SKOLIAD No. 111

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 February, 2009***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our problem set this time around comes from the 2007 Christopher Newport University Math Contest. Only a selection of the problems is presented here. My thanks go to R. Persky, Christopher Newport University, Newport News, VA.

2007 Christopher Newport University Math Contest (selected questions)

1. Find the mid-point of the domain of the function $f(x) = \sqrt{4 - \sqrt{2x+5}}$.

   (A) $\frac{1}{4}$ (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) $\frac{-2}{5}$

2. The sum $a+b$, the product ab and the difference $a^2 - b^2$ for two positive numbers a and b is the same non-zero number. What is b?

   (A) 2 (B) $\frac{1+\sqrt{5}}{2}$ (C) $\sqrt{5}$ (D) $\frac{3-\sqrt{5}}{3}$

3. Let $f(x)$ be a quadratic polynomial with $f(3) = 15$ and $f(-3) = 9$. Find the coefficient of x in $f(x)$.

   (A) 2 (B) 3 (C) 1 (D) -2

4. A pair of fair dice is cast. What is the probability that the sum of the numbers falling uppermost is 7 or 11 if it is known that one of the numbers is a 5?

   (A) $\frac{2}{9}$ (B) $\frac{7}{36}$ (C) $\frac{1}{9}$ (D) $\frac{4}{11}$

5. The number $\sqrt{24 + \sqrt{572}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are whole numbers and $b > a$. What is the value $b - a$?

   (A) 4 (B) -2 (C) 2 (D) 3

6. Let $x = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cdots}}}}$ be the indicated continued fraction. Which one of the following is equal to x?

   (A) $\frac{\sqrt{15}+1}{2}$ (B) $\frac{\sqrt{2}+1}{3}$ (C) $\frac{-3+\sqrt{15}}{2}$ (D) $\frac{-\sqrt{15}-3}{2}$

7. If $f\left(\sqrt{\frac{1+x}{1-x}}\right) = 5x$, find $f(2)$.

   (A) -15 (B) $15\sqrt{-1}$ (C) 3 (D) -4

Université Christopher Newport 2007
Concours de maths (questions individuelles)

1. Trouver le point milieu du domaine de la fonction $f(x) = \sqrt{4 - \sqrt{2x+5}}$.

   (A) $\frac{1}{4}$ (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) $\frac{-2}{5}$

2. La somme $a+b$, le produit ab et la difference a^2-b^2 pour deux nombres positifs a et b est le même nombre non-nul. Que vaut b?

   (A) 2 (B) $\frac{1+\sqrt{5}}{2}$ (C) $\sqrt{5}$ (D) $\frac{3-\sqrt{5}}{3}$

3. Soit $f(x)$ un polynôme quadratique tel que $f(3) = 15$ et $f(-3) = 9$. Trouver le coefficient de x dans $f(x)$.

   (A) 2 (B) 3 (C) 1 (D) -2

4. Une paire de dés honnêtes est lancée. Quelle est la probabilité que la somme des nombres sur le dessus est 7 ou 11 si on sait qu'un des nombres est un 5?

   (A) $\frac{2}{9}$ (B) $\frac{7}{36}$ (C) $\frac{1}{9}$ (D) $\frac{4}{11}$

5. Le nombre $\sqrt{24+\sqrt{572}}$ peut être écrit sous la forme $\sqrt{a}+\sqrt{b}$, où a et b sont des nombres entiers avec $b > a$. Que vaut $b-a$?

   (A) 4 (B) -2 (C) 2 (D) 3

6. Soit la fraction continue $x = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cdots}}}}$. Laquelle des expressions suivantes est égale à x?

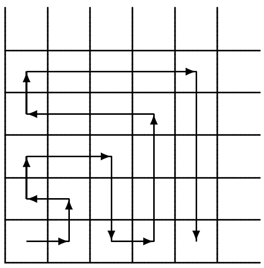
(A) $\dfrac{\sqrt{15}+1}{2}$ (B) $\dfrac{\sqrt{2}+1}{3}$ (C) $\dfrac{-3+\sqrt{15}}{2}$ (D) $\dfrac{-\sqrt{15}-3}{2}$

7. Si $f\left(\sqrt{\dfrac{1+x}{1-x}}\right) = 5x$, trouver $f(2)$.

(A) -15 (B) $15\sqrt{-1}$ (C) 3 (D) -4

Next we give solutions to the National Bank of New Zealand Junior Mathematics Competition 2004 run by the University of Otago with the support of The National Bank of New Zealand [2007 : 386–392].

1 (For year 9 only). Linda starts to write down the natural numbers in the square cells of a very large piece of graph paper. (The graph paper is much larger than shown below.) She starts at the bottom left corner and writes down the numbers using the following arrangement:



17	…				
16	15	14	13		
5	6	7	12		
4	3	8	11		
1	2	9	10		

(The arrangement is suggested in the left diagram; some of the numbers are shown in the right diagram.)

We identify each of the cells using coordinates (a, b), where a is the number of positions to the right, and b is the number of positions up from the bottom. For example, the cell containing the number 1 has the coordinates $(1, 1)$, while the cell containing the number 8 has the coordinates $(3, 2)$.

(a) What are the coordinates of the cell containing the number 15?

(b) Starting with 1, 9, $\dots$, every second number along the bottom row follows a certain pattern. In a few words, or using an algebraic expression, describe these numbers.

(c) The cell containing the number 21 has the coordinates $(5, 5)$. What is the number contained in the cell with coordinates $(6, 6)$? As well, find the number contained in the cell with coordinates $(7, 7)$.

(d) What is the number contained in the cell with coordinates $(20, 20)$?

(e) What are the coordinates of the cell containing the number 2004?

Solutions to (a), (b), *and* (d) *by Jochem van Gaalen, grade 9 student, Medway High School, Arva, ON. Solution to* (c) *by the editor. Official solution to* (e) *modified by the editor.*

(a) The number 15 is placed two squares to the right and 4 squares up, so it is placed in the cell with coordinates $(2, 4)$.

(b) The numbers in these cells are x^2, where x is odd and x is the first coordinate of the square we are looking at.

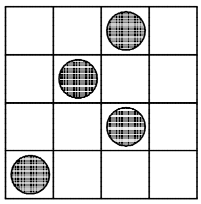
(c) The square with corners $(1, 1)$, $(1, k)$, $(k, 1)$, and (k, k) has k^2 cells in it and when traversed contains all the numbers from 1 to k^2. To reach the cell at (k, k) we arrive from one side of the square and we stop at (k, k), leaving unfilled the $k - 1$ cells on the other side of the square. Since $k - 1$ cells of the square are unfilled, then $k^2 - (k - 1)$ cells are filled. Therefore, $6^2 - (6 - 1) = 31$ is in cell $(6, 6)$ and $7^2 - (7 - 1) = 43$ is in cell $(7, 7)$.

(d) By part (c), the number $20^2 - (20 - 1) = 381$ is in cell $(20, 20)$.

(e) By part (b), the number $45^2 = 2025$ is in the cell $(45, 1)$. It was filled going down, so we subtract from 2025 by the correct amount to go up in the y-coordinates. Since $2025 - 2004 = 21$, the number 2004 is in cell $(45, 22)$.

A correct answer to part (c) *was given by JOCHEM VAN GAALEN, grade 9 student, Medway High School, Arva, ON, but without an explanation for the formula.*

2. The diagram shows a 4×4 grid containing four coins. Imagine that we have enough coins available to place anywhere we like on the grid. However, we would like to place coins so that we do not have three placed anywhere along a line, either horizontally, or vertically, or diagonally.



(a) Imagine that we add one more coin to the given layout. In how many different squares could we place the extra coin so that we would not have three coins placed anywhere along a line?

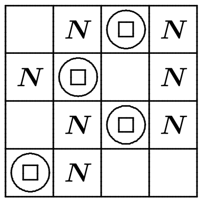
(b) Is it possible to add two more coins into the given layout so that we would not have three coins placed anywhere along a line? If it is possible, show by drawing a diagram where the two extra coins could be placed. If it is not possible, explain why not.

(c) Imagine now that the grid contains no coins at all. What is the smallest number of coins which could be placed onto the grid so that we would not have three placed anywhere along a line, but if we were then to add an extra coin we could not avoid having three placed along a line? Describe, perhaps including a diagram, where the coins would be placed.

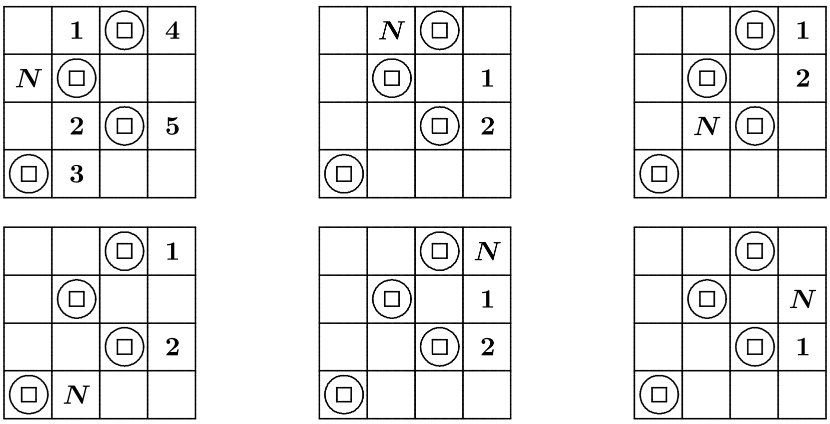
(d) Imagine again that the grid contains no coins at all. What is the largest number of coins which could be placed onto the grid so that nowhere are there three coins placed anywhere along a line? Describe, perhaps including a diagram, where the coins would be placed.

Solutions to (a) *and* (c) *by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON. Solutions to* (b) *and* (d) *by the editor.*

(a) We can look at two coins at a time and eliminate any horizontal, vertical, or diagonal line that the two coins may determine. In the squares that are left over (see the diagram), we put the letter N to show that it is possible to place a new coin there without making three coins that all lie along a line. We see that there are seven squares where a new coin can be placed.



(b) For each of the N's in part (a) we place a new coin there and then we number all the places where a second coin could be added without making three coins along a line *provided that* the second coin is either below the first coin or to the right of it. We find the following six grids, for a total of $5+2+2+2+2+1=14$ solutions.

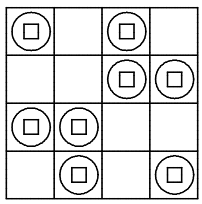


(c) For each square in the diagram, we count how many lines (consisting of 3 or more squares) pass through it. A coin can be on at most 4 lines, so 3 coins can be on at most $3 \cdot 4 = 12$ lines. There are 14 lines in total, so a new coin can be added to any 3. Four coins placed on the 4's in the grid is a smallest solution to the problem. [Ed.: Another solution is to put coins in the four corners.]

3	3	3	3
3	4	4	3
3	4	4	3
3	3	3	3

(d) Clearly any row of the grid can contain at most two coins. Therefore, at most $4 \cdot 2 = 8$ coins can be placed on the grid.

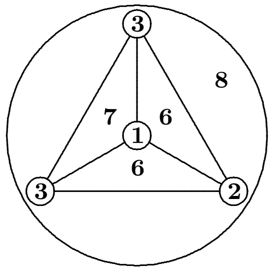
[Ed.: There are many ways to draw such patterns. We will present one that wasn't handed in by our two solvers, who both gave the same right answer but claimed that it was unique.]



Also solved by RUIQI YU, student, Stephen Leacock Collegiate Institute, Toronto, ON (parts (b) *and* (d)*); and JOCHEM VAN GAALEN, grade 9 student, Medway High School, Arva, ON.*

Both solvers gave solutions to part (b) *but neither gave a complete list of all solutions.*

3. The diagram shows an equilateral triangle divided into three smaller triangles. Small circles have been placed on each of the vertices, and positive whole numbers (in this case 3, 3, 2, and 1) have been written inside each small circle. Overall the shape forms four regions: three triangles and an outer region. In each of these regions the sum of the corresponding vertices has been written. For example, the outer region contains the value 8, because ③+③+② $=$ 8.



In this question we shall be investigating what happens when the numbers in the small circles are changed. (Throughout this question, only positive whole numbers will be used.)

(a) Imagine that the number in each one of the small circles is ⑤. What is the total when the numbers inside all four regions are added together?

(b) Find possible numbers inside each of the four small circles so that the sums in the three triangles are 8, 9, and 10, respectively, while the sum of the outer region is 6.

(c) Find possible numbers inside each of the four small circles so that the sums in the three triangles are 8, 9, and 9, respectively. (Do not worry about the sum of the outer region in this part of the question.)

(d) Is your answer to part (c) the only possible answer? If it is, explain why no other answer is possible. If it is not, find another answer.

(e) We have been using positive whole numbers throughout this question. In a few words, or using an algebraic expression, give a general description of the total when the numbers inside all four regions are added together. Explain your reasoning for your description.

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

(a) Let a, b, c, and d be the numbers in the small circles (with b in the centre) and let A, B, C, and D be the numbers in the regions (with D in the outer region). Then we have

$$\begin{aligned} A &= a + b + c\,, \\ B &= a + b + d\,, \\ C &= b + c + d\,, \\ D &= a + c + d\,. \end{aligned}$$

If $a = b = c = d = 5$, then $A = B = C = D = 15$ and their sum is 60.

(b) and (e) We are given that $A = 8$, $B = 9$, $C = 10$, and $D = 6$. From

$$A + B + C + D = 3(a + b + c + d) = 33$$

we have $S = a + b + c + d = 11$. Then $a = S - C = 11 - 10 = 1$, and similarly $b = 5$, $c = 2$, and $d = 3$. In particular, the sum of the numbers in the regions is three times the sum of the numbers in the circles.

(c) and (d) Let $C = 8$ and $A = B = 9$. Then $A - C = a - d = 1$ and $B - C = a - c = 1$, hence, $c = d$. We can have a solution only for c, d in $\{1, 2, 3\}$, otherwise c, $d \geq 4$ and then $b \leq 0$, a contradiction. Therefore, for (a, b, c, d) we have the three solutions $(2, 6, 1, 1)$, $(3, 4, 2, 2)$, and $(4, 2, 3, 3)$.

Also solved by Jochem van Gaalen, grade 9 student, Medway High School, Arva, ON.

4. A class of students votes to select one candidate as their representative on the school council. Their teacher decides on the following voting system:

"You have to rank the three candidates in order: first, second, and third. Your first choice will receive one point, your second choice will receive two points, and your third choice will receive four points. The winner will be the student with the smallest total."

After the voting has been completed, the teacher discovers that there is a problem with this voting system. She explains the problem to the principal:

"The student with the smallest score is Diane, who received 44 points. However, only four people voted for her as their first choice. Next was Belinda with 45 points. She received more first choice votes than anyone else. Colin was in last place with 51 points, and he had more people voting for him as their third choice than voted for the other candidates. It looks as though I will have to announce to the class that Diane is the winner, even though she had the smallest number of people voting for her as first choice."

(a) Show that 20 students took part in the voting.

(b) How many people voted for Belinda as their first choice?

(c) Explain why your answer to (b) is the only possible answer which fits the teacher's description of how the votes were cast.

(d) How many people voted for Belinda as their second choice, and how many people voted for her as their third choice?

Solution to (a) *by Jochem van Gaalen, grade 9 student, Medway High School, Arva, ON. Official solution to* (b) *and* (c). *Solution to* (d) *by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.*

(a) The total number of points received is $44 + 45 + 51 = 140$. A single ballot gives out 7 points (1, 2, and 4 points for the first, second, and third choices, respectively) Therefore, $\frac{140}{7} = 20$ people took part in the voting.

(b) and (c) Four people voted "first" for Diane, leaving 16 "first" votes for Belinda and Colin between them. Belinda and Colin received an odd number of points, so both must have had an odd number of "first" votes. Because Belinda had the highest number of "first" votes, and because both of them received more than Diane's four votes, the only possibilities are Belinda: 11, Colin: 5 *or* Belinda: 9, Colin: 7.

If Belinda received 11 "first" votes, then to obtain a total 45 she must have received $11(1) + 1(2) + 8(4) = 45$. Then Colin would have received $5(1)+7(2)+8(4) = 51$. However this gives Colin the same number of "third" votes as Belinda, which contradicts one of the statements in the question.

If Belinda received 9 "first" votes and Colin received 7 "first" votes, then the totals for the candidates are Belinda: $9(1) + 4(2) + 7(4) = 45$, Colin: $7(1) + 4(2) + 9(4) = 51$, and Diane: $4(1) + 12(2) + 4(4) = 44$. (Diane appears to be the "compromise" candidate, not really supported but not disliked either.) In conclusion, Belinda receiving nine "first" votes is the only solution.

(d) By parts (a) and (b), Belinda has 11 votes other than "first". If all of these are "second" votes, then this makes up 22 points, leaving a shortfall of 14 points to reach her total. Divide the 14 points by 2 to get 7. So 7 "third" votes and 4 "second" votes make up her remaining 36 points. Therefore, Belinda got 4 second choice votes and 7 third choice votes.

Also solved by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON (parts (a), (b), *and* (c)); *and Jochem van Gaalen, grade 9 student, Medway High School, Arva, ON (parts* (b), (c), *and* (d)).

5. Ari has cut some regular pentagons out of cardboard and is joining them together to make a ring (see Figure 1). He has cut them using a template so that they are all the same size.

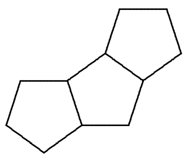


Figure 1

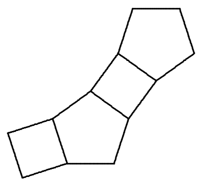


Figure 2

(a) The external angle of a regular pentagon is $72°$. Explain how this value is calculated.

(b) When the ring is complete, how many pentagons will there be?

Next Ari decides to join his pentagons with squares which have the same side length (see Figure 2). He would like to combine them all together to make a new ring with alternating squares and pentagons.

(c) Is it possible for Ari to construct a ring in this way? If it is possible, explain why. If it is not possible, explain why not.

