ÉDITORIAL

Václav Linek

Vous avez probablement déjà reçu, à travers les correspondances, la triste nouvelle que notre coéditeur bien aimé, Jim Totten, est mort le 9 mars 2008. Une nécrologie suit cet édtiorial.

En 2009, on consacra un numéro spécial à Jim. Les propositions de problèmes consacrés à Jim que nous recevons avant le **1er janvier 2009** seront considérées pour un ensemble spécial de problèmes à apparaître dans le numéro spécial.

Les lecteurs peuvent également envoyer des articles consacrés à Jim pour le numéro spécial avant le **1er décembre 2008**. Les lecteurs qui veulent contribuer des projets spéciaux devraient nous consulter aussitôt que possible, au cas où une attention particulière serait requise pour accomplir le projet et pour éviter la duplication (d'autres peuvent avoir des projets semblables à soumettre).

Vue le grand réseau des amis mathématiciens de Jim partout dans le monde, il est fort possible qu'un seul numéro spécial n'est pas suffisant pour contenir tous les propositions et les projets. Dans ce cas nous distribuerons le matériel supplémentaire dans les numéros suivants.

Je vous remercie, lecteurs, de votre appui au cours de cette période difficile. Je suis maintenu à flot par vos mots aimables, et votre encouragement me donne la motivation pour continuer à faire mon travail. Au plaisir de communiquer avec vous dans le prochain futur.

Some of you already know the sad news through correspondence, that our cherished Co-Editor, Jim Totten, passed away on March 9, 2008. An obituary follows this editorial.

In 2009 a special issue will be dedicated to Jim. Problem proposals we receive before **January 1, 2009** and dedicated to Jim will be considered for a special set of problems to appear in the special issue.

Readers may also send articles dedicated to Jim for the special issue before **December 1, 2008**. Readers wishing to contribute special projects should consult with us as soon as possible, in case special attention is required to complete the project and to avoid duplication (others may have similar projects in mind).

Given Jim's extensive network of mathematical friends all over the world, it is quite possible there will be more material than one special issue can hold. In that case we will distribute the extra material in subsequent issues.

I thank you, the readers, for your support during this difficult period. I am buoyed by your kind words, and your encouragement keeps me going. I very much look forward to corresponding with all of you in the future.

IN MEMORIAM

James Edward Totten, 1947–2008

With the sudden death of Jim (James Edward Totten) on March 9, 2008, the Mathematics Community lost someone who was dedicated to mathematics education and to mathematical outreach. Jim was born August 9, 1947 in Saskatoon and raised in Regina. After obtaining his Bachelor's degree at the University of Saskatchewan, Jim then earned a Master's and PhD degree in Mathematics from the University of Waterloo, after which he joined the faculty at Saint Mary's University in Halifax. One of us (Robert) first got to know Jim when he and Jim shared an office at the University of Saskatchewan while Jim visited there during 1978-1979. That was a long cold winter, but Jim's active interest and enthusiasm for mathematics and the teaching of mathematics made the year a memorable one. The next year Jim took up a position at Cariboo College, where he remained as it evolved into the University College of the Cariboo and then eventually into Thompson Rivers University, retiring as Professor Emeritus in 2007.

During his years in Kamloops, Jim was a mainstay of the Cariboo Contest, an annual event which brought students in to the college and which featured a keynote speaker, often drawn from Jim's list of mathematical friends. This once included an invitation to Robert, which featured a talk on public key encryption mostly memorable for the failure of technology at a key moment, much to Jim's amusement.

Jim became a member of the CMS in 1981, and joined the editorial board of ***Crux*** in 1994. When Bruce was looking for someone to succeed him as Editor-in-Chief, there was no doubt in his mind whom to approach. Bruce spent a week in Kamloops staying at Jim's home and working with Jim and Bruce Crofoot to smooth the transition. Jim's attention to detail and care was appreciated by all, particularly those contributing copy that was carefully checked, as Robert gladly confirms from his continued association with Jim through the Olympiad Corner, an association which continued to the end.

Jim loved his Oldtimers' hockey and was an avid golfer. When not playing hockey or golf, he was active with the Kamloops Outdoors Club. Jim was never just a participant, always an active volunteer.

Jim is survived by his loving wife of 40 years, Lynne, his son Dean, daughter-in-law Christie and granddaughter Mikayla of Sechelt, his father Wilf Totten of Edmonton, sister Judy Totten of Regina, sister Josie Laing (Neil) of Onoway, sister-in-law Marilyn Totten of Red Deer, sister-in-law Constance Ladell (David Dahl) of Kamloops, many cousins, family friends and mathematical colleagues. All miss his warmth and love.

To honour the memory of Jim, the Thompson Rivers University Foundation is now accepting contributions for the Jim Totten Scholarship.

As remembered by two of his colleagues from ***Crux***,

Bruce Shawyer and Robert Woodrow

SKOLIAD No. 110

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 October, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Nos questions proviennent ce mois-ci du Concours de Comté, Société Mathématique de Croatie 2007 (niveau secondaire). Nous remercions Hanjš Željko, Université de Zagreb, pour la copie du concours.

Concours de Comté Société Mathématique de Croatie (niveau secondaire, 1er grade) 9 mars 2007

1. Trouver toutes les solutions entières de l'équation $x^2 + 11^2 = y^2$.

2. Dans un cercle avec centre S et rayon $r = 2$, deux rayons SA et SB sont tracés. L'angle entre eux est 45°. Soit K l'intersection de la ligne AB et la perpendiculaire de AS passant par S. Soit L le pied de la hauteur du sommet B dans le triangle ABS. Déterminer l'aire du trapèze $SKBL$.

3. Soient a, b et c trois nombres réels non-nuls donnés. Trouver x, y et z si

$$\frac{ay+bx}{xy} = \frac{bz+cy}{yz} = \frac{cx+az}{zx} = \frac{4a^2+4b^2+4c^2}{x^2+y^2+z^2}\,.$$

4. Soient a et b deux nombres réels positifs tels que $a > b$ et $ab = 1$. Prouvez l'inéquation

$$\frac{a-b}{a^2+b^2} \leq \frac{\sqrt{2}}{4}\,.$$

Déterminer $a + b$ quand l'égalité tient.

5. Le ratio des longueurs des deux côtés d'un rectangle est $12 : 5$. Les diagonales divisent le rectangle en quatre triangles. Les cercles inscrits de deux triangles qui ont un côté commun sont tracés. Soient r_1 et r_2 les rayons. Trouver le ratio $r_1 : r_2$.

County Competition
The Croatian Mathematical Society
(secondary level, 1st grade) March 9, 2007

1. Find all integer solutions to the equation $x^2 + 11^2 = y^2$.

2. In a circle with centre S and radius $r = 2$, two radii SA and SB are drawn. The angle between them is 45°. Let K be the intersection of the line AB and the perpendicular to line AS through point S. Let L be the foot of the altitude from vertex B in ΔABS. Determine the area of trapezoid $SKBL$.

3. Let a, b, and c be given nonzero real numbers. Find x, y, and z if

$$\frac{ay+bx}{xy} = \frac{bz+cy}{yz} = \frac{cx+az}{zx} = \frac{4a^2+4b^2+4c^2}{x^2+y^2+z^2}\,.$$

4. Let a and b be positive real numbers such that $a > b$ and $ab = 1$. Prove the inequality

$$\frac{a-b}{a^2+b^2} \le \frac{\sqrt{2}}{4}\,.$$

Determine $a + b$ if equality holds.

5. The ratio between the lengths of two sides of a rectangle is $12 : 5$. The diagonals divide the rectangle into four triangles. Circles are inscribed in two of them having a common side. Let r_1 and r_2 be their radii. Find the ratio $r_1 : r_2$.

Next we give solutions to the 23^{rd} W.J. Blundon Mathematics Contest given at [2007 : 321-322].

23^{rd} W.J. Blundon Mathematics Contest
Sponsored by the Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland
February 22, 2006

1. If $\log_a x = \log_b y$, show that each is also equal to $\log_{ab} xy$.

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

Let $N = \log_a x = \log_b y$. Then $a^N = x$ and $b^N = y$ or $(ab)^N = xy$. This means that N is also equal to $\log_{ab} xy$.

2. In how many ways can 20 dollars be changed into dimes and quarters, with at least one of each coin used?

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

An amount of 20 dollars is equal to 2000 cents. The last digit being 0, the number of quarters used is even, since the other denomination gives us multiples of 10. The number of nonzero even multiples of 25 which are less than 2000 is 39. Therefore there are 39 ways that 20 dollars can be changed into dimes and quarters, with at least one of each used.

3. If one of the women at a party leaves, then 20% of the people remaining at the party are women. If, instead, another woman arrives at the party, then 25% of the people at the party are women. How many men are at the party?

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

Fractions 20% and 25% are equal to $\frac{1}{5}$ and $\frac{1}{4}$, respectively. Let x be the number of women at the party and let y be the number of people at the party. According to the problem, we have

$$\begin{aligned} x-1 &= \frac{1}{5}(y-1)\,, \\ x+1 &= \frac{1}{4}(y+1)\,. \end{aligned}$$

These equations have the solution $(x,y)=(7,31)$, and therefore the number of men at the party is $y-x=24$.

4. Find two factors of $2^{48}-1$ between 60 and 70.

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

Repeatedly using the formula $x^2-y^2=(x-y)(x+y)$, we have

$$2^{48}-1=(2^{24}+1)(2^{12}+1)(2^6+1)(2^6-1)\,.$$

Since $2^6=64$, two factors of $2^{48}-1$ between 60 and 70 are 63 and 65.

5. The yearly changes in the population census of a town for four consecutive years are, respectively, 25% increase, 25% increase, 25% decrease, and 25% decrease. Find the net percent change to the nearest percent over the four years.

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

Let A be the number of people in the town originally. The number of people after four years is $A(1.25)(1.25)(0.75)(0.75) = 0.87890625A$. Therefore, the net percent change to the nearest percent over four years is a 12% decrease.

6. If $x + y = 5$ and $xy = 1$, find $x^3 + y^3$.

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

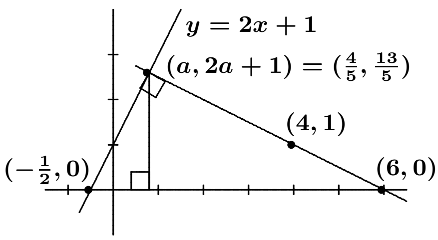
The steps in the solution are

$$\begin{aligned}(x+y)^3 &= 5^3\,,\\ x^3+3x^2y+3xy^2+y^3 &= 125\,,\\ x^3+y^3 &= 125-3xy(x+y)\,,\\ x^3+y^3 &= 125-3(1)(5)\,,\\ x^3+y^3 &= 110\,.\end{aligned}$$

7. The point $(4,1)$ is on the line that passes through the point $(4,1)$ and is perpendicular to the line $y = 2x + 1$. Find the area of the triangle formed by the line $y = 2x + 1$, the given perpendicular line, and the x-axis.

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

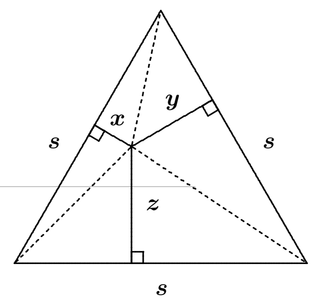
Let $y = mx + b$ be the equation of the unknown line. Since it is perpendicular to the line $y = 2x + 1$ and it passes through the point $(4,1)$, its equation is $y = -\frac{1}{2}x + 3$. From the equations, we get the coordinates of all the vertices and the dimensions of the triangle. Its base is 6.5 units and its height is 2.6 units. Therefore, its area is 8.45 square units.



8. An arbitrary point is selected inside an equilateral triangle. From this point perpendiculars are dropped to each side of the triangle. Show that the sum of the lengths of these perpendiculars is equal to the length of the altitude of the triangle.

Official Solution, modified by the editor.

Let s be the length of the equal sides, let h be the altitude, and let x, y, and z be the three distances from the interior point to the sides of the triangle. Then,



$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}sx + \frac{1}{2}sy + \frac{1}{2}sz \\ &= \frac{1}{2}s(x+y+z)\,.\end{aligned}$$

But we also have

$$\text{Area of triangle} = \frac{1}{2}sh\,.$$

Cancelling $\frac{1}{2}s$ from each expression for the area, we have $x + y + z = h$.

Also solved by RUIQI YU, student, Stephen Leacock Collegiate Institute, Toronto, ON

9. Find all positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad\text{and}\quad x + y^2 - z = 124\,.$$

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

Subtract the first equation from the second and then factor the result to obtain $(y - x)(x + y - 1) = 24$. Trying out all pairs of factors of 24, namely $(1, 24)$, $(2, 12)$, $(3, 8)$, and $(4, 6)$ for each of the two expressions leads to two positive integer solutions for x and y, namely $(x, y) = (3, 6)$ or $(12, 13)$. However, the first solution for x and y leads to a negative z, and must be excluded. Hence $(x, y, z) = (12, 13, 57)$.

10. How many roots are there of the equation $\sin x = \frac{1}{100}x$?

Solution by Ruiqi Yu, student, Stephen Leacock Collegiate Institute, Toronto, ON.

If x is measured in degrees, then we have 3 roots, since in this case the line $y = \frac{x}{100}$ only intersects one period of $y = \sin x$. If x is measured in radians, then the line has a y-value bigger than 1 for x bigger than 100. From $x = 0$ to $x = 100$, the line $y = \frac{x}{100}$ cuts through approximately 15.91 periods of $y = \sin x$. So we see that the line cuts the curve 32 times when x is not negative. By symmetry, it cuts the curve 32 times when x is not positive. But we count $(0, 0)$ twice, hence the line intersects the curve $32+32-1 = 63$ times. Hence there are 63 roots when x is measured in radians.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Ruiqi Yu. Congratulations Ruiqi! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 juillet 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M344. *Proposé par l'Équipe de Mayhem.*

On construit le tableau carré

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

formé des nombres de 1 à 9 écrits en ordre croissant, ligne après ligne. La somme des entiers sur les deux diagonales est 15. Si l'on construit un tableau carré semblable avec les entiers de 1 à $10\,000$, quelle sera la somme des nombres sur chacune des diagonales ?

