SKOLIAD No. 112

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 March, 2009***. A copy of ***MATHEMATICAL MAYHEM* Vol. 6** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Nos questions proviennent ce mois-ci du Concours de Mathématiques des écoles de Colombie-Britannique, niveau secondaire. On présente la finale junior A. Nous remercions Clint Lee du Collège Okanagan pour nous avoir fourni les questionnaires.

CONCOURS DE MATHÉMATIQUES DES ÉCOLES SECONDAIRES DE COLOMBIE BRITANNIQUE, 2008
Finale Junior A
Vendredi 2 mai

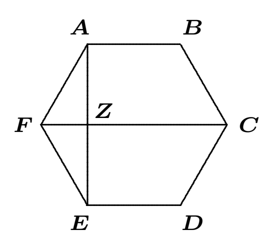
1. Jeeves le valet a été promis un salaire de \$8000 et une voiture pour travailler une année. Jeeves quitte l'emploi après 7 mois de service et reçoit l'auto avec un salaire corrigé au pro-rata de \$1600. La valeur monétaire de l'auto était :

(A) 6400 (B) 7200 (C) 7360 (D) 8000 (E) 15360

2. On rappelle que $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. La valeur maximale de l'entier x tel que 3^x divise $30!$ est :

(A) 30 (B) 14 (C) 13 (D) 10 (E) 4

3. Dans le dessin, $ABCDEF$ est un hexagon régulier. Des segments AE et FC se rencontre à Z. Le ratio de l'aire du triangle FZE à l'aire du quadrilatère $ABCZ$ est :



(A) $1:5$ (B) $1:4$ (C) $4:1$
(D) $5:1$ (E) $1:6$

4. Definissons $\lfloor x \rfloor$ comme étant l'entier le plus grand inférieur à x. Par exemple, $\lfloor 7 \rfloor = 7$, $\lfloor 7.2 \rfloor = 7$, et $\lfloor -5.5 \rfloor = -6$. Si z est un nombre réel tel qu'il ne soit pas un entier, alors la valeur de $\lfloor z \rfloor + \lfloor 1 - z \rfloor$ est :

(A) -1 (B) 0 (C) 1 (D) 2 (E) z

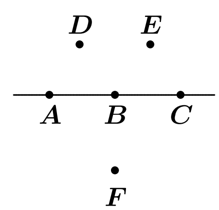
5. Des tests dans chacune des trois matières que sont l'Anatomie, la Biologie et la Chimie, ont été pris par un groupe de 41 étudiants. Le tableau suivant indique combien d'étudiants ont coulé chaque sujet, ainsi que chaque combinaison de sujets :

subject	A	B	C	AB	AC	BC	ABC
# failed	12	5	8	2	6	3	1

(Par exemple, 5 étudiants ont coulé en Biologie, dont 3 ont coulé la Biologie et la Chimie, et seulement 1 des 3 a coulé les trois sujets.) Le nombre d'étudiants qui ont passé les trois sujets est :

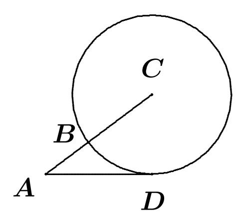
(A) 4 (B) 16 (C) 21 (D) 24 (E) 26

6. Six points, A, B, C, D, E, et F sont disposés pour dans la formation illustrée dans le diagramme, avec A, B, et C sur une même droite. Trois de ces six points sont selectionnés pour former un triangle. Le nombre de triangles qui peuvent être fromés de la sorte est :



(A) 12 (B) 14 (C) 16
(D) 19 (E) 20

7. Dans la figure, C est le centre du cercle et AD est tangent au cercle en D. AC est une droite. Si $\overline{AD} = 10$ et $\overline{AB} = 7$, la longueur de BC est :



(A) $\frac{\sqrt{151}-7}{2}$ (B) $\sqrt{14}$ (C) $\frac{51}{14}$
(D) $\frac{\sqrt{51}}{2}$ (E) $\frac{7}{2}$

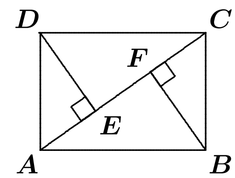
8. Quand on effectue 2008^{2008}, le chiffre des unités du produit final est :

(A) 8 (B) 6 (C) 4 (D) 2 (E) 0

9. On rappelle qu'un nombre premier est un entier plus grand que un qui n'est divisible que par un et lui-même. On considère l'ensemble des nombres à deux chiffres inférieurs à 40 qui sont soit premiers soit divisibles par seulement un nombre premier. Parmi ces nombres, on choisit ceux dont la somme des chiffres forme un nombre premier et dont la différence positive des chiffre est un autre nombre premier. La somme des valeurs des nombres choisis est :

(A) 29 (B) 41 (C) 54 (D) 70 (E) 93

10. Dans la figure, $ABCD$ est un rectangle avec $\overline{AD} = 1$, et les deux segments DE et BF sont perpendiculaires à la diagonale AC. De plus, $\overline{AE} = \overline{EF} = \overline{FC}$. La longueur du côté AB est :



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2
(D) $\sqrt{5}$ (E) 3

BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2008

Junior Final, Part A
Friday May 2

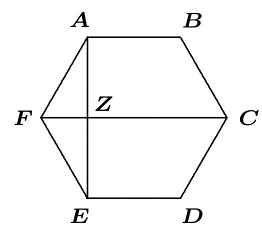
1. Jeeves the valet was promised a salary of $\$8000$ and a car for a year of service. Jeeves left the job after 7 months of service and received the car and $\$1600$ as his correctly prorated salary. The dollar value of the car was:

(A) 6400 (B) 7200 (C) 7360 (D) 8000 (E) 15360

2. Recall that $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. The maximum value of the integer x such that 3^x divides $30!$ is:

(A) 30 (B) 14 (C) 13 (D) 10 (E) 4

3. In the diagram, $ABCDEF$ is a regular hexagon. Line segments AE and FC meet at Z. The ratio of the area of triangle FZE to the area of the quadrilateral $ABCZ$ is:



(A) $1:5$ (B) $1:4$ (C) $4:1$
(D) $5:1$ (E) $1:6$

4. Define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. For example, $\lfloor 7 \rfloor = 7$, $\lfloor 7.2 \rfloor = 7$, and $\lfloor -5.5 \rfloor = -6$. If z is a real number that is not an integer, then the value of $\lfloor z \rfloor + \lfloor 1 - z \rfloor$ is:

(A) -1 (B) 0 (C) 1 (D) 2 (E) z

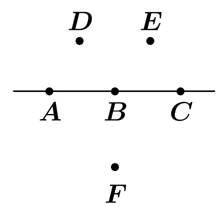
5. Examinations in each of three subjects, Anatomy, Biology, and Chemistry, were taken by a group of 41 students. The following table shows how many students failed in each subject, as well as in the various combinations:

subject	A	B	C	AB	AC	BC	ABC
# failed	12	5	8	2	6	3	1

(For instance, 5 students failed in Biology, among whom there were 3 who failed both Biology and Chemistry, and just 1 of the 3 who failed all three subjects.) The number of students who passed all three subjects is:

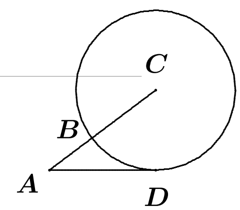
(A) 4 (B) 16 (C) 21 (D) 24 (E) 26

6. Six points, A, B, C, D, E, and F are arranged in the formation shown in the diagram, with A, B, and C on a straight line. Three of these six points are selected to form a triangle. The number of such triangles that can be formed is:



(A) 12 (B) 14 (C) 16
(D) 19 (E) 20

7. In the diagram, C is the centre of the circle and AD is tangent to the circle at D. AC is a straight line. If $\overline{AD} = 10$ and $\overline{AB} = 7$, the length of BC is:



(A) $\dfrac{\sqrt{151}-7}{2}$ (B) $\sqrt{14}$ (C) $\dfrac{51}{14}$
(D) $\dfrac{\sqrt{51}}{2}$ (E) $\dfrac{7}{2}$

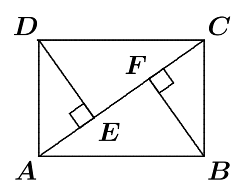
8. When 2008^{2008} is multiplied out, the units digit in the final product is:

(A) 8 (B) 6 (C) 4 (D) 2 (E) 0

9. Recall that a prime number is an integer greater than one that is divisible only by one and itself. Consider the set of two digit numbers less than 40 that are either prime or divisible by only one prime number. From this set select those for which the sum of the digits is a prime number, and the positive difference between the digits is another prime number. The sum of the values of the numbers selected is:

(A) 29 (B) 41 (C) 54 (D) 70 (E) 93

10. In the diagram $ABCD$ is a rectangle with $\overline{AD} = 1$, and both DE and BF perpendicular to the diagonal AC. Further, $\overline{AE} = \overline{EF} = \overline{FC}$. The length of the side AB is:



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2
(D) $\sqrt{5}$ (E) 3

Now we give solutions to the 2007 Maritime Mathematics Competition coordinated by the APICS Mathematics and Statistics Committee and sponsored by the Canadian Mathematical Society. Our apologies for giving the wrong year (2006) for this contest in [2007 : 450, 451].

1. In a 100-metre race, Alice beat Bob by 10 metres and Bob beat Charlie by 20 metres. Assuming that each runner ran at a constant speed, by how much did Alice beat Charlie?

Official Solution.

Let a, b, and c be, respectively, the running speed of Alice, Bob, and Charlie measured in metres per second. Bob runs 90 m in the time it takes Alice to run 100 m, so $\frac{100}{a} = \frac{90}{b}$. Similarly, the time taken by Bob to run 100 m is equal to the time required by Charlie to run 80 m, so $\frac{100}{b} = \frac{80}{c}$. Manipulating these equations gives $\frac{100}{a} = \frac{72}{c}$, so in the time it takes Alice to run 100 m, Charlie runs 72 m. Thus, Alice beats Charlie by 28 m.

Also solved by JOCHEM VAN GAALEN, grade 9 student, Medway High School, Arva, ON.

2. Find positive integers x and y such that $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

Solution by Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia.

We have $\sqrt{x} + \sqrt{y} = \sqrt{2007} = 3\sqrt{223}$. This equation will be true if $\sqrt{x} = \sqrt{223}$ and $\sqrt{y} = 2\sqrt{223} = \sqrt{892}$. Thus, $x = 223$ and $y = 892$ is a solution.

Also solved by JOCHEM VAN GAALEN, grade 9 student, Medway High School, Arva, ON.

3. An expedition to the planet Bizarro finds the following equation scrawled in the dust.

$$3x^2 - 25x + 66 = 0 \qquad \Longrightarrow \qquad x = 4 \quad \text{or} \quad x = 9\,.$$

What base is used for the number system on Bizarro?

Official Solution.

Let b be the base. The given equation may then be written, in base 10, as $3x^2 - (2b+5)x + (6b+6) = 0$. Since $x = 4$ is a root, we have $3(4^2) - (2b+5)(4) + (6b+6) = 0$, which implies that $b = 17$. Similarly, substituting $x = 9$ gives $3(9^2) - (2b+5)(9) + (6b+6) = 0$, which also yields $b = 17$. Therefore, the number system on Bizarro uses base 17.

There was one incorrect solution submitted.

4. Two circles, one of radius 1, the other of radius 2, intersect so that the larger circle passes through the centre of the smaller circle. Find the distance between the two points at which the circles intersect.

Official Solution, modified by the editor.

Let $(0,0)$ and $(2,0)$ be, respectively, the centre of the smaller and larger circle. We then have $x^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$. Solving for the intersection points gives $1 - x^2 = 4 - (x-2)^2$, or $x = 1/4$. Thus, $y^2 = 15/16$ and $y = \pm\sqrt{15}/4$. The required distance is then $2\left(\sqrt{15}/4\right) = \sqrt{15}/2$ units.

Also solved by NATALIA DESY, student, SMP Xaverius 1, Palembang, Indonesia.

5. The positive integers from 1 up to n (inclusive) are written on a blackboard. After one number is erased, the average (arithmetic mean) of the remaining $n-1$ numbers is $46\frac{20}{23}$. Determine n and the number that was erased.

Solution by Natalia Desy, student, SMP Xaverius 1, Palembang, Indonesia.

Let 1, 2, $\ldots$, n be the numbers written on the blackboard and let x be the erased number. Then $(1+2+3+\cdots+n)-x = \left(46\frac{20}{23}\right)(n-1)$, or $x = \dfrac{n(n+1)}{2} - \left(46\frac{20}{23}\right)(n-1)$. Thus, 23 divides $n-1$, so we try $n=24$, 47, 70, and 93. The last number is the only one giving a positive x, namely $x=59$. [*Ed.*: The average of 1, 2, $\ldots$, 93 is 47, so if $n \geq 94$, then the average of the numbers remaining on the blackboard is too large.]

6. Points $P_1(0,1)$, $P_2(0,0)$, $P_3(1,0)$, and $P_4(1,1)$ are the vertices of a square. For $n \geq 5$, let P_n be defined as below, where $r(n)$ is the remainder when n is divided by 8.

$$P_n = \begin{cases} \text{mid-point of } P_{n-3}P_{n-4} & \text{if } r(n) = 1,\ 2, \text{ or } 3, \\ \text{mid-point of } P_{n-4}P_{n-7} & \text{if } r(n) = 4, \\ \text{mid-point of } P_{n-1}P_{n-4} & \text{if } r(n) = 5, \\ \text{mid-point of } P_{n-4}P_{n-5} & \text{if } r(n) = 0,\ 6, \text{ or } 7. \end{cases}$$

Find the coordinates of P_{2007}.

Official Solution

We calculate $P_5\left(\frac{1}{2},1\right)$, $P_6\left(0,\frac{1}{2}\right)$, $P_7\left(\frac{1}{2},0\right)$, $P_8\left(1,\frac{1}{2}\right)$, and $P_9\left(\frac{1}{4},\frac{3}{4}\right)$ It becomes apparent that the points $P_5P_6P_7P_8$ are the vertices of a diamond inside the square determined by $P_1P_2P_3P_4$. Moreover, this pattern of a diamond inside a square repeats with every consecutive 8 points and the dimensions are halved at each stage. Therefore, point P_{15} is at the bottom of the second diamond and has coordinates $\left(\frac{1}{2},\frac{1}{2}-\left(\frac{1}{2}\right)^2\right)$. More generally, P_{8k+7} has coordinates $\left(\frac{1}{2},\frac{1}{2}-\left(\frac{1}{2}\right)^{k+1}\right)$. Now $2007 = 8(250)+7$, so the coordinates of P_{2007} are $\left(\frac{1}{2},\frac{1}{2}-\left(\frac{1}{2}\right)^{251}\right)$.

Also solved by NATALIA DESY, student, SMP Xaverius 1, Palembang, Indonesia.

That brings us to the end of another issue. This month's winner of a past Volume of ***Mayhem*** is Natalia Desy. Congratulations Natalia! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 decembre 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M357. *Proposé par l'Équipe de Mayhem.*

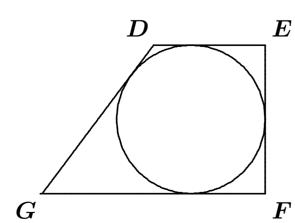
Déterminer tous les nombres réels x satisfaisant $3^{2x+2}+3 = 3^x+3^{x+3}$.

M358. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Combien d'entiers dans la liste 1, 2008, 2008^2, $\dots$, 2008^{2009} sont-ils à la fois des carrés parfaits et des cubes parfaits ?

M359. *Proposé par l'Équipe de Mayhem.*

Un trapèze $DEFG$ est circonscrit à un cercle de rayon 2, comme indiqué dans la figure. Le côté DE mesure 3 et les angles en E et F sont droits. Trouver l'aire du trapèze.



M360. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Déterminer tous les entiers positifs x satisfaisant

$$3^x = x^3 + 3x^2 + 2x + 1 .$$

M361. *Proposé par George Tsapakidis, Agrinio, Grèce.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$ab(a+b-c) \;+\; bc(b+c-a) \;+\; ca(c+a-b) \;\geq\; 3abc\,.$$

M362. *Proposé par Mihály Bencze, Brasov, Roumanie.*

On suppose que x_1, x_2, ..., x_n est une suite d'entiers tels que

$$\big||x_1|+|x_2|+\cdots+|x_n| \;-\; |x_1+x_2+\cdots+x_n|\big| \;=\; 2\,.$$

Montrer qu'au moins un des x_1, x_2, ..., x_n est égal à 1 ou -1.

. . . .

M357. *Proposed by the Mayhem Staff.*

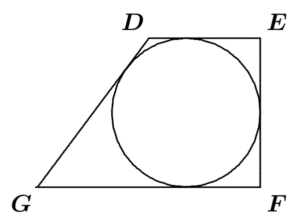
Determine all real numbers x that satisfy $3^{2x+2}+3=3^x+3^{x+3}$.

M358. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

How many integers in the list 1, 2008, 2008^2, ..., 2008^{2009} are simultaneously perfect squares and perfect cubes?

M359. *Proposed by the Mayhem Staff.*

A trapezoid $DEFG$ is circumscribed about a circle of radius 2, as shown in the diagram. The side DE has length 3 and there are right angles at E and F. Determine the area of the trapezoid.



M360. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all positive integers x that satisfy

$$3^x \;=\; x^3+3x^2+2x+1\,.$$

M361. *Proposed by George Tsapakidis, Agrinio, Greece.*

Let a, b, and c be positive real numbers. Prove that

$$ab(a+b-c) \;+\; bc(b+c-a) \;+\; ca(c+a-b) \;\geq\; 3abc\,.$$

M362. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that x_1, x_2, ..., x_n is a sequence of integers such that

$$\big||x_1|+|x_2|+\cdots+|x_n| \;-\; |x_1+x_2+\cdots+x_n|\big| \;=\; 2\,.$$

Prove that at least one of x_1, x_2, ..., x_n equals 1 or -1.

Mayhem Solutions

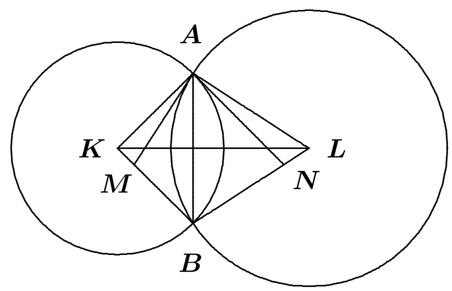
M313. *Proposed by Babis Stergiou, Chalkida, Greece.*

Two circles with centres K and L intersect at points A and B. The tangent at A to the circle centred at L meets segment KB at M and the tangent at A to the circle centred at K meets segment BL at N. Prove that $AB \perp MN$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since AM is the tangent at A to the circle centred at L, we have that $\angle MAB = \angle ALK = \angle KLB$.

Similarly, $\angle NAB = \angle LKB$. Therefore,



$$\begin{aligned} & \angle MAN + \angle MBN \\ = \ & \angle MAB + \angle BAN + \angle MBN \\ = \ & \angle KLB + \angle LKB + \angle MBN \\ = \ & 180^\circ , \end{aligned}$$

because these are the three angles of $\triangle KBL$.

Thus, the quadrilateral $AMBN$ is cyclic, and from this it follows that $\angle MNB = \angle MAB = \angle KLB$. We conclude that MN is parallel to KL. Finally, since $AB \perp KL$, then $AB \perp MN$.