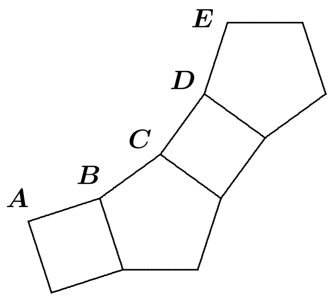
(d) Ari finally decides to construct a ring using regular hexagons (six sides) joined together. (This is not shown in any diagram.) If the hexagons have side length of exactly one unit, what is the area of the shape enclosed inside the ring?

Solutions to (a), (b), *and* (c) *by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON (part* (c) *solution modified by the editor). Solution to* (d) *by the editor.*

(a) The sum of all the external angles of a regular pentagon is $360°$ and they are all equal, hence, the external angle of a regular pentagon is $\frac{360°}{5} = 72°$.

(b) If a ring can be constructed, then the internal figure of the ring is a regular polygon, and the internal angle of the polygon is $72°+72° = 144°$. If n is the number of sides of the polygon, then we have $(n-2)\cdot 180° = n\cdot 144°$, and hence, $n = 10$. Each regular pentagon in the ring shares exactly one side with the inner polygon, so there are 10 pentagons in the ring.

(c) Label 5 consecutive points along what would be the inner side of the ring, as shown in the diagram at right. We have that



$$\angle ABC \;=\; 360° - 90° - 108° \;=\; 162°\,,$$

and that $\triangle ABC$ is isoceles with $AB = BC$, hence, $\angle CAB = \angle BCA = 9°$. By symmetry we also have $\angle DCE = 9°$. It follows that

$$\angle ACE \;=\; 360° - 198° - 18° \;=\; 144°\,.$$

By the result of part (b) we can state that the ring will be completed by alternating 10 squares and 10 pentagons.

(d) If regular hexagons of the same size are used to construct a ring, then the shape included in the ring is an identical regular hexagon. Thus, the area of the inner shape is 6 times the area of an equilateral triangle of side 1, or

$$\text{Area} \;=\; 6\left(\frac{1\cdot\frac{\sqrt{3}}{2}}{2}\right) \;=\; \frac{3\sqrt{3}}{2}\,.$$

There was one incorrect solution submitted.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON. Congratulations Ruiqi! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff ~~members are Monika Khbeis (Ascension of Our Lord~~ Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by **15 November 2008**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M350. *Proposed by the Mayhem Staff.*

Dean rides his bicycle from Coe Hill to Apsley. By distance, one-third of the route is uphill, one-third of the route is downhill, and the rest of the route is on flat ground. Dean rides uphill at an average speed of 16 km/h and on flat ground at an average speed of 24 km/h. If his average speed over the whole trip is 24 km/h, then what is his average speed while riding downhill?

M351. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

Let C be a point on a circle with centre O and radius r. The chord AB is of length r and is parallel to OC. The line AO cuts the circle again at E and it cuts the tangent to the circle at C at the point F. The chord BE cuts OC at L and AL cuts CF at M. Determine the ratio $CF : CM$.

M352. *Proposed by the Mayhem Staff.*

Consider the numbers 37, 44, 51, $\ldots$, 177, which form an arithmetic sequence. A number n is the sum of five distinct numbers from this sequence. How many possible values of n are there?

M353. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all pairs (x, y) of real numbers for which

$$xy + \frac{1}{x} + \frac{1}{y} = \frac{1}{xy} + x + y .$$

M354. *Proposed by the Mayhem Staff.*

Without using a calculating device, determine the prime factorization of $3^{20} + 3^{19} - 12$.

M355. *Proposed by the Mayhem Staff.*

A right circular cone with vertex C has a base with radius 8 and a slant height of 24. Points A and B are diametrically opposite points on the circumference of the base. Point P lies on CB.

(a) If $CP = 18$, determine the shortest path from A through P and back to A that travels completely around the cone.

(b) Determine the position of P on CB that minimizes the length of the shortest path in part (a).

M356. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all pairs (k, n) of positive integers for which

$$k(k+1)(k+2)(k+3) = n(n+1) .$$

. . . .

M350. *Proposé par l'Équipe de Mayhem.*

Daniel se rend à bicyclette de Montignez à Boncourt. Un tiers du chemin est à la montée, un tiers à la descente et le reste au plat. Il grimpe à une vitesse moyenne de 16 km/h et fait du 24 km/h au plat. Si sa vitesse moyenne pour tout le trajet est 24 km/h, quelle est-elle pour la portion descendante ?

M351. *Proposé par Kunal Singh, étudiant, Kendriya Vidyalaya School, Shillong, Inde.*

Soit C un point sur un cercle de centre O et de rayon r. Soit AB une corde de longueur r parallèle à OC. La droite AO coupe de nouveau le cercle en E et elle coupe la tangente au cercle par C au point F. La corde BE coupe OC en L et AL coupe CF en M. Déterminer le rapport $CF : CM$.

M352. *Proposé par l'Équipe de Mayhem.*

On considère la suite arithmétique 37, 44, 51, $\ldots$, 177. Si n désigne la somme de 5 nombres distincts de cette suite, combien de valeurs le nombre n peut-il prendre ?

M353. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Trouver toutes les paires (x, y) de nombres réels pour lesquelles

$$xy + \frac{1}{x} + \frac{1}{y} = \frac{1}{xy} + x + y \,.$$

M354. *Proposé par l'Équipe de Mayhem.*

Sans l'aide d'une calculatrice, trouver la décomposition en facteurs premiers de $3^{20} + 3^{19} - 12$.

M355. *Proposé par l'Équipe de Mayhem.*

Un cône de révolution de sommet C a une base de rayon 8 et une apothème mesurant 24. Soit A et B deux points du cercle de base situés sur un même diamètre, et P un point sur CB.

(a) Si $CP = 18$, trouver le plus court chemin autour du cône partant de A et passant par P pour finir en A.

(b) Trouver la position de P sur CB minimisant la longueur du plus court chemin comme mentionné en (a).

M356. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Trouver toutes les paires (k, n) d'entiers positifs pour lesquelles

$$k(k+1)(k+2)(k+3) = n(n+1)\,.$$

Mayhem Solutions

M301. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

The general term of a sequence is $t_n = n^2 + 20$, for $n \geq 1$. Show that for all $n \geq 1$, the greatest common divisor of t_n and t_{n+1} must be a divisor of 81.

Solution by Geoffrey A. Kandall, Hamden, CT, USA

Let d be any common divisor of t_n and t_{n+1}. Since

$$\begin{aligned} t_{n+1} - t_n &= \left((n+1)^2 + 20\right) - \left(n^2 + 20\right) \\ &= \left(n^2 + 2n + 21\right) - \left(n^2 + 20\right) = 2n + 1\,, \end{aligned}$$

we find that $d \mid (2n+1)$ because $d \mid t_n$ and $d \mid t_{n+1}$.

Also, since $4t_n - (2n+1)(2n-1) = 81$, we find that $d \mid 81$ (because $d \mid t_n$ and $d \mid (2n+1)$). Thus, any common divisor of t_n and t_{n+1} (in particular the greatest common divisor) must divide 81.

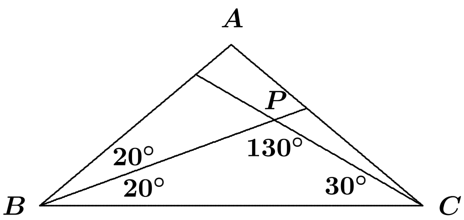
Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; J. SUCK, Essen, Germany; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON.

M302. *Proposed by Babis Stergiou, Chalkida, Greece.*

A triangle ABC has $\angle ABC = \angle ACB = 40^\circ$. If P is a point in the interior of the triangle such that $\angle PBC = 20^\circ$ and $\angle PCB = 30^\circ$, prove that $BP = BA$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Without loss of generality, we may assume that $AB = AC = 1$. Since triangle ABC is isosceles, we have that $BC = 2\cos(40^\circ)$. Also, $\angle BPC = 180^\circ - \angle PBC - \angle PCB$, hence, $\angle BPC = 130^\circ$.



Considering triangle BPC and using the Law of Sines, we obtain the following equivalent ratios:

$$\frac{BP}{\sin(30^\circ)} = \frac{BC}{\sin(130^\circ)},$$
$$\frac{BP}{1/2} = \frac{2\cos(40^\circ)}{\sin(130^\circ)}.$$

We also know that $\sin(130^\circ) = \sin(90^\circ + 40^\circ) = \cos(40^\circ)$, so the last equation simplifies to $BP = 1$. Therefore, $BP = BA$.

Also solved by PAUL BRACKEN and N. NADEAU, University of Texas, Edinburg, TX, USA; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON. There were 5 incorrect or incomplete solutions submitted.

M303. *Proposed by Neven Jurič, Zagreb, Croatia.*

A curious relation among squares states that the sum of $n+1$ consecutive squares, beginning with the square of $n(2n+1)$, is equal to the sum of the squares of the next n consecutive integers. (For example, when $n = 1$ we have $3^2 + 4^2 = 5^2$, and when $n = 2$ we have $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.) Show that this property holds for any $n \geq 1$.

I. *Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.*

We must prove the equality

$$\sum_{j=2n^2+n}^{2n^2+2n} j^2 \;=\; \sum_{j=2n^2+2n+1}^{2n^2+3n} j^2 \,,$$

which is equivalent to

$$\sum_{k=1}^{n+1}(2n^2+2n+1-k)^2 \;=\; \sum_{k=1}^{n}(2n^2+2n+k)^2 \,.$$

For any $n \geq 1$ the required equality follows from the calculation below, where the second equality arises by factoring a difference of squares, and we use the fact that the sum of the first n odd positive integers is n^2:

$$\begin{aligned}
&\sum_{k=1}^{n}(2n^2+2n+k)^2 - \sum_{k=1}^{n}(2n^2+2n+1-k)^2 - (2n^2+n)^2 \\
=\;& \sum_{k=1}^{n}\left((2n^2+2n+k)^2 - (2n^2+2n+1-k)^2\right) - (2n^2+n)^2 \\
=\;& \left(\sum_{k=1}^{n}(2k-1)(4n^2+4n+1)\right) - (2n^2+n)^2 \\
=\;& (4n^2+4n+1)\left(\sum_{k=1}^{n}(2k-1)\right) - (2n^2+n)^2 \\
=\;& (2n+1)^2n^2 - (2n^2+n)^2 \;=\; 0\,.
\end{aligned}$$

II. *Solution submitted independently by Carl Libis, University of Rhode Island, Kingston, RI, USA; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.*

Let S be the sum of the $n+1$ consecutive squares beginning with the square of $n(2n+1)$. Since $\sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$, we find that

$$\begin{aligned}
S \;&=\; \sum_{i=2n^2+n}^{2n^2+2n} i^2 \;=\; \left(\sum_{i=1}^{2n^2+2n} i^2\right) - \left(\sum_{i=1}^{2n^2+n-1} i^2\right) \\
&=\; \tfrac{1}{6}\Big((2n^2+2n)(2n^2+2n+1)(4n^2+4n+1) \\
&\qquad - (2n^2+n-1)(2n^2+n)(4n^2+2n-1)\Big) \\
&=\; \tfrac{1}{6}\Big((16n^6+48n^5+60n^4+40n^3+14n^2+2n) \\
&\qquad - (16n^6+24n^5-10n^3-n^2+n)\Big) \\
&=\; 4n^5+10n^4+\tfrac{25}{3}n^3+\tfrac{5}{2}n^2+\tfrac{1}{6}n\,.
\end{aligned}$$

Similarly, let T be the sum of the n consecutive squares, beginning with the square of $2n^2+2n+1$. Then

$$\begin{aligned} T &= \sum_{i=2n^2+2n+1}^{2n^2+3n} i^2 \;=\; \left(\sum_{i=1}^{2n^2+3n} i^2\right) - \left(\sum_{i=1}^{2n^2+2n} i^2\right) \\ &= \tfrac{1}{6}\Big((2n^2+3n)(2n^2+3n+1)(4n^2+6n+1) \\ &\qquad - (2n^2+2n)(2n^2+2n+1)(4n^2+4n+1)\Big) \\ &= \tfrac{1}{6}\Big((16n^6+72n^5+120n^4+90n^3+29n^2+3n) \\ &\qquad - (16n^6+48n^5+60n^4+40n^3+14n^2+2n)\Big) \\ &= 4n^5+10n^4+\tfrac{25}{3}n^3+\tfrac{5}{2}n^2+\tfrac{1}{6}n\,. \end{aligned}$$

We then have $S=T$ for any $n \geq 1$, which is the required equality.

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DEREK MERRELL and SUE YANG, students, California State University, Fresno, CA, USA; J. SUCK, Essen, Germany; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON.

M304. Corrected. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be real numbers such that both $a+b+c$ and $ab+bc+ca$ are rational numbers, and $a+b+c \neq 0$. Show that $a^4+b^4+c^4$ is a rational number if and only if the product abc is a rational number.

Solution by Miguel Marañón Grandes, student, Universidad de La Rioja, Logroño, La Rioja, Spain.

Let $a+b+c=\frac{p}{q}$ and $ab+bc+ca=\frac{r}{s}$, where p, q, r, and s are integers with $q \neq 0$ and $s \neq 0$. We have

$$\begin{aligned} a^2+b^2+c^2+2(ab+bc+ca) &= (a+b+c)^2 \;=\; \frac{p^2}{q^2}\,; \\ a^2+b^2+c^2 &= \frac{p^2}{q^2} - 2\left(\frac{r}{s}\right) \;=\; \frac{p_1}{q_1}\,, \end{aligned}$$

where p_1 and q_1 are integers, hence, $a^2+b^2+c^2$ is a rational number.

Setting $A=a^2b^2+b^2c^2+c^2a^2$, we have

$$\begin{aligned} a^4+b^4+c^4+2\left(a^2b^2+b^2c^2+c^2a^2\right) &= \left(a^2+b^2+c^2\right)^2 \;=\; \frac{p_1^2}{q_1^2}\,; \\ a^4+b^4+c^4 &= \frac{p_1^2}{q_1^2} - 2A\,. \end{aligned}$$

This means that $a^4+b^4+c^4$ is a rational number if and only if A is a rational number.

Finally we obtain

$$a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a+b+c) \;=\; (ab+bc+ca)^2 \;=\; \frac{r^2}{s^2}\,;$$
$$A \;=\; \frac{r^2}{s^2} \;-\; 2abc\left(\frac{p}{q}\right)\,.$$

Since $\frac{p}{q} = a+b+c \neq 0$, then A is a rational number if and only if abc is a rational number.

Thus, $a^4+b^4+c^4$ is a rational number if and only if abc is a rational number, with $a+b+c \neq 0$, as we wanted to show.

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; CARL LIBIS, University of Rhode Island, Kingston, RI, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; J. SUCK, Essen, Germany; and EDWARD T.H. WANG and KAIMING ZHAO, Wilfrid Laurier University, Waterloo, ON. There were 7 incomplete or incorrect solutions submitted.

Our apologies for any flawed solutions due to the original incorrect version of the problem. A counterexample to the original problem statement is $a = b = \sqrt{2}$ and $c = -2\sqrt{2}$. These give $a+b+c = 0$, $ab+ac+bc = -6$, and $a^4+b^4+c^4 = 72$, but $abc = -4\sqrt{2}$.

M305. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions to the following system of equations:

$$\begin{aligned} \sqrt{x}+\sqrt{y}+\sqrt{z} &= 3\,, \\ x\sqrt{x}+y\sqrt{y}+z\sqrt{z} &= 3\,, \\ x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z} &= 3\,. \end{aligned}$$

Solution by J. Suck, Essen, Germany.

Adding the first and third equations and subtracting twice the second equation, we obtain

$$\begin{aligned} (x^2+1-2x)\sqrt{x} \;&+\; (y^2+1-2y)\sqrt{y} \\ &+\; (z^2+1-2z)\sqrt{z} \;=\; 3+3-2(3)\,, \end{aligned}$$

or

$$(x-1)^2\sqrt{x} \;+\; (y-1)^2\sqrt{y} \;+\; (z-1)^2\sqrt{z} \;=\; 0\,.$$

Each of the three terms is nonnegative, so each must equal zero; that is, $(x-1)^2\sqrt{x} = (y-1)^2\sqrt{y} = (z-1)^2\sqrt{z} = 0$. Therefore, x, y, and z each take the value 0 or 1.

Since $\sqrt{x}+\sqrt{y}+\sqrt{z} = 3$, we must have $x = y = z = 1$, which clearly satisfies all three equations.

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; URZICA SORIN, Grigore Cobalcescu High school, Moinesti, Romania; GEORGE TSAPAKIDIS, Agrinio, Greece; EDWARD T.H. WANG and KAIMING ZHAO, Wilfrid Laurier University, Waterloo, ON; VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON (2 solutions); and TITU ZVONARU, Comănești, Romania. There were 3 solutions submitted that were incomplete or incorrect.

M306. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Find all solutions to the following addition problem, in which each letter represents a distinct digit:

$$\begin{array}{cccccc} & & & T & E & N \\ & & & T & E & N \\ & & N & I & N & E \\ & E & I & G & H & T \\ + & T & H & R & E & E \\ \hline & F & O & R & T & Y \end{array}$$

Solution by Candice Arredondo, student, California State University, Fresno, CA, USA.

The solutions are

$$\begin{array}{lllllllllll}
417 & + & 417 & + & 7871 & + & 18324 & + & 42511 & = & 69540\,, \\
329 & + & 329 & + & 9492 & + & 24863 & + & 36022 & = & 71035\,, \\
308 & + & 308 & + & 8780 & + & 07643 & + & 34200 & = & 51239\,, \\
718 & + & 718 & + & 8281 & + & 12347 & + & 74011 & = & 96075\,, \\
701 & + & 701 & + & 1410 & + & 04267 & + & 76500 & = & 83579\,, \\
713 & + & 713 & + & 3431 & + & 14207 & + & 70611 & = & 89675\,, \\
631 & + & 631 & + & 1013 & + & 30756 & + & 65233 & = & 98264\,, \\
624 & + & 624 & + & 4042 & + & 20756 & + & 65322 & = & 91368\,, \\
148 & + & 148 & + & 8284 & + & 42391 & + & 19644 & = & 70615\,, \\
047 & + & 047 & + & 7174 & + & 41690 & + & 09344 & = & 58302\,.
\end{array}$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA. There was 1 incorrect solution submitted.

