While the author does show one side of the arguments in the text, through thorough footnotes the reader is given the opportunity to investigate other interpretations if so inclined. The book often provides only a historical sketch, but at any point where the reader might ask, "Where can I learn more about this?" the reader is directed to a variety of primary and secondary sources from annotated translated texts to biographies of key figures.

This book may not be ideal for the high school teacher or student for a variety of reasons. While *The Mathematics of the Heavens and the Earth* provides excellent historical background, which could flavour a class on trigonometry, a large portion of the book is devoted to areas which, while important in the historical context, are rarely addressed in the modern setting. These include a thorough depiction of chord and sine tables and their constructions, as well as the development and uses of spherical trigonometry. These are both highly important, especially in the context of astronomy, but neither is investigated in the high school setting, making this book difficult to incorporate into the classroom.

The book itself is well written and provides a solid historical account though a small failing can be found in its layout. The diagram for an example is often found on the page after the corresponding written description. This makes following many of the examples somewhat frustrating. However, the mathematics required to follow the examples are easily at the high school or early university level. Aside from this small inconvenience, in practically every other sense the book flows well and is easy to read.

Overall, this book is an excellent resource for a university student and an interesting read for anyone with a solid grasp of high school mathematics and an interest in history.

Homage to a Pied Puzzler and *Mathematical Wizardry for a Gardner*
Edited by Ed Pegg, Jr., Alan Schoen, and Tom Rodgers
Published by A K Peters, Ltd., 2009
ISBN 978-1-56881-315-8, hardcover, 285+xxii pages, US$49.00; and
ISBN 978-1-56881-447-6, hardcover, 262+xvii pages, US$49.00 (resp.)
Reviewed by **David Ehrens**, *Macalester College, St. Paul, MN, USA*

These books are compiled from the seventh "Gathering for Gardner", also known as G4G7. The presentations, and thus the articles, are on anything related to the types of recreational mathematics made popular by Martin Gardner in his decades of articles for *Scientific American*. This leads to a huge variety of topics, with only the additional theme of the number seven helping to link them all. For example, a map colouring on a torus requires at most seven colours, and a number of people connected that seven to this year's theme and brought in related ideas. One of the colour plates in these well-illustrated books depicts a map (and its dual) that require seven colours; the map is knitted whilst the dual is crocheted!

While there may not be strong connections between the articles, the variety of topics does mean that anyone who enjoys puzzles and games will find many articles to suit their particular interests. Pure math, paper folding, and even the creation and marketing of puzzles are dealt with by a variety of people, tantalizing us with what's possible. Anyone who enjoys the works of Gardner will find a direction to follow in these collections.

Puzzle history is well represented, with the 14-15 sliding block puzzle and the invention claims of Sam Loyd examined. That same puzzle is expanded upon, with a variety of moves suggested instead of the traditional sliding. I grew up with the small plastic versions of the game instead of the original blocks in a box, so the idea of other movements, like jumping or switching, was new to me. Other Loyd puzzles, like the Two Ovals-to-Table problem, are also discussed.

Many articles explore the jump needed to turn something that is mathematically possible into a physical reality. While the mathematician may be intrigued by a game that takes ten thousand moves to finish, this would not make for a popular toy. In Homage to a Pied Puzzler, Adrian Fisher, the creator of Navigati (which first appeared in the Daily Mail in 2006) describes what it takes to create a puzzle for a newspaper audience.

When pondering the audience for this book, I was first struck by the variety of topics. This makes it a fine book for the puzzler who likes the topics that Gardner brought to the forefront. I was also intrigued by the idea that a talented or interested high school student could create physical examples of these mathematical ideas. Many of the paper folding articles seemed to be perfect for these students to do as projects for class. The games and card tricks could also lead to fine demonstrations. The articles in these books could be put to use in the classroom to popularize mathematics just as Gardner's articles popularized mathematics for earlier generations.

Lessons in Play: An Introduction to Combinatorial Game Theory
By Michael H. Albert, Richard J. Nowakowski, and David Wolfe
Published by A K Peters, Ltd., 2007
ISBN 1-56881-277-9, hardcover, 288+xi pages, US$69.00
Reviewed by **Sarah K.M. Aldous**, *South Island School, Hong Kong*

Imagine you are playing a game of Nim with a classmate, colleague or friend: you start with three heaps of pennies. On your turn you can choose one heap and remove any number of pennies. The winner will be the last person to be able to play, and the afternoon coffee will be bought by the loser. You scour your brain. Surely there is a way to describe the optimal strategy, guaranteeing you a free coffee and muffin?

Combinatorial game theory explores the strategies behind two-player games such as Nim in which there are no chance elements (for example, dice rolling or coin flipping). It is a rich area of very interesting mathematics that is easily accessible to students and instructors. This book is a comprehensive introductory textbook designed to be used for a first course in combinatorial game theory at university level. The authors are three respected games researchers who aimed to create a book that can be used by a student alone, by an instructor new to game theory, or by an experienced game theorist.

There have been two other standard texts used for combinatorial game theory courses, *On Numbers and Games* (Conway, 1976, second edition 2001) and *Winning Ways for Your Mathematical Plays* (Berlekamp, Conway, and Guy; 1982, second edition 2001). Naturally, *Lessons in Play* draws heavily on the ideas and material presented in these two books. However, the excellent organisation of *Lessons in Play* lends itself much better to its purpose as a course textbook. In particular, the text is easy to follow and the sections are very well laid out; a motivated student would be able to follow independently the clear explanations. Students reading on their own are also encouraged to proceed with a pencil in hand, trying out ideas through short exercises interspersed in the text. Instructors new to teaching game theory would be helped by notes before each chapter giving ideas about how to structure their lectures. There are also preparation questions before each chapter that could be given to students as an assignment before each section is covered in class. Instructors will also appreciate support from a website, from which a solutions manual is available.

The book has many strengths, the foremost of which is the clear, logical organisation. This will make the text more useful to beginning students and teachers and easier to follow than other books. Furthermore, an appendix lists each combinatorial game mentioned in the book, giving its set of rules and an example. Another strength is the reference to `CGSuite`, a piece of software that performs algebraic manipulation of games. An appendix in the book gives details about (freely) obtaining the software and some examples of how to use it. There is a link from the textbook website to the `CGSuite` page. Also helpful are the occasional discussions in the early chapters of how research into games is done. This will encourage readers to try some research for themselves. Indeed, there are lots of open problems being currently researched in game theory, and some of these are mentioned throughout the book and in the chapter entitled "Further Directions".

The text contains a few errors, and corrections are listed on the book's website. At times the readability could be improved by adding examples or having a definition appear earlier, such as the definition of a "simpler" number. These are very infrequent, however, and they do not detract much from the overall quality of the book. Perhaps the sets of rules could be improved by adding references to the parts of the text that mention them and also to literature so readers could find further analysis of the games that interest them.

Overall, *Lessons in Play* is a welcome addition to the stock of combinatorial game theory textbooks. Students and instructors will find it thorough and easy to use. Reading the book and working through the exercises may help you win your morning coffee games, and will certainly open up areas of further research and reading.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er septembre 2010**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3505. Correction. *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

Les cercles Γ_1 et Γ_2 ont même centre O, et Γ_1 est à l'intérieur de Γ_2. Soit $A \neq O$ un point à l'intérieur de Γ_1 ; un rayon non parallèle à AO et issu de A coupe respectivement Γ_1 et Γ_2 aux points B et C. Soit D le point d'intersection des tangentes aux points B et C des cercles correspondants. Soit E un point sur la droite BC tel que DE soit perpendiculaire à BC. Montrer que $AB = EC$ si et seulement si OA est perpendiculaire à BC.