Also solved by ALPER CAY, Uzman Private School, Kayseri, Turkey; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; GEOFFREY A. KANDALL, Hamden, CT, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; and TITU ZVONARU, Comănești, Romania. There was one incorrect solution submitted.

M314. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a be a real number with $a > 1$. Solve the following equation for x:

$$a^{1/x}x + a^x/x = 2a .$$

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

If x is negative, then (since $a > 1$) both $a^{1/x}x$ and a^x/x are negative and so is their sum. However, $a^{1/x}x + a^x/x = 2a > 0$, a contradiction. Also, $x \neq 0$, because it is in a denominator. Thus, x is positive.

We apply the Arithmetic Mean–Geometric Mean Inequality to obtain

$$2a = a^{1/x}x + a^x/x \geq 2\sqrt{\left(a^{1/x}x\right)\left(a^x/x\right)} = 2\sqrt{a^{x+1/x}} .$$

We have $x + 1/x \geq 2\sqrt{x\,(1/x)} = 2$ (by the AM–GM Inequality again) and $a > 1$; hence, $a^{x+1/x} \geq a^2$. It follows that

$$2\sqrt{a^{x+1/x}} \;\geq\; 2\sqrt{a^2} \;=\; 2a\,.$$

Combining what we have discovered, we have $2a \geq 2\sqrt{a^{x+1/x}} \geq 2a$ so equality must hold in both applications of the AM–GM Inequality. Thus, from the second step, $x = 1/x$, which implies $x^2 = 1$ and $x = 1$, since x is positive.

Upon substituting $x = 1$, the equation $a^{1/x}x + a^x/x = 2a$ is seen to be true. Therefore, $x = 1$ is the only solution of the equation.

Also solved by ARKADY ALT, San Jose, CA, USA; ALPER CAY, LOKMAN GOKCE, and MURAT OZARSLAN, Geomania Problem Group, Kayseri, Turkey; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain (who also proved the result when $0 < a < 1$); D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; URZICA SORIN, Grigore Cobalcescu High School, Moinesti, Romania; GEORGE TSAPAKIDIS, Agrinio, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and TITU ZVONARU, Comăneşti, Romania.

M315. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle. Let D be the intersection of AB with the interior bisector of angle C, and let E be the mid-point of AB. Show that $CD + CE < AC + BC$.

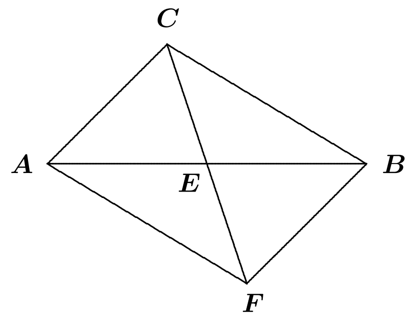
Solution (submitted independently) by Oliver Geupel, Brühl, NRW, Germany and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

We begin by showing that $CD < \frac{1}{2}(AC + BC)$. Let $\angle ACB = 2\theta$ and let $[XYZ]$ denote the area of $\triangle XYZ$. Using the area formula for a triangle, $[XYZ] = \frac{1}{2}XY \cdot YZ\sin(\angle XYZ)$, we successively obtain

$$\begin{aligned}
[ABC] &= [ACD] \;+\; [BCD]\,, \\
\frac{1}{2}AC \cdot BC\sin 2\theta &= \frac{1}{2}AC \cdot CD\sin\theta \;+\; \frac{1}{2}BC \cdot CD\sin\theta\,, \\
2AC \cdot BC\sin\theta\cos\theta &= AC \cdot CD\sin\theta \;+\; BC \cdot CD\sin\theta\,, \\
2AC \cdot BC\cos\theta &= AC \cdot CD \;+\; BC \cdot CD\,, \qquad \text{(since } \theta > 0^\circ) \\
2AC \cdot BC\cos\theta &= CD(AC + BC)\,, \\
CD &= \frac{2AC \cdot BC}{AC + BC}\cos\theta \;\leq\; \frac{2AC \cdot BC}{AC + BC}\,, \\
&= \frac{(AC + BC)^2 - (AC - BC)^2}{2(AC + BC)}\,, \\
&\leq \frac{(AC + BC)^2}{2(AC + BC)} \;=\; \frac{1}{2}(AC + BC)\,.
\end{aligned}$$

Now, construct a parallelogram $AFBC$ by rotating $\triangle ACB$ through 180° about E.

We have $CE = \frac{1}{2}CF$. Applying the triangle inequality to $\triangle CFB$ we obtain



$$CF < BF + CB \;=\; AC + BC\,.$$

Thus, $CE < \frac{1}{2}(AC + BC)$.

Therefore, $CD + CE < \frac{1}{2}(AC + BC) + \frac{1}{2}(AC + BC) = AC + BC$, as required.

Also solved by ALPER CAY, Kayseri, Turkey and MURAT YALCIN, Istanbul, Turkey; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GEORGE TSAPAKIDIS, Agrinio, Greece; and TITU ZVONARU, Comăneşti, Romania. There were 2 incorrect solutions submitted.

M316. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the value of $\sum\limits_{1\le k\le 99} \dfrac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}$.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

We have

$$\begin{aligned}
\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} &= \sum_{k=1}^{99} \frac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k}+\sqrt{k+1})} \\
&= \sum_{k=1}^{99} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}(\sqrt{k}+\sqrt{k+1})(\sqrt{k+1}-\sqrt{k})} \\
&= \sum_{k=1}^{99} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}((k+1)-k)} \;=\; \sum_{k=1}^{99} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}\cdot 1} \\
&= \sum_{k=1}^{99} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \\
&= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \cdots + \left(\frac{1}{\sqrt{99}} - \frac{1}{\sqrt{100}}\right) \\
&= 1 - \frac{1}{\sqrt{100}}\,,
\end{aligned}$$

as the series "telescopes".

Thus, $\sum\limits_{k=1}^{99} \dfrac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} = 1 - \dfrac{1}{\sqrt{100}} = \dfrac{9}{10}$.

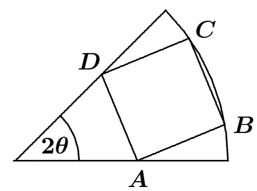
Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; ALPER CAY, LOKMAN GOKCE, and

MURAT OZARSLAN, Geomania Problem Group, Kayseri, Turkey; SHI CHANGWEI, Xi`an City, Shaan Xi Province, China; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; ANGELA DREI, Riolo Terme, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; URZICA SORIN, Grigore Cobalcescu High School, Moinesti, Romania; GEORGE TSAPAKIDIS, Agrinio, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and TITU ZVONARU, Comăneşti, Romania. There was 1 incorrect solution submitted.

Most of the solutions used essentially the same approach. Stanciu and Zvonaru each pointed out that M316 is identical to M218 from 2005.

M317. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Square $ABCD$ is inscribed in a sector of a circle of radius 1 so that there is one vertex on each radius and two vertices on the arc. The angle at the centre is 2θ. Determine the value of θ that results in the square of largest area.



Solution by D. Kipp Johnson, Beaverton, OR, USA.

Let M and N be the mid-points of AD and BC, respectively, and let O be the centre of the circle. In right-angled $\triangle OAM$ we have $\angle MOA = \theta$ and $AM = \frac{1}{2}AB$. Therefore, $OM = AM\cot\theta = \frac{1}{2}AB\cot\theta$, with $0 < \theta \leq \frac{\pi}{2}$ (the square has area 0 if $\theta = 0$).

In the right-angled $\triangle OBN$, we have $OB = 1$, $BN = \frac{1}{2}AB$, and $ON = OM + AB = \frac{1}{2}AB\cot\theta + AB$. By the Pythagorean Theorem, we have successively

$$\begin{aligned} OB^2 &= BN^2 + ON^2\,, \\ 1^2 &= \left(\frac{1}{2}AB\right)^2 + \left(\frac{1}{2}AB\cot\theta + AB\right)^2\,, \\ 1 &= AB^2\left(\frac{1}{4} + \left(\frac{1}{2}\cot\theta + 1\right)^2\right)\,, \\ 1 &= \frac{1}{4}AB^2\left(1 + (\cot\theta + 2)^2\right)\,, \\ AB^2 &= \frac{4}{\cot^2\theta + 4\cot\theta + 5}\,. \end{aligned}$$

Now, the area of the square is $S(\theta) = AB^2 = \dfrac{4}{\cot^2\theta + 4\cot\theta + 5}$. The function $\cot\theta$ is decreasing and non-negative on the interval $0 < \theta \leq \frac{\pi}{2}$. Thus, $S(\theta)$ is an increasing function on this interval and takes its maximum value when $\cot\theta = 0$ and $\theta = \frac{\pi}{2}$.

We note that the corresponding maximum area is $S\left(\frac{\pi}{2}\right) = \frac{4}{5}$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There was one incorrect solution submitted.

M318. *Proposed by Houda Anoun, Bordeaux, France.*

Are there real numbers x and y such that $x^2 + xy = 3$ and $x - y^2 = 2$?

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Assume that (x, y) is a solution to the given system of equations. Substituting $x = y^2 + 2$ into the equation $x^2 + xy - 3 = 0$ we obtain the two equivalent equations

$$\begin{aligned} (y^2+2)^2 + y(y^2+2) - 3 &= 0\,, \\ y^4 + y^3 + 4y^2 + 2y + 1 &= 0\,. \end{aligned}$$

However,

$$\begin{aligned} y^4 + y^3 + 4y^2 + 2y + 1 &= y^2(y^2+y+3) + y^2 + 2y + 1 \\ &= y^2\left(\left(y+\tfrac{1}{2}\right)^2 + \tfrac{11}{4}\right) + (y+1)^2 \;>\; 0\,, \end{aligned}$$

therefore, there is no real solution to the given system of equations.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MIHÁLY BENCZE, Brasov, Romania; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; and TITU ZVONARU, Comănești, Romania. There were 4 incorrect or incomplete solutions submitted.

M319. *Proposed by Dragoljub Milošević, Pranjani, Serbie.*

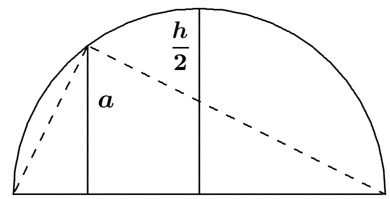
If h and a are the hypotenuse and altitude, respectively, of a right-angled triangle, prove that

$$\frac{a}{h} + \frac{h}{a} \;\geq\; \frac{5}{2}\,.$$

When does equality hold?

I. *Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain.*

In a right-angled triangle, the diagram (on page 334) shows that $a \leq \frac{h}{2}$, since every right-angled triangle can be inscribed in a circle whose diameter is the hypotenuse of the triangle.

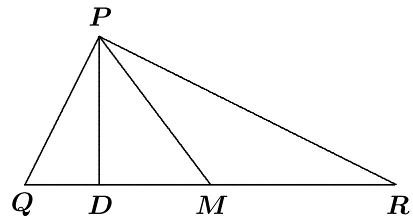


Thus, $2a \le h$, so that $a \le h - a$. Also, $2a \le h$ implies $h \le 2(h-a)$. By multiplying these together, we obtain

$$\begin{aligned} ha &\le 2(h-a)^2 \\ &= 2h^2 - 4ha + 2a^2\,, \end{aligned}$$

hence, $5ha \le 2h^2 + 2a^2$. Dividing both sides by $2ha$, we obtain $\frac{5}{2} \le \frac{h}{a} + \frac{a}{h}$. Equality holds when $2a = h$, which corresponds to a right-angled isosceles triangle.

II. *Solution by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*



Suppose that $\triangle PQR$ has a right angle at P. Consider the median PM. We have $PM = \frac{h}{2}$, since it is the radius of a circle with diameter QR. Let D be the foot of the altitude from P to QR.

Let $\angle PMD = u$. Then we have $\sin u = \frac{PD}{PM}$, implying $\frac{a}{h} = \frac{\sin u}{2}$.

Thus, we have to prove $\frac{\sin u}{2} + \frac{2}{\sin u} \ge \frac{5}{2}$, which is equivalent to $\sin^2 u - 5\sin u + 4 \ge 0$, or $(\sin u - 1)(\sin u - 4) \ge 0$, which is true, since sin $u \le 1$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MIHÁLY BENCZE, Brasov, Romania (2 solutions); ALPER CAY, Uzman Private School, Kayseri, Turkey and SAYGIN DINCER, Ankara, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; MIGUEL MARAÑÓN GRANDES, student, Universidad de La Rioja, Logroño, La Rioja, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece (second solution); SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and TITU ZVONARU, Comăneşti, Romania. There were 2 incorrect solutions submitted.

M320. *Proposed by Mihály Bencze, Brasov, Romania.*

If p and q are any pair of twin primes, show that the numbers $p^4 + 4$ and $q^4 + 4$ are never relatively prime.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

First, notice that

$$\begin{aligned} p^4 + 4 &= \left(p^2+2\right)^2 - 4p^2 = \Big(\left(p^2+2\right) + 2p\Big)\Big(\left(p^2+2\right) - 2p\Big) \\ &= \left(p^2+2p+2\right)\left(p^2-2p+2\right)\,. \end{aligned}$$

Also, since p and q are twin primes, then without loss of generality we may assume that $q = p + 2$. Thus,

$$\begin{aligned} q^4 + 4 &= \left(q^2 + 2q + 2\right)\left(q^2 - 2q + 2\right) \\ &= \left((p+2)^2 + 2(p+2) + 2\right)\left((p+2)^2 - 2(p+2) + 2\right) \\ &= \left(p^2 + 6p + 10\right)\left(p^2 + 2p + 2\right) . \end{aligned}$$

Therefore, $p^2 + 2p + 2$ divides both $p^4 + 4$ and $q^4 + 4$. We also have that $p^2 + 2p + 2 = (p+1)^2 + 1 > 1$ since $p \geq 3$, hence, $p^4 + 4$ and $q^4 + 4$ have a common divisor greater than 1 and they are not relatively prime.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; ARKADY ALT, San Jose, CA, USA; ALPER CAY, Uzman Private School, Kayseri, Turkey; MIGUEL MARAÑÓN GRANDES, student, Universidad de La Rioja, Logroño, La Rioja, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; GEORGE TSAPAKIDIS, Agrinio, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and TITU ZVONARU, Comănești, Romania.

The solver points out that the condition that p and q be twin primes can be relaxed to require that they be any integers that differ by 2. In that case, $p^4 + 4$ and $q^4 + 4$ are not relatively prime even if $p = -1$ and $p^2 + 2p + 2 = 1$.

M321. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Determine all positive integers n and k for which we have

$$\frac{\binom{n}{n-1}^6 + \binom{n-2}{k}^6 + \binom{n+3}{n+1}^3}{3\binom{n-2}{k}^2\binom{n+3}{2}} = n^2 .$$

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

By the Arithmetic Mean–Geometric Mean Inequality, we have

$$\begin{aligned} \frac{\binom{n}{n-1}^6 + \binom{n-2}{k}^6 + \binom{n+3}{n+1}^3}{3} &\geq \sqrt[3]{\binom{n}{n-1}^6\binom{n-2}{k}^6\binom{n+3}{n+1}^3} \\ &= \binom{n}{n-1}^2\binom{n-2}{k}^2\binom{n+3}{n+1} \\ &= n^2\binom{n-2}{k}^2\binom{n+3}{2} , \end{aligned}$$

where equality holds if and only if $\binom{n}{n-1}^6 = \binom{n-2}{k}^6 = \binom{n+3}{n+1}^3$, which is equivalent to $n^2 = \binom{n-2}{k}^2 = \binom{n+3}{2}$. Comparing the inequality we have obtained to the original equation, we see that we need equality in the inequality in order for the original equation to be true.

Now $n^2 = \binom{n+3}{2} = \dfrac{(n+3)(n+2)}{2}$ if and only if $n^2 - 5n - 6 = 0$, or $(n-6)(n+1) = 0$. Thus, since n is a positive integer, $n^2 = \binom{n+3}{n+1}$ if and only if $n = 6$.

Finally, since $n = 6$, the only positive integer k satisfying $n = \binom{n-2}{k}$, or $6 = \binom{4}{k}$, is $k = 2$.

Also solved by MIHÁLY BENCZE, Brasov, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comănești, Romania.

M322. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3abc} + \frac{8abc}{(a+b)(b+c)(c+a)} \;\geq\; 2\,.$$

Solution by George Tsapakidis, Agrinio, Greece.

By the Power Mean Inequality, $\sqrt[3]{\dfrac{a^3+b^3+c^3}{3}} \geq \dfrac{a+b+c}{3}$, which implies that

$$\begin{aligned} 9(a^3+b^3+c^3) &\geq (a+b+c)^3 \\ &= a^3+b^3+c^3+3(a+b)(b+c)(a+c)\,. \end{aligned}$$

Thus, $8(a^3+b^3+c^3) \geq 3(a+b)(b+c)(a+c)$ and

$$\frac{a^3+b^3+c^3}{3abc} \;\geq\; \frac{(a+b)(b+c)(a+c)}{8abc}\,.$$

Finally, since $x + \dfrac{1}{x} \geq 2$ for all $x > 0$, we have

$$\begin{aligned} &\frac{a^3+b^3+c^3}{3abc} + \frac{8abc}{(a+b)(b+c)(a+c)} \\ &\quad\geq\; \frac{(a+b)(b+c)(a+c)}{8abc} + \frac{8abc}{(a+b)(b+c)(a+c)} \;\geq\; 2\,, \end{aligned}$$

as desired.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; APOSTOLIS VERGOS, student, University of Patras, Patras, Greece; TITU ZVONARU, Comănești, Romania; and the proposer. There were 4 incorrect or incomplete solutions submitted.

M323. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions (x, y) to the equation

$$20\sin x - 21\cos x \;=\; 81y^2 - 18y + 30\,.$$

Solution by Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

The given equation is equivalent to each of

$$\begin{aligned} 20\sin x - 21\cos x &= (81y^2 - 18y + 1) + 29\,, &(1)\\ \frac{20}{29}\sin x - \frac{21}{29}\cos x &= \frac{1}{29}(9y-1)^2 + 1\,. &(2)\end{aligned}$$

Let $\theta \in (0, \pi/2)$ denote the unique angle such that $\cos\theta = \dfrac{20}{29}$. Then, since $20^2 + 21^2 = 29^2$, we have $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{20}{29}\right)^2} = \dfrac{21}{29}$. Hence, equation (2) is equivalent to each of

$$\begin{aligned} \cos\theta\sin x - \sin\theta\cos x &= \frac{1}{29}(9y-1)^2 + 1\,, &(3)\\ \sin(x-\theta) &= \frac{1}{29}(9y-1)^2 + 1\,. &(4)\end{aligned}$$

Since the left side of equation (4) is at most 1 and the right side is at least 1, equality holds if and only if $\sin(x-\theta) = 1$ and $y = \dfrac{1}{9}$.

Hence, $x - \theta = \dfrac{\pi}{2} + 2\pi k$, where k is an integer, and all real solutions are given by $(x, y) = \left(\theta + \dfrac{(4k+1)\pi}{2}, \dfrac{1}{9}\right)$, where $k \in \mathbb{Z}$ and $\theta = \cos^{-1}\left(\dfrac{20}{29}\right)$.