Ms. Arredondo described the exhaustive trial and error approach that she took to find all the solutions. Is there a "cleaner" approach? Hess also obtained the same solutions as above, but without the solutions containing leading zeros.

M307. *Proposed by Neven Jurič, Zagreb, Croatia.*

Two 4×4 magic squares have the property that all four of their rows, all four of their columns, and their two diagonals all sum to the same value N. Consider the sum of the four corner elements of each square. Can these sums be different, or must they be the same? (In other words, does the corner sum depend on the square itself, or only on the *magic sum* N?) Either determine the constant sum, or show that these sums can differ.

Almost all submitted solutions were the same.

Let a_{ij} denote the entry in the i^{th} row and j^{th} column. The sum of the four corner elements of a 4×4 magic square is then $a_{11} + a_{14} + a_{41} + a_{44}$. Using the properties of the magic square, we obtain the following

$$\begin{aligned} & 2(a_{11} + a_{14} + a_{41} + a_{44}) \\ = \; & (a_{11} + a_{12} + a_{13} + a_{14}) \; + \; (a_{41} + a_{42} + a_{43} + a_{44}) \\ & + (a_{11} + a_{22} + a_{33} + a_{44}) \; + \; (a_{14} + a_{23} + a_{32} + a_{41}) \\ & - (a_{12} + a_{22} + a_{32} + a_{42}) \; - \; (a_{13} + a_{23} + a_{33} + a_{43}) \\ = \; & N + N + N + N - N - N \\ = \; & 2N \, . \end{aligned}$$

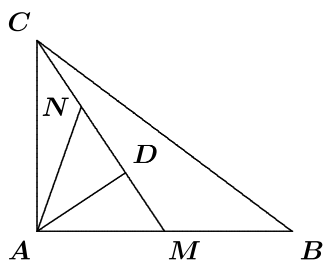
Therefore, the four corner elements of a 4×4 magic square satisfying the given properties have a constant sum of N.

Solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CARL LIBIS, University of Rhode Island, Kingston, RI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC. There was 1 incorrect solution submitted.

M308. *Proposed by Babis Stergiou, Chalkida, Greece.*

Let ABC be a right triangle with $A = 90^\circ$, and let M be the mid-point of side AB. If D is the foot of the perpendicular from A to CM and N is the mid-point of DC, prove that $BD \perp AN$.

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.



We assign coordinates. Let A, B, and C have coordinates $(0,0)$, $(b,0)$, and $(0,c)$, respectively. Then the coordinates of M are $(\frac{b}{2},0)$. Also, the slope of CM is $\frac{-2c}{b}$ and the equation of CM is $y = -\frac{2c}{b}x + c$. Since AD and CM are perpendicular, the slope of AD is the negative reciprocal of $\frac{-2c}{b}$, therefore, the equation of AD is $y = \frac{b}{2c}x$.

Since D is the point of intersection of CM and AD, we can find its coordinates using the equations of CM and AD. Equating values of y, we obtain $-\frac{2c}{b}x + c = \frac{b}{2c}x$ which upon solving for x yields $x = \frac{2bc^2}{b^2+4c^2}$. Therefore, $y = \frac{b}{2c}x = \frac{b^2c}{b^2+4c^2}$ and D has coordinates $\left(\frac{2bc^2}{b^2+4c^2}, \frac{b^2c}{b^2+4c^2}\right)$.

The point N has coordinates $\left(\frac{bc^2}{b^2+4c^2}, \frac{b^2c+2c^3}{b^2+4c^2}\right)$, since it is the midpoint of DC. The slope of AN is $\frac{b^2c+2c^3}{bc^2} = \frac{b^2+2c^2}{bc}$ and the slope of BD is

$$\frac{\left(\frac{b^2c}{b^2+4c^2}\right)}{\left(\frac{2bc^2}{b^2+4c^2} - b\right)} = \frac{b^2c}{2bc^2 - b(b^2+4c^2)} = -\frac{bc}{b^2+2c^2}\,.$$

The product of the slopes is -1, hence, AN and BD are perpendicular.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DANIEL REISZ, Auxerre, France; and D.J. SMEENK, Zaltbommel, the Netherlands.

M309. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all possible non-negative integers x, y, z, and t such that $3^x + 3^y + 3^z + 3^t$ is a perfect cube.

Partial solution independently by Jaclyn Chang, student, Western Canada High School, Calgary, AB; and the proposer.

For any non-negative integers a and b,

$$\begin{aligned}(3^a+3^b)^3 &= (3^a)^3 + 3(3^a)^2(3^b) + 3(3^a)(3^b)^2 + (3^b)^3 \\ &= 3^{3a} + 3^{2a+b+1} + 3^{a+2b+1} + 3^{3b}\,.\end{aligned}$$

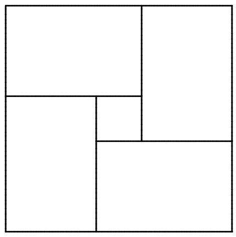
Thus, $x = 3a$, $y = 2a+b+1$, $z = a+2b+1$, and $t = 3b$, and their permutations are solutions for any non-negative integers a and b.

No other solutions were submitted.

The partial answer gives an infinite family of solutions. Are there any more solutions? Can any readers prove that there are not any more solutions? As it turns out, we had intended to ask for an infinite family of solutions rather than all solutions but somehow asked for all solutions – our apologies for this.

M310. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Four congruent rectangles are arranged in a square pattern so that they enclose a smaller square.



Let S be the area of the outer square and Q the area of the inner square. If the area of the outer square is 9 times the area of the inner square, determine the ratio of the sides of the rectangles.

Almost all submitted solutions were the same.

Let x and y be the side lengths of one of the four rectangles. Without loss of generality, we may assume that $x > y$.

Therefore, the outer square has side length $x + y$ and the inner square has side length $x - y$. From the given information $S = 9Q$, we obtain successively

$$\begin{aligned}(x+y)^2 &= 9(x-y)^2\,,\\ x^2+2xy+y^2 &= 9x^2-18xy+9y^2\,,\\ 8x^2-20xy+8y^2 &= 0\,,\\ 2x^2-5xy+2y^2 &= 0\,,\\ (2x-y)(x-2y) &= 0\,.\end{aligned}$$

Since $x > y$, we have $x = 2y$ and the required ratio is $\frac{x}{y} = 2$.

Solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DANIEL REISZ, Auxerre, France; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There was 1 incorrect solution submitted.

M311. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers, and let $m \in \left(0, \frac{1}{4}\right)$. Show that at least one of the following equations has real roots:

$$\begin{aligned}ax^2+bx+cm &= 0\,,\\ bx^2+cx+am &= 0\,,\\ cx^2+ax+bm &= 0\,.\end{aligned}$$

Solution submitted independently by Geoffrey A. Kandall, Hamden, CT, USA; and the proposer.

Assume that none of the three equations has real roots. Thus, each of the quadratic polynomials has a negative discriminant; that is, $b^2-4acm < 0$

and $c^2 - 4abm < 0$ and $a^2 - 4bcm < 0$. Equivalently, $4acm > b^2$ and $4abm > c^2$ and $4bcm > a^2$.

Since all quantities are positive, multiplying across these inequalities yields $64a^2b^2c^2m^3 > a^2b^2c^2$. This implies that $m^3 > \frac{1}{64}$, or $m > \frac{1}{4}$. This is a contradiction, since $0 < m < \frac{1}{4}$. Therefore, at least one of the equations has real roots.

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico.

M312. *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, ON.*

John is negotiating the terms of a mortgage with his bank manager. They have agreed that the loan will be for L dollars and that the annual interest rate will be i.

John says, "I will make payments of P dollars at the end of each year for the next n years. This is more than enough to pay the interest. The excess will reduce the principal outstanding for the next year. At the end of n years, I will arrange a new mortgage for the remaining principal."

The manager responds, "I would like more frequent payments. I suggest payments of $P/4$ each quarter-year with interest rate $i/4$ applied to the previous quarter's balance."

John objects, "But then the effective annual interest rate will be greater than i!"

The manager replies, "Yes, but the amount outstanding at time n will be less!"

John finds this hard to believe. Is it true?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the Mayhem Staff.

We will write the interest rate as a decimal rather than as a percentage (for example, we would write 0.05 rather than 5%). Let J_n be John's amount outstanding after n years under his arrangement and let B_n be the amount outstanding under the banker's arrangement.

Under John's arrangement, the amount outstanding after one year is $J_1 = L(1+i) - P$, since interest accumulates at a rate of i over the year on the initial loan and then John pays P dollars at the end of the year. Similarly, for each $k \geq 1$, the amount outstanding at the end of year $k+1$ will be $J_{k+1} = J_k(1+i) - P$.

Using these equations, we determine that

$$J_n \;=\; L(1+i)^n \;-\; P\left(\frac{(1+i)^n - 1}{i}\right) . \qquad (1)$$

[Ed.: See the note after the solution for a more detailed explanation.]

Similarly, under the banker's arrangement, we determine that

$$\begin{aligned} B_n &= L\left(1+\frac{i}{4}\right)^{4n} - \frac{P}{4}\left(\frac{\left(1+\frac{i}{4}\right)^{4n}-1}{\frac{i}{4}}\right) \\ &= L\left(1+\frac{i}{4}\right)^{4n} - \frac{P}{i}\left(\left(1+\frac{i}{4}\right)^{4n}-1\right), \end{aligned}$$

which we can obtain by using the formula in (1) for J_n, but with $4n$ periods (quarter-years) instead of n periods (years).

Now we have

$$\begin{aligned} J_n - B_n &= \left(L(1+i)^n - P\left(\frac{(1+i)^n-1}{i}\right)\right) \\ &\quad - \left(L\left(1+\frac{i}{4}\right)^{4n} - \frac{P}{i}\left(\left(1+\frac{i}{4}\right)^{4n}-1\right)\right) \\ &= L\left((1+i)^n - \left(1+\frac{i}{4}\right)^{4n}\right) - \frac{P}{i}\left((1+i)^n - \left(1+\frac{i}{4}\right)^{4n}\right) \\ &= \left(\left(1+\frac{i}{4}\right)^{4n} - (1+i)^n\right)\left(\frac{P}{i}-L\right). \end{aligned} \tag{2}$$

In order for the loan to be paid off, we need $P > iL$ (otherwise the payment each year will be less than the total interest). Therefore, $\frac{P}{i} - L > 0$.

Also, $(1+\frac{i}{4})^4 = 1 + 4(\frac{i}{4}) + \cdots$, the remaining terms being positive, hence, $(1+\frac{i}{4})^{4n} > (1+i)^n$. (Since $(1+\frac{i}{4})^4 > 1+i$, the annual interest rate is greater than i.)

Therefore, $J_n - B_n > 0$, as the two factors on the right side of (2) are positive. Thus, the bank manager's statement is true.

[Ed.: Given that $J_1 = L(1+i) - P$ and $J_{k+1} = J_k(1+i) - P$ for $k \geq 1$, we can prove the formula in (1) for J_n either by using Mathematical Induction, or more informally by writing out

$$\begin{aligned} J_1 &= L(1+i) - P, \\ J_2 &= J_1(1+i) - P = L(1+i)^2 - (1+i)P - P, \\ J_3 &= J_2(1+i) - P = L(1+i)^3 - (1+i)^2P - (1+i)P - P, \\ &\vdots \\ J_n &= L(1+i)^n - \big((1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1\big)P, \end{aligned}$$

and using the formula for the sum of a geometric series.

There was 1 incorrect solution submitted.

Problem of the Month

Ian VanderBurgh

Problem 1 (2004 Hypatia Contest)

3 green stones, 4 yellow stones, and 5 red stones are placed in a bag. Two stones of different colours are selected at random. These two stones are then removed and replaced with one stone of the third colour. (Enough extra stones of each colour are kept to the side for this purpose.) This process continues until there is only one stone left in the bag, or all of the stones are the same colour. What is the colour of the stone or stones that remain at the end?

This is the first of three problems that we'll look at this month, all of which deal with piles of stones. However, the problems all require different approaches. To me, this is one of the fascinating things about mathematics – problems that appear to be very similar often require quite different approaches in their solutions.

From the wording of Problem 1, we can deduce that the colour of the stone or stones left at the very end is always the same, regardless of the order in which the stones are removed. Thus, it shouldn't be too difficult to determine that colour. (Just trying one particular order will do it!) The tricky part will be to try to justify why the colour is *always* the same, regardless of what we do.

Let's try a particular order to see what answer we get. We'll make a table to keep track of the colours:

Colours Removed	Colour Added	# Green	# Yellow	# Red
—	—	3	4	5
Green, Yellow	Red	2	3	6
Yellow, Red	Green	3	2	5
Green, Red	Yellow	2	3	4
Yellow, Red	Green	3	2	3
Green, Red	Yellow	2	3	2
Green, Red	Yellow	1	4	1
Green, Red	Yellow	0	5	0

So in this particular ordering of choices, we are left with only yellow stones. Thus, we must always be left with yellow stones.

But why is the colour always yellow at the end? Try looking for a pattern among the numbers in the chart. Need a hint? Try looking at the *parity* of the numbers (that is, whether they are even or odd).

Solution to Problem 1: For simplicity, we'll call each "selection and replacement" a turn. Also, we'll use G to represent the number of green stones, Y to represent the number of yellow stones, and R to represent the number of red stones.

At the beginning, G and R are each odd, and Y is even.

Each turn causes each of G, Y, and R to either increase by 1 or decrease by 1. This means that after each turn, the parity of each of G, Y, and R has changed.

Therefore, after the first turn, the parities of G, R, and Y are even, odd, and even, respectively, regardless of which colours are increased or decreased. Similarly, after the second turn, the respective parities will be odd, even, and odd.

Continuing in this way, we can see that the parities will always be odd, even, and odd; or even, odd, and even. We can also see that the parities of G and R are always the same, and the parity of Y is different from the parity of these two.

Suppose that we reach the state where there are only stones of one colour left. Thus, two of G, Y, and R equal 0. For this to be true, then we must be in the "even, odd, and even" case (because 0 is even). Therefore, G and R are both even (and so equal to 0) and Y is odd. So the final stone or stones in the bag are always yellow.

Parity arguments crop up in unexpected places. This type of argument can often be useful when looking at a problem in which there seem to be a lot of cases to consider. See Problem 3 at the end of this column for a problem requiring a different, but related, argument. Here's a similar problem that needs another neat argument.

Problem 2

Zach has three piles of stones, containing 5, 49, and 51 stones. He can combine any two piles together into one pile and he can also divide a pile containing an even number of stones into two piles of equal size. Can he ever achieve 105 piles, each with 1 stone?

Unless you have a rock garden, this one might be a tad more difficult to simulate than the first one, given that you need 105 stones!

Let's do some investigation. Since none of the piles contains an even number of stones, the rules tell us that initially we have to combine piles rather than separate piles.

If we combine the first two piles, we will have two piles remaining with $5 + 49 = 54$ and 51 stones. If we combine the first and last piles, we will have piles with 56 and 49 stones. If we combine the last two piles, we will have piles with 5 and 100 stones.

Hopefully, it is clear that we don't want to combine again at this stage, otherwise we'd end up with one big pile of 105 stones in each case; since 105 is odd, we'd be unable to go any further. This of course doesn't yet mean that we can't achieve the desired result – it simply means that we've gone down the wrong path.

Let's look at our first option where we had piles with 54 and 51 stones. The only useful move here is to divide the pile containing 54 stones into two piles of $\frac{1}{2}(54) = 27$ stones. So we now have piles of 27, 27, and 51 stones. So we could recombine a pair of piles again. Can you see why this isn't going to lead us anywhere?

Let's look at the second option to see if anything becomes clearer. Starting with 56 and 49, we could split the 56 into two piles of 28, giving piles of 28, 28, and 49 stones. Here we have more choices – combining an odd and an even pile, splitting one or both of the even piles, and so on.

At this point, you could spend quite a lot of time trying different approaches to get down to 105 piles of one stone each without success. This, of course, would make you very frustrated with me and probably convince you that it seems as if the desired outcome is impossible. (But as every intrepid problem solver knows, just because you've tried it for 47 hours, doesn't necessarily mean it won't work!) If you're stuck at this stage, read on!

Solution to Problem 2: On his first move, Zach must combine two piles and obtain piles of 54 and 51 stones, or 56 and 49 stones, or 5 and 100 stones.

In the first case, the number of stones in the pile have a common factor of 3, in the second case, a common factor of 7, and in the third case, a common factor of 5.

Suppose that we are starting now from piles of 54 and 51 stones. The two possible moves are to combine two piles or to divide one pile in half. If two integers are both multiples of 3 and are added together, their sum is also a multiple of 3; if an even integer is a multiple of 3 and is divided in half, each of the two resulting integers will also be a multiple of 3. (Can you see why?) Thus, no matter how many moves we make starting with piles of 54 and 51 stones, each of the piles remaining will always contain a number of stones that is a multiple of 3. Thus, we can never reach 105 piles of 1, as with this starting move, each pile will always contain a number of stones that is a multiple of 3, and 1 is not a multiple of 3.

The argument that we made above about combining or dividing piles whose sizes are multiples of 3 also works for multiples of 5 or 7. (In fact, it works for multiples of any odd number greater than 1.) Thus, starting with 56 and 49, each of the piles will always contain a number of stones that is a multiple of 7, and starting with 5 and 100, each of the piles will always contain a number of stones that is a multiple of 5.

In summary, no matter what first move is made, it is impossible to create 105 piles of 1 stone each, as a common factor is always introduced at the first step that can never be made to disappear.

What do you think? I found this problem quite neat when I saw it for the first time. This argument is in some ways similar and in some ways different to the argument needed for Problem 1. Here is a challenge problem for you to consider. We'll briefly look at the solution next month.

Problem 3 (2004 Hypatia Contest)

3 green stones, 4 yellow stones, and 5 red stones are placed in a bag. This time, two stones of different colours are selected at random, removed and replaced with two stones of the third colour. Show that it is impossible for all of the remaining stones to be the same colour, no matter how many times this process is repeated.

