SKOLIAD No. 121

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 June, 2010**. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Niels Henrik Abel Mathematics Contest, 2008–2009, Second Round. Our thanks go to Øyvind Bakke of the Norwegian University of Science and Technology, Trondheim, Norway, for providing us with this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

Concours mathématique Niels Henrik Abel, 2008–2009

2^e ronde

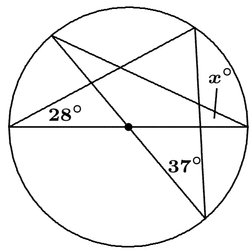
Durée de 100 minutes

1. On dispose les entiers positifs impairs dans un tableau triangulaire tel qu'illustré. Quelle est la somme des entiers dans les sept premières rangées ?

			1
		3	5
	7	9	11
13	15	...	...

2. Un grand panier contient beaucoup d'oeufs. Si on enlève les oeufs deux à la fois, à la toute fin il ne restera qu'un seul oeuf. La même chose se produit si on enlève les oeufs trois à la fois, ou quatre ou cinq ou six à la fois, mais si on enlève les oeufs sept à la fois, on vide le panier complètement. Le panier contenait au moins quel nombre d'oeufs ?

3. Deux des angles d'une étoile à cinq pointes sont $28°$ et $37°$, tel qu'illustré. Tous les sommets se trouvent sur un cercle, où le point indiqué est le centre du cercle. Déterminer la mesure de l'angle x.



4. Lancelot et cinq autres chevaliers sont assis autour d'une table ronde. Or chacun des six chevaliers trouve moyen de se faire ennemi avec ses deux voisins. Combien de manières y a-t-il d'asseoir les six chevaliers de façon à ce que Lancelot garde son siège et qu'aucun des six chevaliers soit assis voisin d'un de ses deux ennemis ?

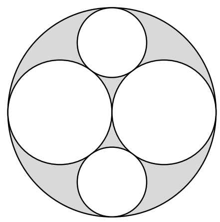
5. La moyenne arithmétique A, de deux nombres réels, x et y, est $\frac{1}{2}(x+y)$ et la moyenne géométrique G, est $\sqrt{xy}$. Déterminer $\frac{y}{x}$ si $3A = 5G$.

6. Une fonction f est telle que pour tout entier positif n, on a

$$f(n+1) \;=\; \frac{f(n)}{1+af(n)}\,,$$

où a est un nombre réel, $f(1) = 1$, et $f(9) = \frac{1}{2009}$. Déterminer a.

7. Le grand cercle a un rayon égal à $\frac{30}{\sqrt{\pi}}$. Les cercles moyens sont tangents l'un à l'autre, au centre du grand cercle. De plus, les cercles moyens sont tangents aux petits cercles, et le grand cercle est tangent à tous les cercles qu'il contient. Déterminer la surface de la région ombragée.



8. Déterminer la somme de tous les entiers positifs n, tels que $2009 + n^2$ est le carré d'un entier positif.

9. Les points $(23, 32)$, $(8, 41)$ et $(17, 45)$ sont les points milieux des côtés d'un triangle. Déterminer la plus grosse valeur possible de $x + y$, où (x, y) est un sommet du triangle.

10. Karine et Mathieu lancent une pièce de monnaie. Chaque fois que la pièce de monnaie montre face, Karine gagne un point et chaque fois qu'elle montre pile, Mathieu gagne un point. La personne gagnante est la première personne à atteindre six points ou aussi atteindre au moins quatre points avec une avance d'au moins trois points. Combien de jeux différents sont-ils possibles ; c'est-à-dire quel est le nombre de suites différentes de lancers de pièce de monnaie, du début jusqu'à la déclaration du gagnant ?

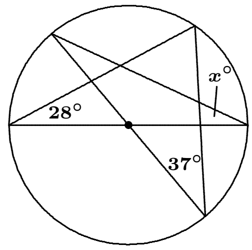
Niels Henrik Abel Mathematics Contest, 2008–2009
2nd Round
100 minutes allowed

1. Arrange the positive odd numbers in a triangular diagram as shown. What is the sum of the numbers in the first seven rows?

$$\begin{array}{cccc} & & & 1 \\ & & 3 & 5 \\ & 7 & 9 & 11 \\ 13 & 15 & \dots & \dots \end{array}$$

2. A large basket contains many eggs. If you remove the eggs two at a time, a single egg remains in the basket. The same happens if you remove the eggs three at a time, or four or five or six at a time, but if you remove the eggs seven at a time, you empty the basket completely. At least how many eggs were there in the basket?

3. Two of the angles in a five-pointed star are $28°$ and $37°$, as shown. All the vertices lie on a circle, and the indicated point is the centre of the circle. Find the measure of angle x.



4. Sir Lancelot and five other knights sit at a round table. All six knights manage to make enemies of both their neighbours. In how many ways can the six knights sit around the table if Sir Lancelot is to keep his seat and no one sits next to either of his new enemies?

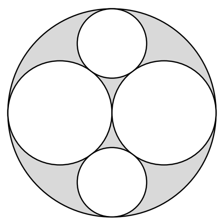
5. The arithmetic mean, A, of two real numbers, x and y, is $\frac{1}{2}(x+y)$ and their geometric mean, G, is $\sqrt{xy}$. Find $\frac{y}{x}$ if $3A = 5G$.

6. A function, f, is such that for each positive integer n,

$$f(n+1) = \frac{f(n)}{1 + af(n)}\,,$$

where a is a real number, $f(1) = 1$, and $f(9) = \frac{1}{2009}$. Find a.

7. The large circle has radius $\frac{30}{\sqrt{\pi}}$. The medium circles are tangent to one another at the centre of the large circle. Moreover, the medium circles are tangent to the small circles, and the large circle is tangent to all the circles it contains. Find the area of the shaded region.



8. Find the sum of all positive integers, n, such that $2009 + n^2$ is the square of a positive integer.

9. The points $(23, 32)$, $(8, 41)$, and $(17, 45)$ are the midpoints of the sides of a triangle. Find the largest possible value of $x + y$ where (x, y) is a vertex of the triangle.

10. Kari and Mons are tossing a coin. Each time they toss heads, Kari earns a point, and each time they toss tails, Mons earns a point. The person who first reaches six points or who has at least four points and leads by at least three points wins. How many different games are possible; that is, how many different sequences of coin tosses end in a win?

Next follow the solutions to the British Columbia Secondary School Mathematics Contest 2007, Final Round, Part B [2009 : 65–68].

1. Joan has a collection of nickels, dimes, and quarters worth $\$2.00$. If the nickels were dimes and the dimes were nickels, the value of the coins would be $\$1.70$. Determine all of the possibilities for the number of nickels, dimes, and quarters that Joan could have.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Suppose Joan has n nickels, d dimes, and q quarters. Then the total value of the coins is $5n + 10d + 25q = 200$, and with the nickels and dimes switched $5d + 10n + 25q = 170$. Subtracting the latter equation from the former yields that $5d - 5n = 30$, so $d = n + 6$.

Substituting $d = n + 6$ into the equation $5n + 10d + 25q = 200$ yields $5n + 10(n + 6) + 25q = 200$, so $15n + 60 + 25q = 200$, so $n = \dfrac{(28 - 5q)}{3}$. Since $n \geq 0$, it follows that $q \leq 5$. You may now try all the possible values for q in turn:

q	0	1	2	3	4	5
$n = \dfrac{(28-5q)}{3}$	$\frac{28}{3}$	$\frac{23}{3}$	6	$\frac{13}{3}$	$\frac{8}{3}$	1

The only integer solutions for (n, d, q) are $(6, 12, 2)$ and $(1, 7, 5)$.

Also solved by OSCAR XIA, student, St. George's School, Vancouver, BC.

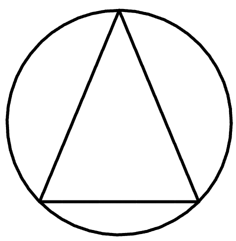
2. A $3 \times 3 \times 3$ cube is formed by stacking $1 \times 1 \times 1$ cubes. Determine the total number of cubes with sides of integral length that are contained in the $3 \times 3 \times 3$ cube.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

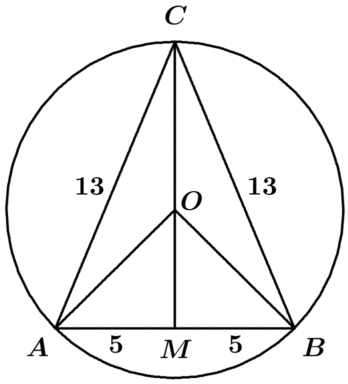
The cube contains one $2 \times 2 \times 2$ cube for each of the 8 vertices in addition to 27 small $1 \times 1 \times 1$ cubes and the $3 \times 3 \times 3$ cube itself. In all 36 cubes.

Also solved by JEREMY TSE, student, Burnaby North Secondary School, Burnaby, BC; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and OSCAR XIA, student, St. George's School, Vancouver, BC.

3. The lengths of the sides of a triangle are 13, 13, and 10. The circumscribed circle of a triangle is the circle that goes through each of the three vertices of the triangle and here has its centre inside the triangle (see the diagram at right). Find the radius of the circumscribed circle.



Solution by Oscar Xia, student, St. George's School, Vancouver, BC.



Let r be the radius of the circle through the vertices A, B, C, and let M be the midpoint of AB. Since $\triangle ABC$ is isosceles, $\angle AMC = 90°$. Therefore, the Pythagorean Theorem applies and $CM = \sqrt{13^2 - 5^2} = 12$. It follows that $MO = 12 - r$. Using the Pythagorean Theorem again, $AM^2 + MO^2 = AO^2$, so we have $5^2 + (12 - r)^2 = r^2$. Solving the equation yields that $r = \frac{169}{24}$.

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

4. The game of End View consists of a tableau with a four by four grid, one additional row at the top and at the bottom, and one additional column on the right and on the left. The letters A, B, and C are placed in the four by four grid in such a way that every letter appears exactly once in each row and each column. This means that there will be exactly one empty square in each row and each column. Letters are placed in the additional rows and columns as hints, at the end of some rows and columns of the four by four grid, to indicate the nearest letter that can be found by reading that row or column of the grid. The diagram below shows the starting tableau and the resulting solution tableau for a game of End View.

	B			C	
C					
A					B
	A		B		

Start

→

	B			C	
	B		A	C	
C	C	B		A	
A		A	C	B	B
	A	C	B		
	A		B		

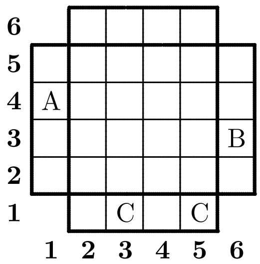
Solution

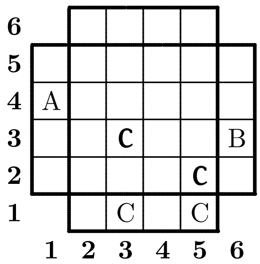
The diagram for another game of End View is shown at right.

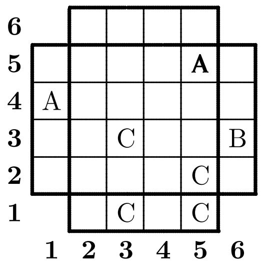
Fill in this tableau with the complete solution. Give a justification of the steps that you used to find the solution.

A					
					B
		C		C	

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.





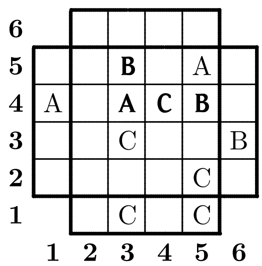


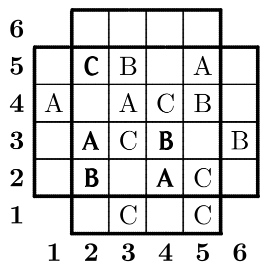
Introduce a coordinate system as in the figures above. Then consider columns 3 and 5. Since each must have each of the three letters, and C must be the closest one to the bottom, only one of the bottom two spaces of these columns can be occupied with C, and the other space not occupied.

Now, in column 5, if C occupied $(5, 3)$ then B would not be the closest letter to the right extra column. Therefore $(5, 2)$ must be C. Then in row 2, there cannot be another C, so in column 3, C must not be at the bottom square, so it must be at $(3, 3)$. We have arrived at the middle figure above.

In row 4, A must be the farthest to the left. Therefore the A in column 5 cannot be at $(5, 4)$. It also cannot be at $(5, 3)$, since B must be the farthest right in row 3. Therefore an A is at $(5, 5)$. We have arrived at the rightmost figure above.

Now in column 3, A cannot be at $(3, 5)$, so it must be at $(3, 4)$. Then B must be at $(3, 5)$. Then in row 4, B and C must be in the rightmost two squares, and since column 5 has a C, the C goes in $(4, 4)$, and the B in $(5, 4)$. See the leftmost figure below.





From here, it is easy to complete the tableau. The last C has to go in $(2, 5)$. In row 3, B must be at $(4, 3)$, and A at $(2, 3)$. Finally, row 2 has A at $(4, 2)$ and B at $(2, 2)$. The finished tableau is shown in the rightmost figure immediately above.

Also solved by OSCAR XIA, student, St. George's School, Vancouver, BC.

5. Determine all of the positive integer solutions, x and y, to the equation

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since both x and y are positive, $\frac{1}{x} > \frac{1}{12}$, so $x < 12$. Solving the equation $\frac{1}{x} - \frac{1}{y} = \frac{1}{12}$ for y yields $y = \frac{12x}{12-x}$. You may now try the possible values for x in turn:

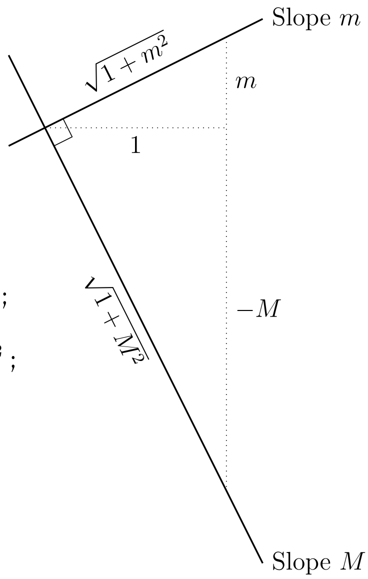
x	1	2	3	4	5	6	7	8	9	10	11
$y = \frac{12x}{12-x}$	$\frac{12}{11}$	$\frac{12}{5}$	4	6	$\frac{60}{7}$	12	$\frac{84}{5}$	24	36	60	132

Thus the only positive integer solutions for (x, y) are $(3, 4)$, $(4, 6)$, $(6, 12)$, $(8, 24)$, $(9, 36)$, $(10, 60)$, and $(11, 132)$.

Also solved by OSCAR XIA, student, St. George's School, Vancouver, BC.

Here is a proof that if perpendicular lines have slopes m and M, then $mM = -1$:

$$\left(\sqrt{1+m^2}\right)^2 + \left(\sqrt{1+M^2}\right)^2 = (m-M)^2\,;$$
$$(1+m^2) + (1+M^2) = m^2 - 2mM + M^2\,;$$
$$2 = -2mM\,;$$
$$mM = -1\,.$$



This concludes another Skoliad. This issue's prize for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON. We are looking forward to the results of our readers' efforts.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **1^er^ mars 2010**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M413. *Proposé par l'Équipe de Mayhem.*

Trouver le nombre d'entiers positifs formés de trois chiffres dont le produit donne 36.

M414. *Proposé par l'Équipe de Mayhem.*

On considère la liste des entiers positifs, rangés en ordre croissant, pouvant être exprimés comme la somme de 21 entiers (non nécessairement positifs). Déterminer le 21^{e} entier de cette liste.

M415. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Les côtés AB et CD d'un trapèze $ABCD$ sont parallèles. Si $AB = 15$, $CD = 30$, $AD = 9$ et $BC = 12$, trouver l'aire du trapèze $ABCD$.

M416. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Montrer que 9 est un diviseur de $10^n + 3(4^{n+2}) + 5$ pour tous les entiers n non négatifs.

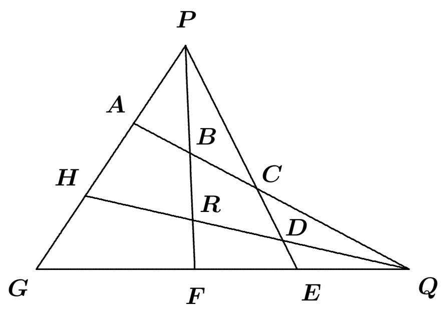
M417. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $M = \{x^2 + 4xy + y^2 : x, y \in \mathbb{Z}\}$. Montrer que le nombre 2022 appartient à M, mais pas le nombre 11.