Also solved by ARKADY ALT, San Jose, CA, USA; ALPER CAY, Uzman Private School, Kayseri, Turkey and IBRAHIM KOSCUOGLU, Armada Private School, Izmir, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; MIGUEL MARAÑÓN GRANDES, student, Universidad de La Rioja, Logroño, La Rioja, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; GEORGE TSAPAKIDIS, Agrinio, Greece; and TITU ZVONARU, Comănești, Romania. There were 6 incorrect or incomplete solutions submitted.

M324. *Proposed by Mihály Bencze, Brasov, Romania.*

Let functions $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) \;=\; 3x - 1 + |2x+1| \qquad \text{and} \qquad g(x) \;=\; \tfrac{1}{5}(3x + 5 - |2x+5|)\,.$$

Prove that $g \circ f = f \circ g$ and $(f \circ f)^{-1} = g \circ g$.

Solution by Brandon Affenzeller and Jonathon Henson, Auburn University Montgomery, Montgomery, AL, USA.

If $x \geq -\frac{1}{2}$, then $|2x+1| = 2x+1$; if $x < -\frac{1}{2}$, then $|2x+1| = -(2x+1)$. If $x \geq -\frac{5}{2}$, then $|2x+5| = 2x+5$; if $x < -\frac{5}{2}$, then $|2x+5| = -(2x+5)$.

Therefore, the given functions can be written in the following form:

$$f(x) = \left\{ \begin{array}{cl} 5x & \text{if } x \geq -\frac{1}{2} \\ x-2 & \text{if } x < -\frac{1}{2} \end{array} \right\} \quad \text{and} \quad g(x) = \left\{ \begin{array}{cl} \frac{1}{5}x & \text{if } x \geq -\frac{5}{2} \\ x+2 & \text{if } x < -\frac{5}{2} \end{array} \right\}$$

If $x < -\frac{1}{2}$, then $x - 2 < -\frac{5}{2}$, and in that case we have the calculation $g\big(f(x)\big) = g(x-2) = (x-2)+2 = x$. If $x \geq -\frac{1}{2}$, then $5x \geq -\frac{5}{2}$, and in that case we have $g\big(f(x)\big) = g(5x) = \frac{1}{5}(5x) = x$.

Hence, $(g \circ f)(x) = x$ for all $x \in \mathbb{R}$.

Similarly, if $x < -\frac{5}{2}$, then $x + 2 < -\frac{1}{2}$, and in that case we have $f\big(g(x)\big) = f(x+2) = (x+2)-2 = x$; and if $x \geq -\frac{5}{2}$, then $\frac{1}{5}x \geq -\frac{1}{2}$, and then we have $f\big(g(x)\big) = f\left(\frac{1}{5}x\right) = 5\left(\frac{1}{5}x\right) = x$. Hence, $f \circ g = g \circ f = i$, where i is the identity function, that is, $i(x) = x$ for all $x \in \mathbb{R}$.

Since $f \circ g = g \circ f = i$, we conclude that $f = g^{-1}$ and $g = f^{-1}$. Hence, $(f \circ f)^{-1} = \left(g^{-1} \circ g^{-1}\right)^{-1} = g \circ g$.

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; MIGUEL MARAÑÓN GRANDES, student, Universidad de La Rioja, Logroño, La Rioja, Spain; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and TITU ZVONARU, Comănești, Romania. There were 2 incorrect or incomplete solutions submitted.

Problem of the Month

Ian VanderBurgh

After a few months of more investigative problems, let's look at something algebraic this month.

Problem 1

Determine all pairs (a, b) of real numbers that satisfy the system of equations

$$\begin{aligned} a + \log a &= b\,, \\ b + \log b &= a\,. \end{aligned}$$

Anyone who has ever played around with equations involving both polynomial terms (like a) and logarithmic terms (like $\log a$) will know that things can get a tad tricky. Having a system of equations involving both of these is undoubtedly much worse than a system with just one of them.

Well, it's actually not so bad. First, we should clarify to what base we are taking logarithms. In fact, it doesn't matter at all, but if it makes you more comfortable, think of $\log a$ as meaning $\log_{10} a$.

Next, we should see if we can find any solutions at all. Often when a system of equations is symmetric (that is, we get the same system if we switch a and b), trying to find solutions with $a = b$ is not a bad idea. Here, if $a = b$, both equations become

$$a + \log a \; = \; a$$

which simplifies to $\log a = 0$ or $a = 1$. Therefore, $(a, b) = (1, 1)$ is a solution, which we can verify by substitution.

At this point, we can try as we might to find another solution, but... there are actually no more solutions! Let's look at two different approaches that show us why.

Solution 1 to Problem 1. One approach that we learn early on when solving systems of equations is to combine the equations somehow. Let's try adding the two equations, because this will allow us to do some cancellation:

$$\begin{aligned} a + \log a + b + \log b &= a + b\,, \\ \log a + \log b &= 0\,, \\ \log(ab) &= 0\,, \\ ab &= 1\,, \end{aligned}$$

hence, $b = \dfrac{1}{a}$.

Substituting this into the first equation, we obtain $a + \log a = \dfrac{1}{a}$ or $\log a = \dfrac{1}{a} - a$.

At this stage we've still got an equation involving both logarithms and powers of a. Here's a neat way to deal with it.

What happens if $a > 1$? In this case, $\log a > 0$ and $\dfrac{1}{a} - a < 0$, which is not possible if the two sides are equal. Thus, a cannot be greater than 1.

What happens if $0 < a < 1$? In this case, $\log a < 0$. What about the right side? In fact, the right side here is positive, so again, there can be no solution.

Thus, $a = 1$ works, but neither $a > 1$ nor $0 < a < 1$ can work, so there is a unique solution $a = 1$ (which gives $(a, b) = (1, 1)$ as above).

Our strategy was to find one solution and then to show that no other solutions are possible. Here's a second way to look at this.

Solution 2 to Problem 1. Let's leave the equations in their original form and examine the cases $a > 1$ and $0 < a < 1$, knowing already that $a = 1$ yields a solution.

If $a > 1$, then $\log a > 0$, so $b = a + \log a > a > 1$. Now, since $b > 1$, then $\log b > 0$ which gives $a = b + \log b > b$. However, this means that $a > b > a$, which is impossible. Thus, a cannot be greater than 1.

If $0 < a < 1$, then $\log a < 0$, so $b = a + \log a < a < 1$. Now, since

$b < 1$, then $\log b < 0$ which gives $a = b + \log b < b$. However, this means that $a < b < a$, which is also impossible.

Therefore, $a = 1$ and $b = 1$ is the only possible solution.

So there are two different approaches to this problem. If you feel ambitious, try solving the next problem from the 2008 Euclid Contest. (You may want to transform it first so that it looks more like Problem 1 above.)

Problem 2.

Determine all real solutions to the system of equations

$$\begin{array}{rcrcl} x & + & \log_{10} x & = & y - 1\,, \\ y & + & \log_{10}(y-1) & = & z - 1\,, \\ z & + & \log_{10}(z-2) & = & x + 2\,, \end{array}$$

and prove that there are no more solutions.

A Loose End. At the end of last month's column, I promised you a solution to the following problem:

Problem 3.

3 green stones, 4 yellow stones, and 5 red stones are placed in a bag. This time, two stones of different colours are selected at random, removed and replaced with two stones of the third colour. Show that it is impossible for all of the remaining stones to be the same colour, no matter how many times this process is repeated.

Solution to Problem 3. We'll use the notation and terminology from last month. Let's suppose that we have G green stones, Y yellow stones and R red stones at a given stage.

Let's consider the remainder when the number of green stones, G, is divided by 3. Let's think about how a turn can change this remainder. Note that after a turn, G becomes either $G - 1$ (if a green stone was removed) or $G + 2$ (if two green stones are added).

Consider the flow chart $\boxed{2 \to 1 \to 0 \to 2}$. After a turn, the remainder upon division by 3 has moved one position to the right. For example, $G = 4$ becomes either $G = 3$ or $G = 6$, so a remainder of 1 becomes a remainder of 0. Those familiar with modular arithmetic can feel free to use this idea formally.

The same thing is true for the yellow and red stones – the remainder when the number of stones after the turn is divided by 3 is one position to the right in the chart from where it was before the turn.

Initially, we have $G = 3$, $Y = 4$, and $R = 5$. Thus, the respective remainders are 0, 1, and 2. After the first turn, the (respective) remainders will be 2, 0, and 1. After the second turn, they will be 1, 2, and 0. After the third turn, they will be 0, 1, and 2, and so forth.

Can we get to a position where the stones are all the same colour? This would mean that two of the numbers G, R, and Y would equal 0 and so give a remainder of 0. But the three remainders are always different, so this is impossible.

THE OLYMPIAD CORNER

No. 272

R.E. Woodrow

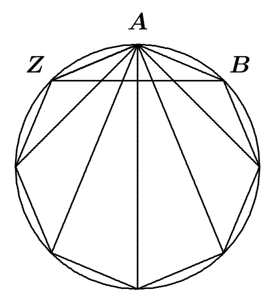
We begin this number with a selection of problems from the Olimpiada Matemática Española 2005, chosen from various sessions of the National and Local Stages of competition. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them.

OLIMPIADA MATEMÁTICA ESPAÑOLA 2005
National Stage
(selected questions)

1. In the plane, is it possible to colour the points with integer coordinates with three colours, in such a way that each colour is found infinitely many times on an infinite number of horizontal lines, and any three points of different colours are never collinear?

2. A triangle is said to be *multiplicative* if the product of the lengths of two of its sides equals the length of the third side.

Let $AB \dots Z$ be a regular polygon with n sides, each of length 1. The $n-3$ diagonals from the vertex A divide the triangle ZAB into $n-2$ smaller triangles. Prove that all of these triangles are multiplicative.



3. Let r, s, u, and v be real numbers. Prove that

$$\min\{r-s^2, s-u^2, u-v^2, v-r^2\} \;\leq\; \frac{1}{4}\,.$$

4. In triangle ABC the sides BC, AC, and AB have lengths a, b, and c, respectively, and a is the arithmetic mean of b and c. Let r and R be the radius of the incircle and circumcircle of ABC, respectively. Prove that:

(a) $0^\circ \;\leq\; \angle BAC \;\leq\; 60^\circ$.

(b) The altitude from A is three times the inradius r.

(c) The distance from the circumcentre of ABC to the side BC is $R-r$.

Local Stage
(selected questions)

5. In triangle ABC we have $\angle BAC = 45°$ and $\angle ACB = 30°$. Let M be the mid-point of the side BC. Prove that $\angle AMB = 45°$ and that $BC \cdot AC = 2\,AM \cdot AB$.

6. Four black marks and five white marks are arbitrarily placed around a circle. If two consecutive marks are of the same colour, we put a new black mark between them and if the two marks are not the same colour, we put a new white mark between them. Then we remove all the previous marks.

Is it possible to obtain nine white marks by repeating this process?

7. Prove that the equation $x^2 + y^2 - z^2 - x - 3y - z - 4 = 0$ has infinitely many integer solutions.

8. On a 10×10 chessboard are placed 41 rooks. Prove that there are at least five rooks among them such that no two of the five attack each other.

Next we give the problems of the 54th Czech Mathematical Olympiad (2004/5), Category B, 10th Class. The D-, S-, and K-series questions correspond to the respective rounds "Domácí", "Školní", and "Krajské". Thanks again go to Felix Recio, Canadian Team Leader to the IMO in Mexico for obtaining them for us.

54th CZECH MATHEMATICAL OLYMPIAD 2004/5
Category B
10th Class

D1. Find all pairs (a, b) of real numbers such that each of the equations

$$\begin{aligned} x^2 + ax + b &= 0\,, \\ x^2 + (2a+1)x + 2b + 1 &= 0\,, \end{aligned}$$

has two distinct real roots and the roots of the second equation are reciprocals of the roots of the first equation.

D2. Let $ABCD$ be a parallelogram. A line through D meets the segment AC in G, the side BC in F, and the line AB in E. The triangles BEF and CGF have the same area. Determine the ratio $|AG| : |GC|$.

D3. Let $k \geq 3$ be an integer. We have k piles of stones with (respectively) 1, 2, $\dots$, k stones in them. At each turn we choose three piles, merge them together, and add one stone (not already in a pile) to the resulting pile. Prove that if after some number of turns only one pile remains, then the number of stones in that pile is not divisible by 3.

D4. Let ABC be a scalene triangle with orthocentre H and circumcentre O. Prove that if $\angle ACB = 60^\circ$, then the bisector of $\angle ACB$ is the perpendicular bisector of OH.

D5. Find all real numbers x such that

$$\frac{x}{x+4} = \frac{5\lfloor x\rfloor - 7}{7\lfloor x\rfloor - 5},$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

D6. In a circle Γ with radius r are inscribed two mutually tangent circles, Γ_1 and Γ_2, each with radius $r/2$. Circle Γ_3 is tangent to Γ_1 and Γ_2 externally and to Γ internally. Circle Γ_4 is tangent to Γ_2 and Γ_3 externally and to Γ internally. Determine the radii of the circles Γ_3 and Γ_4.

S1. We have 54 piles of stones with (respectively) 1, 2, $\ldots$, 54 stones in them. At each turn we choose an arbitrary pile, say consisting of k stones, and remove it along with k stones from each pile which has at least k stones. For example, if we choose the pile with 52 stones at the first turn, then after the turn there will be 53 piles remaining with 1, 2, 3, $\ldots$, 51, 1, 2 stones in them, respectively. Suppose that after some number of turns only one pile remains. How many stones can be in that pile?

S2. Let ABC be a right triangle with $a = |BC|$, $b = |AC|$, and $c = |BC|$ and such that $a < b < c$. Let Q be the mid-point of BC and let S be the mid-point of AB. The line CA meets the perpendicular bisector of AB at R. Prove that $|RQ| = |RS|$ if and only if $a^2 : b^2 : c^2 = 1 : 2 : 3$.

S3. Find all real numbers x such that

$$\left\lfloor \frac{x}{1-x} \right\rfloor = \frac{\lfloor x\rfloor}{1-\lfloor x\rfloor},$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

K1. Circle Γ_1 with radius 1 is externally tangent to circle Γ_2 with radius 2. Each of the circles Γ_1 and Γ_2 is internally tangent to circle Γ_3 with radius 3. Determine the radius of the circle Γ, which is tangent externally to the circles Γ_1 and Γ_2 and internally to the circle Γ_3.

K2. On a public website participants vote for the world's best hockey player of the last decade. The percentage of votes a player receives is rounded off to the nearest percent and displayed on the website. After Jožko votes for Miroslav Šatan, the hockey player's score of 7% remains unchanged. What is the minimum number of people (including Jožko) who voted? (Each participant votes exactly once and for a single player only.)

K3. Let ABC be an acute triangle. Let K and L be the feet of the altitudes from A and B, respectively. Let M be the mid-point of AB and let H be the orthocentre of triangle ABC. Prove that the bisector of $\angle KML$ bisects the line segment HC.

K4. Find all triples of real numbers x, y, z such that

$$\lfloor x\rfloor - y \;=\; 2\lfloor y\rfloor - z \;=\; 3\lfloor z\rfloor - x \;=\; \frac{2004}{2005}\,,$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

Next we give the problems of the three Rounds of the 23rd Iranian Mathematical Olympiad, 2005–2006. Our thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia in 2006, for collecting them for our use.

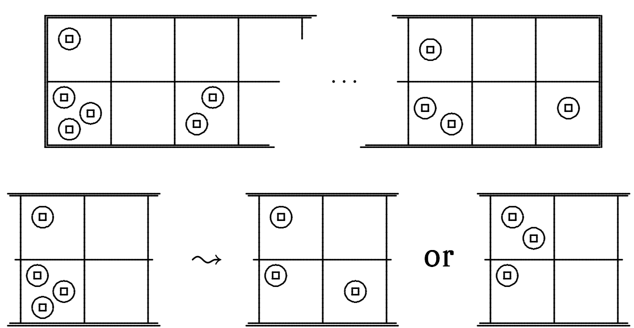
23rd IRANIAN MATHEMATICAL OLYMPIAD
2005–2006
First Round

1. Let n be a positive integer and p a prime number such that $n \mid p-1$ and $p \mid n^3-1$. Show that $4p-3$ is a perfect square.

2. In triangle ABC we have $\angle A = 60^\circ$. Let D be any point on BC. Let O_1 be the circumcentre of ABD and O_2 be the circumcentre of ACD. Let M be the intersection of BO_1 and CO_2 and N be the circumcentre of DO_1O_2. Prove that MN passes through a fixed point.

3. Given are 10^5 points in Euclidean space. If we consider the set of distances between pairs of these points, show that this set has at least 79 elements.

4. Consider a $2\times n$ rectangular grid with $2n$ cells, some of which have coins in them. In each step we choose a cell with more than one coin, then we remove two coins from that cell and put one coin either in the cell immediately above it or in the cell immediately to the right of it. In the beginning there are at least 2^n coins on the grid. Show that by a series of steps, we can always arrange that there will be at least one coin in the right-most upper cell.



5. The segment BC is the diameter of a circle and XY is a chord perpendicular to BC. The points P and M are chosen on XY and CY, respectively, such that $CY \| PB$ and $CX \| MP$. Let K be the intersection of the lines CX and PB. Prove that $PB \perp MK$.

6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $x, y \in \mathbb{R}^+$ we have

$$(x+y)f\big(f(x)y\big) \;=\; x^2 f\big(f(x)+f(y)\big)\,,$$

where $\mathbb{R}^+$ denotes the set of positive real numbers.

Second Round

1. Let $P(x) \in \mathbb{Q}[x]$ be an irreducible polynomial whose degree is an odd number. Let $P(x) \mid \big(Q(x)^2 + Q(x)R(x) + R(x)^2\big)$, where $Q(x)$ and $R(x)$ are polynomials with rational coefficients. Prove that

$$P(x)^2 \mid \big(Q(x)^2 + Q(x)R(x) + R(x)^2\big)\ .$$

2. Let H and O be the orthocentre and the circumcentre of triangle ABC, respectively. Let ω be the circumcircle of ABC and let AO intersect ω at A_1. Let A_1H intersect ω at A' and let AH intersect ω at A''. We define the points B', B'', C', and C'' similarly. Prove that $A'A''$, $B'B''$, and $C'C''$ are concurrent at a point on the Euler line of the triangle ABC.

3. Let a, b, and c be non-negative real numbers. If

$$\frac{1}{a^2+1} \;+\; \frac{1}{b^2+1} \;+\; \frac{1}{c^2+1} \;=\; 2\,,$$

then show that $ab + bc + ca \leq \frac{3}{2}$.

4. Let k be an integer. The sequence $\{a_n\}_{n=0}^{\infty}$ is defined by $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$ by the recursion $a_n = 2ka_{n-1} - (k^2+1)a_{n-2}$. If p is a prime number of the form $4m+3$, prove that

(a) $a_{n+p^2-1} \;\equiv\; a_n \pmod{p}$,

(b) $a_{n+p^3-p} \;\equiv\; a_n \pmod{p^2}$.

5. The sets $A_1, A_2, \ldots, A_{35}$ are such that the intersection of any three of them is a singleton and $|A_k| = 27$ for $1 \leq i \leq 35$. Show that the intersection of $A_1, A_2, \ldots, A_{35}$ is non-empty.