THE OLYMPIAD CORNER

No. 271

R.E. Woodrow

Welcome back from the break. We start this number of the *Corner* with the 20 problems of the 19th Lithuanian Team Contest in Mathematics. Thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico for collecting them.

19th LITHUANIAN TEAM CONTEST IN MATHEMATICS
October 2, 2004

1. Twelve numbers – four 1's, four 5's, and four 6's – are written in some order around a circle. Does there always exist a three-digit number comprised of three neighbouring numbers (its digits can be taken clockwise or counterclockwise) that is divisible by 3?

2. Solve the equation $2\cos(2\pi x) + \cos(3\pi x) = 0$.

3. Solve the equation $3x^{\lfloor x \rfloor} = 13$, where $\lfloor x \rfloor$ denotes the integer part of the number x.

4. Let $0 \le a \le 1$ and $0 \le b \le 1$. Prove the inequality

$$\frac{a}{\sqrt{2b^2+5}} + \frac{b}{\sqrt{2a^2+5}} \le \frac{2}{\sqrt{7}}.$$

5. If a, b, and c are nonzero real numbers, what values can be taken by the expression

$$\frac{a^2-b^2}{a^2+b^2} + \frac{b^2-c^2}{b^2+c^2} + \frac{c^2-a^2}{c^2+a^2}\ ?$$

6. Determine all pairs of real numbers (x, y) such that

$$\begin{aligned} x^6 &= y^4 + 18\,, \\ y^6 &= x^4 + 18\,. \end{aligned}$$

7. Find all triples (m, n, r) of positive integers such that

$$2001^m + 4003^n = 2002^r\,.$$

8. Assume that m and n are positive integers. Prove that, if $mn - 23$ is divisible by 24, then $m^3 + n^3$ is divisible by 72.

9. Is it possible that, for some a, both expressions $\dfrac{1 - 2a\sqrt{35}}{a^2}$ and $a + \sqrt{35}$ are integers?

10. Prove that among any six consecutive positive integers it is always possible to find a number that is relatively prime to the product of the other five integers.

11. What is the greatest value that a product of positive integers can take if their sum is equal to 2004?

12. Positive integers a, b, c, u, v, and w satisfy the system of equations

$$\begin{aligned} a + u &= 21\,, \\ b + v &= 31\,, \\ c + w &= 667\,. \end{aligned}$$

Can abc be equal to uvw?

13. Let u be the real root of the equation $x^3 - 3x^2 + 5x - 17 = 0$, and let v be the real root of the equation $x^3 - 3x^2 + 5x + 11 = 0$. Find $u + v$.

14. Is it possible to write all the integers from 1 to $10\,000$ in the cells of a 100×100 table so that the number in each cell is either smaller or greater than all the numbers written in all the cells having a common side with that cell?

15. Does there exist a polynomial, $P(x)$, with integer coefficients such that for all x in the interval $[\frac{4}{10}, \frac{9}{10}]$ the inequality $|P(x) - \frac{2}{3}| < 10^{-10}$ is valid?

16. Does there exist a positive number a_0 such that all the members of the infinite sequence a_0, a_1, a_2, $\ldots$, defined by the recurrence formula $a_n = \sqrt{a_{n-1} + 1}$, $n \geq 1$, are rational numbers?

17. Let a, b, and c be the sides of a triangle and let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$a^2yz + b^2zx + c^2xy \;\leq\; 0\,.$$

18. Points M and N are on the sides AB and BC of the triangle ABC, respectively. It is given that $\dfrac{AM}{MB} = \dfrac{BN}{NC} = 2$ and $\angle ACB = 2\angle MNB$. Prove that ABC is an isosceles triangle.

19. The two diagonals of a trapezoid divide it into four triangles. The areas of three of them are 1, 2, and 4 square units. What values can the area of the fourth triangle have?

20. The ratio of the lengths of the diagonals of a rhombus is $a : b$. Find the ratio of the area of the rhombus to the area of an inscribed circle.

Next we give the problems of the X Bosnian Mathematical Olympiad. Thanks again go to Felix Recio for collecting them for use in the *Corner*.

X BOSNIAN MATHEMATICAL OLYMPIAD
First Day

1. Let H be the orthocenter of an acute-angled triangle ABC. Prove that the mid-points of AB and CH and the intersection point of the interior bisectors of $\angle CAH$ and $\angle CBH$ are collinear.

2. Let a_1, a_2, and a_3 be nonnegative real numbers with $a_1 + a_2 + a_3 = 1$. Prove that

$$a_1\sqrt{a_2} + a_2\sqrt{a_3} + a_3\sqrt{a_1} \le \frac{1}{\sqrt{3}}.$$

3. Let $n \ge 2$ be an integer, let x_1, x_2, $\ldots$, x_n be positive integers, and let $S_i = x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n$ for $i = 1, 2, \ldots, n$. Find the maximum of the function

$$f(x_1, x_2, \ldots, x_n) = \frac{\gcd(x_1, S_1) + \gcd(x_2, S_2) + \cdots + \gcd(x_n, S_n)}{x_1 + x_2 + \cdots + x_n}$$

if $(x_1, x_2, \ldots, x_n)$ runs through the set of all n-tuples of distinct positive integers.

Second Day

4. Given are a circle and its diameter PQ. Let t be a tangent to the circle, touching it at T, and let A be the intersection of the lines t and PQ. Let p and q be the tangents to the circle at P and Q respectively, and let

$$PT \cap q = \{N\} \quad \text{and} \quad QT \cap p = \{M\}.$$

Prove that the points A, M, and N are collinear.

5. A permutation $(a_1, a_2, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ satisfies the inequality $\dfrac{a_k^2}{a_{k+1}} \le k + 2$ for each $k = 1, 2, \ldots, n-1$. Prove it is the identity.

6. Let a, b, and c be integers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3\,.$$

Prove that abc is a perfect cube.

The Icelandic Mathematical Contest 2004–2005 (Final Round) is the last set of problems we give for this number. Thanks again to Felix Recio for collecting them for our use.

ICELANDIC MATHEMATICAL CONTEST 2004–2005

Final Round

March 12, 2005

1. How many subsets with three elements can be formed from the set $\{1, 2, \ldots, 20\}$ so that 4 is a factor of the product of the three numbers in the subset?

2. Triangle ABC is equilateral, D is a point inside the triangle such that $DA = DB$, and E is a point that satisfies the conditions $\angle DBE = \angle DBC$ and $BE = AB$. How large is the angle $\angle DEB$?

3. Find a three-digit number n that is equal to the sum of all the two-digit numbers that can be formed by using only the digits of the number n. (Note that if a is one of the digits of the number n, then aa is one of the two-digit numbers that can be formed.)

4. Transportation between six cities is such that between any two cities there is either a bus or a train but not both of these. Show that among these six cities there are three cities that are linked either only by buses or only by trains.

5. Determine whether the fraction $\frac{1}{2005}$ can be written as a sum of 2005 different unit fractions. (A unit fraction is a fraction of the form $\dfrac{1}{n}$, where n is a natural number.)

6. Let h be the altitude from A to in an acute triangle ABC. Prove that

$$(b+c)^2 \;\geq\; a^2 + 4h^2\,,$$

where a, b, and c are the lengths of the sides opposite A, B, and C respectively.

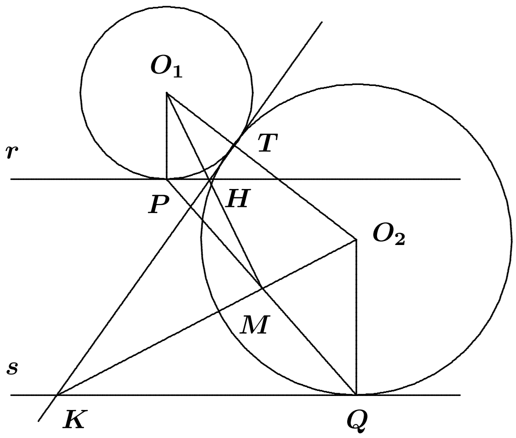
Two solutions arrived to problems we discussed in the April and May numbers of the *Corner*. A solution to problem 1 of the Thai Mathematical Olympiad [2007: 277; 2008: 155-156] was received from Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. He also sent a solution to problem 2 of the 25$^{\text{th}}$ Albanian Mathematical Olympiad [2007: 278; 2008: 220-221].

Next we present a "missed case" in the featured solution to problem 2 of the XX Olimpiadi Italiane Della Matematica, Cesenatico, given at [2007: 149-150; 2008: 83].

2. Let r and s be two parallel lines in the plane, and P and Q two points such that $P \in r$ and $Q \in s$. Consider circles C_P and C_Q such that C_P is tangent to r at P, C_Q is tangent to s at Q, and C_P and C_Q are tangent externally to each other at some point, say T. Find the locus of T when (C_P, C_Q) varies over all pairs of circles with the given properties.

Comment and solution by Jan Verster, Kwantlen University College, BC.

The solution given is missing the case where one of the circles is outside the lines. To handle this case, let the circles be oriented as shown in the diagram, where O_1 and O_2 are the centres of circles C_P and C_Q, respectively. Let H and K be the points where the tangent line common to the two circles crosses the lines r and s, respectively. Let M be the intersection of the lines KO_2 and O_1H. Then $\triangle KTM \cong \triangle KQM$ (as KO_2 bisects the angle at K) and so we have $MT = MQ$ and $\angle KTM = \angle KQM$. Similarly $\triangle MPO_1 \cong \triangle MTO_1$, and thus, we have $MT = MP$ and also $\angle MPO_1 = \angle MTO_1$. By subtracting right angles we obtain $\angle MPH = \angle MTK = \angle KQM$, hence, P, M, and Q are collinear and M is the mid-point of PQ. Thus, T is on a circle with diameter PQ.



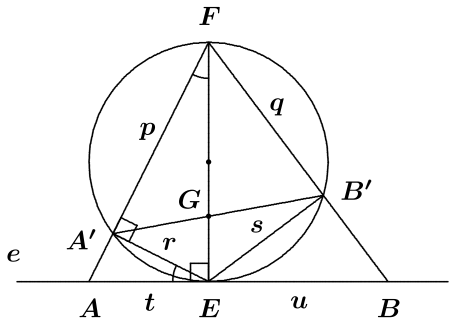
In summary, the locus of points T is the line segment PQ, together with the part of the circle with diameter PQ which is outside the two lines (and if one allows degenerate circles, the four points where the circle touches the lines).

Next we turn to solutions from our readers to problems given in the October 2007 number of the *Corner*. We begin with a solution to Problem 1 of the 2003 Kürschák Competition [2007: 336].

1. Let EF be a diameter of the circle Γ, and let e be the tangent line to Γ at E. Let A and B be any two points of e such that E is an interior point of the segment AB, and $AE \cdot EB$ is a fixed constant. Let AF and BF meet Γ at A' and B', respectively. Prove that all such segments $A'B'$ pass through a common point.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. We give Kandall's solution, modified by the editor.

Let G be the point of intersection of $A'B'$ and EF. Let $FA' = p$, $FB' = q$, $EA' = r$, $EB' = s$, $AE = t$, and $EB = u$. It is given that $tu = k$, where k is a fixed constant.



We have that $\angle EAF$ and $\angle AEF$ are right angles. The pair of angles $\angle FEA'$ and $\angle A'FE$ are supplementary, as are the pair of angles $\angle FEA'$ and $\angle A'EA$. Therefore, $\angle A'FE = \angle A'EA$, and hence, $\triangle EFA' \sim \triangle AEA'$. Consequently, $pt = r \cdot EF$, and by analogy, $qu = s \cdot EF$. Therefore, $pqk = rs \cdot EF^2$.

Now $\angle A'FB'$ and $\angle A'EB'$ are supplementary angles, hence,

$$\frac{FG}{GE} = \frac{[A'FB']}{[A'EB']} = \frac{pq}{rs} = \frac{EF^2}{k},$$

which is a fixed ratio. Thus, all such segments $A'B'$ pass through G.

Next we move to solutions for problems of the Hellenic Mathematical Competitions 2004 given at [2007: 336–337].

1. Find the greatest possible value of the positive real number M such that, for all x, y, $z \in \mathbb{R}$,

$$x^4 + y^4 + z^4 + xyz(x + y + z) \ \geq \ M(xy + yz + zx)^2 .$$

Solved by Michel Bataille, Rouen, France; Andrea Munaro, student, University of Trento, Trento, Italy; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give Bataille's write-up.

The greatest possible value of M is $\frac{2}{3}$.

First assume that

$$x^4 + y^4 + z^4 + xyz(x+y+z) \ \geq \ M(xy+yz+zx)^2 \qquad (1)$$

for all real x, y, and z. In particular, (1) must hold for $x = y = z = 1$, hence, $6 \geq M \cdot 9$ and $M \leq \frac{2}{3}$. To complete the proof, it remains to prove that (1) holds for all x, y, and z once $\frac{2}{3}$ is substituted for M, or equivalently that

$$3(x^4+y^4+z^4) \ \geq \ 2(x^2y^2+y^2z^2+z^2x^2) \ + \ x^2yz + xy^2z + xyz^2 \,. \qquad (2)$$

Now, from the well-known inequality $a^2+b^2+c^2 \geq ab+bc+ca$, we deduce that

$$2(x^4+y^4+z^4) \ \geq \ 2(x^2y^2+y^2z^2+z^2x^2)\,, \qquad (3)$$

and also

$$x^4+y^4+z^4 \ \geq \ (xy)^2+(yz)^2+(zx)^2 \ \geq \ xy\cdot yz + yz\cdot zx + zx \cdot xy\,,$$

that is,

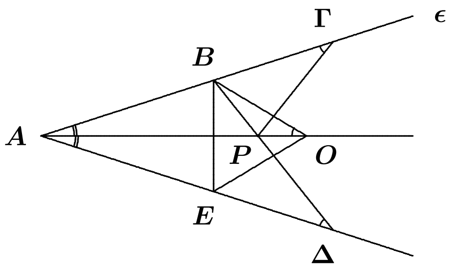
$$x^4+y^4+z^4 \ \geq \ x^2yz + xy^2z + xyz^2\,. \qquad (4)$$

Inequality (2) now follows by addition of (3) and (4).

3. A circle (O,r) and a point A outside the circle are given. From A we draw a straight line ε, different from the line AO, which intersects the circle at B and Γ, with B between A and Γ. Next we draw the symmetric line of ε with respect to the axis AO, which intersects the circle at E and Δ, with E between A and Δ.

Prove that the diagonals of the quadrilateral $B\Gamma\Delta E$ pass through a fixed point; that is, they always intersect at the same point, independent of the position of the line ε.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Andrea Munaro, student, University of Trento, Trento, Italy; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Maragoudakis' solution.



Let $AO \cap B\Delta = \{P\}$. By symmetry, $\angle A\Gamma P = \angle B\Delta A$. Since $\angle B\Delta A = \frac{1}{2}\angle BOE = \angle BOA$, we have $\angle A\Gamma P = \angle BOA$. Thus, triangles $A\Gamma P$ and AOB are similar, so

$$\frac{AP}{AB} = \frac{A\Gamma}{AO}\,, \ \text{ or } \ AP = \frac{AB\cdot A\Gamma}{AO}\,.$$

Since $AB\cdot A\Gamma = AO^2 - r^2$, we have $AP = \dfrac{AO^2-r^2}{AO}$, which is independent of the line ϵ. By symmetry, $E\Gamma$ and AO also intersect at the point P, and the proof is complete.

We finish this number with solutions provided by our readers to some problems of the Vietnamese Mathematical Olympiad 2004 [2007: 337-338].

1. Solve the system of equations

$$\begin{aligned} x^3 + x(y-z)^2 &= 2\,, \\ y^3 + y(z-x)^2 &= 30\,, \\ z^3 + z(x-y)^2 &= 16\,. \end{aligned}$$

Solved by Arkady Alt, San Jose, CA, USA; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Titu Zvonaru, Comănești, Romania. We give the write-up of Maragoudakis.

We note that x, y, and z are positive. We rewrite the equations as

$$\begin{aligned} x^2 + (y-z)^2 &= \frac{2}{x}\,, \\ y^2 + (z-x)^2 &= \frac{30}{y}\,, \\ z^2 + (x-y)^2 &= \frac{16}{z}\,. \end{aligned}$$

We subtract the second equation from the first to obtain

$$(x-y)(x+y) + \big((y-z)-(z-x)\big)\big((y-z)+(z-x)\big) = \frac{2}{x} - \frac{30}{y}\,.$$

Isolating $x - y$ in the above yields the first equation below; the other two equations below follow similarly:

$$\begin{aligned} x - y &= \frac{y - 15x}{xyz}\,, \\ y - z &= \frac{15z - 8y}{xyz}\,, \\ z - x &= \frac{8x - z}{xyz}\,. \end{aligned}$$

The left sides of the above sum to zero, hence, the right sides sum to zero and we obtain $(y - 15x) + (15z - 8y) + (8x - z) = 0$, or $z = \frac{1}{2}(x + y)$. Substituting this into the very first equation, we obtain

$$x^3 + \frac{x(x-y)^2}{4} = 2\,,$$

hence, $(x-y)^2 = \dfrac{8-4x^3}{x}$. The third equation in the original system can be put in the form $(x-y)^2 = \dfrac{16-z^3}{z}$, hence,

$$\frac{8-4x^3}{x} = \frac{16-z^3}{z}\,.$$

It follows that $8(z-2x)+xz(z^2-4x^2) = 0$, which upon factoring becomes $(z-2x)\big(8+xz(z+2x)\big) = 0$. Since x, y, and z are positive, we deduce that $z = 2x$, and hence, $y = 2z - x = 3x$.

Now the very first equation reduces to $x^3 + x(3x-2x)^2 = 2$, or $x = 1$. Therefore, $(x, y, z) = (1, 3, 2)$ is the unique solution to the system.

2. Solve the system of equations

$$\begin{aligned} x^3 + 3xy^2 &= -49\,, \\ x^2 - 8xy + y^2 &= 8y - 17x\,. \end{aligned}$$

Solved by Arkady Alt, San Jose, CA, USA, Pavlos Maragoudakis, Pireas, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; and Titu Zvonaru, Comănești, Romania. We give the write-up of Zvonaru.