M418. *Proposé par Geoffrey A. Kandall, Hamden, CT, É-U.*

Dans la figure, F est sur GE et Q sur le prolongement de GE. De plus, A et H sont sur PG de sorte que QA coupe PF en B et PE en C, et que QH coupe PE en D. Montrer que

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FG} \cdot \frac{GH}{HA} = 1 .$$



M419. *Proposé par Joe Howard, Portales, NM, É-U.*

Soit a, b et c les longueurs des côtés d'un triangle. Montrer que

$$\frac{a(b+c)}{a^2+bc} + \frac{b(c+a)}{b^2+ca} + \frac{c(a+b)}{c^2+ab} \leq 3 .$$

. . . .

M413. *Proposed by the Mayhem Staff.*

Determine the number of three-digit positive integers whose digits have a product of 36.

M414. *Proposed by the Mayhem Staff.*

The positive integers that can be expressed as the sum of 21 consecutive (not necessarily positive) integers are listed in increasing order. Determine the 21^{st} integer in this list.

M415. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

In trapezoid $ABCD$, AB and CD are parallel. If $AB = 15$, $CD = 30$, $AD = 9$, and $BC = 12$, determine the area of the trapezoid.

M416. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Prove that 9 divides $10^n + 3(4^{n+2}) + 5$ for all nonnegative integers n.

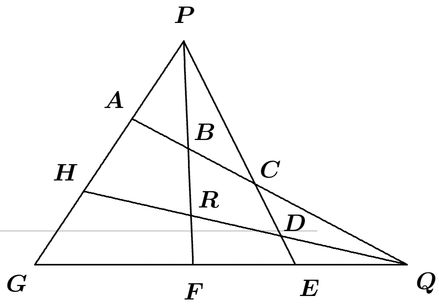
M417. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $M = \{x^2 + 4xy + y^2 : x,\, y \in \mathbb{Z}\}$. Prove that the number 2022 is in M but that the number 11 is not in M.

M418. *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

In the diagram, F lies on GE and Q lies on GE extended. Also, A and H are on PG so that QA intersects PF at B, QA intersects PE at C, and QH intersects PE at D. Prove that

$$\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FG}\cdot\frac{GH}{HA} = 1\,.$$



M419. *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, and c be the side lengths of a triangle. Prove that

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \le 3\,.$$

Mayhem Solutions

M382. *Proposed by the Mayhem Staff.*

Determine all pairs (x, y) of integers for which $4x^2 - y^2 = 480$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

First, we note that if (x, y) is a solution of the given equation with x and y integers, then so are $(x, -y)$, $(-x, y)$, and $(-x, -y)$. Hence, it suffices to find all solutions (x, y) in which $x \ge 0$ and $y \ge 0$.

Since $4x^2$ and 480 are both even, then y^2 is even, so y is even. We thus set $y = 2z$ for some nonnegative integer z. This yields $4x^2 - (2z)^2 = 480$, or $4x^2 - 4z^2 = 480$, or $x^2 - z^2 = 120$, or $(x - z)(x + z) = 2^4 \cdot 3 \cdot 5$.

Next we note that $(x - z) + (x + z) = 2x$, which is even, so $x - z$ and $x + z$ must both be even integers or both odd integers. Since their product is 120 (which is even), then each is even. Also, $x - z \le x + z$ since $z \ge 0$.

We make a chart to summarize the possible values for $x - z$ and $x + z$, knowing that they are even positive integers whose product is 120. We obtain $2x$ by adding $x - z$ and $x + z$, and we recover z by subtracting x from $x + z$:

$x-z$	$x+z$	$2x$	x	z	y
2	60	62	31	29	58
4	30	34	17	13	26
6	20	26	13	7	14
10	12	22	11	1	2

Therefore, the nonnegative solutions are $(31, 58)$, $(17, 26)$, $(13, 14)$, and $(11, 2)$. Thus, the complete integer solution of $4x^2 - y^2 = 480$ consists of the 16 pairs $(\pm 31, \pm 58)$, $(\pm 17, \pm 26)$, $(\pm 13, \pm 14)$, and $(\pm 11, \pm 2)$, where all possible combinations of signs are taken.

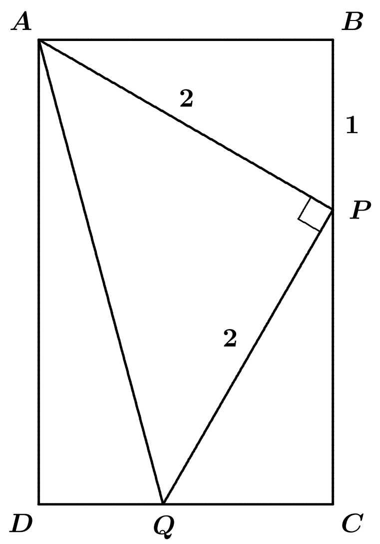
Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were four incorrect and one incomplete solutions submitted.

M383. *Proposed by the Mayhem Staff.*

In rectangle $ABCD$, P is on side BC and Q is on side DC so that $BP = 1$, $AP = PQ = 2$ and $\angle APQ = 90^\circ$. Determine the length of QD.

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

Using the given information, we draw the diagram at right. In the diagram,



$$\angle APQ = \angle ABP = \angle PCQ = 90^\circ, \quad AB = DC, \text{ and } AD = BC.$$

Since $\triangle APB$ has a right angle at B, $AP = 2$, and $BP = 1$, then $\triangle APB$ is a 30°–60°–90° triangle, so $AB = \sqrt{3}$ and $\angle APB = 60^\circ$.

Next, we see that

$$\begin{aligned} \angle QPC &= 180^\circ - \angle APQ - \angle APB \\ &= 180^\circ - 90^\circ - 60^\circ = 30^\circ. \end{aligned}$$

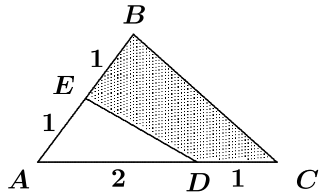
Since $\triangle QPC$ has a 30° angle and a 90° angle, then it is also a 30°–60°–90° triangle. Therefore, $\angle PQC = 60^\circ$ and $QC = \frac{1}{2}QP = 1$.

Lastly, $QD = DC - QC = AB - QC = \sqrt{3} - 1$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; CHRISTOPHER WIRIAWAN, student, Surya Institute, BSD City, Indonesia; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M384. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

In the diagram at right, the point E is on AB and the point D is on AC such that $AE = EB = DC = 1$ and $AD = 2$. Determine the ratio of the area of quadrilateral $BCDE$ to the area of triangle ABC.



Solution by Scott Brown, Auburn University, Montgomery, AL, USA.

We use the notation $[\triangle ABC]$ for the area of $\triangle ABC$ and $[BCDE]$ for the area of quadrilateral $BCDE$.

First, we note that $[\triangle ABC] = [\triangle AED] + [BCDE]$. We will determine the ratio of $[\triangle ABC]$ to $[\triangle AED]$ and use this to determine the required ratio. To do this, we use the property that triangles with equal altitudes have their areas in the same ratio as the lengths of their bases.

Therefore,

$$\frac{[\triangle ABC]}{[\triangle EAD]} = \frac{[\triangle ABC]}{[\triangle ADB]} \cdot \frac{[\triangle ADB]}{[\triangle EAD]} = \frac{AC}{AD} \cdot \frac{AB}{AE} = \frac{3}{2} \cdot \frac{2}{1} = \frac{3}{1}.$$

Thus, $[\triangle EAD]$ is $\frac{1}{3}$ of $[\triangle ABC]$. This implies that $[BCDE]$ is $\frac{2}{3}$ of $[\triangle ABC]$.

In summary, $\dfrac{[BCDE]}{[\triangle ABC]} = \dfrac{2}{3}$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; CHRISTOPHER WIRIAWAN, student, Surya Institute, BSD City, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There was one incomplete solution submitted.

M385. *Proposed by Mihály Bencze, Brasov, Romania.*

The base 10 integer $N = 1\cdots114\cdots44$ starts off with 2009 consecutive digits 1 followed by 4018 consecutive digits 4. Prove that N is not a perfect square.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Considering N modulo 16, we see that $N \equiv 12 \pmod{16}$, as follows:

$$\begin{aligned} 1\cdots114\cdots44 &\equiv 4444 \pmod{16} && \text{(since 10 000 is a multiple of 16)} \\ &\equiv 44 \pmod{16} && \text{(since 400 is a multiple of 16)} \\ &\equiv 12 \pmod{16}\ . \end{aligned}$$

Any integer x is congruent modulo 16 to one of 0, ± 1, ± 2, ± 3, ± 4, ± 5, ± 6, ± 7, or 8; so x^2 is congruent to one of 0, 1, 4, 9, 0, 9, 4, 1, or 0.

Therefore, every perfect square is congruent to 0, 1, 4, or 9 modulo 16, and we conclude that N cannot be a perfect square.

Also solved by MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CHRISTA SOESANTO, student, Surya Institute, BSD City, Indonesia; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incomplete solutions submitted.

M386. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all real numbers x for which

$$\sqrt{2+4x-2x^2} \;+\; \sqrt{6+6x-3x^2} \;=\; x^2-2x+6\,.$$

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

We have

$$\begin{aligned}
&\sqrt{2+4x-2x^2}+\sqrt{6+6x-3x^2}\\
&\quad= \sqrt{4-\left(2x^2-4x+2\right)}+\sqrt{9-\left(3x^2-6x+3\right)}\\
&\quad= \sqrt{4-2(x-1)^2}+\sqrt{9-3(x-1)^2}\\
&\quad\le \sqrt{4-2(0)}+\sqrt{9-3(0)} \;=\; 2+3 \;=\; 5 \;=\; 0+5\\
&\quad\le (x-1)^2+5 \;=\; x^2-2x+6\,.
\end{aligned}$$

For the first expression to actually equal the final expression, it must be that both inequalities are actually equalities, and so $(x-1)^2=0$ or $x=1$.

Thus, the only possible solution to the given equation is $x=1$. We can verify by substitution that $x=1$ is indeed a solution.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; G.C. GREUBEL, Newport News, VA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; FRANCISCA SUSAN, student, Surya Institute, BSD City, Indonesia; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incorrect and one incomplete solutions submitted.

M387. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Temperature can be measured in degrees Fahrenheit (F) or in degrees Celsius (C). The two scales are related by the formula $F = 1.8C + 32$. When a two-digit integer degree temperature in Celsius is converted to Fahrenheit and rounded to the nearest integer degree, it turns out the ones and tens digits of the original Celsius temperature C sometimes switch places to give the rounded Fahrenheit equivalent. Find all two-digit integer values of C for which this occurs.

Solution by the Mayhem Staff.

Consider a two-digit temperature $C = 10a + b$ in degrees Celsius, where a and b are integers with $1 \le a \le 9$ and $0 \le b \le 9$.

The equivalent temperature in degrees Fahrenheit is

$$F = \frac{9}{5}C + 32 = \frac{9}{5}(10a + b) + 32 = 18a + \frac{9}{5}b + 32 .$$

We want the rounded version of this real number to equal $10b+a$. Therefore,

$$\begin{array}{rcccl} 10b + a - \frac{1}{2} & \le & 18a + \frac{9}{5}b + 32 & < & 10b + a + \frac{1}{2}\,; \\ 100b + 10a - 5 & \le & 180a + 18b + 320 & < & 100b + 10a + 5\,; \\ -325 & \le & 170a - 82b & < & -315\,; \\ 315 & < & 82b - 170a & \le & 325\,. \end{array}$$

Since $b \le 9$, then $82b \le 738$. Since $82b - 170a > 315$, then $170a < 82b - 315 < 738 - 315 = 423$, whence $a < \frac{423}{170} = 2\frac{83}{170}$. Since a is an integer, then $a \le 2$. Therefore, we only need to try $a = 1$ and $a = 2$.

If $a = 1$, the inequalities become $315 + 170(1) < 82b \le 325 + 170(1)$ or $485 < 82b \le 495$ or $5\frac{75}{82} < b \le 6\frac{3}{82}$; since b is an integer, then $b = 6$.

Similarly, if $a = 2$, then $b = 8$.

Hence, the two possibilities are $C = 16$ (giving $F = 60.8$, which rounds to $F \approx 61$) and $C = 28$ (giving $F = 82.4$, which rounds to $F \approx 82$).

Also solved by MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; There was one incomplete solution submitted.

Most submitted solutions involved an explicit or implicit complete enumeration of cases from $C = 10$ to $C = 39$ after some examination of bounds.

Problem of the Month

Ian VanderBurgh

What's in a definition? Mathematics is littered with them. Often, we pay attention to them; sometimes we treat them a bit cavalierly. Here are two problems involving geometric sequences. In the second of these problems, the precision of our definition turns out to affect the answer.

Problem 1 (2009 American Invitational Mathematics Examination) Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

Problem 2 (2009 Euclid Contest) If $\log_2 x$, $(1 + \log_4 x)$, and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

So what's a geometric sequence? Many of you will know this already, but by way of reminder, here is our first attempt at a definition:

> **Definition 1**: A geometric sequence is a sequence of numbers in which each term after the first is obtained from the previous term by multiplying by a constant.

Often, we would call the first term in the sequence a and the multiplying factor r, which gives the sequence a, ar, ar^2, ar^3, $\dots$. (You may notice that I've deliberately avoided the phrase "common ratio" – stay tuned!) Let's use this version of the definition to solve the first problem.

Solution to Problem 1 The smallest 3-digit integers have hundreds digit 1. Let's see if any of these integers is *geometric*. Call the tens digit of our candidate number r (note that r is an integer). Since the hundreds digit is 1, the tens digit is r, and the digits form a geometric sequence, then the units digit is r^2. The candidate 3-digit integer is as small as possible when r is as small as possible. Since the digits are distinct, then $r \neq 1$ (otherwise $r = 1$ would give 111) and $r \neq 0$ (otherwise $r = 0$ would give 100). So the smallest candidate occurs when $r = 2$, which yields the integer 124, which must be the smallest 3-digit integer that is geometric.

The largest 3-digit integers have hundreds digit 9. Let's see if any of these integers are geometric. Consider a candidate integer and suppose that the multiplying factor between consecutive digits is R. Then the tens digit is $9R$ and the units digit is $9R^2$. Since we want this integer to be as large as possible, we try the different possibilities for $9R$. If $9R = 9$, then $R = 1$, which would give the integer 999, which violates the condition of distinct digits. If $9R = 8$, then $R = \frac{8}{9}$; in this case, $9R^2 = \frac{64}{9}$, which is not an integer. If $9R = 7$, then $R = \frac{7}{9}$; in this case, $9R^2 = \frac{49}{9}$, which is not an integer. If $9R = 6$, then $R = \frac{2}{3}$, whence $9R^2 = (9R)R = 6R = 4$, which yields the integer 964, which is thus the largest 3-digit integer that is geometric.

Thus, the difference between the largest and smallest 3-digit integers that are geometric is $964 - 124 = 840$. ■

At this point, you're probably wondering about the preamble – the definition doesn't seem to be affecting anything so far. Here's another crack at the definition of a geometric sequence:

> **Definition 2**: A geometric sequence is a sequence of numbers with the property that if a, b, c are consecutive terms, then $b^2 = ac$.

And another one:

Definition 3: A geometric sequence is a sequence of numbers with the property that if a, b, c are consecutive terms, then $\frac{b}{a} = \frac{c}{b}$.

Again, you may wonder what the big deal is all about. So I have a question for you: is 1, 0, 0 a geometric sequence? What do the different versions of the definition tell you?

Solution to Problem 2 First, we express the logarithms in the three terms using a common base, namely the base 2. We obtain:

$$\begin{aligned} 1+\log_4 x &= 1+\frac{\log_2 x}{\log_2 4} = 1+\frac{1}{2}\log_2 x\,; \\ \log_8 4x &= \frac{\log_2 4x}{\log_2 8} = \frac{\log_2 4+\log_2 x}{3} = \frac{2}{3}+\frac{1}{3}\log_2 x\,. \end{aligned}$$

Next, we make the substitution $u = \log_2 x$ to make the next calculations less cumbersome. In terms of u our sequence is thus u, $1+\frac{1}{2}u$, $\frac{2}{3}+\frac{1}{3}u$.

Since this sequence is geometric, then

$$\begin{aligned} \left(1+\tfrac{1}{2}u\right)^2 &= u\left(\tfrac{2}{3}+\tfrac{1}{3}u\right); \\ 3(2+u)^2 &= 4u(2+u) \qquad \text{(multiplying by 12)}\,; \\ 12+12u+3u^2 &= 4u^2+8u\,; \\ 0 &= u^2-4u-12\,; \\ 0 &= (u-6)(u+2)\,; \end{aligned}$$

and so $u = \log_2 x = 6$ or $u = \log_2 x = -2$, hence $x = 64$ or $x = \frac{1}{4}$. ■

So what's the big deal? Let's look at what the sequences are for the two possible values of x.