6. Triangle ABC is given. The point L is on BC and M, N are on the extensions of AB, AC (respectively) such that B is between M and A, C is between N and A, $2\angle AMC = \angle ALC$, and $2\angle ANB = \angle ALB$. If O is the circumcentre of AMN, show that OL is perpendicular to BC.

Third Round

1. Let ABC be a triangle whose circumradius equals the radius of the excircle which is tangent to the side BC. Let this excircle touch the side BC and the lines AC and AB at M, N, and L, respectively. Show that the circumcentre of triangle ABC is the orthocentre of triangle MNL.

2. Let x_1, x_2, $\dots$, x_n be real numbers. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i + x_j| \;\geq\; n\sum_{i=1}^{n}|x_i|\,.$$

3. Let G be a tournament with each edge coloured red or blue. Show that there exists a vertex v of G with the property that, for every other vertex u, there is a mono-coloured directed path from v to u.

4. Given are n points in the plane, no three on a line. If a subset E of these points is such that the members of E are the vertices of a convex polygon containing no other points of E in its interior, then E is a *polite* set. Let c_k the number of polite sets with k points. Show that $\sum\limits_{i=3}^{n}(-1)^i c_i$ depends only on n and not on the configuration of the n points.

5. Let $n > 1$ be an integer, and let the entries of $(a_1, a_2, \dots, a_n)$ be pairwise distince postive integers which are coprime in pairs. Find all such n-tuples for which $(a_1 + a_2 + \cdots + a_n) \mid \left(a_1^i + a_2^i + \cdots + a_n^i\right)$ for $1 \leq i \leq n$.

6. Suppose we have a simple polygon (that is, it does not intersect itself, but it need not be convex). Show that this polygon has a diagonal which is contained completely inside it and divides the perimeter into two parts such that each part has at least one third of the vertices of the polygon.

As a final set of problems for this number we give the 9$^{\text{th}}$ and 10$^{\text{th}}$ Grade of the Romanian Mathematical Olympiad, Final Round, April 15$^{\text{th}}$, 2006. Thanks go to Robert Morewood, Canadian Team Leader to the 47$^{\text{th}}$ IMO in Slovenia in 2006, for collecting them.

ROMANIAN MATHEMATICAL OLYMPIAD
April 15, 2006
Final Round – 9$^{\text{th}}$ Grade

1. (Dan Schwarz) Find the maximum value of $(x^3+1)(y^3+1)$ if x and y are real numbers such that $x + y = 1$.

2. (Manuela Prajea) Let triangles ABC and DBC be such that $AB = BC$, $DB = DC$, and $\angle ABD = 90°$. Let M be the mid-point of BC. Points E, F, and P are interior to the segments AB, MC, and AF, respectively, and $\angle BDE = \angle ADP = \angle CDF$. Prove that P is the mid-point of EF and $DP \perp EF$.

3. (Virgil Nicula) A quadrilateral $ABCD$ is inscribed in a circle of radius r such that there exists a point P on CD satisfying $CB = BP = PA = AB$.

(a) Show that such points A, B, C, D, and P do indeed exist.

(b) Prove that $PD = r$.

4. (Radu Gologan) A tennis tournament with $2n$ players ($n \geq 5$) takes place over 4 days. Each player has exactly one match a day, but it is possible that two players may play each other more than once. Prove that such a tournament can end with exactly one winner, exactly three players in second place, and no player losing all four matches. In that case, how many players won a single match and how many won exactly two matches?

Final Round — 10th Grade

1. (Vasile Pop) Let M be a set with n elements and let $\mathcal{P}(M)$ denote the set of all subsets of M. Find all functions $f : \mathcal{P}(M) \to \{0, 1, 2, \ldots, n\}$, with the following two properties:

(a) $f(A) \neq 0$ for any $A \neq \emptyset$, and

(b) $f(A \cup B) = f(A \cap B) + f(A \triangle B)$, for all A, $B \in \mathcal{P}(M)$, where $A \triangle B = (A \cup B) \backslash (A \cap B)$.

2. (Iurie Boreico) Prove that for all integers $n > 0$ and all a, $b \in (0, \frac{\pi}{4})$ we have

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \geq \frac{\sin^n 2a + \sin^n 2b}{(\sin 2a + \sin 2b)^n}\,.$$

3. (Marius Cavachi) For a real number x let $\lfloor x \rfloor$ be the greatest integer not exceeding x. Prove that the sequence given by $a_n = \lfloor n\sqrt{2} \rfloor + \lfloor n\sqrt{3} \rfloor$, where n is a non-negative integer, contains infinitely many odd numbers and infinitely many even numbers.

4. (Severius Moldoveanu and Costel Chiteş) Let $n \geq 2$ be an integer. Find n pairwise disjoint sets A_1, A_2, $\ldots$, A_n in the Euclidean plane such that

(a) For each circle $\mathcal{C}$ in the plane $A_i \cap \text{Interior}(\mathcal{C}) \neq \emptyset$, $1 \leq i \leq n$.

(b) For each line d in the plane and each A_i, the projection of A_i on d is not all of d.

We now look at solutions from our readers to problems of the 2004 Taiwanese Mathematical Olympiad given at [2007: 339].

1. Let $\mathbb{N}_0$ denote the set of non-negative integers. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(3m + 2n) = f(m) \cdot f(n)$ for all m, $n \in \mathbb{N}_0$.

Solution by Michel Bataille, Rouen, France, modified by the editor.

There are three solutions: the constant functions f_0 and f_1, which map each $k \in \mathbb{N}_0$ to 0 and 1, respectively; and the function f_2 defined by $f_2(0) = 1$ and $f_2(k) = 0$ for all $k \in \mathbb{N}_0 \setminus \{0\}$.

It is easy to check that f_0, f_1, and f_2 are solutions. To show the converse, we will use the following fact: any $u \in \mathbb{N}_0$ with $u \neq 1$ can be expressed as $u = 3m + 2n$ for some m,$n \in \mathbb{N}_0$. If $u = 0$, 2, or 5, then this is clear; for all other u let $u = 3p + r$, $0 \leq r \leq 2$, and write $u = 3(p - r) + 2(2r)$.

Let f be a solution. For each $u \geq 1$ we have

$$f(0) = f(1) = \cdots = f(u) = 1 \implies f(u+1) = 1\,; \qquad (1)$$
$$f(1) = \cdots = f(u) = 0 \implies f(u+1) = 0\,. \qquad (2)$$

To see these write $f(u+1) = f(3m + 2n) = f(m)f(n)$, where m,$n \in \mathbb{N}_0$ are such that $u+1 = 3m+2n$. Then (1) follows since $m \leq u$ and $n \leq u$, so that $f(m)f(n) = 1^2 = 1$, and (2) follows since $1 \leq m \leq u$ or $1 \leq n \leq u$, so that $f(m) = 0$ or $f(n) = 0$, respectively.

From $f(0) = f(3 \cdot 0 + 2 \cdot 0) = f(0)^2$, we obtain $f(0) = 0$ or 1.

Let $a = f(1)$. We have $f(5) = f(3 \cdot 1 + 2 \cdot 1) = a^2$, from which we deduce that $f(25) = f(3 \cdot 5 + 2 \cdot 5) = a^4$. Also

$$\begin{aligned} f(25) &= f(3 \cdot 7 + 2 \cdot 2) = f(7)f(2) = f(3 \cdot 1 + 2 \cdot 2)f(3 \cdot 0 + 2 \cdot 1) \\ &= a^2 f(0)f(2) = a^2 f(0)f(3 \cdot 0 + 2 \cdot 1) = a^3 f(0)\,, \end{aligned}$$

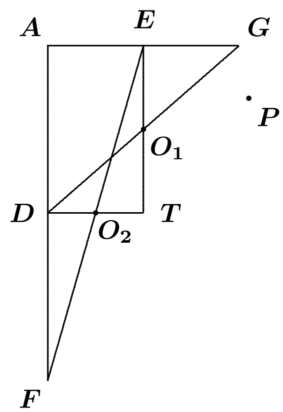
hence $a^4 = a^3 f(0)$.

Now if $f(0) = 0$, then $f(1) = a = 0$, and $f = f_0$ follows by induction using (2). If $f(0) = 1$, then $a^4 = a^3$, hence either $f(1) = 1$ or $f(1) = 0$. In the first case $f = f_1$ follows by induction using (1), in the second case $f = f_2$ follows by induction using (2).

3. Suppose that the points D and E lie on the circumcircle of $\triangle ABC$, ray $\overrightarrow{AD}$ is the interior angle bisector of $\angle BAC$, and ray $\overrightarrow{AE}$ is the exterior angle bisector of $\angle BAC$. Let F be the symmetrical point of A with respect to D, and let G be the symmetrical point of A with respect to E. Prove that, if the circumcircle of $\triangle ADG$ and the circumcircle of $\triangle AEF$ intersect at P, then AP is parallel to BC.

Solution by Titu Zvonaru, Comăneşti, Romania.

Suppose that the parallel to AD and passing through E intersects the parallel to AE and passing through D at the point T. Let O_1 be the circumcentre of triangle ADG and let O_2 be the circumcentre of triangle AEF.



Since O_1 is on the perpendicular bisector of AP, and O_2 is on the perpendicular bisector of AP, it follows that $O_1O_2 \perp AP$.

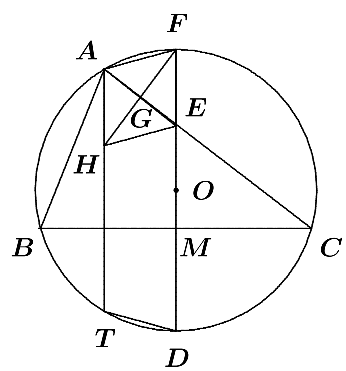
Since $AE \perp AD$, the point O_1 is the mid-point of DG and the mid-point of ET; the point O_2 is the mid-point of EF and the mid-point of DT. It follows that $O_1O_2 \| DE$.

Since AD is the internal bisector of $\angle BAC$, the point D is the mid-point of arc BC; we deduce that DE is a diameter of the circumcircle of $\triangle ABC$, hence $DE \perp BC$.

We have now shown that $O_1O_2 \perp AP$, $O_1O_2 \| DE$, and $DE \perp BC$, hence, AP is parallel to BC.

4. Let O and H be the circumcentre and orthocentre of an acute triangle ABC. Suppose that the bisectrix of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at D, and that the points E and F are symmetrical points of D with respect to BC and O, respectively. If AE and FH intersect at G and if M is the mid-point of BC, prove that GM is perpendicular to AF.

Solution by Titu Zvonaru, Comăneşti, Romania.



The point D is the mid-point of the arc BC, and FD is the perpendicular bisector of BC. If $AB = AC$, then points D, M, O, E, G, A, and F are collinear, so we assume $AB \neq AC$.

We will prove that $AHEF$ is a parallelogram. In that case it follows that G is the mid-point of AE, and since M is the mid-point of DE, we see (in $\triangle ADE$) that $GM \| AD$. Since DF is a diameter, $AD \perp AF$, hence $GM \perp AF$. We offer two demonstrations.

Proof 1: As usual, $a = BC$ and R is the circumradius of $\triangle ABC$. Since $\angle MBD = \frac{1}{2}\angle BAC$, we have that $MD = \frac{a}{2}\tan\frac{A}{2}$. We obtain

$$\begin{aligned} EF &= 2R - 2MD \;=\; 2R - a\tan\frac{A}{2} \\ &= 2R - 2R\cdot\sin A\tan\frac{A}{2} \\ &= 2R\left(1 - 2\sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2}\right) \\ &= 2R\left(1 - 2\sin^2\frac{A}{2}\right) \;=\; 2R\cos A\,. \end{aligned}$$

It is known that $AH = 2R\cos A$; because $AH \perp BC$, $EF \perp BC$, and $AH = EF$, it follows that $AHEF$ is a parallelogram.

Proof 2: Let AH intersect the circumcircle at T.

It is known that T is the symmetric point of H with respect of BC. It follows that the trapezoid $HTDE$ is isosceles. Since the trapezoid $ATDF$ is isosceles, we have

$$\angle HED = \angle EDT = \angle AFE\,,$$

hence $AF \| HE$. However, $AH \| EF$, therefore the quadrilateral $AHEF$ is a parallelogram.

Next, an apology. When I was writing up the material for the May number of the *Corner* I somehow left off Michel Bataille's name in the list of solvers for Problem 5 of the Albanian Mathematical Olympiad (Test 2) discussed at [2007: 278–279, 2008: 222–224]. We recently received an alternate solution by Li Zhou, which we give now.

5. In an acute-angled triangle ABC, let H be the orthocentre, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove that $d_a + d_b + d_c \le 3r$, where r is the radius of the incircle of triangle ABC.

Alternate solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $a = BC$, $b = CA$, and $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Without loss of generality, assume that $a \le b \le c$. Then

$$\begin{aligned} d_a &= CH\cos B \;\le\; CH\cos A \;=\; d_b \\ &= AH\cos C \;\le\; AH\cos B \;=\; d_c\,. \end{aligned}$$

Denote by $[ABC]$ the area of triangle ABC. By Chebyshev's Inequality,

$$sr \;=\; [ABC] \;=\; \frac{1}{2}(ad_a + bd_b + cd_c) \;\ge\; \frac{1}{3}s(d_a + d_b + d_c)\,,$$

completing the proof.

Now we turn to our files of solutions from our readers and to the XXV Brazilian Mathematical Olympiad 2003 given at [2007: 410].

1. Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer n.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Kandall's solution.

The answer is 17.

Let $P(n) = n^2 + 5n + 23$. Since $P(-2) = 17$, we need only show that $P(n) \not\equiv 0 \pmod{p}$ if $p \in \{2, 3, 5, 7, 11, 13\}$.

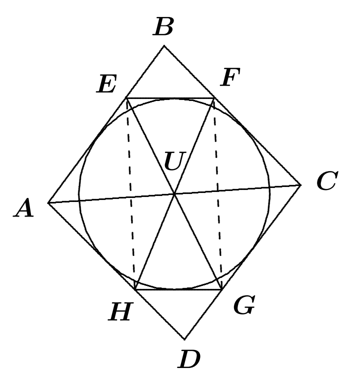
For instance, if $n \equiv 0, 1, 2, 3, 4, 5, 6 \pmod{7}$, then we have respectively $P(n) \equiv 2, 1, 2, 5, 3, 3, 5 \pmod{7}$. Thus, the desired conclusion is true for $p = 7$.

The complete calculation is summarized in the table below.

		$P(n) \pmod{p}$					
n	$P(n)$	2	3	5	7	11	13
0	23	1	2	3	2	1	10
1	29	1	2	4	1	7	3
2	37		1	2	2	4	11
3	47			2	5	3	8
4	59			4	3	4	7
5	73				3	7	8
6	89				5	1	11
7	107					8	3
8	127					6	10
9	149					6	6
10	173					8	4
11	199						4
12	227						6

3. Let $ABCD$ be a rhombus. Let E, F, G, and H be points on the sides AB, BC, CD, and DA, respectively, so that EF and GH are tangent to the incircle of $ABCD$. Show that EH and FG are parallel.

Solved by Michel Bataille, Rouen, France.



The hexagon $AEFCGH$ circumscribes the incircle of the rhombus, so Brianchon's theorem implies that the lines AC, EG, and FH are concurrent. Let U be their common point and let h denote the homothety with centre U which transforms A into C. Since AE and CG are parallel and E, U, and G are collinear, we see that $h(E) = G$. Similarly, $h(H) = F$ and $EH \| FG$ follows.

5. Let $f(x)$ be a real-valued function defined on the positive reals such that

(i) $f(x) < f(y)$ if $x < y$, and

(ii) $f\left(\dfrac{2xy}{x+y}\right) = \dfrac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

Solution by Michel Bataille, Rouen, France.

As f is an increasing function on $(0,\infty)$, either $\lim\limits_{x\to 0^+} f(x) = -\infty$ or $\lim\limits_{x\to 0^+} f(x) = a$ for some real number a. Assume that the latter holds. In the relation

$$f\left(\frac{2xy}{x+y}\right) = \frac{f(x)+f(y)}{2},$$

fix $x > 0$ and let y approach 0^+. Since $\lim\limits_{y\to 0^+} \dfrac{2xy}{x+y} = 0$, it follows that

$$a = \frac{f(x)+a}{2},$$

hence, $f(x) = a$. Consequently, f would be a constant function, contrary to (i). Thus, $\lim\limits_{x\to 0^+} f(x) = -\infty$ and certainly $f(x) < 0$ for some positive x.

Next we turn to solutions of problems of the Second and Third Selection Tests of the 2004 Republic of Moldova, given at [2007: 411–412].

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the relation

$$f(x^3) - f(y^3) = (x^2+xy+y^2)(f(x)-f(y))$$

for all real numbers x and y.

Solved by Michel Bataille, Rouen, France; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. We give the solution of Malikić.

Taking $y = 0$ in the identity yields $f(x^3) - f(0) = x^2(f(x) - f(0))$. Setting $g(x) = f(x) - f(0)$ we have $g(x^3) = x^2 g(x)$. The following are then equivalent

$$\begin{aligned}
f(x^3) - f(y^3) &= (x^2+xy+y^2)(f(x)-f(y)),\\
g(x^3) - g(y^3) &= (x^2+xy+y^2)(g(x)-g(y)),\\
x^2 g(x) - y^2 g(y) &= x^2 g(x) + xy\, g(x) + y^2 g(x)\\
&\quad - x^2 g(y) - xy\, g(y) - y^2 g(y),\\
0 &= xy\, g(x) + y^2 g(x) - x^2 g(y) - xy\, g(y)\\
0 &= (x+y)(y\, g(x) - x\, g(y)). \qquad (1)
\end{aligned}$$

Taking $y = 1$ in equation (1), we must have $(x+1)\big(g(x) - x \cdot g(1)\big) = 0$. Thus, for all $x \in \mathbb{R}\backslash\{-1\}$, we must have $g(x) = xg(1)$, or equivalently $f(x) - f(0) = x\big(f(1) - f(0)\big)$. This means that $f(x) = kx + c$ for all $x \in \mathbb{R}\backslash\{-1\}$, where $k = f(1) - f(0)$ and $c = f(0)$.

By what we have just done, $f(2^3) = f(8) = 8k+c$ and $f(2) = 2k+c$, thus, taking $x = 2$ and $y = -1$ in the identity for f yields

$$8k + c - f(-1) \;=\; 3\big(2k + c - f(-1)\big)\,.$$

Solving for $f(-1)$ we obtain $f(-1) = k(-1) + c$. Finally, we conclude that $f(x) = kx + c$, where k and c are constants.

Conversely, if $f(x) = kx + c$ where k and c are arbitrary constants, then one readily checks that this f satisfies the required identity for all reals x and y.

7. Let ABC be an acute-angled triangle with orthocentre H and circumcentre O. The inscribed and circumscribed circles have radii r and R, respectively. If P is an arbitrary point of the segment $[OH]$, prove that $6r \le PA + PB + PC \le 3R$.

Solution by Arkady Alt, San Jose, CA, USA.

