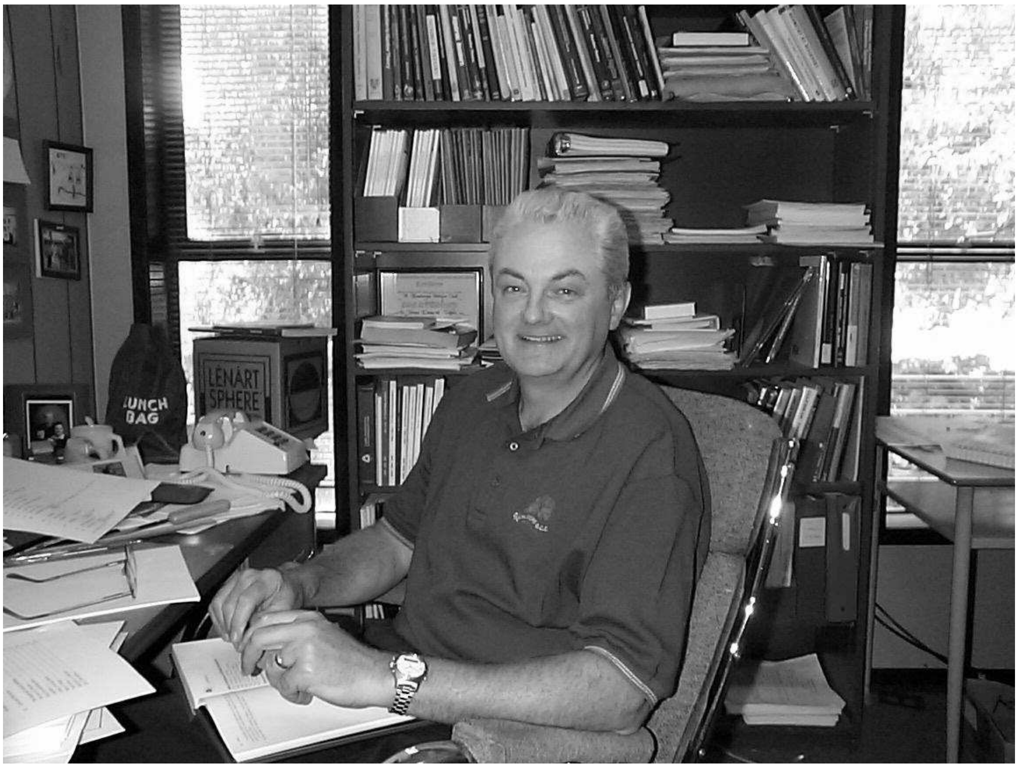
James Edward Totten
1947–2008



A photograph of Jim Totten in his office taken by his colleague Don DesBrisay around the time of Jim's retirement. Jim loved to demonstrate mathematics using models, such as the Lénárt Sphere™ kit for investigating spherical geometry that sits in the background. (Photo courtesy of Don DesBrisay)

IN MEMORIAM
James Edward Totten, 1947–2008

With the sudden death of Jim (James Edward Totten) on March 9, 2008, the Mathematics Community lost someone who was dedicated to mathematics education and to mathematical outreach. Jim was born August 9, 1947 in Saskatoon and raised in Regina. He obtained a B.Sc. from the University of Saskatchewan, and then an M.Sc. in Computer Science and a Ph.D. in Mathematics from the University of Waterloo. After a two year NRC Postdoctoral Fellowship in Tubingen, Germany, he joined the faculty at Saint Mary's University in Halifax. One of us (Robert) first got to know Jim when he and Jim shared an office at the University of Saskatchewan while Jim visited there in 1978-1979. That was a long cold winter, but Jim's active interest and enthusiasm for mathematics and the teaching of mathematics made the year a memorable one. The next year Jim took a position at Cariboo College, where he remained as it evolved into the University College of the Cariboo and then into Thompson Rivers University, retiring as Professor Emeritus in 2007.

During his years in Kamloops, Jim was a mainstay of the Cariboo Contest, an annual event which brought students to the college and which featured a keynote speaker, often drawn from Jim's list of mathematical friends. This once included an invitation to Robert, which featured a talk on public key encryption mostly memorable for the failure of technology at a key moment, much to Jim's amusement.

Jim became a member of the CMS in 1981, and joined the editorial board of ***Crux*** in 1994. When Bruce was looking for someone to succeed him as Editor-in-Chief, there was no doubt in his mind whom to approach. Bruce spent a week in Kamloops staying at Jim's home and working with Jim and Bruce Crofoot to smooth the transition. Jim's attention to detail and care was appreciated by all, particularly those contributing copy that was carefully checked, as Robert gladly confirms from his continued association with Jim through the *Olympiad Corner*, an association which continued to the end.

Jim loved his Oldtimers' hockey and was an avid golfer. When not playing hockey or golf, he was active with the Kamloops Outdoors Club. Jim was never just a participant, always an active volunteer.

Jim is survived by his loving wife of 40 years, Lynne, son Dean, daughter-in-law Christie and granddaughter Mikayla of Sechelt, father Wilf Totten of Edmonton, sister Judy Totten of Regina, sister Josie Laing (Neil) of Onoway, sister-in-law Marilyn Totten of Regina, mother-in-law Joyce Ladell of Swansea Point, brother-in-law Brian Ladell (Iris) of Red Deer, sister-in-law Constance Ladell (David Dahl) of Kamloops, many cousins, family friends, and mathematical colleagues; all miss his warmth and love. He was predeceased by his mother Alice, brother Gerry, and father-in-law Syd Ladell.

To honour the memory of Jim, the Thompson Rivers University Foundation is now accepting contributions for the Jim Totten Scholarship.

As remembered by two of his colleagues from ***Crux***,

Bruce Shawyer and Robert Woodrow

Totten Commemorative Issue

After the sudden and unexpected death of Jim Totten on March 9, 2008, we announced in the May 2008 issue that a special commemorative issue of ***CRUX with MAYHEM*** would appear in Jim's honour. We now have the pleasure of presenting this issue to our readers, for which the members of the editorial board have made special efforts.

This issue has also drawn upon many in Jim's community, including various colleagues at Thompson Rivers University (TRU). The obituary prepared by Bruce Shawyer and Robert Woodrow (that appeared in the May 2008 *CMS Notes*) has been adapted with the assistance of Lynne Totten for publication here. It is preceded by a picture of Jim in his office.

After serving for five years as a Problems Editor for ***CRUX***, Jim became Editor-in-Chief of ***CRUX with MAYHEM*** in 2003 and ran the journal until he began handing over the reins just months before his death last year. He also had an enormous influence in his home province of British Columbia in spreading his love of mathematics through his outreach activities. It is no surprise, then, that this special issue has a distinctive regional emphasis. A piece about Jim's activities and influence follows this editorial, which readers may wish to correlate with the article by Clint Lee on the British Columbia Secondary School Mathematics Contest and Jim's inspiring role in its development. *Skoliad* 118 in this issue features a contest selected by John Ciriani, a founder of the Cariboo contest that Jim helped to develop. The ***Mayhem*** section has a special *Problem of the Month* taken from one of the contests.

The ***Mayhem*** section also has a full-length article by that master of mathematical recreations, Ross Honsberger, who inspired Jim's interest in problem solving while he and Jim were office neighbours at the University of Waterloo. An article by Michel Bataille on the geometry of bicentric quadrilaterals graces the articles section, followed by further recreations in Awani Kumar's article on knight's tours on the surface of a cube.

Serendipitously, we found it fitting to reprint a book review written by Jim himself in the book reviews section. Also in the book reviews is an amusing anecdote of Andy Liu related to Jim.

From our readers' submissions we have compiled 26 special problems dedicated to Jim's memory including 10 in ***Mayhem*** and 16 in ***CRUX***, a dozen of which make a special section and four others that open the usual ***CRUX*** complement. Our solutions section is particularly rich this time around with some hefty problems settled by ***CRUX*** readers.

For those wishing to learn more about Jim we announce here that the proceedings of the May 2009 conference *Sharing Mathematics: A Tribute to Jim Totten* will be published later this year.

Jim Totten's Reach

John Grant McLoughlin

Shane Rollans opened the conference Sharing Mathematics: A Tribute to Jim Totten with these words: "This conference is a tribute to our long-time colleague, Jim Totten ... Jim had at least five passions. Foremost was his family. His wife, Lynne, shared his passion for the outdoors. His son, Dean, shared his passion for golf. His other passions were playing hockey and sharing his love of mathematics, which brings us here."

Jim is known for his work with the British Columbia Secondary School Mathematics Contest (see Clint Lee's article on pages 307-309 for details) and over fourteen years of service to ***Crux***, including nine years as a Problems Editor prior to becoming Editor-in-Chief of ***CRUX with Mayhem*** in 2003. This profile of Jim offers a broader picture of Jim's contributions in mathematics and spirit, mainly as seen through the eyes of his colleagues.

When Jim arrived at Cariboo College in 1979 he promptly organized a Putnam team that he led until 2005. He also started the "Problem of the Week" tradition, something he had done previously at the University of Saskatchewan and St. Mary's University. Each week he would collect solutions, grade them, and post the results complete with a solution. For the first 27 years no problem was repeated, although Jim did allow himself to repost a few of his favourite problems in his final year. Jim compiled 80 of these problems and solutions into a book, *Problems of the Week, Volume VII* in the ATOM (A Taste of Mathematics) Series published by the Canadian Mathematical Society (CMS) in 2007.

Cariboo College became the University College of the Cariboo (UCC) and in 1989 offered degrees with majors in mathematics in association with the University of British Columbia (UBC). Jim then began teaching upper level courses ranging from Linear Programming to Complex Analysis to Graph Theory and Geometry. He also chaired the UCC Dept. of Mathematics and Statistics (1994-1998). UCC later became Thompson Rivers University (TRU), offering an independent degree in 2005.

Rick Brewster shares an unusual perspective on Jim's contributions in Kamloops, ranging from his time as a local high school student, to an undergraduate at UCC, and finally as a colleague of Jim's. Rick writes:

"It is clear that Jim was well on track to a career as a research mathematician with 14 papers from 1974 to 1980, many solo (e.g., Basic properties of restricted linear spaces, *Discrete Math.* **13** (1975), No. 1, 67-74) and others collaborative (e.g., On a class of linear spaces with two consecutive line degrees, *Ars Combin.* **10** (1980), 107-114, co-authored with Lynn Batten). He made a decision to suspend his research programme when he moved to Cariboo in 1979. He was obviously happy (and at peace) with his decision as he found other outlets for sharing his love of mathematics."

[At `http://www.ams.org/mathscinet/search/author.html?mrauthid=298276` are links to a list of Jim's research publications and related information.]

"My overwhelming memory of Jim is of his enthusiasm. I certainly noticed it as the high school kid visiting Cariboo College, but I thought all teachers were keen. As his student in 2nd year though, it was clear Jim was cut from a different cloth. He had very high expectations of us. He pushed us hard on assignments, but his enthusiasm balanced out this work. In other words, he acted as though we would be thrilled to work on tough problems (math and computing science), because problem solving is so much fun. That sort of keenness never disappeared."

"I particularly remember second year Discrete Math (Math 222), the first time it was taught at Cariboo College. Jim was so excited to teach in his area. Late in the course he gave us a talk about finite geometry and the hunt for the projective plane of order 10. I can say I really didn't understand much of what he said that day, but I was struck by his excitement in being able to take us to the 'frontier of mathematics' (his words). This seemed very important to him: namely, that we were part of the mathematical community, and we had the opportunity to contribute. I was in graduate school when the nonexistence of the projective plane of order 10 was established. Upon hearing that result, I remembered Jim's lecture 7 years earlier. Two thoughts: 'Ah! That's what Jim was talking about,' and 'Wow, he had a lot of faith in a group of second year students to share that with us.' "

"While I was teaching Number Theory in my first semester at UCC, Jim showed up in my office one day with a collection of notes (about 30 years old) from a course he taught in graduate school. He was so keen to show me a construction of a certain ring of functions based on number theoretic ideas that he had presented in the (grad) course. It was nice mathematics, but a bit advanced for third years without abstract algebra. Still Jim's enthusiasm was so typical. An opportunity to share should not be wasted."

The Adrien Pouliot Award is a CMS award honouring "significant and sustained contributions to mathematics education in Canada." Members of the TRU department nominated Jim in 2007. In a supporting letter, John Ciriani echoed Rick's sentiment: "He was particularly pleased to develop and teach a course in geometry. Even students who found the course difficult recognized Jim's love of the subject and were able to relate to his enthusiasm for it." John added, "I must mention that Jim derives great pleasure in making presentations in science fairs and schools. He is particularly pleased when he encounters talented students who share his enthusiasm for mathematics. I believe he plans to continue this activity after he retires." In fact, Parkcrest Elementary School in Kamloops made Jim an honourary teaching staff member in 2007 to recognize his long term volunteer service.

Don DesBrisay expresses his respect: "With the math contests, Problem of the Week, mathemagic shows, Putnam, ***Crux***, his boxes of puzzles (many he crafted himself), his journal/book collections, and his vast knowledge, intellect, and wit (excluding some awful puns), Jim kept us all aware of the joy and excitement found in mathematics and in life. All this as well as family, outdoor club activities, golf and hockey!!" Quoting Kirk Evenrude: "Jim went at golf with the same determination and dedication as he did everything.

When I first met him in 1982 his handicap was about 13. In 2007 it was 6. One day he shot 69 at the Kamloops club. ... He had boundless energy when it came to golf. ... It was easy to spot Jim on the course; all you had to do was look for a big white Tilley hat. He loved playing in tournaments and was very competitive, but I think he liked the social part as much as the golf."

Indeed Jim's outreach had many branches. His participation at a 1999 CMS Education Session (organized by Bruce Shawyer and Ed Williams) in St. John's raised the profile of the BC contest. While there Jim visited my class at Memorial University of Newfoundland sharing his wooden puzzles and love of recreational mathematics with an enamored class of future high school math teachers. An entirely different form of mathematical outreach began in 1969 when Jim initiated the weekly pickup hockey games through the Faculty of Mathematics at the University of Waterloo.

Grace and gratitude are the words that come to mind as I reflect upon my collaborations with Jim Totten. Jim combined intellect and heart in a manner that placed the greater good ahead of his own. Jim was one of those exemplary people in the academic community who consciously expressed appreciation for the work of others – a gift in itself. This form of outreach is less visible but equally important. It is not unlike the teacher who, though challenging, manages to create a safe space for making errors and genuinely fumbling with mathematical ideas. Jim was a great teacher. He was honoured with separate awards for teaching and merit, and shortly after his retirement in 2007 with a Professor Emeritus designation. Those present at the March 2008 celebration of Jim's life at the Grand Hall in TRU witnessed the love and outpouring of appreciation for Jim and his family. Students, hockey players, golfers, colleagues, hikers, and family spoke to the breadth of Jim's passions and activities, a taste of which has been offered here.

These closing words are from Fae DeBeck and then, Dennis Acreman:

"Jim's top priority has always been his students, but his enthusiasm for mathematics and his efforts to promote mathematical ideas extend far beyond his own classroom. He has been an inspiration and a model for all in our department to follow. In the past two years I have acted as the local coordinator for the Math Contest and have begun to appreciate at a deeper level Jim's phenomenal contribution in this particular regard. His influence throughout the college/university system in the province cannot be overstated."

"Jim was a wonderful colleague and a good friend. He loved everything about Mathematics and assumed everyone else would too if it was just shared with them. His outreach activities showed true dedication to that goal from ***Crux*** to years of school visits and Problems of the Week. He was a generous person who loved his family and all aspects of his life and we all mourn his loss but are inspired by his example."

John Grant McLoughlin

SKOLIAD No. 118

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by ***1 December, 2009***. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

This special issue of ***CRUX with MAYHEM*** features a selection from a contest that Jim Totten played a pivotal role in developing. The Cariboo College High School Mathematics Contest began in 1973 and has since developed into the British Columbia Secondary School Math Contest. In 1992 a compilation of the problems was published in a book edited by Jim Totten. In the preface he wrote: "One person above all must receive our thanks for all his time spent proof-reading and offering suggestions and encouragement; John Ciriani continues to guide and inspire all of us in this department with his dedication to mathematics and teaching."

We therefore asked John Ciriani to select the problems for this Skoliad. John Ciriani replied: "It was difficult to select a contest which was memorable in terms of Jim's early contributions. In the end I chose the Junior Final 1990. This final contains a problem about golfers in a hurry to get to the course and reflects Jim's love of the game of golf."

Our thanks go to John Grant McLoughlin, University of New Brunswick, for suggesting John Ciriani and for contacting him for us; to John Ciriani, Kamloops, British Columbia, for his choice of contest and the reasons behind the choice; and to Rolland Gaudet, University College of St. Boniface, Winnipeg, MB, for the translation.

Cariboo College High School Mathematics Contest 1990
Junior Final, Part B

1. A boy on a bicycle coasts down from the top of a hill. He covers 4 metres in the first second and in each succeeding second covers 5 metres more than in the previous second. He reaches the bottom of the hill in 11 seconds.

(a) How long is the hill?

(b) What is the boy's average speed in metres per second?

(c) What distance did he cover in the last second?

2. Two golfers, on their way to the course, reached a railway crossing just as a 2.5 km train arrived. Rather than waiting, they decided to go on to the next crossing 1 km along in the direction the train was going. They travelled at 50 km/h while the train travelled at 70 km/h.

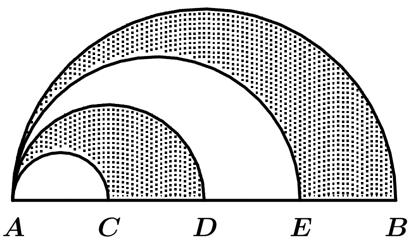
(a) How long did they have to wait for the train to clear the crossing?

(b) Rather than travelling at $50\,\mathrm{km/h}$, how fast would they have had to travel to reach the crossing just as the train was clearing the crossing?

3. A student asks you to choose a number from 1 to 9 and multiply it by 109, then asks you to find the sum of the digits in the product. Knowing the sum of the digits, the student is able to tell you the number with which you began. Explain how this can be done.

4. Suppose you throw 5 darts at a round board with a radius of $25\sqrt{2}\,\mathrm{cm}$. If all 5 darts stick in the board, show that at least two of them must be within $50\,\mathrm{cm}$ of each other.

5. The diameter, AB, of a circle is divided into 4 equal parts by the points C, D, and E. Semicircles are drawn on AC, AD, AE, and AB as shown. Find the ratio of the area of the shaded parts to the area of the unshaded parts.



Concours mathématique du Collège Cariboo 1990
Niveau secondaire, finale junior, partie B

1. Un garçon roule sans pédaler, à partir du haut d'une colline. Il couvre 4 mètres pendant la première seconde ; chaque seconde subséquente, il couvre 5 mètres de plus que pendant la seconde précédente. Il arrive au bas de la colline après 11 secondes.

(a) Quelle est la distance totale parcourue ?

(b) Quelle est la vitesse moyenne du garçon, en mètres à la seconde ?

(c) Quelle est la distance parcourue pendant la dernière seconde ?

2. Deux golfeurs arrivent à un passage à niveau juste au moment où un train de longueur $2,5$ kilomètres s'y présente. Au lieu d'attendre, ils décident de se rendre au prochain passage à niveau, à une distance de 1 kilomètre du premier passage à niveau, dans la direction où le train se dirige. Ils se déplacent à 50 kilomètres à l'heure tandis que le train va à 70 kilomètres à l'heure.

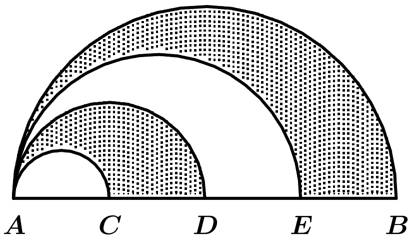
(a) Combien de temps ont-ils à attendre au deuxième passage à niveau, avant que le train ait dégagé le passage ?

(b) Au lieu de se déplacer à 50 kilomètres à l'heure, à quelle vitesse les deux golfeurs auraient-ils besoin de se déplacer afin d'arriver au deuxième passage à niveau justement au moment où le train dégage le passage ?

3. Un étudiant vous demande de choisir un entier de 1 à 9, puis de le multiplier par 109 ; il vous demande alors de lui fournir la somme des chiffres dans ce produit. Connaissant cette somme, l'étudiant peut alors vous dire avec quel entier vous avez commencé. Expliquer comment il le fait.

4. Vous lancez 5 fléchettes vers une cible de rayon $25\sqrt{2}$ centimètres. Les 5 fléchettes frappent la cible. Montrer qu'au moins deux d'entre elles doivent se retrouver à une distance d'au plus 50 centimètres l'une de l'autre.

5. Des demi cercles sont tracés avec AC, AD, AE et AB comme diamètres, où le segment AB est divisé en 4 parties égales par les points C, D et E. Pour le schéma à droite, déterminer le ratio des parties colorées aux parties non colorées.



Next follow the solutions to the British Columbia Secondary School Mathematics Contest, 2008, Junior Final, Part A [2008 : 321–324]. Note that problems 7 and 10 below have been adjusted so that the length of a segment is given as $|XY|$ instead of $\overline{XY}$.

1. Jeeves the valet was promised a salary of $\$8000$ and a car for a year of service. Jeeves left the job after 7 months of service and received the car and $\$1600$ as his correctly prorated salary. The dollar value of the car was:

(A) 6400 (B) 7200 (C) 7360 (D) 8000 (E) 15360

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let x be the cost of the car. The total value of everything that Jeeves receives after seven months equals $\frac{7}{12}$ of what he would have received had he worked for a whole year. Thus $x + 1600 = \frac{7}{12}(x + 8000)$. It follows that $12x + 19200 = 7x + 56000$, so $5x = 36800$. Hence $x = 7360$ and the answer is (C).

Also solved by JOCHEM VAN GAALEN, student, Medway High School, Arva, ON.

2. Recall that $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. The maximum value of the integer x such that 3^x divides $30!$ is:

(A) 30 (B) 14 (C) 13 (D) 10 (E) 4

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

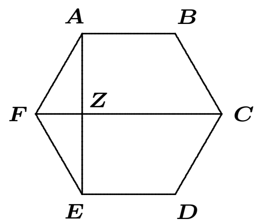
The following table lists the multiples of 3 that are less than or equal to 30 along with the number of times each is divisible by 3.

Multiple	3	6	9	12	15	18	21	24	27	30
Times	1	1	2	1	1	2	1	1	3	1

Thus $30!$ is divisible by three $1+1+2+1+1+2+1+1+3+1$ or fourteen times, and the answer is (B).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

3. In the diagram, $ABCDEF$ is a regular hexagon. Line segments AE and FC meet at Z. The ratio of the area of triangle FZE to the area of the quadrilateral $ABCZ$ is:



(A) $1 : 5$ (B) $1 : 4$ (C) $4 : 1$
(D) $5 : 1$ (E) $1 : 6$

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

Assume without loss of generality that the side length of the hexagon is 2.

Angles inside a regular hexagon are $120°$, so $\angle FEZ = 120°-90° = 30°$ and $\angle EFZ = \frac{1}{2}(120°) = 60°$. Thus $\triangle FZE$ is a $30°$–$60°$–$90°$ triangle with sides 1, 2, and $\sqrt{3}$, namely $|FZ| = 1$, $|EF| = 2$, and $|EZ| = \sqrt{3}$. Thus, $\triangle FZE$ has area $\frac{\sqrt{3}}{2}$.

Now draw a line segment from B to D intersecting CF at the point G. Then $\triangle CGB \cong \triangle FZE$, so the area of $\triangle CGB$ is also $\frac{\sqrt{3}}{2}$. Since $ABGZ$ is a rectangle with sides 2 and $\sqrt{3}$, it has area $2\sqrt{3}$. Hence the area of trapezoid $ABCZ$ is $\frac{\sqrt{3}}{2} + 2\sqrt{3} = \frac{5\sqrt{3}}{2}$.

Therefore, the desired ratio is $\frac{\sqrt{3}}{2} : \frac{5\sqrt{3}}{2} = 1 : 5$, and the answer is (A).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

4. Define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. For example, $\lfloor 7 \rfloor = 7$, $\lfloor 7.2 \rfloor = 7$, and $\lfloor -5.5 \rfloor = -6$. If z is a real number that is not an integer, then the value of $\lfloor z \rfloor + \lfloor 1 - z \rfloor$ is:

(A) -1 (B) 0 (C) 1 (D) 2 (E) z

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let $x = \lfloor z \rfloor$ and $\alpha = z-x$. Then x is an integer and $0 < \alpha < 1$ (since z is not an integer, $\alpha \neq 0$). Evidently, $\lfloor 1-z \rfloor = \lfloor 1-x-\alpha \rfloor = \lfloor -x+(1-\alpha) \rfloor$. Since $0 < 1 - \alpha < 1$, you have that $\lfloor -x + (1 - \alpha) \rfloor = -x$. Therefore, $\lfloor z \rfloor + \lfloor 1 - z \rfloor = x + (-x) = 0$ and the answer is (B).

As our solver points out, the question implies that the value of $\lfloor z \rfloor + \lfloor 1-z \rfloor$ is independent of z (as long as z is not an integer). Thus you can find the answer by simply substituting some value, say 1.5, for z. Of course this does not prove *anything but relies on trust in the formulation of the question.*

5. Examinations in each of three subjects, Anatomy, Biology, and Chemistry, were taken by a group of 41 students. The following table shows how many students failed in each subject, as well as in the various combinations:

subject	A	B	C	AB	AC	BC	ABC
# failed	12	5	8	2	6	3	1

(For instance, 5 students failed in Biology, among whom there were 3 who failed both Biology and Chemistry, and just 1 of the 3 who failed all three subjects.) The number of students who passed all three subjects is:

(A) 4 (B) 16 (C) 21 (D) 24 (E) 26

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The two students who fail Anatomy and Biology include the single student who fails all three subjects. Therefore just one student fails Anatomy and Biology but not Chemistry. Likewise, five students fail Anatomy and Chemistry but not Biology, and two students fail Biology and Chemistry but not Anatomy.

The 12 students who fail Anatomy include

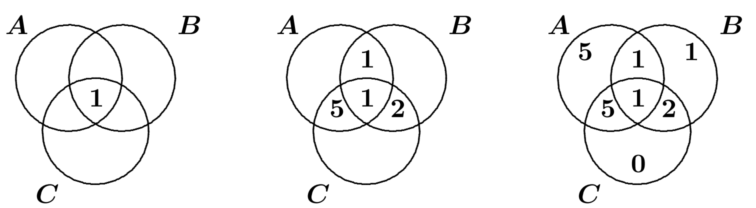
- the single student who fails Anatomy and Biology but not Chemistry; and
- the five students who fail Anatomy and Chemistry but not Biology; and
- the single student who fails all three subjects.

Therefore the number of students who fail Anatomy but pass the other two subjects is $12-1-5-1 = 5$. Likewise, one student fails Biology and passes the other two subjects, and zero students fail Chemistry but pass the other two subjects.

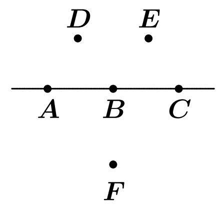
Therefore the total number of students who fail at least one subject is $5+1+0+1+5+2+1 = 15$, whence the number of students who pass all three subjects is $41 - 15 = 26$, and the answer is (E).

Also solved by JOCHEM VAN GAALEN, student, Medway High School, Arva, ON.

This type of question is most easily solved by means of a Venn Diagram. The diagrams below show how you may sequentially enter information into the Venn Diagram.



6. Six points, A, B, C, D, E, and F are arranged in the formation shown in the diagram, with A, B, and C on a straight line. Three of these six points are selected to form a triangle. The number of such triangles that can be formed is:



(A) 12 (B) 14 (C) 16

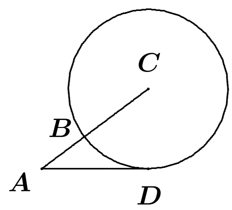
(D) 19 (E) 20

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

There are $\binom{6}{3} = {}_6C_3 = \dfrac{6!}{3!(6-3)!} = 20$ ways to choose three points from the given six. One of these, $\{A, B, C\}$, yields no triangle as A, B, C are collinear. Thus, you can make 19 triangles, and the answer is (D).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

7. In the diagram, C is the centre of the circle and AD is tangent to the circle at D. AC is a straight line. If $|AD| = 10$ and $|AB| = 7$, the length of BC is:



(A) $\dfrac{\sqrt{151}-7}{2}$ (B) $\sqrt{14}$ (C) $\dfrac{51}{14}$

(D) $\dfrac{\sqrt{51}}{2}$ (E) $\dfrac{7}{2}$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since AD is tangent to the circle, $\angle CDA = 90^\circ$. Let r be the radius of the circle. Then $|CD| = |BC| = r$ and $|AC| = |AB| + |BC| = 7 + r$. By the Pythagorean Theorem, $|AD|^2 + |CD|^2 = |AC|^2$, so $10^2 + r^2 = (7+r)^2$. Thus $100 + r^2 = 49 + 14r + r^2$, whence $51 = 14r$, and $r = \dfrac{51}{14}$. Then the answer is (C).