If $x = 64$ (or $u = 6$), the sequence is 6, 4, $\frac{8}{3}$, which is geometric and seems pretty innocuous.

If $x = \frac{1}{4}$ (or $u = -2$), the sequence is -2, 0, 0. Oh dear! Why is this a problem? Which definition are *you* using? This sequence is geometric by Definition 1 and Definition 2, but according to Definition 3 it is NOT geometric. So the choice of definition (that is, one's particular convention) changes the answer to Problem 2. Using Definition 1 or Definition 2, the answer is $x = 64$ or $x = \frac{1}{4}$; but using Definition 3, the answer is $x = 64$ only.

There is a happy ending to this saga, though. Luckily, as the 2009 Euclid Contest was being pre-marked, the markers were alerted to this dilemma of differences of definitions and both versions, with proper justification, were accepted as correct.

So pay attention to definitions – are they completely precise? And think critically about seemingly equivalent definitions – are they really equivalent?

THE OLYMPIAD CORNER

No. 282

R.E. Woodrow

We begin this number with the problems of the Austrian Mathematical Olympiad 2007 National Competition, Final Round, Part 1, written May 17th 2007. Thanks go to Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam, for collecting them for our use.

AUSTRIAN MATHEMATICAL OLYMPIAD 2007
National Competition – Final Round – Part 1
May 17, 2007

1. We are given a 2007×2007 grid. An odd integer is written in each of its cells. Let Z_i be the sum of the numbers in the i^{th} row and S_j the sum of the numbers in the j^{th} column for $1 \le i, j \le 2007$. Furthermore, let $A = \prod_{i=1}^{2007} Z_i$ and $B = \prod_{j=1}^{2007} S_j$. Show that $A + B$ cannot be equal to zero.

2. Determine the largest possible value of $C(n)$ for all positive integers n, such that

$$(n+1)\sum_{j=1}^{n} a_j^2 - \left(\sum_{j=1}^{n} a_j\right)^2 \;\ge\; C(n)\,,$$

holds for all n-tuples $(a_1, a_2, \dots, a_n)$ of pairwise distinct integers.

3. Let $M(n) = \{-1, -2, \dots, -n\}$. For each nonempty subset of $M(n)$ we form the product of the elements. What is the sum of all such products?

4. Let $n > 4$ be an integer. The n-gon $A_0A_1 \dots A_{n-1}A_n$ (with $A_n = A_0$), is inscribed in a circle, is convex, and is such that the lengths of the sides are $A_{i-1}A_i = i$ for $1 \le i \le n$. Let ϕ_i be the angle between the line A_iA_{i+1} and the tangent to the circumcircle of the n-gon at A_i. (Note that the angle between any two lines is at most $90°$.) Determine the value of

$$\Phi \;=\; \sum_{i=0}^{n-1} \phi_i\,.$$

Next we continue with the problems of the two days of Part 2 of the National Competition Final Round Austrian Mathematical Olympiad 2007. Again we thank Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam, for collecting them for us.

AUSTRIAN MATHEMATICAL OLYMPIAD 2007
National Competition – Final Round – Part 2
June 5-6, 2007

1. Determine all nonnegative integers $a < 2007$, for which the congruence $x^2 + a \equiv 0 \pmod{2007}$ has exactly two distinct nonnegative integer solutions smaller than 2007. (In other words, there exist two nonnegative integers u and v each less than 2007, such that u^2+a and v^2+a are divisible by 2007.)

2. Determine all sextuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of nonnegative integers satisfying the following system of equations:

$$x_1x_2(1-x_3) = x_4x_5\,, \qquad x_4x_5(1-x_6) = x_1x_2\,,$$
$$x_2x_3(1-x_4) = x_5x_6\,, \qquad x_5x_6(1-x_1) = x_2x_3\,,$$
$$x_3x_4(1-x_5) = x_6x_1\,, \qquad x_6x_1(1-x_2) = x_3x_4\,.$$

3. Determine all rhombuses with sides of length $2a$, for which a circle exists with the property that each of the four sides of the rhombus intersects the circle producing a chord of length a.

4. Let M be the set of all polynomials $P(x)$ with pairwise distinct integer roots and integer coefficients whose absolute values are all less than 2007. What is the highest degree among all polynomials in M?

5. We are given a convex n-gon with a triangulation, that is, a division into triangles by nonintersecting diagonals. Prove that the n corners of the n-gon can each be labelled by the digits of 2007 such that any quadrilateral composed of two triangles in the triangulation with a common side has corners labelled by digits that sum up to 9.

6. We are given a triangle ABC with circumcentre U. A point P is chosen on the extension of UA beyond A. Let g denote the line symmetric to PB with respect to BA and h the line symmetric to PC with respect to AC. Let the lines g and h intersect at the point Q.

Determine the set of all points Q as P varies on the ray UA beyond A.

The next problems we give are those of the XXI Olimpiadi Italiane della Matematica written at Cesenatico, 11 May 2007. Thanks go to Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam, for collecting them for our use.

XXI OLIMPIADI ITALIANE DELLA MATEMATICA
Cesenatico, 11 May 2007

1. A regular hexagon is given in the plane. For each point P of the plane, let $\ell(P)$ be the sum of the six distances of P from each line determined by a side of the hexagon, and let $v(P)$ be the sum of the six distances of P from the *vertices* of the hexagon.

(a) For which points P of the plane does $\ell(P)$ take its least value?

(b) For which points P of the plane does $v(P)$ take its least value?

2. Polynomials with integer coefficients, $p(x)$ and $q(x)$, are *similar* if they have the same degree and the same coefficients (possibly in different order).

(a) If $p(x)$ and $q(x)$ are similar, prove that $p(2007) - q(2007)$ is even.

(b) Is there an integer $k > 2$ such that $p(2007) - q(2007)$ is divisible by k whenever $p(x)$ and $q(x)$ are similar?

3. Triangle ABC has centroid G, $D \neq A$ is a point on the line AG such that $AG = GD$, and $E \neq B$ is a point on the line GB such that $GB = GE$. The midpoint of AB is M. Prove that the quadrilateral $BMCD$ can be inscribed in a circle if and only if $BA = BE$.

4. On Barbara's birthday Alberto invites her to play a game. Given the numbers 0, 1, 2, $\dots$, 1024, Barbara removes 2^9 of them. Then Alberto removes 2^8 numbers from the ones that remain. Next, Barbara removes 2^7 numbers, and so on, until only two numbers a and b remain. Alberto then gives $|a-b|$ euros to Barbara.

Find the largest amount of euros that Barbara is certain to win, regardless of how Alberto plays.

5. Let x_1, x_2, x_3, $\dots$, be the sequence of integers defined by $x_1 = 2$ and $x_{n+1} = 2x_n^2 - 1$ for $n \geq 1$. Prove that, for every positive integer n, the numbers n and x_n are coprime.

6. For each integer $n \geq 2$, find

(a) the greatest real number c_n such that

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \geq c_n$$

for any positive real n-tuple $(a_1, a_2, \dots, a_n)$ with $a_1 a_2 \cdots a_n = 1$;

(b) the greatest real number d_n such that

$$\frac{1}{1+2a_1} + \frac{1}{1+2a_2} + \cdots + \frac{1}{1+2a_n} \geq d_n$$

for any positive real n-tuple $(a_1, a_2, \dots, a_n)$ with $a_1 a_2 \cdots a_n = 1$.

As a further set of problems for your puzzling pleasure we give the Final Round of the 56th Czech and Slovak Mathematical Olympiad, March 18-21, 2007, edited by Karol Horák and translated by Miroslav Engliš. Thanks go to Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam for collecting them for us.

56th CZECH AND SLOVAK MATHEMATICAL OLYMPIAD
Final Round
March 18-21, 2007

1. A chess piece is arbitrarily placed on a square of a $n \times n$ $(n \geq 2)$ square chessboard. It then alternately makes straight and diagonal moves. A *straight* move is from one square to another one with a common side. A *diagonal* move is from one square to another one with exactly one point in common. Find all n for which there is a sequence of moves, starting with a diagonal move from the original square, such that the piece passes through all the squares of the chessboard and through each square exactly once.

2. In a cyclic quadrangle $ABCD$ let L and M be the incentres of triangles BCA and BCD, respectively. Let R be the intersection of the perpendiculars from the points L and M onto the lines AC and BD, respectively. Show that the triangle LMR is isosceles.

3. Denote by $\mathbb{N}$ the set of all positive integers and consider all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any x, $y \in \mathbb{N}$,

$$f(xf(y)) = yf(x) .$$

Find the least possible value of $f(2007)$.

4. The set M contains all of the integers from 1 to 2007 (inclusive) and if $n \in M$, then M contains the arithmetic progression with first member n and difference $n+1$. Decide whether there must exist a number m such that M contains all integers greater than m.

5. Triangle ABC is acute with $|AC| \neq |BC|$. The points D and E lie on the interiors of the sides BC and AC (respectively) such that $ABDE$ is a cyclic quadrangle, and the diagonals AD and BE intersect at P. If the lines CP and AB are perpendicular, show that P is the orthocentre of triangle ABC.

6. Find all ordered triples (x, y, z) of mutually distinct real numbers which satisfy the set equation

$$\{x, y, z\} = \left\{\frac{x-y}{y-z}, \frac{y-z}{z-x}, \frac{z-x}{x-y}\right\} .$$

A final set of problems for your pleasure over our winter break is the Selected Problems of the 2007 Taiwanese Mathematical Olympiad. Thanks once again are due to Bill Sands, Canadian Team Leader to the 2007 IMO in Vietnam, for collecting them for our use.

2007 TAIWANESE MATHEMATICAL OLYMPIAD
Selected Problems

1. Prove the following statements:

(a) If $0 < a, b \le 1$, then

$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} \le \frac{2}{\sqrt{1+ab}};$$

(b) If $ab \ge 3$, then

$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} \ge \frac{2}{\sqrt{1+ab}}.$$

2. Find all positive integers a, b, c, and d such that

$$2^a = 3^b 5^c + 7^d.$$

3. Given $\triangle ABC$ and its circumcircle, prove that the Simson lines of two diametrically opposite points are perpendicular and intersect on the nine-point circle of the triangle.

4. Let $ABCD$ be a convex quadrilateral. Prove or disprove that there exists a point E in the plane of $ABCD$ such that $\triangle ABE$ is similar to $\triangle CDE$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for all real numbers x and y,

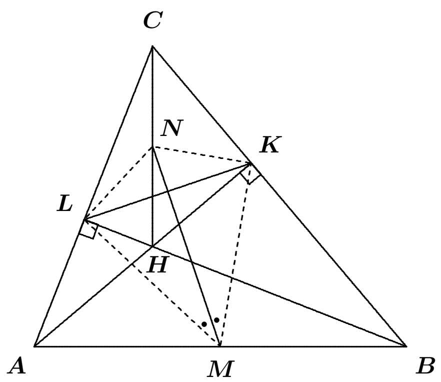
$$f(x)f\big(yf(x)-1\big) = x^2 f(y) - f(x).$$

6. Consider the following variation of the game of Nim. A position consists of k piles of stones, with $n_i \ge 1$ stones in pile i. Two players alternately move by choosing one of the piles, permanently removing one or more stones from that pile, and, optionally, redistributing some (or all) of the remaining stones in that pile to one or more of the other remaining piles. Once a pile is gone, no stones can be added to it, and the player who takes the last stone wins. Determine which vectors of positive integers $(n_1, n_2, \ldots, n_k)$ represent a winning position for the first player and which vectors represent a winning position for the second player.

We now return to the 54th Czech Mathematical Olympiad 2004/2005, Category B, 10th Class [2008 : 342–344] and a solution that could not be fit into the last number of the *Corner*.

K3. Let ABC be an acute triangle. Let K and L be the feet of the altitudes from A and B, respectively. Let M be the midpoint of AB and let H be the orthocentre of triangle ABC. Prove that the bisector of $\angle KML$ bisects the line segment HC.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Titu Zvonaru, Comănești, Romania. We give the solution of Amengual Covas.



Let N be the midpoint of CH. Equivalently, we will prove that MN bisects $\angle KML$.

Triangle ALB has a right angle at L and a median LM of length $\frac{1}{2}AB$. Triangle AKB has a right angle at K and a median KM of length $\frac{1}{2}AB$. Hence, $LM = KM$.

Triangle CLH has a right angle at L and a median LN of length $\frac{1}{2}CH$. Triangle CKH has a right angle at K and a median KN of length $\frac{1}{2}CH$. Hence, $LN = NK$.

Since $LM = KM$ and $LN = NK$, it follows that MN is the perpendicular bisector of segment LK. Since $\triangle KML$ is isosceles, this perpendicular bisector is in fact the bisector of $\angle KML$, as desired.

We also return to one more solution to a problem from the first Round of the 23rd Iranian Mathematical Olympiad [2008 : 345].

6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all x, $y \in \mathbb{R}^+$ we have

$$(x+y)f\big(f(x)y\big) \;=\; x^2 f\big(f(x)+f(y)\big)\,,$$

where $\mathbb{R}^+$ denotes the set of positive real numbers.

Solution by Michel Bataille, Rouen, France.

There is no such function. To prove this, assume that f is a solution. Taking $y = x$ in the functional equation, we obtain the identity

$$f\big(xf(x)\big) \;=\; \frac{x}{2}\cdot f\big(2f(x)\big)\,. \qquad (1)$$

Let a, $b \in \mathbb{R}^+$ be such that $f(a) = f(b)$. Then

$$\begin{aligned}(a+b)f\big(bf(b)\big) &= (a+b)f\big(f(a)b\big) \\ &= a^2 f\big(f(a)+f(b)\big) \;=\; a^2 f\big(2f(b)\big)\end{aligned}$$

and by using (1) it follows that $(a+b)\frac{b}{2} \cdot f\big(2f(b)\big) = a^2\big(f(2f(b)\big)$. Hence $(a+b)b = 2a^2$, or $(a-b)(2a+b) = 0$, and we conclude that f is injective.

Now, let $c = \dfrac{1+\sqrt{5}}{2}$ so that $c+1 = c^2 > 0$. Taking $x = c$ and $y = 1$ in the functional equation yields $(c+1)f\big(f(c)\big) = c^2 f\big(f(c)+f(1)\big)$, or $f\big(f(c)\big) = f\big(f(c)+f(1)\big)$. Since f is injective, we have $f(c) = f(c)+f(1)$, which contradicts $f(1) \in \mathbb{R}^+$. This contradiction completes the proof.

Next we give solutions from our readers to problems of the Second Round of the 23rd Iranian Mathematical Olympiad [2008 : 345].

3. Let a, b, and c be nonnegative real numbers. If

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2\,,$$

then show that $ab+bc+ca \le \frac{3}{2}$.

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Bataille's solution.

Let a, b, and c satisfy the constraint. Then

$$\frac{a^2}{a^2+1} + \frac{b^2}{b^2+1} + \frac{c^2}{c^2+1} = 1$$

(since $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$) and the Cauchy-Schwarz Inequality yields

$$(a+b+c)^2 \le \left(\frac{a^2}{a^2+1} + \frac{b^2}{b^2+1} + \frac{c^2}{c^2+1}\right)\Big((a^2+1)+(b^2+1)+(c^2+1)\Big)\,,$$

or

$$(a+b+c)^2 \le a^2+b^2+c^2+3\,.$$

Finally, $2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2) \le 3$, and the inequality follows.

We look next at solutions from our readers for the Third Round of the 23rd Iranian Mathematical Olympiad given at [2008 : 346].

1. Let ABC be a triangle whose circumradius equals the radius of the excircle which is tangent to the side BC. Let this excircle touch the side BC and the lines AC and AB at M, N, and L, respectively. Show that the circumcentre of triangle ABC is the orthocentre of triangle MNL.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Bataille.

We use the usual notation for the elements of $\triangle ABC$, in particular, O is the circumcentre of ABC and the A-excircle has centre I_a and radius r_a.

First, we show that $LN^2 - LM^2 = ON^2 - OM^2$, which implies that $LO \perp MN$.