From the first equation we have $x \neq 0$, and $y^2 = \dfrac{-x^3-49}{3x}$. Beginning with the second equation we deduce a succession of equations as follows:

$$\begin{aligned} x^2 - 8xy + y^2 &= 8y - 17x\,; \\ x^2 + 17x + y^2 &= 8y(x+1)\,; \\ x^2 + 17x - \left(\frac{x^3+49}{3x}\right) &= 8y(x+1)\,; \\ 2x^3 + 51x^2 - 49 &= 24xy(x+1)\,; \\ (x+1)(2x^2+49x-49) &= 24xy(x+1)\,. \end{aligned}$$

We now make two cases.

Case 1. We have $x = -1$. Then $y^2 = 16$, hence, $(x, y) = (-1, \pm 4)$ are the only two solutions.

Case 2. We have $x \neq -1$. Using the relations obtained so far we deduce a succession of equations as follows:

$$\begin{aligned} 2x^2 + 49x - 49 &= 24xy\,; \\ (2x^2+49x-49)^2 &= 576x^2y^2\,; \\ (2x^2+49x-49)^2 &= 576x^2\left(\frac{-x^3-49}{3x}\right)\,; \\ 196x^4 + 196x^3 + 2205x^2 + 4606x + 2401 &= 0\,; \\ 49(x+1)^2(4x^2-4x+49) &= 0\,. \end{aligned}$$

Since $x \neq -1$, we have $4x^2 - 4x + 49 = 0$, so that $x = \dfrac{1 \pm 4i\sqrt{3}}{2}$.

We obtain y as follows:

$$
\begin{aligned}
y &= \frac{2x^2+49x-49}{24x} \\
&= \frac{2x^2-2x-49+51x}{24x} \\
&= \frac{-\frac{49}{2}-49+51x}{24x} \\
&= \frac{17}{8} - \frac{49}{16x} \\
&= \frac{17}{8} - \frac{49}{8(1\pm 4i\sqrt{3})} = \frac{4\pm i\sqrt{3}}{2}\,.
\end{aligned}
$$

The solutions for (x,y) are $(-1,\pm 4)$ and $\left(\frac{1\pm 4i\sqrt{3}}{2}, \frac{4\pm i\sqrt{3}}{2}\right)$, where a consistent choice of signs is taken in the last pair.

3. Let ABC be a triangle in a plane. The internal angle bisector of $\angle ACB$ cuts the side AB at D.

Consider an arbitrary circle Γ_1 passing through C and D so that the lines BC and CA are not its tangents. This circle cuts the lines BC and CA again at M and N, respectively.

(a) Prove that there exists a circle Γ_2 touching the line DM at M and touching the line DN at N.

(b) The circle Γ_2 from part (a) cuts the lines BC and CA again at P and Q, respectively. Prove that the measures of the segments MP and NQ are constant as Γ_1 varies.

Solved by Andrea Munaro, student, University of Trento, Trento, Italy; and Titu Zvonaru, Comăneşti, Romania. We give Munaro's solution.

(a) Let $\gamma = \frac{1}{2}\angle ACB$. Consider the perpendiculars to DM at M and to DN at N. Clearly they intersect in a point R on Γ_1, where DR is a diameter of the circle. Since $\angle NRD = \gamma = \angle MRD$, then $NR = MR$. Thus, Γ_2 exists, namely Γ_2 is the circle with centre at R and passing through the points M and N.

(b) Using the facts established in part (a), the Law of Sines, and the fact that $DMRC$ is cyclic with a right angle at M, we have

$$
\begin{aligned}
\frac{PC}{\sin\angle PRC} &= \frac{CR}{\sin\angle CPR} = \frac{MR}{\sin\angle RCM} \\
&= \frac{MR}{\sin\angle RDM} = \frac{MR}{\cos\gamma}\,.
\end{aligned}
$$

Since $\angle RMC = \angle RNC$, then triangles MRP and NRQ are congruent and it is easy to show that $\angle PRC = \angle CDN$. Then

$$PC \;=\; \frac{MR}{\cos\gamma}\cdot\sin\angle CDN\,.$$

On the other hand,

$$\frac{CM}{\sin\angle CDM} \;=\; \frac{MD}{\sin\gamma}\,,$$

from which we obtain

$$\begin{aligned} CM &= \frac{MD}{\sin\gamma}\cdot\sin\angle CDM \\ &= \frac{MR\tan\gamma}{\sin\gamma}\cdot\sin\angle CDM \;=\; \frac{MR}{\cos\gamma}\cdot\sin\angle CDM\,. \end{aligned}$$

Hence,

$$\begin{aligned} PC+CM &= \frac{MR}{\cos\gamma}\cdot(\sin\angle CDN+\sin\angle CDM) \\ &= \frac{2MR}{\cos\gamma}\cdot\sin\left(\frac{\angle NDM}{2}\right)\cdot\cos\left(\frac{\angle CDN-\angle CDM}{2}\right) \\ &= 2MR\cdot\cos\left(\frac{\angle CDN-\angle CDM}{2}\right)\,. \end{aligned}$$

However,

$$\begin{aligned} &\angle CDN-\angle CDM \\ &\quad= \big(\angle RDN-(90^\circ-\angle CMD)\big)-(\angle RDM+90^\circ-\angle CMD) \\ &\quad= 2\angle CMD-180^\circ\,. \end{aligned}$$

Then

$$\begin{aligned} PC+CM &= 2MR\cdot\sin\angle CMD \\ &= 2CD\cdot\sin\gamma\cdot\frac{MR}{MD} \\ &= 2CD\cdot\cos\gamma\,, \end{aligned}$$

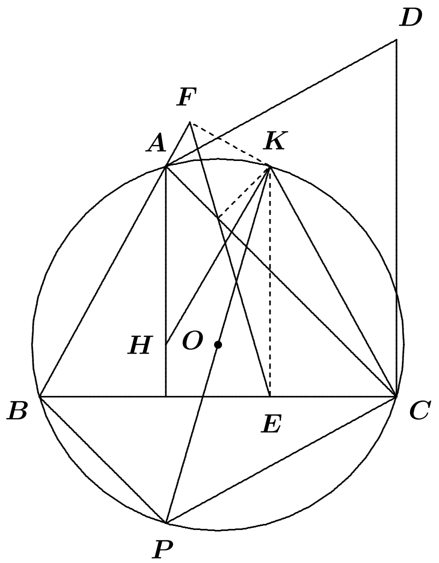
which is a constant. The same holds for NQ, since $MP = NQ$.

4. Given an acute triangle ABC inscribed in a circle Γ in a plane, let H be its orthocentre. On the arc BC of Γ not containing A, take a point P distinct from B and C. Let D be the point such that $\overrightarrow{AD} = \overrightarrow{PC}$. Let K be the orthocentre of triangle ACD, and let E and F be the orthogonal projections of K onto the lines BC and AB, respectively. Prove that the line EF passes through the mid-point of HK.

Solution and note by Michel Bataille, Rouen, France.

Since $\overrightarrow{AD} = \overrightarrow{PC}$, the quadrilateral $ADCP$ is a parallelogram and so $AP \| CD$ and $CP \| AD$. Since we have $AK \perp CD$, we also have $AK \perp AP$. It follows that the circle with diameter KP passes through A. Similarly, this circle passes through C. Finally, this circle is Γ and in particular, K is on Γ. As a result, the line EF is just the Simson line of K relative to $\triangle ABC$ and it is a well-known result that this line bisects the segment joining K to the orthocentre H.

Note. The latter property of the Simson line can be found, for example, in R. Honsberger's book *Episodes in 19th and 20th Century Euclidean Geometry*, MAA, 1995, pp. 43-6.



5. Consider the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ defined by $x_1 = 1$ and

$$x_{n+1} = \frac{(2+\cos 2\alpha)x_n + \cos^2\alpha}{(2-2\cos 2\alpha)x_n + 2 - \cos 2\alpha}$$

for every $n = 1, 2, \ldots$, where α is a real parameter. For each $n = 1, 2, \ldots$, let $y_n = \sum\limits_{k=1}^{n} \frac{1}{2x_k+1}$. Determine all values of α so that the sequence $\{y_n\}_{n=1}^{\infty}$ has a finite limit. Find this limit in these cases.

Solved by Michel Bataille, Rouen, France; and Andrea Munaro, student, University of Trento, Trento, Italy. We give Munaro's solution.

First we prove by induction that

$$\frac{1}{2x_n+1} = \frac{3(3^{n-1}-1)\sin^2\alpha + 1}{3^n}. \tag{1}$$

For $n = 1$ it is clearly true. On the other hand, we have

$$x_{n+1} = \frac{(1+2\cos^2\alpha)x_n + \cos^2\alpha}{4(1-\cos 2\alpha)x_n + 3 - 2\cos^2\alpha} = \frac{\cos^2\alpha(2x_n+1) + x_n}{4x_n + 3 - 2\cos^2\alpha(2x_n+1)}.$$

Hence, $2x_{n+1} + 1 = \dfrac{3(2x_n + 1)}{4x_n + 3 - 2\cos^2\alpha(2x_n + 1)}$ and

$$\begin{aligned}
\frac{1}{2x_{n+1}+1} &= \frac{4x_n + 3 - 2\cos^2\alpha(2x_n+1)}{3(2x_n+1)} \\
&= -\frac{2}{3}\cos^2\alpha + \frac{2(2x_n+1)+1}{3(2x_n+1)} \\
&= -\frac{2}{3}\cos^2\alpha + \frac{2}{3} + \frac{1}{3(2x_n+1)} \\
&= \frac{2}{3}\sin^2\alpha + \frac{1}{3(2x_n+1)}\,.
\end{aligned}$$

Suppose that (1) holds, then

$$\frac{1}{2x_{n+1}+1} = \frac{2}{3}\sin^2\alpha + \frac{3(3^{n-1}-1)\sin^2\alpha + 1}{3^{n+1}} = \frac{3(3^n-1)\sin^2\alpha+1}{3^{n+1}}\,,$$

and the induction proof is complete. Now

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{2x_k+1} &= \sum_{k=1}^{n}\frac{3(3^{k-1}-1)\sin^2\alpha+1}{3^k} \\
&= \left(\sum_{k=1}^{n}\frac{1}{3^k}\right) + \sin^2\alpha\sum_{k=1}^{n}\left(1-\frac{1}{3^{k-1}}\right) \\
&= \frac{1}{2}\left(1-\frac{1}{3^n}\right) + \sin^2\alpha\left(n - \frac{3}{2}\left(1-\frac{1}{3^n}\right)\right) \\
&= \left(1-\frac{1}{3^n}\right)\left(\frac{1}{2}-\frac{3}{2}\sin^2\alpha\right) + n\sin^2\alpha\,.
\end{aligned}$$

If $\sin^2\alpha > 0$, then

$$y_n = \left(1-\frac{1}{3^n}\right)\left(\frac{1}{2}-\frac{3}{2}\sin^2\alpha\right) \geq -1 + n\sin^2\alpha\,,$$

and hence, $y_n \to \infty$ as $n \to \infty$. In the other case, $\sin^2\alpha = 0$, and then we have $y_n = \frac{1}{2}\left(1-\frac{1}{3^n}\right) \to \frac{1}{2}$ as $n \to \infty$. Hence, $\{y_n\}_{n=1}^{\infty}$ has a finite limit if and only if $\alpha = k\pi$, $k \in \mathbb{Z}$, for which the corresponding limit is $\frac{1}{2}$.

6. Find the least value and the greatest value of the expression

$$P = \frac{x^4+y^4+z^4}{(x+y+z)^4}\,,$$

where x, y, and z are positive real numbers satisfying the condition

$$(x+y+z)^3 = 32xyz\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since P is homogeneous, we can assume that $x+y+z=1$. Then subject to conditions $x+y+z=1$ and $xyz=\frac{1}{32}$ we have

$$\begin{aligned} P &= x^4+y^4+z^4 \\ &= 1-4(xy+yz+zx)+2(xy+yz+zx)^2+4xyz \\ &= 2(xy+yz+zx)^2-4(xy+yz+zx)+1+\frac{1}{8} \\ &= 2(1-xy-yz-zx)^2-\frac{7}{8}\,. \end{aligned}$$

Since $xy+yz+zx\le\frac{1}{3}(x+y+z)^2=\frac{1}{3}$, then $1-xy-yz-zx>0$, so

$$\begin{aligned} \min P &= 2\big(1-\max(xy+yz+zx)\big)^2-\frac{7}{8}\,, \\ \max P &= 2\big(1-\min(xy+yz+zx)\big)^2-\frac{7}{8}\,. \end{aligned}$$

Moreover, $xy+yz+zx=\dfrac{1}{32z}+z(1-z)$, since $x+y=1-z$ and $xy=\dfrac{1}{32z}$. Setting $h(z)=\dfrac{1}{32z}+z(1-z)$, we have that

$$\begin{aligned} \min P &= 2\big(1-\max h(z)\big)^2-\frac{7}{8}\,, \\ \max P &= 2\big(1-\min h(z)\big)^2-\frac{7}{8}\,, \end{aligned}$$

where z is constrained by the solvability of the Viète System

$$\begin{aligned} x+y &= 1-z\,, \\ xy &= \frac{1}{32z}\,, \end{aligned}$$

in positive real numbers. That is, $z\in(0,1)$ and z must additionally satisfy the inequality $(1-z)^2-4\cdot\dfrac{1}{32z}\ge 0$. We have

$$(1-z)^2-4\cdot\frac{1}{32z} \;=\; \frac{1}{z}\left(z-\frac{1}{2}\right)\left(z-\frac{3-\sqrt{5}}{4}\right)\left(z-\frac{3+\sqrt{5}}{4}\right)\,,$$

and $0<\dfrac{3-\sqrt{5}}{4}<\dfrac{1}{2}<\dfrac{3+\sqrt{5}}{4}$, thus, for $z\in(0,1)$ the above expression is non-negative for $z\in\left[\dfrac{3-\sqrt{5}}{4},\dfrac{1}{2}\right]$, and we must find $\min h(z)$ and $\max h(z)$ on this interval. We have

$$h'(z) \;=\; \frac{32z^2-64z^3-1}{32z^2} \;=\; -\frac{2}{z^2}\left(z-\frac{1}{4}\right)\left(z-\frac{1-\sqrt{5}}{8}\right)\left(z-\frac{1+\sqrt{5}}{8}\right)\,,$$

hence, $z=\dfrac{1}{4}$ and $z=\dfrac{1+\sqrt{5}}{8}$ are the only roots of h' in the interval of interest. By direct calculation we have $h\left(\dfrac{1}{4}\right)=h\left(\dfrac{1}{2}\right)=\dfrac{5}{16}$ and also that

$h\left(\frac{3-\sqrt{5}}{4}\right) = h\left(\frac{1+\sqrt{5}}{8}\right) = \frac{5\sqrt{5}-1}{32}$, so the minimum and maximum values of $h(z)$ in the interval of interest are $\frac{5}{16}$ and $\frac{5\sqrt{5}-1}{32}$, respectively. Finally, the extreme values of P are

$$\begin{aligned} \min P &= 2\left(1-\frac{5\sqrt{5}-1}{32}\right)^2 - \frac{7}{8} = \frac{383-165\sqrt{5}}{256}, \\ \max P &= 2\left(1-\frac{5}{16}\right)^2 - \frac{7}{8} = \frac{9}{128}. \end{aligned}$$

7. Find all triples of positive integers (x, y, z) satisfying the condition

$$(x+y)(1+xy) = 2^z.$$

Solved by Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give the solution of Bataille.

The solutions are the triples $(1, 2^j-1, 2j)$, $(2^j-1, 1, 2j)$, where j is a positive integer and $(2^k-1, 2^k+1, 3k+1)$, $(2^k+1, 2^k-1, 3k+1)$, where k is an integer with $k \geq 2$.

It is readily checked that these triples are solutions. Conversely, suppose (x, y, z) is a solution. Then $x+y = 2^a$ and $1+xy = 2^b$ for some positive integers a and b. It follows that both x and y are odd. Note that (y, x, z) is also a solution, so we may suppose that $x \leq y$, and we have that $b \geq a$, since $1+xy-(x+y) = (1-x)(1-y) \geq 0$.

If $x = 1$, then $(1+y)^2 = 2^z$ so that $z = 2j$, $1+y = 2^j$ for some positive integer j and $(x, y, z) = (1, 2^j-1, 2j)$.

Now, suppose $3 \leq x \leq y$, in which case $a \geq 3$ and $b \geq 4$. Let $x = 2m+1$ and $y = 2n+1$. From $x+y = 2^a$, $1+xy = 2^b$, we deduce that m and n are of opposite parity and

$$\begin{aligned} mn &= 2^{a-2}(2^{b-a}-1), \\ (m+1)(n+1) &= 2^{a-2}(2^{b-a}+1). \end{aligned}$$

Thus, either one or the other of the following holds:

$$\begin{aligned} (m,n) &= (2^{a-2}, 2^{b-a}-1), \quad (m+1, n+1) = (2^{b-a}+1, 2^{a-2}); \\ (m,n) &= (2^{b-a}-1, 2^{a-2}), \quad (m+1, n+1) = (2^{a-2}, 2^{b-a}+1). \end{aligned}$$

In any case, $b-a = a-2$, so $x+y = 2^a$ and $1+xy = 2^{2a-2}$. As a result, the quadratic polynomial $X^2 - 2^a X + (2^{2a-2}-1)$ has x, y as roots. We recall that $x \leq y$ and set $k = a-1$ to obtain $(x, y, z) = (2^k-1, 2^k+1, 3k+1)$. This completes the proof.

That completes the *Corner* for this month. Send me your nice solutions and generalizations!

BOOK REVIEWS

John Grant McLoughlin

Digital Dice
By Paul J. Nahin, Princeton University Press, 2008
ISBN 978-0-691-12698-2, hardcover, 263+xi pages, US$27.95
Reviewed by ***Amar Sodhi****, Sir Wilfred Grenfell College, Corner Brook, NL*

A collection of twenty-one problems in probability which the reader is invited to solve using Monte Carlo methods surely cannot pique the interest of a person who does not relish computer programming? Not if the book is *Digital Dice*! This book contains more than enough material to interest anyone who enjoys the recreational side of mathematics.