Also solved by JOCHEM VAN GAALEN, student, Medway High School, Arva, ON.

8. When 2008^{2008} is multiplied out, the units digit in the final product is:

(A) 8 (B) 6 (C) 4 (D) 2 (E) 0

Solution by Jochem van Gaalen, student, Medway High School, Arva, ON.

Since the units digit of a product depends only on the units digits of the factors, we only need to consider the units digit of powers of 8. The table shows the first of these.

Power	8^1	8^2	8^3	8^4	8^5	8^6	8^7
Units digit	8	4	2	6	8	4	2

Note that the pattern 8, 4, 2, 6 repeats itself after 8^4 as it must, since only the units digits matter.

Now $2008 \bmod 4 = 0$, that is, the remainder when dividing 2008 by 4 is 0. Thus the units digit of 2008^{2008} is 6 and the answer is (B).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

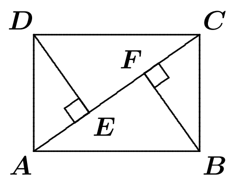
9. Recall that a prime number is an integer greater than one that is divisible only by one and itself. Consider the set of two-digit numbers less than 40 that are either prime or divisible by only one prime number. From this set select those for which the sum of the digits is a prime number, and the positive difference between the digits is another prime number. The sum of the values of the numbers selected is:

(A) 29 (B) 41 (C) 54 (D) 70 (E) 93

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The two-digit primes or powers of primes less than 40 are 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, and 37. Requiring the digit sum to be a prime narrows the list to 11, 16, 23, 25, 29, and 32. Requiring that the difference of the digits be a prime further narrows the list to 16, 25, and 29. The sum of these three numbers is 70, and the answer is (D).

10. In the diagram $ABCD$ is a rectangle with $|AD| = 1$, and both DE and BF perpendicular to the diagonal AC. Further, $|AE| = |EF| = |FC|$. The length of the side AB is:



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2

(D) $\sqrt{5}$ (E) 3

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let x be the common length of AE, EF, and FC. Then $|EC| = 2x$. By the Pythagorean Theorem, $|AE|^2 + |DE|^2 = |AD|^2$, so $x^2 + |DE|^2 = 1^2$, whence $|DE| = \sqrt{1-x^2}$.

Note that $\angle ECD = 90^\circ - \angle EDC = \angle ADE$, so $\triangle DEA \sim \triangle CED$. Therefore,

$$\frac{|EA|}{|DE|} = \frac{|ED|}{|CE|} \iff \frac{x}{\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}}{2x}\,.$$

Thus $2x^2 = 1 - x^2$, so $x^2 = \frac{1}{3}$. Finally, $|AB| = |CD|$ and CD has length $\sqrt{|DE|^2 + |CE|^2} = \sqrt{1 - x^2 + 4x^2} = \sqrt{1+3x^2} = \sqrt{1+3/3} = \sqrt{2}$, and the answer is (A).

That completes another *Skoliad*. This issue's prize for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON. We look forward to receiving more solutions from more readers.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Please send your solutions to the problems in this edition by **15 November 2009**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

Each of the following Mayhem problems is specially dedicated to the memory of Jim Totten, hence the special numbering used below. The numbering of the regular Mayhem problems will resume in subsequent issues.

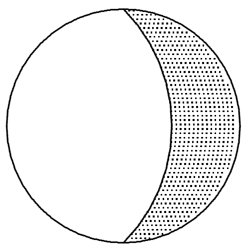
The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

Totten–M1. *Proposed by Shawn Godin, Cairine Wilson Secondary School, Orleans, ON.*

Ancient Egyptians wrote all fractions in terms of distinct unit fractions (that is, in terms of distinct fractions with numerators of 1). For example, instead of writing $\frac{11}{12}$, they would write $\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$. The unit fraction $\frac{1}{2}$ can be written in terms of other unit fractions as $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$. Find an infinite family of unit fractions each of which can be written as the sum of two unit fractions.

Totten–M2. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The boundary of the shadow on the moon is always a circular arc. On a certain day, the moon is seen with the shadow passing through diametrically opposite points. If the centre of the circular arc forming the shadow is on the circumference of the moon, determine the exact proportion of the moon that is not in shadow.



Totten–M3. *Proposed by John Ciriani, Kamloops, BC.*

Prove that the quadratic equation $ax^2 + bx + c = 0$ does not have a rational root if a, b, and c are odd integers.

Totten–M4. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

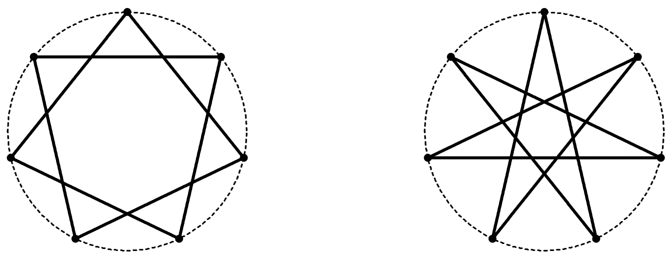
In a survey, some students were asked whether they liked the colour orange. Exactly 2% of the boys in the survey liked orange, while exactly 59% of the girls in the survey liked orange. Altogether, exactly 17% of the students in the survey liked orange. Find the smallest possible number of students in the survey.

Totten–M5. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $a \neq 1$ be a positive real number. Determine all pairs of positive integers (x, y) such that $\log_a x - \log_a y = \log_a(x - y)$.

Totten–M6. *Proposed by Suzanne Feldberg, Thompson Rivers University, Kamloops, BC.*

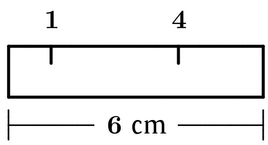
It is widely known how to draw a 5-pointed star quickly. To make it symmetric, one places 5 vertices at $72°$ intervals about a circle and connects the vertices with line segments of equal length without lifting one's pen. By starting from a fixed point and using the same method, one can draw two different (and symmetric) 7-pointed stars without lifting one's pen.



How many different 6-pointed, 8-pointed, or 9-pointed stars can one draw this way? How many different n-pointed stars can one draw this way?

Totten–M7. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm using only 2 marks, as shown in the diagram, at 1 cm and 4 cm, since $2 = 6 - 4$, $3 = 4 - 1$, and $5 = 6 - 1$.



Suppose that an unmarked ruler is known to be exactly 30 cm in length.

(a) Find a way of placing 9 or fewer marks on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm.

(b) Prove that at least 7 marks are needed to be able to exactly measure all integer lengths from 1 cm to 30 cm.

(c)★ Determine the smallest number of marks required on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm.

Totten–M8. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Let T be the set of all ordered triples (a, b, c) of positive integers such that $a < b < c$. We say that two triples (a, b, c) and (u, v, w) are equivalent if $a : b : c = u : v : w$. We use this relation to partition T into equivalence classes. The triple (a, b, c) is *geometric* if $ac = b^2$ (that is, its terms form a geometric sequence) and *harmonic* if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (that is, the reciprocals of its terms form an arithmetic sequence).

(a) Verify that if (a, b, c) is geometric, then all triples equivalent to it are also geometric.

(b) Verify that if (a, b, c) is harmonic, then all triples equivalent to it are also harmonic.

(c) Let G be the set of equivalence classes of geometric triples and H be the set of equivalence classes of harmonic triples. Determine a one-to-one correspondence between G and H.

Totten–M9. *Proposed by Kirk Evenrude, Kamloops, BC.*

A train 900 m long, travelling at 90 km/h, approaches a 100 m long bridge.

(a) How many seconds does it take the train to clear the bridge?

(b) Suppose that, just as the train reaches the bridge, it begins to slow down at the rate of 0.2 m/s^2. Now how long does it take to clear the bridge?

Totten–M10. *Proposed by Nicholas Buck, College of New Caledonia, Prince George, BC.*

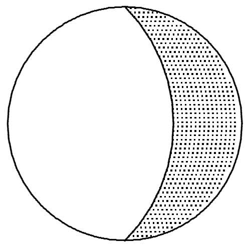
Show that if p is a prime number, and A and B are positive integers such that p divides A, p^2 does not divide A, and p does not divide B, then the Diophantine equation $Ax^2 + By^2 = p^{2008}$ does not have any solutions in positive integers x and y.

Totten–M1. *Proposé par Shawn Godin, École secondaire Cairine Wilson, Orléans, ON.*

Les anciens égyptiens écrivaient toutes leurs fractions en termes de fractions unitaires distinctes, c'est-à-dire de fractions distinctes avec 1 comme numérateur. Par exemple, au lieu d'écrire $\frac{11}{12}$, ils auraient écrit $\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$. La fraction unitaire $\frac{1}{2}$ peut être écrite en termes d'autres fractions unitaires comme $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$. Trouver une famille infinie de fractions unitaires, chacune pouvant être écrite comme somme de deux fractions unitaires.

Totten–M2. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

La frontière de l'ombre sur la lune est toujours un arc de cercle. Un certain jour, on peut constater que l'ombre passe par des points diamétralement opposés. Si le centre de l'arc circulaire formant cette ombre se trouve sur la circonférence de la lune, déterminer la portion exacte de la lune qui n'est pas dans l'ombre.



Totten–M3. *Proposé par John Ciriani, Kamloops, BC.*

Montrer que l'équation quadratique $ax^2+bx+c=0$ n'a pas de racine rationnelle si a, b et c sont des entiers impairs.

Totten–M4. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

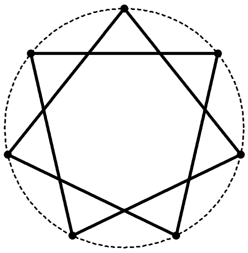
Lors d'un sondage, on a demandé à certains étudiants s'ils aimaient la couleur orange. Chez les garçons exactement 2% ont répondu par oui, tandis que chez les filles, la proportion exacte de oui a été de 59%. Comme au total, exactement 17% des répondants ont dit aimer l'orange, on demande de trouver quel est le plus petit nombre possible de participants au sondage.

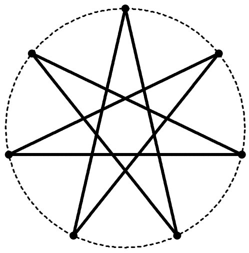
Totten–M5. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $a \neq 1$ un nombre réel positif. Trouver toutes les paires d'entiers positifs (x, y) telles que $\log_a x - \log_a y = \log_a(x-y)$.

Totten–M6. *Proposé par Suzanne Feldberg, Université Thompson Rivers, Kamloops, BC.*

On sait bien comment dessiner rapidement une étoile à 5 sommets. Pour en faire une symétrique, on place les 5 sommets à intervalles de 72° sur un cercle et on relie les sommets par des segments rectilignes de longueur égale sans lever son crayon. En partant d'un point fixé et en utilisant la même méthode, on peut dessiner sans lever son crayon deux étoiles différentes (et symétriques) à 7 sommets.

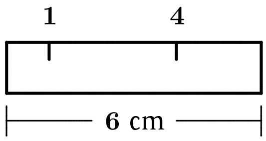




Combien d'étoiles différentes à 6 sommets, 8 sommets, ou 9 sommets peut-on ainsi dessiner? Combien d'étoiles différentes à n sommets peut-on ainsi dessiner?

Totten–M7. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

On utilise une règle sans graduation dont on sait qu'elle mesure exactement 6 cm. Il est possible de mesurer exactement toutes les longueurs entières de 1 cm à 6 cm avec seulement 2 marques, comme indiqué dans la figure ci-contre, à 1 cm et 4 cm, car $2 = 6 - 4$, $3 = 4 - 1$ et $5 = 6 - 1$.



Supposons maintenant qu'on utilise une règle d'exactement 30 cm de long.

(a) Trouver un moyen de placer 9 marques sur la règle afin d'être capable de mesurer exactement toutes les longueurs entières, de 1 cm à 30 cm.

(b) Montrer qu'au moins 7 marques sont nécessaires pour être capable de mesurer exactement toutes les longueurs entières, de 1 cm à 30 cm.

(c)★ Déterminer le plus petit nombre de marques à placer sur la règle afin d'être capable de mesurer exactement toutes les longueurs entières, de 1 cm à 30 cm.

Totten–M8. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Soit T l'ensemble de toutes les triplets ordonnés d'entiers positifs (a, b, c) tels que $a < b < c$. On dit que deux triplets (a, b, c) et (u, v, w) sont équivalents si $a : b : c = u : v : w$. On utilise cette équivalence pour partitionner T en classes d'équivalence. Le triplet (a, b, c) est dit *géométrique* si $ac = b^2$ (c.-à-d. ses éléments forment une suite géométrique) et *harmonique* si $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (c.-à-d. les inverses de ses éléments forment une suite arithmétique).

(a) Vérifier que si (a, b, c) est géométrique, alors tous les triplets qui lui sont équivalents sont aussi géométriques.

(b) Vérifier que si (a, b, c) est harmonique, alors tous les triplets qui lui sont équivalents sont aussi harmoniques.

(c) Soit G l'ensemble des classes d'équivalence de triplets géométriques et H l'ensemble des classes d'équivalence de triplets harmoniques. Trouver une correspondance biunivoque entre G et H.

Totten–M9. *Proposé par Kirk Evenrude, Kamloops, BC.*

Un train de 900 m de long s'approche d'un pont d'une longueur de 100 m à la vitesse de 90 km/h.

(a) En combien de secondes le train traversera-t-il le pont?

(b) Supposons qu'au moment d'atteindre le pont, le train décélère de $0.2\,\mathrm{m/s^2}$. Combien de temps mettra-t-il cette fois pour traverser le pont?

Totten–M10. *Proposé par Nicholas Buck, Collège de New Caledonia, Prince George, BC.*

Montrer que si p est un nombre premier, et si A et B sont des entiers positifs tels que p divise A, p^2 ne divise pas A, et p ne divise pas B, alors l'équation diophantienne $Ax^2 + By^2 = p^{2008}$ n'a aucune solution en entiers positifs x et y.

Mayhem Solutions

M363. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that A is a six-digit positive integer and B is the positive integer formed by writing the digits of A in reverse order. Prove that $A - B$ is a multiple of 9.

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

Since A is a six-digit positive integer, it can be expressed in the form $10^5a + 10^4b + 10^3c + 10^2d + 10^1e + f$, where a is a positive integer and b, c, d, e, and f are nonnegative integers. Since B is formed by writing the digits of A in reverse order, B is of the form $10^5f + 10^4e + 10^3d + 10^2c + 10^1b + a$.

Their difference, $A-B$, can then be simplified as follows:

$$\begin{aligned} A-B &= \left(10^5a+10^4b+10^3c+10^2d+10^1e+f\right) \\ &\quad -\left(10^5f+10^4e+10^3d+10^2c+10^1b+a\right) \\ &= 10^5a-a+10^4b-10^1b+10^3c-10^2c \\ &\quad +10^2d-10^3d+10^1e-10^4e+f-10^5f \\ &= 99999a+9990b+900c-900d-9990e-99999f \\ &= 9(11111a+1110b+100c-100d-1110e-11111f)\,. \end{aligned}$$

Since the digits of A are integers, sums and differences of multiples of these digits are integers too. Thus, the difference of A and B is a multiple of nine.

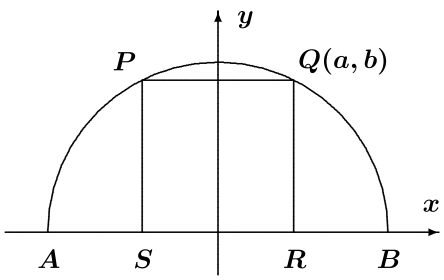
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M364. *Proposed by the Mayhem Staff.*

A semicircle of radius 2 is drawn with diameter AB. The square $PQRS$ is drawn with P and Q on the semicircle and R and S on AB. Is the area of the square less than or greater than one-half of the area of the semicircle?

Solution by Peter Chien, student, Central Elgin Collegiate, St. Thomas, ON, modified by the editor.

Place AB on the x-axis with the midpoint of AB (that is, the centre of the semicircle) at the origin. The full circle has radius 2 and centre $(0,0)$, and so has equation $x^2+y^2=4$.



Let Q have coordinates (a,b), with a and b positive. Since Q is on the semicircle, then $a^2+b^2=4$. Since $PQRS$ is a square, which must sit symmetrically inside the semicircle, then we also have $b=2a$.

Thus, $a^2+(2a)^2=4$, hence $a^2=\frac{4}{5}$ and so $a=\dfrac{2}{\sqrt{5}}=\dfrac{2\sqrt{5}}{5}$. Since the length of one side of $PQRS$ is $b=2a=\dfrac{4\sqrt{5}}{5}$, then the area of the square is $\left(\frac{4\sqrt{5}}{5}\right)^2 = \frac{80}{25} = \frac{16}{5} = 3.2$.

One-half the area of the semicircle is $\frac{1}{2}\times\frac{1}{2}\times\pi\times 2^2 = \pi < 3.2$. The area of the square is therefore greater than one-half the area of the semicircle.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M365. *Proposed by Alexander Gurevich, student, University of Waterloo, Waterloo, ON.*

Let D be the family of lines of the form $y = nx + n^2$, with $n \geq 2$ a positive integer. Let H be the family of lines of the form $y = m$, where $m \geq 2$ is a positive integer. Prove that a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a nonnegative integer.

Solution by Alex Song, student, Elizabeth Ziegler Public School, Waterloo, ON, modified by the editor.

Let $m \geq 2$ be a positive integer. Consider a line $y = m$ from H. Its y-intercept is m. It suffices to prove two things:

(a) if m is prime, then for any positive integer n with $n \geq 2$, the simultaneous equations $y = m$ and $y = nx + n^2$ do not have a solution for x that is a nonnegative integer, and

(b) if m is composite, then there is a positive integer $n \geq 2$ such that the simultaneous equations $y = m$ and $y = nx + n^2$ do have a solution for x that is a nonnegative integer.

Putting this another way, we must prove that if m is prime, then the equation $m = nx + n^2$ does not have a nonnegative integer solution for x for any $n \geq 2$, and if m is composite, then the equation $m = nx + n^2$ does have a nonnegative integer solution for x for some $n \geq 2$.

Assume that m is prime and that n is a positive integer with $n \geq 2$. Suppose also that $m = nx+n^2 = n(x+n)$ has an integer solution for x with $x \geq 0$. Note that m has only two positive divisors, namely m and 1. Since $n \geq 2$, then n must equal m, and so $x+n = 1$, which gives $x = 1-n \leq -1$. Thus, x is not a positive integer or zero. This is a contradiction, so there is no n for which $m = nx + n^2$ has nonnegative integer solutions for x.

Assume next that m is composite. Then $m = pq$ for some integers p and q with $2 \le p \le q$. Thus, we want to find integers n and x with $n \ge 2$ and $x \ge 0$ such that $m = n(x+n) = pq$. Setting $n = p$ and $x + p = q$ satisfies the equation. Here, $x = q - p \ge 0$, so the restrictions on x and n are satisfied. Thus, for some n there is a nonnegative solution for x.

Therefore, a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a nonnegative integer.

Also solved by SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M366. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The roots of the equation $x^3 + bx^2 + cx + d = 0$ are p, q, and r. Find a quadratic equation with roots $(p^2 + q^2 + r^2)$ and $(p + q + r)$.

Solution by Shamil Asgarli, student, Burnaby South Secondary School, Burnaby, BC.

Since p, q, and r are the roots of the cubic equation, we can factor the left side of the equation to get $(x-p)(x-q)(x-r) = 0$. Expanding yields $x^3 - (p+q+r)x^2 + (pq+qr+rp)x - pqr = 0$. Comparing coefficients with the original equation, we obtain $p + q + r = -b$ while $pq + qr + rp = c$.

Since $(p+q+r)^2 = p^2 + q^2 + r^2 + 2(pq + qr + rp)$, then we have $(-b)^2 = p^2 + q^2 + r^2 + 2c$, so $p^2 + q^2 + r^2 = b^2 - 2c$.

A possible quadratic equation with the desired roots is therefore

$$\Big(x - (b^2 - 2c)\Big)\Big(x - (-b)\Big) \;=\; 0\,,$$

or $x^2 + \big(b - b^2 + 2c\big)x + \big(2bc - b^3\big) = 0$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ROBERT SHEETS, Southeast Missouri State University, Cape Girardeau, MO, USA; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M367. *Proposed by George Tsapakidis, Agrinio, Greece.*

For the positive real numbers a, b, and c we have $a + b + c = 6$. Determine the maximum possible value of $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

Solution by José Hernández Santiago, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico.

Applying the Arithmetic Mean–Geometric Mean Inequality to positive real numbers a, b, and c we obtain $abc \le \left(\frac{a+b+c}{3}\right)^3 = 8$ and consequently $\sqrt{abc} \le 2\sqrt{2}$. (Equality holds here if and only if $a = b = c$ and so if and only if $a = b = c = 2$.)

We can then further conclude that

$$\begin{aligned} a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} &= \sqrt{a}\sqrt{abc} + \sqrt{b}\sqrt{abc} + \sqrt{c}\sqrt{abc} \\ &= \sqrt{abc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \\ &\le 2\sqrt{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) . \end{aligned} \tag{1}$$

Now,

$$\begin{aligned} \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 &= a + b + c + 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right) \\ &= 6 + 2\left(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}\right) , \end{aligned}$$

since $a + b + c = 6$.

Applying the AM–GM Inequality once more yields $2\sqrt{xy} \le x + y$, thus we obtain $2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc}) \le a + b + a + c + b + c = 2(a + b + c)$. (Again, equality holds if and only if $a = b = c = 2$.)

Hence, $\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \le 6 + 2(a + b + c) = 18$ and consequently $\sqrt{a} + \sqrt{b} + \sqrt{c} \le \sqrt{18} = 3\sqrt{2}$.

From (1), it follows that $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \le \left(2\sqrt{2}\right)\left(3\sqrt{2}\right) = 12$, so the maximum possible value is 12, which we have seen is achieved when $a = b = c = 2$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SHAMIL ASGARLI, student, Burnaby South Secondary School, Burnaby, BC; GEORGIOS BASDEKIS, student, 1st High School of Karditsa, Karditsa, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M368. *Proposed by J. Walter Lynch, Athens, GA, USA.*

An infinite series of positive rational numbers $a_1 + a_2 + a_3 + \cdots$ is the fastest converging infinite series with a sum of 1, $a_1 = \frac{1}{2}$, and each a_i having numerator 1. (By "fastest converging", we mean that each term a_n is successively chosen to make the sum $a_1 + a_2 + \cdots + a_n$ as close to 1 as possible.) Determine a_5 and describe a recursive procedure for finding a_n.

Solution by Robert Sheets, Southeast Missouri State University, Cape Girardeau, MO, USA.

Let s_n be the n^{th} partial sum of the infinite series. Then $a_1 = \frac{1}{2}$ and $s_1 = \frac{1}{2}$. Since all terms of the infinite series are positive, none of the terms can be negative or zero, thus no partial sum can be greater than or equal to 1. We therefore need $a_2 < 1 - s_1 = \frac{1}{2}$. Seeking the largest such a_2, we find that $a_2 = \frac{1}{3}$ since a_2 is a fraction of positive integers with numerator equal to 1.

Then $s_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ which gives $a_3 < \frac{1}{6}$, whence $a_3 = \frac{1}{7}$. Similarly, we find that $s_3 = \frac{41}{42}$, giving $a_4 = \frac{1}{43}$, and $s_4 = \frac{1805}{1806}$, giving $a_5 = \frac{1}{1807}$.

Next, we describe a recursive procedure to find a_n. We conjecture that if $a_n = \frac{1}{k}$, then $s_n = \frac{k(k-1)-1}{k(k-1)}$ and $a_{n+1} = \frac{1}{k(k-1)+1}$. (Note that $2(1)+1=3$, $3(2)+1=7$, $7(6)+1=43$, and $43(42)+1=1807$, so these equations hold for $n = 1$, 2, 3, and 4.)

We prove these recursive relations by induction. We have already verified the base cases above. Suppose that for some positive integers n and k we have $a_n = \frac{1}{k}$, $s_n = \frac{k(k-1)-1}{k(k-1)}$, and $a_{n+1} = \frac{1}{k(k-1)+1}$. We prove that the relations will also hold for s_{n+1} and a_{n+2}. Let $t = k(k-1)+1$; this means that $a_{n+1} = \frac{1}{t}$ and $s_n = \frac{t-2}{t-1}$. Then

$$\begin{aligned} s_{n+1} &= s_n + a_{n+1} \\ &= \frac{t-2}{t-1} + \frac{1}{t} = \frac{t^2-2t+t-1}{t(t-1)} \\ &= \frac{t^2-t-1}{t(t-1)} = \frac{t(t-1)-1}{t(t-1)}\,. \end{aligned}$$

Finally, since $a_{n+2} < 1 - s_{n+1} = \frac{1}{t(t-1)}$ and a_{n+2} is the largest fraction with numerator 1 satisfying this property, we find that $a_{n+2} = \frac{1}{t(t-1)+1}$, as required.

Therefore, the result holds by induction and for all positive integers n, if $a_n = \frac{1}{k}$ for some positive integer k, then $a_{n+1} = \frac{1}{k(k-1)+1}$.

Also solved by ARKADY ALT, San Jose, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON. There was one incomplete solution submitted.

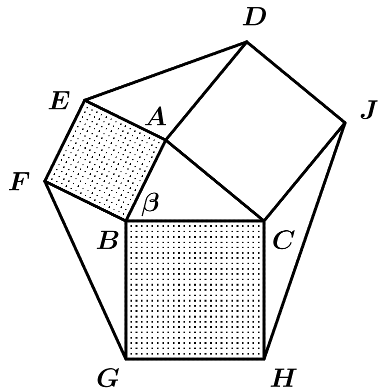
Hess noted that the sequence of denominators appearing in the series is A000058 *in the* On-Line Encyclopedia of Integer Sequences (`http://www.research.att.com/~njas/sequences/`) *and is known as Sylvester's sequence. It is a curious fact that two consecutive terms of Sylvester's sequence differ by a square, since the number k in the sequence is followed by k^2-k+1, yielding a difference of $(k^2-k+1)-k=(k-1)^2$.*

Problem of the Month

Ian VanderBurgh

Among Jim Totten's many interests was his involvement in mathematics outreach. In particular, Jim was a driving force behind the Cariboo College High School Mathematics Contest. He edited a volume of problems taken from those contests written between 1973 and 1992. This month, we look at one of the problems from this volume.

Problem (1989 Cariboo College High School Mathematics Contest, Senior Final Round, Part B) In the figure, $AEFB$, $BGHC$, and $ACJD$ are squares constructed on the sides of $\triangle ABC$. If the (combined) area of the two shaded squares equals the area of the rest of the figure, show that the area of $\triangle ABC$ equals the area of $\triangle FBG$ and then find the number of degrees in $\angle ABC$.



This problem actually appears to be the "poster child" for this contest, as it appears on the cover of the compilation book and appears as a kind of logo elsewhere on the web.

Before we look at the solution, there is one really useful formula upon which we should agree. If in $\triangle XYZ$ we have $XY = z$ and $YZ = x$, then the area of $\triangle XYZ$ is equal to $\frac{1}{2}xz\sin(\angle XYZ)$. This formula is a great alternative to the standard area formula "$\frac{1}{2}bh$" if all you have is one angle of a triangle and the lengths of the two sides enclosing it. We'll derive this formula after the solution to the problem.

Solution Let $BC = a$, $AC = b$, and $AB = c$. Since $AEFB$ is a square, then $BF = AE = c$. Since $BGHC$ is a square, then $BG = CH = a$. Since $ACJD$ is a square, then $AD = CJ = b$.