M345. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

L'aire d'un triangle isocèle ABC est $q\sqrt{15}$. Si $AB = 2BC$, exprimer le périmètre du triangle ABC en fonction de q.

M346. *Proposé par l'Équipe de Mayhem.*

Déterminer le nombre de chiffres que comporte l'entier 2^{80} sans l'aide d'une calculatrice.

M347. *Proposé par l'Équipe de Mayhem.*

Quatre nombres entiers positifs a, b, c et d sont tels que

$$\begin{aligned}(a+b+c)d &= 420\,,\\ (a+c+d)b &= 403\,,\\ (a+b+d)c &= 363\,,\\ (b+c+d)a &= 228\,.\end{aligned}$$

Trouver ces quatre nombres.

M348. *Proposé par l'Équipe de Mayhem.*

Le périmètre d'un secteur de cercle est 12 (le périmètre inclut les deux rayons et l'arc). Déterminer le rayon du cercle qui maximise l'aire de ce secteur.

M349. *Proposé par l'Équipe de Mayhem.*

(a) Trouver toutes les paires ordonnées d'entiers (x, y) avec $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$.

(b) Combien y a-t-il de paires ordonnées d'entiers (x, y) avec

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}\,?$$

. . . .

M344. *Proposed by the Mayhem staff.*

Consider the square array

$$\begin{array}{ccc}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{array}$$

formed by listing the numbers 1 to 9 in order in consecutive rows. The sum of the integers on each diagonal is 15. If a similar array is constructed using the integers 1 to $10\,000$, what is the sum of the numbers on each diagonal?

M345. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The area of isosceles $\triangle ABC$ is $q\sqrt{15}$. Given that $AB = 2BC$, express the perimeter of $\triangle ABC$ in terms of q.

M346. *Proposed by the Mayhem Staff.*

Without using a calculator, find the number of digits in the integer 2^{80}.

M347. *Proposed by the Mayhem Staff.*

Four positive integers a, b, c, and d are such that

$$\begin{aligned} (a+b+c)d &= 420\,, \\ (a+c+d)b &= 403\,, \\ (a+b+d)c &= 363\,, \\ (b+c+d)a &= 228\,. \end{aligned}$$

Find the four integers.

M348. *Proposed by the Mayhem Staff.*

The perimeter of a sector of a circle is 12 (the perimeter includes the two radii and the arc). Determine the radius of the circle that maximizes the area of the sector.

M349. *Proposed by the Mayhem Staff.*

(a) Find all ordered pairs of integers (x, y) with $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$.

(b) How many ordered pairs of integers (x, y) are there with

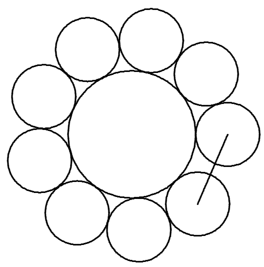
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}\,?$$

Mayhem Solutions

M294. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Nine circles of radius $1/2$ are externally tangent to a circle of radius 1 and are tangent to one another, as shown.

Determine the distance between the centres of the first and last of the circles of radius $1/2$.



Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam, adapted by the editor.

Label the centre of the first circle of radius $\frac{1}{2}$ with A, the centre of the second circle with B, the centre of the last (ninth) circle with C, and the centre of the circle of radius 1 with O. We need to determine the length of AC. We know that AO, BO, and CO are each equal to $1 + \frac{1}{2} = \frac{3}{2}$, because the smaller circles are tangent to the larger circle, and $AB = \frac{1}{2} + \frac{1}{2} = 1$, because the smaller circles with centres A and B are tangent.

Let $\angle AOB = \alpha$. Then $\angle AOC = 360^\circ - 8\alpha$, because there are 8 pairs of circles determining an angle of α at O. Applying the Law of Cosines to $\triangle AOC$ and using $\cos\theta = \cos(360^\circ - \theta)$, we have

$$\begin{aligned} AC^2 &= AO^2 + CO^2 - 2AO\cdot CO\cos\angle AOC \\ &= \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\cos(360^\circ - 8\alpha) \\ &= \frac{9}{2}(1-\cos 8\alpha)\,. \end{aligned}$$

To find $\cos 8\alpha$, we first find $\cos\alpha$ using the Law of Cosines in $\triangle AOB$:

$$\begin{aligned} AB^2 &= AO^2 + BO^2 - 2AO\cdot BO\cos\angle AOB\,, \\ 1^2 &= \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\cos\alpha\,, \end{aligned}$$

and solving for $\cos\alpha$ gives $\cos\alpha = \frac{7}{9}$. Therefore,

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 = 2\left(\frac{7}{9}\right)^2 - 1 = \frac{17}{81}\,, \\ \cos 4\alpha &= 2\cos^2 2\alpha - 1 = 2\left(\frac{17}{81}\right)^2 - 1 = -\frac{5983}{6561}\,, \\ \cos 8\alpha &= \cos^2 4\alpha - 1 = 2\left(\frac{5983}{6561}\right)^2 - 1 = \frac{28545857}{43046721}\,, \end{aligned}$$

and hence

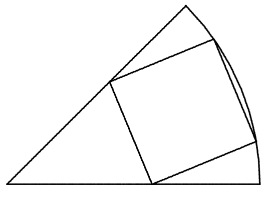
$$AC = \sqrt{\frac{9}{2}(1-\cos 8\alpha)} = \sqrt{\frac{9}{2}\left(1 - \frac{28545857}{43046721}\right)} = \frac{1904\sqrt{2}}{2187}\,.$$

Also solved by HASAN DENKER, Istanbul, Turkey; ANGELA DREI, Riolo Terme, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and J. SUCK, Essen, Germany. There was 1 incorrect solution submitted.

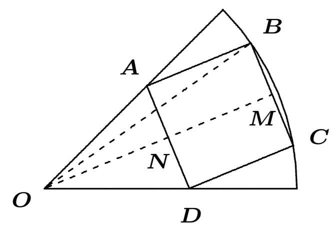
M295. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Square $ABCD$ is inscribed in one-eighth of a circle of radius 1 so that there is one vertex on each radius and two vertices on the arc.

Determine the exact area of the square in the form $\dfrac{a+b\sqrt{c}}{d}$, where a, b, c, and d are integers.



Solved independently by Hasan Denker, Istanbul, Turkey; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; R. Laumen, Deurne, Belgium; and Ricard Peiró, IES "Abastos", Valencia, Spain.



Let N and M represent the midpoints of AD and BC, respectively. In right triangle OAN, we have $AN = \frac{1}{2}BC$ and $\angle AON = \frac{\pi}{8}$. Therefore, $\frac{\frac{1}{2}BC}{ON} = \tan\frac{\pi}{8}$. We calculate $\tan\frac{\pi}{8}$. Since

$$1 = \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}},$$

we let $x = \tan\frac{\pi}{8}$ and we simplify to obtain $x^2 + 2x - 1 = 0$. Now x is positive, so by the quadratic formula $\tan\frac{\pi}{8} = x = \frac{-2+\sqrt{8}}{2} = \sqrt{2}-1$.

Thus $\frac{\frac{1}{2}BC}{ON} = \sqrt{2}-1$ and $ON = \frac{BC}{2\sqrt{2}-2}$, and by the Pythagorean Theorem in triangle BOM we have

$$OB^2 = BM^2 + OM^2 = BM^2 + (ON+NM)^2\,.$$

Since $OB = 1$, $NM = BC$, $BM = \frac{1}{2}BC$, and $ON = \frac{BC}{2\sqrt{2}-2}$, we obtain

$$\begin{aligned} 1 &= \frac{1}{4}BC^2 + \left(\frac{BC}{2\sqrt{2}-2} + BC\right)^2 = \frac{1}{4}BC^2 + \left(\frac{2\sqrt{2}-1}{2\sqrt{2}-2}\right)^2 BC^2 \\ &= \frac{1}{4}BC^2 + \left(\frac{(2\sqrt{2}-1)(2\sqrt{2}+2)}{4}\right)^2 BC^2 \\ &= \frac{1}{4}BC^2 + \left(\frac{11+6\sqrt{2}}{4}\right) BC^2 = \left(\frac{6+3\sqrt{2}}{2}\right) BC^2\,, \end{aligned}$$

hence the area is $BC^2 = \frac{2}{6+3\sqrt{2}}$, which upon rationalizing is $\frac{2-\sqrt{2}}{3}$.

Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and J. SUCK, Essen, Germany.

M296. *Proposed by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.*

Let n be a positive integer. In the Cartesian plane, consider the points $a_k = (k,n)$ and $b_k = (k,0)$ for $k = 1, 2, \ldots, n$. We connect each pair a_k, b_k by a straight (vertical) line segment. Then we draw an arbitrary finite number of horizontal line segments, each connecting two adjacent vertical line segments, such that no one point on any vertical segment is the end-point of two horizontal segments.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Define a map from A to B as follows: starting from a_i, travel down the segment until you meet the end-point of a horizontal segment, go to the other end-point of that segment, and continue on down the new vertical line, repeating this until there are no more horizontal segments to meet, finally ending at b_j for some j. Show that no two points of A map to the same point of B.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

The process described is uniquely reversible, so that starting from b_j gets you to a_i if going from a_i got you to b_j. Therefore, when going in reverse, there is no possibility of splitting paths from b_j to a_i, and therefore no two points in A can map to the same point in B.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; and the proposer.

M297. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Numbers such as 34543 and 713317 whose digits can be reversed without changing the number are called *palindromes*. Show that all four-digit palindromes are multiples of 11.

Solved independently by Angela Drei, Riolo Terme, Italy; Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; R. Laumen, Deurne, Belgium; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India

Every four-digit palindrome can be written in the form $abba$, where a and b are integers, such that $1 \leq a \leq 9$ and $0 \leq b \leq 9$. We can then conclude that

$$abba \;=\; 1000a + 100b + 10b + a \;=\; 1001a + 110b \;=\; 11(91a + 10b)\,.$$

Thus, every four-digit palindrome is a multiple of 11.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan.

M298. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

(a) Given that a number is a four-digit palindrome, what is the probability that the number is a multiple of 99?

(b) Given that a four-digit number is a multiple of 99, what is the probability that the number is a palindrome?

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

(a) A four-digit palindrome has the form $abba$, where a and b are integers with $1 \leq a \leq 9$ and $0 \leq b \leq 9$. There are therefore $9 \cdot 10 = 90$ four-digit palindromes. By Problem M297, we know that all four-digit palindromes are multiples of 11. For a four-digit palindrome, being a multiple of 99 is therefore equivalent to being a multiple of 9.

A positive integer is a multiple of 9 if and only if the sum of its digits is also a multiple of 9, so the four-digit palindromes which are multiples of 99 are precisely those for which $2(a+b)$ is a multiple of 9, that is, $2(a+b)$ equals 18 or 36. A complete list of these palindromes is 1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118, 9009, and 9999. Thus the probability is $\frac{10}{90} = \frac{1}{9}$.

(b) There are a total of 91 four-digit multiples of 99, namely the multiples from $99(11) = 1089$ to $99(101) = 9999$. By part (a), we know that among these there are ten four-digit palindromes. Therefore the required probability is $\frac{10}{91}$.

Also solved by ANGELA DREI, Riolo Terme, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There were 2 incorrect solutions submitted.

M299. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Let a, b, and c be positive real numbers with $ab+bc+ca=3$. Prove that

$$\frac{ab}{c^2+1} + \frac{bc}{a^2+1} + \frac{ca}{b^2+1} \;\geq\; \frac{3}{2}\,.$$

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Using the Arithmetic Mean-Geometric Mean Inequality, we have

$$(abc)^{2/3} \;=\; \sqrt[3]{(ab)(bc)(ac)} \;\leq\; \frac{ab+bc+ca}{3} \;=\; 1\,,$$

hence $abc \leq 1$. The desired inequality is equivalent to

$$\left(ab - \frac{ab}{c^2+1}\right) + \left(bc - \frac{bc}{a^2+1}\right) + \left(ca - \frac{ca}{b^2+1}\right) \;\leq\; \frac{3}{2}\,,$$

which is the same as

$$\frac{abc^2}{c^2+1} + \frac{bca^2}{a^2+1} + \frac{cab^2}{b^2+1} \;\leq\; \frac{3}{2}\,.$$

We have $(x-1)^2 \geq 0$, hence $x^2+1 \geq 2x$, and hence $\dfrac{1}{x^2+1} \leq \dfrac{1}{2x}$ when x is positive. Thus,

$$\frac{abc^2}{c^2+1} + \frac{bca^2}{a^2+1} + \frac{cab^2}{b^2+1} \;\leq\; \frac{abc^2}{2c} + \frac{bca^2}{2a} + \frac{cab^2}{2b} \;=\; \frac{3}{2}abc \;\leq\; \frac{3}{2}\,.$$

Equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA (two solutions); JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (second solution); and SHI CHANGWEI, Xi'an City, Shaan Xi Province, China. There were 2 incorrect solutions submitted.