Observing that I_a, L, B, and M are on the circle with diameter BI_a, we see that $\angle LI_aM = B$. Since $I_aL = I_aM = r_a = R$, it follows that $LM = 2R\sin\frac{B}{2}$. Similarly, since $\angle LI_aN = 180^\circ - A$, it follows that $LN = 2R\cos\frac{A}{2}$. As a result,

$$LN^2 - LM^2 \;=\; 4R^2\left(\cos^2\frac{A}{2} - \sin^2\frac{B}{2}\right) \;=\; 2R^2(\cos A + \cos B)\,.$$

Now, the area of $\triangle ABC$ is $r_a(s-a) = R(s-a)$ as well as $\frac{abc}{4R}$, hence $2R^2 = \frac{abc}{b+c-a}$. Using the Law of Cosines, this leads to

$$LN^2 - LM^2 \;=\; \frac{a(b^2+c^2-a^2) + b(c^2+a^2-b^2)}{2(b+c-a)}\,. \qquad (1)$$

On the other hand, since $BM = s-c$ and $CM = s-b$, we have that $(s-b)\overrightarrow{MB} + (s-c)\overrightarrow{MC} = \overrightarrow{0}$, and so $a\overrightarrow{OM} = (s-b)\overrightarrow{OB} + (s-c)\overrightarrow{OC}$. It follows that

$$\begin{aligned} a^2OM^2 &= (s-b)^2R^2 \;+\; (s-c)^2R^2 \;+\; 2(s-b)(s-c)R^2\cos 2A \\ &= R^2\big((s-b)^2 + (s-c)^2 + 2(s-b)(s-c)\big) \\ &\quad - 4R^2(s-b)(s-c)\sin^2 A \\ &= R^2a^2 \;-\; (s-b)(s-c)a^2\,, \end{aligned}$$

and so $OM^2 = R^2 - (s-b)(s-c)$.

Similarly, from $\overrightarrow{AN} = \frac{s}{b}\overrightarrow{AC}$ we obtain $b\overrightarrow{ON} = s\overrightarrow{OC} - (s-b)\overrightarrow{OA}$ and a similar calculation yields $ON^2 = R^2 + s(s-b)$. As a result,

$$ON^2 - OM^2 \;=\; (s-b)(s+s-c) \;=\; \frac{(a+b)(c+a-b)}{2}\,. \qquad (2)$$

A comparison of the righthand sides of (1) and (2) yields the desired equality $LN^2 - LM^2 = ON^2 - OM^2$. We obtain $NO \perp LM$ by exchanging the roles of L and N, and the result follows.

5. Let $n > 1$ be an integer, and let the entries of $(a_1, a_2, \dots, a_n)$ be pairwise distinct positive integers which are coprime in pairs. Find all such n-tuples for which $(a_1 + a_2 + \cdots + a_n) \mid (a_1^i + a_2^i + \cdots + a_n^i)$ for $1 \le i \le n$.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We prove that there are no solutions.

First we will show by induction that for each positive integer m

$$\left(\sum_{i=1}^{n} a_k\right) \Bigg| \left(\sum_{k=1}^{n} a_k^m\right) . \qquad (1)$$

This holds for $m = 1, 2, \dots, n$ by hypothesis. Now assume that (1) holds for $m = M - n, M - n + 1, \dots, M - 1$, and consider the polynomial $P(x) = x^{M-n} \prod_{k=1}^{n} (x - a_k)$. It can be written as $P(x) = x^M + \sum_{m=M-n}^{M-1} b_m x^m$, where the coefficients b_m are integers. We obtain

$$0 = \sum_{k=1}^{n} P(a_k) = \sum_{k=1}^{n} a_k^M + \sum_{m=M-n}^{M-1} b_m \left(\sum_{k=1}^{n} a_k^m\right)$$

hence, by the induction hypothesis, $\left(\sum_{i=1}^{n} a_k\right) \Big| \left(\sum_{k=1}^{n} a_k^M\right)$, which completes the induction.

Next we will prove that

$$\left(\sum_{k=1}^{n} a_k\right) \Bigg| \left(n^2 - n\right) . \qquad (2)$$

Let p^ℓ be a prime power dividing $\sum_{k=1}^{n} a_k$. If p divides any a_k, say $p \mid a_1$, then by Euler's Theorem we obtain $a_k^{\varphi(p^\ell)} \equiv 1 \pmod{p^\ell}$ for $2 \le k \le n$; therefore, applying (1), we conclude that $0 \equiv \sum_{k=1}^{n} a_k^{\varphi(p^\ell)} \equiv n - 1 \pmod{p^\ell}$. Otherwise, if p does not divide any p_k, we obtain from (1) and Euler's Theorem that $0 \equiv \sum_{k=1}^{n} a_k^{\varphi(p^\ell)} \equiv n \pmod{p^\ell}$. Consequently, $p^\ell \mid n(n-1)$, which proves (2).

Since $\sum_{k=1}^{n} a_k \ge 1 + 2 + \cdots + n = \frac{1}{2}n(n+1) > \frac{1}{2}n(n-1)$, it follows from (2) that $\sum_{k=1}^{n} a_k = (n^2 - n)$. Let $f(1) = 1$, and for each integer $k \ge 2$ let

$f(k)$ be the highest prime factor of k. If $n \geq 8$, then $\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} f(a_k) \geq 1+2+3+5+7+11+13+\cdots+(2n-1) = n^2-7 > n^2-n$, a contradiction. Since $f(a_1)+f(a_2)+\cdots+f(a_n) \leq n(n-1)$, the remaining cases are easily eliminated:

n	$n(n-1)$	$\{f(a_i)\}_{i=1}^{n}$	$\{a_i\}_{i=1}^{n}$	(1) fails for
3	6	1, 2, 3	the same	$m = 2$
4	12	1, 2, 3, 5	none	—
5	20	1, 2, 3, 5, 7	1, 2^2, 3, 5, 7	$m = 4$
6	30	1, 2, 3, 5, 7, 11	none	—
7	42	1, 2, 3, 5, 7, 11, 13	the same	$m = 6$

Next we turn to solutions of a problem of the Romanian Mathematical Olympiad 2006, Final Round, 9th Grade, given at [2008 : 346–347].

1. (Dan Schwarz) Find the maximum value of $(x^3+1)(y^3+1)$ if x and y are real numbers such that $x+y=1$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's solution.

The maximum value is 4.

Set $p = xy$, then we have

$$\begin{aligned}(x^3+1)(y^3+1) &= (xy)^3 + (x+y)^3 - 3xy(x+y) + 1 \\ &= p^3 - 3p + 2 \;=\; 4 + (p+1)^2(p-2)\,.\end{aligned}$$

Now, $xy \leq \dfrac{(x+y)^2}{4} = \dfrac{1}{4}$, hence $p - 2 < 0$. It then follows that $(x^3+1)(y^3+1) \leq 4$ with equality if and only if $p = xy = -1$ (in addition to $x+y=1$); that is, if and only if x, y are the numbers $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$.

We next look at readers' solutions to problems of the Final Round of the Romanian Mathematical Olympiad 2006, 10th Grade, given at [2008 : 347].

1. (Vasile Pop) Let M be a set with n elements and let $\mathcal{P}(M)$ denote the set of all subsets of M. Find all functions $f : \mathcal{P}(M) \to \{0, 1, 2, \ldots, n\}$, with the following two properties:

(a) $f(A) \neq 0$ for any $A \neq \emptyset$, and

(b) $f(A \cup B) = f(A \cap B) + f(A \triangle B)$, for all A, $B \in \mathcal{P}(M)$, where $A \triangle B = (A \cup B) \backslash (A \cap B)$.

Solution by Michel Bataille, Rouen, France.

The function which maps each subset of M to its cardinality satisfies the requirements, and we will show that there are no other solutions.

Let f satisfy (a) and (b). If A, $B \in \mathcal{P}(M)$ and $A \subsetneq B$, then

$$f(B) \;=\; f(A \cup B) \;=\; f(A \cap B) + f(A \triangle B) \;=\; f(A) + f(B \backslash A)\,,$$

hence $f(B \backslash A) > 0$ by (a) because $B \backslash A \neq \emptyset$. It follows that $f(B) > f(A)$.

Now, let $M = \{m_1, m_2, \ldots, m_n\}$ and observe that since f is "strictly increasing", we have

$$0 \;\leq\; f(\emptyset) \;<\; f\big(\{m_1\}\big) \;<\; f\big(\{m_1, m_2\}\big) \;<\; \cdots \;<\; f(M) \;\leq\; n\,.$$

The $n+1$ images under f are distinct integers in $\{0,\,1,\,2,\,\ldots,\,n\}$; hence $f(\emptyset) = 0$, $f\big(\{m_1\}\big) = 1$, $f\big(\{m_1,\,m_2\}\big) = 2$, $\ldots$, $f(M) = n$. It follows (since subsets may be relabelled) that the image $f(A)$ of any subset A of M equals the cardinality of A.

2. (Iurie Boreico) Prove that for all integers $n > 0$ and all a, $b \in \big(0, \frac{\pi}{4}\big)$ we have

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \;\geq\; \frac{\sin^n 2a + \sin^n 2b}{(\sin 2a + \sin 2b)^n}\,.$$

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give Bataille's version.

[*Ed.: The notations $S_x = \sin x$ and $C_x = \cos x$ will be employed for typographical reasons.*]

Without loss of generality, we assume $n \geq 2$ and $a > b$ (equality holds for $n = 1$ or $a = b$ and a, b play symmetric roles). We want to prove

$$\big(S_a^n + S_b^n\big)\big(S_{2a}^n + S_{2b}^n + T_2\big) \;\geq\; \big(S_{2a}^n + S_{2b}^n\big)\big(S_a^n + S_b^n + T_1\big)\,, \qquad (1)$$

where

$$T_1 \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_a^{n-k} S_b^k \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_b^{n-k} S_a^k\,;$$
$$T_2 \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_{2a}^{n-k} S_{2b}^k \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_{2b}^{n-k} S_{2a}^k\,.$$

Inequality (1) is equivalent to $L_a + L_b \geq R_a + R_b$ where

$$L_a \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_{2a}^{n-k} S_{2b}^k S_a^n\,; \qquad L_b \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_{2b}^{n-k} S_{2a}^k S_b^n\,;$$
$$R_a \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_a^{n-k} S_b^k S_{2a}^n\,; \qquad R_b \;=\; \sum_{k=1}^{n-1} \binom{n}{k} S_b^{n-k} S_a^k S_{2b}^n\,.$$

Now, using the formula $\sin 2x = 2\sin x \cos x$, (or $S_{2x} = 2S_xC_x$) we have

$$L_a - R_a = 2^n\left(\sum_{k=1}^{n-1}\binom{n}{k}(S_a^{2n-k}S_b^kC_a^{n-k})(C_b^k - C_a^k)\right),$$
$$L_b - R_b = 2^n\left(\sum_{k=1}^{n-1}\binom{n}{k}(S_b^{2n-k}S_a^kC_b^{n-k})(C_a^k - C_b^k)\right).$$

It follows that

$$\frac{1}{2^n}(L_a + L_b - (R_a + R_b))$$
$$= \sum_{k=1}^{n-1}\binom{n}{k}(C_b^k - C_a^k)\frac{S_a^kS_b^k}{2^{n-k}}\left[(S_aS_{2a})^{n-k} - (S_bS_{2b})^{n-k}\right].$$

Since $a > b$, we have $C_b^k - C_a^k > 0$ and $(S_aS_{2a})^{n-k} > (S_bS_{2b})^{n-k}$ for each k, and so $L_a + L_b - (R_a + R_b) > 0$, as desired.

Next we move to the November 2008 number of the *Corner* and solutions from our readers to problems of the 2005/6 British Mathematical Olympiad, Round 1, given at [2008 : 408].

1. Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. Is the converse true?

Solution by Titu Zvonaru, Comăneşti, Romania.

Since $n-1$ and $n+1$ are both prime, n is even. If $n = 6k+2$, then $n+1 = 3(2k+1)$ is not prime. If $n = 6k+4$, then $n-1 = 3(2k+1)$ is not prime. Therefore, $n = 6k$ and $n^2(n^2+16) = 36k^2(36k^2+16) = 144k^2(9k^2+4)$ is divisible by 144.

If $n = 5k+1$, then $n-1 = 5k$ is not prime. If $n = 5k+4$, then $n+1 = 5(k+1)$ is not prime. If $n = 5k$, then n^2 is divisible by 5. If $n = 5k+2$ or $n = 5k+3$, then $n^2 = 5j+4$ where $j = 5k^2+6k+1$ is an integer, hence n^2+16 is divisible by 5.

By the preceding two results, $n^2(n^2+16)$ is divisible by $5\cdot 144 = 720$.

The converse is not true. For example, if $n = 48$, then $n^2(n^2+16)$ is divisible by 720, but $n+1 = 49$ is not prime.

2. Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

(i) In how many ways can he split them into two teams of six?

(ii) In how many ways can he split them into three teams of four?

Solution by Oliver Geupel, Brühl, NRW, Germany.

(i) To set up the first team, one member of each pair has to be selected. There are two possible choices for each of the six pairs, thus 2^6 choices in total. Finally, the order of the two teams is irrelevant; hence a factor of $\frac{1}{2}$ applies. We conclude that the teams can be arranged in $2^5 = 32$ ways.

(ii) To set up the first team, Adrian can first choose four of the six pairs and then pick one person from each selected pair. This can be done in $\binom{6}{4}\cdot 2^4$ ways. Let a, b, c, d be the four remaining members of the selected four pairs, and let S, T be the two pairs that were not chosen for the first team. To build the second team, Adrian has to choose one member from each of the two pairs S and T and two extra persons from a, b, c, d. This can be done in $2^2\cdot\binom{4}{2}$ ways. Finally, the order of the three teams is irrelevant; hence a factor of $\frac{1}{3!}$ applies. Therefore, the teams can be arranged in $\frac{1}{3!}\binom{6}{2}\binom{4}{2}2^6 = 960$ ways.

3. In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

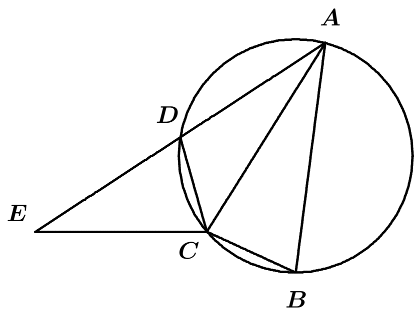
Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain; and Titu Zvonaru, Comănești, Romania. We give the solution of Barroso Campos.

If $CE = CA$, then $\triangle AEC$ is isosceles, hence

$$\angle CED = \angle CAD = \angle CAB\,.$$

Also $\angle CDA = 180^\circ - \angle CBA$ so $\angle DEC$ and $\triangle BAC$ are congruent and $DE = AB$.

Since $\angle DAC = \angle CAB$ we have $CD = CB$. If $DE = AB$, then we have $DE = AB$, $CD = CB$, and $\angle EDC = \angle ABC$, hence $CE = CA$.



Now we continue with solutions to problems of Round 2, 2005/6 British Mathematical Olympiad given at [2008 : 409].

1. Find the minimum possible value of $x^2 + y^2$ given that x and y are real numbers satisfying

$$xy(x^2 - y^2) \;=\; x^2 + y^2 \quad \text{and} \quad x \neq 0\,.$$

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give the write-up of Bataille.

We show that the required minimum is 4.

Let x, y satisfy the constraint. Then $x^2 + y^2 \neq 0$, and there is an angle θ such that $(\cos\theta, \sin\theta) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$.

The given equation $xy(x^2 - y^2) = x^2 + y^2$ can then be rewritten as $\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = \frac{1}{x^2+y^2}$, that is, $\sin 4\theta = \frac{4}{x^2+y^2}$. Since $\sin 4\theta \leq 1$, we obtain $x^2 + y^2 \geq 4$. To complete the proof, we observe that for $(x, y) = \left(2\cos\frac{\pi}{8}, 2\sin\frac{\pi}{8}\right)$ the constraint is satisfied and $x^2 + y^2 = 4$.

3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that $\angle BPC + \angle BAC = 180^\circ$.

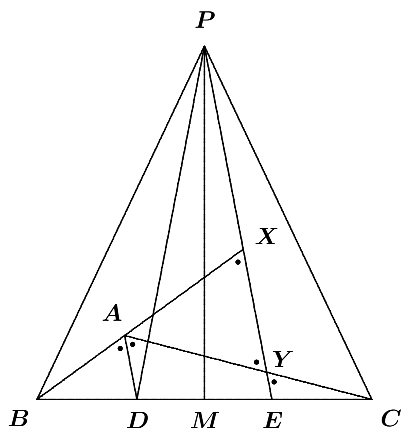
Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give the solution by Amengual Covas.

Let the bisector of $\angle A$ meet BC at D and extend XY to meet BC at E.

Triangle XAY is isosceles since $AX = BX - BA = CA - CY = AY$, hence $\angle AXY = \angle AYX$. The exterior angle of $\triangle XAY$ at A is $\angle A$, so $\angle A = \angle AXY + \angle AYX = 2\angle AXY$. Thus, $\angle AXY = \frac{1}{2}\angle A = \angle BAD$.