Since $\angle ABC = \beta$ and $\angle ABF = \angle CBG = 90°$, it then follows that $\angle FBG = 360° - \beta - 90° - 90° = 180° - \beta$.

We first need to prove that $\triangle ABC$ and $\triangle FBG$ have equal areas. From the formula in the preamble, the area of $\triangle ABC$ is $\frac{1}{2}(BC)(AB)\sin(\angle ABC)$ or $\frac{1}{2}ac\sin\beta$. Similarly, the area of $\triangle FBG$ is $\frac{1}{2}(BG)(FB)\sin(\angle FBG)$ or $\frac{1}{2}ac\sin(180° - \beta)$.

But $\sin\beta = \sin(180° - \beta)$ for any angle β, so the two areas are equal.

While it may not be immediately obvious why this helps, we can stall a bit by noting that we can use the same argument to conclude that the area of

$\triangle EAD$ and the area of $\triangle HCJ$ are each equal to the area of $\triangle ABC$. Can you see why?

At this point, we should probably use the piece of information that we were given, namely, that the combined area of square $AEFB$ and square $BGHC$ equals the area of the rest of the figure. We use the short-hand $|AEFB|$ to denote the area of figure $AEFB$. Thus, we are told that

$$|AEFB| \;+\; |BGHC| \;= \\ |FBG| \;+\; |ABC| \;+\; |EAD| \;+\; |HCJ| \;+\; |ACJD|\,.$$

Using some of what we know so far, this becomes

$$c^2 + a^2 = 4|ABC| + b^2\,,$$

or

$$c^2 + a^2 = 4\left(\frac{1}{2}ac\sin\beta\right) + b^2\,.$$

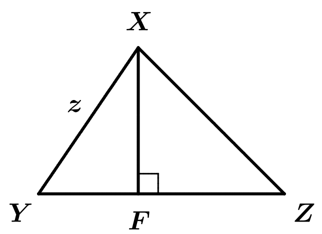
Remember, we're trying to find β. We seem to have too many other pieces of information floating around to have any hope of doing this. But at this point, the amazing pattern recognition abilities of the brain might kick in. This equation looks somewhat similar to a law that we often use. This might prompt us to try applying that law. Do you see what I'm getting at?

Applying the Law of Cosines in $\triangle ABC$ gives $b^2 = a^2 + c^2 - 2ac\cos\beta$. Substituting this into the last equation, we obtain

$$\begin{aligned} c^2 + a^2 &= 2ac\sin\beta + (a^2 + c^2 - 2ac\cos\beta)\,; \\ 2ac\cos\beta &= 2ac\sin\beta\,; \\ \cos\beta &= \sin\beta\,, \end{aligned}$$

since $ac > 0$. Since $\cos\beta = \sin\beta$, and β is an angle in a triangle, then $\beta = 45^\circ$, and we are done. ∎

To me, this was quite surprising. Well, the answer itself wasn't so surprising since these problems almost always have 30°, 45°, or 60° as an answer. But, it was surprising to me that the angle β was completely determined from the given information while no other information (side lengths or angles) can be determined.



Before wrapping up the column this month, we should go back and look at the formula from the preamble. Suppose that $\angle XYZ$ is acute. Drop a perpendicular from X to F on YZ.

Then the area of $\triangle XYZ$ is equal to $\frac{1}{2}(YZ)(XF)$. But $YZ = x$ and we also have $XF = XY\sin(\angle XYZ) = z\sin(\angle XYZ)$, so the area equals $\frac{1}{2}xz\sin(\angle XYZ)$, as required. Can you prove this in the case that angle Y is obtuse or a right angle?

The Tanker Problem

Ross Honsberger

When I walk my puppy with his retractable leash, he is forever getting behind, catching up and running past, cutting across in front, and getting behind again. When I was having my after-lunch nap the other day, this reminded me of those old "patrol boat circling an ocean liner" problems and I mused over the following situation.

A security patrol boat repeatedly circles a supertanker that is a gigantic rectangular box 450 metres long and 50 metres across. The ocean is calm and the tanker travels at a constant speed along a straight path. The patrol boat goes up the left side, across the front, down the right side, and across the back, and keeps doing it over and over.

The patrol boat travels in only two directions of the compass – when going parallel to the path of the tanker, it travels in straight lanes parallel to the tanker, one on each side at a distance of 25 metres from it, and when crossing in front or behind, it goes straight across perpendicular to the path of the tanker.

Neglecting the dimensions of the patrol boat (that is, considering it to be represented geometrically by a point) and given that it goes constantly at twice the speed of the tanker and that its turns are instantaneous, *what is the shortest distance that the patrol boat must travel in completing one cycle around the tanker*?

I felt the problem would be an easy recreation with a pleasing analysis and I expected a solution along the following lines.

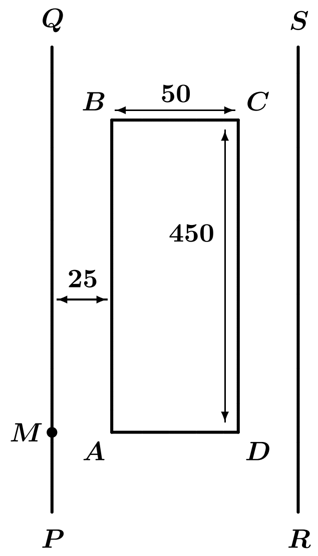


Figure 1

Let the tanker and the lanes of the patrol boat be labeled as in Figure 1 and let the boat travel towards Q when it is in lane PQ.

Consider a cycle that begins at M on PQ; that is, to begin the boat is abreast of the stern AD of the tanker. While the boat travels 900 m up PQ, the tanker will have advanced 450 m and the boat will be abreast of the front BC of the tanker at the point E on PQ (Figure 2).

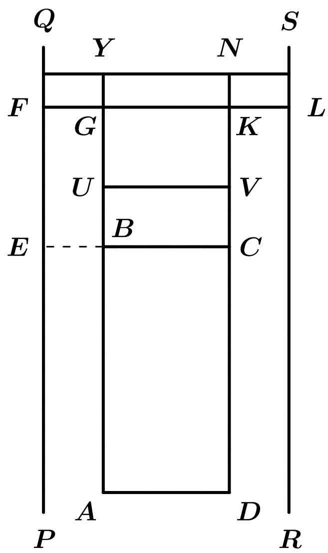


Figure 2

If the boat were to start crossing between its lanes at this point, it would crash into the tanker at a point 12.5 m down the side. In order to cross in front of the tanker, the boat must get $25 + 50 = 75$ m across the gap to the right hand edge of the tanker before the tanker advances to its level. Therefore, it needs to have a lead of 37.5 m when it starts its turn, for the tanker will advance 37.5 m while the boat is going across this 75 m.

In order to get a 37.5 m lead, the boat must continue up PQ another 75 m beyond E (while the boat is doing this the tanker advances 37.5 m to yield a net lead of 37.5 m).

Summarizing, the boat and the tanker are abreast at E and BC; as the boat advances the 75 m to F, the tanker advances 37.5 m to UV; the boat turns at F and while going the 75 m along FK, the tanker advances 37.5 m to GK. Thus, the boat and the tanker just miss each other at K. And while the boat completes the remaining 25 m along KL to its other lane, the tanker takes a 12.5 m lead at YN.

Thus, when the boat turns down SR it is already 12.5 m down that side of the tanker. Now, the boat need go only to a point H that is 12.5 m from the far end before making its turn across the back (while it crosses the 25 m gap toward the tanker, the tanker moves ahead this 12.5 m and the boat just misses the bottom right corner of the tanker). Hence the boat needs to cover a distance of only $450 - 12.5 - 12.5 = 425$ m along this side before turning. Since it goes twice as fast as the tanker, while the boat covers two-thirds of this distance, the tanker moves ahead the other third, and so the boat need travel down that side only two-thirds of 425 m, that is, a distance of $\frac{850}{3}$ m.

Now, from just missing the tanker at its bottom right corner, the boat still has to go 75 m to get back to lane PQ. While the boat is doing this, the tanker gets ahead 37.5 m, and so, in order to catch up and complete its cycle, the patrol boat must go another 75 m up PQ to get abreast of the tanker.

Altogether, then, the patrol boat must travel a grand total of

$$900 + 75 + 100 + \frac{850}{3} + 100 + 75 \;=\; 1533\frac{1}{3}\,\mathrm{m}\,.$$

This solution blithely assumes that the patrol boat and the tanker harmlessly slide by each other when they arrive simultaneously at the tanker's right hand corners. However, it makes a great deal more sense to interpret such an event as a collision: both vessels can hardly move into the same position at the same time without fatal consequences for the patrol boat. Fortunately, this nagging uncertainty is easily removed by adding the condition that *the two vessels are never to get closer to each other than* 25 m.

This is readily accommodated by having the boat get a lead of 62.5 m at F, before turning along the dangerous 75 m stretch FK between the lanes. This gives the boat a net lead of $62.5 - 37.5 = 25$ m when it reaches the right hand side of the tanker, thus avoiding any possibility of a collision and providing the 25 m separation at this critical juncture.

In order to get 62.5 m ahead, the boat must go another 125 m up PQ beyond E to a point F (Figure 3). Thus, when the patrol boat reaches F, the tanker is at UV, and the patrol boat is 62.5 m ahead.

Now the boat turns along FL, and when it reaches L, the tanker has advanced 37.5 m to IT, putting the tanker 25 m behind. While the boat proceeds 25 m along LG to the other lane, the tanker cuts its lead to 12.5 m at WN.

Thus, when the boat turns down SR, it is already 12.5 m ahead. In order to remain 25 m from the other end of the tanker when it goes back across the first lane, it will need to go farther down the side to a point H that is 12.5 m beyond the other end. Then, when it turns and closes the 25 m stretch to the nearer side of the tanker, the tanker will have advanced another 12.5 m to yield a separation of 25 m between them. Hence the patrol boat must cover $450 + 12.5 + 12.5 = 475$ m along this side before making its turn. As before, it needs to travel down that side only two-thirds of this distance, that is, $\frac{950}{3}$ m.

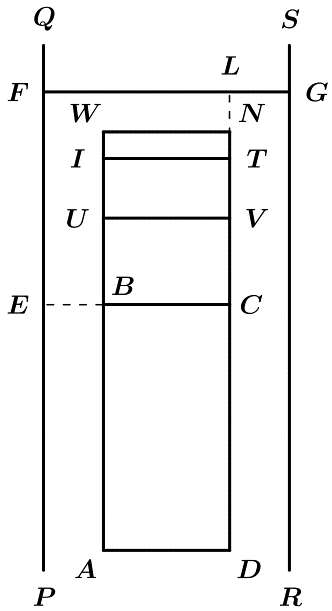


Figure 3

Thus, when passing the right rear corner of the tanker, the boat is already 25 m behind, and still has 75 m of the gap to negotiate to get back to lane PQ. While the boat is doing this, the tanker gets ahead another 37.5 m, for a total of 62.5 m, and so, in order to catch up and complete its cycle, the patrol boat must go another 125 m up PQ to get abreast of the tanker.

Altogether, then, the patrol boat must travel a grand total of

$$900 + 125 + 100 + \frac{950}{3} + 100 + 125 = 1666\frac{2}{3}\text{ m}.$$

This seemed most satisfactory until the brilliant solution of my colleague Larry Rice (retired from the University of Toronto Schools and the University of Waterloo) revealed another unwarranted assumption in my solution. Since the patrol boat had a 25 m lead when it reached the right hand edge of the tanker, it went unchallenged that the minimum 25 m distance between the boat and the front right corner of the tanker would be maintained as the boat completed the final 25 m of the crossing to its other lane.

Unfortunately this is not true; it is not hard to show that the vessels do get closer together than 25 m when the boat is completing the crossing.

So, how much of a lead would the boat have to have when it passes the right hand edge of the tanker in order to maintain a 25 m separation? Since 25 m isn't enough, perhaps a lead of 27 m would do? Again, it is not difficult to see that 27 m is not enough. So, how about 28 m? Bingo: 28 m is sufficient, with a little to spare. Consequently, any lead greater than 28 m would also suffice. But we seek the shortest cycle of the boat around the tanker. Thus we need to determine the smallest acceptable lead. We know it's some value, $25 + y$, between 27 and 28 m. Here y is not a variable, but a well-defined fixed value.

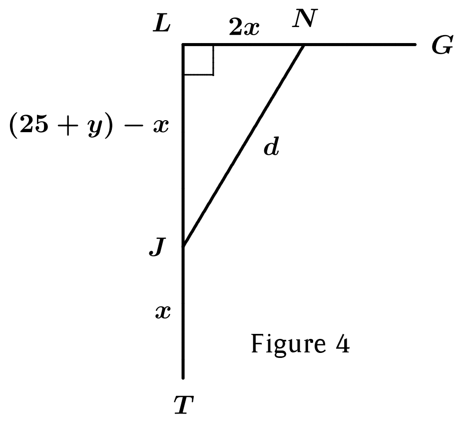


Figure 4

In Figure 4, let the boat pass the right hand edge of the tanker with a lead TL of $(25+y)$ metres. As the tanker advances x metres to J on TL, the boat advances $2x$ metres along LG to the point N. We want to determine the smallest positive value y such that the segment $d = JN$ is at least 25 m for all x between 0 and 12.5.

By the Pythagorean Theorem, we have

$$\begin{aligned} d^2 &= (2x)^2 + (25 + y - x)^2 \\ &= 4x^2 + (25 + y)^2 - 2x(25 + y) + x^2 \\ &= 5x^2 + y^2 + 50y + 625 - 50x - 2xy\,, \end{aligned}$$

and for $d^2 \geq 625$, we need

$$5x^2 + y^2 + 50y - 50x - 2xy \;\geq\; 0\,.$$

Since the only variable here is x, we may write

$$f(x) \;=\; 5x^2 + y^2 + 50y - 50x - 2xy\,,$$

where we require $f(x) \geq 0$ for $0 \leq x \leq 12.5$.

Differentiating, we get

$$f'(x) \;=\; 10x - 50 - 2y \;=\; 2(5x - 25 - y)\,,$$

which vanishes for $x = 5 + \frac{1}{5}y$. Since the graph of $f(x)$ is a parabola opening upwards, its minimum occurs when $f'(x) = 0$.

Hence

$$\begin{aligned}\min f(x) &= 5\left(5+\frac{1}{5}y\right)^2 + y^2 + 50y - 50\left(5+\frac{1}{5}y\right) - 2y\left(5+\frac{1}{5}y\right)\\ &= \frac{4}{5}y^2 + 40y - 125\,.\end{aligned}$$

Now, we don't want this minimum to be greater than 0, for in that case d would always *exceed* 25 and a lead of $25 + y$ would be greater than it need be. So for minimum y, we require the minimum to equal 0, and

$$\frac{4}{5}y^2 + 40y - 125 = 0\,,$$

or

$$4y^2 + 200y - 625 = 0\,.$$

The number y being positive, we obtain

$$\begin{aligned}y &= \frac{-200+\sqrt{200^2-4(4)(-625)}}{2(4)} = \frac{-200+\sqrt{50000}}{8}\\ &= \frac{25}{2}\sqrt{5}-25 \approx 2.95085\,,\end{aligned}$$

or 2.951 to three decimal places of accuracy.

Thus a lead of 27.951 would assure a universal 25 m separation and provide a shortest cycle to a reasonable degree of accuracy.

We have seen that an additional 125 m beyond E gives the boat a 25 m lead when it reaches the right hand edge of the tanker. Hence a lead of 27.951 m would require the boat to go a further $2(2.951) = 5.902$ m for a total of 130.902 m up its left lane before starting its turn across the front. Moreover, its lead is now $12.5 + 2.951 = 15.451$ m when it starts down the right lane.

To produce the same separation at the other end of the tanker, the boat needs to go this same distance, 15.451 m, farther down this lane than the end of the tanker, for a total passage of $450+2(15.451) = 480.902$ m. As before, the boat needs to cover only two-thirds of this distance, namely 320.601 m. After returning the 100 m to its left lane, the boat would be another 50 m behind, for a total of 65.451 m, and therefore needs to go another 130.902 m up PQ in order to complete the cycle. Altogether, then, the boat travels approximately a total of

$$900 + 130.902 + 100 + 320.601 + 100 + 130.902 = 1682.405 \text{ m}\,.$$

Finally, let's consider Larry's wonderful solution – a completely different approach.

The unknown distances in the cycle are those travelled by the patrol boat in the lanes PQ and RS; the distances travelled in crossing between the lanes is always $100 + 100 = 200$ m. Larry obtains these unknown distances from the basic formula

$$\text{distance} = \text{speed} \times \text{time}\,.$$

If we denote the speed of the tanker by v, then the speed of the boat is $2v$, and it remains only to determine the times t_1 and t_2 that the boat spends in lanes PQ and RS respectively, since the distance travelled in lane PQ will be $(2v)t_1$ and the distance travelled in lane RS will be $(2v)t_2$.

Knowing only the *ratio* of the speeds of the boat and the tanker, you might well wonder how he is going to obtain anything of value about these times. Larry very ingeniously enlists the aid of the captain of the tanker in this matter.

To this end, consider the observations of the captain as he watches the boat circle his tanker (Figure 5). First, he sees the boat run up lane PQ from E to F. At F, the boat turns across the lanes in front of him and, as it proceeds, it keeps getting closer to him because the tanker is moving ahead. Upon reaching the other lane, the boat proceeds along it and eventually turns back to lane PQ. As it crosses back, the tanker, still going forward, moves into the lead. Finally, the boat gets abreast of the tanker up lane PQ. Thus the captain sees the boat cycle the tanker along the sides of a trapezoid $EFGH$.

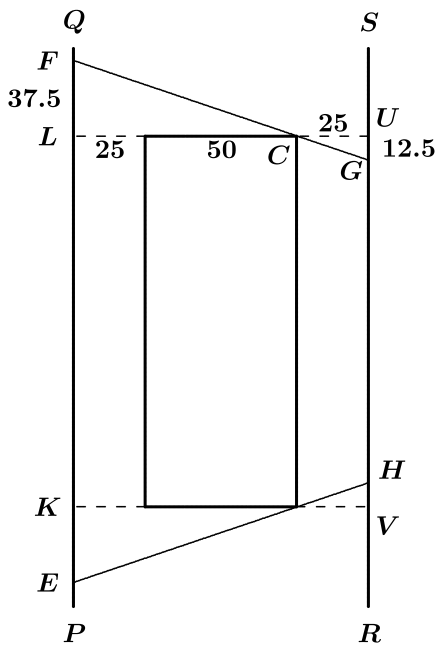


Figure 5

Between two points on a slanted side of the trapezoid, the boat moves twice as far toward the other lane as it does toward, or away from, the tanker, implying that the slopes of these sides are $\pm\frac{1}{2}$. Thus the trapezoid is isosceles and enjoys the symmetry of such a figure.

Since the slope of FG is $-\frac{1}{2}$ and $LC = 75$ m, it follows that $LF = 37.5$ m. The symmetrical part EK is therefore also 37.5 m, making the total length of side $EF = 450 + 2(37.5) = 525$ m. Similarly, since $CU = 25$ m, then $UG = HV = 12.5$ m, making the length of side $GH = 450 - 2(12.5) = 425$ m.

Now, time equals distance divided by speed, so the time t_1 that it takes the boat to traverse EF is 525 divided by the boat's speed along EF. While the boat goes at speed of $2v$ relative to the ocean, the captain sees it proceed up EF at the speed at which it overtakes the tanker; that is, at the relative speed of $2v - v = v$. Hence the captain calculates $t_1 = \dfrac{525}{v}$.

Along side GH, the captain observes the boat is going at a relative speed of $2v + v = 3v$, and so he calculates $t_2 = \frac{425}{3v}$.

It follows that the distance the boat travels up and down the lanes is

$$2v\left(\frac{525}{v}\right) + 2v\left(\frac{425}{3v}\right) = 1050 + \frac{850}{3} = 1333\frac{1}{3}\text{ m}.$$

Adding in the 200 m travelled in crossing between lanes, we obtain the same earlier grand total of $1533\frac{1}{3}$ m for the first case, in which we generously allowed the vessels to slide by each other without incident.

Allowing an additional 25 m buffer at each crossing between the lanes would increase each of the parallel sides of the trapezoid by 50 m and yield the corresponding times $t_1 = \frac{575}{v}$ and $t_2 = \frac{475}{3v}$.

The resulting cycle, then, would have a total length of

$$2v\left(\frac{575}{v}\right) + 2v\left(\frac{475}{3v}\right) + 200 = 1150 + 316\frac{2}{3} + 200 = 1666\frac{2}{3}\text{ m},$$

as found earlier.

Finally, (Figure 6), a universal separation of 25 m, would add an extra $2(12.5)(\sqrt{5}) \approx 25(2.236) = 55.9$ m to each of the parallel sides of the original trapezoid $EFGH$.

This is because the right-angled $\triangle XYC$ has legs in the ratio $1 : 2$. Can you see why?

Thus, $t_1 = \frac{580.9}{v}$ and $t_2 = \frac{480.9}{3v}$ which yields a cycle length of approximately

$$\begin{aligned} & 2(580.9) + 2\left(\frac{480.9}{3}\right) + 200 \\ = & \ 1161.8 + 320.6 + 200 \\ = & \ 1682.4\text{ m}, \end{aligned}$$

in close agreement with our previous result.

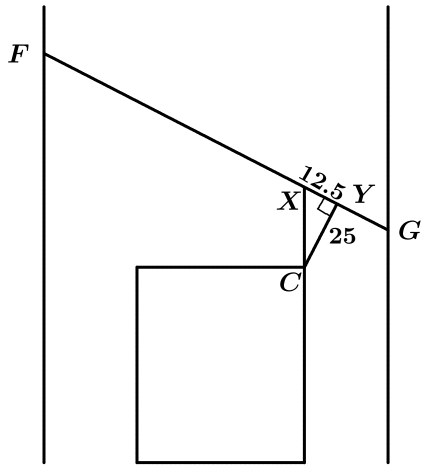


Figure 6

Ross Honsberger
326 Dale Crescent
Waterloo, ON, N2J 3Y3
rosshonsberger@rogers.com

THE OLYMPIAD CORNER

No. 279

R.E. Woodrow

After a much needed break, Joanne Canape has agreed to tackle transforming my scribbles into LaTeX again. Welcome back!

We begin this number with problems of the 37th Austrian Mathematical Olympiad of 2006. Thanks go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for our use.

37th AUSTRIAN MATHEMATICAL OLYMPIAD
Regional Competition (Qualifying Round)
April 27, 2006

1. Let $0 < x < y$ be real numbers and

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad \text{and} \quad Q = \sqrt{\frac{x^2+y^2}{2}}$$

be the harmonic, geometric, arithmetic, and quadratic means of x and y, respectively. It is well known that $H < G < A < Q$ holds. Order the intervals $[H,G]$, $[G,A]$, and $[A,Q]$ by length.

2. Let $n > 1$ be an integer and a a real number. Determine all real solutions $(x_1, x_2, \ldots, x_n)$ of the following system of equations:

$$\begin{aligned} x_1 + ax_2 &= 0\,, \\ x_2 + a^2x_3 &= 0\,, \\ x_3 + a^3x_4 &= 0\,, \\ &\vdots \\ x_{n-1} + a^{n-1}x_n &= 0\,, \\ x_n + a^nx_1 &= 0\,. \end{aligned}$$

3. In a nonisosceles triangle ABC, w is the bisector of the external angle at C. The extension of AB intersects w in D. Let k_A be the circumcircle of the triangle ADC, and k_B the circumcircle of the triangle BDC. Furthermore, let t_A be the tangent of k_A at A, t_B be the tangent of k_B at B, and P be the point in which these two tangents intersect.

Assume that the two points A and B are given. Determine the set of all points $P = P(C)$, such that ABC is acute-angled but not isosceles.

4. Let $\{h_n\}_{n=1}^{\infty}$ be a harmonic sequence of positive rational numbers. In other words, each h_n is the harmonic mean of its neighbours:

$$h_n \;=\; \frac{2h_{n-1}h_{n+1}}{h_{n-1}+h_{n+1}}\,.$$

Prove that if some term h_j of the sequence is the square of a rational number, then the sequence contains an infinite number of terms h_k that are each squares of rational numbers.

National Competition (Final Round, Part 1)

May 21, 2006

1. Let k and n be integers with $k \geq 2$, $n > 10^k$, and the decimal expansion of n ending with exactly k zeros. Give the best possible lower bound (in terms of $k = k(n) \geq 2$) for the number of ways to represent n as the difference of squares of two nonnegative integers.

2. Prove that the sequence $\left\{\dfrac{(n+1)^n n^{2-n}}{7n^2+1}\right\}_{n=0}^{\infty}$ is strictly increasing.

3. The incircle of triangle ABC touches the lines BC and AC at D and E, respectively. Prove that if AD and BE are of the same length, then the triangle is isosceles.

4. Let $\lfloor u \rfloor$ denote the greatest integer less than or equal to the real number u and let $\{u\} = u - \lfloor u \rfloor$. Let $f(x) = \lfloor x^2 \rfloor + \{x\}$ for all positive real numbers x. Find an infinite arithmetic progression of distinct positive rational numbers with denominator 3 (after cancellation) which do not lie in the image of f.

National Competition (Final Round, Part 2)

Day 1 (May 31, 2006)

1. Find the number of nonnegative integers $n \leq N$ with the property that the decimal expansion of some multiple of n contains only the digits 2 and 6 (not necessarily the same number of each).

2. Prove that

$$3(a+b+c) \;\geq\; 8\sqrt[3]{abc} \;+\; \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

for all positive real numbers a, b, and c. Determine when equality holds.

3. Given triangle ABC, let point R be on the extension of AB beyond B with $BR = BC$, and let point S be on the extension of AC beyond C with $CS = CB$. Let the diagonals of $BRSC$ intersect in the point A', and construct the points B' and C' similarly. Prove that the area of the hexagon $AC'BA'CB'$ is the sum of the areas of triangles ABC and $A'B'C'$.

Day 2 (June 1, 2006)

4. Determine all rational numbers x such that $1 + 105 \cdot 2^x$ is the square of a rational number.

5. Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(-f(x)) = f(f(x)) = f(x)^2 .$$

(A function f is monotonic if either $f(a) \leq f(b)$ for all $a < b$ or $f(a) \geq f(b)$ for all $a < b$.)

6. Let A be a nonzero integer. Find all integer solutions of the following system of equations:

$$\begin{aligned} x + y^2 + z^3 &= A\,, \\ \frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} &= \frac{1}{A}\,, \\ xy^2z^3 &= A^2\,. \end{aligned}$$

Next we give the Brazilian Mathematical Olympiad 2005. Thanks again go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for our use.

Brazilian Mathematical Olympiad 2005
First Day (October 22, 2005)

1. A positive integer is a *palindrome* if reversing its digits leaves it unchanged (for example, 481184, 131, and 2 are palindromes). Find all pairs (m, n) of positive integers such that $\underbrace{111\ldots1}_{m \text{ ones}} \times \underbrace{111\ldots1}_{n \text{ ones}}$ is a palindrome.

2. Determine the smallest real number C such that

$$C\left(x_1^{2005} + x_2^{2005} + \cdots + x_5^{2005}\right) \geq x_1x_2x_3x_4x_5\left(x_1^{125} + x_2^{125} + \cdots + x_5^{125}\right)^{16}$$

for all positive real numbers x_1, x_2, x_3, x_4, and x_5.

3. A square is contained in a cube if each of its points is on a face or in the interior of the cube. Determine the largest ℓ such that there exists a square of side ℓ contained in a cube of edge length 1.

Second Day (October 23, 2005)

4. We have four charged batteries, four uncharged batteries, and a radio which needs two charged batteries to work. We do not know which batteries are charged and which ones are uncharged. What is the least number of attempts that suffices to make sure the radio will work? (An attempt consists of putting two batteries in the radio and checking if the radio works or not).

5. Let ABC be an acute triangle and let F be its Fermat point, that is, the interior point of ABC such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each of the triangles ABF, BCF, and CAF, draw its Euler line, that is, the line connecting its circumcentre and its centroid.

Prove that these three lines are concurrent.

6. Let b be an integer and let a and c be positive integers. Prove that there exists a positive integer x such that

$$a^x + x \equiv b \pmod{c} ,$$

that is, prove there exists a positive integer x such that c divides $a^x + x - b$.

Next we look at the problems of the 4$^{\text{th}}$ Grade Croatian Mathematical Olympiad, written April 26–29, 2006. Thanks again go to Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for our use.