M300. *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

Let ABC be an arbitrary triangle. Let D and E be points on the sides AC and AB, respectively, and let P be the point of intersection of BD and CE. If $AE : EB = r$ and $AD : DC = s$, determine the ratio of areas $[ABC] : [PBC]$ in terms of r and s.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Let AP meet BC at point K. Starting with Ceva's Theorem applied to $\triangle ABC$ and substituting the given ratios, we obtain

$$\frac{AE}{EB}\cdot\frac{BK}{KC}\cdot\frac{CD}{DA} = 1 \quad\Longrightarrow\quad r\cdot\frac{BK}{KC}\cdot\frac{1}{s} = 1 \quad\Longrightarrow\quad \frac{KC}{BK} = \frac{r}{s},$$

and hence

$$\frac{CB}{BK} = \frac{KC+BK}{BK} = \frac{r}{s}+1 = \frac{r+s}{s}.$$

Applying Menelaus' Theorem to the transversal BPD and $\triangle AKC$, and substituting the expression just obtained, we have

$$\begin{aligned}\frac{AD}{DC}\cdot\frac{CB}{BK}\cdot\frac{KP}{PA} = 1 \quad&\Longrightarrow\quad s\cdot\frac{r+s}{s}\cdot\frac{KP}{PA} = 1\\ &\Longrightarrow\quad \frac{PA}{KP} = r+s,\end{aligned}$$

so it follows that

$$\frac{KA}{KP} = \frac{PA+KP}{KP} = r+s+1.$$

Since $\dfrac{x}{y} = \dfrac{z}{t}$ implies $\dfrac{x}{y} = \dfrac{z}{t} = \dfrac{x+z}{y+t}$, we combine this with the previous results to obtain

$$\begin{aligned} r+s+1 &= \frac{KA}{KP} = \frac{[\triangle BKA]}{[\triangle BKP]} = \frac{[\triangle KAC]}{[\triangle KPC]}\\ &= \frac{[\triangle BKA]+[\triangle KAC]}{[\triangle BKP]+[\triangle KPC]} = \frac{[\triangle ABC]}{[\triangle PBC]},\end{aligned}$$

where $[\triangle BKA]$ denotes the area of $\triangle BKA$, and so forth. This shows that $[\triangle ABC] : [\triangle PBC] = r+s+1$.

Also solved by HASAN DENKER, Istanbul, Turkey; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; J. SUCK, Essen, Germany; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.

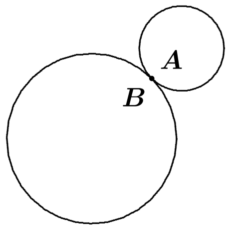
Problem of the Month

Ian VanderBurgh

As spring arrives in most parts of the country, it's time to dust off our bicycles. This problem is a bit reminiscent of the gears and wheels of a bicycle.

Problem (1996 Cayley Contest)

Two circular discs have radii 8 and 28. The larger disc is fixed while the smaller disc rolls around the outside of the larger disc. In their original positions, point A on the smaller disc coincides with point B on the larger disc. The least number of rotations that the small disc makes about its centre before A and B again coincide is



(A) 5 (B) 2 (C) 9 (D) 4.5 (E) 3.5

A friend of mine asked me about this problem from the archives a couple of months ago. It stumped me! Here's what I did initially:

> As the smaller circle rolls around the larger circle, we assume that no slipping occurs. As a result, the length of circumference travelled as measured on the smaller circle equals that as measured on the larger circle.
>
> Let's suppose that the smaller circle travels m times around its circumference while rolling n times around the larger circle. Since we want points A and B to line up at the end of the process, then each of m and n should be integers.
>
> The length of circumference travelled along the smaller circle is $m2\pi(8) = 16\pi m$ and the length travelled along the larger circle is $n2\pi(28) = 56\pi n$.
>
> For these lengths to be equal, $16\pi m = 56\pi n$ or $2m = 7n$. Since m and n are positive integers and we want m to be as small as possible, then $m = 7$ and $n = 2$.
>
> Therefore, the smaller circle rotates 7 times around its centre before the first time that A and B coincide.

Wait... 7 is not one of the possible answers! Is the question wrong? Did we do something wrong mathematically? Is our logic wrong?

The question is correct. In fact, this is indeed the first time that A and B coincide again, but somehow 7 is the wrong answer. The thing that we haven't taken into account is that as the smaller circle rolls, its motion around the larger circle produces extra rotations around its centre.

Here are three ways to look at this problem. It's up for you to decide how valid these solutions are and which convinces you the most.

Solution 1. $7 + 2 = 9$.

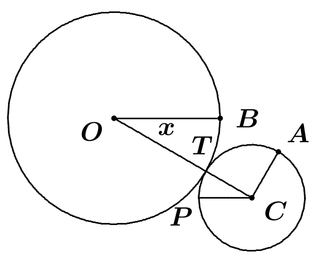
That's the shortest solution to a POTM in my time here! What is this trying to say? The smaller circle rotates 7 times while rolling 2 times about the centre of the larger circle. So perhaps the smaller circle rotates an extra 2 times around its centre, for a total of 9 times. Do you buy this? (Actually, a good point here is that the total number of rotations of the smaller circle should be at least 7, so given the possible answers, (C) 9 must be correct.)

Solution 2. When two circles are tangent (like the smaller and larger circles), the line joining their centres passes through the point of tangency.
Thus, the distance between the centres is the sum of the radii, or $8+28 = 36$. Therefore, the centre of the smaller circle is always a distance of 36 from the centre of the larger circle. Hence, the centre of the smaller circle travels around a circle of radius 36.
The centre of the smaller circle moves 2 times around this circle, so travels a total distance of $2(2\pi(36)) = 144\pi$.
The circumference of the smaller circle is $2\pi(8) = 16\pi$, and so while the centre of this circle moves a distance of 144π cm, it must rotate $\frac{144\pi}{16\pi} = 9$ times.

This feels better than Solution 1, but still leaves a somewhat uncertain feeling. The ratio of distance travelled by the centre to the number of rotations works on a flat surface, but should it work rolling around a circular surface?

The last solution is slightly less intuitive, but a bit more mathematical.

Solution 3. First, we turn the diagram so that the line joining the centres is horizontal. We don't really have to do this, but you might find it easier to visualize.
Next let's suppose that the smaller circle has rolled so that the point of contact between the circles is at an angle of x below the horizontal. (We'll treat the angles as measured in radians, but feel free to think in degrees if that's easier for you.) Call the centre of the larger circle O, the centre of the smaller circle C, the point of tangency T; we label the end-point of the radius to the left of C as P. We have defined $\angle TOB = x$. Since OB is parallel to PC, then we also have $\angle TCP = x$.



We now use the fact that the length of arc BT equals the length of arc TA (by equal distances rolled). The length of arc BT as a fraction of the entire circumference of the circle with centre O equals the fraction that $\angle TOB$ is of the angle associated with a full revolution. Thus, the length of arc BT is

$$\frac{\angle TOB}{2\pi}(2\pi(TO)) .$$

Similarly, the length of arc TA is

$$\frac{\angle TCA}{2\pi}\left(2\pi(TC)\right).$$

Since these arcs are equal in length

$$\frac{\angle TOB}{2\pi}\left(2\pi(TO)\right) = \frac{\angle TCA}{2\pi}\left(2\pi(TC)\right),$$

while in degrees this equation is

$$\frac{\angle TOB}{360^\circ}\left(2\pi(TO)\right) = \frac{\angle TCA}{360^\circ}\left(2\pi(TC)\right).$$

The radian version of this equation becomes

$$\frac{x}{2\pi}\left(2\pi(28)\right) = \frac{\angle TCA}{2\pi}\left(2\pi(8)\right),$$

while the degree version becomes

$$\frac{x}{360^\circ}\left(2\pi(28)\right) = \frac{\angle TCA}{360^\circ}\left(2\pi(8)\right).$$

Both of these give $28x = 8\angle TCA$, whence $\angle TCA = \frac{7}{2}x$. Therefore, we have

$$\angle PCA = \angle TCP + \angle TCA = \frac{9}{2}x.$$

What does this tell us? When the smaller circle has rolled twice around the circumference of the larger circle, we'll have $x = 4\pi$, so this tells us that the total angle travelled by B around C is $\frac{9}{2}(4\pi) = 18\pi$. Thus, the smaller circle makes 9 complete rotations around its centre.

I find this problem pretty fascinating because of its counterintuitive nature. I hope that these different solutions gave you something to think about!

Notes from the Mayhem Editor

Calling all proposers! Do you have a neat problem that you'd like to see in Mayhem? We would appreciate receiving your problem proposals. They can be sent by email to: `mayhem-editors@cms.math.ca` or by regular mail to: Ian VanderBurgh; CEMC, University of Waterloo; 200 University Ave. W.; Waterloo, ON, Canada; N2L 3G1. Please keep in mind that we are looking for problems at the level of those that have appeared so far in Volume 34.

Some of Mayhem's loyal followers have noticed that a few favourite problems from the past have reappeared in recent issue(s). While this may have been accidental, we believe that this is perfectly acceptable in Mayhem, as its audience is intended to be students in secondary and post-secondary schools, who tend to move on after a few years.

THE OLYMPIAD CORNER

No. 270

R.E. Woodrow

The Canadian summer break is coming up. I would like to encourage readers to use this time to send me solutions to problems from the *Corner* to build up my file for future numbers of the *Corner*. For your problem pleasure we first give the twenty problems of the Mathematical Competition Baltic Way 2004. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

MATHEMATICAL COMPETITION BALTIC WAY 2004

November 7, 2004

1. Let a_1, a_2, a_3, $\ldots$ be a sequence of non-negative real numbers such that for each $n \geq 1$ both $a_n + a_{2n} \geq 3n$ and $a_{n+1} + n \leq 2\sqrt{(n+1)a_n}$ hold.

(a) Prove that $a_n \geq n$ for each $n \geq 1$.

(b) Give an example of such a sequence.

2. Let $P(x)$ be a polynomial with non-negative coefficients. Prove that if

$$P\left(\frac{1}{x}\right) P(x) \;\geq\; 1$$

for $x = 1$, then the same inequality holds for each positive x.

3. Let p, q, and r be positive real numbers and let n be a positive integer. If $pqr = 1$, prove that

$$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \;\leq\; 1\,.$$

4. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of real numbers with arithmetic mean X. Prove that there is a positive integer K such that the arithmetic mean of each of the sets

$$\{x_1, x_2, \ldots, x_K\}\,,\ \{x_2, x_3, \ldots, x_K\}\,,\ \ldots\,,\ \{x_{K-1}, x_K\}\,,\ \{x_K\}$$

is not greater than X.

5. For integers k and n let $(k)_{2n+1}$ be the multiple of $2n+1$ closest to k. Determine the range of the function $f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k$.

6. A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three adjacent faces. The sum of these products is 1001. What is the sum of the six numbers on the faces?

7. Find all sets X consisting of at least two positive integers such that for every pair m,$n \in X$, where $n > m$, there exists $k \in X$ such that $n = mk^2$.

8. Let $f(x)$ be a non-constant polynomial with integer coefficients. Prove that there is an integer n such that $f(n)$ has at least 2004 distinct prime factors.

9. A set S of $n - 1$ natural numbers is given, where $n \geq 3$. There exist at least two elements in this set whose difference is not divisible by n. Prove that it is possible to choose a non-empty subset of S so that the sum of its elements is divisible by n.

10. Is there an infinite sequence of prime numbers p_1, p_2, p_3, $\ldots$ such that $|p_{n+1} - 2p_n| = 1$ for each $n \geq 1$?

11. An $m \times n$ table is given with $+1$ or -1 written in each cell. Initially there is exactly one -1 in the table and all the other cells contain a $+1$. A move consists of choosing a cell containing -1, replacing this -1 by a 0, and then multiplying all the numbers in the neighbouring cells by -1 (two cells are neighbouring if they share a side). For which (m, n) can a sequence of such moves always reduce the table to all zeroes, regardless of which cell contains the initial -1?

12. There are $2n$ different numbers in a row. By one move we can exchange any two numbers, or cyclically permute any three numbers, that is, choose a, b, and c and replace b by a, c by b, and a by c. What is the minimum number of moves that suffices to arrange the numbers in increasing order?

13. The 25 member states of the European Union set up a committee with the following rules.

(a) The committee shall meet every day.

(b) At each meeting, at least one member state shall be represented.

(c) At any two different meetings, a different set of member states shall be represented.

(d) The set of states represented at the n^{th} meeting shall include, for every $k < n$, at least one state that was represented at the k^{th} meeting.

For how many days can the committee have its meetings?

14. A pile of one, two, or three nuts is *small*, while a pile of four or more nuts is *large*. Two persons play a game, starting with a pile of n nuts. A player moves by taking a large pile of nuts and splitting it into two non-empty piles (either pile can be large or small). If a player cannot move, he loses. For which values of n does the first player have a winning strategy?

15. A circle is divided into 13 segments, numbered consecutively from 1 to 13. Five fleas called A, B, C, D, and E sit in the segments 1, 2, 3, 4, and 5, respectively. A flea can jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at a time, and two fleas cannot occupy the same segment. After some jumps, the fleas are back in the segments 1, 2, 3, 4, and 5, but possibly in some other order than they started in. Which orders are possible?

16. Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of C onto the diameter is Q. Prove that QC bisects $\angle AQB$.

17. Consider a rectangle with sides of lengths 3 and 4, and on each side pick an arbitrary point that is not a corner. Let x, y, z, and u be the lengths of the sides of the quadrilateral spanned by these points. Prove that

$$25 \;\leq\; x^2 + y^2 + z^2 + u^2 \;\leq\; 50\,.$$

18. A ray emanating from the vertex A of the triangle ABC intersects the side BC at X and the circumcircle of ABC at Y. Prove that

$$\frac{1}{AX} + \frac{1}{XY} \geq \frac{4}{BC}\,.$$

19. In triangle ABC let D be the mid-point of BC and let M be a point on the side BC such that $\angle BAM = \angle DAC$. Let L be the second intersection point of the circumcircle of triangle CAM with AB, and let K be the second intersection point of the circumcircle of triangle BAM with the side AC. Prove that $KL \parallel BC$.

20. Three circular arcs w_1, w_2, and w_3 with common end-points A and B are on the same side of the line AB, and w_2 lies between w_1 and w_3. Two rays emanating from B intersect these arcs at M_1, M_2, M_3 and K_1, K_2, K_3, respectively. Prove that

$$\frac{M_1M_2}{M_2M_3} = \frac{K_1K_2}{K_2K_3}\,.$$

We now give the XIX Olimpiada Iberoamericana de Matemáticas 2004. Thanks again go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for the *Corner*.