Let $\overrightarrow{PO} = t\overrightarrow{HO}$, $t \in [0,1]$ and let $X \in \{A,\, B,\, C\}$. Then

$$\begin{aligned}\overrightarrow{PX} &= \overrightarrow{PO} + \overrightarrow{OX} \;=\; t\,\overrightarrow{HO} + \overrightarrow{OX} \\ &= t\left(\overrightarrow{HX} + \overrightarrow{XO}\right) \;+\; \overrightarrow{OX} \;=\; (1-t)\,\overrightarrow{OX} \;+\; t\,\overrightarrow{HX}\,.\end{aligned}$$

Since $|\overrightarrow{PX}| = |(1-t)\overrightarrow{OX} + t\overrightarrow{HX}| \le (1-t)\,|\overrightarrow{OX}| + t\,|\overrightarrow{HX}|$, we have

$$\begin{aligned}PA + PB + PC &= \sum_{\text{cyclic}} |\overrightarrow{PA}| \;\le\; \sum_{\text{cyclic}} \Big((1-t)\,|\overrightarrow{OA}| + t\,|\overrightarrow{HA}|\Big) \\ &= 3(1-t)R + t\sum_{\text{cyclic}} HA\,.\end{aligned}$$

For any vertex X, $HX = 2R\cos X$. Also, $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$ and Euler's Inequality, $R \ge 2r$, holds. Thus,

$$\begin{aligned}PA + PB + PC &\le 3(1-t)R \;+\; 2Rt(\cos A + \cos B + \cos C) \\ &= 3(1-t)R \;+\; t(2R + 2r) \\ &\le 3(1-t)R \;+\; t(2R + R) \;=\; 3R\,.\end{aligned}$$

Next we prove the inequality $6r \le PA + PB + PC$ for *any* interior point P in the acute-angled triangle ABC.

For each vertex X let R_X be the distance from P to X. Let h_a, h_b, and h_c be the heights of the triangle to the corresponding side, and let d_a, d_b, and d_c be the distances from P to the corresponding side.

Since $R_A + d_a \geq h_a$ we have $\sum_{\text{cyclic}} (R_a + d_a) \geq \sum_{\text{cyclic}} h_a$. By the Erdös–Mordell Inequality in the form $\sum_{\text{cyclic}} d_a \leq \frac{1}{2} \sum_{\text{cyclic}} R_A$ and the preceding inequality we have $\frac{3}{2} \sum_{\text{cyclic}} R_A \geq \sum_{\text{cyclic}} h_a$, or equivalently $\frac{2}{3} \sum_{\text{cyclic}} h_a \leq \sum_{\text{cyclic}} R_A$.

Since

$$h_a + h_b + h_c \;=\; 2F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \;\geq\; 2F\left(\frac{9}{a+b+c}\right) \;=\; \frac{9F}{2s} \;=\; 9r\,,$$

where F is the area of triangle ABC, we finally obtain

$$6r \;\leq\; \frac{2}{3}(h_a + h_b + h_c) \;\leq\; R_a + R_b + R_c\,.$$

Equality occurs if and only if P is the circumcenter and $a = b = c$.

9. For all positive real numbers a, b, and c, prove the inequality

$$\left|\frac{4(a^3 - b^3)}{a+b} + \frac{4(b^3 - c^3)}{b+c} + \frac{4(c^3 - a^3)}{c+a}\right| \;\leq\; (a-b)^2 + (b-c)^2 + (c-a)^2\,.$$

Solved by Arkady Alt, San Jose, CA, USA.

Let $G(a,b,c) = (a-b)^2 + (b-c)^2 + (c-a)^2$ and

$$F(a,b,c) \;=\; \frac{4(a^3 - b^3)}{a+b} \;+\; \frac{4(b^3 - c^3)}{b+c} \;+\; \frac{4(c^3 - a^3)}{c+a}\,.$$

It suffices to prove that $F(a,b,c) \leq G(a,b,c)$ for all positive real numbers a, b, and c. Indeed, if $F(a,b,c) < 0$ then under this assumption we have

$$|F(a,b,c)| \;=\; -F(a,b,c) \;=\; F(b,a,c) \;\leq\; G(b,a,c) \;=\; G(a,b,c)\,.$$

For positive a and b we have $\frac{4b^2}{a+b} \geq 3b - a$, since this is equivalent to $4b^2 \geq 3b^2 - a^2 + 2ab$, and hence to $(a-b)^2 \geq 0$. We now have

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{4(a^3 - b^3)}{a+b} \;&=\; 4\sum_{\text{cyclic}} \frac{a^3 + b^3}{a+b} \;-\; 2\sum_{\text{cyclic}} \frac{4b^3}{a+b} \\
&\leq\; 4\sum_{\text{cyclic}} (a^2 - ab + b^2) \;-\; 2\sum_{\text{cyclic}} b(3b - a) \\
&=\; \sum_{\text{cyclic}} (4a^2 - 4ab + 4b^2 - 6b^2 + 2ab) \\
&=\; \sum_{\text{cyclic}} (4a^2 - 2ab - 2b^2) \;=\; \sum_{\text{cyclic}} (a-b)^2\,.
\end{aligned}$$

10. Determine all the polynomials $P(X)$ with real coefficients which satisfy the relation

$$(x^3+3x^2+3x+2)P(x-1) \;=\; (x^3-3x^2+3x-2)P(x)$$

for every real number x.

Solved by Arkady Alt, San Jose, CA, USA. Comment by Michel Bataille, Rouen, France.

This problem was one of the problems of the 2003 Vietnamese Mathematical Olympiad. A solution appeared in this journal at [2007: 90–91].

11. Let ABC be an isosceles triangle with $AC = BC$, and let I be its incentre. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The straight lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the straight lines DF and GE intersect on the circumcircle of the triangle ABC.

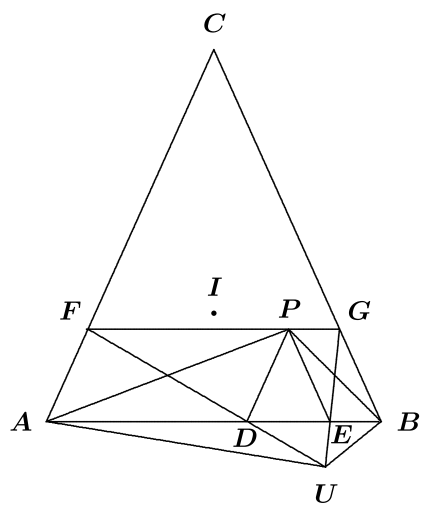
Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain.

For convenience let

$$\angle CAB \;=\; \alpha\,,\ \angle ACB \;=\; \gamma\,,$$
$$\angle DFP \;=\; \xi\,,\ \angle GPB \;=\; \omega\,.$$

We then have

$$\begin{aligned}
\angle APB &= \angle AIB \;=\; \alpha+\gamma\,,\\
\angle APF &= 180^\circ-(\gamma+\alpha)-\omega\\
&= \alpha-\omega\,,\\
\angle GBP &= \alpha-\omega\,,\\
\angle PAF &= \omega\,.
\end{aligned}$$



Thus, $GBEP \sim FPDA$, so that $\angle EGP = \angle DFA = 180^\circ-\alpha-\xi = \alpha+\gamma-\xi$.

Let U be the intersection of the lines EG and FD. We then have

$$\begin{aligned}
\angle GUF &= 180 \;-\; \angle DFP \;-\; \angle EGP\\
&= (2\alpha+\gamma) \;-\; \xi \;-\; (\alpha+\gamma-\xi) \;=\; \alpha\,.
\end{aligned}$$

Now, $UBGD$ is inscribable, since $\angle DUG = \angle DBG = \alpha$. Also, $PGBD$ is inscribable, since $\angle DPG+\angle GBD = 180^\circ$. Thus, $PGBUD$ is inscribable and $\angle GUB = \angle GPB = \omega$. Similarly, $\angle FUA = \angle FPA = \alpha-\omega$ and $\angle AUB = 2\alpha$; hence, U is on the circumcircle of triangle ABC.

That completes this number of the *Corner*. Send solutions soon!

BOOK REVIEWS

John Grant McLoughlin

Calculus Gems: Brief Lives and Memorable Mathematics
By George F. Simmons, Mathematical Association of America, 2007
ISBN 978-0-88385-561-4, hardcover, 355+xv pages, US$49.95
Reviewed by **Robert D. Poodiack**, *Norwich University, Northfield, VT, USA*

This recent reissue of Simmons' 1992 book is a wonderful source of tidbits and examples for teachers and professors of calculus. However, many teachers may end up using the first part of the book more than the second.

In the introduction to *Calculus Gems*, Simmons talks about an impulse to "humanize" calculus in order to bring students closer to the subject. The 33 biographies that comprise the first part of the book accomplish his goal and entice novice readers and professors alike. Many current calculus books give early motivation by describing how Archimedes solved the area problem (how to calculate the area of a non-standard shaped region) and the tangent problem (how to compute the slope of a line tangent to a curve, given only the point of tangency) by using what we would call limiting processes.

Part A of Simmons' book expands the motivational material into a full history of calculus. Simmons begins with Thales' work on geometry and proving theorems and finishes with Weierstrass' work on the foundations of calculus, a period of 2400 years. The life stories are fascinating – this volume would make a wonderful history book by itself – but even better is the way Simmons weaves the development of calculus concepts through the biographies. Most students know that Newton and Leibniz are the fathers of calculus, but they'll be amazed how far back some of their basic concepts date. As much as our students want to blame 17^{th} century mathematicians for their woes, Leibniz and Newton were only rephrasing problems solved 18 centuries before by Archimedes. Students who give these stories a chance will find much to enjoy, not only in the writing but also in the parallels to their experience. Hey, these guys all had trouble with this stuff too!

The second part of the book consists of "Memorable Mathematics," expansions upon topics discussed in Part A. Simmons intends this section to follow along with a typical calculus curriculum. (Six sections even have attached problems for students.) He says in the introduction that "Many of my students have found these 'nuggets' interesting and eye-opening." That they are, but I'm guessing that Simmons had some pretty advanced students. Certainly many topics that turn up in calculus books are present. The sections that include sequences and series, the cycloid, or integration are all standard material, but are presented in a zestful, easy-to-read manner. However, calculus professors may want to relegate some of the number-theoretic material – proofs of the transcendence of π and e for instance – more to the "mention in passing" category. These topics are also presented quite clearly, and the transcendence proofs are indeed based on calculus – quite different from ones I had seen before.

We note that this book is a straight reprint of the 1992 original. The only updating was to Simmons' biography. The chapter on Fermat reveals that his Last Theorem, alas, has not yet been proved.

Calculus Gems stands as an excellent historical study for calculus students and teachers alike that divides down the middle. While the students will enjoy the biographies in Part A, the faculty may get more from the memorable mathematics in Part B. *Calculus Gems* should start many good classroom conversations about calculus.

From Zero to Infinity: What Makes Numbers Interesting (Fiftieth Anniversary Edition)

By Constance Reid, published by A.K. Peters, 2006

ISBN 1-56881-273-6, softcover, 188+xvii pages, US$19.95

Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB*

The 2006 publication of the fiftieth anniversary of this 1955 classic gives rise to another discussion of interesting numbers. Constance Reid is the sister of Julia Robinson (the logician noted for her work on Hilbert's tenth problem). A phone call from Julia in 1952 inspired the writing. Julia's husband, Raphael M. Robinson, had discovered more perfect numbers.

The book has 12 chapters. The initial chapter, Chapter 0, is titled Zero, Chapter 1 is titled One, and so forth until Chapter 9, titled Nine. The other two chapters are Euler's Number and Aleph-Zero. The first ten chapters are each ten to fifteen pages in length, whereas, the two concluding chapters are each twenty pages in length. The chapter contents appear thematic as with binary arithmetic (Chapter 2) or perfect numbers (Chapter 6), for example. A curious feature is the inclusion of a challenge (with answers written upside down) at the end of each chapter. Some challenges enriched the appreciation of the numbers; however, others seemed to be forced by the need for consistency in style. An unfortunate error is the mismatch between page numbers of chapters in the Table of Contents and those actually in the book.

References to number theory, modular arithmetic, or particular mathematical terms are sprinkled throughout the book. Even so, the first ten chapters of the book are accessible to high school students. Familiarity with undergraduate mathematics would enrich the value of the two concluding chapters in which calculus, series, and infinite sets figure prominently. A class at any level could avail of the book as a source of learning. Chapters can serve as independent studies or starting points for discussion. Chapter 0 is a good place to start for those who have taken this special number for granted.

Overall, I recommend this book to those who wish to see more of the beauty in mathematics. Much can be learned through the surprises and rich connections offered in this gem.

The Sum of a Cube and a Fourth Power

Thomas Mautsch and Gerhard J. Woeginger

1 Introduction

The book *Problems in Elementary Number Theory* (published in Romanian) by Paul Radovici-Mărculescu [2] contains the following nice problem:

Problem A. Prove that the number 19^{19} cannot be written as the sum of a cube and a fourth power.

A closely related problem was discussed in the summer of 2006 on the German-language Usenet puzzle newsgroup `de.rec.denksport`:

Problem B. Do there exist 20 consecutive integers, that can all be written as the sum of a cube and a fourth power?

Both problems can be settled quite easily by working modulo 13. A cube leaves one of the residues 0, 1, 5, 8, or 12 modulo 13, while a fourth power leaves one of the residues 0, 1, 3, or 9 modulo 13. It follows that the sum of a cube and a fourth power leaves one of the residues 0, 1, $\dots$, 6; or 8, 9, $\dots$, 12 modulo 13. Thus, the sum of a cube and a fourth power never leaves a residue of 7 modulo 13. This settles Problem A, since the residue of 19^{19} is 7 modulo 13. It also settles Problem B in the negative, since at least one of 20 consecutive integers has a residue of 7 modulo 13.

Working modulo 13 is a good idea in these problems because modulo 13 there are only a few residue classes that are cubes and fourth powers. But is there anything special about the number 13? Aren't there other moduli that would work equally well? In this article, we will answer these questions. Further, we will show that working modulo any other integer n (which is not a multiple of 13) will not solve these problems.

Theorem 1 For each positive integer n that is not divisible by 13, the congruence $x^3 + y^4 \equiv r \pmod{n}$ in x and y is solvable for all integers r.

2 Our Tool Kit

The proposition below is a consequence of the Chinese Remainder Theorem.

Proposition 1 Let r be an integer, and let n_1, n_2, $\dots$, n_ℓ be integers that are relatively prime in pairs. If $x^3 + y^4 \equiv r \pmod{n}$ is solvable in integers for each $n \in \{n_1, n_2, \dots, n_\ell\}$, then it is solvable for $n = n_1 n_2 \cdots n_\ell$.

We will also make use of the following case of Fermat's Little Theorem.

Proposition 2 For all integers x, we have $x^3 \equiv x \pmod 3$.

The next proposition is folklore (and a trivial case of Hensel's Lemma). It can be derived, for instance, from the results in Chapter 4 of the book by Ireland and Rosen [1].

Proposition 3 Let p be a prime, and let k and r be integers, $k \not\equiv 0 \pmod p$ and $r \not\equiv 0 \pmod p$, such that $z^k \equiv r \pmod p$ is solvable in integers. Then $z^k \equiv r \pmod{p^b}$ is solvable in integers for all $b \geq 1$.

Our main tool is a special case of a famous result of André Weil [4].

Proposition 4 (A. Weil) Let p be a prime, let k_1 and k_2 be positive integers, and let r be an integer with $r \not\equiv 0 \pmod p$. Then the number N of solutions (x, y) of the congruence $x^{k_1} + y^{k_2} \equiv r \pmod p$ with $0 \leq x,\ y \leq p-1$ satisfies

$$|N - p| \;\leq\; \big(\gcd(k_1, p-1) - 1\big) \cdot \big(\gcd(k_2, p-1) - 1\big) \cdot \sqrt{p}\,.$$

3 The Proof of the Main Theorem

Now let us turn to the proof of Theorem 1. By Proposition 1, we need only prove the main theorem for prime powers $n = p^b$, where p is a prime other than 13. We distinguish four main cases.

Case 1. $n = 2^b$. The pair $(x, y) = (1, r-1)$ is a solution of the congruence $x^3 + y^4 \equiv r \pmod 2$. Since it also satisfies the condition $x \not\equiv 0 \pmod 2$, by Proposition 3 the congruence $x^3 \equiv r - y^4 \pmod{2^b}$ with $y = r - 1$ is solvable for all integers $b \geq 1$.

Case 2. $n = 3^b$. By Proposition 2 the pair $(x, y) = (r-1, 1)$ is a solution of the congruence $x^3 + y^4 \equiv r \pmod 3$. Since it also satisfies the condition $y \not\equiv 0 \pmod 3$, by Proposition 3 the congruence $y^4 \equiv r - x^3 \pmod{3^b}$ with $x = r - 1$ is solvable for all integers $b \geq 1$.

Case 3. $n = p$ for some prime $p \geq 5$ and p $\neq 13$. Suppose for the sake of contradiction that there is an integer r for which $x^3 + y^4 \equiv r \pmod p$ has no solution. Then $r \not\equiv 0 \pmod p$, and Proposition 4 with $N = 0$ yields the inequality

$$\sqrt{p} \;\leq\; \big(\gcd(3, p-1) - 1\big) \cdot \big(\gcd(4, p-1) - 1\big)\,.$$

If $\gcd(3, p-1) = 1$, then we obtain the contradiction $p \leq 0$. If we have $\gcd(4, p-1) \leq 2$, then we obtain the contradiction $p \leq 4$. The remaining possibility is that $\gcd(3, p-1) = 3$ and $\gcd(4, p-1) = 4$, in which case $p \equiv 1 \pmod{12}$. Furthermore the displayed inequality yields $p \leq 36$. The only prime with these properties is $p = 13$, which yields the final contradiction. Therefore, $x^3 + y^4 \equiv r \pmod{p}$ is solvable for every integer r.

Case 4. $n = p^b$ for some prime $p \geq 5$ and $p \neq 13$. The discussion of Case 3 shows that for every r, there exist x_0 and y_0 with $x_0^3 + y_0^4 \equiv r \pmod{p}$.

(a) If $x_0 \not\equiv 0 \pmod{p}$, then the congruence $x^3 \equiv r - y^4 \pmod{p^b}$ with $y = y_0$ is solvable by Proposition 3.

(b) If $y_0 \not\equiv 0 \pmod{p}$, then the congruence $y^4 \equiv r - x^3 \pmod{p^b}$ with $x = x_0$ is solvable by Proposition 3.

(c) If $x_0 \equiv 0 \pmod{p}$ and $y_0 \equiv 0 \pmod{p}$, then $r \equiv 0 \pmod{p}$. In this case, the congruence $x_0^3 + y_0^4 \equiv r \equiv 0 \pmod{p}$ possesses another solution $(x_0, y_0) = (-1, 1)$, which reduces the proof to the case $x_0 \not\equiv 0 \pmod{p}$ in part (a) above.

This completes the analysis of the last case.

Thus, in all four cases, there is a solution to $x^3 + y^4 \equiv r \pmod{p^b}$. The proof of Theorem 1 is complete.