Croatian Mathematical Olympiad 2006
National Competition
4$^{\text{th}}$ Grade

1. Prove that three tangents to a parabola always form the sides of a triangle whose altitudes intersect on the directrix of the parabola.

2. Let k and n be positive integers. Prove that

$$\left(n^4 - 1\right)\left(n^3 - n^2 + n - 1\right)^k \;+\; (n+1)n^{4k-1}$$

is divisible by $n^5 + 1$.

3. The circles Γ_1 and Γ_2 intersect at the points A and B. The tangent line to Γ_2 through the point A meets Γ_1 again at C and the tangent line to Γ_1 through A meets Γ_2 again at D. A half-line through A, interior to the angle $\angle CAD$, meets Γ_1 at M, meets Γ_2 at N, and meets the circumcircle of $\triangle ACD$ at P. Prove that $|AM| = |NP|$.

4. Six islands are connected by the Ferryboat and the Hydrofoil Boat companies. Any pair of islands is connected, in both directions, by exactly one of these two companies. Prove that it is possible to tour four of the islands using exactly one company. That is, prove that there are four islands A, B, C, and D and one company whose boats sail on the lines $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, $D \leftrightarrow A$.

Next we give the problems of the Balkan Mathematical Olympiad 2006 written at Nicosia, Cyprus. Again we thank Robert Morewood, Canadian Team Leader to the IMO in Slovenia, for collecting them for the *Corner*.

Balkan Mathematical Olympiad 2006
Nicosia, Cyprus

1. *(Greece)* Let a, b, and c be real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

2. *(Greece)* Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The lines through points A, B, C and parallel to the line m intersect the circumcircle of triangle ABC again at the points A_1, B_1, C_1, respectively. Prove that the lines A_1E, B_1F, and C_1D are concurrent.

3. *(Romania)* Find all triples of positive rational numbers (m, n, p) such that each of the numbers

$$m + \frac{1}{np}, \qquad n + \frac{1}{pm}, \qquad p + \frac{1}{mn}$$

is an integer.

4. *(Bulgaria)* Let m be a fixed positive integer. For each positive integer a let the sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_0 = a$ and for $n \geq 0$ the recursion

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even}, \\ a_n + m & \text{otherwise}. \end{cases}$$

Find all values of a such that the sequence is periodic.

As a final set of problems for this number we give the problems of the Finnish Mathematical Olympiad 2006, Final Round. Thanks again go to Robert Morewood, Canadian Team Leader at the IMO in Slovenia for collecting them.

Finnish Mathematical Olympiad 2006
Final Round (February 3, 2006)

1. Determine all pairs (x, y) of positive integers such that

$$x + y + xy = 2006 .$$

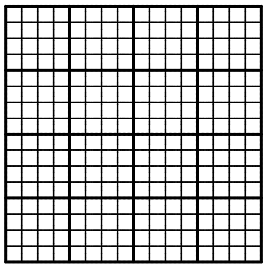
2. For all real numbers a, prove that

$$3\left(1 + a^2 + a^4\right) \geq \left(1 + a + a^2\right)^2$$

3. The numbers p, $4p^2 + 1$, and $6p^2 + 1$ are primes. Determine p.

4. Prove that if two medians of a triangle are perpendicular, then the triangle whose sides are congruent to the medians of the original triangle is a right triangle.

5. The game of *Nelipe* is played on a 16×16 grid. At each turn a player picks a number from $\{1, 2, \ldots, 16\}$ and writes it in one of the squares of the grid. At each stage of the game the numbers in each row, column, and in every one of the 16 four by four subsquares must be different. A player loses if he or she has no legal move. Which player wins, if both play with an optimal strategy?

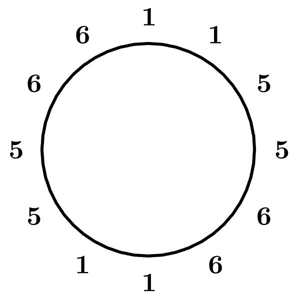


Next we turn to solutions from our readers to problems from the September 2008 number of the *Corner*, from the 19th Lithuanian Team Contest in Mathematics written October 2, 2004 [2008: 282–284].

1. Twelve numbers – four 1's, four 5's, and four 6's – are written in some order around a circle. Does there always exist a three-digit number comprised of three neighbouring numbers (its digits can be taken clockwise or counterclockwise) that is divisible by 3?

Solved by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Titu Zvonaru, Comăneşti, Romania. We give the response of Grant McLoughlin.

The arrangement at right shows that there is **not** always a three digit number comprised of three neighbouring numbers that is divisible by 3.



2. Solve the equation $2\cos(2\pi x) + \cos(3\pi x) = 0$.

Solved by George Apostolopoulos, Messolonghi, Greece; and George Tsapakidis, Agrinio, Greece. We give the write-up of Tsapakidis.

Using the double and triple angle formulas, the equation can be written equivalently as

$$\begin{aligned} 4\cos^3(\pi x) + 4\cos^2(\pi x) - 3\cos(\pi x) - 2 &= 0\,, \\ \big[2\cos(\pi x) + 1\big]\big[2\cos^2(\pi x) + \cos(\pi x) - 2\big] &= 0\,. \end{aligned}$$

Therefore $\cos(\pi x) = -\frac{1}{2}$ or $2\cos^2(\pi x) + \cos(\pi x) - 2 = 0$, hence, either $\cos(\pi x) = -\frac{1}{2}$ or $\cos(\pi x) = \frac{\sqrt{17}-1}{4}$. So, either $\cos(\pi x) = \cos\frac{2\pi}{3}$ or $\cos(\pi x) = \cos\theta$, where $\theta = \arccos\left(\frac{\sqrt{17}-1}{4}\right)$. Thus,

$$\pi x = 2k\pi \pm \frac{2\pi}{3} \qquad \text{or} \qquad \pi x = 2k\pi \pm \arccos\left(\frac{\sqrt{17}-1}{4}\right)\,, \quad k \in \mathbb{Z}\,,$$

and hence

$$x = 2k \pm \frac{2}{3} \qquad \text{or} \qquad x = 2k \pm \frac{1}{\pi}\arccos\left(\frac{\sqrt{17}-1}{4}\right)\,, \quad k \in \mathbb{Z}\,.$$

3. Solve the equation $3x^{\lfloor x\rfloor} = 13$, where $\lfloor x\rfloor$ denotes the integer part of the number x.

Solved by Pavlos Maragoudakis, Pireas, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write-up.

We show that the only solution is $x = \frac{\sqrt{39}}{3}$.

Since 0^0 is undefined, $x \neq 0$.

If $x < 0$, then $\lfloor x\rfloor = -n$ for some integer $n \geq 1$. Thus, $3x^{\lfloor x\rfloor} = \frac{3}{x^n}$. If n is odd, then $\frac{3}{x^n} < 0$. If n is even, then $n \geq 2$ implies $x < -1$ or $-x > 1$. Thus, $\frac{3}{x^n} = \frac{3}{(-x)^n} < 3$. In either case, $3x^{\lfloor x\rfloor} \neq 13$. Hence, it remains to consider $x > 0$.

If $x \leq 2$, then $x^{\lfloor x\rfloor} \leq 4$ and if $x \geq 3$, then $x^{\lfloor x\rfloor} \geq 27$. Hence, $2 < x < 3$. Let $x = 2 + r$ where $0 < r < 1$. Then $3x^{\lfloor x\rfloor} = 13$ becomes $3(2+r)^2 = 3\big(r^2+4r+4\big) = 13$, and we have $3r^2+12r-1 = 0$. Solving, we obtain $r = \frac{-12 \pm \sqrt{156}}{6} = -2 \pm \frac{\sqrt{39}}{3}$. Since $r > 0$, we reject $r = -2 - \frac{\sqrt{39}}{3}$. Hence, $r = -2 + \frac{\sqrt{39}}{3}$ and it follows that $x = \frac{\sqrt{39}}{3}$.

4. Let $0 \leq a \leq 1$ and $0 \leq b \leq 1$. Prove the inequality

$$\frac{a}{\sqrt{2b^2+5}} + \frac{b}{\sqrt{2a^2+5}} \leq \frac{2}{\sqrt{7}}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.

Since $a^2 \le 1$, we have $2b^2+5 \ge 2b^2+2a^2+3$. Similarly, we have $2a^2+5 \ge 2b^2+2a^2+3$ and it follows that

$$\frac{a}{\sqrt{2b^2+5}} + \frac{b}{\sqrt{2a^2+5}} \le \frac{a+b}{\sqrt{2b^2+2a^2+3}}\,.$$

Therefore, it suffices to show that $\sqrt{7}(a+b) \le 2\sqrt{2b^2+2a^2+3}$. Squaring and rearranging terms, this is equivalent to $12ab \le (a-b)^2+12$, which is certainly true since $12ab \le 12$ and $(a-b)^2+12 \ge 12$. The result follows. Clearly, equality holds if and only if $a=b=1$.

5. If a, b, and c are nonzero real numbers, what values can be taken by the expression

$$\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\ ?$$

Solution by Titu Zvonaru, Comăneşti, Romania.

We have $-1 < \dfrac{a^2-b^2}{a^2+b^2} < 1$, since $-2a^2 < 0 < 2b^2$.

Setting $\alpha = a^2b^2+b^2c^2+c^2a^2$, we obtain

$$\begin{aligned}
&\left(a^2-b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)+\left(b^2-c^2\right)\left(a^2+b^2\right)\left(c^2+a^2\right)\\
&\qquad+\ \left(c^2-a^2\right)\left(a^2+b^2\right)\left(b^2+c^2\right)\\
&=\ \left(a^2-b^2\right)\left(c^4+\alpha\right)+\left(b^2-c^2\right)\left(a^4+\alpha\right)+\left(c^2-a^2\right)\left(b^4+\alpha\right)\\
&=\ \left(a^2-b^2\right)c^4+\left(b^2-c^2\right)a^4+\left(c^2-a^2\right)b^4\\
&=\ a^4b^2-a^2b^4-a^4c^2+b^4c^2+a^2c^4-b^2c^4\\
&=\ a^2b^2\left(a^2-b^2\right)-c^2\left(a^4-b^4\right)+c^4\left(a^2-b^2\right)\\
&=\ \left(a^2-b^2\right)\left(a^2b^2-a^2c^2-b^2c^2+c^4\right)\\
&=\ \left(a^2-b^2\right)\left[a^2\left(b^2-c^2\right)-c^2\left(b^2-c^2\right)\right]\\
&=\ -\left(a^2-b^2\right)\left(b^2-c^2\right)\left(c^2-a^2\right)\ .
\end{aligned}$$

It follows that

$$\left|\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\right| = \left|\frac{a^2-b^2}{a^2+b^2}\cdot\frac{b^2-c^2}{b^2+c^2}\cdot\frac{c^2-a^2}{c^2+a^2}\right| < 1\,.$$

Let $k \in (-1,1)$ be a real number. Then, setting $a^2 = b^2\left(\dfrac{1+k}{1-k}\right)$, we obtain

$$\lim_{c\to 0}\left(\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\right) = k+1-1 = k\,.$$

It follows that $\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{b^2-c^2}{b^2+c^2}+\dfrac{c^2-a^2}{c^2+a^2}$ takes all real values in $(-1,1)$.

[*Ed.: It seems the solver assumes a basic fact about connectedness, namely, that the image of a connected set under a continuous, real-valued function is a connected subset of the real line, that is, an interval. Thus, the expression in a, b, and c achieves all values between any two of its given values, and since the image contains points arbitrarily close to* 1 *and* -1 *the conclusion follows.*]

6. Determine all pairs of real numbers (x, y) such that

$$\begin{aligned} x^6 &= y^4+18\,, \\ y^6 &= x^4+18\,. \end{aligned}$$

Solved by George Apostolopoulos, Messolonghi, Greece; Oliver Geupel, Brühl, NRW, Germany; George Tsapakidis, Agrinio, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Apostolopoulos.

Observe that $x \neq 0$ and $y \neq 0$. Since $x^6-y^4=18$ and $y^6-x^4=18$, we have

$$\begin{aligned} x^6-y^6+x^4-y^4 &= 0\,, \\ \left(x^2\right)^3-\left(y^2\right)^3+\left(x^2\right)^2-\left(y^2\right)^2 &= 0\,, \\ \left(x^2-y^2\right)\left(x^4+x^2y^2+y^4\right)+\left(x^2-y^2\right)\left(x^2+y^2\right) &= 0\,, \\ \left(x^2-y^2\right)\left(x^4+x^2y^2+y^4+x^2+y^2\right) &= 0\,. \end{aligned}$$

However, $x^4+x^2y^2+y^4+x^2+y^2>0$, hence $x^2=y^2$ and $x^6-x^4-18=0$. We put $x^2=w$, then the equation in x becomes $w^3-w^2-18=0$, or $(w-3)\left(w^2+2w+6\right)=0$, hence $w=3$ since $w^2+2w+6=x^4+2x^2+6$ is positive for all real numbers x.

Now, $x^2=w=3$, hence $x=\pm\sqrt{3}$ and there are four solutions $(x,y)=(\pm\sqrt{3},\pm\sqrt{3})$.

7. Find all triples (m, n, r) of positive integers such that

$$2001^m \;+\; 4003^n \;=\; 2002^r\,.$$

Solution by Titu Zvonaru, Comăneşti, Romania.

We will find all triples of *nonnegative* integers satisfying the equation.

Since $3^{2k}=9^k\equiv 1 \pmod 8$ and $3^{2k+1}=9^k\cdot 3\equiv 3 \pmod 8$, we have that

$$2001^m+4003^n \;\equiv\; 1+3^n \;\equiv\; 2,4 \pmod 8\,.$$

If $r\geq 3$, then $2002^n\equiv 0 \pmod 8$, hence $r\leq 2$ and we have $0\leq n<r\leq 2$.

If $r = 2$ and $n = 0$, then the equation becomes $2001^m + 1 = 2002^2$, hence $2001^m = 2002^2 - 1$, hence $2001^m = 2001 \cdot 2003$, hence we have $2001^{m-1} = 2003$ and we see that there are no solutions in this case.

If $r = 2$ and $n = 1$, then we obtain $2001^m + 4003 = 2002^2$, hence $2001^m = (2001 + 1)^2 - (2 \cdot 2001 + 1)$, hence $2001^m = 2001^2$ and $m = 2$.

If $r = 1$ and $n = 0$, then $2001^m + 1 = 2002$, hence $m = 1$.

Therefore, the solutions are $(m, n, r) \in \{(1, 0, 1), (2, 1, 2)\}$, of which only $(m, n, r) = (2, 1, 2)$ consists entirely of positive integers.

8. Assume that m and n are positive integers. Prove that, if $mn - 23$ is divisible by 24, then $m^3 + n^3$ is divisible by 72.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give Wang's version.

By assumption, $mn = 24k + 23$, $k \in \mathbb{Z}$; hence $mn \equiv 7 \pmod 8$.

Clearly, m and n are both odd. Thus, $m \equiv 1$, 3, 5, or $7 \pmod 8$ and similarly for n. All of the solutions to $mn \equiv 7 \pmod 8$ are given by $(m, n) \equiv (1, 7)$, $(7, 1)$, $(3, 5)$, or $(5, 3) \pmod 8$, where the congruence $(m, n) \equiv (a, b) \pmod d$ means that $m \equiv a \pmod d$ and $n \equiv b \pmod d$. In the first two cases, we have $m^3 + n^3 \equiv 1 + 343 = 344 \equiv 0 \pmod 8$ and in the other two cases, we have $m^3 + n^3 \equiv 27 + 125 = 152 \equiv 0 \pmod 8$. Hence,

$$8 \mid \left(m^3 + n^3\right) . \tag{1}$$

It remains to show that

$$9 \mid \left(m^3 + n^3\right) . \tag{2}$$

Since $mn = 24k + 23$, we have $mn \equiv 2 \pmod 3$, which implies that $(m, n) \equiv (2, 1)$ or $(1, 2) \pmod 3$. Thus, $m + n \equiv 0 \pmod 3$. Since $m^3 + n^3 = (m + n)\big((m + n)^2 - 3mn\big)$, the relation (2) follows.

The conclusion now follows from (1) and (2).

9. Is it possible that, for some a, both expressions $\dfrac{1 - 2a\sqrt{35}}{a^2}$ and $a + \sqrt{35}$ are integers?

Solved by Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Cománeşti, Romania. We give Geupel's write-up.

Yes, this is possible. Let $m = \dfrac{1 - 2a\sqrt{35}}{a^2}$ and $n = a + \sqrt{35}$; then $m = 1$ and $n = \pm 6$ if $a = \pm 6 - \sqrt{35}$.

We prove additionally that there are no other solutions.

If $m = 0$, then $a = \dfrac{\sqrt{35}}{70}$ and $n = \dfrac{71\sqrt{35}}{70}$ is not an integer. Hence, $|m| \geq 1$ because m is an integer, and then $|1 - 2a\sqrt{35}| \geq a^2$.

If $1 - 2a\sqrt{35} \ge a^2$, then $n^2 = \left(a + \sqrt{35}\right)^2 \le 36$ and thus $|n| \le 6$. Otherwise, $-\left(1 - 2a\sqrt{35}\right) \ge a^2$ and $a^2 - 2\sqrt{35}a + 1 \le 0$. Therefore, $2\sqrt{35} - \sqrt{34} \le n \le 2\sqrt{35} + \sqrt{34}$, hence $7 \le n \le 17$. Altogether, we have $-6 \le n \le 17$. The 24 cases can now be eliminated using a calculator.

[*Ed.: Substitute* $a = n - \sqrt{35}$ *into* $ma^2 = 1 - 2a\sqrt{35}$ *and simplify to obtain* $m\left(n^2 + 35\right) - 71 = (m-1)2n\sqrt{35}$. *Since each side is a rational number,* $n = 0$ *or* $m = 1$*, which quickly leads to the unique solution.*]

11. What is the greatest value that a product of positive integers can take if their sum is equal to 2004?

Solved by George Apostolopoulos, Messolonghi, Greece; and Oliver Geupel, Brühl, NRW, Germany. We first give Apostolopoulos' solution.

Let $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2004$, where each α_j is a positive integer. We want to maximize the product $\alpha_1\alpha_2\cdots\alpha_n$.

If $\alpha_k \ge 4$ for some k, then replacing α_k by 2 and $\alpha_k - 2$ leaves the sum the same and can only increase the product, since $2(\alpha_k - 2) \ge a_k$ for $a_k \ge 4$. Therefore, the maximum product is achieved when each a_k is either 2 or 3. Now, $2 + 2 + 2 = 3 + 3$ but $2^3 < 3^2$, so the product is maximized by taking as many 3's in the sum as possible while leaving at most two 2's. Since $2004 = 3 \cdot 668 + 0 \cdot 2$, the largest possible product is 3^{668}.

Next we give Geupel's solution to a different reading of the problem.

We prove that the greatest value a product of two positive integers x and y can take if their sum is equal to a fixed positive integer n, is $\lfloor \frac{n^2}{4} \rfloor$. In the special case $n = 2004$ the result is $1002^2 = 1004004$. Indeed, we have $xy = x(n-x) = -\left(x - \frac{n}{2}\right)^2 + \frac{n^2}{4}$. For even n, we obtain $xy \le \frac{n^2}{4}$, where equality holds if and only if $x = y = \frac{n}{2}$. For odd n we have $xy \le \frac{n^2-1}{4}$, where equality holds if and only if $\{x, y\} = \left\{\frac{n-1}{2}, \frac{n+1}{2}\right\}$.

12. Positive integers a, b, c, u, v, and w satisfy the system of equations

$$\begin{aligned} a + u &= 21\,, \\ b + v &= 31\,, \\ c + w &= 667\,. \end{aligned}$$

Can abc be equal to uvw?

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write-up.

Yes. If $a = 2{\cdot}7$, $u = 7$, $b = 3{\cdot}5$, $v = 2^4$, $c = \left(2^3\right){\cdot}29$, and $w = 3{\cdot}5{\cdot}29$, then the equations are satisfied and $abc = uvw = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 29 = 48720$.

13. Let u be the real root of the equation $x^3 - 3x^2 + 5x - 17 = 0$, and let v be the real root of the equation $x^3 - 3x^2 + 5x + 11 = 0$. Find $u + v$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Díaz-Barrero.

For real numbers x and t let $f(x) = x^3 - 3x^2 + 5x$ and $g(t) = t^3 + 2t$. Then $f(x) = (x-1)^3 + 2(x-1) + 3$ and g is an odd, increasing, and bijective function, as can be easily checked. Furthermore,

$$\begin{aligned} g(u-1) &= (u-1)^3 + 2(u-1) = f(u) - 3 = 17 - 3 = 14\,, \\ g(v-1) &= (v-1)^3 + 2(v-1) = f(v) - 3 = -11 - 3 = -14\,. \end{aligned}$$

We have $g(u-1) = -g(v-1) = g(1-v)$ because g is odd, and moreover since g is bijective, we deduce that $u - 1 = 1 - v$, hence $u + v = 2$.

15. Does there exist a polynomial, $P(x)$, with integer coefficients such that for all x in the interval $\left[\frac{4}{10}, \frac{9}{10}\right]$ the inequality $|P(x) - \frac{2}{3}| < 10^{-10}$ is valid?

Solution by Oliver Geupel, Brühl, NRW, Germany.

We give such a polynomial explicitly. Let $P(x) = \dfrac{2}{3}\left[1 - \left(1 - 3x^4\right)^n\right]$ and note that $P(x)$ has integer coefficients. For $0.4 \le x \le 0.9$ we have

$$-0.9683 = 1 - 3(0.9)^4 \le 1 - 3x^4 \le 1 - 3(0.4)^4 = 0.9232\,,$$

hence $\left|1 - 3x^4\right| \le 0.9683$. Therefore, the condition $\left|P(x) - \dfrac{2}{3}\right| < 10^{-10}$ is satisfied for $0.4 < x < 0.9$ if $\dfrac{2}{3}(0.9683)^n < 10^{-10}$. It now suffices to choose $n = \left\lfloor \dfrac{1 - \log_{10} 3 - \log_{10} 2}{1 - \log_{10} 9.683} \right\rfloor + 1$. (Using a calculator, we find that $n = 59$.)

16. Does there exist a positive number a_0 such that all the members of the infinite sequence a_0, a_1, a_2, $\ldots$, defined by the recurrence formula $a_n = \sqrt{a_{n-1} + 1}$, $n \ge 1$, are rational numbers?

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove by contradiction that there is no such a_0. For each $n \ge 0$ let $a_n = \frac{p_n}{q_n}$, where p_n and q_n are coprime positive integers. From

$$\frac{p_n^2}{q_n^2} = a_n^2 = a_{n-1} + 1 = \frac{p_{n-1} + q_{n-1}}{q_{n-1}}\,,$$

it follows that

$$q_{n-1} p_n^2 = q_n^2 \left(p_{n-1} + q_{n-1}\right)\,. \tag{1}$$

Then $q_{n-1} \mid q_n^2(p_{n-1}+q_{n-1})$ and the numbers q_{n-1} and $p_{n-1}+q_{n-1}$ are coprime, hence $q_{n-1} \mid q_n^2$.

On the other hand, (1) yields $q_n^2 \mid q_{n-1}p_n^2$ and p_n and q_n are coprime, hence $q_n^2 \mid q_{n-1}$.

It follows that $q_n^2 = q_{n-1}$ for all $n \geq 0$. We conclude that $q_n = 1$ for each n. The a_n are therefore positive integers. It is readily checked that $a_n > 1$ and $a_n > a_{n+1}$ for all $n > 0$. This contradiction completes the proof.

17. Let a, b, and c be the sides of a triangle and let x, y, and z be real numbers such that $x+y+z=0$. Prove that

$$a^2yz + b^2zx + c^2xy \leq 0\,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the write-up of Amengual Covas.

More generally, let α, β, γ, x, y, and z be real numbers and suppose that $x+y+z=0$.

For real numbers u and v we have $(u+v)^2 \leq 2(u^2+v^2)$, hence

$$\begin{aligned}
&(\beta+\gamma)^2yz + (\gamma+\alpha)^2zx + (\alpha+\beta)^2xy \\
&\quad\leq\; 2\big[(\beta^2+\gamma^2)yz + (\gamma^2+\alpha^2)zx + (\alpha^2+\beta^2)xy\big] \\
&\quad=\; 2\left[\alpha^2x(y+z) + \beta^2y(z+x) + \gamma^2z(x+y)\right] \\
&\quad=\; -2\left(\alpha^2x^2+\beta^2y^2+\gamma^2z^2\right) \;\leq\; 0\,.
\end{aligned}$$

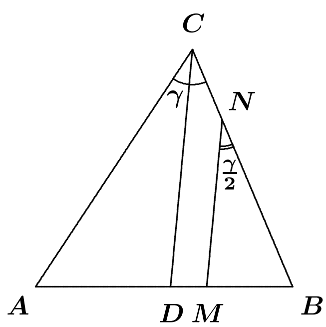
Equality occurs only if $\alpha=\beta=\gamma$ and $\alpha x = \beta y = \gamma z = 0$.

To establish the proposed inequality, we make use of the linear transformation $a=\beta+\gamma$, $b=\gamma+\alpha$, and $c=\alpha+\beta$, where α, β, and γ are uniquely determined positive numbers and apply the above inequality.

Equality holds only if the triangle is equilateral and $x=y=z=0$.

18. Points M and N are on the sides AB and BC of the triangle ABC, respectively. It is given that $\dfrac{AM}{MB} = \dfrac{BN}{NC} = 2$ and $\angle ACB = 2\angle MNB$. Prove that ABC is an isosceles triangle.

Solved by George Apostolopoulos, Messolonghi, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Smeenk.

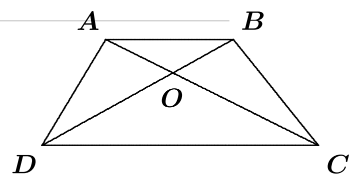


Let $\gamma = \angle ACB$, so that $\angle MNB = \frac{1}{2}\gamma$. The line through C parallel to NM meets AB at D. Then $2:3 = BN:BC = BM:BD$, hence $BD = \frac{3}{2}BM = \frac{1}{2}AB$. Since the bisector of $\angle ACB$ is also a median, $CA = CB$.

19. The two diagonals of a trapezoid divide it into four triangles. The areas of three of them are 1, 2, and 4 square units. What values can the area of the fourth triangle have?

Solved by George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Tsapakidis.

Let $[XYZ]$ be the area of triangle XYZ. Then $[ABD] = [ABC]$, as the two triangles have equal altitudes from their common base AB. Therefore,



$$[OAD] + [OAB] \;=\; [OBC] + [OAB]\,,$$

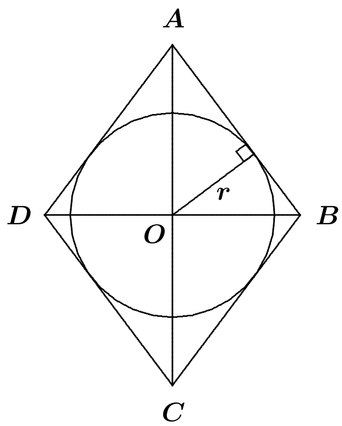
hence $[OAD] = [OBC]$.

Triangles OAB and OCD are similar, so we have $\dfrac{OA}{OC} = \dfrac{OB}{OD}$, that is $\dfrac{OA}{OC}\cdot\dfrac{OD}{OB} = 1$, and hence $\dfrac{[OAB]}{[OBC]}\cdot\dfrac{[ODC]}{[OBC]} = 1$, so $[OBC]^2 = [OAB][OBC]$. It follows that $[OBC] = 2 = [OAD]$, $[OAB] = 1$, and $[ODC] = 4$, that is, the fourth triangle has area 2 square units.

20. The ratio of the lengths of the diagonals of a rhombus is $a : b$. Find the ratio of the area of the rhombus to the area of an inscribed circle.

Solved by George Apostolopoulos, Messolonghi, Greece; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Tsapakidis.