XIX OLIMPIADA IBEROAMERICANA DE MATEMÁTICAS

Castellón, España

September 17–26, 2004

1. Some cells of a 1001×1001 board are coloured according to the following rules.

(a) If two cells share a common side, then at least one of them is coloured.

(b) In every set of six consecutive cells in a row or in a column, there are at least two neighbouring cells that are coloured.

Determine the minimum number of cells that are coloured.

2. Let A be a fixed exterior point with respect to a given circle with centre O and radius r. Let M be a point on the circle and let N be diametrically opposite to M with respect to O. Find the locus of the centres of the circles passing through A, M, and N, as the point M is varied on the circle.

3. Let n and k be positive integers such that n is odd or n and k are even. Prove there exist two integers a and b such that $\gcd(a, n) = \gcd(b, n) = 1$ and $k = a + b$.

4. Find all pairs of positive integers (a, b) such that a and b each have two digits, and such that $100a + b$ and $201a + b$ are perfect squares with four digits each.

5. In a scalene triangle ABC, the interior bisectors of the angles A, B, and C meet the opposite sides at points A', B', and C' respectively. Let A'' be the intersection of BC with the perpendicular bisector of AA', let B'' be the intersection of AC with the perpendicular bisector of BB', and let C'' be the intersection of AB with the perpendicular bisector of CC'. Prove that A'', B'', and C'' are collinear.

6. Given a subset H of the plane, a point P of the plane is said to be a *cut point* of H if there are four points A, B, C, and D in H such that the lines AB and CD are distinct and meet at P. If H_0 is a finite subset of the plane, then define the sets H_1, H_2, H_3, $\ldots$ recursively by taking H_{j+1} to be the union of H_j and the set of cut points of H_j, $j \geq 0$.

Prove that, if the union of the H_j is finite, then $H_j = H_1$ for each j.

As a third group of problems we next give the Qualification Round of the Swedish Mathematical Contest 2004/2005. Thanks again to Felix Recio for obtaining them for our use.

SWEDISH MATHEMATICAL CONTEST 2004/2005
Qualification Round
October 5, 2004

1. The cities A, B, C, D, and E are connected by straight roads (more than two cities may lie on the same road). The distance from A to B, and from C to D, is 3 km. The distance from B to D is 1 km, from A to C it is 5 km, from D to E it is 4 km, and finally, from A to E it is 8 km. Determine the distance from C to E.

2. Linda writes the four positive integers a, b, c, and d on a piece of paper. Since she is amused by arithmetic, she adds the numbers in pairs, obtaining the sums $a+b$, $a+c$, $\dots$, $c+d$, but she forgets to write down one of the possible sums. The five sums she obtains are 7, 11, 12, 18, and 23. Which sum did Linda forget? What are the positive integers a, b, c, d?

3. Determine the greatest and the least value of

$$\frac{mn}{(m+n)^2},$$

if m and n are positive integers, each not greater than 2004.

4. Let k and n be integers with $1 < k < n$. If a set of n real numbers has the property that the mean value of any k of them is an integer, show that all n numbers are integers.

5. At all points with integer coordinates in the plane there are vertical nails of altitude h and of negligible width. When a spherical balloon is dropped it will burst once it touches one of the nails. A balloon is *doomed* if it bursts no matter where it is dropped. What is the radius, R, of the smallest doomed balloon?

6. Let $2n$ (where $n \geq 1$) points lie in the plane so that no straight line contains more than two of them. Paint n of the points blue and paint the other n points yellow. Show that there are n segments, each with one blue end-point and one yellow end-point, such that each of the $2n$ points is an end-point of one of the n segments and none of the segments have a point in common.

The Final Round problems of the Swedish Mathematical Contest 2004/2005 complete the set. Thanks are due to Felix Recio for obtaining them.

SWEDISH MATHEMATICAL CONTEST 2004/2005
Final Round
November 20, 2004

1. Two circles in the plane of the same radius R intersect at a right angle. How large is the area of the region which lies inside both circles?

2. The coins in a certain country have the values 1, 2, 3, 4, and 5. Nisse is buying a pair of shoes. When he is about to pay he tells the salesman that he has 100 coins, but that he does not know how many there are of each kind. "Wonderful, then you have even money," says the salesman. How much did the shoes cost, and how did the salesman know that Nisse had even money?

3. The function f satisfies $f(x) + xf(1-x) = x^2$ for all real numbers x. Determine the function f.

4. If $\tan v = 2v$ and $0 < v < \frac{\pi}{2}$, then does it follow that $\sin v < \frac{20}{21}$?

5. A square of integer side n, where $n \geq 2$, is divided into n^2 squares of side 1. Next, $n-1$ straight lines are drawn so that the interior of each of the small squares (the boundary not being included in the interior) is intersected by at least one of the straight lines.

(a) Give an example which shows that this can be achieved for some $n \geq 2$.

(b) Show that among the $n-1$ straight lines there are two lines which intersect in the interior of the square of side n.

6. Show that any convex, n-sided polygon of area 1 contains a quadrilateral of area at least $\frac{1}{2}$.

Our last group of problems is the Abel Competition 2004–2005, Norway Final. Thanks again to Felix Recio for collecting them for the *Corner*.

ABEL COMPETITION 2004–2005
Norway Final
March 10, 2005

1. (a) A positive integer m is *triangular* if $m = 1 + 2 + \cdots + n$ for some integer $n > 0$. Show that m is triangular if $8m + 1$ is a perfect square.

(b) The base of a pyramid is a right-angled triangle with sides of integer lengths. The height of the pyramid is also an integer. Show that the volume of the pyramid is an even integer.

2. (a) There are nine small fish in a cubic aquarium with a side of 2 metres, entirely filled with water. Show that at any given time there can be found two fish whose distance apart is not greater than $\sqrt{3}$ metres.

(b) Let A be the set of all points in space with integer coordinates. Show that for any nine points in A, there are at least two points among them such that the mid-point of the segment joining the two is also a point in A.

3. (a) Let $\triangle ABC$ be isosceles with $AB = AC$, and let D be the mid-point of BC. The points P and Q lie respectively on the segments AD and AB such that $PQ = PC$ and $Q \neq B$. Show that $\angle PQC = \frac{1}{2}\angle BAC$.

(b) Let $ABCD$ be a rhombus with $\angle BAD = 60^\circ$. Let F, G, and H be points on the segments AD, CA, and DC respectively such that $DFGH$ is a parallelogram. Show that $\triangle BHF$ is equilateral.

4. (a) Let a, b, and c be positive real numbers. Prove that

$$(a+b)(a+c) \;\geq\; 2\sqrt{abc(a+b+c)}\,.$$

(b) Let a, b, and c be real numbers such that $ab+bc+ca > a+b+c > 0$. Prove that $a+b+c \geq 3$.

I seem to have misfiled a solution and comment by Miguel Amengual Covas to Problem 1 of the Thai Mathematical Olympiad (see the last number of the *Corner*). He noted that the answer, $\angle B = \frac{1}{2}\angle A = \frac{1}{2}70^\circ = 35^\circ$, follows from Problem #2559 [2000 : 305] and its solution [2001 : 466-467].

Also, two letters arrived from Pavlos Maragoudakis after we had finalized the April *Corner*. One letter included a solution to Problem 8 (not discussed in the April *Corner*) of the Selected Problems of the Thai Mathematical Olympiad 2003, given at [2007 : 278].

8. Find all primes p such that $p^2 + 2543$ has less than 16 distinct positive divisors.

Solution by Pavlos Maragoudakis, Pireas, Greece.

If $p = 2$ then $p^2+2543 = 2547 = 3^2\cdot 283$ which has $(2+1)(1+1) = 6$ positive divisors.

If $p = 3$ then $p^2 + 2543 = 2552 = 2^3 \cdot 11 \cdot 29$ which has precisely $(3+1)(1+1)(1+1) = 16$ positive divisors.

If $p > 3$ then $p \equiv \pm 1 \pmod 3$ so $p^2 + 2543 \equiv 2544 \equiv 0 \pmod 3$. Also p is odd, so $p^2 \equiv 1 \pmod 8$ and $p^2 + 2543 \equiv 0 \pmod 8$. Thus $3 \mid (p^2 + 2543)$ and $8 \mid (p^2 + 2543)$, hence $p^2 + 2543 = 24x$ where $x > 24$. The numbers 1, 2, 3, 4, 6, 8, 12, 24, x, $2x$, $3x$, $4x$, $6x$, $8x$, $12x$, and $24x$ are then 16 divisors of $p^2 + 2543$, that are distinct and positive.

So $p = 2$ is the only prime such that p^2+2543 has less than 16 distinct positive divisors.

Next we revisit the 25th Albanian Mathematical Olympiad, Test 1 and Test 2, given at [2007: 278–279]. We start with solutions to Test 1.

1. There are 20 pupils in a village school. Any two of them have the same grandfather. Show that there exists a grandfather who has at least 14 grandchildren.

Comment by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Solved by Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Cománeşti, Romania. We give the comment of Amengual Covas.

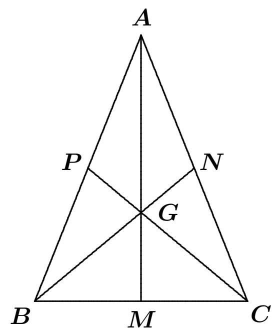
The problem was proposed in the Tournament of Towns, Tournament 16, Junior Questions, Autumn 1994, O Level. A solution can be found in: P.J. Taylor and A.M. Storozhev, *Tournament of Towns 1993–1997, Book 4*, Australian Mathematical Trust, 1998, p. 57.

2. Let M, N, and P be the respective mid-points of sides BC, CA, and AB of triangle ABC, and let G be the intersection point of its medians. Prove that if $BN = \frac{\sqrt{3}}{2}AB$ and $BMGP$ is a cyclic polygon, then triangle ABC is equilateral.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Tsapakidis.

Since $BMGP$ is a cyclic polygon we obtain

$$\begin{aligned} AG \cdot AM &= AP \cdot AB\,, \\ \tfrac{2}{3}AM \cdot AM &= \tfrac{1}{2}AB \cdot AB\,, \\ AM^2 = \tfrac{3}{4}AB^2 &= BN^2\,, \\ AM &= BN\,, \end{aligned}$$



hence $AC = BC$. The segment CP is also an altitude, that is $\angle GPB = 90^\circ$, and since opposite angles of a cylic quadrilateral add to 180°, we also have $\angle GMB = 90^\circ$. Therefore AM is also an altitude, hence $AB = AC$, which together with $AC = BC$ shows that triangle ABC is equilateral.

3. Let x_k and y_k (for $k = 1, 2, \dots, n$) be positive real numbers that satisfy $kx_ky_k \geq 1$.

(a) Prove that $\sum\limits_{k=1}^{n} \dfrac{x_k - y_k}{x_k^2 + y_k^2} \leq \frac{1}{4}n\sqrt{n+1}$.

(b) When does equality hold in part (a)?

Solved by Michel Bataille, Rouen, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bataille's solution.

(a) First note that for $k = 1, 2, \ldots, n$ we have

$$\frac{x_k - y_k}{x_k^2 + y_k^2} \;\le\; \frac{|x_k - y_k|}{|x_k - y_k|^2 + 2x_k y_k} \;\le\; \frac{t_k}{t_k^2 + \frac{2}{k}}\,,$$

where we have set $t_k = |x_k - y_k|$.

But $t_k^2 + \frac{2}{k} \ge 2t_k\sqrt{\frac{2}{k}}$, and hence $\dfrac{t_k}{t_k^2 + \frac{2}{k}} \le \dfrac{\sqrt{k}}{2\sqrt{2}}$. It follows that

$$\sum_{k=1}^{n} \frac{x_k - y_k}{x_k^2 + y_k^2} \;\le\; \frac{1}{2\sqrt{2}}(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n})\,. \qquad (1)$$

From an inequality of means,

$$\left(\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n}\right)^2 \;\le\; \frac{1 + 2 + \cdots + n}{n}\,, \qquad (2)$$

hence

$$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \;\le\; n\sqrt{\frac{n+1}{2}}\,.$$

Returning to (1), we obtain

$$\sum_{k=1}^{n} \frac{x_k - y_k}{x_k^2 + y_k^2} \;\le\; \frac{1}{2\sqrt{2}} \cdot n\sqrt{\frac{n+1}{2}} \;=\; \frac{1}{4}n\sqrt{n+1}\,.$$

(b) The inequality is strict in (2) if $n \ge 2$, for then $\sqrt{1}$, $\sqrt{2}$, $\ldots$, $\sqrt{n}$ are distinct. Thus, equality does not hold in (a) if $n \ge 2$. If $n = 1$ and equality holds, then

$$\frac{\sqrt{2}}{4} \;=\; \frac{x_1 - y_1}{x_1^2 + y_1^2} \;\le\; \frac{t_1}{t_1^2 + 2x_1 y_1} \;\le\; \frac{t_1}{t_1^2 + 2} \;\le\; \frac{\sqrt{2}}{4}\,,$$

so that equality holds throughout. It follows that

$$x_1 y_1 \;=\; 1 \quad \text{and} \quad \frac{\sqrt{2}}{4} \;=\; \frac{x_1 - y_1}{(x_1 - y_1)^2 + 2}\,,$$

hence $x_1 - y_1 = \sqrt{2}$. It is now easy to deduce that equality holds if and only if $n = 1$ and

$$x_1 \;=\; \frac{\sqrt{2} \pm \sqrt{6}}{2}\,, \qquad y_1 \;=\; \frac{-\sqrt{2} \pm \sqrt{6}}{2}\,.$$

4. Find prime numbers p and q such that $p^2 - p + 1 = q^3$.

Solution by Titu Zvonaru, Comănești, Romania.