The problems selected by Nahin are intrinsically interesting. Determining an ideal placement of police patrol cars on a stretch of highway, computing the expected number of stops that an elevator makes on its way up in an office tower, and deriving an optimal dating strategy are three examples that spring to mind. There are also problems involving coin flipping, voting procedures, and random walks as well as a couple of paradoxes which are bound to please. All problems are well referenced, enabling one to pursue a particular topic in more detail. Of particular interest is *Parrondo's Paradox*. Discovering that one can construct a winning game by randomly switching between two losing games is mind boggling to say the least! Rather than use either Monte Carlo methods or pen and paper analysis to investigate this problem, I simply enjoyed myself by using 100-sided "digital-dice" to play Parrondo's game many times and found that I did indeed come out ahead.

The material is very well presented and can be followed by a reader with just a modicum of knowledge of basic probability. The lengthy introduction provided by the author carefully shows how both exact methods and Monte Carlo methods can be used to solve a variety of problems. One such problem, a famous one in geometric probability, involves computing the chance that three points randomly chosen inside a rectangle are the vertices of an obtuse triangle.

Nahin has a warm and witty style of writing which makes his book a pleasure to read. By scattering interesting anecdotes, historical facts, and mathematical insights throughout the book he does cater to an audience who have no desire to write computer code to solve a probability problem. However, even the most pure mathematician might see the worthiness of using Monte Carlo methods for some of the problems. This is an enjoyable book for anyone who likes probability problems and/or solving problems using methods of simulation.

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 March 2009**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3351. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $AB > AC$. Let P be a point on the line AB beyond A such that $AP + PC = AB$. Let M be the mid-point of BC, and let Q be the point on the side AB such that $CQ \perp AM$. Prove that $BQ = 2AP$.

3352. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a right-angled triangle with right angle at A. Let I be the incentre of $\triangle ABC$, and let D and E be the intersections of BI and CI with AC and AB, respectively. Prove that

$$\frac{BI \cdot ID}{CI \cdot IE} = \frac{AB}{AC}.$$

3353. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle all of whose side lengths are positive integers.

(a) Determine all such triangles where one angle has twice the measure of a second angle.

(b) Determine all such triangles where two medians are perpendicular.

3354. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{n^2+k^2}{n^2}\right)^{k^3/n^4}.$$

3355. *Proposed by Todor Yalamov, Sofia University, Sofia, Bulgaria.*

For the triangle ABC let $(x,y)_{ABC}$ denote the line which intersects the union of the segments AB and BC in X and the segment AC in Y such that

$$\frac{\widetilde{AX}}{AB+BC} = \frac{AY}{AC} = \frac{x\cdot AB + y\cdot BC}{(x+y)(AB+BC)},$$

where $\widetilde{AX}$ is either the length of the segment AX if X lies between A and B, or the sum of the lengths of the segments AB and BX if X lies between B and C. Prove that the three lines $(x,y)_{ABC}$, $(x,y)_{BCA}$, and $(x,y)_{CBA}$ intersect in a point dividing the segment NI in the ratio $x:y$, where N is the Nagel point and I the incentre of $\triangle ABC$.

3356. *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let $f:[0,\infty)\to\mathbb{R}$ be integrable on $[0,1]$ and have period 1 (that is, $f(x+1)=f(x)$ for all $x\in[0,\infty)$). If $\{x_n\}_{n=1}^{\infty}$ is any strictly increasing, unbounded sequence with $x_0=0$ for which $(x_{n+1}-x_n)\to 0$, denote

$$r(n) = \max\{k\in\mathbb{N}\mid x_k\le n\}.$$

(a) Prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{r(n)}(x_k-x_{k-1})f(x_k) = \int_0^1 f(x)\,dx.$$

(b) Prove that

$$\lim_{n\to\infty}\frac{1}{\ln n}\sum_{k=1}^{n}\frac{f(\ln k)}{k} = \int_0^1 f(x)\,dx.$$

3357. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let a be a real number such that $-1<a\le 1$. Prove that

$$\int_0^1 \frac{x+a}{x^2+2ax+1}\ln(1-x)\,dx = \tfrac{1}{2}\ln^2\left(2\sin\frac{\theta}{2}\right)+\frac{\theta^2}{8}-\frac{\theta\pi}{4}+\frac{\pi^2}{24},$$

where θ is the unique solution in $(0,\pi]$ of the equation $\cos\theta=-a$.

3358. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The interior bisector of $\angle BAC$ of triangle ABC meets BC at D. Suppose that

$$\frac{1}{BD^2}+\frac{1}{CD^2} = \frac{2}{AD^2}.$$

Prove that $\angle BAC = 90^\circ$.

3359. *Proposed by Ray Killgrove, Vista, CA, USA and David Koster, University of Wisconsin, La Crosse, WI, USA.*

Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n^2 + n + 1$. Find a subsequence $\{b_n\}_{n=1}^{\infty}$ such that $b_1 = a_1$, $b_2 = a_2$, $b_3 > a_3$, every pair of terms from this subsequence are relatively prime, and there are primes which divide no term of the subsequence.

3360. *Proposed by Michel Bataille, Rouen, France.*

For complex numbers a, b, and c, not all zero, let $\mathcal{N}(a,b,c)$ denote the number of solutions $(z_1, z_2, z_3) \in \mathbb{C}^3$ to the system:

$$\begin{aligned} z_1 z_3 &= a\,, \\ z_1 z_2 + z_2 z_3 &= b\,, \\ z_1^2 + z_2^2 + z_3^2 &= c\,. \end{aligned}$$

For which a, b, and c does $\mathcal{N}(a,b,c)$ attain its minimal value?

3361. *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of triangle ABC meet the sides CA and AB at E and F, respectively. For which points P of the line segment EF do the areas of $\triangle EBC$, $\triangle PBC$, and $\triangle FBC$ form an arithmetic progression?

3362. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Prove that

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx \;<\; \frac{\pi^2}{12}\,.$$

. . . .

3351. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit ABC un triangle avec $AB > AC$. Soit P un point sur la droite AB, au-delà de A et tel que $AP + PC = AB$. Soit M le point milieu de BC, et soit Q le point sur le côté AB tel que $CQ \perp AM$. Montrer que $BQ = 2AP$.

3352. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit ABC un triangle rectangle avec son angle droit en A. Soit I le centre du cercle inscrit du triangle ABC, et soit D et E les intersections respectives de BI et CI avec AC et AB. Montrer que

$$\frac{BI \cdot ID}{CI \cdot IE} = \frac{AB}{AC}\,.$$

3353. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle dont la longueur de chacun de ses côtés est un entier positif.

(a) Trouver tous les triangles qui, en plus, ont un angle de mesure double d'un autre angle.

(b) Trouver tous les triangles qui, en plus, ont deux médianes perpendiculaires.

3354. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Evaluer

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{n^2+k^2}{n^2}\right)^{k^3/n^4}.$$

3355. *Proposé par Todor Yalamov, Université de Sofia, Sofia, Bulgarie.*

Dans un triangle ABC désignons par $(x,y)_{ABC}$ la droite qui coupe la réunion des segments AB et BC en X et le segment AC en Y de sorte que

$$\frac{\widetilde{AX}}{AB+BC} = \frac{AY}{AC} = \frac{x\cdot AB+y\cdot BC}{(x+y)(AB+BC)},$$

où $\widetilde{AX}$ est soit la longueur du segment AX si X est entre A et B, ou la somme des longueurs des segments AB et BX si X est entre B et C. Montrer que les trois droites $(x,y)_{ABC}$, $(x,y)_{BCA}$ et $(x,y)_{CBA}$ se coupent en un point qui divise le segment NI dans le rapport $x:y$, où N est le point de Nagel et I le centre du cercle inscrit du triangle ABC.

3356. *Proposé par Cristinel Mortici, Valahia Université de Targoviste, Roumanie.*

Soit $f:[0,\infty)\to\mathbb{R}$ une fonction intégrable sur $[0,1]$ et de période 1 (c-à-d., $f(x+1)=f(x)$ pour tout $x\in[0,\infty)$). Pour toute suite non bornée, strictement croissante $\{x_n\}_{n=1}^{\infty}$, avec $x_0=0$ et telle que $(x_{n+1}-x_n)\to 0$, on pose

$$r(n) = \max\{k\in\mathbb{N} \mid x_k\le n\}.$$

(a) Montrer que

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{r(n)}(x_k-x_{k-1})f(x_k) = \int_0^1 f(x)\,dx.$$

(b) Montrer que

$$\lim_{n\to\infty}\frac{1}{\ln n}\sum_{k=1}^{n}\frac{f(\ln k)}{k} = \int_0^1 f(x)\,dx.$$

3357. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

Soit a un nombre réel tel que $-1 < a \leq 1$. Montrer que

$$\int_0^1 \frac{x+a}{x^2+2ax+1} \ln(1-x)\,dx \;=\; \frac{1}{2}\ln^2\left(2\sin\frac{\theta}{2}\right) + \frac{\theta^2}{8} - \frac{\theta\pi}{4} + \frac{\pi^2}{24}\,,$$

où θ est la seule solution de l'équation $\cos\theta = -a$ dans $(0, \pi]$.

3358. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans le triangle ABC, la bissectrice intérieure de l'angle BAC coupe BC en D. On suppose que

$$\frac{1}{BD^2} + \frac{1}{CD^2} \;=\; \frac{2}{AD^2}\,.$$

Montrer que l'angle $BAC = 90^\circ$.

3359. *Proposé par Ray Killgrove, Vista, CA, É-U et David Koster, Université de Wisconsin, La Crosse, WI, É-U*

Soit la suite $\{a_n\}_{n=1}^{\infty}$ de terme général $a_n = n^2 + n + 1$. Trouver une sous-suite $\{b_n\}_{n=1}^{\infty}$ telle que $b_1 = a_1$, $b_2 = a_2$, $b_3 > a_3$, chaque paire de termes de cette sous-suite soient relativement premiers, et qu'il existe des nombres premiers qui ne divisent aucun terme de cette sous-suite.

3360. *Proposé par Michel Bataille, Rouen, France.*

Pour des nombres complexes (a, b, c), non tous nuls, notons $\mathcal{N}(a, b, c)$ le nombre de solutions $(z_1, z_2, z_3) \in \mathbb{C}^3$ du système :

$$\begin{aligned} z_1 z_3 &= a\,, \\ z_1 z_2 + z_2 z_3 &= b\,, \\ z_1^2 + z_2^2 + z_3^2 &= c\,. \end{aligned}$$

Trouver les nombres a, b et c pour lesquels $\mathcal{N}(a, b, c)$ atteint sa valeur minimale.

3361. *Proposé par Michel Bataille, Rouen, France.*

Le cercle inscrit du triangle ABC est tangent aux côtés CA et AB en E et F respectivement. Pour quels points P du segment de droite EF les aires des triangles EBC, PBC et FBC forment-elles un progression arithmétique ?

3362. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Montrer que

$$\int_0^1 \sqrt[3]{\frac{\ln(1+x)}{x}}\,dx \int_0^1 \sqrt[3]{\frac{\ln^2(1+x)}{x^2}}\,dx \;<\; \frac{\pi^2}{12}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We have received late solutions to problem 3221 from the following solvers: Ateneo Problem Solving Group; John G. Heuver; Thanos Magkos; and Pavlos Maragoudakis. Our apologies to Chip Curtis, Missouri Southern State University, Joplin, MO, USA, for a lost batch of correct solutions to problems 3239, 3241, and 3243–3248.

3251. [2006 : 297,299] *Proposed by Michel Bataille, Rouen, France.*

Let u_1, u_2, and u_3 be any real numbers. Prove that

$$\frac{1}{6}\sum_{i=1}^{3}\left[\cos^2(u_i-u_{i+1})+\cos^2(u_i+u_{i+1})\right] \geq (\cos u_1\cos u_2\cos u_3)^2+(\sin u_1\sin u_2\sin u_3)^2\,,$$

where the subscripts in the summation are taken modulo 3.

Solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

Since $\cos(x+y)=\cos x\cos y-\sin x\sin y$ for real numbers x and y, it follows that for each i we have

$$\cos^2(u_i-u_{i+1})+\cos^2(u_i+u_{i+1}) = 2(\cos^2 u_i\cos^2 u_{i+1}+\sin^2 u_i\sin^2 u_{i+1})\,,$$

where subscripts are taken modulo 3. Since $0\leq\cos^2 u_i\leq 1$ and also $0\leq\sin^2 u_i\leq 1$ for each i, we have

$$\begin{aligned}
&\frac{1}{6}\sum_{i=1}^{3}\left[\cos^2(u_i-u_{i+1})+\cos^2(u_i+u_{i+1})\right]\\
&= \frac{1}{6}\sum_{i=1}^{3}2(\cos^2 u_i\cos^2 u_{i+1}+\sin^2 u_i\sin^2 u_{i+1})\\
&= \frac{1}{3}\sum_{i=1}^{3}(\cos^2 u_i\cos^2 u_{i+1}+\sin^2 u_i\sin^2 u_{i+1})\\
&\geq \frac{1}{3}\sum_{i=1}^{3}(\cos^2 u_i\cos^2 u_{i+1}\cos^2 u_{i+2}+\sin^2 u_i\sin^2 u_{i+1}\sin^2 u_{i+2})\\
&= \frac{1}{3}\cdot 3(\cos^2 u_1\cos^2 u_2\cos^2 u_3+\sin^2 u_1\cos^2 u_2\sin^2 u_3)\\
&= (\cos u_1\cos u_2\cos u_3)^2+(\sin u_1\cos u_2\sin u_3)^2\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; IAN JUNE L. GARCES and WINFER C. TABARES, Ateneo de Manila University, Quezon City, The Philippines; ATENEO PROBLEM SOLVING GROUP, Ateneo de Manila University, Quezon City, The Philippines; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PAVLOS MARAGOUDAKIS, Pireas, Greece; ANDREA MUNARO, student, University of Trento, Trento, Italy; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Ros proved two generalizations of this inequality. For $n > 2$ and $u_1, u_2, \dots, u_n$ real numbers,

$$\frac{1}{2n}\sum_{i=1}^{n}[\cos^2(u_i-u_{i+1})+\cos^2(u_i+u_{i+1})] \;\geq\; \left(\prod_{i=1}^{n}\cos u_i\right)^2+\left(\prod_{i=1}^{n}\sin u_i\right)^2$$

and

$$\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}[\cos^2(u_i-u_j)+\cos^2(u_i+u_j)] \;\geq\; \left(\prod_{i=1}^{n}\cos u_i\right)^2+\left(\prod_{i=1}^{n}\sin u_i\right)^2,$$

where subscripts in the summation are taken modulo n. Equality holds if and only if $u_1 \equiv \cdots \equiv u_n \equiv 0 \pmod{\pi}$, or $u_1 \equiv \cdots \equiv u_n \equiv \frac{\pi}{2} \pmod{\pi}$.

Benito, Ciaurri, Fernández, and Roncal proved another generalization. For n and u_i $(1 \leq i \leq n)$ as above,

$$\frac{1}{2n}\sum_{i=1}^{n}[\cos^2(u_i-u_{i+1})+\cos^2(u_i+u_{i+1})] \;\geq\; \left(\prod_{i=1}^{n}\cos u_i\right)^{4/n}+\left(\prod_{i=1}^{n}\sin u_i\right)^{4/n},$$

where subscripts in the summation are taken modulo n.

3252. [2007 : 297,300] *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{S}$ be a set of complex 2×2 matrices such that, for all A, B, $C \in \mathcal{S}$, we have $ABCAB = C$.

(a) Show that $(ABC)^n = A^nB^nC^n$ for all positive integers n and all matrices A, B, $C \in \mathcal{S}$.

(b) Give an example of such a set $\mathcal{S}$ containing at least three matrices with two of them non-commuting.

Solution by Michael Parmenter, Memorial University of Newfoundland, St. John's, NL.

(a) The condition $ABCAB = C$ implies $ABCABC = C^2$, which, up to symmetry, is the same as $CABCAB = B^2$. Also, $ABCAB = C$ implies

$CABCAB = C^2$, so that $B^2 = C^2$ for any two matrices $B,C \in S$. Now, $B^2 = C^2$ implies $B^2C = C^3$ and $CB^2 = C^3$, so that B^2 commutes with C for any two matrices $B,C \in S$. Applying the condition $ABCAB = C$ for $A = B = C$, we obtain $A^5 = A$ for every $A \in S$.

Let k be any positive integer. Using the above observations, we have

$$(ABC)^{2k} \;=\; \left((ABC)^2\right)^k \;=\; (C^2)^k \;=\; C^{2k}$$

and

$$\begin{aligned} A^{2k}B^{2k}C^{2k} &= (A^2)^k(B^2)^k(C^2)^k \;=\; (C^2)^{3k} \;=\; C^{6k} \\ &= C^kC^{5k} \;=\; C^k(C^5)^k \;=\; C^kC^k \;=\; C^{2k}\,. \end{aligned}$$

Thus, $(ABC)^{2k} = A^{2k}B^{2k}C^{2k}$. Using this result and the established commutativity, we obtain

$$\begin{aligned} (ABC)^{2k+1} &= (ABC)^{2k}ABC \;=\; A^{2k}B^{2k}C^{2k}ABC \\ &= A^{2k}AB^{2k}BC^{2k}C \;=\; A^{2k+1}B^{2k+1}C^{2k+1}\,. \end{aligned}$$

Therefore, $(ABC)^n = A^nB^nC^n$ for all positive integers n.