Let r be the radius of the circle inscribed in the rhombus. The required ratio is then



$$\begin{aligned}
\frac{4\cdot\frac{1}{2}OA\cdot OB}{\pi r^2} \;&=\; \frac{2}{\pi}\cdot OA\cdot OB\cdot\frac{1}{r^2}\\
=\;& \frac{2}{\pi}\cdot OA\cdot OB\left(\frac{1}{OA^2}+\frac{1}{OB^2}\right)\\
=\;& \frac{2}{\pi}\left(\frac{OB}{OA}+\frac{OA}{OB}\right) \;=\; \frac{2}{\pi}\left(\frac{b}{a}+\frac{a}{b}\right)\\
=\;& \frac{2(a^2+b^2)}{\pi ab}\,,
\end{aligned}$$

where $\frac{1}{r^2} = \frac{1}{OA^2}+\frac{1}{OB^2}$ holds since r is the altitude of the right triangle OAB.

That completes the *Corner* for this special issue in honour of Jim Totten. The Editorial Board decided to stay with "business as usual" for the *Olympiad Corner*. Having worked with Jim over many years, I suspect that is what he would have opted for.

BOOK REVIEWS

Amar Sodhi

For this special issue we feature book reviews by Andy Liu and John Grant McLoughlin, former Book Reviews Editors (1991-1998 and 2002-2008, respectively) who each knew Jim Totten for an extended period.

The following sentiment is from Andy: "Jim and I went back way before he became formally involved with ***Crux***. His father lived in Edmonton, and he came to visit twice a year. Whenever he was in town, we would get together for a Chinese dim-sum. One day, he asked me why I was no longer doing the review column. Actually, I had already stepped down for a couple of years. However, I replied flippantly I had just been fired by the new editor, little knowing that I was sitting across the table from the new editor himself. He was very upset, and despite repeated apology, it remained a sore point with him. So to set the record straight, I now state publicly that I was never fired by Jim Totten as the Reviews Editor for ***Crux***, and in his memory, it is only fitting that I contribute a Book Review to this special issue."

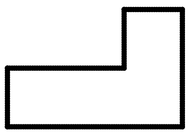
All-Star Mathlete Puzzles
By Dick Hess, Sterling Publishing Co. Inc., New York, 2009
ISBN 978-1-4027-5528-6, softcover, 94+ii pages, US$6.95
Reviewed by **Andy Liu**, *University of Alberta, Edmonton, AB*

Sterling has published a wide range of mathematical puzzle books. At one end are the definitive treatises like Jerry Slocum's *The Tangram Book*. At the other end are rather prosaic offerings. Nevertheless, they do serve a purpose in gradually attracting new enthusiasts. They offer the novices an easier path into the hobby than more serious works like Rodolfo Kurchan's *Mesmerizing Math Puzzles*, Serhiy Grabarchuk's *The New Puzzle Classics*, and the present volume.

Dick Hess is well known to the readers of ***CRUX with MAYHEM*** as a regular contributor. His intriguing problems always come with an aura of something out of the ordinary. Here are a couple of samples.

Problem 89(A) Find an expression equal to 88 which uses each of the digits 1, 2, and 3 exactly once. You may use a combination of the operations addition, subtraction, multiplication, division, exponents, roots, concatenation, decimals, repeating decimals, factorials, and brackets.

Problem 44 The figure at right consists of four unit squares joined edge to edge. Find a polygon, not necessarily convex, such that five non-overlapping copies can fit inside this figure, and cover more than 98% of its area.



It should be mentioned that Dick Hess is an avid tennis fan, and there are no books by him without problems related to that game. To find out more about these problems, and the solution to the two problems above, buy the book! It is inexpensive and very highly recommended.

John adds: "The second edition of Ravi Vakil's book *A Mathematical Mosaic* has been published. The original edition blended biography, problems, and mathematical connections, in an effective manner. This sentiment is captured in the 1997 ***CRUX with MAYHEM*** review by Jim Totten, so we therefore reprint Jim's review in its entirety preceded by a brief review of the second edition. As I was the Book Reviews Editor during Jim's tenure as Editor-in-Chief, the opportunity to here accompany a review by Jim is an honour. ~~Jim took a keen interest in the books that were reviewed in~~ ***CRUX with MAYHEM***. Indeed, Jim wrote enthusiastically about Stewart Coffin's *Geometric Puzzle Design* in a review published in April 2008.

A Mathematical Mosaic: Patterns & Problem Solving (New Expanded Edition)
By Ravi Vakil, Brendan Kelly Publishing, Burlington, ON, 2008
ISBN 978-1-895997-28-6, softcover, 288 pages, US$19.95
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB*

I highly recommend the new and expanded edition of *A Mathematical Mosaic: Patterns & Problem Solving*, but only if you do not have a copy of the 1996 edition already. Quoting from the foreword: "(T)here are few changes here and there from the first edition, but this book remains essentially the same one that appeared in 1996." That comment reflects this reviewer's take on the book.

The book is about 30 pages longer than the original version. Select additions of sections such as "Higher-dimensional versions of Platonic Solids" within the Combinatorics Chapter, insertion of mathematical portraits, and inclusion of a more thorough index account for this difference.

Other changes reflect updating of biographical information and annotated references. Also, Ravi Vakil is now a well established algebraic geometer. However, with this in mind, the October 1997 ***CRUX with MAYHEM*** review by Jim Totten is still relevant today. It is noteworthy that Ravi Vakil is a founding co-editor of ***Mathematical Mayhem*** which became part of this journal in 1997, the same year as the publication of Jim's review.

A Mathematical Mosaic
By Ravi Vakil, published by Brendan Kelly Publishing Inc., 1996,
2122 Highview Drive, Burlington, ON L7R 3X4
ISBN 1-895997-04-6, softcover, 253+ pages, US$16.95 plus handling.
Reviewed by **Jim Totten**, *University College of the Cariboo.*

So, you have a group of students who have decided they want the extra challenge of doing some mathematics competitions. You want a source of problems which will pique the students' interest, and which also lead to further exploration. The problem source should lend itself well to independent work. The question is: where do you find the appropriate level enrichment material? Many of us have already tried to answer this question and have a collection of such problem books. Well, here is a book to be added to your collection!

It is certainly a problem book, but it is much more than that. The author at one moment guides the reader through some very nice mathematical developments, and throws out problems as they crop up in the development, and in the next moment uses a problem as a starting point for some interesting mathematical development.

With a few exceptions the problems in this book are not new, nor are the solutions. They are, however, well organized, both by topic and by level of mathematical maturity needed. Answers are NOT always provided; instead there is often simply a solution strategy or hint given, and occasionally there is simply a reference to some other source for a full-blown treatment. Even when answers are provided, they are not tucked away at the end of the book, but rather they are worked into another topic (usually later in the book, but not always), where they become part of the development of another topic or problem.

The author is a PhD candidate in pure mathematics at Harvard University (at the time the book was written). Being still very young, he knows how to speak to today's teenagers. His sense of humour and general puckishness is present throughout: just when you are lulled into some serious computation in probability, he deviously throws a trick question at you, that has a totally non-obvious answer (non-obvious, that is, until you CAREFULLY re-read the question).

Many mathematics books published today include short biographies on famous mathematicians through history, especially those whose names come up in the theory developed in the book. This book is no exception. But what is unique about this book is the inclusion of Personal Profiles of young mathematicians from several countries that he has met at International Mathematical Olympiads (IMOs) in the past. The profiles are quite diverse, which means that most bright students could find one to identify with and to use as a role model. The author and those he profiles have taken a risk in doing this: they have tried to predict some of the important mathematicians in the early part of the next century. It should be interesting to follow their careers and see if those predictions can come true, or if by placing them in the spotlight, they find too much pressure to deal with.

The problems range from puzzles that elementary school children can do to problems that provide training for Putnam candidates (toward the end of the book). There are many cross-references and connections between seemingly unrelated problems from different areas of mathematics, connections that most students would be unable to make. Many of these connections are new to this reviewer. However, once made these connections are quite clear.

As for his credentials, Ravi Vakil placed among the top five in the Putnam competition in all four of his undergraduate years at the University of Toronto. Before that he won two gold medals and a silver medal in IMOs and coached the Canadian team to the IMO from 1989 to 1995.

The British Columbia Secondary School Mathematics Contest

Clint Lee

In 1973 John Ciriani of Cariboo College (which became Thompson Rivers University, TRU) initiated the Cariboo College High School Mathematics Contest with the help of colleagues in the Cariboo College Mathematics and Statistics Department. The intent of the contest was to enable high school students in the Cariboo College region with an interest in mathematics to get local support and recognition for their interest and abilities in mathematics. In 1979 Jim Totten joined the department and quickly became an enthusiastic contributor to and supporter of the contest. In 1992 he compiled a book containing all of the problems from the contest papers over this period. He also co-edited (with Leonard Janke) a separate book of solutions.

Meanwhile, two other institutions in British Columbia developed high school mathematics contests. In 1977, at the suggestion of Robin Insley, the Mathematics Department at the College of New Caledonia (CNC), consisting of Phil Beckman, Robin Insley, Peter Trushell, and myself, developed a high school mathematics contest for students in the Central Interior region served by CNC, inspired by the successful contest being run at Cariboo College. It ran successfully and independently from its inception until 1994. In 1990, shortly after I joined the Mathematics Department at Okanagan College (OC) a third high school mathematics contest was initiated for students in the OC region which ran independently from 1991 until 1994.

The 1993 meeting of the British Columbia Committee in Undergraduate Programs in Mathematics (BCCUPM, now BCCUPMs) was held at Selkirk College in Castlegar. There Jim Totten approached members of the OC and CNC mathematics departments with a suggestion that the three institutions pool their resources to offer a single mathematics contest. The contest would serve the three colleges as well as any others in BC interested in such a contest, with the name, as suggested by Jim, the British Columbia Colleges High School Mathematics Contest. This title was intended to make explicit that the contest was intended for colleges, who interacted directly with their regions, rather than universities. Both OC and CNC enthusiastically agreed. Jim then approached other institutions and found further support for the idea. Over the years Colleges evolved into University Colleges and then into Universities, so the original name became inappropriate and was changed to the British Columbia Secondary School Mathematics Contest (BCSSMC).

It took a year to organize the combined contest with the first being held in 1995. Jim's idea was that Cariboo College would prepare the contest papers, with contributions from the other institutions, and all participating institutions would use these for their local contests. The 1995 and

1996 contests were handled this way. The combined contest, following the Cariboo model, consists of Preliminary and Final Rounds. Each round has a Junior level for grades 8 to 10 and a Senior level for grades 11 and 12. The Preliminary Round consisting of twelve multiple choice questions is written and marked at the individual schools in early March. Based on these results, each school typically chooses two to four students to participate in the Final Round at the sponsoring post-secondary institution in early May. The Final Round consists of ten multiple choice questions and five questions requiring written answers, all of which are marked by a panel of faculty from the sponsoring institution. In addition to the morning contest writing, Final Round activities usually include a morning session for teachers, lunch for all participants, an afternoon activity for the students during marking, and an award ceremony. Examples of afternoon student activities are lectures by local/invited speakers, sports activities involving mathematics, or scavenger hunts. The prizes awarded to winners range from scholarship prizes to cash or book prizes, depending on the sponsoring institution. Note that copies of the contest papers from 1999 to the present, with solutions, are available at the BCSSMC website at `http://people.okanagan.bc.ca/clee/BCSSMC`.

Following the success of the 1995 and 1996 contests, it was suggested that the only improvement would be to have members of more institutions be directly involved in the contest preparation process. For the 1997 contest Don DesBrisay, Kirk Evenrude, and Jack Bradshaw from Cariboo College; Nicholas Buck and Edward Dobrowolski from CNC; Dave Murray, John Grant McLoughlin, and Clint Lee from OC; Jim Bailey from College of the Rockies (formerly East Kootenay Community College) and Wayne Matthews from Camosun College; met at Cariboo College in Kamloops in August 1996 for the inaugural Math Contest Brainstorming Session. Jim Totten had made most of the local arrangements, including billets and a party at the end of the first day for all of the participants. However, he was absent due to winning an opportunity to play in a pro-am golf event at the Greater Vancouver Open. A year later people came together in Kamloops again, before shifting to the Kalamalka Campus of Okanagan College in Vernon where the session was hosted by myself in 1998. Since 1999, the Brainstorming Sessions have been held in conjunction with the BCCUPMs articulation meetings in mid-May, either before or after the main meeting. The Brainstorming Sessions include about 8 to 16 representatives from a cross-section of institutions across the province. Participants usually bring some problems that they have prepared ahead of time and materials for generating additional problems as needed. The session is divided into Junior and Senior groups, each of which prepares rough drafts of Preliminary and Final round papers. Drafts are later typeset into tentative versions of the contest papers. These are reviewed and revised over several months by as many as twenty reviewers. Finished forms are then produced including solutions to all of the problems that ultimately appear on the contest papers.

From the beginning, until he took the position of Editor-in-Chief of ***CRUX with Mayhem***, Jim participated actively in these sessions sharing his infectious enthusiasm for mathematics and its manifestations in all areas. Jim's humour contributed to the success and the enjoyment experienced by all participants. He contributed problems and solutions for the contest, typeset either the contest papers or the solutions, and participated in the review of the contest papers as they evolved into their finished form. His abilities as a proofreader, his attention to detail, and his skills with wording problems clearly were invaluable to all of those involved in the process.

Also during this period, Jim was active in recruiting institutions to participate in the contest. He made himself available to any institution, especially smaller institutions just getting started with the contest, to help during the final round in any capacity, especially as a speaker. In 2004 it was suggested that The Pacific Institute for the Mathematical Sciences (PIMS) might be willing to support some of these activities. Rick Brewster approached PIMS for such support and as a result PIMS provides support for a range of the province-wide activities associated with the contest. Jim's recruiting activities brought the number of participating institutions up to a maximum of twelve in one year, with a core of at least ten institutions. The institutions that have participated in the contest over the years are: TRU in Kamloops, CNC in Prince George, OC in Kelowna, Langara College in Vancouver, Capilano University in North Vancouver, Camosun College in Victoria, University of the Fraser Valley in Abbotsford, North Island College in Campbell River, Malaspina University in Nanaimo, College of the Rockies in Cranbrook, Selkirk College in Castlegar, Northwest Community College in Prince Rupert, UBC Okanagan in Kelowna, and Douglas College in New Westminster. Approximately 2500 grade 8 to 12 students participate in the Preliminary Round of the Contest each year and 500 participate again in the Final Round.

While Jim was Editor-in-Chief of ***CRUX with Mayhem***, he scaled back his involvement with the contest. During this period I took over many of the responsibilities that Jim had undertaken. I typeset the drafts of the contest papers, managed the evolution of the contest papers, saw to the distribution of the papers to the participating institutions, and oversaw and typeset solutions. Jim continually remained available to provide advice and contribute his reviewing skills in doing a final run through of the contest papers. Shortly before his death, he contacted me and indicated that now that he was winding down his Editor-in-Chief duties and had retired from TRU, he was looking forward to increasing his involvement with the contest.

It is clear that without the inspiration and efforts of Jim Totten, the British Columbia Secondary School Mathematics Contest would not exist today. His enthusiasm for and contributions to the contest were key in developing the feelings of involvement and ownership of all of the participating institutions.

Clint Lee
Okanagan College
Vernon, BC
clee@okanagan.bc.ca

A Duality for Bicentric Quadrilaterals

Michel Bataille

In memory of Jim Totten

Bicentric quadrilaterals are convex quadrilaterals that have both an incircle and a circumcircle. As every problem solver probably knows, such a quadrilateral is easily constructed from its future incircle. Below, we review this property that we formulate as an *internal* theorem about the determination of a bicentric quadrilateral. The purpose of this note is to state and prove an *external* theorem which is, in a sense, the dual of the internal one.

Internal Theorem A bicentric quadrilateral is uniquely determined by three of its sides tangent to a given circle.

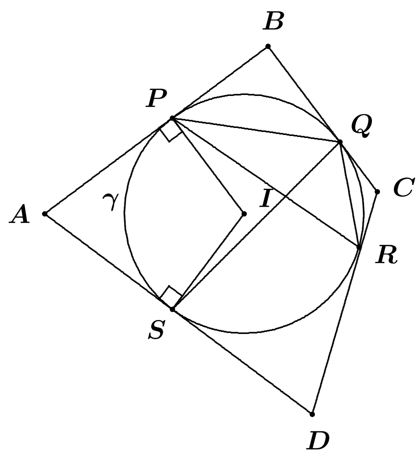
This results immediately from the following well-known characterization:

> Let $PQRS$ be a convex quadrilateral inscribed in a circle γ. The tangents to γ at P, Q, R, and S are the sidelines of a bicentric quadrilateral if and only if QS is perpendicular to PR.

For completeness, here is a short proof, in which $\angle(\ell, \ell')$ denotes the directed angle between lines ℓ and ℓ', measured modulo π.

Proof: Let I be the centre of γ and A, B, C, and D be the vertices of the quadrilateral formed by the tangents, as in the figure. Observing that A, P, I, and S are concyclic, we have

$$\begin{aligned}\angle(AB, AD) &= \angle(AP, AS) \\ = \angle(IP, IS) &= 2\angle(QP, QS)\,.\end{aligned}$$



Similarly,

$$\angle(CB, CD) = 2\angle(PQ, PR)\,,$$

so that $\angle(AB, AD) = \angle(CB, CD)$ if and only if $\angle(QP, QS) = \angle(PQ, PR)$ mod $\frac{\pi}{2}$, that is, if and only if $QS \perp PR$ (since QS and PR are not parallel). ■

We are guided to the next theorem by considering the circumcircle to be the dual of the incircle, and recalling the duality between a tangent to a circle and its point of tangency.

External Theorem A bicentric quadrilateral is uniquely determined by three of its vertices lying on a given circle.

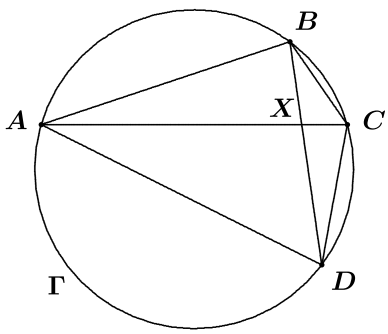
As above, the proof rests upon a characterization of the bicentric quadrilateral and will provide a construction of this quadrilateral.

We begin by showing that the conjunction of two simple properties distinguishes bicentric quadrilaterals among all quadrilaterals inscribed in a given circle. The following lemma is inspired by (and is an extension of) a problem posed in this journal (see [2], [3]).

Lemma Let $ABCD$ be a convex quadrilateral inscribed in a circle Γ with diagonals AC and BD meeting at X. Then $ABCD$ is bicentric if and only if

(a) B and D are on the same side of the perpendicular bisector of AC, and

(b) $BX = \left(\cos^2\frac{B}{2}\right) BD$.



Proof: First, we assume that $ABCD$ is bicentric. Then,

$$BA - BC = DA - DC \qquad (1)$$

so that $BA-BC$ and $DA-DC$ certainly have the same sign and (a) holds.

Let us denote area by $[\cdot]$ and for simplicity let $AB = a$, $BC = b$, $CD = c$, $DA = d$, and $AC = e$. Observing that $D = \angle ADC = \pi - B$, we have

$$\frac{BX}{BD} = \frac{[ABX]}{[ABD]} = \frac{[CBX]}{[CBD]} = \frac{[ABC]}{[ABCD]} = \frac{ab\sin B}{ab\sin B + cd\sin D} = \frac{ab}{ab+cd}\,.$$

Furthermore, from the Law of Cosines we deduce that

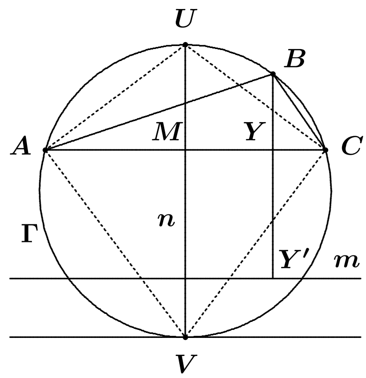
$$\cos B \;=\; \frac{a^2+b^2-e^2}{2ab} \;=\; \frac{e^2-c^2-d^2}{2cd} \;=\; \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\,.$$

Squaring each side of (1), we obtain $a^2+b^2-c^2-d^2 = 2(ab-cd)$ and so $\cos B = \dfrac{ab-cd}{ab+cd} = 2\cdot\dfrac{BX}{BD} - 1$ and (b) follows.

Conversely, if (b) holds, then $(a-b)^2 = (c-d)^2$ follows readily from the above calculations. Since $a-b$ and $d-c$ have the same sign by part (a), we have $a-b = d-c$, that is, $AB+CD = BC+DA$, which implies that $ABCD$ has an incircle (see [1]). ∎

For a proof of the external theorem, consider a triangle ABC with circumcircle Γ. We are looking for a point D on Γ such that $ABCD$ is bicentric. Let Y be the foot of the perpendicular from B to AC, and let Y' be the point such that $\overrightarrow{BY} = \left(\cos^2\frac{B}{2}\right)\overrightarrow{BY'}$; equivalently, $\overrightarrow{YY'} = \left(\tan^2\frac{B}{2}\right)\overrightarrow{BY}$.

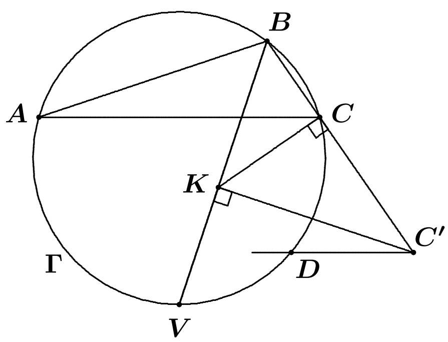
The lemma implies that the desired point D must lie both on Γ and on the line m through Y' perpendicular to BY and on the same side as B of the perpendicular bisector n of AC. It remains to show that m necessarily contains a point of Γ. To this end, let n intersect AC at M and Γ at U and V, where U and B are on the same arc AC.



Now $\angle AUC = B$ and $AVCU$ is cyclic, hence $\angle AVC = \pi - B$ and we have the relations $UM = \frac{1}{2}\left(\cot\frac{B}{2}\right)AC$ and $VM = \frac{1}{2}\left(\tan\frac{B}{2}\right)AC$. Observing that $BY \le UM$, we obtain

$$YY' \le \left(\tan^2\frac{B}{2}\right)\cdot\frac{1}{2}\left(\cot\frac{B}{2}\right)AC = \frac{1}{2}\left(\tan\frac{B}{2}\right)AC = VM\,.$$

It follows that m lies between AC and the tangent to Γ at V. In other words, except if $B = U$, m intersects Γ in two points, one of which is the desired point D. The lemma then ensures that $ABCD$ is bicentric.



To conclude, the figure at right shows how to quickly construct D using K on the bisector BV of $\angle ABC$ with KC perpendicular to BC, since $\frac{BC}{BK} = \cos\frac{B}{2} = \frac{BK}{BC'}$ implies that $\frac{BC}{BC'} = \cos^2\frac{B}{2}$.

Acknowledgment. The author would like to thank the referee for his careful reading of the manuscript and for a suggestion which greatly improved the original proof of the external theorem.

References

[1] N. Altshiller-Court, *College Geometry*, Dover reprint (2007), p. 135.

[2] D.J. Smeenk, Problem 2027, ***Crux Mathematicorum***, **21** (1995) p. 90; Solution in **22** (1996), p. 94.

[3] Problem 3211, ***Crux Mathematicorum with Mathematical Mayhem***, **33** (2007) p. 43; Solution in **34** (2008), p. 61.

Michel Bataille
12 rue Sainte-Catherine
76000 Rouen, France
michelbataille@wanadoo.fr

A Study of Knight's Tours on the Surface of a Cube

Awani Kumar

1 Introduction

The classical puzzle of a knight's tour was, is, and will always be fascinating. The reason is its simplicity as well as its complexity, which have endless charm. For centuries, the traditional study of the knight's tour was mostly confined to square boards. Later, Schubert [1], Gibbins [2], and Stewart [3] delved into the knight's tour puzzle in 3-dimensional space. More recently, Kumar [4] looked into the possibilities of knight's tours in cubes and cuboids having magic properties. Awani Kumar, Francis Gaspalou, and Guenter Stertenbrink have discovered a magic knight's tour inside a $4 \times 4 \times 4$ cube. However, a lot remains to be explored and discovered even on its surface. A tour of a knight on a square board is called a *magic tour* if the sum of the numbers in each row and column is the same (the *magic constant*). Perusal of the literature reveals that knight's tours have been constructed on the surfaces of cubes smaller or larger than $4 \times 4 \times 4$ but, astonishingly, not on a $4 \times 4 \times 4$ cube itself. We know that a knight's tour is not possible on a 4×4 board, but is it possible on the surface of a $4 \times 4 \times 4$ cube? Will such tours be open or closed? Can they have magic properties? What about magic tours on the surfaces of larger cubes? We shall look at these questions.

2 Knight's tours on surfaces of small cubes

The reader can visualize the knight's move on the face of a cube in a much better way when it is unfolded, as shown in Figure 1. On a conventional board a knight can cover only up to eight squares. However, this need not be the case when it is moving on the surface of a cube or cuboid. In fact, it can cover up to 10 squares, as shown in Figure 2a. Some readers may find this strange because the consecutive knight's moves are not equidistant, as on a conventional board. Well, don't worry. It is because a cube can be unfolded along its edges in many different ways. Look at Figure 2b and the knight's move to the square labelled "6" is clear. If it is ever unclear, then appropriately unfold a cube along its edges using a pair of scissors!

A knight's tour is even possible on the smallest possible cube measuring $1 \times 1 \times 1$, as shown in Figure 3. The discerning reader will observe that here the numbers are arranged as on a conventional die. That is, the numbers on opposite faces add up to 7.

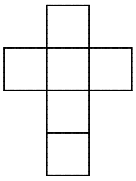


Figure 1

Figure 2a

		10	1		
			6		
9		Kt.		5	
4	8			2	
		7	3		

Figure 2b

					10
				6	1
9		Kt.		5	
4	8			2	
		7	3		

Figure 3

	1	
2	3	5
	6	
	4	

Figure 4

		1	22		
		14	10		
15	11	24	2	13	9
18	3	12	8	23	5
		16	4		
		7	20		
		21	17		
		19	6		

Figure 5

		1	22		
		14	10		
15	11	23	2	13	9
18	3	12	8	24	5
		16	4		
		7	20		
		21	17		
		19	6		

Figure 6

		2	8		
		17	23		
4	15	1	7	9	22
11	20	18	24	6	13
		5	12		
		14	19		
		21	3		
		16	10		

Figure 7

		2	24		
		7	17		
13	23	1	9	21	3
6	8	18	22	16	10
		15	19		
		12	4		
		14	20		
		11	5		

Figure 8

				63	90	11	30				
				10	31	62	91				
				89	64	29	12				
				32	9	92	61				
51	22	33	88	7	28	65	94	13	60	79	42
36	85	50	23	66	93	8	27	78	43	16	57
21	52	87	34	25	6	95	68	59	14	41	80
86	35	24	49	96	67	26	5	44	77	58	15
				1	48	69	76				
				70	73	4	45				
				47	2	75	72				
				74	71	46	3				
				19	54	39	82				
				38	83	18	55				
				53	20	81	40				
				84	37	56	17				

Figure 9

				87	66	11	30				
				10	31	86	67				
				65	88	29	12				
				32	9	68	85				
21	76	33	64	89	28	7	70	13	84	55	42
62	35	24	73	8	69	90	27	54	43	16	81
75	22	63	34	25	92	71	6	83	14	41	56
36	61	74	23	72	5	26	91	44	53	82	15
				93	46	1	52				
				4	49	94	45				
				47	96	51	2				
				50	3	48	95				
				19	78	57	40				
				60	37	18	79				
				77	20	39	58				
				38	59	80	17				

Watkins [5] and Pickover [6] have given a knight's tour on the surface of a $2 \times 2 \times 2$ cube. Figure 4 and Figure 5 are two examples of closed and open tours, respectively, on the surface of a $2 \times 2 \times 2$ cube. There are millions of such tours and, therefore, these are of little interest. However, tours having magic properties are a different story. Figure 6 and Figure 7 are open and closed tours, respectively, having magic properties. In these tours, the sum of the numbers on each face is 50, the *magic sum*. There are thousands of such tours. In fact, $2 \times 2 \times 2$ is the smallest cube on which such interesting tours with magic faces are possible. But how can such tours be constructed? Well, the secret lies in the systematic movement of the knight over the faces of the cube. Altogether, there are 24 squares. Put them into four groups: 1 to 6, 7 to 12, 13 to 18, and 19 to 24. Start from any face and complete the knight's tour in such a way that each face has numbers from all the groups. With a little patience and perseverance, tours with magic properties can be constructed.