3514. *Proposé par Michel Bataille, Rouen, France.*

Soit m un nombre réel positif et a, b et c des nombres réels tels que

$$a(a-b)+b(b-c)+c(c-a) \;=\; m\,.$$

Quel est le domaine de $ab(a-b)+bc(b-c)+ca(c-a)$?

3515. *Proposé par José Luis Díaz-Barrero et Josep Rubió-Massegú, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit x, y et z trois nombres réels positifs. Montrer que

$$\left(\frac{\{x\}^2}{y}+\frac{[x]^2}{z}\right) + \left(\frac{\{y\}^2}{z}+\frac{[y]^2}{x}\right) + \left(\frac{\{z\}^2}{x}+\frac{[z]^2}{y}\right) \;\geq\; \frac{x^2+y^2+z^2}{x+y+z}\,,$$

où $\lfloor a \rfloor$ est le plus grand entier n'excédant pas a, et $\{a\} = a - \lfloor a \rfloor$.

3516. *Proposé par János Bodnár, Budapest, Hongrie.*

Soit P et Q deux points intérieurs du triangle ABC. Soit AA', BB' et CC' trois céviennes passant par P. La droite par A' parallèle à AQ coupe respectivement les droites BQ et CQ aux points L et M'. La droite par B' parallèle à BQ coupe respectivement les droites CQ et AQ aux points M et N'. La droite par C' parallèle à CQ coupe respectivement les droites AQ et BQ aux points N et L'.

Est-ce vrai que les triangles LMN et $L'M'N'$ ont même aire ?

3517. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Y a-t-il un triangle scalène ABC pour lequel il existe un point P dans le plan de ABC tel que, pour chaque droite ℓ par P, la somme des carrés des distances de A, B et C à ℓ est constante ?

3518. *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

Montrer que si n et i sont des entiers avec $0 \leq i \leq n$, alors

$$0 \; < \; 1 - \frac{C_i}{2^{2i+1}} - \left\{ 10^{(2i+1)n} \cdot \sqrt{10^{2n} - 1} \right\} \; < \; \frac{1}{10^{2n}} \,,$$

où $\{x\}$ désigne la partie fractionnaire du nombre réel x et C_i le i^{e} nombre de Catalan, $C_i = \dfrac{1}{i+1}\dbinom{2i}{i}$, $i \geq 0$.

3519. *Proposé par Nguyen Duy Khanh, étudiant, Université des Sciences de Hanoi, Hanoi, Vietnam.*

Deux triangles ABC et $A'B'C'$ ont les aires respectives S et S'. Soit w_a, w_b et w_c les longueurs des bissectrices internes des angles de ABC aux côtés BC, AC, AB, respectivement ; on définit w'_a, w'_b et w'_c de manière analogue. Montrer si oui ou non, on a $w_a w'_a + w_b w'_b + w_c w'_c \geq 3\sqrt{3SS'}$.

3520. *Proposé par Ricardo Barroso Campos, Université de Séville, Séville, Espagne.*

Construire un triangle ABC tel que la droite passant pas les centres des cercles inscrit et circonscrit soit parallèle au côté AB.

3521. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Soit x_1, x_2, $\dots$, x_n des nombres réels dans l'intervalle $[e, \infty)$ et pour chaque indice k soit $e_k = \dfrac{x_1 + x_2 + \cdots + x_k}{x_k}$. Montrer que

$$x_1^{e_1} + \; x_2^{e_2} + \; \cdots \; + \; x_n^{e_n} \;\; \geq \;\; nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n \,.$$

3522. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Si a, b, c et d sont des nombres réels positifs tels que $abcd = 1$, montrer que

$$\left(1 + \frac{a}{b}\right)^{cd} \left(1 + \frac{b}{c}\right)^{da} \left(1 + \frac{c}{d}\right)^{ab} \left(1 + \frac{d}{a}\right)^{bc} \;\geq\; 2^{\left(\frac{16}{a^2 + b^2 + c^2 + d^2}\right)} \,.$$

3523. *Proposé par Slavko Simic, Institut de Mathématiques SANU, Belgrade, Serbie.*

Soit $f : \mathbb{R} \to \mathbb{R}$ une fonction continûment différentiable. Résoudre l'équation fonctionnelle

$$f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) = p(x-y)\big(f'(x) - f'(y)\big),$$

où p est un paramètre réel indépendant de x, y.

3524. *Proposé par Titu Zvonaru, Comănești, Roumanie.*

Soit $a_1, a_2, \ldots, a_{n+1}$ des nombres réels positifs satisfaisant la condition $a_{n+1} = \min\{a_1, a_2, \ldots, a_{n+1}\}$. Montrer que

$$\begin{aligned} & a_1^{n+1} + a_2^{n+1} + \cdots + a_{n+1}^{n+1} - (n+1)a_1a_2\cdots a_{n+1} \\ & \quad \geq (n+1)a_{n+1}\Big[(a_1 - a_{n+1})^n + (a_2 - a_{n+1})^n + \cdots + (a_n - a_{n+1})^n \\ & \qquad - n(a_1 - a_{n+1})(a_2 - a_{n+1})\cdots(a_n - a_{n+1})\Big]. \end{aligned}$$

3525. *Proposé par un proposeur anonyme.*

Soit $0 \leq a, b \leq 1$. Montrer que $\dfrac{a+b}{1+ab} \leq \left(\sqrt{a} + \sqrt{b} - \sqrt{ab}\right)^2$.

3526. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sum_{\text{cyclique}} \frac{a}{\sqrt{a^2 + 2(b+c)^2}} \geq 1.$$

. . . .

3505. Correction. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

The circles Γ_1 and Γ_2 have a common centre O, and Γ_1 lies inside Γ_2. The point $A \neq O$ lies inside Γ_1; a ray not parallel to AO that starts at A intersects Γ_1 and Γ_2 at the points B and C, respectively. Let tangents to corresponding circles at the points B and C intersect at the point D. Let E be a point on the line BC such that DE is perpendicular to BC. Prove that $AB = EC$ if and only if OA is perpendicular to BC.

3514. *Proposed by Michel Bataille, Rouen, France.*

Let m be a positive real number and a, b, c real numbers such that

$$a(a-b) + b(b-c) + c(c-a) = m.$$

What is the range of $ab(a-b) + bc(b-c) + ca(c-a)$?

3515. *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x, y, and z be positive real numbers. Prove that

$$\left(\frac{\{x\}^2}{y}+\frac{\lfloor x\rfloor^2}{z}\right) + \left(\frac{\{y\}^2}{z}+\frac{\lfloor y\rfloor^2}{x}\right) + \left(\frac{\{z\}^2}{x}+\frac{\lfloor z\rfloor^2}{y}\right) \geq \frac{x^2+y^2+z^2}{x+y+z},$$

where $\lfloor a\rfloor$ is the greatest integer not exceeding a, and $\{a\} = a - \lfloor a\rfloor$.