(b) We first give a general approach to finding such an example. Let A be any matrix with $A^4 = I$ and $A^2 \neq I$, where I is the identity matrix. Let $B = A^3$ and let C be any matrix such that $C^2 = A^2$ and $AC \neq CA$. Then $AB = BA = A^4 = I$. It is easy to show that the condition $XYZXY = Z$ holds for any permutation (X, Y, Z) of the matrices A, B, and C:

$$\begin{aligned} ABCAB &= ICI \;=\; C\,, \\ BACBA &= ICI \;=\; C\,, \\ ACBAC &= ACIC \;=\; AC^2 \;=\; A^3 \;=\; B\,, \\ CABCA &= CICA \;=\; C^2A \;=\; A^3 \;=\; B\,, \\ BCABC &= BCIC \;=\; BC^2 \;=\; A^3A^2 \;=\; A\,, \\ CBACB &= CICB \;=\; C^2B \;=\; A^2A^3 \;=\; A\,. \end{aligned}$$

One particular example is

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad B = A^3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} i & x \\ 0 & -i \end{bmatrix},$$

where $x \neq 0$. Clearly, A and C do not commute, because

$$AC - CA = \begin{bmatrix} 0 & 2xi \\ 0 & 0 \end{bmatrix}.$$

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; PETER Y. WOO, Biola University, La Mirada, CA, USA (part (a) *only); and the proposer.*

Both Barbara and Parmenter noted that the result in part (a) *is true in any semigroup.*

3253. [2007 : 297, 300] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) \;<\; e^2 + \pi^2\,.$$

I. *Essentially similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Tom Leong, Brooklyn, NY, USA; Andrea Munaro, student, University of Trento, Trento, Italy; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and the proposer.*

By the AM-GM Inequality, we have

$$\begin{aligned} \log_e(e^\pi - 1)\log_e(e^\pi + 1) &\le \left(\frac{\log_e(e^\pi - 1) + \log_e(e^\pi + 1)}{2}\right)^2 \\ = \;\; \frac{1}{4}\Big(\log_e\big((e^\pi - 1)(e^\pi + 1)\big)\Big)^2 \;&=\; \frac{1}{4}\big(\log_e(e^{2\pi} - 1)\big)^2 \\ < \;\; \frac{1}{4}\big(\log_e(e^{2\pi})\big)^2 \;&=\; \pi^2\,. \end{aligned} \tag{1}$$

Similarly, we have

$$\log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) \;<\; e^2\,. \tag{2}$$

The result follows by adding (1) and (2).

II. *Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.*

For $a > 1$ and $b > 0$, we have

$$\begin{aligned} &\log_a(a^b - 1)\log_a(a^b + 1) \\ &\quad= \big(b + \log_a(1 - a^{-b})\big)\big(b + \log_a(1 + a^{-b})\big) \\ &\quad= b^2 + b\log_a(1 - a^{-2b}) + \log_a(1 - a^{-b})\log_a(1 + a^{-b}) \;<\; b^2\,, \end{aligned}$$

since b and $1 - a^{-2b}$ are positive, $1 - a^{-b} < 1$, and $1 + a^{-b} > 1$. From the inequality above, we conclude that

$$\sum_{i=1}^{n} \log_{a_i}(a_i^{b_i} - 1)\log_{a_i}(a_i^{b_i} + 1) \;\le\; \sum_{i=1}^{n} b_i^2\,,$$

where $a_i > 1$ and $b_i > 0$ for $i = 1, 2, \ldots, n$.

The proposed inequality is the special case when $n = 2$, $a_1 = b_2 = e$ and $a_2 = b_1 = \pi$.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK, PAULA KOCA, and ANDREW SIEFKER, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France;

RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; ASHLEY TANGEMAN and PETRUS MARTINS, students, California State University, Fresno, CA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3254. [2007 : 298, 300] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a convex figure in the plane. A *diametrical chord* AB of $\mathcal{C}$ parallel to the direction vector $\overrightarrow{v}$ is a chord of $\mathcal{C}$ of maximal length parallel to the direction vector $\overrightarrow{v}$.

Prove that if every diametrical chord of $\mathcal{C}$ bisects the area enclosed by $\mathcal{C}$, then $\mathcal{C}$ must be centro-symmetric.

Solution by P.C. Hammer and T. Jefferson Smith from 1964, adapted by the editor.

What follows is a simplified version of the proof of Theorem 2.4 in [3]. In that work the authors prove that *any* convex planar body is centrally symmetric provided that each line bisecting the area is a diametral line. (Hammer and Smith use the words *diametral* and *centrally symmetric* rather than the equivalent but less common *diametrical* and *centro-symmetric*.)

Because in every direction there is exactly one line that bisects the given area, our assumption that every diametral chord is area bisecting implies that there is a unique diametral chord in every direction. The Hammer and Smith result is therefore stronger since it applies also to centrally symmetric regions whose boundary contains line segments (for which points of parallel sides are joined by parallel diametral chords, one of which bisects the area). In order to avoid a page of technical arguments, we will further restrict our result by assuming that the boundary of the convex region, denoted by $\mathcal{C}$, is a differentiable curve. For such boundaries our assumption that an area-bisecting chord is diametral implies that the tangent lines at its ends are parallel. We treat the plane as a vector space and let $u(\theta) = (\cos\theta, \sin\theta)$ be a unit vector function; then $u'(\theta) = (-\sin\theta, \cos\theta) = u(\theta + \frac{1}{2}\pi)$. Let $m(\theta)$ be the unique diametral line parallel to the direction of $u(\theta)$.

Then there exists a unique real number $p(\theta)$ such that $m(\theta)$ is representable as

$$\{x:\ x \cdot u'(\theta) = p(\theta)\} \;=\; \{x:\ x = p(\theta)u'(\theta) + tu(\theta),\ t \in \mathbb{R}\}\,. \qquad (1)$$

Note that $p(\theta)u'(\theta)$ is the foot of the perpendicular from the origin to $m(\theta)$. Because we assume that $\mathcal{C}$ is differentiable, it follows easily that so is $p(\theta)$. We now represent $\mathcal{C}$ by a function $x(\theta)$ in the following way. Each line $m(\theta)$ intersects the boundary in two points, one of which is given by

$$x(\theta) \;=\; p(\theta)u'(\theta) \;+\; f(\theta)u(\theta)\,, \qquad (2)$$

where $f(\theta)$ is assigned a value that makes it continuous. With that assignment then, since $p(\theta+\pi) = -p(\theta)$ while $u(\theta+\pi) = -u(\theta)$, the other point of $m(\theta) = m(\theta+\pi)$ on the boundary would be

$$x(\theta+\pi) \;=\; p(\theta)u'(\theta) \;-\; f(\theta+\pi)u(\theta)\,. \tag{3}$$

Subtracting (3) from (2) gives us $x(\theta) - x(\theta+\pi) = \big(f(\theta) + f(\theta+\pi)\big)u(\theta)$. Therefore, we choose the function $f(\theta)$ so that $f(\theta) + f(\theta+\pi) > 0$, whence

$$f(\theta) \;+\; f(\theta+\pi) \;=\; |x(\theta) - x(\theta+\pi)| \tag{4}$$

for this choice. With this notation the condition that the tangents at the ends of a diametral chord are parallel becomes

$$|x'(\theta+\pi)|\cdot x'(\theta) \;=\; -|x'(\theta)|\cdot x'(\theta+\pi)\,. \tag{5}$$

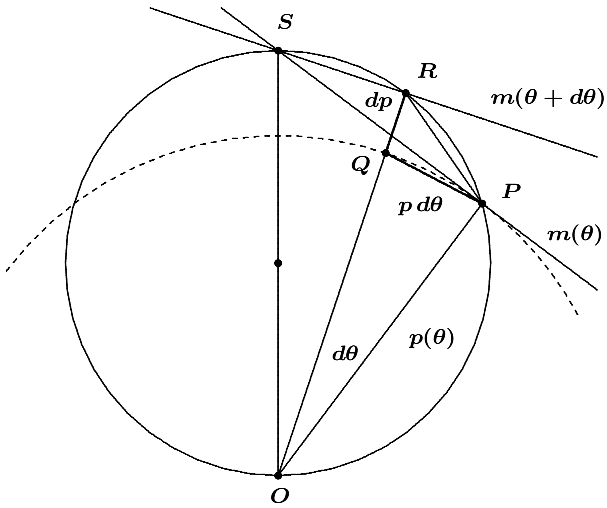
We must prove that $x'(\theta+\pi) = -x'(\theta)$. This will require two simple observations about neighbouring diametral chords. First, *in the limit*, they intersect at

$$s(\theta) \;=\; p(\theta)u'(\theta) \;-\; p'(\theta)u(\theta)\,. \tag{6}$$

More precisely, the diametral line $m(\theta)$ (whose points, according to (1) with $x = (x_1, x_2)$, satisfies the equation $x_1 \sin\theta - x_2\cos\theta + p(\theta) = 0$) intersects the diametral line $m(\theta+h)$ in the point

$$\left(\frac{p(\theta)\cos(\theta+h) - p(\theta+h)\cos\theta}{\sin h}, \frac{p(\theta)\sin(\theta+h) - p(\theta+h)\sin\theta}{\sin h}\right)\,.$$

The limit of these intersection points on $m(\theta)$ as $h \to 0$ is $s(\theta)$ in (6). Alternatively, one can avoid such calculations by referring to the diagram below,



where $m(\theta+h) = SR$ intersects $m(\theta) = SP$ at S, while R and P are the feet of the perpendiculars from the origin O. The claim to be verified is that the distance PS from the foot of the perpendicular P to the point of intersection S approaches $p'(\theta)$: Because of the right angles at R and P, these points lie on the circle whose diameter is OS. The circle with centre O and radius $p = p(\theta) = OP$ (which is tangent to SP at P) intersects OR at Q (as in the figure). Thus, for small values of $h = d\theta$ we see that PQ is approximately $p\,d\theta$, and $QR = dp$. As $h \to 0$, RP approaches the tangent to circle OPS at P, so that $\angle RPS \to \angle POR$. Consequently, $\dfrac{dp}{p\,d\theta} \to \dfrac{SP}{p}$, so that the limit satisfies

$$SP \;=\; \frac{dp}{d\theta} \;=\; p'(\theta)\,,$$

as claimed.

The second required observation is that as $h \to 0$, the point where $m(\theta + h)$ intersects $m(\theta)$ approaches the midpoint $\frac{1}{2}\big(x(\theta) + x(\theta + \pi)\big)$ of the chord of $\mathcal{C}$ along $m(\theta)$. This is an immediate consequence of the fact that two area bisecting chords, namely $m(\theta+h)$ and $m(\theta)$, divide the region bounded by $\mathcal{C}$ into four sectors such that opposite sectors have the same area. [Ed.: The intersecting lines are becoming the sides of isosceles triangles with equal areas and vertex angles: If the area bisecting chords AB and $A'B'$ intersects at X, then $\angle A'XA = \angle B'XB$ while $XA' \to XA$ as $A' \to A$ and $XB' \to XB$ as $B' \to B$.] This observation and equations (2), (3), and (5) yield

$$\begin{aligned} s(\theta) = \frac{1}{2}\big(x(\theta) \;+\; x(\theta+\pi)\big) \;&=\; p(\theta)u'(\theta) \;+\; \frac{1}{2}\big(f(\theta) - f(\theta+\pi)\big)u(\theta) \\ &=\; p(\theta)u'(\theta) \;-\; p'(\theta)u(\theta)\,, \end{aligned}$$

from which we conclude that $-p'(\theta) = \frac{1}{2}\big(f(\theta) - f(\theta+\pi)\big)$, or

$$p'(\theta) \;+\; f(\theta) \;=\; f(\theta+\pi) \;-\; p'(\theta)\,. \tag{7}$$

By taking the derivatives of the expressions in equations (2) and (3), we note that the coefficient of $u'(\theta)$ in $x'(\theta)$ is $p'(\theta) + f(\theta)$, while in $x'(\theta+\pi)$ it is $-\big(f(\theta+\pi) - p'(\theta)\big)$. Setting the coefficients of $u'(\theta)$ equal in equation (5) therefore gives us

$$\big(p'(\theta) + f(\theta)\big)|x'(\theta+\pi)| \;=\; \big(f(\theta+\pi) - p'(\theta)\big)|x'(\theta)|\,.$$

Because $p'(\theta) + f(\theta)$ is strictly positive (compare formula (2) with (6) and recall that the midpoint of each chord is interior to $\mathcal{C}$), this last equation together with the equality in (7) implies that

$$|x'(\theta+\pi)| \;=\; |x'(\theta)|\,.$$

Equation (5) now says that $x'(\theta) = -x'(\theta+\pi)$. On integration we find that $\frac{1}{2}\big(x(\theta) + x(\theta+\pi)\big)$ is a constant, namely the midpoint of every diametral chord, which completes the proof.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The editors are grateful for the help and the references provided by Paul Goodey, University of Oklahoma, Norman, OK; and Horst Martini University of Technology, Chemnitz, Germany. By coincidence, each has recently published a paper that discusses the claim in our problem, see [1] *and* [2] *where related theorems are proved and further references provided. In* [3] *the authors prove that* $\mathcal{C}$ *will also be centrally symmetric if each diametral line bisects the circumference. They trace their theorems back to a 1921 work of Konrad Zindler* [4].

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3255. [2007 : 298, 300] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that, as the points A, B, C move over the surface of an ellipsoid centred at O while the lines OA, OB, OC stay mutually perpendicular, the plane ABC remains tangent to a fixed sphere.

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be the equation of the given ellipsoid in an orthonormal coordinate system; that is, the centre of the ellipsoid is $O(0,0,0)$ and its axes are along the x-, y-, and z-axes. If the vectors from the origin to the moving points are $\overrightarrow{OA}=|\overrightarrow{OA}|(a_1,a_2,a_3)$, $\overrightarrow{OB}=|\overrightarrow{OB}|(b_1,b_2,b_3)$, and $\overrightarrow{OC}=|\overrightarrow{OC}|(c_1,c_2,c_3)$, then the condition that the lines OA, OB, and OC stay mutually perpendicular implies the equations

$$\begin{aligned}\sum_{i=1}^{3}a_i^2 &= \sum_{i=1}^{3}b_i^2 = \sum_{i=1}^{3}c_i^2 = 1\,,\\ \sum_{i=1}^{3}a_ib_i &= \sum_{i=1}^{3}b_ic_i = \sum_{i=1}^{3}c_ia_i = 0\,.\end{aligned} \tag{1}$$

The points A, B, and C lie on the ellipse, hence, their coordinates satisfy the equation of the ellipse:

$$\begin{aligned}\frac{1}{|\overrightarrow{OA}|^2} &= \frac{a_1^2}{a^2}+\frac{a_2^2}{b^2}+\frac{a_3^2}{c^2}\,,\\ \frac{1}{|\overrightarrow{OB}|^2} &= \frac{b_1^2}{a^2}+\frac{b_2^2}{b^2}+\frac{b_3^2}{c^2}\,,\\ \frac{1}{|\overrightarrow{OC}|^2} &= \frac{c_1^2}{a^2}+\frac{c_2^2}{b^2}+\frac{c_3^2}{c^2}\,.\end{aligned} \tag{2}$$

If we define the matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

then the first set of equations in (1) imply that $MM^t = I$. That is, M is an orthogonal matrix. This means that we also have $M^tM = I$, so that

$$a_1^2 + b_1^2 + c_1^2 \;=\; a_2^2 + b_2^2 + c_2^2 \;=\; a_3^2 + b_3^2 + c_3^2 \;=\; 1\,. \qquad (3)$$

Combining the equations in (2) and (3), we deduce that

$$\begin{aligned} &\frac{1}{|\overrightarrow{OA}|^2} + \frac{1}{|\overrightarrow{OB}|^2} + \frac{1}{|\overrightarrow{OC}|^2} \\ &= \frac{a_1^2+b_1^2+c_1^2}{a^2} + \frac{a_2^2+b_2^2+c_2^2}{b^2} + \frac{a_3^2+b_3^2+c_3^2}{c^2} \\ &= \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\,. \end{aligned}$$

Letting d denote the distance from O to the plane ABC, we recognize the left side of the preceding set of equations to be $\frac{1}{d^2}$. [*Ed.:* Although Demis includes a short proof of this claim, it can be found in standard references: the distance from the origin to the plane $px + qy + rz = 1$ is $\frac{1}{\sqrt{p^2+q^2+r^2}}$, where $\frac{1}{p}$, $\frac{1}{q}$, and $\frac{1}{r}$ are the distances from the origin to the points where the coordinate axes intersect the plane.] The equation tells us that $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ is a constant. In other words,

$$d = \frac{abc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}\,,$$

and we conclude that the plane ABC remains tangent to the sphere with centre O and radius d.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Although all the submitted solutions were essentially the same, only the treatment of Benito, Ciaurri, Fernández, and Roncal explicitly dealt with the n-dimensional version of the problem – the result (and the proof) is independent of dimension; it applies to an n-dimensional ellipsoid. The n-dimensional version of our problem can be found as exercise 15.7.20 in Marcel Berger's Geometry, *volume 2 (Springer 1987). His solution, which is also similar to our featured solution, appears in the accompanying volume by Berger et al.,* Problems in Geometry *(Springer Problem Books in Mathematics, 1984), problem 15.4, pages 227-229. After their proof the authors investigate the dual problem: the polarity with respect to the fixed sphere in our problem takes A, B, C to mutually orthogonal planes that intersect in the pole of the plane ABC (which is the point where the plane is tangent to the sphere). Because A, B, C lie on an ellipsoid, their polar planes are tangent to an ellipsoid, the polar body of the ellipsoid we started with. If, instead, we start with that new ellipsoid, we get the dual result:*

> *The locus of those points common to three mutually orthogonal planes that are tangent to an ellipsoid is a sphere, concentric with the ellipsoid.*

The dual result seems to be better known, and it comes with a variety of proofs. The sphere so obtained is variously known as the orthoptic sphere, *Monge's* sphere, *or the* director sphere.

3256. [2007 : 298, 300] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

A *bicentric* quadrilateral (also called a *chord-tangent* quadrilateral) is a quadrilateral that is simultaneously inscribed in one circle and circumscribed about another.

Let $ABCD$ be a bicentric quadrilateral in which there are no parallel sides. Suppose that the circumscribed and inscribed circles of $ABCD$ have centres O and I, respectively. Let AC meet BD at E. Join the points of tangency on the opposite sides of the quadrilateral, thus obtaining two lines which intersect at a point T.

Prove that O, E, T, and I are collinear. When do the points E and T coincide? (Compare 2978 [2004 : 429, 432; 2005 : 470–472].)

Solution by Michel Bataille, Rouen, France.