Before proceeding any further, let us prove the following theorem.

Theorem For odd n, a knight's tour on the surface of an $n \times n \times n$ cube cannot have a magic sum on its faces.

Proof: Assume there is a magic sum. Altogether, there are $6n^2$ numbers, which sum up to $3n^2(6n^2 + 1)$. The magic sum (the sum on each face) is thus $\frac{n^2(6n^2+1)}{2}$. Since n is odd, the numerator is odd. Thus, the sum of the numbers on each face is not an integer, a contradiction. ■

3 Knight's tours on the surface of a $4 \times 4 \times 4$ cube

Professional and amateur mathematicians have been studying the knight's tour inside a $4 \times 4 \times 4$ cube for over a century but, astonishingly, no one has looked for them on its surface! There are billions of tours on the surface of a $4 \times 4 \times 4$ cube, and consequently these are of little interest. However, tours having magic properties are, again, a different story. We have seen that the faces of a $2 \times 2 \times 2$ cube can have magic properties and this is true for tours on a $4 \times 4 \times 4$ cube too. In fact, this is true for all even-sided cubes. Later, we will see that a doubly-even cube (say, $8 \times 8 \times 8$) can have both rows and columns magic but a singly-even cube (say, $6 \times 6 \times 6$) has only the rows or the columns (but not both) magic. More interesting and challenging is the construction of tours in which all rows and columns have magic properties. Figure 8 is one such tour in which five faces are magic in rows and columns and only 4 lines (2 rows and 2 columns) are just a hair's breadth away from the magic constant. The discerning reader must have observed that these tours are made up of regular quads. Here also the numbers are arranged in four groups: 1 to 24, 25 to 48, 49 to 72, and 73 to 96. All these groups are represented in each row and each column of the faces. The same holds true for the four 2×2 squares on each of the faces. Figure 9 shows a reentrant tour. Curiously, it can be converted to a magic tour of two knights by interchanging the squares 46 and 48. One knight covers the squares 1 to 48 and the other covers 49 to 96. It is a four-fold cyclic almost magic tour. That is, it retains its almost magic properties (44 magic lines) when starting from the square 25, 49, or 73. Readers are encouraged to improve on this.

4 Knight's tours on the surface of a $6 \times 6 \times 6$ cube

The knight's tour on the surface of $6 \times 6 \times 6$ cube has gotten scantily little attention. Perusal of the literature reveals that, in spite of billions of possible tours, only Watkins [5] has given a solution found by Arden Rzewnicki and Jesse Howard. The author has enumerated 88 semimagic tours (only rows or columns are magic) on a 6×6 board and by carefully selecting and linking these tours, one can obtain hundreds of reentrant tours on the surface of a $6 \times 6 \times 6$ cube. One such example is shown in Figure 10. Since it is a closed tour, by selecting a suitable starting point (square 19 as 1, square 20 as 2, and so on), we can get a tour having up to 12 magic lines with magic constant 651. In fact, by choosing any of the six starting point squares 19, 55, 91, 127, 163, or 199, we can get tours having 12 magic lines. Jelliss [7] has proved that there cannot be magic tours on singly-even boards (that is, 6×6, 10×10, and so forth). But what about these on the surfaces of singly-even cubes? The author conjectures that they do not exist. The best the author could achieve is shown in Figure 11. All six faces there are semimagic and altogether there are 40 magic lines. So, the magic sum has been achieved up to 55%. Readers are challenged to improve on this.

5 Knight's tours on the surface of an $8 \times 8 \times 8$ cube

This is the oldest problem of the lot, over 200 years old. Dudeney [8] writes that the problem was raised by Vandermonde, an 18$^{\text{th}}$ century musician and mathematician. Dudeney gave a solution by completing a tour on a face and then proceeding to the next face. Subsequent writers, namely Petkovic [9], Pickover [6], and Watkins [5] have merely reproduced Dudeney's solution, giving an erroneous impression it is the only possible tour. In fact, there are trillions of such tours. Using powerful computers and intelligent programs, the international team of Mackay, Meyrignac, and Stertenbrink [10] enumerated all of the 280 magic tours on the 8×8 board. By judiciously selecting and linking these tours, one can get hundreds of tours and by rearranging the knight's tour, we can get magic tours on all six faces (magic constant $= 1540$). Figure 12 is one such tour; all its rows and columns are magic. The diagonal sums are twice the magic constant on two of the faces but, in spite of intense effort, the author could not obtain any diagonally magic face or a reentrant magic tour. Readers are encouraged to look for them.

						213	190	187	204	201	196						
						186	205	212	197	188	203						
						191	214	189	202	195	200						
						206	185	198	211	182	209						
						215	192	183	208	199	194						
						184	207	216	193	210	181						
69	46	59	44	57	52	1	10	35	28	25	12	172	147	180	157	174	145
60	43	68	53	66	37	34	29	2	11	8	27	179	156	173	146	163	158
47	70	45	58	51	56	3	36	9	26	13	24	148	171	162	175	152	167
42	61	54	67	38	65	30	33	20	5	16	7	155	178	153	166	159	164
71	48	63	40	55	50	21	4	31	18	23	14	170	149	176	161	168	151
62	41	72	49	64	39	32	19	22	15	6	17	177	154	169	150	165	160
						88	93	80	95	82	105						
						73	102	89	104	79	96						
						92	87	94	81	106	83						
						101	74	103	90	97	78						
						86	91	76	99	84	107						
						75	100	85	108	77	98						
						124	109	128	137	122	111						
						129	138	123	110	127	136						
						116	125	130	139	112	121						
						131	140	117	126	135	144						
						118	115	142	133	120	113						
						141	132	119	114	143	134						

Figure 10

						123	116	101	94	125	118						
						100	95	124	117	102	93						
						115	122	99	108	119	126						
						96	107	112	121	92	103						
						111	114	105	98	109	120						
						106	97	110	113	104	91						
195	30	25	32	197	184	1	212	201	10	13	210	136	81	142	75	144	73
24	189	196	185	26	33	18	11	2	211	202	9	141	90	135	82	127	76
29	194	31	190	183	198	3	200	213	12	209	14	134	137	80	143	74	83
188	23	186	27	34	19	214	17	4	205	8	203	89	140	87	130	77	128
193	28	21	36	191	182	199	206	215	6	15	208	86	133	138	79	84	131
22	187	192	181	20	35	216	5	16	207	204	7	139	88	85	132	129	78
						180	165	38	47	178	49						
						37	52	179	50	39	46						
						164	173	166	45	48	177						
						53	44	51	172	169	40						
						174	163	42	167	176	171						
						43	54	175	170	41	168						
						160	157	162	55	60	57						
						155	68	159	58	149	62						
						158	161	156	61	56	59						
						67	154	69	148	63	150						
						70	147	152	65	72	145						
						153	66	71	146	151	64						

Figure 11

								167	170	183	220	185	222	195	198								
								182	219	168	171	194	197	186	223								
								169	166	217	184	221	188	199	196								
								218	181	172	165	200	193	224	187								
								179	216	201	208	173	164	189	210								
								204	207	180	213	192	209	174	161								
								215	178	205	202	163	176	211	190								
								206	203	214	177	212	191	162	175								
39	54	331	348	333	350	35	50	15	354	17	384	11	358	23	378	242	159	144	225	240	255	146	129
330	347	38	53	36	51	334	351	18	383	14	355	22	379	10	359	227	142	241	160	145	130	239	256
55	40	345	332	349	34	49	336	353	16	381	20	357	12	377	24	158	243	226	143	254	237	132	147
346	329	56	37	52	335	352	33	382	19	356	13	380	21	360	9	141	228	157	244	131	148	253	238
41	58	327	344	323	62	337	48	3	366	29	372	5	364	25	376	156	245	140	229	152	249	236	133
328	343	42	57	340	45	322	63	32	369	4	365	28	373	8	361	139	230	153	248	135	234	149	252
59	326	341	44	61	324	47	338	367	2	371	30	363	6	375	26	246	155	232	137	250	151	134	235
342	43	60	325	46	339	64	321	370	31	368	1	374	27	362	7	231	138	247	154	233	136	251	150
								66	83	302	317	300	315	70	87								
								303	318	67	84	69	86	299	314								
								320	65	82	301	316	297	88	71								
								81	304	319	68	85	72	313	298								
								94	291	80	305	296	311	74	89								
								79	306	93	292	73	90	295	312								
								290	95	308	77	92	293	310	75								
								307	78	289	96	309	76	91	294								
								102	283	268	117	288	97	270	115								
								267	118	101	284	269	116	287	98								
								120	103	282	265	100	285	114	271								
								281	266	119	104	113	272	99	286								
								106	121	264	277	260	127	112	273								
								263	280	105	124	109	274	257	128								
								122	107	278	261	276	259	126	111								
								279	262	123	108	125	110	275	258								

Figure 12

6 Conclusion and an Acknowledgement

We have seen that a knight's tour is possible on the surfaces of cubes of various sizes. Cubes of size $4 \times 4 \times 4$ and larger can have magic rows and magic columns on their faces.

The author has felt that in India, getting the references is more difficult than discovering a magic tour on a cube. The author is grateful to Takaya Iwamoto for providing photocopies of [2] and [3].

References

[1] H. Schubert (1904); Mathematische Mussestunden Eine Sammlung von Geduldspielen, Kunststücken und Unterhaltungs-aufgaben mathematischer Natur. (Leipzig), p. 235-7, $4 \times 4 \times 4$ closed tours, p. 238, $3 \times 4 \times 6$ closed tour. [Mathematical Association Library, Leicester University]

[2] N.M. Gibbins (1944); Chess in 3 and 4 dimensions, *Mathematical Gazette*, May 1944, pp. 46-50.

[3] J. Stewart (1971); Solid Knight's Tours, *Journal of Recreational Mathematics* Vol. 4, No. 1, January 1971, p. 1.

[4] A. Kumar (2006); Studies in Tours of the Knight in Three Dimensions, *The Games and Puzzles Journal* #43 (`electronic`), January-April 2006.

[5] J.J. Watkins (2005); *Across the Board, The Mathematics of Chessboard Problems*, Princeton University Press, 2005, pp. 10-11, pp. 88-89, pp. 93-94.

[6] C.A. Pickover (2004); *The Zen of Magic Squares, Circles, and Stars*, Princeton University Press, 2004, pp. 214-216.

[7] G.P. Jelliss (2003); Existence Theorems on Magic Tours, *The Games and Puzzles Journal* #25 (`electronic`), January-March 2003.

[8] H.E. Dudeney (1970); *Amusements in Mathematics*, Dover Publications, New York, 1970, p. 103, p. 229.

[9] M. Petkovic (1997); *Mathematics and Chess*, Dover Publications, New York, 1997, p. 53, p. 64.

[10] G. Stertenbrink (2003); Computing Magic Knight Tours, `http://magictour.free.fr/`

Awani Kumar
2/11-C, Vijayant Khand Gomti Nagar
Lucknow 226 010
India
awanieva@gmail.com

TOTTEN PROBLEMS

The following problems have all been dedicated by the proposers to the lasting memory of Jim Totten. Solutions should arrive no later than **1 March 2010**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

TOTTEN–01. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let H be the orthocentre of triangle ABC and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of $\angle BAC$. If X is the circumcentre of triangle APB and if Y is the orthocentre of triangle APC, prove that the length of XY is equal to the circumradius of triangle ABC.

TOTTEN–02. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k \geq 2$ be an integer and let $f : [0,\infty) \to \mathbb{R}$ be a bounded continuous function. If x is a positive real number, find the value of

$$\lim_{n\to\infty} \sqrt[k]{n} \int_0^x \frac{f(t)}{(1+t^k)^n}\, dt\,.$$

TOTTEN–03. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Find the limit

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\left(\frac{n}{\dfrac{1}{x_1} + \cdots + \dfrac{1}{x_n}}\right) dx_1 \ldots dx_n\,.$$

TOTTEN–04. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Suppose that $0 < a < b$, $m_0 = \sqrt{ab}$, and $m_1 = \dfrac{a+b}{2}$. If $x \geq 0$, prove that

$$\frac{x}{m_1(x+m_1)} \;\leq\; \frac{1}{b-a}\log\frac{b(x+a)}{a(x+b)} \;\leq\; \frac{x}{m_0(x+m_0)}\,.$$

TOTTEN–05. *Proposed by Michel Bataille, Rouen, France.*

Let I be the incentre of triangle ABC. Let the point A' be such that $\overrightarrow{AA'} = (\cos A)\overrightarrow{AI}$, and let points B' and C' be defined similarly. Find the radius of the circle passing through A', B', and C' and locate its centre.

TOTTEN–06. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Jim and three of his buddies played a round of golf. As usual, Jim won the game. In fact, he beat every two of his three buddies, in the following sense. Let his three buddies be A, B, and C and let a_i be A's score on hole i, for all $1 \le i \le 18$, and similarly define b_i and c_i. Set $S_{ab} = \sum_{i=1}^{18} \min(a_i, b_i)$, and similarly define S_{ac} and S_{bc}. Then Jim's total score was less than S_{ab}, S_{ac}, and S_{bc}. However, Jim's score was more than $S_{abc} = \sum_{i=1}^{18} \min(a_i, b_i, c_i)$. Jim's score was 72. What was the minimum possible score of any of his buddies?

TOTTEN–07. *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove or disprove that

(a) $1 \le \dfrac{a}{1-ab} + \dfrac{b}{1-bc} + \dfrac{c}{1-ca} \le \dfrac{3\sqrt{3}}{2}$,

(b) $1 \le \dfrac{a}{1+ab} + \dfrac{b}{1+bc} + \dfrac{c}{1+ca} \le \dfrac{3\sqrt{3}}{4}$.

TOTTEN–08. *Proposed by Richard Hoshino, Government of Canada, Ottawa, ON.*

In triangle ABC suppose that $AB < AC$. Let D and M be the points on side BC for which AD is the angle bisector and AM is the median. Let F be on side AC so that AD is perpendicular to DF. Finally, let E be the intersection of AM and DF. Prove that $AB \cdot DE + AB \cdot DF = AC \cdot EF$.

TOTTEN–09. *Proposed by Richard Hoshino, Government of Canada, Ottawa, ON.*

Let n and k be integers with $n \ge 2$ and $k \ge 0$. Consider n dinner guests sitting around a circular table. Let $g_n(k)$ be the number of ways that k of these n guests can be chosen so that no two chosen guests are sitting next to one another. To illustrate, $g_6(0) = 1$, $g_6(1) = 6$, $g_6(2) = 9$, $g_6(3) = 2$, and $g_6(4) = 0$ for all $k \ge 4$. For each $n \ge 2$, let

$$f_n(x) \;=\; \sum_{k \ge 0} g_n(k) x^k \,.$$

For example, $f_6(x) = 1+6x+9x^2+2x^3 = (1+2x)(1+4x+x^2)$. Determine all n for which $(1 + 2x)$ is a factor of $f_n(x)$.

TOTTEN–10. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Determine all triangles ABC whose side lengths are positive integers and such that $\cos C = \frac{4}{5}$.

TOTTEN–11. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x, y, and z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{8\sqrt{3}}{9} \leq \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)\left(\frac{1}{\sqrt{z}} - \sqrt{z}\right).$$

(b) ★. Let $n \geq 2$ and let x_1, x_2, $\ldots$, x_n be positive real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove or disprove that

$$\left(\frac{n-1}{\sqrt{n}}\right)^n \leq \prod_{k=1}^{n}\left(\frac{1}{\sqrt{x_k}} - \sqrt{x_k}\right).$$

TOTTEN–12. *Proposed by Mihály Bencze, Brasov, Romania.*

Let w, x, y, and z be positive real numbers with $w+x+y+z = wxyz$, and let

$$f(x) = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{x^3}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{x^3}}}.$$

Prove that $\sqrt[3]{wxy} + \sqrt[3]{xyz} + \sqrt[3]{yzw} + \sqrt[3]{zwx} \geq f(w) + f(x) + f(y) + f(z)$.

. . . .

TOTTEN–01. *Proposé par Cosmin Pohoaţă, Collège National Tudor Vianu, Bucarest, Roumanie.*

Soit H l'orthocentre du triangle ABC et soit P la seconde intersection du cercle circonscrit du triangle AHC avec la bissectrice intérieure de l'angle BAC. Si X est le centre du cercle circonscrit du triangle APB et Y l'orthocentre du triangle APC, montrer que la longueur de XY est égale au rayon du cercle circonscrit du triangle ABC.

TOTTEN–02. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $k \geq 2$ un entier et $f : [0,\infty) \to \mathbb{R}$ une fonction continue et bornée. Pour un nombre réel positif x, trouver la valeur de

$$\lim_{n\to\infty} \sqrt[k]{n} \int_0^x \frac{f(t)}{(1+t^k)^n}\, dt.$$

TOTTEN–03. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $f : [0,1] \to \mathbb{R}$ une fonction continue. Trouver la limite

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\left(\frac{n}{\dfrac{1}{x_1} + \cdots + \dfrac{1}{x_n}}\right) dx_1 \ldots dx_n\,.$$

TOTTEN–04. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

On suppose que $0 < a < b$, $m_0 = \sqrt{ab}$ et $m_1 = \dfrac{a+b}{2}$. Si $x \geq 0$, montrer que

$$\frac{x}{m_1(x+m_1)} \;\leq\; \frac{1}{b-a}\log\frac{b(x+a)}{a(x+b)} \;\leq\; \frac{x}{m_0(x+m_0)}\,.$$

TOTTEN–05. *Proposé par Michel Bataille, Rouen, France.*

Soit I le centre du cercle inscrit du triangle ABC. Soit A' le point tel que $\overrightarrow{AA'} = (\cos A)\overrightarrow{AI}$, et soit B' et C' les points définis de manière analogue. Trouver le rayon du cercle passant par A', B' et C' ainsi que la position de son centre.

TOTTEN–06. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Jim et trois de ses copains ont joué une ronde de golf. Comme d'habitude, Jim a gagné la partie. En fait, il a battu chaque paire de ses trois copains, dans le sens suivant. Soit A, B et C ses trois copains et soit a_i le score de A au trou i pour tout $1 \leq i \leq 18$; on définit de même b_i et c_i. Soit $S_{ab} = \sum\limits_{i=1}^{18} \min(a_i, b_i)$, et de manière analogue, S_{ac} et S_{bc}. Le score total de Jim a été plus petit que S_{ab}, S_{ac} et S_{bc}. Par contre, son résultat de 72 dépassait $S_{abc} = \sum\limits_{i=1}^{18} \min(a_i, b_i, c_i)$. Quel était le score minimal possible de chacun de ses copains ?

TOTTEN–07. *Proposé par Šefket Arslanagić et Faruk Zejnulahi, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit a, b et c trois nombres réels non négatifs tels que $a^2 + b^2 + c^2 = 1$. Etudier la validité de

(a) $1 \leq \dfrac{a}{1-ab} + \dfrac{b}{1-bc} + \dfrac{c}{1-ca} \leq \dfrac{3\sqrt{3}}{2}$,

(b) $1 \leq \dfrac{a}{1+ab} + \dfrac{b}{1+bc} + \dfrac{c}{1+ca} \leq \dfrac{3\sqrt{3}}{4}$.

TOTTEN–08. *Proposé par Richard Hoshino, Gouvernement du Canada, Ottawa, ON.*

Supposons que dans le triangle ABC, on a $AB < AC$. Soit D et M les points sur le côté BC pour lesquels AD est la bissectrice et AM la médiane. Soit F le point sur le côté AC tel que AD soit perpendiculaire à DF. Soit finalement E l'intersection de AM et DF. Montrer que $AB \cdot DE + AB \cdot DF = AC \cdot EF$.

TOTTEN–09. *Proposé par Richard Hoshino, Gouvernement du Canada, Ottawa, ON.*

Soit n et k deux entiers tels que $n \geq 2$ et $k \geq 0$. On a n hôtes à dîner, assis autour d'une table ronde. Soit $g_n(k)$ le nombre de possibilités que k de ces n hôtes puissent être choisis de telle sorte qu'aucune paire d'hôtes choisis soit assis l'un à côté de l'autre. Par exemple, $g_6(0) = 1$, $g_6(1) = 6$, $g_6(2) = 9$, $g_6(3) = 2$, et $g_6(4) = 0$ pour tout $k \geq 4$. Pour tout $n \geq 2$, soit

$$f_n(x) \;=\; \sum_{k\geq 0} g_n(k)x^k\,.$$

Par exemple, $f_6(x) = 1+6x+9x^2+2x^3 = (1+2x)(1+4x+x^2)$. Déterminer tous les n pour lesquels $(1+2x)$ est un facteur de $f_n(x)$.

TOTTEN–10. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Trouver tous les triangles ABC dont la longueur des côtés sont des entiers positifs et tels que $\cos C = \frac{4}{5}$.

TOTTEN–11. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

(a) Soit x, y et z trois nombres réels positifs tels que $x + y + z = 1$. Montrer que

$$\frac{8\sqrt{3}}{9} \;\leq\; \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)\left(\frac{1}{\sqrt{z}} - \sqrt{z}\right)\,.$$

(b) $\bigstar$. Soit $n \geq 2$ et soit $x_1, x_2, \ldots, x_n$ n nombres réels positifs tels que $x_1 + x_2 + \cdots + x_n = 1$. Etudier la validité de

$$\left(\frac{n-1}{\sqrt{n}}\right)^n \;\leq\; \prod_{k=1}^{n}\left(\frac{1}{\sqrt{x_k}} - \sqrt{x_k}\right)\,.$$

TOTTEN–12. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit w, x, y et z des nombres réels positifs avec $w+x+y+z = wxyz$, et soit

$$f(x) \;=\; \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{x^3}}} \;+\; \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{x^3}}}\,.$$

Montrer que $\sqrt[3]{wxy}+\sqrt[3]{xyz}+\sqrt[3]{yzw}+\sqrt[3]{zwx} \;\geq\; f(w)+f(x)+f(y)+f(z)$.

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 March 2010**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

Problems 3451, 3452, 3453, *and* 3454 *below are dedicated by the proposers to the memory of Jim Totten.*

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3451. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $(X, <\cdot,\cdot>)$ be a real or complex inner product space and let x, y, and z be nonzero vectors in X. Prove that

$$\sum_{\text{cyclic}} \left| \frac{<z,x><x,y>}{||x||} \right|^{1/2} \leq \sum_{\text{cyclic}} \left(\frac{||x||}{||y||\,||z||} \right)^{1/2} |<y,z>| \,.$$

3452. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Prove the following and generalize these results.

(a) $\tan^2 36^\circ + \tan^2 72^\circ = 10$,

(b) $\tan^4 36^\circ + \tan^4 72^\circ = 90$,

(c) $\tan^6 36^\circ + \tan^6 72^\circ = 850$,

(d) $\tan^8 36^\circ + \tan^8 72^\circ = 8050$.

3453. *Proposed by Scott Brown, Auburn University, Montgomery, AL, USA.*

Triangle ABC has side lengths $a = BC$, $b = AC$, $c = AB$; and altitudes h_a, h_b, h_c from the vertices A, B, C, respectively. Prove that

$$8\left(\sum_{\text{cyclic}} h_a^2(h_b+h_c)\right) + 16h_ah_bh_c \leq 3\sqrt{3}\left(\sum_{\text{cyclic}} a^2(b+c)\right) + 6\sqrt{3}abc\,.$$

3454. *Proposed by Richard Hoshino, Government of Canada, Ottawa, ON.*

Let a, b, c, and d be positive real numbers such that $a+b+c+d=1$. Prove that

$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b} \;\geq\; \frac{1}{8}\,.$$

3455. *Proposed by Michel Bataille, Rouen, France.*

Find the minimum value of $x^2+y^2+z^2$ over all triples (x,y,z) of real numbers such that

$$13x^2+40y^2+45z^2-36xy+12yz+24xz \;\geq\; 2009$$

and characterize all the triples at which the minimum is attained.

3456. *Proposed by Michel Bataille, Rouen, France.*

Given a triangle ABC with circumcircle Γ, let circle Γ' centred on the line BC intersect Γ at D and D'. Denote by Q and Q' the projections of D and D' on the line AB, and by R and R' their projections on AC; assume that none of these projections coincide with a vertex of the triangle.

Show that if Γ' is orthogonal to Γ, then $\dfrac{BQ}{BQ'}=\dfrac{CR}{CR'}$. Does the converse hold?

3457. *Proposed by Michel Bataille, Rouen, France.*

Let $A_n=\sum\limits_{j=0}^{\lfloor n/2\rfloor}\dfrac{(-3)^j}{2j+1}a_j$, where $a_j=\sum\limits_{k=2j}^{n}\dbinom{k}{2j}\dfrac{k+1}{2^k}$ and n is a positive integer. Prove that $A_1+A_2+\cdots+A_n\geq n$ with equality for infinitely many n.

3458. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Determine the central angle of a sector, such that the square drawn with one vertex on each radius of the sector and two vertices on the circumference, has area equal to the square of the radius of the sector.

3459. *Proposed by Zafar Ahmed, BARC, Mumbai, India.*

Let a, b, c and p, q, r be positive real numbers. Prove that if $q^2\leq pr$ and $r^2\leq pq$, then

$$\frac{a}{pa+qb+rc}+\frac{b}{pb+qc+ra}+\frac{c}{pc+qa+rb} \;\leq\; \frac{3}{p+q+r}\,.$$

When does equality hold?

3460. *Proposed by Tran Quang Hung, student, Hanoi National University, Vietnam.*

The triangle ABC has circumcentre O, orthocentre H, and circumradius R. Prove that

$$3R - 2OH \;\leq\; HA + HB + HC \;\leq\; 3R + OH$$

3461. *Proposed by Tran Quang Hung, student, Hanoi National University, Vietnam.*

Let I be the incentre of triangle ABC and let A', B', and C' be the intersections of the rays AI, BI, and CI with the respective sides of the triangle. Prove that

$$IA + IB + IC \;\geq\; 2(IA' + IB' + IC')\,.$$

3462. *Proposed by Sotiris Louridas, Aegaleo, Greece.*

Let x, y, and z be positive real numbers such that

$$\left(x^3 + z^3 - y^3\right)\left(y^3 + x^3 - z^3\right)\left(z^3 + y^3 - x^3\right) \;>\; 0\,.$$

Prove that

$$\left(x^3 + y^3 + z^3 + 3xyz\right) \prod_{\text{cyclic}} \left(x^3 + y^3 - z^3 + xyz\right) \leq 3 \prod_{\text{cyclic}} \sqrt[3]{x^4\left(x^2 + yz\right)^4}\,.$$

. . . .

3451. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $(X, < \cdot, \cdot >)$ un espace vectoriel réel ou complexe muni d'un produit scalaire et soit x, y et z des vecteurs non nuls de X. Montrer que

$$\sum_{\text{cyclique}} \left|\frac{< z, x >< x, y >}{||x||}\right|^{1/2} \leq \sum_{\text{cyclique}} \left(\frac{||x||}{||y||\,||z||}\right)^{1/2} |< y, z >|\,.$$

3452. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Montrer les égalités suivantes et généraliser ces résultats.

(a) $\tan^2 36^\circ + \tan^2 72^\circ = 10$,

(b) $\tan^4 36^\circ + \tan^4 72^\circ = 90$,

(c) $\tan^6 36^\circ + \tan^6 72^\circ = 850$,

(d) $\tan^8 36^\circ + \tan^8 72^\circ = 8050$.

3453. *Proposé par Scott Brown, Université Auburn, Montgomery, AL, USA.*

Soit $a = BC$, $b = AC$, $c = AB$ les longueurs des côtés du triangle ABC, de hauteurs respectives h_a, h_b, h_c issues des sommets A, B, C. Montrer que

$$8\left(\sum_{\text{cyclique}} h_a^2(h_b+h_c)\right) + 16h_ah_bh_c \;\le\; 3\sqrt{3}\left(\sum_{\text{cyclique}} a^2(b+c)\right) + 6\sqrt{3}abc\,.$$

3454. *Proposé par Richard Hoshino, Gouvernement du Canada, Ottawa, ON.*

Soit a, b, c et d des nombres réels non négatifs avec $a+b+c+d=1$. Montrer que

$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b} \;\ge\; \frac{1}{8}\,.$$

3455. *Proposé by Michel Bataille, Rouen, France.*

Trouver la valeur minimale de $x^2+y^2+z^2$ pour tous les triplets (x, y, z) de nombres réels tels que

$$13x^2+40y^2+45z^2-36xy+12yz+24xz \;\ge\; 2009$$

et caractériser tous les triplets pour lesquels le minimum est atteint.