The angles $\angle AXY$ and $\angle BAD$ are equal, hence $AD \| XY$ and

$$\frac{DE}{BD} = \frac{AX}{AB}. \tag{1}$$



Since AD bisects $\angle A$, we have $\frac{DC}{BD} = \frac{AC}{AB}$, from which we obtain

$$\frac{DC - BD}{BD} = \frac{CA - AB}{AB} = \frac{CA - CY}{AB} = \frac{AY}{AB} = \frac{AX}{AB}. \tag{2}$$

From (1) and (2) we have $DE = DC - BD$. Also $DE = DC - EC$, so $BD = EC$.

Let M be the midpoint of BC. Then M is also the midpoint of DE and $\triangle PDE$ is isosceles, since $DM = BM - BD = MC - EC = ME$. Because $\angle PED = \angle ECY + \angle CYE = \angle BCA + \angle AYX = \angle C + \frac{1}{2}\angle A$, each of the base angles at D and E is $\angle C + \frac{1}{2}\angle A$. Consequently,

$$\angle DPE = \angle B - \angle C \qquad \text{and} \qquad \angle DPM = \frac{1}{2}(\angle B - \angle C).$$

Then,

$$\begin{aligned} PM &= DM \cdot \cot \angle DPM \;=\; (BM - BD) \cdot \cot \tfrac{1}{2}(\angle B - \angle C) \\ &= \left(\frac{a}{2} - \frac{ac}{b+c}\right) \cdot \cot \frac{1}{2}(\angle B - \angle C) \\ &= \frac{a(b-c)}{2(b+c)} \cdot \cot \frac{1}{2}(\angle B - \angle C) \end{aligned}$$

where a, b, and c are the sides of $\triangle ABC$ in the usual order.

We make the substitution $\dfrac{b-c}{b+c} = \dfrac{\tan\frac{1}{2}(\angle B - \angle C)}{\tan\frac{1}{2}(\angle B + \angle C)}$ to obtain

$$PM \;=\; \frac{a}{2\tan\frac{1}{2}(\angle B + \angle C)}\,,$$

or

$$\tan\frac{1}{2}(\angle B + \angle C) \;=\; \frac{a/2}{PM} \;=\; \frac{BM}{PM}\,.$$

Also $\tan(\angle BPM) = \dfrac{BM}{PM}$; so

$$\tan\frac{1}{2}(\angle B + \angle C) \;=\; \tan(\angle BPM) \;=\; \tan\frac{1}{2}(\angle BPC)\,.$$

From this it follows that $\angle BPC = \angle B + \angle C$, and therefore we have $\angle BPC + \angle BAC = (\angle B + \angle C) + \angle A = 180^\circ$, as desired.

We now finish with readers' solutions to problems of the Bulgarian National Olympiad 2006 given at [2008 : 409–410].

1. (Aleksandar Ivanov) Consider the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 2$. Find the number of subsets B of A, such that if the sum of two elements of A is a power of 2 then exactly one of them belongs to B.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $\mathcal{B}_n$ be the collection of all admissible subsets B where $n \geq 2$. Then $\mathcal{B}_2 = \{\{1\}, \{1, 2\}, \{1, 2, 4\}, \{1, 4\}, \{2, 3\}, \{2, 3, 4\}, \{3\}, \{3, 4\}\}$, hence, $|\mathcal{B}_2| = 8$.

Consider the bipartite graph with the two sets of nodes $\mathcal{B}_n$ and $\mathcal{B}_{n+1}$, where a node $B \in \mathcal{B}_n$ is adjacent to a node $B' \in \mathcal{B}_{n+1}$ if and only if $B \subset B'$. For $2^n + 1 \leq k \leq 2^{n+1} - 1$ we have $k \in B'$ if and only if $k - 2^n \in B$. Thus, the elements of B' except 2^{n+1} are uniquely determined from B, whereas 2^{n+1} may or may not be a member of B'. Therefore, each node in $\mathcal{B}_n$ has degree 2, while each node in $\mathcal{B}_{n+1}$ has degree 1, hence, $|\mathcal{B}_{n+1}| = 2|\mathcal{B}_n|$.

From $|\mathcal{B}_2| = 8$ and the recursion $|\mathcal{B}_{n+1}| = 2|\mathcal{B}_n|$ we conclude that there are exactly $|\mathcal{B}_n| = 2^{n+1}$ subsets with the desired property.

2. (Oleg Mushkarov, Nikolai Nikolov) Let $\mathbb{R}^+$ be the set of all positive real numbers and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that

$$f(x+y) - f(x-y) \;=\; 4\sqrt{f(x)f(y)}$$

for all $x > y > 0$.

(a) Prove that $f(2x) = 4f(x)$ for all $x \in \mathbb{R}^+$.

(b) Find all such functions.

Solution by Michel Bataille, Rouen, France, modified by the editor.

(a) If $a > b > 0$, then

$$\begin{aligned} f(a) - f(b) &= f\left(\frac{a+b}{2} + \frac{a-b}{2}\right) - f\left(\frac{a+b}{2} - \frac{a-b}{2}\right) \\ &= 4\sqrt{f\left(\frac{a+b}{2}\right) f\left(\frac{a-b}{2}\right)} \;>\; 0\,, \end{aligned}$$

hence f is increasing. Since f is also bounded below by 0, f has a (finite) limit as x approaches 0 from the right. Let $l_0 = \lim\limits_{x\to 0+} f(x)$. Note that $l_0 \geq 0$.

Since f is increasing, f has a limit from each side at each $a \in \mathbb{R}^+$, and

$$\lim_{x\to a^-} f(x) \;=\; l_a \;\leq\; f(a) \;\leq\; r_a \;=\; \lim_{x\to a^+} f(x)\,.$$

Setting $x = 2y$ in the given relation and letting $y \to 0^+$, we obtain $l_0 - l_0 = 4\sqrt{l_0^2}$, hence $l_0 = 0$. Now setting $x = a$ and letting $y \to 0^+$ in the same relation yields $r_a - l_a = 4\sqrt{f(a)l_0} = 0$, thus f is continuous at a.

From $f(a+x) - f(a-x) = 4\sqrt{f(a)f(x)}$, we deduce $f(2a) = 4f(a)$ (by continuity at a and $l = 0$). Since a is arbitrary, the result follows.

(b) The only solutions are the functions $x \mapsto kx^2$, where $k > 0$.

Let f be any solution. Then $f(nx) = n^2 f(x)$ holds for $n = 1, 2$. Assume that it holds for $n = 1, 2, \ldots, m$ where $m \geq 2$ is an integer. Then, from $f\big((m+1)x\big) - f\big((m-1)x\big) = 4\sqrt{f(mx)f(x)}$, we obtain the relation $f\big((m+1)x\big) = (m-1)^2 f(x) + 4mf(x) = (m+1)^2 f(x)$.

Thus, by induction, $f(nx) = n^2 f(x)$ for all positive integers n and all $x > 0$. It follows that $f\left(\frac{p}{q}\cdot x\right) = \frac{p^2}{q^2} f(x)$ for all positive integers p, q. Lastly, if $r \in (0,\infty)$, then there is a rational sequence $\{r_n\}_{n=1}^{\infty}$ converging to r, and by the continuity of f we deduce $f(rx) = r^2 x$. In particular, $f(r) = r^2 f(1)$ for all $r \in (0,\infty)$. This completes the proof.

The last *Corner* for this Volume of ***CRUX with MAYHEM*** is now complete. Send me your nice solutions and generalizations in the New Year!

BOOK REVIEWS

Amar Sodhi

A Certain Ambiguity: A Mathematical Novel
By Gaurav Suri and Hartosh Singh Bal, Princeton University Press, 2007
ISBN-13: 978-069112-709-5, hardcover, 292 pages, US$27.95
Reviewed by ***Mark Taylor****, Halifax, NS*

A Certain Ambiguity is subtitled *A Mathematical Novel*. Mathematical it certainly is, and it is novel, but a mathematical novel... .

The book opens with Ravi Kapoor recalling the day his mathematician grandfather (bauji) gave him a calculator. The gift was to initiate bauji's plan to "get Ravi passionate about mathematics" and together with the gift came an arithmetical teaser that set him along the way.

Unfortunately, the day after the plan was set in motion, bauji died and young Ravi was abandoned to a school system dedicated to rote learning and the accumulation of facts. Although Ravi's grades were excellent his schooling was, in his words, a joyless endeavour. However, the high grades and a bequest from bauji (eventually) enabled Ravi to enter Stanford. The young Kapoor's undergraduate years were initially those of a dilettante; he dabbled in this and that but no subject had lasting interest for him. His eventual major, Economics, was chosen to satisfy his father who felt it would make Ravi attractive to a wide range of corporate recruiters.

Just as it seems that our boy is destined to become an acolyte in the service of Mammon, he meets Dr. Nico Aliprantis, mathematician, jazz saxophonist manqué and teacher extraordinaire. Ravi is invited to join Nico's Math 208 class "Thinking About Infinity", and this is where the story really begins. The authors use Aliprantis to draw us into a mathematical feast. We are fed tasty morsel after tasty morsel that serve to addict the neophyte and bring a smile to the lips of the cognoscente. From Zeno to infinite sums, counting to Cantor, the infinity of prime numbers and the irrationality of the square root 2.

The table is well set and the servers, Nico, Ravi, and other members of the Math 208 class lay out the dishes and anticipate our needs like all good wait staff. However, if the food does not suit the reader's palate there is little nourishment in the characters, that is until the authors introduce a new literary and mathematical thread. Ravi discovers that as a young man his grandfather, formally known as Vijay Sahnis (VS), had spent some time in a New Jersey prison.

The incarceration came as news to Vijay Sahnis' surviving family and Ravi set about to unravel the mystery. He tracks down a transcript of discussions between VS and a respected New Jersey judge, John Taylor. Using transcripts and newspaper articles together with some augmentation, the authors develop a situation and characters that hold the reader's attention while a new mathematical line is developed. The new line introduces geometry and the idea of formal axioms.

Very soon the Judge and VS are discussing Euclid's fifth postulate and this leads them to non-Euclidean geometries and eventually into the very nature of mathematics. The dialogue between VS and the Judge is interwoven with observations from members of the Math 208 class and this allows the introduction of the Continuum Hypothesis and mention of the works of Gödel and Cohen.

Suri and Bal succeed admirably in describing and explaining some beautiful mathematical results in such a way that they are accessible to people with little formal training in the discipline.

A personal quibble: The book contains over a dozen "Journal Entries" of one sort or another ascribed to various mathematicians with the authors' acknowledgement that the contents of many are either apocryphal or fictitious (there is no attempt to deceive the reader – each entrant is explained in the notes at the end of the book).

For my taste, the time and effort spent in constructing most of the journal entries would have been better employed in providing the reader with some good recipes. In one part in the book Nico invites a group of his students to a simple dinner consisting of a Greek salad, lamb marinated in a garlic sauce, and homemade pita bread. The preparation of the meal is described but not in detail. Recipes would have been most appropriate. Ravi might have reciprocated Nico's generosity with an Indian meal; perhaps pakoras, dhall, rice, chappati, a good chicken dish or rogan gosht, brinjal, bindi, and kheer. A good kheer recipe is hard to come by.

Finally the question: To read or not to read?

Imagine you are in a doctor's or dentist's waiting room sifting through dog eared copies of *Readers Digest*, *MacLean's*, *Time*, *Field and Stream*, *Ecum Secum Hog Breeders Quarterly*, etc., when you come across a copy of ***Crux with Mayhem***. If you pick it up and scan through it and find anything of interest in it, I daresay you will find more than a little to enjoy in *A Certain Ambiguity*.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er juin 2010**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3488. *Proposé par Pham Huu Duc, Ballajura, Australie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab} \le \sqrt{\frac{a^{-1}+b^{-1}+c^{-1}}{a+b+c}}\,.$$

3489. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit n un entier non négatif. Montrer que

$$\frac{1}{2^{n-1}}\sum_{k=0}^{n}\sqrt{k}\binom{2n}{k} \le \sqrt{n\left(2^{2n}+\binom{2n}{n}\right)}\,.$$

3490. *Proposé par Michael Rozenberg, Tel-Aviv, Israël.*

Soit a, b et c trois nombres réels non négatifs tels que $a+b+c=1$. Montrer que

(a) $\sqrt{9-32ab}+\sqrt{9-32ac}+\sqrt{9-32bc} \ge 7$;

(b) $\sqrt{1-3ab}+\sqrt{1-3ac}+\sqrt{1-3bc} \ge \sqrt{6}$.

3491. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Soit a_1, a_2, …, a_{n+1} des nombres réels positifs où $a_{n+1}=a_1$. Montrer que

$$\sum_{i=1}^{n}\frac{a_i^4}{(a_i+a_{i+1})(a_i^2+a_{i+1}^2)} \ge \frac{1}{4}\sum_{i=1}^{n}a_i\,,$$

3492$\bigstar$. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit P un point à l'intérieur d'un tétraèdre $ABCD$ de sorte que $\angle PAB$, $\angle PBC$, $\angle PCD$ et $\angle PDA$ soient tous égaux à $\arccos\sqrt{\frac{2}{3}}$. Montrer que $ABCD$ est un tétraèdre régulier et que P est son centre de gravité.

3493. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Soit AB_0C_0 un triangle rectangle d'hypoténuse $c = AB_0$ et de côtés $a = B_0C_0$ et $b = C_0A$. On définit les points B_i sur AB_0 et C_i sur AC_0 de sorte que B_iC_i soit perpendiculaire au côté AC_0 et tangent au cercle inscrit du triangle rectangle précédent $AB_{i-1}C_{i-1}$. Trouver $S = \sum\limits_{i=1}^{\infty} AC_i$ en fonction de a, b et c.

3494. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier, $n > 1$ et, pour chaque $k = 1, 2, \ldots, n$, soit

$$\sigma(n,k) \;=\; \sum_{1 \le i_1 < \cdots < i_k \le n} i_1 i_2 \cdots i_k \,.$$

Montrer que

$$\sum_{k=1}^{n} \frac{\ln n}{n+1-k} \cdot \sigma(n,k) \;\sim\; (n+1)! \;\sim\; \sum_{k=1}^{n} \frac{n+1-k}{\ln n} \cdot \sigma(n,k)\,,$$

où $f(n) \sim g(n)$ signifie $\dfrac{f(n)}{g(n)} \to 1$ si $n \to \infty$.

3495. *Proposé par Cosmin Pohoaţă, Collège National Tudor Vianu, Bucarest, Roumanie.*

Soit a, b, c trois nombres réels positifs avec $a + b + c = 2$. Montrer que

$$\frac{1}{2} + \sum_{\text{cyclique}} \frac{a}{b+c} \;\le\; \sum_{\text{cyclique}} \frac{\left(a^2+bc\right)}{b+c} \;\le\; \frac{1}{2} + \sum_{\text{cyclique}} \frac{a^2}{b^2+c^2}\,.$$

3496. *Proposé par Elias C. Buissant des Amorie, Castricum, Pays-Bas.*

Montrer la validité des équations suivantes :

(a) $\tan 72^\circ = \tan 66^\circ + \tan 36^\circ + \tan 6^\circ$.

(b)★ $\tan 84^\circ = \tan 78^\circ + \tan 72^\circ + \tan 60^\circ$;

[*Ed. : Le proposeur a ajouté six autres identités de la forme* $f(\theta) = \sum\limits_{i=1}^{4} \tan k_i\theta = 0$ *pour* $k_i \in \mathbb{Z}$ *et* $\theta = 2\pi/n$ *avec* $n|360$*, ici non incluses par manque d'espace.*]

3497. *Proposé par Salem Malikić, étudiant, Collège de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit P un point à l'intérieur d'un triangle ABC et désignons par r le rayon du cercle inscrit de ABC. Montrer que $\max\{AP, BP, CP\} \ge 2r$.

3498. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit F_n le n^{e} nombre de Fibonacci, c'est-à-dire $F_0 = 0$, $F_1 = 1$, et $F_n = F_{n-1} + F_{n-2}$ pour $n \geq 2$. Montrer que, pour tout entier positif n, on a

$$\sqrt{\frac{F_{n+3}}{F_n}} + \sqrt{\frac{F_n + F_{n+2}}{F_{n+1}}} > 1 + 2\left(\sqrt{\frac{F_n}{F_{n+3}}} + \sqrt{\frac{F_{n+1}}{F_n + F_{n+2}}}\right).$$

3499★. *Proposé par Bernardo Recamán, Institut Alberto Merani, Bogota, Colombie.*

Dans un bâtiment de n étages, numérotés de 1 à n, on a un certain nombre d'ascenseurs qui s'arrêtent aux étages 1 et n, et peut-être à d'autres étages. Pour chaque n, trouver le nombre minimal d'ascenseurs requis dans le bâtiment correspondant pour que deux étages quelconques soient reliés par au moins un ascenseur express.

Par exemple, si $n = 6$, neuf ascenseurs suffisent : $(1, 6)$, $(1, 3, 4, 6)$, $(1, 5, 6)$, $(1, 4, 6)$, $(1, 2, 4, 5, 6)$, $(1, 2, 5, 6)$, $(1, 2, 6)$, $(1, 3, 5, 6)$ et $(1, 2, 3, 6)$.