We will prove that the only solution to $p^2 - p + 1 = q^3$ where p is a prime number and q is a natural number is $p = 19$ and $q = 7$ (giving $19^2 - 19 + 1 = 343 = 7^3$).

We have
$$p(p-1) \;=\; (q-1)(q^2+q+1)\,, \tag{1}$$
thus $p \mid (q-1)$ or $p \mid (q^2+q+1)$. If $p \mid (q-1)$, then $p < q$, which leads to the contradiction $p^2 - p + 1 = q^3 > p^3$. It follows that $p \mid (q^2+q+1)$, that is,
$$q^2+q+1 \;=\; kp\,. \tag{2}$$
By (1) we obtain
$$p-1 \;=\; k(q-1)\,. \tag{3}$$
By (2) and (3) we obtain $q^2+q+1 = k(kq-k+1)$, or equivalently
$$q^2 + (1-k^2)q + (k^2-k+1) \;=\; 0\,.$$
Since q is an integer, the discriminant $\Delta = (k^2-1)^2 - 4(k^2-k+1)$ of the quadratic equation is a perfect square. Also, since $\Delta < (k^2-1)^2$ and $\Delta \equiv (k^2-1)^2 \pmod 2$, it follows that
$$\begin{aligned}\Delta \;\le\; (k^2-3)^2 \quad&\iff\quad (k^2-1)^2 - (k^2-3)^2 \;\le\; 4(k^2-k+1)\\ &\iff\quad 4(k^2-2) \;\le\; 4(k^2-k+1)\,,\end{aligned}$$
and hence $k \le 3$. If $k = 2$, then $\Delta = -3$, which is not a perfect square. However, if $k = 3$, then $\Delta = 6^2$, and we obtain $q = 7$ and $p = 19$.

We now turn to our readers' solutions to Test 2 of the Albanian Mathematical Olympiad.

2. Prove the inequality
$$\frac{1}{\sqrt{a+\frac{1}{b}+0.64}} + \frac{1}{\sqrt{b+\frac{1}{c}+0.64}} + \frac{1}{\sqrt{c+\frac{1}{a}+0.64}} \;\ge\; 1.2\,,$$
where $a > 0$, $b > 0$, $c > 0$, and $abc = 1$.

Solved by Arkady Alt, San Jose, CA, USA; D. Kipp Johnson, Beaverton, OR, USA; and Pavlos Maragoudakis, Pireas, Greece. We give Johnson's write-up.

We first make the standard substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, and $c = \frac{z}{x}$, for suitable positive numbers x, y, and z. Then
$$\frac{1}{\sqrt{a+\frac{1}{b}+0.64}} \;=\; \frac{1}{\sqrt{\frac{x}{y}+\frac{z}{y}+0.64}} \;=\; \frac{\sqrt{y}}{\sqrt{x+0.64y+z}}\,,$$

and the desired inequality becomes

$$\sum_{\text{cyclic}} \frac{\sqrt{y}}{\sqrt{x+0.64y+z}} \;\geq\; 1.2\,.$$

This inequality is homogeneous of degree zero, so without loss of generality we let $x+y+z=1$, where x, y, and z are in the interval $(0,1)$, and our goal is to prove that

$$\sum_{\text{cyclic}} \frac{\sqrt{y}}{\sqrt{x+y+z-0.36y}} \;=\; \sum_{\text{cyclic}} \frac{\sqrt{y}}{\sqrt{1-0.36y}} \;=\; \sum_{\text{cyclic}} \frac{5\sqrt{y}}{\sqrt{25-9y}} \;\geq\; 1.2\,.$$

Now for $0<x<1$ we have the succession of equivalent inequalities

$$\begin{aligned} \frac{5\sqrt{x}}{\sqrt{25-9x}} &\geq \frac{6x}{5}\,, \\ 25 &\geq 6\sqrt{x}\sqrt{25-9x}\,, \\ 625 &\geq 36x(25-9x)\,, \\ 0 &\leq 9x^2-25x+\frac{625}{36}\,, \end{aligned}$$

which are all true, as the last inequality is just $(3x-\frac{25}{6})^2 \geq 0$. Adding three such inequalities we have

$$\begin{aligned} &\frac{5\sqrt{x}}{\sqrt{25-9x}} + \frac{5\sqrt{y}}{\sqrt{25-9y}} + \frac{5\sqrt{z}}{\sqrt{25-9z}} \\ &\geq \frac{6x}{5} + \frac{6y}{5} + \frac{6z}{5} = \frac{6(x+y+z)}{5} = 1.2\,, \end{aligned}$$

and our goal is achieved. This completes the proof.

3. Solve the following equation in integers:

$$y^2 \;=\; 1+x+x^2+x^3+x^4\,.$$

Comments by Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. Zvonaru notes that this was problem M242. A solution appears at [2007 : 141]. *Now we give Bataille's comment.*

This is an old problem set by T.H. Gronwall and solved by A.A. Bennett in *Amer. Math. Monthly*, Vol 33, No. 5, May 1926, pp. 291–2. The solver showed that the solutions are the following six pairs (x,y):

$$(-1,\pm 1)\,,\;\; (0,\pm 1)\,,\;\; (3,\pm 11)\,.$$

4. Prove that for any integer $n \geq 2$, the number 2^n-1 is not divisible by n.

Solution by Michel Bataille, Rouen, France.

Suppose on the contrary that n divides $2^n - 1$. Since $2^n - 1$ is odd, n and all its prime factors are odd as well. Let p be one of these prime factors. There exist positive integers k, l such that $n = kp$ and $2^n = 1 + klp$. Recalling that $2^p \equiv 2 \pmod{p}$ (by Fermat's Little Theorem), we obtain

$$2^k \equiv 1 \pmod{p} . \qquad (1)$$

Now, 2 belongs to the multiplicative group $\mathbb{Z}/p\mathbb{Z} - \{0\}$ and the order of 2 in this group is an integer $m > 1$ which divides the order $p - 1$ of the group and, from (1), also divides k. Thus, choosing a prime divisor q of m, we get a prime divisor of n satisfying $q < p$. Repeating with q what we have just done with p and continuing this way, we would construct an infinite sequence of prime divisors of n, which is clearly impossible. This contradiction completes the proof.

5. In an acute-angled triangle ABC, Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove that $d_a + d_b + d_c \leq 3r$, where r is the radius of the incircle of triangle ABC.

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Amengual Covas.

Let a, b, and c be the lengths of the sides opposite A, B, and C, respectively, and let $s = \frac{1}{2}(a + b + c)$ be the semiperimeter. Let R be the circumradius of triangle ABC, and let O and A' be the circumcentre and the mid-point of the side BC, respectively.

Letting h_a be the length of the altitude from A to BC in $\triangle ABC$, and using the well-known relation $AH = 2 \cdot OA'$, we deduce that

$$d_a = h_a - AH = h_a - 2 \cdot OA' = h_a - 2R\cos A .$$

By symmetry, we have $d_b = h_b - 2R\cos B$ and $d_c = h_c - 2R\cos C$. It follows that

$$d_a + d_b + d_c = h_a + h_b + h_c - 2R(\cos A + \cos B + \cos C) .$$

We have the relation $h_a = \dfrac{2rs}{a}$ and its symmetric analogues, and also we have the well-known relations

$$\begin{aligned} abc &= 4Rrs , \\ ab + bc + ca &= s^2 + r^2 + 4Rr , \\ \cos A + \cos B + \cos C &= \frac{R + r}{R} . \end{aligned}$$

Using these we deduce that

$$d_a + d_b + d_c \;=\; \frac{s^2 + r^2 - 4Rr}{2R}\,,$$

and hence the given inequality is equivalent to

$$s^2 + r^2 - 4R^2 \;\le\; 6Rr\,. \tag{1}$$

To establish the validity of (1), we make use of the linear transformation $a = y + z$, $b = z + x$, and $c = x + y$, where x, y, and z are uniquely determined positive numbers. Ater applying the transformation, (1) becomes

$$(x+y+z)^2 \;+\; \frac{xyz}{x+y+z} \;-\; \frac{(x+y)^2(y+z)^2(z+x)^2}{4xyz(x+y+z)} \;\le\; 6\cdot\frac{(x+y)(y+z)(z+x)}{4(x+y+z)}\,,$$

or equivalently

$$4xyz(x+y+z)^3 \;+\; 4x^2y^2z^2 \;-\; (x+y)^2(y+z)^2(z+x)^2 \;\le\; 6xyz(x+y)(y+z)(z+x)\,.$$

We substitute $x^3+y^3+z^3+3(x+y)(y+z)(z+x)$ for $(x+y+z)^3$, simplify, and obtain

$$4xyz(x^3+y^3+z^3) \;+\; 6xyz(x+y)(y+z)(z+x) \;+\; 4x^2y^2z^2 \;\le\; [(x+y)(y+z)(z+x)]^2\,.$$

We substitute $2xyz + \sum_{\text{sym}} x^2y$ for $(x+y)(y+z)(z+x)$ (the symmetric sum is over the six permutations of x, y, and z) to obtain

$$4xyz(x^3+y^3+z^3) \;+\; 6xyz\left(2xyz + \sum_{\text{sym}} x^2y\right) \;+\; 4x^2y^2z^2$$
$$\le\; \left(2xyz + \sum_{\text{sym}}\right)^2 x^2y \;=\; \left(\sum_{\text{sym}} x^2y\right)^2 \;+\; 4xyz\left(\sum_{\text{sym}} x^2y\right) \;+\; 4x^2y^2z^2\,.$$

We have the expansion

$$\left(\sum_{\text{sym}} x^2y\right)^2 \;=\; \left(\sum_{\text{sym}} x^4y^2\right) \;+\; 2xyz(x^3+y^3+z^3) \;+\; 2xyz\left(\sum_{\text{sym}} x^2y\right) \;+\; 2(x^3y^3+y^3z^3+z^3x^3) \;+\; 6x^2y^2z^2\,,$$

which we substitute into the above and then simplify to obtain

$$2xyz(x^3+y^3+z^3) \;+\; 6x^2y^2z^2 \;\le\; \left(\sum_{\text{sym}} x^4y^2\right) \;+\; 2(x^3y^3+y^3z^3+z^3x^3)\,.$$

However, this is equivalent to

$$2\left[(xy)^3+(yz)^3+(zx)^3-3x^2y^2z^2\right] \\ +\ x^4(y-z)^2\ +\ y^4(z-x)^2\ +\ z^4(x-y)^2\ \geq\ 0\,,$$

which is true, since $(xy)^3+(yz)^3+(zx)^3 \geq 3x^2y^2z^2$ holds by the AM-GM Inequality.

Equality occurs only if $x = y = z$, that is, if and only if the triangle ABC is equilateral.

Next we turn to solutions from our readers to some problems of the 44th Ukrainian Mathematical Olympiad, 11th Form, Final Round, given at [2007 : 279–280].

1. (V.M. Leifura) Solve the equation

$$\arcsin\lfloor\sin x\rfloor\ =\ \arccos\lfloor\cos x\rfloor\,,$$

where $\lfloor a\rfloor$ is the greatest integer not exceeding a.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

For convenience, we will write $\sin^{-1}x$ for $\arcsin x$.

Let $\theta = \sin^{-1}\lfloor\sin x\rfloor = \cos^{-1}\lfloor\cos x\rfloor$. From the definitions of the inverse trigonometric functions, we have $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \theta \leq \pi$. Hence, $0 \leq \theta \leq \frac{\pi}{2}$. We consider four cases.

Case 1. If $-1 \leq \sin x < 0$, then $\theta = \sin^{-1}(-1) = -\frac{\pi}{2}$, a contradiction.

Case 2. If $0 < \sin x < 1$, then $\theta = \sin^{-1}(0) = 0$, hence $\lfloor\cos x\rfloor = \cos\theta = 1$ and $\cos x = 1$. Thus $\sin^2 x + \cos^2 x > 1$, a contradiction.

Case 3. If $\sin x = 0$, then $\cos x = 1$ as in Case 2 above, which implies that $x = 2k\pi$, $k \in \mathbb{Z}$. Conversely, if $x = 2k\pi$, $k \in \mathbb{Z}$, then $\sin^{-1}\lfloor\sin x\rfloor = 0$ and also $\cos^{-1}\lfloor\cos x\rfloor = 0$.

Case 4. If $\sin x = 1$, then $\theta = \sin^{-1}(1) = \frac{\pi}{2}$; hence $\lfloor\cos x\rfloor = \cos\theta = \cos\left(\frac{\pi}{2}\right) = 0$ and $0 \leq \cos x < 1$. Evidently $\cos x = 0$; otherwise we have $\sin^2 x + \cos^2 x > 1$, a contradiction. Solving $\sin x = 1$ and $\cos x = 0$ we find that $x = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$. Conversely, if $x = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$, then $\sin^{-1}\lfloor\sin x\rfloor = \sin^{-1}(1) = \frac{\pi}{2}$ and $\cos^{-1}\lfloor\cos x\rfloor = \cos^{-1}(0) = \frac{\pi}{2}$. Therefore, the solution set of the given equation is

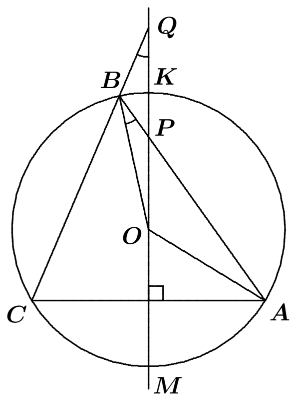
$$S = \left\{2k\pi\,, \frac{\pi}{2} + 2k\pi :\ k \in \mathbb{Z}\right\}\,.$$

2. (V.V. Lymanskiy) The acute-angled triangle ABC is given. Let O be the centre of its circumcircle. The perpendicular bisector of the side AC intersects the side AB and the line BC at the points P and Q, respectively. Prove that $\angle PQB = \angle PBO$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give Maragoudakis' solution.