3516. *Proposed by János Bodnár, Budapest, Hungary, Budapest, Hungary.*

Let P and Q be interior points of triangle ABC. Let AA', BB', and CC' be three concurrent cevians through P. The line through A' parallel to AQ intersects the lines BQ and CQ at points L and M', respectively. The line through B' parallel to BQ intersects the lines CQ and AQ at points M and N', respectively. The line through C' parallel to CQ intersects the lines AQ and BQ at points N and L', respectively.

Is it true that triangles LMN and $L'M'N'$ have the same area?

3517. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Is there a scalene triangle ABC for which there exists a point P in the plane of ABC such that, for each line ℓ through P, the sum of the squares of the distances of A, B, and C to ℓ is constant?

3518. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Prove that if n and i are integers with $0 \leq i \leq n$, then

$$0 < 1 - \frac{C_i}{2^{2i+1}} - \left\{10^{(2i+1)n}\cdot\sqrt{10^{2n}-1}\right\} < \frac{1}{10^{2n}},$$

where $\{x\}$ denotes the fractional part of the real number x and C_i is the i^{th} Catalan number, $C_i = \frac{1}{i+1}\binom{2i}{i}$, $i \geq 0$.

3519. *Proposed by Nguyen Duy Khanh, student, Hanoi University of Science, Hanoi, Vietnam.*

Two triangles ABC and $A'B'C'$ have areas S and S', respectively. Let w_a, w_b, w_c be the lengths of the internal angle bisectors of ABC to the sides BC, AC, AB, respectively, and define w'_a, w'_b, w'_c similarly. Prove or disprove that $w_aw'_a + w_bw'_b + w_cw'_c \geq 3\sqrt{3SS'}$.

3520. *Proposed by Ricardo Barroso Campos, University of Seville, Seville, Spain.*

Construct a triangle ABC such that the line through the incentre and the circumcentre is parallel to the side AB.

3521. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $[e, \infty)$ and for each index k let $e_k = \dfrac{x_1 + x_2 + \cdots + x_k}{x_k}$. Prove that

$$x_1^{e_1} + x_2^{e_2} + \cdots + x_n^{e_n} \geq nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n .$$

3522. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

If a, b, c, and d are positive real numbers satisfying $abcd = 1$, prove that

$$\left(1+\frac{a}{b}\right)^{cd}\left(1+\frac{b}{c}\right)^{da}\left(1+\frac{c}{d}\right)^{ab}\left(1+\frac{d}{a}\right)^{bc} \geq 2^{\left(\frac{16}{a^2+b^2+c^2+d^2}\right)} .$$

3523. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Solve the functional equation

$$f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) = p(x-y)\big(f'(x) - f'(y)\big) ,$$

where p is a real parameter independent of x, y.

3524. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers satisfying the condition $a_{n+1} = \min\{a_1, a_2, \ldots, a_{n+1}\}$. Prove that

$$\begin{aligned} &a_1^{n+1} + a_2^{n+1} + \cdots + a_{n+1}^{n+1} - (n+1)a_1a_2\cdots a_{n+1} \\ &\quad\geq (n+1)a_{n+1}\Big[(a_1 - a_{n+1})^n + (a_2 - a_{n+1})^n + \cdots + (a_n - a_{n+1})^n \\ &\qquad - n(a_1 - a_{n+1})(a_2 - a_{n+1})\cdots(a_n - a_{n+1})\Big] . \end{aligned}$$

3525. *Proposed by an anonymous proposer.*

Let $0 \leq a, b \leq 1$. Prove that $\dfrac{a+b}{1+ab} \leq \left(\sqrt{a} + \sqrt{b} - \sqrt{ab}\right)^2$.

3526. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 + 2(b+c)^2}} \geq 1 .$$

SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

We acknowledge a correct solution to problem 3354 by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Due to switched LaTeX commands, we instead incorrectly credited Gottfried Perz, Pestalozzigymnasium, Graz, Austria. We further acknowledge a correct solution to problem 3412 by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. Our apologies for these errors.

3415. [2009 : 108, 111] *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.*

Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \leq \sqrt[3]{3(3 + a + b + c + ab + bc + ca)}\,.$$

Comment: Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina informs us that this problem has appeared as problem O98 in the journal *Mathematical Reflections*, Vol. 5 (2008), pp. 38-39, by the same proposer. Three solutions are given there using a variety of techniques.

Solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3416. [2009 : 108, 111] *Proposed by Michel Bataille, Rouen, France.*

Let the sequence (a_n) be defined by $a_0 = 6$ and the recursion

$$a_{n+1} = \frac{1}{13}\left(8a_n\sqrt{3a_n^2 + 13} - 6a_n^2 - 13\right)$$

for $n \geq 0$. Prove that each a_n is a positive integer, and that $a_n^2 - a_{n+1}$ is divisible by 13 for each $n \geq 0$.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Set

$$f(x) = \frac{2}{13}x\left(4\sqrt{3x^2 + 13} - 3x\right) - 1\,;$$

by hypothesis, $a_{n+1} = f(a_n)$. Let

$$\omega = 2 + \sqrt{3}\,, \qquad \overline{\omega} = 2 - \sqrt{3}\,;$$
$$\alpha = \frac{1}{2}\left(4 + \sqrt{3}\right), \qquad \overline{\alpha} = \frac{1}{2}\left(4 - \sqrt{3}\right).$$

As an aid to calculating, we observe that $\omega\overline{\omega} = 1$, $\alpha\overline{\alpha} = \frac{13}{4}$, and $\alpha - \overline{\alpha} = \sqrt{3}$.

Part 1. Define increasing sequences of *positive integers* (u_n) and (v_n) by

$$u_0 = 4\,,\ \ u_1 = 11\,,\ \ u_{n+2} = 4u_{n+1} - u_n\,;$$
$$v_0 = 1\,,\ \ v_1 = 6\,,\ \ v_{n+2} = 4v_{n+1} - v_n\,.$$

Solving for u_n and v_n, we obtain

$$\begin{aligned} u_n &= \alpha\omega^n + \overline{\alpha}\,\overline{\omega}^n\,, \\ v_n &= \frac{1}{\sqrt{3}}\left(\alpha\omega^n - \overline{\alpha}\,\overline{\omega}^n\right). \end{aligned} \tag{1}$$

From (1) we easily deduce that $3v_n^2 + 13 = u_n^2$, so that $u_n = \sqrt{3v_n^2 + 13}$; hence, $f(v_n) = \frac{2}{13}v_n(4u_n - 3v_n) - 1$. Using (1) again we have

$$\begin{aligned} 4u_n - 3v_n &= \alpha\left(4 - \sqrt{3}\,\right)\omega^n + \overline{\alpha}\left(4 + \sqrt{3}\,\right)\overline{\omega}^n \\ &= 2\alpha\overline{\alpha}\left(\omega^n + \overline{\omega}^n\right) \;=\; \frac{13}{2}\left(\omega^n + \overline{\omega}^n\right). \end{aligned}$$

Hence,

$$\begin{aligned} f(v_n) &= \frac{2}{13} \cdot \frac{1}{\sqrt{3}}\left(\alpha\omega^n - \overline{\alpha}\,\overline{\omega}^n\right) \cdot \frac{13}{2}(\omega^n + \overline{\omega}^n) - 1 \\ &= \frac{1}{\sqrt{3}}\left(\omega^n + \overline{\omega}^n\right)\left(\alpha\omega^n - \overline{\alpha}\,\overline{\omega}^n\right) - 1 \\ &= \frac{1}{\sqrt{3}}\left(\alpha\omega^{2n} - \overline{\alpha}\,\overline{\omega}^{2n}\right). \end{aligned} \tag{2}$$

For $n \geq 0$ define the *positive integer* $b_n = v_{2^n}$. Using (2) and (1) we have $f(b_n) = f(v_{2^n}) = \frac{1}{\sqrt{3}}\left(\alpha\omega^{2^{n+1}} - \overline{\alpha}\,\overline{\omega}^{2^{n+1}}\right) = v_{2^{n+1}} = b_{n+1}$. Finally, from $a_0 = b_0 = 6$, $a_{n+1} = f(a_n)$, and $b_{n+1} = f(b_n)$, induction yields that $a_n = b_n$ for $n \geq 0$, and the result follows.