3500. *Proposé par Paul Bracken, Université du Texas, Edinburg, TX, É-U.*

On définit la fonction $f(a) = \sum_{k=1}^{\infty} \frac{\ln k}{k(k+a)}$ pour $a \in (-1, \infty)$, et on pose $\beta = -f(1) + \frac{1}{2} f\left(\frac{1}{4}\right) - \frac{1}{2} f\left(-\frac{1}{4}\right)$. Montrer que

$$\prod_{k=1}^{\infty} \frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}} = 2^{-\frac{3}{2}\ln(2)+1-\gamma} \cdot e^{\beta},$$

où γ est la constante d'Euler.

. . . .

3488. *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{a}{2a^2 + bc} + \frac{b}{2b^2 + ca} + \frac{c}{2c^2 + ab} \leq \sqrt{\frac{a^{-1} + b^{-1} + c^{-1}}{a + b + c}}.$$

3489. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a nonnegative integer. Prove that

$$\frac{1}{2^{n-1}} \sum_{k=0}^{n} \sqrt{k}\binom{2n}{k} \leq \sqrt{n\left(2^{2n} + \binom{2n}{n}\right)}.$$

3490. *Proposed by Michael Rozenberg, Tel-Aviv, Israel.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=1$. Prove that

(a) $\sqrt{9-32ab}+\sqrt{9-32ac}+\sqrt{9-32bc}\geq 7$;

(b) $\sqrt{1-3ab}+\sqrt{1-3ac}+\sqrt{1-3bc}\geq\sqrt{6}$.

3491. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers where $a_{n+1}=a_1$. Prove that

$$\sum_{i=1}^{n}\frac{a_i^4}{(a_i+a_{i+1})(a_i^2+a_{i+1}^2)}\;\geq\;\frac{1}{4}\sum_{i=1}^{n}a_i\,,$$

3492★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let P be a point in the interior of tetrahedron $ABCD$ such that each of $\angle PAB$, $\angle PBC$, $\angle PCD$, and $\angle PDA$ is equal to $\arccos\sqrt{\frac{2}{3}}$. Prove that $ABCD$ is a regular tetrahedron and that P is its centroid.

3493. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let AB_0C_0 be a right triangle with hypotenuse $c=AB_0$ and legs $a=B_0C_0$ and $b=C_0A$. Define the points B_i on AB_0 and C_i on AC_0 so that B_iC_i is perpendicular to the leg AC_0 and tangent to the incircle of the previous right triangle $AB_{i-1}C_{i-1}$. Find $S=\sum\limits_{i=1}^{\infty}AC_i$ in terms of a, b, and c.

3494. *Proposed by Michel Bataille, Rouen, France.*

Let $n>1$ be an integer and for each $k=1, 2, \ldots, n$ let

$$\sigma(n,k)\;=\;\sum_{1\leq i_1<\cdots<i_k\leq n} i_1i_2\cdots i_k\,.$$

Prove that

$$\sum_{k=1}^{n}\frac{\ln n}{n+1-k}\cdot\sigma(n,k)\;\sim\;(n+1)!\;\sim\;\sum_{k=1}^{n}\frac{n+1-k}{\ln n}\cdot\sigma(n,k)\,,$$

where $f(n)\sim g(n)$ means that $\dfrac{f(n)}{g(n)}\to 1$ as $n\to\infty$.

3495. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let a, b, c be positive real numbers with $a+b+c=2$. Prove that

$$\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c}\;\leq\;\sum_{\text{cyclic}}\frac{(a^2+bc)}{b+c}\;\leq\;\frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}\,.$$

3496. *Proposed by Elias C. Buissant des Amorie, Castricum, the Netherlands.*

Prove the following equations:

(a) $\tan 72^\circ = \tan 66^\circ + \tan 36^\circ + \tan 6^\circ$.

(b)★ $\tan 84^\circ = \tan 78^\circ + \tan 72^\circ + \tan 60^\circ$;

[*Ed.: The proposer gave six more relations of the form* $f(\theta) = \sum_{i=1}^{4} \tan k_i\theta = 0$ *for* $k_i \in \mathbb{Z}$ *and* $\theta = 2\pi/n$ *with* $n|360$, *not included here for lack of space.*]

3497. *Proposed by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.*

Let P be a point in the interior of triangle ABC, and let r be the inradius of ABC. Prove that $\max\{AP, BP, CP\} \geq 2r$.

3498. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let F_n be the n^{th} Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For each positive integer n, prove that

$$\sqrt{\frac{F_{n+3}}{F_n}} + \sqrt{\frac{F_n + F_{n+2}}{F_{n+1}}} > 1 + 2\left(\sqrt{\frac{F_n}{F_{n+3}}} + \sqrt{\frac{F_{n+1}}{F_n + F_{n+2}}}\right).$$

3499★. *Proposed by Bernardo Recamán, Instituto Alberto Merani, Bogotá, Colombia.*

A building has n floors numbered 1 to n and a number of elevators all of which stop at both floors 1 and n, and possibly other floors. For each n, find the least number of elevators needed in this building if between any two floors there is at least one elevator that connects them non-stop.

For example, if $n = 6$, nine elevators suffice: $(1,6)$, $(1,5,6)$, $(1,4,6)$, $(1,3,4,6)$, $(1,2,4,5,6)$, $(1,2,5,6)$, $(1,2,6)$, $(1,3,5,6)$, and $(1,2,3,6)$.

3500. *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

Define the function $f(a) = \sum_{k=1}^{\infty} \frac{\ln k}{k(k+a)}$ for $a \in (-1,\infty)$, and set $\beta = -f(1) + \frac{1}{2}f\left(\frac{1}{4}\right) - \frac{1}{2}f\left(-\frac{1}{4}\right)$. Prove that

$$\prod_{k=1}^{\infty} \frac{(2k-1)^{\frac{1}{2k}}}{(2k)^{\frac{1}{2k-1}}} = 2^{-\frac{3}{2}\ln(2)+1-\gamma} \cdot e^{\beta},$$

where γ is Euler's constant.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We belatedly acknowledge a correct solution to #3340 by "Solver X", dedicated to the memory of Jim Totten, which we had previously classified as incorrect. Our apologies.

3389. [2008 : 483, 486] *Proposed by Mihály Bencze, Brasov, Romania.*

For $a \in \mathbb{R}$ define a sequence (x_n) by $x_0 = a$ and $x_{n+1} = 4x_n - x_n^2$ for all $n \geq 0$. Prove that there exist infinitely many $a \in \mathbb{R}$ such that the sequence (x_n) is periodic.

Similar solutions by George Apostolopoulos, Messolonghi, Greece and Michel Bataille, Rouen, France.

For each positive integer p, let $\theta_p = \dfrac{2\pi}{2^p - 1}$ and $a_p = 2(1 - \cos\theta_p)$. Clearly, $a_j \neq a_k$ for $j \neq k$, so it suffices to show that the sequence (x_n) is periodic for $a = a_p$. If $x_0 = a_p$, then $x_n = 2\big[1 - \cos(2^n\theta_p)\big]$ holds for $n = 0$. Moreover, if we assume $x_n = 2\big[1 - \cos(2^n\theta_p)\big]$ to be true, then using the formula $2\cos^2 y = 1 + \cos 2y$, we have

$$\begin{aligned} x_{n+1} &= 8\big[1 - \cos(2^n\theta_p)\big] - 4\big[1 - \cos(2^n\theta_p)\big]^2 \\ &= 4 - 4\cos^2(2^n\theta_p) \;=\; 2\big[1 - \cos(2^{n+1}\theta_p)\big]\,. \end{aligned}$$

Thus, for all nonnegative integers n, we have $x_n = 2\big[1 - \cos(2^n\theta_p)\big]$, and so

$$\begin{aligned} x_{n+p} - x_n &= 2\big[\cos(2^n\theta_p) - \cos(2^{n+p}\theta_p)\big] \\ &= 4\sin\Big(2^{n-1}\theta_p(2^p - 1)\Big)\sin\Big(2^{n-1}\theta_p(2^p + 1)\Big) \;=\; 0\,, \end{aligned}$$

where the latter equality follows from $\sin(2^{n-1}\theta_p(2^p - 1)) = \sin(2^n\pi) = 0$. This shows that (x_n) is p-periodic.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Peter Y. Woo, Biola University, La Mirada, CA, USA sketched the iterates of the function $f(x) = 4x - x^2$ and indicated that periodic points are found by intersecting the graph of the line $y = x$ with the graphs of these iterates, the n^{th} iterate having 2^{n-1} "bumps" on it.

The function $f(x) = 4x - x^2$ is known as the logistic function, *and its dynamics have been extensively studied. For example, see chapter 10 of the book by Heinz-Otto Peitgen, Harmut Jürgens, and Deitmar Saupe,* Chaos and Fractals: New Frontiers of Science, *2^{nd} ed., Springer.*

3390. [2008 : 483, 486] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that if A, B, C, and D are the solutions of

$$X^2 = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix},$$

then $A^{2007} + B^{2007} + C^{2007} + D^{2007} = O$, where O is the 2×2 zero matrix.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We generalize as follows: Let $M \in M_2(\mathbb{C})$ have two distinct nonzero eigenvalues in $\mathbb{C}$. Then the equation $X^2 = M$ has exactly four distinct solutions A, B, C, $D \in M_2(\mathbb{C})$ and moreover, if m is any odd positive integer, then $A^m + B^m + C^m + D^m = 0$.

Proof. First we show that if M is a diagonal matrix with distinct diagonal entries, then any solution to $X^2 = M$ is also a diagonal matrix. Indeed, let

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix}$$

with $\lambda \neq \mu$. It follows immediately that $a^2 \neq d^2$, hence $a + d \neq 0$. Thus $b = c = 0$.

Secondly, if $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^2$, then $a^2 = \lambda$ and $d^2 = \mu$. Thus, if ω and σ are choices of square roots of λ and μ, the equation $X^2 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ has exactly four distinct roots : $\begin{pmatrix} \pm\omega & 0 \\ 0 & \pm\sigma \end{pmatrix}$. [*Ed: Recall that λ and μ are nonzero.*]

Now let M be any 2×2 matrix with distinct nonzero eigenvalues λ, μ. Then, there exists an invertible matrix V such that $V^{-1}MV = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. It is easy to see that X is a solution to $X^2 = M$ if and only if $Y = V^{-1}XV$ is a solution to $Y^2 = V^{-1}MV = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. By the first part it follows that $X^2 = M$ has exactly four solutions, namely

$$V \begin{pmatrix} \pm\omega & 0 \\ 0 & \pm\sigma \end{pmatrix} V^{-1}.$$

Then

$$\begin{aligned} &A^m + B^m + C^m + D^m \\ &\qquad = V \begin{pmatrix} 2\omega^m - 2\omega^m & 0 \\ 0 & 2\sigma^m - 2\sigma^m \end{pmatrix} V^{-1} = 0. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE and ROGER

ZARNOWSKI, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; REBECCA EVERDING, student, Southeast Missouri State University, Cape Girardeau, MO, USA; CODY GUINAN, student, Southeast Missouri State University, Cape Girardeau, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; WILLIAM McNEARY, Charleston, MO, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; NANCY MUELLER and SETH STAHLHEBER, Southeast Missouri State University, Cape Girardeau, MO, USA; JENNIFER PAJDA, student, Southeast Missouri State University, Cape Girardeau, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete solution submitted.

Some solvers observed that the problem is solved once it is known that $X^2 = M$ has exactly four roots, for then the roots can be grouped in pairs $\pm X$.

Barbara remarked that in general if R is a unitary ring where 2 is not a zero divisor, N is a positive odd integer, and $S = \{x \in R : x^2 = \theta\}$ is finite for some $\theta \in R$, then $\sum_{x\in S} x^N = 0$.

3391. [2008 : 483, 486] *Proposed by Michel Bataille, Rouen, France.*

Let $ABCD$ be a convex quadrilateral such that AC and BD intersect in right angles at P, and let I, J, K, and L be the midpoints of AB, BC, CD, and DA, respectively. Show that the circles (PIJ), (PJK), (PKL), and (PLI) are congruent if and only if $ABCD$ is cyclic.

I. *Solution by Václav Konečný, Big Rapids, MI, USA.*

The midpoint quadrilateral $IJKL$ has sides parallel to the diagonals of $ABCD$, whence it is a rectangle. Because $ABCD$ is convex, the point P lies inside the rectangle. Moreover, because I and J are the midpoints of BA and BC, the line IJ is the perpendicular bisector of BP and, therefore, $\angle PIJ = \angle JIB$. Since $IJ \| AC$, we have $\angle JIB = \angle CAB$, whence

$$\angle PIJ = \angle CAB\,.$$

Similarly, JK is the perpendicular bisector of CP and

$$\angle PKJ = \angle CDB\,,$$

because they both equal $\angle CKJ$. Furthermore, since P is inside the rectangle (so that $\angle PIJ < \angle LIJ$ and $\angle PKJ < \angle LKJ$), both $\angle PIJ$ and $\angle PKJ$ are acute. Because two triangles with a common side that subtend acute angles have equal circumradii if and only if those angles are equal, we deduce that

$$\angle PIJ = \angle PKJ \iff (PIJ) \text{ and } (PJK) \text{ are congruent.}$$

Because $ABCD$ is convex, the vertices A and D lie on the same side of the line BC, whence

$$\angle CAB = \angle CDB \iff ABCD \text{ is cyclic.}$$

We conclude that if just the two circles (PIJ) and (PJK) are congruent, then $ABCD$ must be cyclic. Because there was nothing special about the circumcircles (PIJ) and (PJK), we have as a converse, if $ABCD$ is cyclic, then any two consecutive circumcircles of the chain (PIJ), (PJK), (PKL), (PLI) are congruent and, consequently, all four are congruent.

II. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands, expanded by the editor.*

The circles (PIJ), (PJK), (PKL), and (PLI) are the nine-point (or Feuerbach) circles of triangles ABC, BCD, CDA, and DAB, respectively (because each contains the midpoints of two sides and the foot of an altitude); thus if A, B, C, D are four points in any order on a circle for which AC and BD intersect orthogonally at P, the radius of each of the nine-point circles equals half the radius of the large circle. The converse is not so obvious—without using the convexity of $ABCD$, all we can conclude from the congruence of the circles (PIJ), (PJK), (PKL), and (PLI) is that the circles (ABC), (BCD), (CDA), and (DAB) are themselves congruent. Indeed, if two from the latter set of four circles are distinct, then these four circles are related as in ŢiŢeica's theorem: If three congruent circles pass through a common point, then their other three intersection points lie on a fourth circle of the same radius and, moreover, the four intersection points form an orthocentric quadrilateral (meaning each point is the orthocentre of the triangle formed by the other three). (See problem 3337 [2009 : 191-192], where that theorem is discussed and references are provided.) Thus, either the points A, B, C, D form an orthocentric quadrilateral, or they must lie on a circle. Since an orthocentric quadrilateral can never be convex (exactly one of the four triangles formed by three of the four points must be acute with the fourth point—the orthocentre—inside), we see that convexity forces $ABCD$ to be cyclic.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3392. [2008 : 484, 486] *Proposed by Michel Bataille, Rouen, France.*

Let A, B, C, D, and E be concyclic with V and W on the lines AB and AD, respectively. Show that if the line CE, the parallel to CB through V, and the parallel to CD through W are concurrent, then triangles EVB and EWD are similar. Does the converse hold?

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

[*Ed.: Note that the problem has been carefully worded to allow the points A through E to be any five distinct points on a circle (in any order). Because of this, we must use directed angles to avoid numerous special cases.*] Let P be the point where the parallel to CB through V intersects the parallel to CD through W. We shall prove a modified version of the result: P lies on CE if and only if the triangles EVB and EWD are *directly* similar. Because $V \in BA$, $W \in DA$, and A, B, D, and E lie on a circle, we have

$$\angle EBV \;=\; \angle EBA \;=\; \angle EDA \;=\; \angle EDW. \tag{1}$$

Furthermore, because C lies on that same circle, $\angle CBA = \angle CDA$. However, because $PV \| CB$ and $PW \| CD$, we have $\angle PVA = \angle CBA$ and $\angle PWA = \angle CDA$, whence $\angle PVA = \angle PWA$ and, therefore, the points A, V, P, and W also lie on a circle. Note that "P lies on CE" means that E is on the transversal CP of the parallel lines WP and CD. Thus,

$$P \in CE \iff \angle DCE = \angle WPE\,.$$

But, $\angle DCE = \angle DAE = \angle WAE$, whence

$$P \in CE \iff E \text{ lies on the circle containing } A,\ V,\ P,\ W.$$

This, in turn, is equivalent to $\angle AWE = \angle AVE$. Since $\angle DWE = \angle AWE$ and $\angle BVE = \angle AVE$, we deduce finally that

$$P \in CE \iff \angle DWE = \angle BVE\,. \tag{2}$$

From (1) and (2) we now have two pairs of equal corresponding angles, which completes the proof that P lies on CE if and only if the triangles EVB and EWD are *directly* similar.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer, who provided two solutions.