In triangle BOA we have $OA = OB$. Thus

$$\begin{aligned}\angle PBO &= \frac{180^\circ - \angle BOA}{2} = \frac{180^\circ - \widehat{BKA}}{2} \\ &= \frac{180^\circ - \widehat{BK} - \widehat{AK}}{2} \\ &= \frac{180^\circ - \widehat{BK} - \widehat{CK}}{2} \\ &= \frac{\widehat{KM} - \widehat{BK} - \widehat{CK}}{2} \\ &= \frac{\widehat{CM} - \widehat{BK}}{2} = \angle PQB\,.\end{aligned}$$



6. (O.O. Malakhov) Find the sum of the real roots of the equation

$$x + \frac{x}{\sqrt{x^2-1}} = 2004\,.$$

Solved by Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; and Vedula N. Murty, Dover, PA, USA. We give Bataille's solution.

The required sum is 2004.

Let $a = 2004$. It is readily seen that the real roots of the given equation are those of $P(x) = 0$ satisfying $0 < x < a$, where

$$P(x) = (x^2-1)(a-x)^2 - x^2 = x^4 - 2ax^3 + (a^2-2)x^2 + 2ax - a^2\,.$$

The key observation is that the polynomial $P(a-x)$ is just the same as $P(x)$ (this is easily checked). Hence, if x_0 is a solution to $P(x) = 0$, the same is true of $a - x_0$.

Now, $P(0) < 0$, $P(a/2) > 0$, $P(a) < 0$, hence $P(x)$ has at least two real roots in $(0,a)$. If in addition $P(x)$ had two positive or complex conjugate roots, the product of the roots would be positive. However, the product of the roots is $-a^2 < 0$, therefore $P(x)$ must have a negative root, say x_1. Then $a - x_1$ is also a root of $P(x)$ and $a - x_1 > a$. It follows that the four roots of $P(x)$ are real, with exactly two of them in $(0,a)$, of the form x_2, $a - x_2$ for some $x_2 \in (0,a)$. These real numbers are the roots of the given equation and their sum is $a = 2004$, as announced.

7. (V.M. Radchenko) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(x^2y + f(x + y^2)\big) = x^3 + y^3 + f(xy)$ for all $x, y \in \mathbb{R}$?

Solved by Michel Bataille, Rouen, France; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Bataille.

There is no such function. Assume on the contrary that $f : \mathbb{R} \to \mathbb{R}$ satisfies the given identity. Then, taking $x = 0$ in the identity, we have

$$f\big(f(y^2)\big) \;=\; y^3 \;+\; f(0)\,,$$

and taking $y = 0$ we have

$$f\big(f(x)\big) \;=\; x^3 + f(0)\,.$$

However, with $x = 1$ the latter gives $f\big(f(1)\big) = 1 + f(0)$, while with $y = -1$ the former yields $f\big(f(1)\big) = -1 + f(0)$. Thus, we have the contradiction $1 + f(0) = -1 + f(0)$, and the result follows.

8. (V.A. Yasinskiy) Let a, b, and c be positive real numbers such that $abc \geq 1$. Prove that $a^3 + b^3 + c^3 \geq ab + bc + ca$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Vedula N. Murty, Dover, PA, USA; George Tsapakidis, Agrinio, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Titu Zvonaru, Comănești, Romania. We give Murty's write-up.

We have the two known inequalities

$$\begin{aligned} a^2 + b^2 + c^2 &\geq ab + bc + ca\,, \\ a^3 + b^3 + c^3 &\geq \frac{(a+b+c)(a^2+b^2+c^2)}{3}\,, \end{aligned}$$

from which we obtain

$$a^3 + b^3 + c^3 \;\geq\; \frac{(a+b+c)(ab+bc+ca)}{3}\,.$$

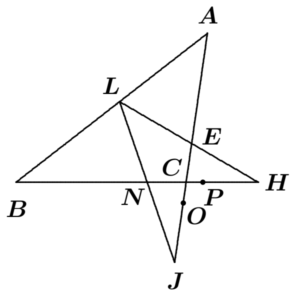
From the AM-GM Inequality and the given condition, $abc \geq 1$, we have

$$a + b + c \;\geq\; 3(abc)^{1/3} \;\geq\; 3\,.$$

The desired inequality now follows by combining the last two inequalities.

10. (I.P. Nagel) Let ω be the inscribed circle of the triangle ABC. Let L, N, and E be the points of tangency of ω with the sides AB, BC, and CA, respectively. Lines LE and BC intersect at the point H, and lines LN and AC intersect at the point J (all the points H, J, N, E lie on the same side of the line AB). Let O and P be the mid-points of the segments EJ and NH, respectively. Find $S(HJNE)$ if $S(ABOP) = u^2$ and $S(COP) = v^2$. (Here $S(\mathcal{F})$ is the area of figure $\mathcal{F}$).

Solution by Titu Zvonaru, Comăneşti, Romania.



As usual we write $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$.

It is well known that $AL = AE = s - a$, $BL = BN = s - b$, and $CN = CE = s - c$.

Assume that $c > b$. In order that the points H, J, N, and E all lie on the same side of the line AB, we must have $c > a$.

By Menelaus' Theorem applied to $\triangle ABC$ and transversal HEL, we have

$$\frac{HC}{HB} \cdot \frac{LB}{LA} \cdot \frac{EA}{EC} = 1 \Rightarrow \frac{HC}{HC+a} = \frac{s-c}{s-b} \Rightarrow HC = \frac{a(s-c)}{c-b},$$

and by symmetry we have $JC = \frac{b(s-c)}{c-a}$. Therefore we deduce

$$\begin{aligned} NH &= NC + CH = (s-c) + \frac{a(s-c)}{c-b} \\ &= \frac{(s-c)(c-b+a)}{c-b} = \frac{2(s-b)(s-c)}{c-b}, \end{aligned}$$

and by symmetry we have $EJ = \frac{2(s-a)(s-c)}{c-a}$. Using the expressions we have obtained so far, we have

$$\begin{aligned} S(HJNE) &= \frac{NH \cdot JE \cdot \sin C}{2} \\ &= \frac{2(s-a)(s-b)(s-c)^2}{(c-b)(c-a)} \cdot \sin C, \end{aligned} \tag{1}$$

$$\begin{aligned} CP &= NP - NC = \frac{1}{2}NH - NC \\ &= \frac{(s-b)(s-c)}{c-b} - (s-c) \\ &= \frac{(s-c)(s-b-c+b)}{c-b} = \frac{(s-c)^2}{c-b}, \end{aligned}$$

and by symmetry, $CO = \frac{(s-c)^2}{c-a}$. It follows that

$$S(COP) = v^2 = \frac{(s-c)^4 \sin C}{2(c-b)(c-a)}. \tag{2}$$

Continuing with our calculations,

$$\begin{aligned} AO &= AC + CO = b + \frac{(s-c)^2}{c-a} \\ &= \frac{-c(2s-c-b)+s^2-ab}{c-a} = \frac{s^2-ab-ac}{c-a} \\ &= \frac{s^2-\big(a(b+c)\big)}{c-a} = \frac{s^2-2as+a^2}{c-a} = \frac{(s-a)^2}{c-a}, \end{aligned}$$

and by symmetry $BP = \frac{(s-b)^2}{c-b}$, hence

$$S(ABOP) = u^2 = \frac{(s-a)^2(s-b)^2}{(c-a)(c-b)} \cdot \frac{\sin C}{2}. \qquad (3)$$

Combining (1), (2), and (3) we obtain the desired result

$$\begin{aligned} S(HJNE) &= \frac{2(s-a)(s-b)(s-c)^2}{(c-b)(c-a)} \cdot \sin C \\ &= 4\sqrt{\frac{(s-a)^2(s-b)^2 \cdot \sin C}{2(c-b)(c-a)}} \cdot \sqrt{\frac{(s-c)^4 \sin C}{2(c-b)(c-a)}} = 4uv. \end{aligned}$$

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.

BOOK REVIEW

John Grant McLoughlin

Problems of the Week
By Jim Totten, Canadian Mathematical Society, 2007
ISBN 0-919558-16-X, coilbound, 60 pages, $12
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB*

The author needs no introduction to the ***CRUX*** readership. Jim Totten completes his tenure as Editor with this issue of the journal. As the Book Reviews Editor, I am taking the opportunity to acknowledge a recently published work of Jim Totten – another way of thanking Jim for his contributions to problem solving within ***CRUX*** and beyond. The publisher, the Canadian Mathematical Society, will surely support me taking such editorial liberty within the review.

Jim Totten taught at Thompson Rivers University (formerly known as the University College of the Cariboo) from 1979 to 2007. Each week of the academic semesters featured a different posted problem with minimal repetition of individual problems over the years. This publication features a selection of 80 such problems used prior to Fall 1986. In fact, some were first used by Jim at Saint Mary's University from 1976-1979.

The problems take a range of forms spanning an array of topics as noted in the Table of Contents. A helpful "Index of Problems by subject matter" can guide solvers toward topics of interest. The subjects (and subheadings) are: Algebra (Equations, Functions, Word Problems, Other), Analysis, Combinatorics, Geometry (Triangles, Circles, Other), Logic, Numbers, Probability, and Recreation. Many problems are cross-referenced to two subjects.

Problems of the Week is Volume VII in the series *ATOM (A Taste of Mathematics)*. Publications in the series are intended for keen high school students, mathematics teachers, and problem solvers who wish to engage with accessible material. The *ATOM* materials lend themselves to independent study or collaborative projects. This book is no exception as it grows out of collaboration with students though the inclusion of detailed solutions allows one to learn much math independently from problems that may appear beyond one's grasp.

I am reminded of my undergraduate studies at the University of Waterloo, particularly in two separate courses, commonly referred to as the "100 Problem Courses." Many of the problems introduced to me by Dean Hoffman or Ross Honsberger appear in this book. The introduction clarifies that this is not coincidental, as Jim writes in the dedication: "For showing me that not only was it acceptable to be excited and enthusiastic about mathematical problem-solving, but that it was to be strongly encouraged, I am dedicating this book to Ross Honsberger."

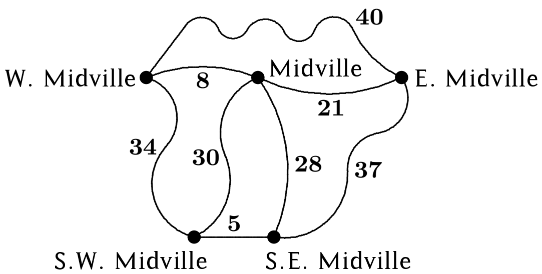
A few problems are stated here to pique the interest of problem solvers.

3. In a certain classroom, there are 5 rows with 5 seats per row arranged in a square. Each student is to change her seat by going either to the seat immediately in front or behind her, or immediately to the left or right. (Of course, not all possibilities are open to all students.) Determine whether this can be done, beginning with a full class of students.

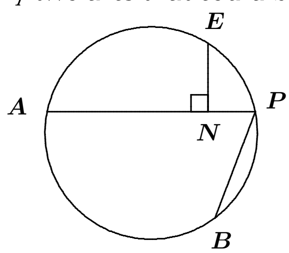
Try to generalize to a rectangular array and find the conditions under which such a change of seats can be managed.

16. The three sides and height of a triangle are four consecutive integers. What is the area of the triangle?

48. The Ramada County Department of Highways has just resurfaced the county roads, and now the yellow stripe down the middle of the road must be repainted. The truck used for this purpose is very inefficient as far as gas consumption is concerned, and thus the Department would like to have the truck travel the shortest distance possible. A road map of the county is shown (with distances given in kilometres). The county truck is garaged in Midville, and it must return there when the job is done. How many kilometres must it travel and what route should it take?



61. Given any two points A and B on the circumference of a circle, and E the mid-point of the arc AB (note that there are really two arcs that could be called AB; it does not matter which one we choose as long as the rest of the discussion is assumed to pertain only to points and arcs lying on the arc AB that we chose). Let P be any point on the arc EB and construct EN perpendicular to AP with N on the chord AP. Prove that $AN = NP + PB$ (we are dealing only with magnitudes of line segments here).



This book is an excellent addition to a library, a departmental coffee room, or the math teaching office in a school. Those who like problem solving will enjoy this compact resource, whether a high school student aspiring to learn advanced math or a seasoned mathematician who likes a challenge.

As the Book Reviews Editor through Jim's entire editorial tenure, I wish to express my gratitude for the privilege of working with Jim in this capacity.

Addendum. The review was sent to Jim less than two weeks before his sudden passing. As noted on the inside front cover, Jim is a Co-Editor of this issue. Indeed the loss to the mathematical community is great. The spirit of Jim Totten is remembered fondly in this review.

The Proof of Three Open Inequalities

Vasile Cîrtoaje

The inequalities below were conjectured in *Crux Mathematicorum* [2] and on *Mathlinks Forum* [5] (N.B. inequalities (1) and (2) are incorrectly stated on p. 106 of [2]):

Conjecture. Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers which satisfy $x_1 + x_2 + \cdots + x_n \geq n$. If $p > 1$, then

$$\sum_{\text{cyclic}} \frac{1}{x_1^p + x_2 + \cdots + x_n} \leq 1\,, \tag{1}$$

$$\sum_{\text{cyclic}} \frac{x_1}{x_1^p + x_2 + \cdots + x_n} \leq 1\,, \tag{2}$$

$$\sum_{\text{cyclic}} \frac{x_1^p - x_1}{x_1^p + x_2 + \cdots + x_n} \geq 0\,, \tag{3}$$

where each sum is over the n cyclic permutations of the variables x_i.