3456. *Proposé par Michel Bataille, Rouen, France.*

Étant donné un triangle ABC et Γ son cercle circonscrit, soit Γ' un cercle centré sur la droite BC et coupant Γ en D et D'. Désignons par Q et Q' les projections de D et D' sur la droite AB, et soit R et R' leurs projections sur AC ; supposons qu'aucune de ces projections ne coincide avec un sommet du triangle.

Montrer que si Γ' est orthogonal à Γ, alors $\dfrac{BQ}{BQ'} = \dfrac{CR}{CR'}$. La réciproque est-elle valide ?

3457. *Proposé par Michel Bataille, Rouen, France.*

Soit $A_n = \sum\limits_{j=0}^{\lfloor n/2\rfloor} \dfrac{(-3)^j}{2j+1}a_j$, où $a_j = \sum\limits_{k=2j}^{n} \dbinom{k}{2j}\dfrac{k+1}{2^k}$ et n est un entier positif. Montrer que $A_1 + A_2 + \cdots + A_n \ge n$ où l'on a égalité pour une infinité de n.

3458. *Proposé by Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Déterminer l'angle au centre d'un secteur tel que le carré, construit avec un sommet sur chaque rayon du secteur et deux sommets sur la circonférence, possède une aire égale au carré du rayon du secteur.

3459. *Proposé par Zafar Ahmed, BARC, Mumbai, Inde.*

Soit a, b, c et p, q, r des nombres réels positifs. Montrer que si $q^2 \leq pr$ et $r^2 \leq pq$, alors

$$\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \leq \frac{3}{p+q+r}\,.$$

Quand y a-t-il égalité ?

3460. *Proposé par Tran Quang Hung, étudiant, Université Nationale de Hanoi, Vietnam.*

Soit O le centre du cercle circonscrit du triangle ABC, de rayon R , et H son orthocentre. Montrer que

$$3R - 2OH \;\leq\; HA + HB + HC \;\leq\; 3R + OH$$

3461. *Proposed by Tran Quang Hung, étudiant, Université Nationale de Hanoi, Vietnam.*

Soit I le centre du cercle circonscrit du triangle ABC et soit A', B' et C' les intersections des rayons AI, BI et CI avec les côtés respectifs du triangle. Montrer que

$$IA + IB + IC \;\geq\; 2(IA' + IB' + IC')\,.$$

3462. *Proposé par Sotiris Louridas, Aegaleo, Grèce.*

Soit x, y et z trois nombres réels positifs tels que

$$\left(x^3+z^3-y^3\right)\left(y^3+x^3-z^3\right)\left(z^3+y^3-x^3\right) \;>\; 0\,.$$

Montrer que

$$\begin{aligned}\left(x^3+y^3+z^3+3xyz\right) \prod_{\text{cyclique}} \left(x^3+y^3-z^3+xyz\right) \\ \leq\; 3 \prod_{\text{cyclique}} \sqrt[3]{x^4\left(x^2+yz\right)^4}\,.\end{aligned}$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2557. [2000 : 304] *Proposed by Gord Sinnamon, University of Western Ontario, London, ON and Hans Heinig, McMaster University, Hamilton, ON.*

(a) Show that for all positive sequences $\{x_i\}$ and all integers $n > 0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \;\leq\; 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}.$$

(b)★ Does the above inequality remain true without the factor of 2?

(c)★ [Proposed by the editors] What is the minimum constant c that can replace the factor 2 in the above inequality?

Solution to part (c) *by Li Chao, student, High School Affiliated to Renmin University of China, Beijing, China.*

The given inequality is equivalent to

$$\sum_{i=1}^{n}\binom{n+2-i}{2}x_i \;<\; c\sum_{i=1}^{n}\frac{1}{x_i}\left(\sum_{j=1}^{i} x_j\right)^2.$$

Making the substitution

$$s_k \;=\; \sum_{i=1}^{k} x_i\,, \quad k = 0\,,1\,,2\,,\,\ldots\,,n$$

and then simplifying puts the inequality into another equivalent form:

$$\sum_{i=1}^{n}(n+1-i)s_i \;<\; c\sum_{i=1}^{n}\frac{s_i^2}{s_i - s_{i-1}}.$$

By the Cauchy–Schwartz Inequality we have

$$\begin{aligned}
\left(\sum_{i=1}^{n}(n+1-i)s_i\right)^2 \;&\leq\; \left(\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)\left(\sum_{i=1}^{n}(n+1-i)^2(s_i-s_{i-1})\right)\\
&=\; \left(\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)\left(\sum_{i=1}^{n}(2n+1-2i)s_i\right)\\
&<\; \left(\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)\left(2\sum_{i=1}^{n}(n+1-i)s_i\right).
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}(n+1-i)s_i \;<\; 2\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\,,$$

so that taking $c = 2$ ensures that the inequality holds for all positive real numbers $x_1, x_2, \ldots, x_n$.

For each $n > 4$ we will now give a strictly increasing sequence of numbers $s_0, s_1, \ldots, s_n$ such that $s_0 = 0$ and

$$\lim_{n\to\infty}\frac{\left(\displaystyle\sum_{i=1}^{n}(n+1-i)s_i\right)}{\left(\displaystyle\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)} \;=\; 2$$

(this is equivalent to specifying $x_1, x_2, \ldots, x_n$). Take $n > 4$, let $s_0 = 0$ and

$$\begin{aligned} s_i &= \frac{(n-1)(n-2)}{(n-i)(n-i-1)}\,, \quad 1 \le i \le n-2\,; \\ s_n &= 2s_{n-1} \;=\; 4s_{n-2}\,. \end{aligned}$$

Then we have

$$\begin{aligned} &\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}} \\ &= 1 + \sum_{i=2}^{n-2}\frac{\dfrac{(n-1)^2(n-2)^2}{(n-i)^2(n-i-1)^2}}{(n-1)(n-2)\left[\dfrac{1}{(n-i)(n-i-1)}-\dfrac{1}{(n-i+1)(n-i)}\right]} \\ &\quad + 6(n-1)(n-2) \\ &= 1 + \frac{(n-1)(n-2)}{2}\sum_{i=2}^{n-2}\left(\frac{2}{n-i-1}-\frac{1}{n-i}\right) \\ &\quad + 6(n-1)(n-2) \\ &= 1 + \frac{(n-1)(n-2)}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-3}+\left(1-\frac{1}{n-2}\right)\right] \\ &\quad + 6(n-1)(n-2) \\ &= 1 + \frac{(n-1)(n-2)}{2}\big[\ln n + O(1)\big] \;+\; 6(n-1)(n-2) \\ &= \frac{1}{2}n^2\ln n + O(n^2)\,, \end{aligned}$$

where the notation $f(n) = O\big(g(n)\big)$ here means that there is a positive constant C such that $|f(n)| \le C|g(n)|$ holds for sufficiently large n, and we have used the fact that $\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\ln n$ tends to Euler's constant, γ, as n tends to infinity.

We also have

$$\begin{aligned}
& \sum_{i=1}^{n}(n+1-i)s_i \\
= & \sum_{i=1}^{n-2}(n-i+1)\frac{(n-1)(n-2)}{(n-i)(n-i-1)} \\
& \quad + 4(n-1)(n-2) \\
= & (n-1)(n-2)\sum_{i=1}^{n-2}\left(\frac{2}{n-i-1}-\frac{1}{n-i}\right) \\
& \quad + 4(n-1)(n-2) \\
= & (n-1)(n-2)\left[1+\frac{1}{2}+\cdots+\frac{1}{n-2}+\left(1-\frac{1}{n-1}\right)\right] \\
& \quad + 4(n-1)(n-2) \\
= & (n-1)(n-2)\left[\ln n+O(1)\right] \;+\; 4(n-1)(n-2) \\
= & n^2\ln n+O(n^2)\,.
\end{aligned}$$

We now compute

$$\begin{aligned}
\lim_{n\to\infty}\frac{\left(\displaystyle\sum_{i=1}^{n}(n+1-i)s_i\right)}{\left(\displaystyle\sum_{i=1}^{n}\frac{s_i^2}{s_i-s_{i-1}}\right)} &= \lim_{n\to\infty}\frac{n^2\ln n+O(n^2)}{\frac{1}{2}n^2\ln n+O(n^2)} \\
&= \lim_{n\to\infty}\frac{2+2O(n^2)/n^2\ln n}{1+2O(n^2)/n^2\ln n} \;=\; 2\,,
\end{aligned}$$

as desired.

Therefore, the minimum value of the constant c is 2.

No other solutions to part (c) *were submitted.*

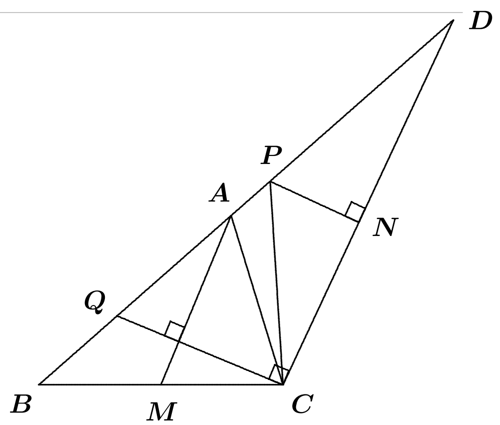
3351. [2008 : 298, 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $AB > AC$. Let P be a point on the line AB beyond A such that $AP+PC=AB$. Let M be the midpoint of BC, and let Q be the point on the side AB such that $CQ\perp AM$. Prove that $BQ=2AP$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Taichi Maekawa, Takatsuki City, Osaka, Japan.

Let D be the point on the line PB beyond P such that $PD = PC$. Since

$$\begin{aligned} AD &= AP + PD \\ &= AP + PC = AB, \end{aligned}$$



A is the midpoint of the segment BD. It follows that $AM || DC$, because M is the midpoint of BC. Hence $\angle DCQ = 90°$. Let $PN \perp DC$ with point N on the line CD. Then $DN = NC$, because triangle CDP is isosceles. Since N is the midpoint of CD and $PN \parallel QC$, it follows that P is the midpoint of QD. Thus $PQ = PD = PC$. Hence

$$AP + AQ = PQ = PC = AB - AP,$$

and therefore,

$$2AP = AB - AQ = BQ,$$

as claimed.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHRIS BROYLES and MATTHEW STEIN, Southeast Missouri State University, Cape Girardeau, MO, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany (two solutions); JOHN G. HEUVER, Grande Prairie, AB; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3352. [2008 : 298, 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a right-angled triangle with right angle at A. Let I be the incentre of $\triangle ABC$, and let D and E be the intersections of BI and CI with AC and AB, respectively. Prove that

$$\frac{BI \cdot ID}{CI \cdot IE} = \frac{AB}{AC}.$$

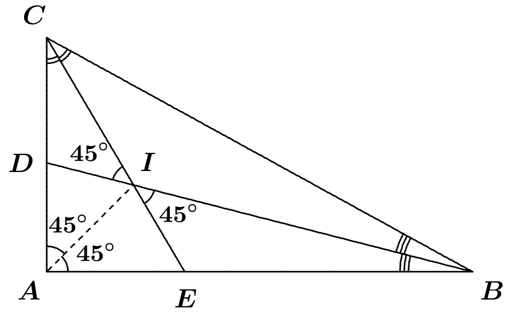
Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India.

Since AI bisects the right angle at A, we have

$$\angle CID = \frac{1}{2}\angle B + \frac{1}{2}\angle C = \frac{1}{2}\bigl(180^\circ - \angle A\bigr) = 45^\circ = \angle CAI\,.$$

Hence, triangles CAI and CID are similar, so that

$$\frac{ID}{CI} = \frac{AI}{AC}\,.$$



Likewise, triangles BAI and BIE are similar, and therefore,

$$\frac{BI}{IE} = \frac{AB}{AI}\,.$$

The result now follows by multiplying across the last two equations.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3353. [2008 : 298, 301] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle all of whose side lengths are positive integers.

(a) Determine all such triangles where one angle has twice the measure of a second angle.

(b) Determine all such triangles where two medians are perpendicular.

Composite of solutions by Roy Barbara, Lebanese University, Fanar, Lebanon and Michel Bataille, Rouen, France.

We denote by α, β, and γ the angles opposite the sides $BC = a$, $CA = b$, and $AB = c$ respectively.

(a) Without loss of generality we take $\gamma = 2\alpha$ and show that the solutions are the triangles with

$$a \;=\; dn^2\,, \qquad b \;=\; d\big(m^2 - n^2\big)\,, \qquad c \;=\; dmn\,, \tag{1}$$

where d, m, and n are positive integers, m is coprime to n, and $n < m < 2n$.

We use an equivalence proved in the April 2006 issue (see [2006 : 159]) of this journal: A triangle satisfies $\gamma = 2\alpha$ if and only if $c^2 = a(a+b)$.

Now suppose that a, b, and c are the sides of a triangle with $\gamma = 2\alpha$. Let $d = \gcd(a,b)$. Then $a = da'$ and $b = db'$ with a' coprime to b'. Clearly, d also divides c, so $c = dc'$.

Since a' and b' are coprime, then a' and $a' + b'$ are also coprime. Since $(c')^2 = a'(a' + b')$, it follows that there exist coprime positive integers m and n such that $a' = n^2$ and $a' + b' = m^2$.

Thus, (1) holds for positive integers d, m, and n with m coprime to n and $n < m$.

Finally, since ABC is a triangle, $a + c > b$, hence $2n^2 > m(m-n)$, hence $m < 2n$.

Conversely, if (1) holds for positive integers d, m, and n with the aforementioned restrictions, then the inequality $n < m < 2n$ ensures that a, b, and c are the side lengths of a triangle and also the relation $c^2 = a(a+b)$ holds, which ensures that $\gamma = 2\alpha$.

(b) Without loss of generality, we restrict the problem to the case where the medians from B and C are perpendicular.

First we show that the positive real numbers a, b, and c are the sides of a triangle ABC where the medians from B and C are perpendicular if and only if

$$b^2 + c^2 \;=\; 5a^2 \quad\text{and}\quad bc \;>\; 2a^2\,. \tag{2}$$

If G is the centroid of ABC, then BG is perpendicular to CG if and only if $BC^2 = BG^2 + CG^2$, or

$$\frac{1}{9}\left(2a^2 + 2c^2 - b^2\right) \;+\; \frac{1}{9}\left(2a^2 + 2b^2 - c^2\right) \;=\; a^2\,,$$

that is, $b^2 + c^2 = 5a^2$.

In addition, $1 > \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{2a^2}{bc}$, hence $bc > 2a^2$.

Conversely, if the conditions in (2) hold, then $(b+c)^2 = 5a^2 + 2bc > a^2$ and $(b-c)^2 = 5a^2 - 2bc < a^2$. Hence, $|b - c| < a < b + c$ and there exists a triangle ABC with sides $BC = a$, $AC = b$, and $AB = c$. The relation $b^2 + c^2 = 5a^2$ holds for this triangle, so the medians from B and C are perpendicular.

Thus the problem amounts to finding the positive integers a, b, and c that satisfy (2). Because of the homogeneity of the conditions, we need only find the solutions such that $\gcd(a,b,c) = 1$; the general solution is then obtained by multiplying these *primitive* solutions by positive integers.

Now let a, b, c be a primitive solution. Then there exist coprime positive integers m and n for which $\dfrac{b-a}{2a-c} = \dfrac{2a+c}{b+a} = \dfrac{m}{n}$, and thus

$$\begin{aligned} a(2m+n) - nb - mc &= 0\,, \\ a(m-2n) + mb - nc &= 0\,. \end{aligned}$$

Solving for a, b, and c yields

$$\begin{aligned} a &= \lambda\left(m^2+n^2\right)\,, \\ b &= \lambda\left(n^2-m^2+4mn\right)\,, \\ c &= 2\lambda\left(m^2-n^2+mn\right)\,, \end{aligned}$$

where λ is a positive rational number. The condition $bc > 2a^2$ reduces to

$$\left(m^2-n^2\right)\left(3mn+2n^2-2m^2\right) \;>\; 0\,,$$

which holds if and only if $n < m < 2n$.

Writing $\lambda = \dfrac{u}{v}$, where u and v are positive coprime integers, we obtain

$$\begin{aligned} va &= u\left(m^2+n^2\right)\,, \\ vb &= u\left(n^2-m^2+4mn\right)\,, \\ vc &= 2u\left(m^2-n^2+mn\right)\,. \end{aligned}$$

Now, $\gcd(u,v) = 1$ and $\gcd(a,b,c) = 1$, hence $u = 1$ and

$$(a,b,c) \;=\; \left(\frac{m^2+n^2}{v}, \frac{n^2-m^2+4mn}{v}, \frac{2\left(m^2-n^2+mn\right)}{v}\right)\,, \tag{3}$$

where $v = \gcd\left(m^2+n^2, n^2-m^2+4mn, 2\left(m^2-n^2+mn\right)\right)$.

Conversely, it is easy to show that a triple such as in (3) with m and n coprime and $n < m < 2n$ is a primitive solution.

If p is prime, k is a positive integer, and p^k divides v, then p^k divides both m^2+n^2 and $2\left(n^2-m^2+4mn\right) + 2\left(m^2-n^2+mn\right) = 10mn$.

However, p does not divide n or m, since otherwise it would divide both n and m, and so p^k divides 10. Hence, $p = 2$ or $p = 5$ and $k = 1$, which implies that $v \in \{1,\ 2,\ 5,\ 10\}$.

If v is even then, since v divides m^2+n^2, m, n must be odd.

If 5 divides v then 5 divides $\left(m^2+n^2\right)+\left(n^2-m^2+4mn\right) = 2n(n+2m)$ and $\left(n^2-m^2+4mn\right) - \left(m^2+n^2\right) = 2m(2n-m)$. Since 5 does not divide m or n, then 5 divides both $n+2m$ and $2n-m$. Note that $n+2m = 5k$ and $2n-m = 5l$ is equivalent to $m = 2k-l$ and $n = k+2l$, and that in this case $n < m < 2n$ becomes $0 < 3l < k$.

A complete description of the primitive solutions can now be given:

- $(a,b,c) \;=\; \left(\dfrac{m^2+n^2}{10}, \dfrac{n^2-m^2+4mn}{10}, \dfrac{2(m^2-n^2+mn)}{10}\right)$, where $m = 2k-l$, $n = k+2l$; $\gcd(k,l) = 1$; k, l are odd; and $0 < 3l < k$.

- $(a,b,c) = \left(\frac{m^2+n^2}{5}, \frac{n^2-m^2+4mn}{5}, \frac{2(m^2-n^2+mn)}{5}\right)$, where $m = 2k-l$, $n = k+2l$; $\gcd(k,l) = 1$; k, l have opposite parity; and $0 < 3l < k$.

- $(a,b,c) = \left(\frac{m^2+n^2}{2}, \frac{n^2-m^2+4mn}{2}, \frac{2(m^2-n^2+mn)}{2}\right)$, where $\gcd(m,n) = 1$; m, n are odd; $n+2m$ is not divisible by 5; and $0 < n < m < 2n$.

- $(a,b,c) = \left(m^2+n^2, n^2-m^2+4mn, 2(m^2-n^2+mn\right)$, where $\gcd(m,n) = 1$; m, n have opposite parity; $n+2m$ is not divisible by 5; and $0 < n < m < 2n$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (part (a) only); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, CA, USA (part (a) only); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) only); RODOLFO LARREA and LUZ RONCAL, Logroño, Spain (part (b) only); MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India (part (a) only); and TITU ZVONARU, Cománeşti, Romania (part (a) only). There were three incomplete solutions submitted.

Modak remarks that Theorem 44 of [1] gives all integer solutions to $b^2+c^2 = 5a^2$, while Janous refers to [2] where solutions to this equation are obtained by factoring over the Gaussian integers and using the fact that 5 is a sum of two squares.

Geupel notes that Problem 129(a) of [3] asks for the smallest integral triangle for which one angle is twice another.

References

[1] L.E. Dickson, *Introduction to the Theory of Numbers*, Dover Publ., 1957.

[2] L.J. Mordell, *Dophantine Equations*, Academic Press, London and New York, 1969.

[3] D.O. Shklarsky, N.N. Khentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book*, Dover Publ., New York, 1962.

3354. [2008 : 298, 301] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{n^2+k^2}{n^2}\right)^{k^3/n^4}.$$

Composite of nearly identical solutions submitted by all the solvers whose names appear below.

The function $f(x) = x^3 \ln\left(1+x^2\right)$ is continuous and hence Riemann integrable on $[0, 1]$. Using integration by parts, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{n^2+k^2}{n^2}\right)^{k^3/n^4} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\ln\left(1+\left(\frac{k}{n}\right)^2\right)$$

$$= \int_0^1 x^3\ln\left(1+x^2\right)\,dx = \frac{1}{4}x^4\ln\left(1+x^2\right)\Big|_0^1 - \frac{1}{2}\int_0^1\frac{x^5}{1+x^2}\,dx$$

$$= \frac{1}{4}\ln 2 - \frac{1}{2}\int_0^1\left(x^3-x+\frac{x}{1+x^2}\right)dx$$

$$= \frac{1}{4}\ln 2 - \frac{1}{2}\left(\frac{1}{4}x^4-\frac{1}{2}x^2+\frac{1}{2}\ln\left(1+x^2\right)\right)\Big|_0^1 = \frac{1}{8}\,.$$

Solvers: GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; OVIDIU FURDUI, Campia Turzii, Cluj, Romania; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3355. [2008 : 299, 301] *Proposed by Todor Yalamov, Sofia University, Sofia, Bulgaria.*

For the triangle ABC let $(x, y)_{ABC}$ denote the line which intersects the union of the segments AB and BC in X and the segment AC in Y such that

$$\frac{\widetilde{AX}}{AB+BC} = \frac{AY}{AC} = \frac{x\cdot AB+y\cdot BC}{(x+y)(AB+BC)}\,,$$

where $\widetilde{AX}$ is either the length of the segment AX if X lies between A and B, or the sum of the lengths of the segments AB and BX if X lies between B and C. Prove that the three lines $(x, y)_{ABC}$, $(x, y)_{BCA}$, and $(x, y)_{CBA}$ intersect in a point dividing the segment NI in the ratio $x : y$, where N is the Nagel point and I the incentre of $\triangle ABC$.

Solution outline by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

As usual we let $a = BC$, $b = CA$, $c = AB$, and $s = \dfrac{a+b+c}{2}$. Set $t = \dfrac{x}{x+y}$; thus, $1 - t = \dfrac{y}{x+y}$, and the ratio of interest becomes a function of t, namely

$$f(t) \;=\; \frac{tc + (1-t)a}{a+c} \;=\; \frac{\widetilde{AX}_t}{a+c} \;=\; \frac{AY_t}{b}\,,$$

where X_t is the point of $AB \cup BC$, and Y_t of AC, that correspond to our parameter t. In this notation, $(x,y)_{ABC} = X_tY_t$. Of course, when $a = c$, $f(t)$ is constant and $(x,y)_{ABC}$ is the line NI for all x, y. This is consistent with what we are to prove unless $a = b = c$; when $\triangle ABC$ is equilateral we have $N = I$ (so there is no line NI), and our three lines intersect at that point. Let us therefore assume that $a \neq c$, so that $f(t)$ is not constant and the lines NI and $(x,y)_{ABC}$ intersect.

We shall see that $(x,y)_{ABC}$ is the line in the family of lines parallel to the bisector of $\angle B$ (where B is the middle vertex in the subscript) that divides the segment NI in the ratio $t:(1-t) = x:y$. Note that as a consequence, $(x,y)_{ABC}$ and $(x,y)_{CBA}$ represent the same line, so the proposer probably intended $(x,y)_{CAB}$ for his third line. For this problem we are concerned with the domain $0 \le t \le 1$; in particular,

(a) $AY_1 = bf(1) = \dfrac{bc}{a+c}$; $CY_1 = b - AY_1 = \dfrac{ab}{a+c}$; Y_1 is the foot of the bisector of $\angle CBA$ (because it divides the side AC in the ratio $c:a$);

(b) $AY_0 = bf(0) = \dfrac{ab}{a+c}$; $CY_0 = \dfrac{bc}{a+c}$;

(c) $X_1 = B$ (because $f(1) = \dfrac{c}{a+c} = \dfrac{AX_1}{a+c}$);

(d) $\widetilde{AX}_0 = (a+c)f(0) = a$: when $c \ge a$, X_0 lies on AB (whence $AX_0 = a$ and $BX_0 = c - a$); otherwise, when $a \ge c$, X_0 lies on BC (whence $CX_0 = c$ and $BX_0 = a - c$).

By items (a) and (c), the line X_1Y_1 bisects $\angle CBA$ and, therefore, it passes through the incentre I. We next will see that the line X_0Y_0 passes through the Nagel point N. To determine N we use the points P, Q, R where the excircles meet the sides BC, CA, AB of $\triangle ABC$, whence

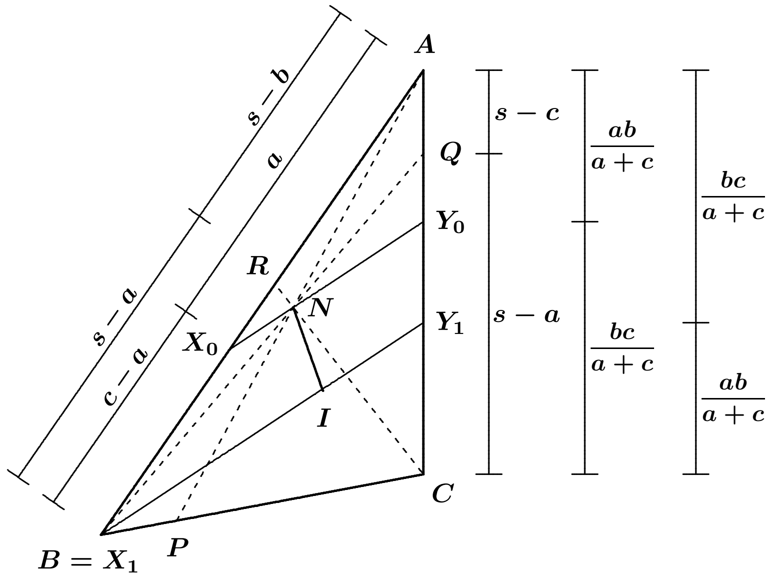
$$BR = CQ = s-a\,, \quad AR = CP = s-b\,, \quad \text{and} \quad AQ = BP = s-c\,.$$

The Nagel point is defined to be the point common to AP, BQ, and CR. Apply Menelaus' theorem to transversal NCR of $\triangle BQA$ to deduce that

$$\frac{BN}{NQ} \;=\; \frac{CA}{QC}\cdot\frac{RB}{AR} \;=\; \frac{b}{s-a}\cdot\frac{s-a}{s-b} \;=\; \frac{b}{s-b}\,. \tag{1}$$

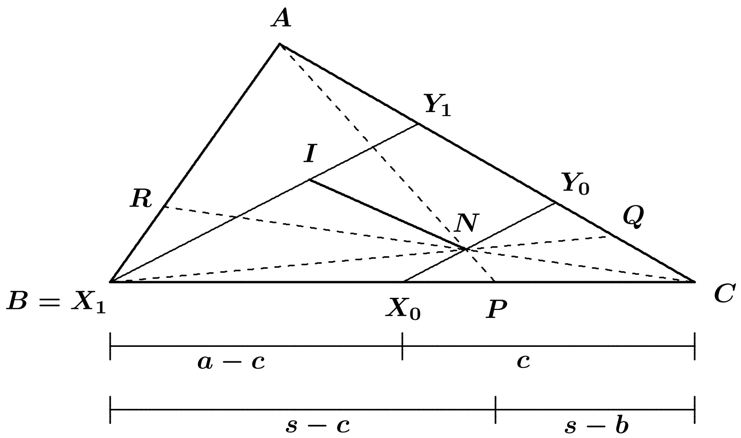
Let $N' = X_0Y_0 \cap BQ$; we want to show that $N' = N$. Here we will need the length

$$QY_0 \;=\; |AY_0 - AQ| \;=\; \left|\frac{ab}{a+c} - (s-c)\right| \;=\; \frac{|a-c|(s-b)}{a+c}\,.$$



When $c > a$ we apply Menelaus' theorem to transversal $N'Y_0X_0$ of $\triangle BQA$ to get

$$\frac{BN'}{N'Q} = \frac{Y_0A}{QY_0} \cdot \frac{X_0B}{AX_0} = \frac{ab}{a+c} \cdot \frac{a+c}{(c-a)(s-b)} \cdot \frac{c-a}{a} = \frac{b}{s-b}.$$



Otherwise, when $a > c$ we apply Menelaus' theorem to transversal $N'Y_0X_0$ of $\triangle BCQ$ to get

$$\frac{BN'}{N'Q} = \frac{Y_0C}{QY_0} \cdot \frac{X_0B}{CX_0} = \frac{bc}{a+c} \cdot \frac{a+c}{(a-c)(s-b)} \cdot \frac{a-c}{c} = \frac{b}{s-b}.$$

In both cases $\frac{BN'}{N'Q}$ equals the value of $\frac{BN}{NQ}$ in equation (1), from which we conclude that $N = N'$, and X_0Y_0 intersects IN at N, as claimed.