4 Final Remarks

It is well known that if k_1 and k_2 are positive integers and p is a prime, then $x^{k_1} + y^{k_2} \equiv r \pmod{p}$ is solvable for all integers r *except* for finitely many primes p. Indeed, by Proposition 4 the exceptional primes satisfy

$$\sqrt{p} \leq \left(\gcd(k_1, p-1) - 1\right) \cdot \left(\gcd(k_2, p-1) - 1\right).$$

From this the reader may verify that each of the following enumerations of instances $x^{k_1} + y^{k_2} \not\equiv r \pmod{p}$ is complete for the respective exponents k_1 and k_2.

$$\begin{aligned}
x^2 + y^5 &\not\equiv 7 \pmod{11}\ ; \\
x^2 + y^9 &\not\equiv 13, 14 \pmod{19}\ ,\ x^2 + y^9 \not\equiv \pm 14 \pmod{37}\ ; \\
x^3 + y^5 &\not\equiv \pm 12 \pmod{31}\ ; \\
x^3 + y^7 &\not\equiv \pm 13, \pm 19 \pmod{43}\ ; \\
x^3 + y^8 &\not\equiv 7 \pmod{13}\ ; \\
x^4 + y^5 &\not\equiv 7 \pmod{11}\ ,\ x^4 + y^5 \not\equiv \pm 6, \pm 12 \pmod{41}\ ; \\
x^4 + y^7 &\not\equiv 5, 9, 10, 14, 27 \pmod{29}\ ; \\
x^5 + y^7 &\not\equiv \pm 8, \pm 10, \pm 11 \pmod{71}\ ; \\
x^5 + y^8 &\not\equiv 7 \pmod{11}\ , \\
x^5 + y^8 &\not\equiv \pm 6, 8, \pm 12, 20, 22, 26, 31 \pmod{41}\ ; \\
x^7 + y^8 &\not\equiv 5, 9, 10, 14, 27 \pmod{29}\ , \\
x^7 + y^8 &\not\equiv \pm 21, \pm 23, \pm 54 \pmod{113}\ .
\end{aligned}$$

Eckard Specht [3] poses the following nice puzzle in the Problem Corner of the *Mathematical Gazette*:

Problem C. Determine all integer solutions (x, y) of $x^3 + y^7 = 2004^{2008}$.

The 2005 USA Mathematical Olympiad consisted of six challenging problems. Here is the second problem from this competition:

Problem D. (2005 USAMO) Prove that the system

$$\begin{aligned} x^6 + x^3 + x^3y + y &= 147^{157} \\ x^3 + x^3y + y^2 + y + z^9 &= 157^{147} \end{aligned}$$

has no solutions in integers x, y, and z.

And here is a minor variation on the theme of this article:

Problem E. Prove that there are infinitely many integers which cannot be written in the form $x^3 - y^4$, where x and y are integers.

We encourage the reader to settle these problems along the lines indicated above. For problem D, it is convenient to first add the two equations.

References

[1] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, 1992.

[2] P. Radovici-Mărculescu, *Probleme de Teoria Elementară a Numerelor*, Editura Tehnică, Bucharest, 1983.

[3] E. Specht, Problem 91.K., *Mathematical Gazette* **91** (2007) p. 561.

[4] A. Weil, Numbers of solutions of equations in finite fields. *Bulletin of the American Mathematical Society* **55** (1949) pp. 497–508.

Thomas Mautsch
Ettenhauserstrasse 42
CH-8062 Wetzikon ZH
Switzerland
`mautsch@ethz.ch`

Gerhard J. Woeginger
Dept. of Mathematics and Computer Science
TU Eindhoven
P.O. Box 513, NL-5600 MB Eindhoven
The Netherlands
`gwoegi@win.tue.nl`

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er avril 2009**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3363. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit ABC un triangle avec $\angle ACB = 90^\circ + \frac{1}{2}\angle ABC$. Soit M le point milieu de BC. Montrer que $\angle AMC < 60^\circ$.

3364. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit ABC un triangle avec $\angle BAC = 120^\circ$ et $AB > AC$. Soit M le point milieu de BC. Montrer que $\angle MAC > 2\angle ACB$.

3365. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Un carré $ABCD$ est inscrit dans un cercle Γ. Soit P un point sur la petite portion de l'arc AD de Γ, et soit E et F les intersections respectives de AD avec PB et PC. Montrer quer

$$AE \cdot DF \;=\; 2\big([PAE] + [PDF]\big)\,,$$

où $[KLM]$ désigne l'aire du triangle KLM.

3366. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

Soit $\{x\}$ la partie fractionnaire du nombre réel x ; à savoir, $\{x\} = x - \lfloor x \rfloor$, où $\lfloor x \rfloor$ est la partie entière de x. Calculer

$$\int_0^1 \left\{\frac{1}{x}\right\}^4 dx\,.$$

3367. *Proposé par Li Zhou, Polk Community College, Winter Haven, FL, É-U.*

Soit $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$ un polynôme à coefficients entiers, où $a_n > 0$ et $\sum\limits_{k=1}^{n} a_k = 1$. Existe-t-il, oui ou non, une infinité de paires d'entiers positifs (k, ℓ) tels que $p(k+1) - p(k)$ et $p(\ell+1) - p(\ell)$ soient relativement premiers.

3368. *Proposé par Neven Jurič, Zagreb, Croatie.*

Soit m un entier, $m \geq 2$, et soit $A = [A_{ij}]$ une matrice en blocs, de dimensions $2^m \times 2^m$, avec $A_{ij} \in M_{4,4}(\mathbb{N})$ pour $1 \leq i,\, j \leq 2^{m-2}$ et définie par $A_{ij} = 2^m B_{ij} + C_{ij}$, où

$$B_{ij} \;=\; \begin{bmatrix} 2^m-4i+4 & 4i-4 & 4i-4 & 2^m-4i+4 \\ 4i-3 & 2^m-4i+3 & 2^m-4i+3 & 4i-3 \\ 4i-2 & 2^m-4i+2 & 2^m-4i+2 & 4i-2 \\ 2^m-4i+1 & 4i-1 & 4i-1 & 2^m-4i+1 \end{bmatrix},$$

$$\text{et}\quad C_{ij} \;=\; \begin{bmatrix} 4-4j & 4j-2 & 4j-1 & 1-4j \\ 4j-3 & 3-4j & 2-4j & 4j \\ 4j-3 & 3-4j & 2-4j & 4j \\ 4-4j & 4j-2 & 4j-1 & 1-4j \end{bmatrix}.$$

Montrer que la matrice A est un carré magique d'ordre 2^m.

3369. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit $A_1A_2A_3A_4$ un tétraèdre qui contient le centre O de sa sphère circonscrite comme point intérieur. Soit ρ_i la distance de O à la face opposée au sommet A_i. Si R est le rayon de la sphère circonscrite, montrer que

$$\frac{4}{3}R \;\geq\; \sum_{i=1}^{4} \rho_i\,.$$

3370. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit a_i et b_i des nombres réels positifs avec $1 \leq i \leq k$, et soit n un entier positif. Montrer que

$$\left(\sum_{i=1}^{k} a_i^{\frac{1}{n}}\right)^{n} \;\leq\; \left(\sum_{i=1}^{k} \frac{a_i}{b_i}\right)\left(\sum_{i=1}^{k} b_i^{\frac{1}{n-1}}\right)^{n-1}.$$

3371. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit ABC un triangle de côtés respectifs a, b et c, et soit M un de ses points intérieur. Les droites AM, BM et CM coupent respectivement les côtés opposés aux points A_1, B_1 et C_1. Les droites passant par M et perpendiculaires aux côtés coupent respectivement BC, CA et AB en A_2, B_2, et C_2. Soit p_1, p_2 et p_3 les distances respectives de M aux côtés BC, CA et AB. Montrer que

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} \;=\; \frac{(ap_1+bp_2)(bp_2+cp_3)(cp_3+ap_1)}{8a^2b^2c^2}\left(\frac{a}{p_1}+\frac{b}{p_2}+\frac{c}{p_3}\right),$$

où $[KLM]$ désigne l'aire du triangle KLM.

3372. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Si x, y, $z \geq 0$ et $xy + yz + zx = 1$, montrer que

(a) $$\frac{1}{\sqrt{2x^2+3yz}} + \frac{1}{\sqrt{2y^2+3zx}} + \frac{1}{\sqrt{2z^2+3xy}} \geq \frac{2\sqrt{6}}{3} \;;$$

(b)★ $$\frac{1}{\sqrt{x^2+yz}} + \frac{1}{\sqrt{y^2+zx}} + \frac{1}{\sqrt{z^2+xy}} \geq 2\sqrt{2} \;.$$

3373. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Montrer que, quels que soient les nombres réels positifs x, y, z et t,

$$(x+y)(x+z)(x+t)(y+z)(y+t)(z+t) \;\geq\; 4xyzt(x+y+z+t)^2 \,.$$

3374. *Proposé par Pham Huu Duc, Ballajura, Australie.*

Montrer que, quels que soient les nombres réels positifs a, b et c,

$$\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab} \;\leq\; \frac{a+b+c}{2\sqrt[3]{abc}} \,.$$

3375. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

Soit p un entier non négatif et x un nombre réel quelconque. Evaluer la somme

$$\sum_{n=1}^{\infty} (-1)^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^{n+p}}{(n+p)!} \right) \,.$$

....

3363. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle ACB = 90^\circ + \frac{1}{2}\angle ABC$. Let M be the mid-point of BC. Prove that $\angle AMC < 60^\circ$.

3364. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle BAC = 120^\circ$ and $AB > AC$. Let M be the mid-point of BC. Prove that $\angle MAC > 2\,\angle ACB$.

3365. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A square $ABCD$ is inscribed in a circle Γ. Let P be a point on the minor arc AD of Γ, and let E and F be the intersections of AD with PB and PC, respectively. Prove that

$$AE \cdot DF \;=\; 2\big([PAE] + [PDF]\big) \,,$$

where $[KLM]$ denotes the area of triangle KLM.

3366. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let $\{x\}$ denote the fractional part of the real number x; that is, $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}^4 dx\,.$$

3367. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$ be a polynomial with integer coefficients, where $a_n > 0$ and $\sum\limits_{k=1}^{n} a_k = 1$. Prove or disprove that there are infinitely many pairs of positive integers (k, ℓ) such that $p(k+1) - p(k)$ and $p(\ell+1) - p(\ell)$ are relatively prime.

3368. *Proposed by Neven Jurič, Zagreb, Croatia.*

Let m be an integer, $m \geq 2$, and let $A = [A_{ij}]$ be a block matrix of dimension $2^m \times 2^m$ with $A_{ij} \in M_{4,4}(\mathbb{N})$ for $1 \leq i,\ j \leq 2^{m-2}$, defined by $A_{ij} = 2^m B_{ij} + C_{ij}$, where

$$B_{ij} \;=\; \begin{bmatrix} 2^m - 4i + 4 & 4i - 4 & 4i - 4 & 2^m - 4i + 4 \\ 4i - 3 & 2^m - 4i + 3 & 2^m - 4i + 3 & 4i - 3 \\ 4i - 2 & 2^m - 4i + 2 & 2^m - 4i + 2 & 4i - 2 \\ 2^m - 4i + 1 & 4i - 1 & 4i - 1 & 2^m - 4i + 1 \end{bmatrix},$$

$$\text{and} \quad C_{ij} \;=\; \begin{bmatrix} 4 - 4j & 4j - 2 & 4j - 1 & 1 - 4j \\ 4j - 3 & 3 - 4j & 2 - 4j & 4j \\ 4j - 3 & 3 - 4j & 2 - 4j & 4j \\ 4 - 4j & 4j - 2 & 4j - 1 & 1 - 4j \end{bmatrix}.$$

Show that matrix A is a magic square of order 2^m.

3369. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2A_3A_4$ be a tetrahedron which contains the centre O of its circumsphere as an interior point. Let ρ_i be the distance from O to the face opposite vertex A_i. If R is the radius of the circumsphere, prove that

$$\frac{4}{3}R \;\geq\; \sum_{i=1}^{4} \rho_i\,.$$

3370. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let a_i and b_i be positive real numbers for $1 \leq i \leq k$, and let n be a positive integer. Prove that

$$\left(\sum_{i=1}^{k} a_i^{\frac{1}{n}}\right)^n \;\leq\; \left(\sum_{i=1}^{k} \frac{a_i}{b_i}\right)\left(\sum_{i=1}^{k} b_i^{\frac{1}{n-1}}\right)^{n-1}.$$

3371. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with a, b, and c the lengths of the sides opposite the vertices A, B, and C, respectively, and let M be an interior point of $\triangle ABC$. The lines AM, BM, and CM intersect the opposite sides at the points A_1, B_1, and C_1, respectively. Lines through M perpendicular to the sides of $\triangle ABC$ intersect BC, CA, and AB at A_2, B_2, and C_2, respectively. Let p_1, p_2, and p_3 be the distances from M to the sides BC, CA, and AB, respectively. Prove that

$$\frac{[A_1B_1C_1]}{[A_2B_2C_2]} = \frac{(ap_1+bp_2)(bp_2+cp_3)(cp_3+ap_1)}{8a^2b^2c^2}\left(\frac{a}{p_1}+\frac{b}{p_2}+\frac{c}{p_3}\right),$$

where $[KLM]$ denotes the area of triangle KLM.

3372. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

If x, y, $z \geq 0$ and $xy+yz+zx=1$, prove that

(a) $$\frac{1}{\sqrt{2x^2+3yz}} + \frac{1}{\sqrt{2y^2+3zx}} + \frac{1}{\sqrt{2z^2+3xy}} \geq \frac{2\sqrt{6}}{3};$$

(b)★ $$\frac{1}{\sqrt{x^2+yz}} + \frac{1}{\sqrt{y^2+zx}} + \frac{1}{\sqrt{z^2+xy}} \geq 2\sqrt{2}.$$

3373. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let x, y, z, and t be positive real numbers. Prove that

$$(x+y)(x+z)(x+t)(y+z)(y+t)(z+t) \geq 4xyzt(x+y+z+t)^2.$$

3374. *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b and c be positive real numbers. Prove that

$$\frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab} \leq \frac{a+b+c}{2\sqrt[3]{abc}}.$$

3375. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let p be a non-negative integer and x any real number. Find the sum

$$\sum_{n=1}^{\infty}(-1)^n\left(e^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^{n+p}}{(n+p)!}\right).$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3282. Correction. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let $A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial with complex coefficients. Suppose that $a_1 = -a_0$, and that the zeroes $z_1, z_2, \ldots, z_n$ of $A(z)$ are distinct, non-zero complex numbers. Prove that

$$\sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} = 0 .$$

Counterexample by Oliver Geupel, Brühl, NRW, Germany.

The statement is false. As a counterexample consider the polynomial $A(z) = z^2 + \frac{1}{2}z - \frac{1}{2} = (z+1)\left(z - \frac{1}{2}\right)$, where $z_1 = -1$ and $z_2 = \frac{1}{2}$. We obtain

$$\begin{aligned} \sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} &= \frac{1}{e} \cdot \frac{1}{-1 - \frac{1}{2}} + \frac{e^{\frac{1}{2}}}{\frac{1}{4}} \cdot \frac{1}{\frac{1}{2}+1} \\ &= \frac{2\left(4e^{\frac{3}{2}} - 1\right)}{3e} \neq 0 . \end{aligned}$$

Problem 3282 was originally misstated in [2007 : 429, 431]. *Michel Bataille, Rouen, France gave a counterexample to that earlier version, as did Geupel. The correction of 3282 appeared in* [2008 : 239, 241]. *Our apologies to all parties for not spotting the error initially.*

3263. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1; \\ L_0 &= 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad \text{for } n \geq 1. \end{aligned}$$

Prove that for each positive integer n,

$$L_n L_{n+1} \leq 2 + \left(\sum_{k=1}^{n} L_k F_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} .$$

I. *Solution by Oliver Geupel, Brühl, NRW, Germany.*

For each positive integer n we have $L_nL_{n+1} = 2 + \sum_{k=1}^{n} L_k^2$, as can be verified by induction. Since $L_k \geq 1$ and $F_{2k} \geq F_k$ for each k, the inequality thus follows:

$$\begin{aligned} L_nL_{n+1} &= 2 + \sum_{k=1}^{n} L_k^2 \leq 2 + \sum_{k=1}^{n} \left(\sqrt{L_kF_{2k}} \cdot \frac{L_k^2}{\sqrt{F_k}} \right) \\ &\leq 2 + \sum_{k=1}^{n} \left(\left(\sum_{j=1}^{n} L_jF_{2j} \right)^{\frac{1}{2}} \frac{L_k^2}{\sqrt{F_k}} \right) \\ &= 2 + \left(\sum_{k=1}^{n} L_kF_{2k} \right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} \,. \end{aligned}$$

Equality holds if and only if $n = 1$.

II. *Solution by Arkady Alt, San Jose, CA, USA.*

The inequality can be strengthened. For each $k \geq 1$ let $a_k = L_k^{\frac{2}{3}}F_k^{\frac{1}{3}}$ and let $b_k = L_k^{\frac{4}{3}}F_k^{\frac{-1}{3}}$. Applying Hölder's Inequality

$$\sum_{k=1}^{n} a_kb_k \leq \left(\sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q \right)^{\frac{1}{q}}$$

for $p = 3$ and $q = \frac{3}{2}$, we successively obtain

$$\begin{aligned} \sum_{k=1}^{n} L_k^2 &\leq \left(\sum_{k=1}^{n} L_k^2F_k \right)^{\frac{1}{3}} \left(\sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} \right)^{\frac{2}{3}} , \\ \left(\sum_{k=1}^{n} L_k^2 \right)^{\frac{2}{3}} &\leq \left(\sum_{k=1}^{n} L_k^2F_k \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} \right) . \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=1}^{n} L_k^2 &= \sum_{k=1}^{n} L_k(L_{k+1} - L_{k-1}) \\ &= \sum_{k=1}^{n} (L_kL_{k+1} - L_kL_{k-1}) = L_nL_{n+1} - 2 \end{aligned}$$

and it is well known that $L_kF_{2k} = L_k^2F_k$ for each k, we then have

$$(L_nL_{n+1} - 2)^{\frac{3}{2}} \leq \left(\sum_{k=1}^{n} L_kF_{2k} \right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}} \,.$$

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU, Comănești, Romania; and the proposer.

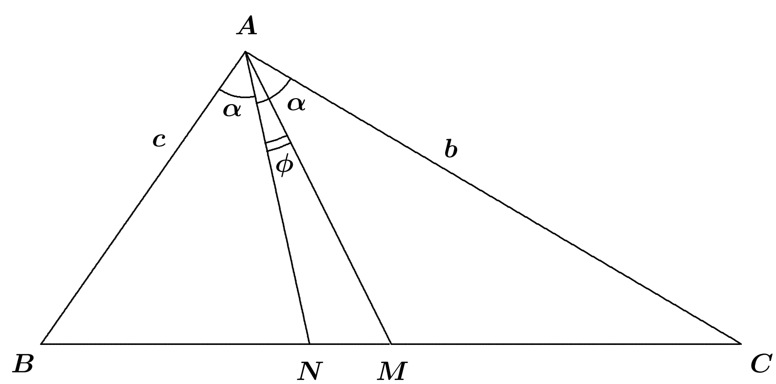
Janous obtained $L_n L_{n+1} \leq \left(\sum_{k=1}^{n} L_k F_2 k\right)^{1/2} \cdot L_{n+1}$ *for* $n \geq 2$, *which is also stronger than the desired inequality.*

3264. [2007 : 366, 368] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let M be the mid-point of BC in $\triangle ABC$, and let the interior angle bisector of $\angle BAC$ meet BC at N. Prove that $\angle BAC = 90^\circ + \angle MAN$ if and only if $b/c = 1 - 2\cos A$.