Under the more restrictive condition $x_1x_2\cdots x_n \geq 1$, these inequalities had already been stated and proved in [1], [2], and [4], respectively. On the other hand, the inequality (3) under the condition $x_1x_2\cdots x_n \geq 1$ generalizes the following inequality from the IMO (let $n = 3$ and $p = \frac{5}{2}$):

Problem 3 (IMO–2005). Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0\,.$$

In this paper, we prove each of these three inequalities by means of the LCF-Theorem, as it is stated in [1] and [3]:

Theorem 1 (LCF-Theorem) Let $f(u)$ be a function on an interval I, which is concave for $u \leq s$, $s \in I$. If

$$(n-1)f(x) + f(y) \leq nf(s)$$

for all $x, y \in I$ such that $x \leq y$ and $(n-1)x + y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all $x_1, x_2, \ldots, x_n \in I$ which satisfy $\frac{x_1 + x_2 + \cdots + x_n}{n} \leq s$.

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1 Proof of Inequality (1)

Having in view the expression on the left-hand side of (1), it is easy to see that we need only consider the case $x_1 + x_2 + \cdots + x_n = n$. Then, we have to show that

$$\frac{1}{x_1^p - x_1 + n} + \frac{1}{x_2^p - x_2 + n} + \cdots + \frac{1}{x_n^p - x_n + n} \le 1 \tag{4}$$

for $x_1 + x_2 + \cdots + x_n = n$. By Lemma 1 below, it suffices to prove (4) for all $x_i \ge a = p^{\frac{1}{1-p}}$ which satisfy $x_1 + x_2 + \cdots + x_n = n$. To do this, we apply the LCF-Theorem to the function $f(u) = \dfrac{1}{u^p - u + n}$ defined on $I = [a, \infty)$, for the case $s = 1$.

First, we must show that f is concave on $[a, 1]$. The second derivative is

$$f''(u) = \frac{p(p+1)u^{2p-2} - p(p+3)u^{p-1} - np(p-1)u^{p-2} + 2}{(u^p - u + n)^3}.$$

If

$$g(u) = p(p+1)u^{2p-2} - p(p+3)u^{p-1} + 2 < 0$$

for $a \le u \le 1$, then $f''(u) < 0$, and hence $f(u)$ is concave on $[a, 1]$. Indeed, setting $t = pu^{p-1}$ implies $1 \le t \le p$, and hence

$$\begin{aligned} pg(u) &= (p+1)t^2 - p(p+3)t + 2p \\ &= (p+1)(t-1)(t-p) + (1-p)(t+p) < 0. \end{aligned}$$

By the LCF-Theorem, it suffices to show that

$$\frac{n-1}{x^p - x + n} + \frac{1}{y^p - y + n} \le 1$$

for any $a \le x \le 1 \le y$ and $(n-1)x + y = n$. Since this inequality is trivial for $x = y = 1$, assume next that $x < 1 < y$, and write the desired inequality in succession as follows

$$\begin{aligned} y^p - y + n &\ge \frac{x^p - x + n}{x^p - x + 1}, \\ y^p - y &\ge \frac{(n-1)(x - x^p)}{x^p - x + 1}, \\ \frac{y^p - y}{y - 1} &\ge \frac{x - x^p}{(1-x)(x^p - x + 1)}. \end{aligned} \tag{5}$$

Let $h(y) = \dfrac{y^p - y}{y - 1}$. By the weighted AM-GM Inequality, we have

$$h'(y) = \frac{(p-1)y^p + 1 - py^{p-1}}{(y-1)^2} > 0.$$

Therefore, $h(y)$ is strictly increasing, and since $y-1 = (n-1)(1-x) \geq 1-x$, that is, $y \geq 2-x$, we have

$$h(y) \;\geq\; h(2-x) \;=\; \frac{(2-x)^p - 2 + x}{1-x}\,.$$

Then, it suffices to prove that

$$(2-x)^p - 2 + x \;\geq\; \frac{x - x^p}{x^p - x + 1}\,.$$

This inequality is equivalent to each of the following

$$\begin{aligned} (2-x)^p + x - 1 &\geq \frac{1}{x^p - x + 1}\,, \\ (1+t)^p - t &\geq \frac{1}{(1-t)^p + t}\,, \\ (1-t^2)^p + t(1+t)^p &\geq 1 + t^2 + t(1-t)^p\,, \end{aligned}$$

where $t = 1-x$ and $0 < t \leq 1$. By Bernoulli's Inequality, we have

$$(1-t^2)^p + t(1+t)^p \;\geq\; 1 - pt^2 + t(1+pt) \;=\; 1+t\,.$$

Thus, it suffices to show that $t(1-t) \geq t(1-t)^p$. However, this is clearly true, and the proof is completed. Equality in (1) occurs if and only if we have $x_1 = x_2 = \cdots = x_n = 1$.

Lemma 1. Let $p > 1$, and let x_1, x_2, $\ldots$, x_n be nonnegative real numbers. If (4) holds for all $x_i \geq p^{\frac{1}{1-p}}$ such that $x_1 + x_2 + \cdots + x_n = n$, then it holds for all $x_i \geq 0$ such that $x_1 + x_2 + \cdots + x_n = n$.

Proof: Let $a = p^{\frac{1}{1-p}}$, and let $f(x) = \dfrac{1}{x^p - x + n}$. From

$$f'(x) \;=\; \frac{1 - px^{p-1}}{(x^p - x + n)^2}\,,$$

it follows that $f'(a) = 0$, $f'(x) > 0$ for $0 \leq x < a$, and $f'(x) < 0$ for $x > a$. Thus, $f(x)$ is strictly increasing on $[0, a]$ and strictly decreasing on $[a, \infty)$. We have to prove that $f(x_1) + f(x_2) + \cdots + f(x_n) \leq 1$ for all $x_i \geq 0$ such that $x_1 + x_2 + \cdots + x_n = n$. Without loss of generality, assume that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. If $x_1 \geq a$, then the conclusion follows immediately by hypothesis. Otherwise, since $a < 1$, there exists an integer $k \in \{1, 2, \ldots, n-1\}$ such that

$$0 \leq x_1 \leq \cdots \leq x_k < a \leq x_{k+1} \leq \cdots \leq x_n\,.$$

Let $y_i = a$ for $i = 1$, 2, $\ldots$, k, and let $y_i = x_i$ for $i = k+1$, $\ldots$, n. We have $y_i \geq a$ and $y_i \geq x_i$ for $i = 1$, 2, $\ldots$, n; and $f(y_i) > f(x_i)$ for each $i = 1$, 2, $\ldots$, k, hence

$$y_1 + y_2 + \cdots + y_n \;>\; x_1 + x_2 + \cdots + x_n \;=\; n$$

and

$$f(y_1) + f(y_2) + \cdots + f(y_n) \; > \; f(x_1) + f(x_2) + \cdots + f(x_n) \, .$$

Therefore, it suffices to show that $f(y_1) + f(y_2) + \cdots + f(y_n) \leq 1$ for all $y_i \geq a$ with $y_1 + y_2 + \cdots + y_n > n$. By hypothesis, this inequality is true for all $y_i \geq a$ with $y_1 + y_2 + \cdots + y_n = n$. Since f is decreasing on $[a, \infty)$, the inequality is also true for all $y_i \geq a$ such that $y_1 + y_2 + \cdots + y_n > n$. ■

Remark 1. If $0 \leq p < 1$ and $x_1 + x_2 + \cdots + x_n \leq n$, then the inequality in (1) reverses. We can prove this using the AM-HM Inequality together with Jensen's Inequality:

$$\begin{aligned} \sum_{\text{cyclic}} \frac{1}{x_1^p + x_2 + \cdots + x_n} \; &\geq \; \frac{n^2}{\sum\limits_{\text{cyclic}} (x_1^p + x_2 + \cdots + x_n)} \\ &= \; \frac{n^2}{\sum\limits_{i=1}^{n} x_i^p \; + \; (n-1) \sum\limits_{i=1}^{n} x_i} \\ &\geq \; \frac{n^2}{n \left(\frac{1}{n} \sum\limits_{i=1}^{n} x_i \right)^p \; + \; (n-1) \sum\limits_{i=1}^{n} x_i} \; \geq \; 1 \, . \end{aligned}$$

2 Proof of Inequality (2)

Without loss of generality we can assume $x_i > 0$ for each i. Furthermore, we need only consider the case $x_1 + x_2 + \cdots + x_n = n$. Indeed, if we set $r = \frac{x_1 + x_2 + \cdots + x_n}{n}$ and $y_i = \frac{x_i}{r}$ for $i = 1, 2, \ldots, n$ (where $r \geq 1$ and $y_1 + y_2 + \cdots + y_n = n$), then (2) becomes

$$\sum_{\text{cyclic}} \frac{y_1}{r^{p-1} y_1^p + y_2 + \cdots + y_n} \; \leq \; 1 \, ,$$

and it suffices to prove this for $r = 1$; that is, to prove that

$$\frac{x_1}{x_1^p - x_1 + n} \; + \; \frac{x_2}{x_2^p - x_2 + n} \; + \; \cdots \; + \; \frac{x_n}{x_n^p - x_n + n} \leq 1 \qquad (6)$$

for $x_1 + x_2 + \cdots + x_n = n$. We consider two cases.

Case 1. $1 < p \leq n + 1$. By Bernoulli's Inequality,

$$x_1^p = \big(1 + (x_1 - 1)\big)^p \; \geq \; 1 + p(x_1 - 1) \, ,$$

and hence $x_1^p - x_1 + n \; \geq \; n - p + 1 + (p-1)x_1 \; \geq \; 0$. Consequently, it suffices to show that

$$\sum_{i=1}^{n} \frac{x_i}{n - p + 1 + (p-1)x_i} \; \leq \; 1 \, .$$

For $p = n+1$, this inequality is an equality. Otherwise, for $1 < p < n+1$, we use

$$\frac{(p-1)x_i}{n-p+1+(p-1)x_i} = 1 - \frac{n-p+1}{n-p+1+(p-1)x_i}$$

to rewrite the inequality as

$$\sum_{i=1}^{n} \frac{1}{n-p+1+(p-1)x_i} \geq 1 \,.$$

We then prove it by means of the AM-HM Inequality:

$$\sum_{i=1}^{n} \frac{1}{n-p+1+(p-1)x_i} \geq \frac{n^2}{\sum\limits_{i=1}^{n} \big(n-p+1+(p-1)x_i\big)} = 1 \,.$$

Case 2. $p > n+1$. By Lemma 2 below, it suffices to prove (6) for all $x_i \geq a = (\frac{n}{p-1})^{\frac{1}{p}}$ such that $x_1 + x_2 + \cdots + x_n = n$. We will apply the LCF-Theorem to the function $f(u) = \dfrac{u}{u^p - u + n}$ defined on $I = [a, \infty)$, for the case $s = 1$.

First, we must show that f is concave on $[a, 1]$. The second derivative has the expression

$$f''(u) = \frac{-p(p-1)u^{p-1}(u^p - u + n) + 2(pu^{p-1} - 1)\big((p-1)u^p - n\big)}{(u^p - u + n)^3} \,.$$

Since $(p-1)u^p - n \geq (p-1)a^p - n = 0$ for $u \geq a$, if

$$g(u) = -p(p-1)u^{p-1}(u^p - u + n) + 2pu^{p-1}\big((p-1)u^p - n\big) < 0$$

for $a \leq u \leq 1$, then $f''(u) < 0$, and hence $f(u)$ is concave on $[a, 1]$. Indeed, we have

$$\begin{aligned} \frac{g(u)}{pu^{p-1}} &= (p-1)(u^p + u) - n(p+1) \\ &\leq 2(p-1) - 2(p+1) = -4 < 0 \,. \end{aligned}$$

By the LCF-Theorem, it suffices to show that

$$\frac{(n-1)x}{x^p - x + n} + \frac{y}{y^p - y + n} \leq 1$$

for any $a \leq x \leq 1 \leq y$ and $(n-1)x + y = n$. For $x = y = 1$, equality occurs. Assume now that $x < 1 < y$, and write the desired inequality successively as follows

$$y^p - y + n \geq \frac{y(x^p - x + n)}{x^p - nx + n} \,, \tag{7}$$

$$y^p - y \ \geq \ \frac{(n-1)x(x-x^p)}{x^p - nx + n}\,. \tag{8}$$

Inequality (5) states that $y^p - y \geq \dfrac{(n-1)(x-x^p)}{x^p - x + 1}$. Therefore, it suffices to show that

$$\frac{1}{x^p - x + 1} \ \geq \ \frac{x}{x^p - nx + n}\,.$$

This inequality is true, because $x < 1$ and $x^p - nx + n > x^p - x + 1$. Equality in (2) occurs if and only if $x_1 = x_2 = \cdots = x_n = 1$.

Lemma 2. Let $p > n+1$, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. If (6) holds for all $x_i \geq (\frac{n}{p-1})^{\frac{1}{p}}$ such that $x_1 + x_2 + \cdots + x_n = n$, then it holds for all $x_i \geq 0$ such that $x_1 + x_2 + \cdots + x_n = n$.