Note that in the statement of the problem the triangles are only required to be similar, not directly similar as all but one of the solvers assumed. The exception was Konečný, who joked that it was obvious that the converse does **not** *hold because of the way the question was worded. He and Bataille both provided counterexamples that require a picture. For an easier counterexample, start by choosing the points A and E on opposite ends of a diameter, and complete the configuration with $P \in CE$. Then, from the featured solution, we know that the triangles EVB and EWD are directly similar; moreover they have right angles at the corresponding vertices B and D. Reflect V in the line EB to the point V'. Then $EV'B$ and EVB are oppositely oriented congruent triangles, so that the triangles $EV'B$ and EWD are, indeed, similar triangles; however, the parallel to CB through V' is different from the parallel to CB through V so that it could not intersect the parallel to CD through W at a point of the line CE.*

3393. Correction. [2008 : 484, 486; 2009 : 108, 110] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let ABC be a triangle with $a = BC$, $b = AC$, $c = AB$, and semiperimeter s. Prove that

$$\frac{y+z}{x}\cdot\frac{A}{a(s-a)}+\frac{z+x}{y}\cdot\frac{B}{b(s-b)}+\frac{x+y}{z}\cdot\frac{C}{c(s-c)}\ \geq\ \frac{9\pi}{s^2}\,,$$

where the angles A, B, and C are measured in radians and x, y, and z are any positive real numbers.

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

By the AM–GM Inequality we have $\frac{y+z}{2}\geq\sqrt{yz}$, $\frac{z+x}{2}\geq\sqrt{xz}$, and $\frac{x+y}{2}\geq\sqrt{xy}$; hence $(y+z)(z+x)(x+y)\geq 8xyz$. Applying the AM–GM Inequality once more and then using the preceding inequality, we have

$$\begin{aligned}
&\frac{y+z}{x}\cdot\frac{A}{a(s-a)}+\frac{z+x}{y}\cdot\frac{B}{b(s-b)}+\frac{x+y}{z}\cdot\frac{C}{c(s-c)}\\
\geq\ & 3\sqrt[3]{\frac{(y+z)(z+x)(x+y)}{xyz}\cdot\frac{ABC}{abc(s-a)(s-b)(s-c)}}\\
\geq\ & 6\sqrt[3]{\frac{ABC}{abc(s-a)(s-b)(s-c)}}\,. \qquad (1)
\end{aligned}$$

Now, using the inequality $\prod\left(\frac{3A}{\pi}\right)\geq\frac{2r}{R}$ (see 6.59, p. 188 of [1]) and $2s^2\geq 27Rr$ (see 5.12, p. 52 of [2]); and also using the well-known relations $abc=4RF$, $F=rs$, and $F^2=s(s-a)(s-b)(s-c)$, where F is the area of triangle ABC, we have

$$\begin{aligned}
\frac{ABC}{abc(s-a)(s-b)(s-c)}\ &\geq\ \frac{\left(\frac{2r\pi^3}{27R}\right)}{4Rrs\left(\frac{r^2s^2}{s}\right)}\ =\ \frac{\pi^3}{54R^2r^2s^2}\\
&\geq\ \frac{\pi^3}{54s^2}\cdot\frac{27^2}{4s^4}\ =\ \frac{27\pi^3}{8s^6}\,. \qquad (2)
\end{aligned}$$

The desired inequality now follows directly from (1) and (2).

Equality holds if and only if $A+B+C=\frac{\pi}{3}$ and $x=y=z$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

References

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović, and P.M. Vasić, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969.

[2] D.S. Mitrinović, J.E. Pecarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1989.

3394. [2008 : 484, 486] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let $ABCD$ be a tetrahedron with h_A and m_A the lengths of the altitude and the median from vertex A to the opposite face BCD, respectively. If V is the volume of the tetrahedron, prove that

$$(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \;\geq\; \frac{128}{\sqrt{3}}V\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we consider the sum of altitudes. Let $[XYZ]$ denote the area of triangle XYZ. By the AM–HM Inequality we obtain

$$\begin{aligned} h_A + h_B + h_C + h_D &= 3V\left(\frac{1}{[BCD]} + \frac{1}{[CDA]} + \frac{1}{[DAB]} + \frac{1}{[ABC]}\right) \\ &\geq 3V \cdot \frac{16}{[BCA] + [CDA] + [DAB] + [ABC]}\,. \end{aligned}$$

Next we compute the median. By Weitzenböck's inequality we have $BC^2 + CD^2 + DB^2 \geq 4\sqrt{3}[BCD]$, with similar inequalities for the other three faces, hence,

$$\begin{aligned} &AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \\ &\qquad \geq\; 2\sqrt{3}([ABC] + [BCD] + [CDA] + [DAB])\,. \end{aligned}$$

We also have

$$\begin{aligned} 9m_A^2 &= \left|\overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D} - 3\overrightarrow{A}\right|^2 \\ &= 3\left(\left|\overrightarrow{B} - \overrightarrow{A}\right|^2 + \left|\overrightarrow{C} - \overrightarrow{A}\right|^2 + \left|\overrightarrow{D} - \overrightarrow{A}\right|^2\right) \\ &\qquad - \left|\overrightarrow{C} - \overrightarrow{B}\right|^2 - \left|\overrightarrow{D} - \overrightarrow{C}\right|^2 - \left|\overrightarrow{B} - \overrightarrow{D}\right|^2 \\ &= 3\left(AB^2 + AC^2 + AD^2\right) - BC^2 - CD^2 - DB^2\,, \end{aligned}$$

with cyclic variants holding for the other three medians. Therefore,

$$\begin{aligned} &9\left(m_A^2 + m_B^2 + m_C^2 + m_D^2\right) \\ &\qquad = 4\left(AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2\right) \\ &\qquad \geq 8\sqrt{3}([ABC] + [BCD] + [CDA] + [DAB])\,. \end{aligned}$$

Finally we put everything together:

$$(h_A + h_B + h_C + h_D)\left(m_A^2 + m_B^2 + m_C^2 + m_D^2\right) \;\geq\; 48V \cdot \frac{8\sqrt{3}}{9} \;=\; \frac{128}{\sqrt{3}}V\,.$$

This completes the proof. Equality holds if and only if the tetrahedron is regular, as can be seen from the condition for equality in Weitzenböck's inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania (two solutions); CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLEH FAYNSHTEYN, Leipzig, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU, Comănești, Romania; and the proposer. One incomplete solution was submitted.

By employing the Hadwiger–Finsler Inequality, Bencze proved that

$$(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \;\geq\; \frac{128}{\sqrt{3}}V + \frac{32r}{9}\sum (AB - AC)^2\,,$$

where r is the inradius of the tetrahedron and the sum is over the 12 pairs of edges of the tetrahedron that share a common vertex.

Janous proved that $(h_A h_B h_C h_D)^{1/4}(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq \frac{32}{\sqrt{3}}V$, from which the proposed inequality follows on account of the AM–GM Inequality.

3395. [2008 :484, 486] *Proposed by Taichi Maekawa, Takatsuki City, Osaka, Japan.*

Let triangle ABC have orthocentre H and circumradius R. Prove that $4R^3 - (l^2 + m^2 + n^2)R - lmn = 0$, where $AH = l$, $BH = m$, and $CH = n$.

Solution by Michel Bataille, Rouen, France.

The relation $4R^3 - \left(l^2 + m^2 + n^2\right)R - lmn = 0$ holds only for acute-angled or right-angled triangles. We show, more generally, that

$$4R^3 - \left(l^2 + m^2 + n^2\right)R - \epsilon lmn \;=\; 0\,, \qquad (1)$$

where $\epsilon = -1$ if one of the angles of $\triangle ABC$ is obtuse and $\epsilon = +1$ otherwise.

Let O be the circumcentre of $\triangle ABC$ and A', B', C' the midpoints of the sides BC, CA, AB, respectively. It is well known and easy to prove that $l = 2OA'$, $m = 2OB'$, and $n = 2OC'$.

If $A \leq 90^\circ$, then $\angle BOC = 2A$ and $OA' = R\cos A$; but if $A > 90^\circ$, then we have $\angle BOC = 360^\circ - 2A$ and $OA' = R\cos(180^\circ - A) = -R\cos A$. Thus,

$$\epsilon lmn \;=\; 8R^3 \cos A \cos B \cos C\,.$$

Using the well-known trigonometric identity for the angles of a triangle,

$$\cos^2 A + \cos^2 B + \cos^2 C \;=\; 1 - 2\cos A \cos B \cos C\,,$$

we obtain

$$\begin{aligned}\left(l^2 + m^2 + n^2\right)R &= 4R^3\left(\cos^2 A + \cos^2 B + \cos^2 C\right)\\ &= 4R^3\left(1 - 2\cos A \cos B \cos C\right).\end{aligned}$$

The relation (1) follows.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comănești, Romania; and the proposer.

Alt, Bataille, Geupel, Janous, Schlosberg, and Zvonaru each noted that the extra condition is required and either suggested adding a hypothesis or provided a counterexample for obtuse-angled triangles. The other solvers assumed the triangle is acute-angled.

3396. [2008 : 484, 487] *Proposed by Neven Jurič, Zagreb, Croatia.*

Let n be a positive integer, and for i, j, and k in $\{1, 2, \dots, n\}$ let

$$a_{ijk} = 1 + \operatorname{mod}(k-i+j-1,n) + n\operatorname{mod}(i-j+k-1,n) + n^2\operatorname{mod}(i+j+k-2,n),$$

where $\operatorname{mod}(a,n)$ is the residue of a modulo n in the range $0, 1, \dots, n-1$. For which n is the cube with entries a_{ijk} a magic cube? (Here "magic" means that the sum of a_{ijk} is constant if two indices are fixed and the third index varies, and also the sums along the great diagonals of the cube are equal to this constant.)

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We will prove that the cube is magic for all n and that the magic sum is $S_n = \frac{n(n^3+1)}{2}$. Let $L_n = \{0, 1, 2, \dots, n-1\}$, and for convenience write $b_{ijk} = k-i+j-1$, $c_{ijk} = i-j+k-1$, and $d_{ijk} = i+j+k-2$. Then

$$a_{ijk} = 1 + \operatorname{mod}(b_{ijk},n) + n\operatorname{mod}(c_{ijk},n) + n^2\operatorname{mod}(d_{ijk},n).$$

For fixed j and k, the n numbers b_{ijk}, $1 \le i \le n$, are pairwise incongruent modulo n, since $k - i_1 + j - 1 \equiv k - i_2 + j - 1 \pmod{n}$ implies $i_1 = i_2$. Hence, $\{\operatorname{mod}(b_{ijk},n) : 1 \le i \le n\} = L_n$. Similarly, $\{\operatorname{mod}(c_{ijk},n) : 1 \le i \le n\} = \{\operatorname{mod}(d_{ijk},n) : 1 \le i \le n\} = L_n$.

It follows that

$$\begin{aligned}\sum_{i=1}^{n} a_{ijk} &= n + (1+n+n^2)\sum_{l=0}^{n-1} l \\ &= n + (1+n+n^2)\frac{n(n-1)}{2} \\ &= n + \frac{n(n^3-1)}{2} = \frac{n(n^3+1)}{2} = S_n.\end{aligned}$$

Analogously, $\sum_{j=1}^{n} a_{ijk} = S_n$ and $\sum_{k=1}^{n} a_{ijk} = S_n$.

Now we compute the four diagonal sums. Since $b_{iii} = c_{iii} = i - 1$, we have $\{\text{mod}(b_{iii}, n) \colon 1 \le i \le n\} = \{\text{mod}(c_{iii}, n) \colon 1 \le i \le n\} = L_n$. Hence, $\sum_{i=1}^{n} \text{mod}(b_{iii}, n) = \sum_{i=1}^{n} \text{mod}(c_{iii}, n) = \sum_{l=0}^{n-1} l = \frac{n(n-1)}{2}$. Next, we have $d_{iii} = 3i - 2$. If $3 \nmid n$, then $\{\text{mod}(d_{iii}, n) \colon 1 \le i \le n\} = L_n$ so $\sum_{i=1}^{n} \text{mod}(d_{iii}, n) = \frac{n(n-1)}{2}$.
If $3|n$, then $n = 3m$ for some positive integer m. Hence,

$$\begin{aligned}\sum_{i=1}^{n} \text{mod}(d_{iii}, n) &= \sum_{i=1}^{n} \text{mod}(3i-2, n) = 3\sum_{i=1}^{m} \text{mod}(3i-2, 3m) \\ &= 3\sum_{i=1}^{m}(3i-2) = 3\left(\frac{3m(m+1)}{2} - 2m\right) \\ &= \frac{3m(3m-1)}{2} = \frac{n(n-1)}{2},\end{aligned}$$

as before.

Therefore, the sum along the main diagonal is

$$\begin{aligned}\sum_{i=1}^{n} a_{iii} &= \sum_{i=1}^{n}\left(1 + \text{mod}(b_{iii}, n) + n \text{ mod } (c_{iii}, n) + n^2 \text{ mod } (d_{iii}, n)\right) \\ &= n + \left(1 + n + n^2\right)\frac{n(n-1)}{2} = \frac{n(n^3+1)}{2} = S_n\,.\end{aligned}$$

Next we consider the diagonal $\{(n+1-i, i, i) \colon 1 \le i \le n\}$. For notational convenience, let $i^* = n + 1 - i$. Then

$$\begin{aligned}b_{i^*ii} &= i - (n+1-i) + i - 1 = 3i - n - 2\,, \\ c_{i^*ii} &= (n+1-i) - i + i - 1 = -i + n\,, \\ d_{i^*ii} &= (n+1-i) + i + i - 2 = i + n - 1\,.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} a_{i^*ii} &= \sum_{i=1}^{n}\big(1 + \text{mod}(3i-n-2, n) \\ &\qquad + n \text{ mod } (-i+n, n) + n^2 \text{ mod } (i+n-1, n)\big) \\ &= n + \left(1 + n + n^2\right)\sum_{l=0}^{n-1} l = \frac{n(n^3+1)}{2} = S_n\,,\end{aligned}$$

by the same argument as above.

The sums of the entries in the cells of $\{(i, n+1-i, i) | 1 \le i \le n\}$ and $\{i, i, n+1-i) | 1 \le i \le n\}$ can be calculated in a similar manner to yield the same value S_n.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and the proposer. There was one partly incorrect solution.

The proposer and the editor had tacitly assumed that the entries of the cube must be $1, 2, \ldots, n^3$ in which case the cube would be magic if and only if n is odd. This was proved by the proposer. Apparently, neither Curtis nor Geupel made this assumption in their solutions.

3397. [2008 : 485, 487] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Evaluate

$$\lim_{n\to\infty} \frac{1}{n^2} \int_0^n \frac{\sqrt{n^2 - x^2}}{2 + x^{-x}}\, dx \,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let I denote the above integral. Making the substitution $x = nt$, we have

$$\begin{aligned} \frac{1}{n^2} I &= \int_0^1 \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}}\, dt \\ &= \int_0^{\frac{1}{\sqrt{n}}} \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}}\, dt \;+\; \int_{\frac{1}{\sqrt{n}}}^1 \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}}\, dt \,. \end{aligned} \tag{1}$$

Observe that for $0 < t < 1$, we have

$$0 \;<\; \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}} \;<\; \frac{1}{2}\,, \tag{2}$$

and for $\frac{1}{\sqrt{n}} < t < 1$ we have $\sqrt{n} < nt$ and $t^t < 1$, so that

$$\frac{1}{2 + \sqrt{n}^{(-\sqrt{n})}} \;<\; \frac{1}{2 + (nt)^{-nt}} \;<\; \frac{1}{2 + n^{-n}}\,. \tag{3}$$

From (2) we obtain

$$0 \;<\; \int_0^{\frac{1}{\sqrt{n}}} \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}} dt \;<\; \frac{1}{2\sqrt{n}} \tag{4}$$

and from (3) we obtain

$$\begin{aligned} \frac{1}{2 + \sqrt{n}^{(-\sqrt{n})}} \int_{\frac{1}{\sqrt{n}}}^1 \sqrt{1-t^2}\, dt \;&<\; \int_{\frac{1}{\sqrt{n}}}^1 \frac{\sqrt{1-t^2}}{2 + (nt)^{-nt}}\, dt \\ &<\; \frac{1}{2 + n^{-n}} \int_{\frac{1}{\sqrt{n}}}^1 \sqrt{1-t^2}\, dt \,. \end{aligned} \tag{5}$$

Letting n tend to infinity in (5), we see that the integral in the middle approaches $\frac{1}{2}\int_0^1 \sqrt{1-t^2}dt$. Since $\int_0^1 \sqrt{1-t^2}dt = \frac{\pi}{4}$, we obtain from (1), (4), and (5) that

$$\lim_{n\to\infty} \frac{1}{n^2}\int_0^n \frac{\sqrt{n^2-x^2}}{2+x^{-x}}\,dx \;=\; 0+\frac{1}{2}\cdot\frac{\pi}{4} \;=\; \frac{\pi}{8}\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One solution was submitted with an incomplete justification of the calculations.