Part 2. For $n \geq 0$ set $\theta_n = \frac{1}{12}\left(\omega^{2^{n+1}} + \overline{\omega}^{2^{n+1}} - 2\right)$. Since θ_n is the solution of the recursion $\theta_{n+1} = 12\theta_n^2 + 4\theta_n$ with initial condition $\theta_0 = 1$, it is a *positive integer*. As $a_n = v_{2^n}$, then from (1) we have

$$a_n \;=\; \frac{1}{\sqrt{3}}\left(\alpha\omega^{2^n} - \overline{\alpha}\,\overline{\omega}^{2^n}\right).$$

It follows that $a_n^2 - a_{n+1} = \frac{13}{12}(\omega^{2^{n+1}} + \overline{\omega}^{2^{n+1}} - 2) = 13\theta_n$, whence 13 divides $a_n^2 - a_{n+1}$, as desired.

Also solved by ARKADY ALT, San Jose, CA, USA; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incomplete solution submitted.

Many solvers began with the general Pell equation $B^2 - 3A^2 = 13$. With u_n and v_n as defined in the featured solution above, we see that the pair $B = u_n$, $A = v_n$, provides an infinite family of solutions to this Pell equation. Deiermann began instead with the equation $B^2 - dA^2 = N$, where $d > 0$ is a nonsquare integer and $N > 0$ is an integer for which the pair $B = B_0$, $A = A_0$ is the minimal positive solution for an equivalence class of solutions, while the pair B_1 and A_1 is the next largest solution. He proved that if (a_n) is defined by $a_0 = A_1$ and

$$a_{n+1} = \frac{1}{N}\left(2\sqrt{dA_0^2 + N} \cdot a_n \sqrt{da_n^2 + N} - 2A_0 da_n^2 - A_0 N\right)$$

for $n \geq 0$, then each a_n is a positive integer and $a_n^2 - A_0 a_{n+1}$ is divisible by N for each $n \geq 0$. (In our problem $N = 13$, $d = 3$, $A_0 = 1$, $B_0 = 4$, $A_1 = 6$, and $B_1 = 11$.) Hess proved a similar generalization.

3417. [2009 : 109, 111] *Proposed by Michel Bataille, Rouen, France.*

Let $S_p(n) = 1^p + 2^p + \cdots + n^p$. Prove that

$$\sum_{n=1}^{\infty} \frac{\left(S_{-1}(n)\right)^2}{S_1(n)} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_3(n)}\right) + 2\left(\sum_{n=1}^{\infty} \frac{S_{-1}(n)}{S_1(n)}\right).$$

Solution by Joel Schlosberg, Bayside, NY, USA.

Let $H_n = 1^{-1} + 2^{-1} + \cdots + n^{-1}$ be the n^{th} harmonic number. Then $H_n = H_{n-1} + \frac{1}{n}$. As usual let $\zeta(k) = \sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ be the Riemann zeta function. A known identity due to Euler (see Jonathan Sondow and Eric W. Weisstein "Harmonic Number." `http://mathworld.wolfram.com/HarmonicNumber.html`) states that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3). \tag{1}$$

It is well known that H_n is asymptotic to $\ln n$ and that $\ln n = o(n^k)$ for any positive k. Therefore,

$$\lim_{n\to\infty} \frac{H_n}{n+1} = 0 \tag{2}$$

and

$$\lim_{n\to\infty} \frac{H_n^2}{n+1} = 0. \tag{3}$$

By (2),

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{H_n}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} H_n \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \left(H_1 \cdot \frac{1}{1} + \sum_{n=2}^{N} \frac{1}{n}(H_n - H_{n-1}) - \frac{H_N}{N+1}\right) \\
&= \lim_{N\to\infty} \left(\frac{1}{1^2} + \sum_{n=2}^{N} \frac{1}{n^2}\right) - \lim_{N\to\infty} \frac{H_N}{N+1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)\,. \qquad (4)
\end{aligned}$$

By (1),

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} &= \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} = \sum_{n=2}^{\infty} \frac{H_n - \frac{1}{n}}{n^2} = \sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3) - \zeta(3) = \zeta(3)\,. \qquad (5)
\end{aligned}$$

By (1), (4), and (5),

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)^2} &= \sum_{n=1}^{\infty} H_n \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)}\right) \\
&= \sum_{n=1}^{\infty} \frac{H_n}{n^2} + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} - 2\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} \\
&= 2\zeta(3) + \zeta(3) - 2\zeta(2) = 3\zeta(3) - 2\zeta(2)\,. \qquad (6)
\end{aligned}$$

By (1) and (3),

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{H_n^2}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} H_n^2 \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \left(H_1^2 \cdot \frac{1}{1} + \sum_{n=2}^{N} \frac{1}{n}(H_n^2 - H_{n-1}^2) - \frac{H_N^2}{N+1}\right) \\
&= \lim_{N\to\infty} \left(1 + \sum_{n=2}^{N} \frac{1}{n}(H_n + H_{n-1})(H_n - H_{n-1})\right) - \lim_{N\to\infty} \frac{H_N^2}{N+1} \\
&= \lim_{N\to\infty} \left(1 + \sum_{n=2}^{N} \frac{1}{n}\left(2H_n - \frac{1}{n}\right)\left(\frac{1}{n}\right)\right)
\end{aligned}$$

$$= \lim_{N\to\infty}\left(2+2\sum_{n=2}^{N}\frac{H_n}{n^2}\right) - \lim_{N\to\infty}\left(1+\sum_{n=2}^{N}\frac{1}{n^3}\right)$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n^3} = 2\cdot 2\zeta(3) - \zeta(3) = 3\zeta(3)\,. \qquad (7)$$

Note that $S_{-1}(n) = H_n$. By the well-known formulas for the sums of the first n positive integers and the first n cubes,

$$S_1(n) = \frac{n(n+1)}{2} \qquad \text{and} \qquad S_3(n) = \frac{n^2(n+1)^2}{4}\,. \qquad (8)$$

By (4), (6), and (8),

$$\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{S_{-1}(n)}{S_3(n)}\right) + 2\left(\sum_{n=1}^{\infty}\frac{S_{-1}(n)}{S_1(n)}\right)$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+1)^2} + 4\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)}$$
$$= 2(3\zeta(3) - 2\zeta(2)) + 4\zeta(2) = 6\zeta(3)\,. \qquad (9)$$

By (7),

$$\sum_{n=1}^{\infty}\frac{(S_{-1}(n))^2}{S_1(n)} = 2\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)} = 6\zeta(3)\,,$$
$$\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{S_{-1}(n)}{S_3(n)}\right) + 2\left(\sum_{n=1}^{\infty}\frac{S_{-1}(n)}{S_1(n)}\right) = 6\zeta(3)\,,$$

hence the two sides are equal with a common value of $6\zeta(3)$.