It remains to observe that $X_0Y_0 \parallel X_1Y_1$: when $c > a$ (that is, when X_0 lies on AB),

$$AX_0 : AX_1 \;=\; a : c \;=\; \frac{ab}{a+c} : \frac{bc}{a+c} \;=\; AY_0 : AY_1\,;$$

when $a > c$ (and X_0 lies on BC),

$$CX_0 : CX_1 \;=\; c : a \;=\; \frac{bc}{a+c} : \frac{ab}{a+c} \;=\; CY_0 : CY_1\,.$$

Finally, because X_t divides the segment X_0X_1 in the ratio $t : (1-t)$ while Y_t divides Y_0Y_1 in that same ratio, X_tY_t is parallel to both X_0Y_0 and X_1Y_1 for all t. It follows that X_tY_t meets the segment NI in a point that divides it in that same ratio $t : (1-t) = x : y$, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Geupel remarked that the Spieker centre is the midpoint of NI (on the line $X_{1/2}Y_{1/2}$) while the centroid of the triangle is a trisector of NI (on the line $X_{2/3}Y_{2/3}$).

3356. [2008 : 299, 301] *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let $f : [0,\infty) \to \mathbb{R}$ be integrable on $[0,1]$ and have period 1 (that is, $f(x+1) = f(x)$ for all $x \in [0,\infty)$). If $\{x_n\}_{n=0}^{\infty}$ is any strictly increasing, unbounded sequence with $x_0 = 0$ for which $(x_{n+1} - x_n) \to 0$, denote

$$r(n) \;=\; \max\{k \in \mathbb{N} \mid x_k \le n\}\,.$$

(a) Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{r(n)} (x_k - x_{k-1}) f(x_k) \;=\; \int_0^1 f(x)\,dx\,.$$

(b) Prove that

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{f(\ln k)}{k} \;=\; \int_0^1 f(x)\,dx\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $I = \int_0^1 f(x)\,dx$ and

$$S_n \;=\; \sum_{k=r(n)+1}^{r(n+1)} (x_k - x_{k-1}) f(x_k)\,.$$

We first prove that

$$\lim_{n\to\infty} S_n = I\,. \tag{1}$$

Consider the following partition of the interval $[n, n+1]$

$$n \;<\; x_{r(n)+1} \;<\; x_{r(n)+2} \;<\; \cdots \;<\; x_{r(n+1)} \;\le\; n+1\,,$$

where the lengths of the resulting intervals between the points are denoted by Δ_1, Δ_2, ..., Δ_m and $m = r(n+1) - r(n) + 1$. Consider the Riemann sum $R_n = \sum_{k=1}^{m} f(x_k^*)\Delta_k$, where x_k^* is the right endpoint of the k^{th} interval. Because f is bounded and $(x_{k+1} - x_k) \to 0$, we obtain

$$S_n \;=\; R_n + \big(n - x_{r(n)}\big) f(x_{r(n)+1}) - \big(n+1-x_{r(n+1)}\big) f(n+1) \to I$$

as $n \to \infty$.

Next we prove the limit in part (a) in the following equivalent form:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} S_k \;=\; I\,. \tag{2}$$

Let $\epsilon > 0$ be given. By (1) there is an integer N such that $|S_k - I| < \frac{\epsilon}{2}$ whenever $k \ge N$. Let $S = \left|\sum_{k=0}^{N-1}(S_k - I)\right|$. Whenever $n > \max\left\{N,\ \frac{2S}{\epsilon}\right\}$ we have

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=0}^{n-1} S_k - I\right| \;&\le\; \frac{1}{n}\left|\sum_{k=0}^{N-1}(S_k - I)\right| \;+\; \frac{1}{n}\left|\sum_{k=N}^{n-1}(S_k - I)\right| \\ &\le\; \frac{S}{n} \;+\; \frac{n-N}{n}\cdot\frac{\epsilon}{2} \;<\; \epsilon\end{aligned}$$

and (2) is proved.

For $B > 0$, let $r(B) = \max\{k \mid x_k \le B\}$. We next prove that

$$\lim_{B\to\infty} \frac{1}{B}\sum_{k=1}^{r(B)} (x_k - x_{k-1}) f(x_k) \;=\; I\,. \tag{3}$$

Let $n = \lfloor B \rfloor$, $U_n = \sum_{k=0}^{n-1} S_k$, and $V_n = \sum_{k=r(n)+1}^{r(B)} (x_k - x_{k-1}) f(x_k)$. Then $\sum_{k=1}^{r(B)} (x_k - x_{k-1}) f(x_k) \;=\; U_n + V_n$ and by (2) we have $\lim_{n\to\infty} \frac{1}{n} U_n \;=\; I$; therefore, it suffices to prove that $\frac{1}{B}(U_n + V_n) - \frac{1}{n}U_n \to 0$ as $B \to \infty$. But this follows from the fact that V_n is bounded and the calculation

$$\frac{1}{B}(U_n + V_n) - \frac{1}{n}U_n \;=\; \frac{V_n}{B} - \left(\frac{B - \lfloor B \rfloor}{B}\right)\left(\frac{U_n}{n}\right) \to 0 - 0\cdot I \;=\; 0\,.$$

Finally, we prove the limit in part (b). Let

$$T_n \;=\; \frac{1}{\ln n}\sum_{k=1}^{n}\ln\left(1+\frac{1}{k-1}\right)f(\ln k)\,; \qquad W_n \;=\; \frac{1}{\ln n}\sum_{k=1}^{n}\frac{f(\ln k)}{k}\,.$$

We must show that $W_n \to I$. It follows from (3) with $x_k = \ln k$, $(k > 0)$ and $n = e^B$ that $T_n \to I$. Therefore, it suffices to prove $T_n - W_n \to 0$. Let $\epsilon > 0$ be given and let $|f| < C$, where C is a constant. Note that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = e$, so there is an integer K such that

$$\left|\ln\left(1+\frac{1}{k-1}\right)^k - 1\right| \;<\; \frac{\epsilon}{2C}$$

whenever $k > K$. Moreover, K can be chosen so that $H_K > H_n - \ln n$ for each n, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n^{th} Harmonic number. Let

$$T \;=\; \left|\sum_{k=1}^{K}\frac{1}{k}\left(\ln\left(1+\frac{1}{k-1}\right)^k - 1\right)f(\ln k)\right|\,.$$

Now if $n > \max\{K,\, e^{2T/\epsilon}\}$, then

$$\begin{aligned}|T_n - W_n| \;&\le\; \frac{T}{\ln n} \;+\; \frac{1}{\ln n}\left|\sum_{k=K+1}^{n}\frac{1}{k}\left(\ln\left(1+\frac{1}{k-1}\right)^k - 1\right)f(\ln k)\right| \\ &<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon\,.\end{aligned}$$

This completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France (*part* (a) *only*)*; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India* (*part* (a) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA* (*part* (a) *only*)*; and the proposer. There were two incomplete solutions submitted to part* (b).

The fact that $S_n \to I$ *as* $n \to \infty$ *implies that* $\frac{1}{n}\sum\limits_{k=0}^{n-1} S_k \to I$ *as* $n \to \infty$ *is part of the theory of Cesàro sums, which some solvers quoted directly.*

3357. [2008 : 172, 175] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let a be a real number such that $-1 < a \le 1$. Prove that

$$\int_0^1 \frac{x+a}{x^2+2ax+1}\ln(1-x)\,dx \;=\; \frac{1}{2}\ln^2\left(2\sin\frac{\theta}{2}\right) + \frac{\theta^2}{8} - \frac{\theta\pi}{4} + \frac{\pi^2}{24}\,,$$

where θ is the unique solution in $(0,\pi]$ of the equation $\cos\theta = -a$.

Solution by Michel Bataille, Rouen, France.

Let

$$f(a) \;=\; \int_0^1 \frac{x+a}{x^2+2ax+1}\ln(1-x)\,dx\,.$$

For $a \in (-1,1]$ and $x \in [0,1)$, we have

$$\left|\frac{x+a}{x^2+2ax+1}\ln(1-x)\right| \;\le\; \frac{|\ln(1-x)|}{1+x} \;\le\; |\ln(1-x)|\,.$$

Since $|\ln(1-x)| = -\ln(1-x)$ is integrable on $[0,1)$, we see that $f(a)$ is well-defined and continuous on $(-1,1]$. [*Ed.: It seems one must also compute* $\left|\dfrac{x+a}{x^2+2ax+1} - \dfrac{x+b}{x^2+2bx+1}\right| = \dfrac{|b-a|\,(1-x^2)}{(x^2+2ax+1)\,(x^2+2bx+1)}$ *and note that* $x^2+2ax+1$ *on* $[0,1)$ *is bounded below by* 1 *if* $0 \le a \le 1$ *or by* $1-a^2$ *if* $-1 < a < 0$*, and then combine this with the salient observation above.*]
For $s \in (0,1)$ we have

$$\begin{aligned}
&\int_0^{1-s} \frac{2(x+a)}{x^2+2ax+1}\ln(1-x)\,dx \\
&\quad= \int_0^{1-s} \ln(1-x)\cdot d\big[\ln\left(x^2+2ax+1\right)\big] \\
&\quad= \ln s\cdot\ln\big((1-s)^2+2a(1-s)+1\big) \;+\; \int_0^{1-s}\frac{\ln\left(x^2+2ax+1\right)}{1-x}\,dx \\
&\quad= \ln s\cdot\ln\left(s^2-(2+2a)s+2+2a\right) \\
&\qquad+ \int_u^1 \frac{\ln\left(u^2-(2+2a)u+2+2a\right)}{u}\,du \\
&\quad= \ln s\cdot\ln(2+2a) + \ln s\cdot\ln\left(1-s+\frac{s^2}{2+2a}\right) \\
&\qquad+ \int_s^1 \frac{\ln(2+2a)}{u}\,du \;+\; \int_s^1 \frac{\ln\left(1-u+\dfrac{u^2}{2+2a}\right)}{u}\,du \\
&\quad= \ln s\cdot\ln(2+2a) + \varepsilon(s) - \ln s\cdot\ln(2+2a) \\
&\qquad+ \int_s^1 \frac{\ln\left(1-u+\dfrac{u^2}{2+2a}\right)}{u}\,du\,,
\end{aligned}$$

where $\varepsilon(s) \to 0$ as $s \to 0^+$.

It follows that

$$2f(a) \;=\; \int_0^1 \frac{\ln\left(1-u+\dfrac{u^2}{2+2a}\right)}{u}\,du$$

or $2f(a) = g(\theta)$ where

$$g(\theta) \;=\; \int_0^1 \frac{\ln\left(1 - u + \dfrac{u^2}{4\sin^2(\theta/2)}\right)}{u}\,du\,. \tag{1}$$

First, we consider the case $a = 1$ that is, $\theta = \pi$. Making the substitution $u = 2v$, we have

$$f(1) \;=\; \frac{1}{2}\int_0^1 \frac{\ln\left(1 - u + \dfrac{u^2}{4}\right)}{u}\,du \;=\; \int_0^{\frac{1}{2}} \frac{\ln(1-v)}{v}\,dv\,.$$

Next, integrating by parts and then using the well-known identity

$$\int_0^1 \frac{\ln(1-t)}{t}\,dt \;=\; -\int_0^1 \left(\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}\right) dt \;=\; -\sum_{n=1}^{\infty} \frac{1}{n^2}$$

we obtain

$$\begin{aligned}
f(1) &= (\ln 2)^2 + \int_0^{\frac{1}{2}} \frac{\ln(v)}{1-v}\,dv \\
&= (\ln 2)^2 + \int_0^1 \frac{\ln(v)}{1-v}\,dv \;-\; \int_{\frac{1}{2}}^1 \frac{\ln(v)}{1-v}\,dv \\
&= (\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt \;-\; \int_0^{\frac{1}{2}} \frac{\ln(1-t)}{t}\,dt \\
&= (\ln 2)^2 - \frac{\pi^2}{6} - f(1)\,,
\end{aligned}$$

so that $f(1) = \frac{1}{2}(\ln 2)^2 - \dfrac{\pi^2}{12}$, in agreement with the given formula.

To determine $g(\theta)$ for $\theta \in (0,\pi)$, we differentiate under the integral sign in equation (1):

$$g'(\theta) \;=\; -\frac{\cos(\theta/2)}{\sin(\theta/2)} \int_0^1 \frac{u}{u^2 - 4u\sin^2(\theta/2) + 4\sin^2(\theta/2)}\,du \tag{2}$$

(To justify this, note that if $\beta \in \left(0, \dfrac{\pi}{2}\right)$ and $\theta \in [\beta, \pi - \beta]$, then for all $u \in [0,1]$,

$$\left|\frac{\cos(\theta/2)}{\sin(\theta/2)} \cdot \frac{u}{u^2 - 4u\sin^2(\theta/2) + 4\sin^2(\theta/2)}\right| \;\le\; \frac{u\cot(\beta/2)}{u^2 + 4\sin^2(\beta/2)(1-u)}\,,$$

and the dominating function is integrable on $[0,1]$.) [*Ed.: One may rewrite the integrand in* (1) *as the integral of its partial derivative with respect to* θ*, then change the order of integration by Fubini's Theorem by the solver's remark, then differentiate each side with respect to* θ *to obtain* (2) .]

Writing the numerator u as $\frac{1}{2}\big(2u - 4\sin^2(\theta/2)\big) + 2\sin^2(\theta/2)$, the calculation of the integral in (2) is straightforward and gives

$$g'(\theta) \;=\; 2\cdot\frac{\frac{1}{2}\cos(\theta/2)}{\sin(\theta/2)}\ln\left(2\sin\frac{\theta}{2}\right)+\frac{\theta}{2}-\frac{\pi}{2}\,.$$

It follows that

$$g(\theta) \;=\; \big(\ln(2\sin(\theta/2))\big)^2 \;+\; \frac{\theta^2}{4}-\frac{\pi\theta}{2}+C$$

for some constant C. This constant is $\frac{\pi^2}{12}$, easily determined using the continuity of g and the already found value of $f(1) = g(\pi)$. Finally,

$$f(a) \;=\; \frac{1}{2}g(\theta) \;=\; \frac{1}{2}\ln^2\left(2\sin\frac{\theta}{2}\right) \;+\; \frac{\theta^2}{8}-\frac{\theta\pi}{4}+\frac{\pi^2}{24}\,.$$

Also solved by the proposer. There was one incorrect submission.

3358. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

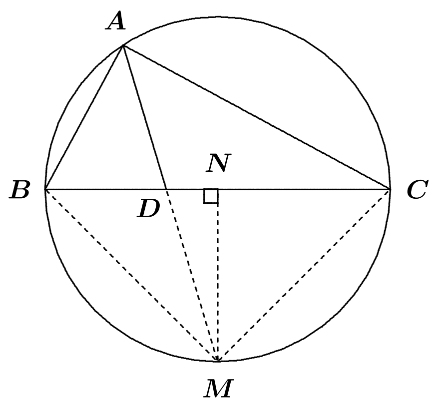
The interior bisector of $\angle BAC$ of triangle ABC meets BC at D. Suppose that

$$\frac{1}{BD^2}+\frac{1}{CD^2} \;=\; \frac{2}{AD^2}\,.$$

Prove that $\angle BAC = 90^\circ$.

I. *Solution by the proposer.*

Let M be the second intersection of AD with the circumcircle of $\triangle ABC$ and let N be the midpoint of BC. Since $\angle BAM = \angle MAC$, we have that $BM = MC$ and $MN \perp BC$. Since $BD\cdot CD = AD\cdot DM$, we obtain



$$\begin{aligned}\frac{2}{AD^2} &= \frac{1}{BD^2}+\frac{1}{CD^2}\\ &= \frac{BD^2+CD^2}{BD^2\cdot CD^2}\\ &= \frac{BD^2+CD^2}{AD^2\cdot DM^2}\,,\end{aligned}$$

hence,

$$2DM^2 \;=\; BD^2+CD^2\,. \qquad (1)$$

Since N is the midpoint of BC,

$$BD^2+CD^2 \;=\; 2\left(DN^2+BN^2\right)\,.$$

Thus, we have from equation (1) that

$$DM^2 = DN^2 + BN^2 . \qquad (2)$$

Since $\angle DNM = 90^\circ$, we have $DM^2 = DN^2 + MN^2$, and comparing this with (2) yields $BN^2 = MN^2$, that is, $BN = MN$. We have now deduced that $\angle MBN = 45^\circ$.

Therefore,

$$\angle BAC = 2\angle MAC = 2\angle MBC = 2\angle MBN = 90^\circ .$$

II. *Solution by Oliver Geupel, Brühl, NRW, Germany.*

We shall prove the following generalization:

$$\frac{2}{AD^2} - \frac{1}{BD^2} - \frac{1}{CD^2} \begin{cases} < 0 & \text{if} \quad A < 90^\circ , \\ = 0 & \text{if} \quad A = 90^\circ , \\ > 0 & \text{if} \quad A > 90^\circ . \end{cases}$$

Let $a = BC$, $b = CA$, $c = AB$, $p = BD$, $q = CD$, and $w = AD$. It is a well-known fact that each interior angle bisector divides the opposite side in the ratio of the other two sides. Hence, $p = \dfrac{ac}{b+c}$ and $q = \dfrac{ab}{b+c}$. Another common formula is $w^2 = \dfrac{4b^2c^2\cos^2(A/2)}{(b+c)^2}$. Using these relations as well as the Law of Cosines, we derive

$$\begin{aligned}
& \frac{1}{\dfrac{1}{p^2} + \dfrac{1}{q^2}} - \frac{1}{\dfrac{2}{w^2}} \\
= \;& \frac{1}{\dfrac{(b+c)^2}{a^2c^2} + \dfrac{(b+c)^2}{a^2b^2}} - \frac{2b^2c^2\cos^2\dfrac{A}{2}}{(b+c)^2} \\
= \;& \frac{b^2c^2}{(b+c)^2}\left(\frac{a^2}{b^2+c^2} - 2\cos^2\frac{A}{2}\right) \\
= \;& \frac{b^2c^2}{(b+c)^2} \cdot \frac{b^2+c^2-2bc\cos A - \big(b^2+c^2\big)(1+\cos A)}{b^2+c^2} \\
= \;& -\frac{b^2c^2\cos A}{b^2+c^2} ,
\end{aligned}$$

which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, *France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; NANCY MUELLER and SETH STAHLHEBER, Southeast Missouri State University, Cape Girardeau, MO, USA; K.C. SANDEEP, student, Southeast Missouri State University, Cape Girardeau, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; BIKRAM KUMAR SITOULA, student, Southeast Missouri State University, Cape Girardeau, MO, USA; SOUTHEAST MISSOURI STATE UNIVERSITY MATH CLUB; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Cománeşti, Romania. There were two incorrect solutions submitted.*

3359. [2008 : 300, 302] *Proposed by Ray Killgrove, Vista, CA, USA and David Koster, University of Wisconsin, La Crosse, WI, USA.*

Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n^2 + n + 1$. Find a subsequence $\{b_n\}_{n=1}^{\infty}$ such that $b_1 = a_1$, $b_2 = a_2$, $b_3 > a_3$, every pair of terms from this subsequence are relatively prime, and there are primes which divide no term of the subsequence.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Define $\{b_n\}$ by $b_1 = a_1 = 3$, $b_2 = a_2 = 7$, and for $n > 1$ let

$$b_{n+1} = \left(\prod_{k=1}^{n} b_k\right)^2 + \left(\prod_{k=1}^{n} b_k\right) + 1 .$$

Then clearly $\{b_n\}$ is a subsequence of $\{a_n\}$ and $b_3 > a_3$. Since we have $b_n \equiv 1 \pmod{b_m}$ for $m < n$, we see that b_m and b_n are relatively prime. Finally, since all the b_n's are odd, it follows that the prime number 2 divides no term of $\{b_n\}$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposers.

Most solutions were similar to the one featured above.

Geupel remarks that computations of the residues of n^2+n+1 modulo p for $1 \le n < p$ show that no prime $p \in \{2, 5, 11\}$ divides any term of the sequence $\{a_n\}$.

A prime $p \neq 3, 7$ can divide at most one term b_m, $m \ge 3$, of the sequence $\{b_n\}$ in the featured solution, so by deleting at most term from that sequence the prime p can be avoided. Thus, any finite set of primes not containing 3 and 7 can be avoided by a subsequence of $\{a_n\}$ beginning with $b_1 = a_1$ and $b_2 = a_2$.

3360. [2008 : 300, 302] *Proposed by Michel Bataille, Rouen, France.*

For complex numbers a, b, and c, not all zero, let $\mathcal{N}(a,b,c)$ denote the number of solutions $(z_1,z_2,z_3) \in \mathbb{C}^3$ to the system:

$$\begin{aligned} z_1z_3 &= a\,, \\ z_1z_2 + z_2z_3 &= b\,, \\ z_1^2 + z_2^2 + z_3^2 &= c\,. \end{aligned}$$

For which a, b, and c does $\mathcal{N}(a,b,c)$ attain its minimal value?

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

It is straightforward to see that if (z_1,z_2,z_3) is a solution then

$$\begin{aligned} (z_1+z_2+z_3)^2 &= 2a+2b+c\,, \\ (z_1-z_2+z_3)^2 &= 2a-2b+c\,. \end{aligned}$$

Let A be a square root of $2a+2b+c$ and B be a square root of $2a-2b+c$. We then obtain the system of linear equations:

$$\begin{aligned} z_1+z_2+z_3 &= \pm A\,, \\ z_1-z_2+z_3 &= \pm B\,. \end{aligned}$$

Solving yields

$$\begin{aligned} (z_1+z_3,z_2) \;=\; & \left(\frac{A+B}{2},\frac{A-B}{2}\right),\left(\frac{A-B}{2},\frac{A+B}{2}\right), \\ & \left(\frac{-A+B}{2},\frac{-A-B}{2}\right),\left(\frac{-A+B}{2},\frac{-A-B}{2}\right). \qquad (1) \end{aligned}$$

The system of two equations $z_1+z_3 = \dfrac{\pm A \pm B}{2}$ and $z_1z_3 = a$ always has a solution. When such a solution is paired with the corresponding z_2 a solution to the original system is obtained. Also we note that if (z_1,z_2,z_3) is a solution to the system, then $-(z_1,z_2,z_3)$ is a different solution to the system (since $(a,b,c) \neq (0,0,0)$), thus the system has at least two solutions. We claim that this is in fact the minimal value of $\mathcal{N}(a,b,c)$.

If the terms on the right side of (1) are distinct, then $\mathcal{N}(a,b,c) \geq 4$. The only way for two of those terms to be equal is if $A = 0$ or $B = 0$.

If $A = 0$ and $B \neq 0$, then $B^2 = -4b$ and $(z_1+z_3,z_2) = \pm\left(\frac{B}{2},-\frac{B}{2}\right)$. Since $B \neq 0$, in order for the original system to have two solutions, the system $z_1+z_3 = \pm\dfrac{B}{2}$; $z_1z_3 = a$ must have only one solution; that is, $z_1 = z_3$, and hence $-b = \left(\pm\dfrac{B}{2}\right)^2 = 4a$. Therefore, $b = -4a$, and since $A = 0$ we have $c = 6a$ and $(z_1,z_2,z_3) = \pm(\alpha,-2\alpha,\alpha)$, where α is a square root of $a \neq 0$.

If $A = B = 0$, then $b = 0$, $c = -2a$, and $(z_1, z_2, z_3) = \pm(\alpha, 0, -\alpha)$, where α is a square root of $-a \neq 0$.

If $A \neq 0$ and $B = 0$, then $A^2 = 4b$ and $(z_1 + z_3, z_2) = \pm\left(\frac{A}{2}, \frac{A}{2}\right)$. Since $A \neq 0$, in order for the original system to have two solutions, the system $z_1 + z_3 = \pm\frac{A}{2}$; $z_1 z_3 = a$ must have only one solution, and hence $b = \left(\pm\frac{A}{2}\right)^2 = 4a$. Therefore, $b = 4a$, and since $B = 0$ we have $c = 6a$ and $(z_1, z_2, z_3) = \pm(\alpha, 2\alpha, \alpha)$, where α is a square root of $a \neq 0$.

In summary, the minimal value of $\mathcal{N}(a, b, c)$ is two, which is attained for triples of the form $(a, 4a, 6a)$, $(a, -4a, 6a)$, or $(a, 0, -2a)$, where a is a nonzero complex number.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ROY BARBARA, Lebanese University, Fanar, Lebanon; and TITU ZVONARU, Comăneşti, Romania. There was one incomplete solution submitted.

3361. [2008 : 300, 302] *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of triangle ABC meet the sides CA and AB at E and F, respectively. For which points P of the line segment EF do the areas of $\triangle EBC$, $\triangle PBC$, and $\triangle FBC$ form an arithmetic progression?

Composite of solutions by Oliver Geupel, Brühl, NRW, Germany and by Titu Zvonaru, Comăneşti, Romania.

Let e, f, and p be the distances from the points E, F, and P, respectively, to the line BC. The areas of the triangles EBC, PBC, and FBC form an arithmetic progression if and only if their respective altitudes e, p, and f satisfy

$$2p = e + f\,.$$

If $AB = AC$, then the lines BC and EF are parallel, whence each point P on EF satisfies the desired condition. Otherwise, only the midpoint of EF has the desired property. (This last claim follows immediately from the more familiar theorem that *if EFF' is a triangle with P on side EF, P' on side EF', and $PP' \parallel FF'$, then P is the midpoint of EF if and only if $FF' = 2PP'$*. To prove the claim from the triangle theorem, let the line through E that is parallel to BC meet at F' and P' our altitudes to BC from F and P, respectively. Then P is the midpoint of EF if and only if $FF' = 2PP'$, if and only if $(FF' + e) + e = 2PP' + 2e$, if and only if $f + e = 2p$.)

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Observe that the incircle plays almost no role in the solution—the conclusion holds for any segment EF that lies entirely on one side of the line BC. However, when E and F are the points of tangency of the incircle as in our problem, then $AF = AE$ so that the location of P for which the areas form an arithmetic progression can alternatively be described as lying on the bisector of $\angle BAC$, or as the foot of the perpendicular from A to EF.

3362. [2008 : 172,175] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Prove that

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx \;<\; \frac{\pi^2}{12}\,.$$

Similar solutions by Kee-Wai Lau, Hong Kong, China; Ovidiu Furdui, Campia Turzii, Cluj, Romania; and Missouri State University Problem Solving Group, Springfield, MO, USA.

By Hölders inequality, we have

$$\begin{aligned}\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx &< \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{1/3}\left(\int_0^1 1^{3/2}dx\right)^{2/3}\\ &= \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{1/3}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx &< \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{2/3}\left(\int_0^1 1^{3}dx\right)^{1/3}\\ &= \left(\int_0^1 \frac{\ln(1+x)}{x}\,dx\right)^{2/3}.\end{aligned}$$

It now follows that

$$\begin{aligned}&\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx\\ &\quad< \int_0^1 \frac{\ln(1+x)}{x}\,dx\\ &\quad= \int_0^1 \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n-1}}{n}\,dx\\ &\quad= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{n-1}\,dx\\ &\quad= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} \;=\; \frac{\pi^2}{12}\,,\end{aligned}$$

where the integral and the sum can be interchanged since the sum converges uniformly on $[0,1]$, and $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^2} = \dfrac{\pi^2}{12}$ follows by straightforward manipulations from the well-known formula $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, and KARL HAVLAK, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Both Geupel and the proposer identified the last integral computed above as essentially a value of the dilogarithm function, one form of which is $\mathrm{Li}_2(x) = \int_x^0 \ln(1-t)/t\,dt$.