Solution independently submitted by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands.

Let $AB = c$, $AC = b$, $\angle BAC = 2\alpha$, $\angle MAN = \phi$ and let $[ABC]$ denote the area of $\triangle ABC$. The statement is trivially true if $b = c$.



We assume that $b > c$, since the proof is similar if $b < c$. We have $[AMB] = [AMC]$, hence

$$\frac{1}{2}c \cdot AM \cdot \sin(\alpha + \phi) \;=\; \frac{1}{2}b \cdot AM \cdot \sin(\alpha - \phi)\ ,$$

or

$$\frac{b}{c} \;=\; \frac{\sin(\alpha + \phi)}{\sin(\alpha - \phi)}\,.$$

First, let us suppose that $2\alpha = 90^\circ + \phi$. Then $\phi = 2\alpha - 90^\circ$ and we obtain

$$\begin{aligned}\frac{b}{c} \;&=\; \frac{\sin(3\alpha - 90^\circ)}{\sin(-\alpha + 90^\circ)} \;=\; \frac{-\cos 3\alpha}{\cos\alpha} \;=\; \frac{3\cos\alpha - 4\cos^3\alpha}{\cos\alpha}\\ &=\; 3 - 4\cos^2\alpha \;=\; 3 - 4\left(\frac{1+\cos 2\alpha}{2}\right) \;=\; 1 - 2\cos A\,.\end{aligned}$$

Conversely, let us suppose that $\frac{b}{c} = 1 - 2\cos A$. From the above calculation, we have

$$1 \;-\; 2\cos 2\alpha \;=\; \frac{-\cos 3\alpha}{\cos\alpha}\,,$$

hence,

$$\frac{\sin(\alpha+\phi)}{\sin(\alpha-\phi)} \;=\; \frac{-\cos 3\alpha}{\cos\alpha}\,.$$

We now obtain the succession

$$\begin{aligned}
\sin(\alpha+\phi)\cos\alpha \;+\; \sin(\alpha-\phi)\cos 3\alpha &= 0\,,\\
\frac{1}{2}\big[\sin(2\alpha+\phi)+\sin\phi\big] \;-\; \frac{1}{2}\big[\sin(2\alpha+\phi)-\sin(4\alpha-\phi)\big] &= 0\,,\\
\sin\phi+\sin(4\alpha-\phi) &= 0\,,\\
2\sin\left(\frac{\phi+(4\alpha-\phi)}{2}\right)\cos\left(\frac{\phi-(4\alpha-\phi)}{2}\right) &= 0\,,\\
\cos(\phi-2\alpha)\sin 2\alpha &= 0\,,\\
\cos(2\alpha-\phi) &= 0\,,\\
2\alpha-\phi &= 90^\circ\,,
\end{aligned}$$

because $0 < 2\alpha < 180^\circ$ and $0 < 2\alpha - \phi < 180^\circ$. This completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (one direction only); ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; ANDREA MUNARO, student, University of Trento, Trento, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3265. [2007 : 366, 369] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $ABCD$ be a trapezoid with $AB \parallel CD$ for which $AD = CD$ and $AC = BC$, and let E be the intersection of AC and BD. Let x, y, z denote the measures of angles ABC, BDC, AED, respectively. Show that $y \le 30^\circ$,

$$\tan y \;=\; \frac{2\tan x}{3+\tan^2 x}\,,\quad \text{and} \quad \tan z \;=\; \frac{2\sin x+\sin 3x}{2\cos x+\cos 3x}\,.$$

Solution by Andrea Munaro, student, University of Trento, Trento, Italy.

Since $AB||CD$ and triangles ABC and CAD are isosceles, we have

$$x \;=\; \angle ABC \;=\; \angle BAC \;=\; \angle ACD \;=\; \angle CAD\,.$$

Consequently, $AB = 2AC\cos x$ and $AC = 2AD\cos x$. Combining these we get

$$AB \;=\; 4\,AD\cos^2 x\,.$$

Again because $AB||CD$, we have $y = \angle BDC = \angle DBA$. By the Law of Sines applied to $\triangle ABD$ we have $\dfrac{AB}{\sin(2x+y)} = \dfrac{AD}{\sin y}$, which after substitution becomes

$$\begin{aligned} 4\cos^2 x\sin y &= \sin(2x+y) \\ &= 2\sin x\cos x\cos y + (\cos^2 x - \sin^2 x)\sin y\,, \end{aligned}$$

or

$$(3\cos^2 x + \sin^2 x)\sin y = 2\sin x\cos x\cos y\,.$$

Since $y < x < 90^\circ$, this equation becomes

$$\tan y = \frac{2\sin x\cos x}{3\cos^2 x + \sin^2 x} \;=\; \frac{2\tan x}{3+\tan^2 x}\,.$$

Let $p = \tan x$ so that $\tan y = \dfrac{2p}{3+p^2}$. Since y is acute, $y \le 30^\circ$ is equivalent, in turn, to each of the following inequalities:

$$\begin{aligned} \frac{\sqrt{3}}{3} &\ge \frac{2p}{3+p^2}\,, \\ \sqrt{3}p^2 - 6p + 3\sqrt{3} &\ge 0\,, \\ 3p^2 - 6\sqrt{3}p + 9 &= (\sqrt{3}p-3)^2 \ge 0\,. \end{aligned}$$

Indeed $(\sqrt{3}p-3)^2 \ge 0$, so that we have $y \le 30^\circ$, as desired.

Since z is an external angle of $\triangle CDE$ whose opposite internal angles are x and y, we have $z = x+y$, whence

$$\begin{aligned} \tan z &= \frac{\tan x + \tan y}{1-\tan x\tan y} = \frac{\tan x + \left(\dfrac{2\tan x}{3+\tan^2 x}\right)}{1-\tan x\left(\dfrac{2\tan x}{3+\tan^2 x}\right)} = \frac{\tan^3 x + 5\tan x}{3-\tan^2 x} \\ &= \frac{\sin^3 x + 5\sin x\cos^2 x}{4\cos^3 x - \cos x} = \frac{5\sin x - 4\sin^3 x}{4\cos^3 x - \cos x} = \frac{2\sin x + \sin 3x}{2\cos x + \cos 3x}\,. \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3266. [2007 : 366, 369] *Proposed by Michel Bataille, Rouen, France.*

Find all positive integers n with the following property: whenever a and b are integers such that $ab+1$ is a multiple of n, then $a+b$ is also a multiple of n.

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

We wish to characterize those n such that $ab+1 \equiv 0 \pmod{n}$ implies that $a+b \equiv 0 \pmod{n}$. The first congruence implies that a and b are units in the ring $\mathbb{Z}_n$, and that $b = -a^{-1}$ in that ring. Substituting into the second congruence we have $a - a^{-1} \equiv 0 \pmod{n}$ or $a^2 \equiv 1 \pmod{n}$. In other words, all of the non-identity elements in $\mathbb{Z}_n^\times$, the group of units of $\mathbb{Z}_n$, have order 2. The following facts are well known:

$$\begin{array}{rcll} \mathbb{Z}_{st}^\times & \cong & \mathbb{Z}_s^\times \times \mathbb{Z}_t^\times & \text{(for } s,\, t \text{ relatively prime)}\,, \\ \mathbb{Z}_{p^k}^\times & \cong & \mathbb{Z}_{p^{k-1}(p-1)} & \text{(for } p \text{ an odd prime)}\,, \\ \mathbb{Z}_{2^k}^\times & \cong & \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}} & \text{(for } k \geq 2)\,, \\ |\mathbb{Z}_2^\times| & = & 1 \quad \text{and} \quad |\mathbb{Z}_1^\times| \;=\; 1\,. & \end{array}$$

By the Fundamental Theorem of Finitely Generated Abelian Groups, in order for every non-identity element of $\mathbb{Z}_n^\times$ to have order 2, we must have $\mathbb{Z}_n^\times \cong (\mathbb{Z}_2)^m$ for some non-negative integer m. By the results above, the only prime powers $n = p^k$ having this property are $n = 2$, 3, 4, 8. Including the trivial case $n = 1$ and using the first fact about products above, we have that $n \in \{1,\, 2,\, 3,\, 4,\, 6,\, 8,\, 12,\, 24\}$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon (2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.

Malikić mentions that Problem 11.3 on page 64 of the book Bulgarian Mathematical Competitions 1997-2002 *is very similar to the proposed problem. It asks to find all positive integers n such that if a and b are positive integers for which n divides a^2b+1, then n divides a^2+b. The answer is all positive divisors of 240.*

3267. [2007 : 367, 369] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a non-equilateral triangle with circumcentre O and incentre I. Let X, Y, Z be the mid-points of BC, CA, AB, respectively. If $\pi(P)$ represents the projection of a point P onto the line OI, and $\sigma_{MN}(P)$ represents the reflection of the point P in the line MN, prove that

$$\sigma_{YZ} \circ \pi(A) \;=\; \sigma_{ZX} \circ \pi(B) \;=\; \sigma_{XY} \circ \pi(C)\,.$$

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain.

By virtue of the symmetry in the roles played by the three vertices, it suffices to prove that $\sigma_{YZ} \circ \pi(A) = \sigma_{ZX} \circ \pi(B)$. In fact, we shall see that our result holds for any line through the circumcentre, not just for OI. Specifically, let d be any diameter of the circumcircle, and denote the projections of A and B onto d by $A' = \pi(A)$ and $B' = \pi(B)$; I shall prove that $\sigma_{YZ}(A') = \sigma_{ZX}(B')$. To avoid special cases we shall use directed angles (where $\angle PQR$ represents that angle through which the line PQ must be rotated counterclockwise about Q in order to coincide with QR).

Because $\angle OZA = \angle OYA = \angle OA'A = 90^\circ$, the points Y, Z, and A' lie on the circle with OA as diameter. In the same way (since we also have $\angle OZB = \angle OXB = \angle OB'B = 90^\circ$), the points X, Z, and B' lie on the circle with OB as diameter. Consequently,

$$\begin{aligned}
\angle A'ZB' &= \angle A'ZO + \angle OZB' = \angle A'AO + \angle OBB' \\
&= (90^\circ - \angle AOA') + (90^\circ - \angle B'OB) \\
&= 180^\circ - (\angle AOA' + \angle B'OB) \\
&= \angle BOA = 2\angle BCA \\
&= 2\angle YZX .
\end{aligned}$$

We conclude that the rotation ρ with centre Z that maps A' to B' can be regarded as the composition of reflections σ_{YZ} and σ_{ZX}, whence

$$\sigma_{ZX}(\sigma_{YZ}(A')) = (\sigma_{ZX} \circ \sigma_{YZ})(A') = \rho(A') = B' .$$

Since $\sigma_{ZX}^{-1} = \sigma_{ZX}$, we have $\sigma_{YZ}(A') = \sigma_{ZX}(B')$, as desired.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

The point we found to be the common image under the three reflections is called the orthopole of d. More generally, if perpendiculars are dropped from the vertices of $\triangle ABC$ to points A', B', and C' on an arbitrary line d, then the perpendiculars from A', B', C' to the opposite sides BC, CA, AB of the triangle are concurrent at a point called the orthopole of d for the triangle. *The line d goes through O (as in our problem) if and only if its orthopole lies on the 9-point circle of $\triangle ABC$. (This follows quickly from the observation that the circle on diameter AO is taken by the reflection in YZ to the 9-point circle—the circumcircle of the midpoint triangle XYZ.) Moreover, if $d = OI$ as in the original statement of the problem, then its orthopole is the Feuerbach point (the point where the incircle is tangent to the 9-point circle). Both Bataille and Woo proved this, but the proof can be found in references that explore triangle geometry such as Nathan Altshiller Court's* College Geometry, *pp. 287–291. In addition to these comments, Demis added the easily proved observation that when the orthopole lies on the 9-point circle of the given triangle, then its Simson line with respect to the midpoint triangle $\triangle XYZ$ connects the points where the lines joining A', B', C' to the orthopole intersect the mirrors YZ, ZX, XY; furthermore, d is its Steiner line (the line parallel to the Simson line that passes through the orthocentre, namely O, of $\triangle XYZ$).*

3268. *Proposed by Bill Sands and John Wiest, University of Calgary, Calgary, AB.*

You are given an infinite sequence of cards C_1, C_2, ..., on each of which is written an infinite series of non-negative real numbers which sums to 1.

(a) Prove that there is a reordering D_1, D_2, ... of the cards such that the series $\sum_{i=1}^{\infty} d_{ii}$ converges, where d_{ii} is the i^{th} term of the series on card D_i.

(b)★ Is there necessarily a reordering such that $\sum_{i=1}^{\infty} d_{ii} \le 1$?

[*Ed:* Compare with problem 2620 [2002 : 127; 2005 : 319–326].]

Solution to part (a) *by the first proposer.*

We prove that, for any real number $r > 1$, we can find such a permutation so that $S = d_{11} + d_{22} + d_{33} + \cdots$ is at most r.

Given $r > 1$, choose a positive integer n so that $2^{-n+1} < r - 1$. To choose D_i, we will need the already chosen cards D_1, D_2, ..., D_{i-1}, the family $\mathcal{C}_i = \mathcal{C} - \{D_1, D_2, \ldots, D_{i-1}\}$ of cards not yet chosen, the first card E_i of $\mathcal{C}_i$ under the ordering inherited from $\mathcal{C}$, and a positive integer t_i (defined recursively below). The initial values of these parameters are $\mathcal{C}_1 = \mathcal{C}$, $E_1 = C_1$, and $t_1 = 1$.

There are two ways in which each D_i could be chosen.

(1) If the i^{th} entry on E_i is less than $2^{-(n+t_i)}$, we let $D_i = E_i$, and set $t_{i+1} = t_i + 1$.

(2) Otherwise, we set $t_{i+1} = t_i$, and let a_i be the infimum of the set of i^{th} numbers on the cards in $\mathcal{C}_i$. Then there exists a card in $\mathcal{C}_i$ whose i^{th} number is less than $a_i + 2^{-(n+i)}$, and we let D_i be that card. Thus the i^{th} number on D_i is at most $2^{-(n+i)}$ greater than the i^{th} number on any other card in $\mathcal{C}_i$

This procedure defines the sequence D_1, D_2, D_3, ... of cards, with each D_i having been chosen using either option (1) or option (2). Now partition the set of positive integers into two sets N_1 and N_2, where

$$N_i \;=\; \{j : D_j \text{ was chosen using option } (i)\}\,.$$

Note that each use of option (1) increases the value of t_i by 1. Thus,

$$\sum_{i \in N_i} (i^{\text{th}} \text{ number from } D_i) \;<\; \sum_{i=1}^{\infty} 2^{-(n+i)} \;=\; 2^{-n}\,.$$

On the other hand, the value of t_i does not change whenever option (2) is used. Also, for any card C_i, since the sum of the entries of C_i is 1,

$$\lim_{j \to \infty} (j^{\text{th}} \text{ entry on } C_i) \;=\; 0\,.$$

It follows that any card C_i, once it becomes the first card in some $\mathcal{C}_j$, must eventually be chosen by option (1) if it is not chosen through option (2). Thus, the sequence D_1, D_2, D_3, $\dots$ constructed by the above algorithm is a permutation of the C_i's.

Also, for every k,

$$\begin{aligned}\sum_{i\in N_2,\, i\le k} (i^{\text{th}} \text{ number from } D_i) &\le \sum_{i=1}^{k} \Big((i^{\text{th}} \text{ number from } D_k) + 2^{-(n+i)}\Big) \\ &< 1 + 2^{-n}\,,\end{aligned}$$

since card D_k was eligible for selection whenever D_i was selected, for every $i \le k$, and the sum of all entries in D_k is 1.

Thus, by our two estimates, every partial sum S_k of the series S satisfies $S_k < 2^{-n} + \big(1 + 2^{-n}\big)$. Since $S_k < r$ for all k, $S \le r$, as claimed.

Part (a) also solved by Oliver Geupel, Brühl, NRW, Germany. There were 2 incorrect solutions submitted.

No solutions were submitted for part (b). Part (b) remains open.

3269. [2007 : 367, 369] *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\exp\left(\frac{2^n}{n+1}\right)\sum_{k=1}^{n}\frac{k}{\exp\binom{n}{k}} \;\ge\; \binom{n+1}{2}\,.$$

Composite of essentially similar solutions by Michel Bataille, Rouen, France; and Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

The given inequality is equivalent to

$$\sum_{k=1}^{n}\frac{k}{\binom{n+1}{2}}\exp\left(-\binom{n}{k}\right) \;\ge\; \exp\left(-\frac{2^n}{n+1}\right)\,. \tag{1}$$

Note that the function $f(x) = \exp(-x)$ is convex.

Let $w_k = k/\binom{n+1}{2}$ for $k = 1, 2, \dots, n$. Then $w_k \ge 0$ for each k and $w_1 + w_2 + \cdots + w_n = 1$. Hence, by Jensen's Inequality, we have

$$\sum_{k=1}^{n} w_k f\left(\binom{n}{k}\right) \;\ge\; f\left(\sum_{k=1}^{n} w_k\binom{n}{k}\right)\,,$$

or

$$\sum_{k=1}^{n}\frac{k}{\binom{n+1}{2}}\exp\left(-\binom{n}{k}\right) \;\ge\; \exp\left(-\frac{2}{n(n+1)}\sum_{k=1}^{n} k\binom{n}{k}\right)\,. \tag{2}$$

We have $n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}$, which follows upon differentiating $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k}x^k$. Setting $x = 1$ yields

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}\,. \tag{3}$$

The inequality in (1) now follows by substituting (3) into (2).

Also solved by MIHÁLY BENCZE, Brasov, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous replaces the binomial coefficient $\binom{n}{k}$ with a function f defined on $\{1, 2, \dots, n\}$ such that $f(k) = f(n-k)$, thereby generalizing the result.

3270. [2007 : 367, 370] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let k and ℓ be two straight lines, and let P be any point equidistant from them. Let A and B be the orthogonal projections of P onto k and ℓ, respectively. Prove that, for any $M \in k$ and $N \in \ell$, the following statements are equivalent:

(i) $PN \perp BM$;

(ii) $PM \perp AN$;

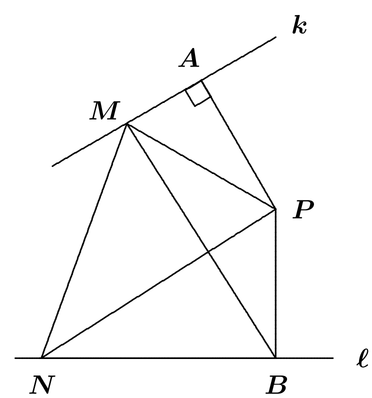
(iii) $MN^2 = AM^2 + BN^2$.

Solution by Titu Zvonaru, Comănești, Romania.

The following is well known and easy to prove.

Lemma Let $ABCD$ be a quadrilateral. Then the diagonals AC and BD are perpendicular if and only if $AB^2 + CD^2 = BC^2 + DA^2$.