Also solved by ARKADY ALT, San Jose, CA, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3418. [2009 : 109, 111] *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $\mathcal{I}(\phi)$ be the set of all antiderivatives of a continuous function ϕ.

(a) Determine the continuous function $f\colon I_p \subset \mathbb{R} \to \mathbb{R}\backslash\{0\}$ such that $f(0) = 1$ and $f^{-p} \in \mathcal{I}(f)$, where p is an odd natural number and the interval I_p contains zero and is maximal for the given properties of f.

(b) Prove that $p = q$ if and only if $I_p = I_q$.

Solution by Michel Bataille, Rouen, France.

(a) Let I be an interval containing 0 and let $f : I \to \mathbb{R}\backslash\{0\}$ be such that $f(0) = 1$ and $f^{-p} \in \mathcal{I}(f)$. Then f^{-p} is differentiable on the interval I and $(f^{-p})'(x) = f(x)$ for all $x \in I$.

Since p is an odd natural number, the function $h(x) = x^p$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. Moreover, its inverse is differentiable except at 0. The composition $h \circ f^{-1}(x) = f^{-p}(x)$ is differentiable, thus the function $f^{-1}(x) = \dfrac{1}{f(x)}$ is differentiable, and so is the function f. Note also that $f(x) > 0$ for all $x \in I$ (since f is continuous, does not vanish, and $f(0) > 0$).

Now let $p = 2m - 1$, where m is a positive integer. Since the function f is differentiable,

$$f(x) \;=\; (f^{-p})'(x) \;=\; -pf^{-p-1}(x)f'(x) \;=\; -(2m-1)\frac{f'(x)}{f(x)^{2m}}\,,$$

so that f satisfies

$$(2m-1)f'(x) + \big(f(x)\big)^{2m+1} \;=\; 0 \quad (x \in I)\,.$$

An easy calculation shows that the function $g(x) = f(x)^{-m}$ satisfies

$$g'(x)g(x) \;=\; \frac{m}{2m-1}\,.$$

Integrating and substituting $g(0) = 1$, it follows that $\big(g(x)\big)^2 = \dfrac{2mx}{2m-1} + 1$. Thus, $\dfrac{2mx}{2m-1} + 1 > 0$ and $g(x) = \Big(\dfrac{2mx}{2m-1} + 1\Big)^{1/2}$ for $x \in I$. We conclude that

$$I \subset I_p \;=\; \Big(\frac{-p}{p+1}, \infty\Big) \qquad \text{and} \qquad f(x) = \Big(\frac{(p+1)x}{p} + 1\Big)^{-\frac{1}{p+1}}\,. \qquad (1)$$

Conversely, it is easy to check that the function f defined by (1) on I_p satisfies

$$(f^{-p})'(x) \;=\; \left[\Big(\frac{(p+1)x}{p} + 1\Big)^{\frac{p}{p+1}}\right]' \;=\; f(x)$$

for all $x \in I_p$.

(b) $p = q$ is equivalent to $\frac{p}{p+1} = \frac{q}{q+1}$, hence it is equivalent to $I_p = I_q$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposers.

The differentiability of the function f was not addressed by any of the other solvers.

3419. [2009 : 109, 111] *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be positive real numbers.

(a) Prove that $\sum\limits_{\text{cyclic}} \sqrt{\dfrac{a^2+4bc}{b^2+c^2}} \;\geq\; 2+\sqrt{2}$.

(b) Prove that $\sum\limits_{\text{cyclic}} \sqrt{\dfrac{a^2+bc}{b^2+c^2}} \;\geq\; 2+\dfrac{1}{\sqrt[3]{2}}$.

Solution to (a) *by Oliver Geupel, Brühl, NRW, Germany; and the proposer.*

We will show that the inequality holds even if one of a, b, c is zero. Without loss of generality, assume that $a \geq b \geq c \geq 0$.

Case 1. If $4b^3 \geq a^2c$, then we have

$$\frac{a^2+4bc}{b^2+c^2} - \frac{a^2}{b^2} \;=\; \frac{c(4b^3-a^2c)}{b^2(b^2+c^2)} \;\geq\; 0\,,$$
$$\frac{b^2+4ca}{c^2+a^2} - \frac{b^2}{a^2} \;=\; \frac{c(4a^3-b^2c)}{a^2(c^2+a^2)} \;\geq\; 0\,,$$

and

$$\frac{c^2+4ab}{a^2+b^2} \;\geq\; \frac{4ab}{a^2+b^2}\,.$$

The result now follows after several applications of the AM–GM Inequality:

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{\frac{a^2+4bc}{b^2+c^2}} \;&\geq\; \frac{a}{b}+\frac{b}{a}+2\sqrt{\frac{ab}{a^2+b^2}} \\
&=\; \left(1-\frac{1}{\sqrt{2}}\right)\frac{a^2+b^2}{ab} \;+\; \left(\frac{a^2+b^2}{\sqrt{2}ab} \;+\; 2\sqrt{\frac{ab}{a^2+b^2}}\;\right) \\
&\geq\; 2\left(1-\frac{1}{\sqrt{2}}\right)+2\sqrt[4]{\frac{2(a^2+b^2)}{ab}} \\
&\geq\; 2\left(1-\frac{1}{\sqrt{2}}\right)+2\sqrt{2} \;=\; 2+\sqrt{2}\,.
\end{aligned}$$

Case 2. If $4b^3 \leq a^2c$, then $a \geq 2b$ and we have

$$\frac{a^2+4bc}{b^2+c^2} - \frac{a^2+4b^2}{2b^2} \;=\; \frac{(b-c)[a^2(b+c)-4b^2(b-c)]}{2b^2(b^2+c^2)} \;\geq\; 0\,.$$

Hence,

$$\sum_{\text{cyclic}} \sqrt{\frac{a^2+4bc}{b^2+c^2}} \;\geq\; \sqrt{\frac{a^2+4b^2}{2b^2}} \;+\; \frac{b}{a} \;+\; 2\sqrt{\frac{ab}{a^2+b^2}}\,.$$

Setting $x = \frac{a}{b} \geq 2$, we need to prove that

$$f(x) = \sqrt{\frac{x^2}{2} + 2} + \frac{1}{x} + 2\sqrt{\frac{x}{x^2+1}} \geq 2 + \sqrt{2}\,.$$

Since $x \geq 2$, we have

$$\begin{aligned} x(x^2+1) - (x+1)^2 &= x^3\left(1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}\right) \\ &\geq x^3\left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8}\right) = \frac{1}{8}x^3 > 0\,; \end{aligned}$$

hence $\sqrt{x(x^2+1)} > x + 1$, and then

$$\begin{aligned} f'(x) &= \frac{x}{2\sqrt{\frac{x^2}{2}+2}} - \frac{1}{x^2} - \frac{x^2-1}{(x^2+1)\sqrt{x(x^2+1)}} \\ &> \frac{1}{2\sqrt{\frac{1}{2}+\frac{2}{x^2}}} - \frac{1}{x^2} - \frac{x^2-1}{(x^2+1)(x+1)} \\ &\geq \frac{1}{2} - \frac{1}{x^2} - \frac{x-1}{x^2+1} = \frac{x^2-4}{4x^2} + \frac{(x-2)^2+1}{4(x^2+1)} > 0\,. \end{aligned}$$

Thus, $f(x)$ is an increasing function on $[2, \infty)$, and therefore,

$$f(x) \geq f(2) = \frac{5}{2} + 2\sqrt{\frac{2}{5}} > 2 + \sqrt{2}\,.$$

There was one incorrect solution submitted.