3398. [2008 : 485, 487] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given the equation

$$\left\lfloor \frac{n}{10} \right\rfloor \;+\; \left(n - 10\left\lfloor \frac{n}{10} \right\rfloor\right)\cdot 10^{\lfloor \log_{10} n \rfloor} \;=\; \frac{2n}{3}\,,$$

(a) show that $n = 5294117647058823$ is a solution,

(b) ★ find all other positive integer solutions of the equation.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We will prove that all solutions are given by

$$n \;=\; \frac{3b(10^{16k}-1)}{17}\,,$$

where k is a positive integer and b is any integer with $1 \le b \le 5$.

The solution in (a) is obtained by setting $k = 1$ and $b = 3$.

Each positive integer n has an unique representation $n = 10a + b$, where a and b are nonnegative integers and $b \le 9$. Let $l + 1$ be the number of decimal digits of n. It is easy to check that there is no solution with $n \le 9$, hence we can assume that $l \ge 1$. Then, the number n is a solution if and only if

$$a + 10^l b \;=\; \frac{2}{3}(10a+b)\,,$$

or equivalently

$$a \;=\; \frac{(3\cdot 10^l - 2)b}{17}\,,$$

where

$$10^{l-1} \le a < 10^l \quad \text{and} \quad 0 \le b \le 9\,.$$

For $l = 1$ we get $28b = 17a$, which is impossible because 17 is not a divisor of $28b$ unless b is zero. Therefore, we have $l \ge 2$. The inequality involving a will now be satisfied if and only if $1 \le b \le 5$.

[*Ed:* $10^{l-1} \le a < 10^l \iff 10^{l-1} \le \dfrac{(3\cdot 10^l - 2)b}{17} \le 10^l - 1 \iff 17\cdot 10^{l-1} \le (3\cdot 10^l - 2)b \le 17\cdot(10^l - 1)$. *Now* $17\cdot 10^{l-1} < 30\cdot 10^{l-1} - 2$ *because* $l \ge 2$, *and since* b *is a nonnegative integer the first inequality is equivalent to* $b \ge 1$.

The second inequality becomes $b \le \dfrac{17\cdot(10^l - 1)}{3\cdot 10^l - 2} = 5 + \left(\dfrac{2\cdot 10^l - 7}{3\cdot 10^l - 2}\right)$. *Since* b *is an integer, this is equivalent to* $b \le 5$.]

Thus, n is a solution to the given equation if and only if

$$3\cdot 10^l \equiv 2 \pmod{17} \qquad (1)$$

and $1 \le b \le 5$.

By Fermat's Little Theorem, $10^{16+l} \equiv 10^{16} \pmod{17}$, and an easy computation shows that the only solution l of (1) with $1 \le l \le 16$ is $l = 15$. Thus, (1) is equivalent to $l \equiv 1 \pmod{16}$, or $l + 1 = 16k$.

Putting everything together, we have that n is a solution if and only if

$$\begin{aligned} n &= 10a + b = 10\cdot\left(\frac{3\cdot 10^l - 2}{17}\right)b + b \\ &= \frac{3\cdot(10^{l+1} - 3)}{17}b = \frac{3b(10^{16k} - 1)}{17}, \end{aligned}$$

where $k \ge 1$ and $1 \le b \le 5$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and ALBERT STADLER, Herrliberg, Switzerland.

Part (a) was also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, WA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution submitted.

The proposer remarked that $\frac{1}{3}\cdot 5294117647058823 = 1764705882352941$, *that is, the decimal digits of the first solution to the equation are rotated five places to the right upon multiplication by* $\frac{1}{3}$

3399. [2008 : 485, 487] *Proposed by Vincenţiu Rădulescu, University of Craiova, Craiova, Romania.*

Prove that there does not exist a positive, twice differentiable function $f : [0, \infty) \to \mathbb{R}$ such that $f(x)f''(x) + 1 \le 0$ for all $x \ge 0$.

Comments by Michel Bataille, Rouen, France; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Bataille indicated this problem was posed in the *College Mathematics Journal* by the same proposer in January, 2008. A solution and a generalization appeared in the January, 2009 issue of that journal (problem 869, pp. 60-61).

Janous indicated that this problem was also posed in the Swiss journal *Elemente der Mathematick* in the problem section of Heft 4, 2008, by the same proposer.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

Barbara showed that if f is negative and satisfies the given inequality, then $-f$ is positive and also satisfies the inequality. Hence, the condition "f positive" can be omitted. On the other hand, by re-scaling, the number 1 can be replaced by ϵ. The result can be reformulated as: Let $\epsilon > 0$. There does not exist a twice differentiable function $f : [0,\infty) \to \mathbb{R}$ such that $f(x)f''(x) + \epsilon \leq 0$ for all $x \geq 0$.

Barbara further observed that ϵ cannot be omitted. The function $f(x) = \log(x+2)$ satisfies $f(x)f''(x) < 0$ for all $x \geq 0$. This leads to another reformulation of this result: let $f : [0,\infty) \to \mathbb{R}$ be a twice differentiable function such that $f(x)f''(x) < 0$ (or ≤ 0) for all $x \geq 0$. Then, $f(x)f''(x)$ tends to zero as x tends to infinity.

3400. [2008 : 485, 487] *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

For positive integers m and k let $(m)_k = m(1+10+10^2+\cdots+10^{k-1})$, for example, $(1)_2 = 11$ and $(3)_4 = 3333$. Find all real numbers α such that

$$\left\lfloor 10^n\sqrt{(1)_{2n}+\alpha}\right\rfloor \;=\; (3)_{2n} \;-\; \left\lfloor\frac{5-9\alpha}{6}\right\rfloor$$

holds for each positive integer n, where $\lfloor x\rfloor$ is the greatest integer not exceeding x.

Solution by the proposer.

The answer for α is the union of 16 closed and disjoint intervals:

$$\alpha \in \bigcup_{m=-8}^{7}\left[\frac{m^2-66m-11}{100}\,,\,\frac{5-6m}{9}\right]. \tag{1}$$

To prove this, we first let $n=1$. Then

$$\left\lfloor 10\sqrt{11+\alpha}\right\rfloor \;=\; 33-\left\lfloor\frac{5-9\alpha}{6}\right\rfloor. \tag{2}$$

Set $m = \left\lfloor\frac{5-9\alpha}{6}\right\rfloor$. Then

$$m \;\leq\; \frac{5-9\alpha}{6} \;<\; m+1\,,$$

which is equivalent to

$$-\frac{6m+1}{9} \;<\; \alpha \;\leq\; \frac{5-6m}{9}\,. \tag{3}$$

From (2) we obtain $33 - m \le 10\sqrt{11+\alpha} < 34 - m$. Solving for α yields

$$\frac{m^2 - 66m - 11}{100} \;\le\; \alpha \;<\; \frac{m^2 - 68m + 56}{100}\,. \tag{4}$$

It is easy to check that

$$\frac{5 - 6m}{9} \;<\; \frac{m^2 - 68m + 56}{100}$$

and

$$-\frac{6m + 1}{9} \;<\; \frac{m^2 - 66m - 11}{100}\,.$$

Then, from (3) and (4), it follows that

$$\frac{m^2 - 66m - 11}{100} \;\le\; \alpha \;\le\; \frac{5 - 6m}{9}\,. \tag{5}$$

From (5) we obtain successively

$$\begin{aligned} \frac{m^2 - 66m - 11}{100} \;&\le\; \frac{5 - 6m}{9}\,, \qquad (6)\\ (3m+1)^2 \;&\le\; 600\,,\\ |3m+1| \;&\le\; \left\lfloor 10\sqrt{6} \right\rfloor \;=\; 24\,, \end{aligned}$$

from which we see that inequality (6) is strict for $m = -8$, -7, $\dots$, 7. Thus, the equality (2) may only hold for values of α satisfying the inequality (5) for $m = -8$, -7, $\dots$, 7; these are precisely the values of α covered in (1).

We shall now prove that the given equality does indeed hold for these values of α. We note that the given equality is equivalent to each of the following double inequalities:

$$(3)_{2n} - m \;\le\; 10^n\sqrt{(1)_{2n} + \alpha} \;<\; (3)_{2n} - m + 1\,,$$

$$-m \;\le\; 10^n\sqrt{(1)_{2n} + \alpha} \;-\; (3)_{2n} \;<\; -m + 1\,,$$

$$-m \;\le\; 10^n\sqrt{\frac{10^{2n} - 1}{9} + \alpha} \;-\; \left(\frac{10^{2n} - 1}{3}\right) \;<\; -m + 1\,,$$

$$-1 - 3m \;\le\; 10^n\sqrt{10^{2n} + 9\alpha - 1} \;-\; 10^{2n} \;<\; 2 - 3m\,,$$

$$-1 - 3m \;\le\; \frac{10^n(9\alpha - 1)}{\sqrt{10^{2n} + 9\alpha - 1} \;+\; 10^n} \;<\; 2 - 3m\,,$$

$$-1 - 3\left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor \;\le\; \frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^{2n}}} \;+\; 1} \;<\; 2 - 3\left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor\,. \tag{7}$$

Note that the double inequality (7) holds for $\alpha = \frac{1}{9}$, and that

$$\frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^2}} + 1} \leq \frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^{2n}}} + 1} < \frac{9\alpha - 1}{2}$$

holds (consider separately the cases $\alpha > \frac{1}{9}$ and $\alpha < \frac{1}{9}$). Therefore, in order to prove (7), it suffices to show that

$$-1 - 3\left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor \leq \frac{9\alpha - 1}{\sqrt{1 + \frac{9\alpha - 1}{10^2}} + 1} \tag{8}$$

and

$$\frac{9\alpha - 1}{2} \leq 2 - 3\left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor . \tag{9}$$

Inequality (8) can be written as

$$m \geq -\frac{30\alpha + \sqrt{11 + \alpha}}{10 + 3\sqrt{11 + \alpha}} ,$$

which simplifies to $m \geq 33 - 10\sqrt{11 + \alpha}$. This last inequality is equivalent to the left inequality of (5). Inequality (9) simplifies to

$$\left\lfloor \frac{5 - 9\alpha}{6} \right\rfloor \leq \frac{5 - 9\alpha}{6} ,$$

which is obvious.

Remark. The given equality holds asymptotically for any value of α. This can be easily proved by fixing α and letting $n \to \infty$ in inequality (7).

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany. One incomplete solution and two incorrect solutions were submitted.

The proposer mentioned that some problems on equalities involving the integer part function and a parameter had appeared earlier [1], [2].

References

[1] I. Bluskov, Problem 1650, ***Crux Mathematicorum*** Vol. 17 (1991) p. 141

[2] I. Bluskov, Problem 3216, *Mathematics in School* (Russian) Vol. 3 (1988) p. 60

YEAR END FINALE

As I enter my final year as Editor-in-Chief of ***CRUX with MAYHEM***, the call has been put out by the Canadian Math Society for a new Editor-in-Chief, with nominations being taken by the CMS Publications Committee. Any potential nominee should know that (if needed) I am willing to serve as a Co-Editor for all of 2011 and to serve as an Editor-at-Large beyond that.

This past year JOHAN RUDNICK took over the reins from GRAHAM WRIGHT as Executive Director of the CMS and also as Managing Editor of ***CRUX***. I appreciate very much Johan's interest in ***CRUX*** and our discussions, and also Graham's continuing support in his role as Consultant towards the end of the year. The last Publications Committee meeting I attended marked the end of Graham's term, so it is now finally safe to say, "Graham, all the best for you in your retirement!"

DENISE CHARRON and LAURA ALYEA have done a wonderful job this past year administering ***CRUX*** at the CMS headquarters in Ottawa.

Our past ***CRUX*** editor BRUCE SHAWYER continues to contribute to ***CRUX*** and is an inspiration for me. I also thank our past ***CRUX*** editor BILL SANDS for his sound advice and fantastic proof reading (it is my theory that in a previous life Bill was an eagle that learned to read). My colleague and past CMO Chair TERRY VISENTIN has also lent his experience to proofing the copy.

In the New Year the ***CRUX*** board warmly welcomes two new Problems Editors: DZUNG MINH HA of Ryerson University and JONATAN ARONSSON of the University of Manitoba. I hope they find the problems moderating every bit as fascinating as I do. This past year LILY YEN and MOGENS LEMVIG HANSEN came on board as ***Skoliad*** Editors. Thank you Lily and Mogens for your wonderful contributions, and for generously agreeing to continue editing ***Skoliad*** beyond your first year.

ILIYA BLUSKOV and MARIA TORRES are stepping down at the end of 2009 as Problems Editors, and I thank them for their service. Iliya, it has been a pleasure and fun as well, may you find every design that you search for (except those that do not exist). Maria, good luck with your future projects and I am grateful for your generous offer to help with Spanish translations in the future. Gracias!

JOHN GRANT McLOUGHLIN is stepping down as Member-at-Large to take up editing the Education section of the *CMS Notes*. John, best of luck in the future and thank you for your incredible effort for the JIM TOTTEN special issue this past year, which made it that much more special.

JEFF HOOPER, our Associate Editor, has pulled my bacon out of the fire more than once this past year, and I am most grateful for it! IAN VANDERBURGH, our ***Mayhem*** Editor, continues to outpace this Editor-in-Chief as he churns out yet another Problem of the Month column. Ian, let's see if I can catch up this year! A thank you is due to ROBERT WOODROW for managing the Herculean task of assembling the Olympiad Corner this past year. This was also AMAR SODHI's first year as Book Reviews Editor, and a prolific one judging from the long list of reviews in the index that follows. I am ever grateful for the work of my colleague and Articles Editor, JAMES CURRIE, who has kept ***CRUX*** with a healthy supply of articles which makes it that much easier to complete an issue.

I thank CHRIS FISHER for his myriad contributions to ***CRUX*** beyond his duties as Problems Editor. Chris' sense of humour is infectious and brings some much needed levity into the process of editing. I thank EDWARD WANG for continuing as Problems Editors this past year despite a very high workload, and for his perseverance during a reallocation of support staff within his department.

I thank MONIKA KHBEIS and ERIC ROBERT for their continuing service on the ***Mayhem*** staff. I thank JEAN-MARC TERRIER for translating the problems that appear in ***CRUX with MAYHEM***, and ROLLAND GAUDET for translations of the ***Skoliad*** contests. So great was Jean-Marc's zeal for translating this past year that he crazily requested that copy be sent over the holiday, to which this editor instantly responded with the required certificate of insanity!

I thank JILL AINSWORTH and JOANNE CANAPE at the University of Calgary; TAO GONG, JUNE ALEONG, and LOUIS MOUSTERAKIS at Wilfrid Laurier University, for their support in producing LaTeXfiles. A big thank you goes to Mr. MATHIAS PIELAHN, the ***CRUX*** journal assistant, for his help with managing the mountains of information we receive as input.

Thanks go to JUDI BORWEIN, MICHAEL DOOB, CRAIG PLATT, and STEVE LA ROCQUE for technical support and putting ***CRUX*** up on the net. TAMI EHRLICH and the people at Thistle Printing continue to print fine quality issues of ***CRUX***, which are always satisfying to receive in the mail.

The Department of Mathematics and Statistics at the University of Winnipeg continues to provide support, and I thank the Dean of Science, ROD HANLEY, for keeping up the commitment of the University of Winnipeg.

I especially thank LYNNE TOTTEN, for her kind words about our efforts on the JIM TOTTEN special issue, and for providing a sense of perspective when it was most helpful.

I thank my wife CHARLENE for her support during this past year, and special friends PETER ARPIN and RANDALL PYKE for their special help with ***CRUX***.

Though this last year has seen a global recession, our subscriptions have remained almost steady. This speaks to how tremendously ***CRUX*** is valued by you, the readers. I thank all of you for your support of ***CRUX*** in these difficult financial times, and I thank you even more for all your wonderful mathematical contributions, which is the stuff that ***CRUX*** is made of.

It is an honour to partake of your contributions almost each and every day, and a wonderful source of enrichment for me.

I wish everyone happiness, peace, and fulfillment in the New Year.

Václav (Vazz) Linek

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