By the Lemma, the hypotheses, and the Pythagorean Theorem, we have a succession of equivalent statements as follows:



$$\begin{aligned}
&PN \perp BM\,;\\
PM^2 + BN^2 &= MN^2 + PB^2\,;\\
PM^2 + BN^2 &= MN^2 + PA^2\,;\\
PM^2 + BN^2 &= MN^2 + PM^2 - AM^2\,;\\
AM^2 + BN^2 &= MN^2\,.
\end{aligned}$$

Similarly,

$$PM \perp AN \iff AM^2 + BN^2 = MN^2\,,$$

which completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3271. [2007 : 367, 370] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, and c be real numbers. Prove that $|a+b|+|b+c|+|c+a| \leq 2$ if and only if $|a| \leq 1$, $|b| \leq 1$, $|c| \leq 1$, and $|a+b+c| \leq 1$.

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak and Paula Koca, Angelo State University, San Angelo, TX, USA (submitted jointly).

First, assume that $|a+b|+|b+c|+|c+a| \leq 2$. Then from

$$\begin{aligned} 2\,|a+b+c| &= |(a+b)+(b+c)+(c+a)| \\ &\leq |a+b|+|b+c|+|c+a| \;\leq\; 2\,, \end{aligned}$$

we deduce that $|a+b+c| \leq 1$. Also,

$$\begin{aligned} 2\,|a| &= |(a+b)-(b+c)+(c+a)| \\ &\leq |a+b|+|b+c|+|c+a| \;\leq\; 2\,, \end{aligned}$$

which implies that $|a| \leq 1$. Similarly, $|b| \leq 1$ and $|c| \leq 1$.

Conversely, assume that $|a| \leq 1$, $|b| \leq 1$, $|c| \leq 1$ and $|a+b+c| \leq 1$. Then either $a+b$, $b+c$, and $c+a$ are all negative or all non-negative, *or* two of them are in one of these categories and the third is in the opposite category.

In the former case, we have

$$\begin{aligned} |a+b|+|b+c|+|c+a| &= |(a+b)+(b+c)+(c+a)| \\ &= 2\,|a+b+c| \;\leq\; 2\,. \end{aligned}$$

In the latter case we may assume, without loss of generality, that $a+b$ is in one category while $b+c$ and $c+a$ are in the opposite category. Then

$$\begin{aligned} & |a+b|+|b+c|+|c+a| \\ =\; & |a+b|+|(b+c)+(c+a)| \\ =\; & \big|(a+b)-\big((b+c)+(c+a)\big)\big| \;=\; 2\,|c| \;\leq\; 2\,. \end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, *Rouen, France; SARAH BURNHAM, Auburn Univeristy Montgomery, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina, JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.*

3272. [2007 : 368, 370] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Characterize all natural numbers a and b such that $a \mid (b^2+1)$ and $b \mid (a^2+1)$.

Comment by Michel Bataille, Rouen, France.

This problem was proposed at the South African Mathematics Olympiad, Section B, September 1995. A solution by Pierre Bornsztein appeared in this journal [2001 : 490]. Bornsztein also gave a reference to a generalization of the problem.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; BINGJIE WU, student, High School Affiliated to Fudan University, Shanghai, China; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3273. [2007 : 368, 370] *Proposed by Virgil Nicula, Bucharest, Romania.*

On the sides of triangle ABC are mounted isosceles triangles BMC, CNA, and APB with $MB = MC$, $NC = NA$, and $PA = PB$. If $\angle BMC + \angle CNA + \angle APB = 360^\circ$, prove that the angles of $\triangle MNP$ are independent of $\triangle ABC$.

I. *Composite of similar solutions by Michel Bataille, Rouen, France and by Oliver Geupel, Brühl, NRW, Germany.*

We will show that $\angle PMN = \frac{1}{2}\angle BMC$, $\angle MNP = \frac{1}{2}\angle CNA$, and $\angle NPM = \frac{1}{2}\angle APB$. Let ρ_M be the rotation about M that takes C to B, let ρ_P be the rotation about P that takes B to A, and let ρ_N be the rotation about N that takes A to C. The product $\rho_N \circ \rho_P \circ \rho_M$ fixes the point C, and is therefore a rotation through the angle $\angle BMC + \angle CNA + \angle APB = 360^\circ$, namely, the identity. It follows that $\rho_M^{-1} = \rho_N \circ \rho_P$. Let $P' = \rho_M^{-1}(P)$; then

$$P' \;=\; \rho_M^{-1}(P) \;=\; \rho_N\big(\rho_P(P)\big) \;=\; \rho_N(P)\,.$$

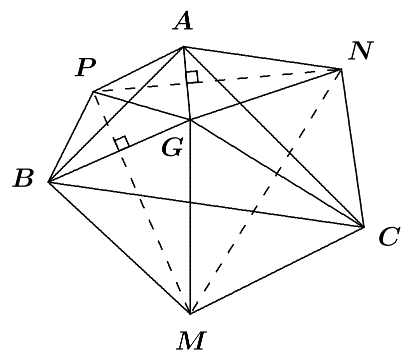
Since $MP = MP'$ and $NP = NP'$, we see that

$$\angle PMN = \frac{1}{2}\angle PMP' = \frac{1}{2}\angle BMC\,.$$

We deduce $\angle MNP = \frac{1}{2}\angle CNA$ and $\angle NPM = \frac{1}{2}\angle APB$ at once, by the symmetry in the roles of P, M, and N.

II. *Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.*

Let G be the reflection of A in the line NP. Then we have $PN \perp AG$, $GN = AN$, and $GP = AP$, whence $GN = NC$ and $GP = PB$. Thus, $\angle AGN = 90^\circ - \frac{1}{2}\angle ANG$ and we also have $\angle NGC = 90^\circ - \frac{1}{2}\angle GNC$; hence,



$$\angle AGC = 180^\circ - \frac{1}{2}\angle CNA\,.$$

Similarly,

$$\angle AGB = 180^\circ - \frac{1}{2}\angle APB\,.$$

It follows that

$$\begin{aligned}\angle BGC &= 360^\circ - \angle AGB - \angle AGC = \frac{1}{2}(\angle CNA + \angle APB)\\ &= \frac{1}{2}(360^\circ - \angle BMC) = 180^\circ - \frac{1}{2}\angle BMC\,.\end{aligned}$$

Because M is defined to be equidistant from B and C, the last equation implies that it is the circumcentre of $\triangle BGC$, whence $MG = MB = MC$. Thus the triangles PGM and PBM are congruent, so $\angle GMP = \angle BMP$. This, in turn, implies that $MP \perp GB$ which, together with $PN \perp AG$ (established earlier), yields $\angle MPN = 180^\circ - \angle AGB$. We conclude that

$$\angle MPN = 180^\circ - (180^\circ - \frac{1}{2}\angle APB) = \frac{1}{2}\angle APB\,.$$

Similarly, we obtain $\angle PMN = \frac{1}{2}\angle BMC$ and $\angle MNP = \frac{1}{2}\angle CNA$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

A disguised version of our problem appears as Theorem 3.35 on page 63 of Geometry Revisited *by H.S.M. Coxeter and S.L. Greitzer:*

> *If similar triangles QCB, CRA, BAS are erected externally on the sides of any triangle ABC, their circumcentres M, N, P form a triangle similar to the three triangles.*

Note, from the order in which the vertices of the similar triangles are named, that the angles at Q, R, S are not *corresponding angles of these triangles; instead, they sum to* 180°. *Our* *second featured proof resembles that of Coxeter and Greitzer, but our notation could have been simplified by using the circumcircles of the three outside triangles. Demis' proof establishes that those three circumcircles meet at his point G.*

Geupel comments that in the special case $\angle BMC = \angle CNA = \angle APB = 120^\circ$, the triangle MNP is the Outer Napoleon Triangle. He describes his proof as similar to the proof of Napoleon's theorem found in Arthur Engel's Problem-Solving Strategies *(Springer 1998), p. 309 ff. (and in many other standard references). Although the statement of our problem suggests that the three isosceles triangles are mounted outside the given triangle ABC, our first featured proof is also valid when the triangles are mounted inside (so that the three new triangles have an orientation opposite that of the given triangle).*

3274. [2007 : 368, 370] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, and c be non-negative real numbers. Prove that

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2} \;\geq\; \frac{a+b+c}{3}\,.$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The required inequality is equivalent to

$$\begin{aligned} &3a^3(2b^2+c^2)(2c^2+a^2) \;+\; 3b^3(2c^2+a^2)(2a^2+b^2)\\ &\quad+\;\; 3c^3(2a^2+b^2)(2b^2+c^2)\\ &\quad-\;\; (a+b+c)(2a^2+b^2)(2b^2+c^2)(2c^2+a^2) \;\geq\; 0\,. \end{aligned} \tag{1}$$

Let $F(a,b,c)$ be the polynomial on the left-hand side of inequality (1). Expanding, we obtain

$$\begin{aligned} F(a,b,c) \;=\;& (b^2c^5+c^2a^5+a^2b^5) \;+\; 2(b^5c^2+c^5a^2+a^5b^2)\\ &+\; 2(b^4c^3+c^4a^3+a^4b^3) \;-\; 2(b^3c^4+c^3a^4+a^3b^4)\\ &-\; 2abc(bc^3+ca^3+ab^3) \;-\; 4abc(b^3c+c^3a+a^3b)\\ &+\; 3a^2b^2c^2(a+b+c)\,. \end{aligned}$$

By symmetry, we can assume that $a \leq b$ and $a \leq c$. We have (with the help of a computer algebra system, if necessary)

$$\begin{aligned} F(a,b,c) \;-\; F(a,c,b) \;=\;& \\ &(b-a)(c-a)(b-c)(ab^3+a^3b\\ &\;+ac^3+a^3c+bc^3+b^3c\\ &\;+4abc^2+4ab^2c+4a^2bc+5a^2b^2+5a^2c^2+5b^2c^2)\,. \end{aligned}$$

Therefore, $F(a,b,c) \leq F(a,c,b)$ if and only if $b \leq c$. To minimize $F(a,b,c)$, we can assume that $a \leq b \leq c$. We also have

$$F(a,b,b) \;=\; 3b^2(a^2-ab+b^2)(a+b)(b-a)^2$$

and

$$F(a,b,c) \;-\; F(a,b,b) \;=\; (c-a)\,G(a,b,c)\,,$$

where $G(a,b,c)$ is a homogeneous polynomial of degree 6. It therefore suffices to prove that $G(a,b,c) \geq 0$. Set $x = \frac{b}{a}$, then $x \geq 1$. We have

$$\begin{aligned} G(a,b,b) &= b(b-a)(a+b)(7b^3 - 16ab^2 + 14a^2b - 2a^3) \\ &= b^4(b-a)(a+b)\,g(x)\,, \end{aligned}$$

where

$$\begin{aligned} g(x) &= 7x^3 - 16x^2 + 14x - 2 \\ &= 7(x-1)^3 + 5(x-1)^2 + 3(x-1) + 3 \;\geq\; 0\,. \end{aligned}$$

Also

$$G(a,b,c) \;-\; G(a,b,b) \;=\; (c-b)\,H(a,b,c)\,,$$

where $H(a,b,c)$ is a homogeneous polynomial of degree 5. It therefore suffices to prove that $H(a,b,c) \geq 0$. However,

$$\begin{aligned} H(a,b,b) &= 6b^5 - 16ab^4 + 8a^2b^3 + 15a^3b^2 - 8a^4b + a^5 \\ &= b^5\,h(x)\,, \end{aligned}$$

where

$$\begin{aligned} h(x) &= 6x^5 - 16x^4 + 8x^3 + 15x^2 - 8x^3 + 1 \\ &= 6(x-1)^5 + 14(x-1)^4 + 4(x-1)^3 \\ &\quad + 3(x-1)^2 + 12(x-1) + 6 \;\geq\; 0\,. \end{aligned}$$

Also

$$H(a,b,c) \;-\; H(a,b,b) \;=\; (c-b)\,K(a,b,c)\,,$$

where

$$\begin{aligned} K(a,b,c) &= 2b^4 - 2a^4 - 6ab^3 + 6a^3b + 2a^3c + b^3c \\ &\quad + 3a^2b^2 + 2a^2c^2 + b^2c^2 - 2ab^2c + 2a^2bc\,. \end{aligned}$$

It therefore suffices to prove that $K(a,b,c) \geq 0$. However,

$$\begin{aligned} K(a,b,b) &= 4b^4 - 8ab^3 + 7a^2b^2 + 8a^3b - 2a^4 \\ &= b^4\,k(x)\,, \end{aligned}$$

where

$$\begin{aligned} k(x) &= 4x^4 - 8x^3 + 7x^2 + 8x - 2 \\ &= 4(x-1)^4 + 8(x-1)^3 + 7(x-1)^2 + 14(x-1) + 9 \;\geq\; 0\,. \end{aligned}$$

Also

$$K(a,b,c) \;-\; K(a,b,b) \;=\; (c-b)\,L(a,b,c)\,,$$

where

$$L(a,b,c) \;=\; 2a^3+2b^3-2ab^2+4a^2b+2a^2c+b^2c\,.$$

It therefore suffices to prove that $L(a,b,c) \geq 0$. However,

$$\begin{aligned} L(a,b,b) &= 3b^3-2ab^2+6a^2b+2a^3 \\ &= b^3\,\ell(x)\,, \end{aligned}$$

where

$$\begin{aligned} \ell(x) &= 3x^3-2x^2+6x+2 \\ &= 3(x-1)^3+7(x-1)^2+11(x-1)+9 \;\geq\; 0\,. \end{aligned}$$

Finally,

$$L(a,b,c) \;-\; L(a,b,b) \;=\; (c-b)(2a^2+b^2) \;\geq\; 0\,,$$

hence, $L(a,b,c) \geq 0$, as desired.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous shows that the inequality $\dfrac{a^3}{\mu a^2+b^2}+\dfrac{b^3}{\mu b^2+c^2}+\dfrac{b^3}{\mu c^2+a^2} \geq \dfrac{a+b+c}{\mu+1}$ *holds whenever* $\frac{1}{3}\left(16-2\sqrt{61}\right) \leq \mu \leq \frac{1}{11}\left(\sqrt{1057}-10\right)$ *and* a, b, c *are non-negative real numbers. The given inequality corresponds to* $\mu=1$. *He asks for the largest range on* μ *for which the parametrized inequality holds for all non-negative real numbers* a, b, *and* c.

3275. [2007 : 368, 370] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let x, y, and z be non-negative real numbers satisfying $x+y+z=3$, and let $0 \leq r \leq 8$. Prove that

$$\frac{1}{xy^2+r}+\frac{1}{yz^2+r}+\frac{1}{zx^2+r} \;\geq\; \frac{3}{1+r}\,.$$

Solution by the proposer.

Clearing denominators and combining like terms yields the equivalent inequality

$$3r^2 \;\geq\; 3p^3+r(r-2)A \;+\; (2r-1)B\,, \tag{1}$$

where $p=xyz$, $A=xy^2+yz^2+zx^2$, and $B=x^2y+y^2z+z^2x$. We have

$$p \leq 1\,, \qquad 3p \leq A \leq 4-p\,, \qquad 3p \leq B \leq 4-p\,,$$

where the upper bounds on A and B are proved in the Lemma below and the other bounds follow from the AM–GM Inequality.

Case 1. $0 \leq r \leq \frac{1}{2}$. Since $A \geq 3p$ and $B \geq 3p$, it suffices to show that

$$3r^2 \;\geq\; 3p^3 \;+\; 3r(r-2)p \;+\; 3(2r-1)p^2\,.$$

This inequality is equivalent to $3(p-1)(p+r)^2 \le 0$, which is clearly true.

Case 2. $\frac{1}{2} < r \le 2$. Since $A \ge 3p$ and $B \le 4-p$, it suffices to show that

$$3r^2 \;\ge\; 3p^3 \;+\; 3r(r-2)p \;+\; 3(2r-1)p(4-p)\,.$$

This inequality is equivalent to

$$(p-1)\left(3p^2+2(2-r)p+3r^2\right) \;\le\; 0\,,$$

which is clearly true.

Case 3. $2 < r \le 7$. Since $A \le 4-p$ and $B \le 4-p$, it suffices to show that

$$3r^2 \;\ge\; 3p^3 \;+\; r(r-2)(4-p) \;+\; (2r-1)p(4-p)\,.$$

This inequality is equivalent to

$$(p-1)\left(3p^2-(2r-4)p+8r-r^2\right) \;\le\; 0\,.$$

The last inequality can be rewritten in the form

$$(p-1)\big((1-p)(7-3p)+(7-r)(2p+r-1)\big) \;\le\; 0\,,$$

and it is clearly true.

Case 4. $7 < r \le 8$. We write equation (1) in the equivalent form

$$3r^2 \;\ge\; 3p^3 \;+\; \big(r(r-2)-(2r-1)p\big)A \;+\; (2r-1)p(A+B)\,.$$

By Schur's Inequality, $(x+y+z)^3+9xyz \ge 4(x+y+z)(xy+yz+zx)$, hence,

$$A+B \;\le\; \frac{27-3p}{4}\,.$$

Since $r(r-2)-(2r-1)p \ge r(r-2)-(2r-1) = r^2-4r+1 > 0$, $A \le 4-p$, and the above inequality holds, it suffices to show that

$$3r^2 \;\ge\; 3p^3 \;+\; \big(r(r-2)-(2r-1)p\big)(4-p) \;+\; (2r-1)p\left(\frac{27-3p}{4}\right)\,,$$

or (by extracting the numerator of the above)

$$(p-1)\left(12p^2+(2r+11)p+4r(8-r)\right) \;\le\; 0\,.$$

The last inequality is true and completes the proof.

Lemma If x, y, and z are non-negative real numbers satisfying $x+y+z=3$, then

$$xy^2 \;+\; yz^2 \;+\; zx^2 \;+\; xyz \;\le\; 4\,.$$

Proof: We write the inequality in the homogeneous form

$$4(x+y+z)^3 \;\ge\; 27(xy^2+yz^2+zx^2+xyz)\,.$$

Without loss of generality let $x = \min\{x,\, y,\, z\}$. Substituting $y = x + a$ and $z = x + b$ (where $a \geq 0$ and $b \geq 0$), the homogeneous inequality becomes

$$9(a^2 - ab + b^2)x \;+\; (2a - b)^2(a + 4b) \;\geq\; 0\,,$$

which is clearly true. Equality holds for $(x, y, z) = (1, 1, 1)$ and also when $\{x,\, y,\, z\} = \{0,\, 1,\, 2\}$.

Also solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA. There was one incorrect solution submitted.

The proposer remarks that, by setting $(x, y, z) = (0, 1, 2)$ in the inequality and then isolating r, one sees that $0 \leq r \leq 8$ is the largest range over which the inequality holds.

Autumn is here in those northern latitudes where ***Crux with Mayhem*** resides. The winter deadlines for the Jim Totten special issue are approaching: **December 1, 2008** for articles and **January 1, 2009** for problem proposals. Please note that we are also soliciting problems from our readers for the ***Mayhem*** section as well, so send them along! A trickle of proposals has already come in, and the spring of 2009 will see the arrival of the special issue.

In the meantime, we would occasionally like to grace our pages with a Contributor Profile. Please contact us if you were approached previously for a ***Crux*** profile, or if you would like to recommend someone for a profile.

As a final note, we ask that material submitted to ***Crux with Mayhem*** include the full name of each contributor, that is, both a first name and a last name.

Václav (Vazz) Linek