Geupel credits the proposer and his publication [1] *at the MathLinks website for his solution to* (a). *The proposer submitted a solution to part* (b) *as well; however, his argument relies on a graphical computational package result, which has not been verified otherwise. Geupel remarked that the author has published a solution to part* (b) *in* [3]. *Geupel also mentioned a similar inequality,*

$$\sum_{\text{cyclic}} \sqrt{\frac{a^2+bc}{b^2+c^2}} \geq \sqrt{\frac{2(a^2+b^2+c^2)}{ab+bc+ca}} + \frac{1}{\sqrt{2}}\,,$$

proved by the proposer in [2].

References

[1] Vo Quoc Ba Can, `www.mathlinks.ro/viewtopic.php?t=197674`,
download file `toanhocmuonmaumain.pdf` (in Vietnamese), Problem 1.46, pages 39f.

[2] Vo Quoc Ba Can, *One more inequality*,
`http://www.mathlinks.ro/viewtopic.php?t=186338`

[3] Vo Quoc Ba Can, *Some new results*,
`http://www.mathlinks.ro/viewtopic.php?t=186334&start=25`

3420. [2009 : 109, 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\prod_{k=1}^{n}\left(\frac{(k+1)^2}{k(k+2)}\right)^{k+1} \;<\; n+1 \;<\; \prod_{k=1}^{n}\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1}.$$

Similar solutions by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam and Oliver Geupel, Brühl, NRW, Germany.

We recognize that the product on the left is a telescoping product, thus

$$\prod_{k=1}^{n}\left(\frac{(k+1)^2}{k(k+2)}\right)^{k+1} \;=\; \frac{2(n+1)^{n+2}}{(n+2)^{n+1}} \;=\; \frac{2(n+1)}{\left(1+\frac{1}{n+1}\right)^{n+1}}.$$

By the Bernoulli Inequality, we have

$$\left(1+\frac{1}{n+1}\right)^{n+1} \;>\; 1+(n+1)\frac{1}{n+1} \;=\; 2,$$

which proves the left inequality.

Again by the Bernoulli Inequality, we have

$$\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1} \;=\; \left(1+\frac{1}{k(k+1)}\right)^{k+1} \;>\; \left(1+\frac{1}{k}\right) \;=\; \frac{k+1}{k}.$$

Hence,

$$\prod_{k=1}^{n}\left(\frac{k^2+k+1}{k(k+1)}\right)^{k+1} \;>\; \prod_{k=1}^{n}\frac{k+1}{k} \;=\; n+1.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3421. [2009 : 109, 112] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $abc \le 1$. Prove that

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} \;\ge\; \frac{3}{2}.$$

Solution by Arkady Alt, San Jose, CA, USA.

After the substitution $(a, b, c) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$, the inequality becomes equivalent to

$$\frac{y^2}{x(y+1)} + \frac{z^2}{y(z+1)} + \frac{x^2}{z(x+1)} \geq \frac{3}{2},$$

where x, y, and z are positive real numbers with $xyz \geq 1$.

Since by the Cauchy-Schwartz Inequality we have

$$\big((xy+x)+(yz+y)+(zx+z)\big)\left(\frac{y^2}{x(y+1)} + \frac{z^2}{y(z+1)} + \frac{x^2}{z(x+1)}\right) \geq (x+y+z)^2,$$

it suffices to prove that

$$\frac{(x+y+z)^2}{xy+yz+zx+x+y+z} \geq \frac{3}{2},$$

or

$$2(x+y+z)^2 \geq 3(xy+yz+zx)+3(x+y+z).$$

However, the last inequality follows immediately from

$$(x+y+z)^2 \geq 3(xy+yz+zx),$$

and

$$\begin{aligned}(x+y+z)^2 &= (x+y+z)(x+y+z)\\ &\geq 3\sqrt[3]{xyz}(x+y+z)\\ &\geq 3(x+y+z).\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect solution and two incomplete solutions submitted.

3422. [2009 : 110, 112] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c \leq 1$. Prove that

$$\frac{a}{a^3+a^2+1} + \frac{b}{b^3+b^2+1} + \frac{c}{c^3+c^2+1} \leq \frac{27}{31}.$$

Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

Set $a = \frac{x}{3}$, $b = \frac{y}{3}$, and $c = \frac{z}{3}$. Then x, y, and z are positive real numbers satisfying $x + y + z \le 3$, and the inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{x}{x^3 + 3x^2 + 27} \le \frac{3}{31}.$$

By the AM-GM Inequality, we have

$$x^3 + 1 + 1 \ge 3x\,, \quad \text{and } 3\big(x^2 + 1\big) \ge 6x\,.$$

Therefore, it suffices for us to prove that

$$\sum_{\text{cyclic}} \frac{x}{9x + 22} \le \frac{3}{31} \iff 3 - \sum_{\text{cyclic}} \frac{22}{9x + 22} \le \frac{27}{31}\,.$$

By the Cauchy–Schwarz Inequality, $\Big(\sum_{\text{cyclic}} \frac{1}{9x + 22}\Big)\Big(\sum_{\text{cyclic}} 9x + 22\Big) \ge 9$, so it follows that

$$\begin{aligned}\sum_{\text{cyclic}} \frac{22}{9x + 22} &\ge \frac{198}{9(x + y + z) + 66} \\ &\ge \frac{198}{27 + 66} \\ &= \frac{66}{31}\,,\end{aligned}$$

which settles the inequality on the right of the double implication above.

Equality holds if and only if $x = y = z = 1$, or $a = b = c = \frac{1}{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Janous defined the numbers $x_I = 1.100203739\ldots$ and $x_M = 0.657298106\ldots$, the respective (unique) real roots of $3x^4 + 3x^3 + x^2 - 6x - 3$ and $2x^3 + x^2 - 1$, and he showed that if a, b, and c are positive reals with $a + b + c = K$ and $0 < K/3 \le x_I$, then

$$\sum_{\text{cyclic}} \frac{a}{a^3 + a^2 + 1} \le \begin{cases} \dfrac{27K}{K^3 + 3K^2 + 27}\,, & \text{if } 0 < K/3 \le x_M\,, \\ \dfrac{3x_M}{x_M^3 + x_M^2 + 1}\,, & \text{if } x_M < K/3 \le x_I\,. \end{cases}$$

3423. [2009 : 110, 112] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let $n \geq 2$ be an integer and $x_1, x_2, \ldots, x_n$ positive real numbers such that $x_1 + x_2 + \cdots + x_n = 2n$. Prove that

$$\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{x_j}{\sqrt{x_i^3+1}}\right) \;\geq\; \frac{2n(n-1)}{3}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $x_i = 2t_i$, $1 \leq i \leq n$. Then the given inequality is equivalent to

$$\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{t_j}{\sqrt{1+8t_i^3}}\right) \geq \frac{n(n-1)}{3}\,, \qquad (1)$$

where each t_i is positive and $t_1 + t_2 + \cdots + t_n = n$.