First we show the following:

> *If the points A, B, C, and D are arranged so that the lines AB, BC, CD, and DA are tangent to a circle* [Ed.: or, more generally, to a conic] γ *at P, Q, R, and S, respectively, then the point of intersection T of PR and QS is also the point of intersection of AC and BD.*

This result is just a special case of Brianchon's Theorem. [Ed.: See the comments after the list of solvers.] However, the following proof seems more appropriate here. Let AB and CD meet at U, and let AD and BC meet at V. Then PR is the polar of U with respect to γ, and QS is the polar of V. It follows that UV is the polar of T. But $X = PS \cap QR$ and $W = RS \cap PQ$ are conjugates of T, hence, the polar UV of T passes through W and X. Moreover, the polar RS of D and the polar PQ of B pass through W, hence, the polar of W is BD, and so BD passes through T. Similarly, the polar of X is AC and passes through T. The result follows. Note that the argument is projective, so that it is easily adapted if any of U, V, X, or W is at infinity; in other words, the proof remains valid should any or all of the sides of the quadrilaterals $ABCD$ and $PQRS$ be parallel.

We now suppose that γ is a circle with centre I and that A, B, C, and D are, in addition, on a circle Γ with centre O. Since AC and BD meet at T, the polar of T with respect to Γ passes through U and V, hence, is the line UV. As a result, UV is the polar of T with respect to Γ as well as to γ. Because lines that are conjugate with respect to a circle are perpendicular, we conclude that E (which, as we have seen, coincides with T), O, and I are on the perpendicular to UV through T.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Léon Anne served as répétiteur *at the Collège Louis-le-Grand. His proof that $E = T$ appeared in Les* Nouvelles Annales *1842, page 186; 1844, pages 28 and 485; it was reproduced in paragraph 1274 (pages 563-564) in* [4].

Our problem appears in paragraph 1275 (page 565) in [4] *under the heading "Newton's Theorem," with a proof similar to that of our first featured solution. No explicit reference to Newton's work appears there. The result appeared before in* **Crux Mathematicorum** [1989 : 226] *as an unused I.M.O. proposal. Other references, such as* [3]*, page 2, seem to refer to the (restricted) assertion that $E = T$ for a circle as Newton's theorem. Proofs (of this restricted theorem) in* [1]*, page 79; and* [5]*, page 100, apply Brianchon's theorem (If a hexagon is circumscribed about a conic, the three diagonals are concurrent) to the degenerate hexagons $APBCRD$ and $ABQCDS$ to conclude that AC, PR, BD, and QA all pass through the same point $E = T$. In Problem 39 of* [2]*, pages 188-191, there is a discussion of properties of chord-tangent quadrilaterals. We thank Curtis, Demis, Heuver (he refers to problem 40 of* [6]*), Malikić, and Konečný for the helpful references.*

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3257. [2007 : 298, 300] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Find the number of ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 13\}$ such that $|A \cup B|$ is even.

Essentially similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela; and Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; modified by the editor.

The solution provided is for $S = \{1, 2, \ldots, n\}$, the given problem being the special case $n = 13$. If A is a subset of S with $|A| = k$, $0 \le k \le n$, and if B is a subset of S such that $A \cup B = S$, then B is the union of $S - A$ and an arbitrary subset C of A. There are $\binom{n}{k}$ such subsets A and there are

2^k possibilities for the subset C, hence, the number of such pairs (A, B) is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n .$$

It is clear that the number of ordered pairs (A, B) of subsets of S such that $|A \cup B|$ is even, is $E_n = \sum_{k \text{ even}} \binom{n}{k} 3^k$. To evaluate E_n observe that

$$\sum_{k=0}^{n} \binom{n}{k} 3^k \;=\; (1+3)^n \;=\; 4^n , \tag{1}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k \;=\; (1-3)^n \;=\; (-2)^n . \tag{2}$$

Adding (1) and (2) yields $2E_n = 4^n + (-2)^n$, hence, $E_n = 2^{2n-1} + (-1)^n 2^{n-1}$. The required number of pairs is $E_{13} = 2^{25} - 2^{12} = 33550336$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; WHITNEY BULLOCK and DEBORAH SALAS-SMITH, students, California State University, Fresno CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; EDMUND SWYLAN, Riga, Latvia; BINGJIE WU, student, High School Affiliated to Fudan University, Shanghai, China; and the proposer.

Geupel generalized the result by showing that the number of q-tuples $(B_1, \ldots, B_q)$ of subsets of $\{1, \ldots, N\}$ such that $|B_1 \cup \cdots \cup B_q|$ is even is $\frac{1}{2}\left(2^{qN} + (-1)^N (2-2^q)^N\right)$. The Missouri State University Problem Solving Group showed that if $|B_1 \cup \cdots \cup B_q|$ is restricted to be a multiple of d, then the number of q-tuples is $\frac{1}{d} \sum_{t=0}^{d-1} \left(1 + (2^q - 1)\omega^t\right)^N$, where ω is a primitive d^{th} root of unity.

3258★. [2007 : 298, 300] *Proposed by Alper Cay, Uzman Private School, Kayseri, Turkey.*

Let ABC be a triangle with $\angle ABC = 80^\circ$. Let BD be the angle bisector of $\angle ABC$ with D on AC. If $AD = DB + BC$, determine $\angle A$, using a purely geometric argument.

Solution. The Problem Editor of *Math Horizons*, Andy Liu, informed us that this problem appeared recently as problem 201 in the April 2006 issue of *Math Horizons*, page 32. A solution (showing that $\angle A = 20^\circ$) by David Rhee and Jerry Lo was published in the November 2006 issue, pages 41–42.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3259. [2007 : 298, 301] *Proposed by Neven Jurič, Zagreb, Croatia.*

Is it possible to find a cubic polynomial P such that, for any positive integer n, the polynomial $\underbrace{P \circ P \circ \cdots \circ P}_{n \text{ times}}$ has exactly 3^n distinct real roots? Find one, if possible, or show that none exists.

I. *Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela, modified slightly by the editor.*

Let $P(x) = x^3 - 3x$. Then P has three distinct real roots, namely 0 and $\pm\sqrt{3}$. Since $P'(x) = 3x^2 - 3 = 3(x+1)(x-1)$ we see that P is strictly increasing on $(-\infty, -1]$ and $[1, \infty)$, and strictly decreasing on $[-1, 1]$.

Note that $P(-2) = -2$, $P(-1) = 2$, $P(1) = -2$, and $P(2) = 2$, so it follows that $P([-2,-1]) = P([-1,1]) = P([1,2]) = [-2,2]$.

Now we prove by induction that $P^n = P \circ P \circ \cdots \circ P$ (the n-fold composite) has exactly 3^n distinct real roots, all of which lie in the interval $(-2,2)$. This is clearly true for $n = 1$. Suppose the claim holds for some $n \geq 1$. Note that for $x \in (-2,-1)$, the polynomial P takes each value in $(-2,2)$ exactly once. The same is true for $x \in (-1,1)$ and for $x \in (1,2)$. Therefore, $P^{n+1}(x) = P^n\big(P(x)\big)$ has exactly 3^n distinct real roots in each of the intervals $(-2,-1)$, $(-1,1)$, and $(1,2)$, hence, P^{n+1} has 3^{n+1} distinct real roots in the interval $(-2\,,2)$. These are all the roots of P^{n+1}, because P^{n+1} is of degree 3^{n+1}. The induction is complete.

II. *Solution by the Missouri State University Problem Solving Group.*

More generally, we show that for any positive integer d, there exists a polynomial P of degree d such that for any positive integer n, the n-fold composite $P^n = P \circ P \circ \cdots \circ P$ has exactly d^n distinct real roots.

Let $P(x) = T_d(x) = \cos(d\cos^{-1}x)$ be the Chebyshev Polynomial of degree d of the first kind. It follows from the definition (and is well known) that T_d has the d distinct real roots $\cos\left(\dfrac{(2k+1)\pi}{2d}\right)$, $k = 0, 1, \ldots, d-1$, and that $T_a \circ T_b = T_{ab}$. Therefore, $T_{d^n} = T_d \circ T_d \circ \cdots \circ T_d$ has d^n distinct real roots. In particular, if we take $d = 3$, then

$$\begin{aligned} P(x) &= T_3(x) = \left(x - \cos\left(\frac{\pi}{6}\right)\right)\left(x - \cos\left(\frac{\pi}{2}\right)\right)\left(x - \cos\left(\frac{5\pi}{6}\right)\right) \\ &= x\left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right) = \frac{1}{4}(4x^3 - 3x) \end{aligned}$$

is a solution to the proposed problem.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3260. [2007 : 298, 301] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b be distinct positive real numbers such that $(a-1)(b-1) \geq 0$. Prove that

$$a^b + b^a \;\geq\; 1 + ab + (1-a)(1-b)\cdot\min\{1,\, ab\}\,.$$

Solution by Tom Leong, Brooklyn, NY, USA.

The numbers a and b need not be distinct, as the proof will show.

First suppose that $0 < a < 1$. Then we also have $0 < b < 1$. Put $a = 1-r$ and $b = 1-s$, where $0 < r < 1$ and $0 < s < 1$. Since in this case $\min\{1,\, ab\} = ab$, the right-hand side of the inequality is

$$1 + (1-r)(1-s) + rs(1-r)(1-s) \;=\; 1 + (1-r)(1-s)(1+rs)\,.$$

We have $(1-r)^s \leq 1 - rs$ and $(1-s)^r \leq 1 - rs$ by Bernoulli's Inequality. Since $\dfrac{1}{1-rs} > 1 + rs$, we obtain

$$\begin{aligned} a^b + b^a \;&=\; \frac{1-r}{(1-r)^s} + \frac{1-s}{(1-s)^r} \\ &\geq\; \frac{1-r}{1-rs} + \frac{1-s}{1-rs} \\ &=\; 1 + \frac{(1-r)(1-s)}{1-rs} \\ &>\; 1 + (1-r)(1-s)(1+rs) \\ &=\; 1 + ab + (1-a)(1-b)ab\,. \end{aligned}$$

Next suppose that $a \geq 1$. Then we also have $b \geq 1$. Put $a = 1 + r$ and $b = 1 + s$, where $r \geq 0$ and $s \geq 0$. Since in this case $\min\{1,\, ab\} = 1$, the right-hand side of the inequality is

$$1 + (1+r)(1+s) + rs \;=\; 2rs + r + s + 2\,.$$

We have $(1+r)^{1+s} \geq 1 + r(1+s)$ and $(1+s)^{1+r} \geq 1 + s(1+r)$ by Bernoulli's Inequality. Adding the two inequalities, we have $a^b + b^a \geq 2rs + r + s + 2$. Equality holds for $a = b = 1$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer.

3261. [2007 : 299, 301] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$\begin{array}{llll} F_0 = 0\,, & F_1 = 1\,, & \text{and} \quad F_{n+1} = F_n + F_{n-1}\,, & \text{for } n \geq 1; \\ L_0 = 2\,, & L_1 = 1\,, & \text{and} \quad L_{n+1} = L_n + L_{n-1}\,, & \text{for } n \geq 1. \end{array}$$

Prove that

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \leq \frac{4}{\pi}\arctan(\beta)\left(\arctan(\beta)+\frac{1}{3}\right),$$

where $\beta = \frac{1}{2}(\sqrt{5}-1)$.

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The following relations between the Fibonacci and Lucas numbers

$$L_{2n} + L_{2n+2} = 5F_{2n+1} \quad \text{and} \quad L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2\,,$$

are well known and easy to check. From these we have

$$\frac{1}{F_{2n+1}} = \frac{L_{2n}+L_{2n+2}}{L_{2n}L_{2n+2}-1} = \frac{\dfrac{1}{L_{2n}} + \dfrac{1}{L_{2n+2}}}{1 - \dfrac{1}{L_{2n}}\dfrac{1}{L_{2n+2}}}\,,$$

so that

$$\begin{aligned} \arctan\left(\frac{1}{F_{2n+1}}\right) &= \arctan\left(\frac{\dfrac{1}{L_{2n}} + \dfrac{1}{L_{2n+2}}}{1 - \dfrac{1}{L_{2n}}\dfrac{1}{L_{2n+2}}}\right) \\ &= \arctan\left(\frac{1}{L_{2n}}\right) + \arctan\left(\frac{1}{L_{2n+2}}\right). \end{aligned}$$

Applying the inequality $xy \leq \frac{1}{4}(x+y)^2$, we obtain

$$\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right) \leq \frac{1}{4}\left[\arctan\left(\frac{1}{F_{2n+1}}\right)\right]^2,$$

therefore,

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right) \arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \;\leq\; \frac{1}{4} \sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right) .$$

Using the relation $F_{2n+2} - F_{2n} = F_{2n+1}$ and the well known and easy to check formula $F_{2n}F_{2n+2} + 1 = F_{2n+1}^2$, we have

$$\frac{1}{F_{2n+1}} \;=\; \frac{F_{2n+2} - F_{2n}}{F_{2n}F_{2n+2} + 1} \;=\; \frac{\frac{1}{F_{2n}} - \frac{1}{F_{2n+2}}}{1 + \frac{1}{F_{2n}}\,\frac{1}{F_{2n+2}}} ,$$

and then

$$\begin{aligned} \arctan\left(\frac{1}{F_{2n+1}}\right) &= \arctan\left(\frac{\frac{1}{F_{2n}} - \frac{1}{F_{2n+2}}}{1 + \frac{1}{F_{2n}}\,\frac{1}{F_{2n+2}}}\right) \\ &= \arctan\left(\frac{1}{F_{2n}}\right) - \arctan\left(\frac{1}{F_{2n+2}}\right) . \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{4} \sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right) &= \frac{1}{4} \sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{F_{2n}}\right) - \arctan\left(\frac{1}{F_{2n+2}}\right)\right] \\ &= \frac{1}{4} \arctan\left(\frac{1}{F_2}\right) = \frac{1}{4} \arctan 1 = \frac{\pi}{16} . \end{aligned}$$

Thus, the sum of the given series does not exceed $\frac{\pi}{16} \approx 0.196$, which improves the proposed upper bound, because

$$\frac{4}{\pi} \arctan(\beta)\Big(\arctan(\beta) + \frac{1}{3}\Big) \approx 0.625 .$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous also improved the proposed upper bound.

3262. [2007 : 299, 301] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let m be an integer, $m \geq 2$, and let a_1, a_2, $\dots$, a_m be positive real numbers. Evaluate the limit

$$L_m \;=\; \lim_{n\to\infty} \frac{1}{n^m} \int_1^e \prod_{k=1}^{m} \ln\big(1 + a_k x^n\big)\, dx .$$

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain, modified by the editor.

For each integer $m \geq 1$ we will show that

$$L_m \;=\; (-1)^{m+1}m! \;+\; e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\,. \tag{1}$$

First note that for $x \geq 1$, we have

$$xa_k^{1/n} \;\leq\; (1+a_kx^n)^{1/n} \;\leq\; x(1+a_k)^{1/n}\,. \tag{2}$$

Since $a_k^{1/n}$ and $(1+a_k)^{1/n}$ each converge to 1 as $n\to\infty$, it follows from the above that $(1+a_kx^n)^{1/n}$ converges to x as $n\to\infty$, thus,

$$\lim_{n\to\infty}\frac{\ln(1+a_kx^n)}{n} \;=\; \lim_{n\to\infty}\ln(1+a_kx^n)^{1/n} \;=\; \ln x\,. \tag{3}$$

Taking logarithms across the last inequality in (2), we obtain

$$\frac{\ln(1+a_kx^n)}{n} \;\leq\; \ln x \;+\; \frac{\ln(1+a_k)}{n} \;\leq\; \ln x \;+\; \ln(1+a_k)\,,$$

from which it follows that

$$\prod_{k=1}^{m}\frac{\ln(1+a_kx^n)}{n} \;\leq\; \prod_{k=1}^{n}\big(\ln x+\ln(1+a_k)\big)\,.$$

By Lebesgue's Dominated Convergence Theorem, we may bring the limit inside the integral; then we apply (3) as follows

$$\begin{aligned} L_m &= \int_1^e \lim_{n\to\infty}\prod_{k=1}^{n}\frac{\ln(1+a_kx^n)}{n}\,dx \\ &= \int_1^e \prod_{k=1}^{n}\lim_{n\to\infty}\frac{\ln(1+a_kx^n)}{n}\,dx \\ &= \int_1^e (\ln x)^m\,dx\,. \end{aligned} \tag{4}$$

Next we integrate by parts to derive the recurrence relation

$$L_m \;=\; e - mL_{m-1}\,. \tag{5}$$

Finally, we use induction on m to show that (with the appropriate initial condition) the solution to the reccurence in (5) is given by (1).

The case when $m=1$ is clear, since the right side of (1) is 1 and from (4) we have $L_1=\int_1^e \ln x\,dx = 1$.

Suppose (1) holds for some $m \geq 1$. Then using (5) we have

$$\begin{aligned} L_m &= e - m\left\{(-1)^m(m-1)! + e\sum_{k=0}^{m-1}(-1)^k\frac{(m-1)!}{(m-1-k)!}\right\} \\ &= e + (-1)^{m+1}m! + e\sum_{k=0}^{m-1}(-1)^{k+1}\frac{m!}{(m-1-k)!} \\ &= (-1)^{m+1}m! + e + e\sum_{k=1}^{m}(-1)^k\frac{m!}{(m-k)!} \\ &= (-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\,, \end{aligned}$$

and our proof is complete.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN and N. NADEAU, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was 1 incorrect solution submitted.

Janous notes the interesting fact that L_m can be expressed in terms of D_m, the number of derangements of $1, 2, \ldots, m$. (A permutation σ of $1, 2, \ldots, m$ is called a derangement *if $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, m$.) Since it is well known that $D_m = m! \sum_{k=0}^{m}(-1)^k\frac{1}{k!}$, we see that $L_m = (-1)^{m+1}m! + (-1)^m eD_m$.*

The proposer remarked that his proposal was a generalization of the following problem which appeared in the Romanian journal Gazeta *in 2000:*

Compute $\lim_{n\to\infty}\frac{1}{n^2}\int_1^e \ln(1+x^n)\ln(1+2x^n)\,dx$.

Both he and Bracken and Nadeau pointed out the interesting fact that the answer is completely independent of the a_k's given.