Since $\sqrt{1+8t_i^3} \leq 1 + 2t_i^2$ is equivalent to $1 + 8t_i^3 \leq 1 + 4t_i^2 + 4t_i^4$, or $2t_i \leq 1 + t_i^2$, which is clearly true, we then have

$$\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{t_j}{\sqrt{1+8t_i^3}}\right) \;\geq\; \sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{t_j}{1+2t_i^2}\right)\,.$$

Hence, to establish (1), it suffices to show that

$$\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{t_j}{1+2t_i^2}\right) \;\geq\; \frac{n(n-1)}{3}\,. \qquad (2)$$

Now,

$$\begin{aligned}
\sum_{j=1}^{n}\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{t_j}{1+2t_i^2}\right) &= \sum_{j=1}^{n}\left(-\frac{t_j}{1+2t_j^2} + \sum_{i=1}^{n}\frac{t_j}{1+2t_i^2}\right)\\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{t_j}{1+2t_i^2}\right) - \sum_{j=1}^{n}\frac{t_j}{1+2t_j^2}\\
&= \sum_{i=1}^{n}\frac{n}{1+2t_i^2} - \sum_{j=1}^{n}\frac{t_j}{1+2t_j^2}\\
&= \sum_{i=1}^{n}\frac{n-t_i}{1+2t_i^2}\,.
\end{aligned}$$

Thus, (2) becomes

$$\sum_{i=1}^{n} \frac{n - t_i}{1 + 2t_i^2} \;\geq\; \frac{n(n-1)}{3}\,. \tag{3}$$

To prove (3) we show that the inequality below holds for all positive real numbers x and all positive integers n, with equality if and only if $x = 1$:

$$\frac{n - x}{1 + 2x^2} \;\geq\; \left(\frac{7n-4}{9}\right) \;-\; \left(\frac{4n-1}{9}\right) x\,. \tag{4}$$

Note that (4) is equivalent, in succession, to

$$\begin{aligned} 9(n - x) &\geq \Big[(7n - 4) - (4n - 1)x\Big]\left(1 + 2x^2\right), \\ 0 &\leq (8n - 2)x^3 - (14n - 8)x^2 + (4n - 10)x + (2n + 4)\,, \\ 0 &\leq \Big[(4n - 1)x + (n + 2)\Big](x - 1)^2\,, \end{aligned}$$

and clearly the last inequality is true.

Using (4) and the fact that $t_1 + t_2 + \cdots + t_n = n$, we then have

$$\begin{aligned} \sum_{i=1}^{n} \frac{n - t_i}{1 + 2t_i^2} &\geq \sum_{i=1}^{n} \left(\frac{7n-4}{9} - \left(\frac{4n-1}{9}\right) t_i\right) \\ &= \frac{(7n-4)n}{9} - \frac{(4n-1)n}{9} \;=\; \frac{n(3n-3)}{9} \;=\; \frac{n(n-1)}{3}\,, \end{aligned}$$

establishing (3) and completing our proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3424. Correction. [2009 : 110, 112; 233, 235] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

For a positive integer m, let σ be the permutation of $\{0, 1, 2, \ldots, 2m\}$ defined by $\sigma(2i) = i$ for each $i = 0, 1, 2, \ldots, m$ and $\sigma(2i - 1) = m + i$ for each $i = 1, 2, \ldots, m$. Prove that there exists a positive integer k such that $\sigma^k = \sigma$ and $1 < k \leq 2m + 1$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

If $0 \leq i \leq m$, then $\sigma^{-1}(i) = 2i$; and if $m + 1 \leq i \leq 2m$, then $\sigma^{-1}(i) = 2(i - m) - 1 \equiv 2i \pmod{2m+1}$. Therefore, we have

$$\sigma^{-1}(i) \;\equiv\; 2i \pmod{2m+1} \tag{1}$$

for each $i \in \{0, 1, \ldots, 2m\}$.

By Euler's theorem, we have

$$2^{\phi(2m+1)} \equiv 1 \pmod{2m+1} , \tag{2}$$

where ϕ denotes Euler's totient function.

Using (1) and (2), we then have

$$\sigma^{-\phi(2m+1)}(i) \;\equiv\; 2^{\phi(2m+1)}i \;\equiv\; i \pmod{2m+1} ,$$

and it follows that $\sigma^{\phi(2m+1)}(i) = i$ for each i.

Hence, $\sigma^{\phi(2m+1)+1} = \sigma$, and since $2 \le \phi(2m+1)+1 \le 2m+1$, an appropriate choice for k would be $k = \phi(2m+1)+1$.

Also solved by MICHEL BATAILLE, Rouen, France; EDMUND SWYLAN, Riga, Latvia; and the proposer.

3425. [2009 : 110, 112] *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

For real $x \neq -1$, let $f(x) = \dfrac{e^x}{x+1}$. Prove that if $f(x) = f(y)$ for some $x \neq y$, then

$$\left(\sqrt{x+1} - \sqrt{y+1}\right)^2 \;\ge\; \ln f(y) .$$

Solution by Albert Stadler, Herrliberg, Switzerland.

The function $f(x)$ is negative and decreasing on $(-\infty,-1)$, positive and decreasing on $(-1,0]$, and positive and increasing on $[0,\infty)$. So, if $f(x) = f(y)$ for some $x \neq y$, then we can assume that $-1 < x \le 0 \le y$. Upon setting $u = x+1$ and $v = y+1$, the problem statement becomes:

Let $g(u) = \dfrac{e^u}{u}$, $u > 0$. Prove that if $g(u) = g(v)$ for some $u \neq v$, then

$$(\sqrt{u} - \sqrt{v})^2 \;\ge\; v - 1 - \ln v . \tag{1}$$

Thus, $g(1) = e$ is a local minimum value, and $0 < u \le 1 \le v$ with $u \neq v$.

Case 1. Here we assume that $v \ge 2$. Then $0 < u < 0.5$, since $g(2) > 3.6$ and $g(0.5) < 3.3$. From $g(u) = g(v)$ we obtain $\dfrac{1}{u} \le \dfrac{e^u}{u} \le \dfrac{e^v}{v} \le \dfrac{e^{0.5}}{u}$, and hence $ve^{-v} \le u \le ve^{0.5-v}$. We claim that

$$ve^{-v} + 1 + \ln v \;\ge\; 2ve^{\frac{1}{4}-\frac{v}{2}} . \tag{2}$$

holds for $v \ge 2$. Indeed, the inequality (2) holds for $v = 2$ and also

$$\begin{aligned} & \frac{d}{dv}\left(ve^{-v} + 1 + \ln v - 2ve^{\frac{1}{4}-\frac{v}{2}}\right) \\ = \;& -(v-1)e^{-v} + \frac{1}{v} - (v-2)e^{\frac{1}{4}-\frac{v}{2}} \\ = \;& \frac{e^{-v}}{v}\left(v(v-2)e^{\frac{1}{4}+\frac{v}{2}} + e^v - v(v-1)\right) \;\ge\; 0 , \end{aligned}$$

since $e^v \geq v(v-1)$ for $v > 0$; inequality (2) follows from these two facts.

We conclude that $u + 1 + \ln v \geq ve^{-v} + 1 + \ln v \geq 2ve^{\frac{1}{4}-\frac{v}{2}} \geq 2\sqrt{uv}$, which is equivalent to (1). This concludes Case 1.

Case 2. Here we assume that $1 \leq v \leq 2$. We will need two lemmas in order to settle this case.

Lemma 1 If $0 \leq t \leq 0.8$, then

$$1 - \sqrt{1-t} \;\geq\; \sqrt{\frac{t^2 + 0.5t - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1}}\,. \tag{3}$$

Proof: Both sides of (3) are positive, so we obtain an equivalent inequality by squaring and rearranging terms:

$$1 - t - 2\sqrt{1-t} \;\geq\; \sqrt{\frac{-1 - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1}}\,,$$

so we want to show that

$$\Psi(t) = 2\ln(t^2 + 0.5t + 1) - 2t^3 + t^2 - t + 4 - (4t^2 + 2t + 4)\sqrt{1-t} \geq 0\,.$$

We have $\Psi(0) = 0$ and

$$\frac{\Psi'(t)}{t} \;=\; \frac{(-12t^3 + 2t^2 + 12t - 11)\sqrt{1-t} + 5(2t-1)(2t^2 + t + 2)}{(2t^2 + t + 2)\sqrt{1-t}}\,,$$

so we are left to prove that the numerator is nonnegative for $0 < t \leq 0.8$.

For $0 \leq t \leq 0.6$ we have $11 - 12t^3 - 2t^2 - 12t$ and $\sqrt{1-t} \geq 1 - \frac{2}{3}t$, so that

$$\begin{aligned}
& (-12t^3 + 2t^2 + 12t - 11)\sqrt{1-t} + 5(2t-1)(2t^2 + t + 2) \\
\geq\;& (-12t^3 + 2t^2 + 12t - 11)(1 - \frac{2}{3}t) + 5(2t-1)(2t^2 + t + 2) \\
=\;& \frac{1}{3}(24t^4 + 28t^3 + 18t^2 - 13t + 3) \;\geq\; \frac{1}{3}(28t^2 - 13t + 3) \;>\; 0\,,
\end{aligned}$$

and the last inequality holds for $t > 0$.

For $0.6 \leq t \leq 0.8$ we have

$$\begin{aligned}
& \left|(-12t^3 + 2t^2 + 12t - 11)\sqrt{1-t} + 5(2t-1)(2t^2 + t + 2)\right| \\
\geq\;& 5(2t-1)(2t^2 + t + 2 - \left|12t^3 + 2t^2 + 12t - 11\right| \\
\geq\;& \begin{cases} 3.32 - 2.49 > 0\,, & \text{if } 0.6 \leq t \leq 0.7\,, \\ 7.63 - 6.1 \;> 0\,, & \text{if } 0.7 \leq t \leq 0.8\,. \end{cases}
\end{aligned}$$

This concludes the proof of Lemma 1. ∎

Let $t = \frac{v-u}{v}$. Then $0 < t < 1$, and we deduce from $\frac{e^u}{u} = \frac{e^v}{v}$ that $v - u = \ln\left(\frac{v}{u}\right)$, or equivalently

$$t \;=\; -\frac{1}{v}\ln(1-t) \;=\; \frac{1}{v}\sum_{k=1}^{\infty}\frac{t^k}{k}\,. \tag{4}$$

Dividing by t we obtain

$$v-1 \;=\; \sum_{k=2}^{\infty}\frac{t^{k-1}}{k}\,. \tag{5}$$

The function $\varphi(x) = \sum_{k=2}^{\infty}\frac{x^{k-1}}{k}$ satisfies $\varphi(0) = 1$, is strictly increasing on $(0,1)$, and $\varphi(x) \to \infty$ as $x \to 1$. Thus, for each $v \geq 1$, there is a unique $t = t(v)$ that satisfies (4) and $t(v)$ strictly decreases to 0 as v tends to 1. We now estimate $t(v)$ from below.

Lemma 2 If $1 \leq v \leq 2$, then $t = \frac{v-u}{v}$ satisfies $t^2 + \frac{t}{2} + 1 \geq v$, and this latter inequality holds if and only if $t \geq -\frac{1}{4} + \frac{1}{4}\sqrt{1+16(v-1)}$.

Proof: Since $g(0.4) > g(2)$, we have $0 < t(2) < 0.8$. Then we obtain that $t = \frac{v-u}{v} < \frac{2-0.4}{2} = 0.8$. We deduce from (4) and (5) that

$$\begin{aligned}
v-1 \;&=\; \sum_{k=2}^{\infty}\frac{t^{k-1}}{k} \\
&=\; \frac{t}{2} \;+\; t^2\sum_{k=3}^{19}\frac{t^{k-3}}{k} \;+\; t^{18}\sum_{k=20}^{\infty}\frac{t^{k-19}}{k} \\
&\leq\; \frac{t}{2} \;+\; t^2\sum_{k=3}^{19}\frac{t^{k-3}}{k} \;-\; t^{18}\ln(1-t) \\
&=\; \frac{t}{2} \;+\; t^2\sum_{k=3}^{19}\frac{t^{k-3}}{k} \;+\; t^{19}v \;\leq\; \frac{t}{2} \;+\; t^2\,,
\end{aligned}$$

since $\sum_{k=3}^{19}\frac{0.8^{k-3}}{k} + 2(0.8)^{17} < 1$. This yields $t^2 + \frac{t}{2} + 1 \geq v$. By straightforward calculations using the quadratic formula, we see that $t^2 + \frac{t}{2} - (v-1) \geq 0$ is equivalent to $t \geq \frac{-0.5+\sqrt{0.25+4(v-1)}}{2} = -\frac{1}{4} + \frac{1}{4}\sqrt{1+16(v-1)}$. ∎

Now $\phi(x) = \frac{x-1-\log x}{x}$ is monotonically increasing for $x > 1$. Then

$$\frac{t^2+0.5t-\ln(t^2+0.5t+1)}{t^2+0.5t+1} \;\geq\; \frac{v-1-\log v}{v}\,,$$

because $t^2 + \frac{t}{2} + 1 \geq v$ by Lemma 2. Finally, by using Lemma 1, we obtain

$$\begin{aligned}
\sqrt{v} - \sqrt{u} &= \sqrt{v}\left(1 - \sqrt{\frac{v-u}{v}}\right) \\
&= \sqrt{v}(1 - \sqrt{1-t}) \\
&\geq \sqrt{v}\sqrt{\frac{t^2 + 0.5t - \ln(t^2 + 0.5t + 1)}{t^2 + 0.5t + 1}} \\
&\geq \sqrt{v}\sqrt{\frac{v - 1 - \log v}{v}} \\
&= \sqrt{v - 1 - \log v}\,,
\end{aligned}$$

and squaring yields inequality (1).

This concludes Case 2, and all is proved.

One incomplete solution was received which used the method of Lagrange multipliers but without beforehand establishing the existence of extrema.

The proposer commented that he discovered the result by "some other considerations" and that he sought an elementary solution to the problem.