Proof: Let $a = (\dfrac{n}{p-1})^{\frac{1}{p}}$, and let $f(x) = \dfrac{x}{x^p - x + n}$. From

$$f'(x) = \frac{n - (p-1)x^p}{(x^p - x + n)^2}\,,$$

it follows that $f'(a) = 0$, $f'(x) > 0$ for $0 \leq x < a$, and $f'(x) < 0$ for $x > a$. Consequently, $f(x)$ is strictly increasing on $[0, a]$ and strictly decreasing on $[a, \infty)$. We have to prove that $f(x_1) + f(x_2) + \cdots + f(x_n) \leq 1$ for all $x_i \geq 0$ such that $x_1 + x_2 + \cdots + x_n = n$. Without loss of generality, assume that $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$. If $x_1 \geq a$, then the conclusion follows by hypothesis. Otherwise, since $a < 1$, there exists an integer $k \in \{1, 2, \ldots, n-1\}$ such that

$$0 \leq x_1 \ \leq \ \cdots \ \leq \ x_k < a \ \leq \ x_{k+1} \ \leq \ \cdots \ \leq \ x_n\,.$$

Let $y_i = a$ for $i = 1, 2, \ldots, k$, and $y_i = x_i$ for $i = k+1, k+2, \ldots, n$. We have $y_i \geq a$ and $y_i \geq x_i$ for each i; and $f(y_i) > f(x_i)$ for $i = 1, 2, \ldots, k$, hence

$$y_1 + y_2 + \cdots + y_n \ > \ x_1 + x_2 + \cdots + x_n \ = \ n$$

and

$$f(y_1) + f(y_2) + \cdots + f(y_n) \ > \ f(x_1) + f(x_2) + \cdots + f(x_n)\,.$$

Therefore, it suffices to show that $f(y_1) + f(y_2) + \cdots + f(y_n) \leq 1$ for all $y_i \geq a$ such that $y_1 + y_2 + \cdots + y_n > n$. By hypothesis, this inequality is true for all $y_i \geq a$ such that $y_1 + y_2 + \cdots + y_n = n$. Since f is decreasing on $[a, \infty)$, the inequality is also true for all $y_i \geq a$ such that $y_1 + y_2 + \cdots + y_n > n$. ■

Remark 2. If $0 \leq p < 1$ and $x_1 + x_2 + \cdots + x_n \geq n$, then the inequality in (2) reverses. By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{x_1}{x_1^p + x_2 + \cdots + x_n} &\geq \frac{\left(\sum\limits_{i=1}^n x_i\right)^2}{\sum\limits_{\text{cyclic}} x_1(x_1^p + x_2 + \cdots + x_n)} \\
&= \frac{\left(\sum\limits_{i=1}^n x_i\right)^2}{\sum\limits_{i=1}^n x_i^{p+1} + \left(\sum\limits_{i=1}^n x_i\right)^2 - \sum\limits_{i=1}^n x_i^2} \\
&= 1 + \frac{\sum\limits_{i=1}^n x_i^2 - \sum\limits_{i=1}^n x_i^{p+1}}{\sum\limits_{i=1}^n x_i^{p+1} + \left(\sum\limits_{i=1}^n x_i\right)^2 - \sum\limits_{i=1}^n x_i^2},
\end{aligned}$$

and it suffices to show that $\sum\limits_{i=1}^n x_i^2 - \sum\limits_{i=1}^n x_i^{p+1} \geq 0$. This inequality follows by the Power Mean Inequality:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n x_i^2 &\geq \left(\frac{1}{n}\sum_{i=1}^n x_i^{p+1}\right)^{\frac{2}{p+1}} \\
&\geq \left(\frac{1}{n}\sum_{i=1}^n x_i^{p+1}\right)\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^{1-p} \geq \frac{1}{n}\sum_{i=1}^n x_i^{p+1}.
\end{aligned}$$

3 Proof of Inequality (3)

Inequality (3) is equivalent to

$$\sum_{\text{cyclic}} \frac{1}{x_1^p + x_2 + \cdots + x_n} \leq \frac{n}{x_1 + x_2 + \cdots + x_n}.$$

For $x_1 + x_2 + \cdots + x_n = n$, this inequality immediately follows from (1). Otherwise, if $x_1 + x_2 + \cdots + x_n > n$, then we set $r = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$, and $y_i = \dfrac{x_i}{r}$ for $i = 1, 2, \ldots, n$, so that $r > 1$ and $y_1 + y_2 + \cdots + y_n = n$. Then the inequality becomes

$$\sum_{\text{cyclic}} \frac{1}{r^{p-1}y_1^p + y_2 + \cdots + y_n} \leq 1.$$

This inequality follows from (1), since

$$\sum_{\text{cyclic}} \frac{1}{r^{p-1}y_1^p + y_2 + \cdots + y_n} \leq \sum_{\text{cyclic}} \frac{1}{y_1^p + y_2 + \cdots + y_n} \leq 1.$$

Equality in (3) occurs if and only if $x_1 = x_2 = \cdots = x_n = 1$.

4 New Conjectures

Readers can try to extend the inequalities (1) and (2) in different directions. One of these directions is to find a condition as weak as possible on the positive real number p such that the inequality below holds for any nonnegative real numbers $x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \cdots + x_n \geq n$:

$$\sum_{\text{cyclic}} \frac{x_1^2}{x_1^p + x_2 + \cdots + x_n} \leq 1\,.$$

For $n \geq 2$, we conjecture that $p \geq n$ is such a sufficient condition. Moreover, we claim that

$$p \geq 1 + \frac{\ln 2}{\ln n - \ln(n-1)}$$

is a necessary and sufficient condition for the inequality to be true.

On the other hand, the study of the reverse inequality for $n \geq 2$ and $x_1 + x_2 + \cdots + x_n \geq n$, is also an interesting problem for readers. We conjecture that the reverse inequality is true for $0 \leq p \leq \frac{9}{5}$.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er novembre 2008**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

Oliver Geupel nous a signalé une erreur dans l'énoncé du problème #3282 [2007 : 429, 431], proposé par José Luis Díaz-Barrero et Pantelimon George Popescu. Nous nous excusons de cette erreur.

3282. Correction. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne et Pantelimon George Popescu, Bucarest, Roumanie.*

Soit $A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ un polynôme unitaire à coefficients complexes. On suppose que $a_1 = -a_0$ et que les zéros $z_1, z_2, \ldots, z_n$ de $A(z)$ sont des nombres complexes non nuls et distincts. Montrer que

$$\sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j} = 0\,.$$

3338. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Un quadrilatère circulaire convexe $ABCD$ possède un cercle inscrit de centre I. Soit P le point d'intersection de AC et BD. Montrer que l'on a $AP : CP = AI^2 : CI^2$.

3339. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit Γ_1 et Γ_2 deux cercles sans points communs et situés à l'extérieur l'un de l'autre. Soit ℓ_1 et ℓ_2 les tangentes externes communes à Γ_1 et Γ_2. Soit A et B les points d'intersection de ℓ_1 avec Γ_1 et Γ_2; C et D ceux de ℓ_2 avec Γ_1 et Γ_2. Notons M et N les points milieu AB et CD, et soit P et Q les points d'intersection de NA et NB avec Γ_1 et Γ_2, différents de A et B. Montrer que CP, DQ et MN sont concourants.

3340. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un triangle ABC, la bissectrice de l'angle BAC coupe le cercle circonscrit en un second point D. Supposons que $AB^2 + AC^2 = 2AD^2$. Montrer que AD et BC se coupent à $45°$.

3341. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit ABC un triangle quelconque, de côtés a, b et c; montrer que $\sqrt{3}(R_a + R_b + R_c) \le a + b + c$, où R_a, R_b et R_c sont les distances respectives du centre du cercle inscrit du triangle ABC aux sommets A, B et C.

3342. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit respectivement r et R le rayon des cercles inscrit et circonscrits du triangle ABC. Montrer que

$$2 \sum_{\text{cyclique}} \sin\frac{A}{2}\sin\frac{B}{2} \;\le\; 1 + \frac{r}{R}\,.$$

3343. *Proposé par Stan Wagon, Macalester College, St. Paul, MN, É-U.*

Si l'on supprime toutes les factorielles dans la série de Maclaurin de $\sin x$, on obtient la série de $\arctan x$. Supposons alors qu'on en supprime seulement une sur deux. La série ainsi obtenue a-t-elle une forme fermée? C'est-à-dire, peut-on trouver la fonction dont la série de Maclaurin soit

$$x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7} + \frac{x^9}{9!} - \frac{x^{11}}{11} + \cdots\ ?$$

3344. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit n un entier positif, $n \ge 4$, et soit $a_1, a_2, \ldots, a_n$ des nombres réels positifs tels que $a_1 + a_2 + \cdots + a_n = n$. Montrer que

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \;\ge\; \frac{3}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2 - n\right).$$

3345. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit a, b, c et d des nombres réels positifs tels que $a + b + c + d = 4$. Montrer que

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \;\ge\; 2\,.$$

3346. *Proposé par Bin Zhao, étudiant, YunYuan HuaZhong Université de Technologie et Science, Wuhan, Hubei, Chine.*

Dans un triangle ABC, montrer que

$$\pi \sum_{\text{cyclique}} \frac{1}{A} \;\ge\; \left(\sum_{\text{cyclique}} \sin\frac{A}{2}\right)\left(\sum_{\text{cyclique}} \csc\frac{A}{2}\right).$$

3347. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $A_1A_2A_3A_4$ un quadrilatère convexe. Soit B_i un point sur A_iA_{i+1} pour $i \in \{1, 2, 3, 4\}$, les indices étant pris modulo 4, de sorte que

$$\frac{B_1A_1}{B_1A_2} = \frac{B_3A_4}{B_3A_3} = \frac{A_1A_4}{A_2A_3} \quad \text{et} \quad \frac{B_2A_2}{B_2A_3} = \frac{B_4A_1}{B_4A_4} = \frac{A_1A_2}{A_3A_4}\,.$$

Montrer que $B_1B_3 \perp B_2B_4$ si et seulement si $A_1A_2A_3A_4$ est un quadrilatère circulaire.

3348. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Dans un triangle acutangle ABC, soit respectivement A_1, B_1 et C_1 des points sur les côtés BC, CA et AB, de sorte que $\angle AC_1B_1$, $\angle BC_1A_1$ et $\angle ACB$ soient égaux. Soit respectivement M, N et P les centres des cercles circonscrits des triangles AC_1B_1, BA_1C_1 et CB_1A_1. Montrer que AM, BN et CP sont concourants si et seulement si AA_1, BB_1 et CC_1 sont les hauteurs du triangle ABC.

3349. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$6\prod_{\text{cyclique}} \frac{a^3+1}{a^2+1} \;\geq\; \max\left\{\sum_{\text{cyclique}} \frac{a(1+bc)(a^2+1)}{a^3+1}\,,\, \sum_{\text{cyclique}} \frac{ab(1+c)(a^2b^2+1)}{a^3b^3+1}\right\}.$$

3350. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit x, y et z trois nombres réels positifs tels que $x + y + z = 1$. Montrer que

$$\frac{yz}{1+x} + \frac{zx}{1+y} + \frac{xy}{1+z} \;\leq\; \frac{1}{4}\,.$$

.·.

3282. Correction. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let $A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a monic polynomial with complex coefficients. Suppose that $a_1 = -a_0$, and that the zeroes $z_1, z_2, \ldots, z_n$ of $A(z)$ are distinct, non-zero complex numbers. Prove that

$$\sum_{k=1}^{n} \frac{e^{z_k}}{z_k^2} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k - z_j} \;=\; 0\,.$$

3338. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex cyclic quadrilateral $ABCD$ has an incircle with centre I. Let P be the intersection of AC and BD. Prove that $AP : CP = AI^2 : CI^2$.

3339. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let Γ_1 and Γ_2 be two non-intersecting circles each lying in the exterior of the other. Let ℓ_1 and ℓ_2 be the common external tangents to Γ_1 and Γ_2. Let ℓ_1 meet Γ_1 and Γ_2 at A and B, respectively, and let ℓ_2 meet Γ_1 and Γ_2 at C and D, respectively. Let M and N be the mid-points of AB and CD, respectively, and let P and Q be the intersections of NA and NB with Γ_1 and Γ_2, respectively, different from A and B. Prove that CP, DQ, and MN are concurrent.

3340. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The bisector of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at a second point D. Suppose that $AB^2 + AC^2 = 2AD^2$. Prove that the angle of intersection of AD and BC is 45°.

3341. *Proposed by Arkady Alt, San Jose, CA, USA.*

For any triangle ABC with sides of lengths a, b, and c, prove that $\sqrt{3}(R_a + R_b + R_c) \le a + b + c$, where R_a, R_b, and R_c are the distances from the incentre of $\triangle ABC$ to the vertices A, B, and C, respectively.

3342. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r and R be the inradius and circumradius of $\triangle ABC$, respectively. Prove that

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\sin\frac{B}{2} \;\le\; 1 + \frac{r}{R}\,.$$

3343. *Proposed by Stan Wagon, Macalester College, St. Paul, MN, USA.*

If the factorials are deleted in the Maclaurin series for $\sin x$, one obtains the series for $\arctan x$. Suppose instead that one alternates factorials in the series. Does the resulting series have a closed form? That is, can one find an elementary expression for the function whose Maclaurin series is

$$x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7} + \frac{x^9}{9!} - \frac{x^{11}}{11} + \cdots ?$$

3344. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let n be a positive integer, $n \ge 4$, and let a_1, a_2, $\ldots$, a_n be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \;\ge\; \frac{3}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2 - n\right).$$

3345. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, c, and d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{1+b^2c}+\frac{b}{1+c^2d}+\frac{c}{1+d^2a}+\frac{d}{1+a^2b} \;\geq\; 2\,.$$

3346. *Proposed by Bin Zhao, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China.*

Given triangle ABC, prove that

$$\pi \sum_{\text{cyclic}} \frac{1}{A} \;\geq\; \left(\sum_{\text{cyclic}} \sin\frac{A}{2}\right)\left(\sum_{\text{cyclic}} \csc\frac{A}{2}\right)\,.$$

3347. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $A_1A_2A_3A_4$ be a convex quadrilateral. Let B_i be a point on A_iA_{i+1} for $i \in \{1,\,2,\,3,\,4\}$, where the subscripts are taken modulo 4, such that

$$\frac{B_1A_1}{B_1A_2}=\frac{B_3A_4}{B_3A_3}=\frac{A_1A_4}{A_2A_3} \quad\text{and}\quad \frac{B_2A_2}{B_2A_3}=\frac{B_4A_1}{B_4A_4}=\frac{A_1A_2}{A_3A_4}\,.$$

Prove that $B_1B_3 \perp B_2B_4$ if and only if $A_1A_2A_3A_4$ is a cyclic quadrilateral.

3348. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be an acute-angled triangle. Let A_1, B_1, and C_1 be points on the sides BC, CA, and AB, respectively, such that the angles $\angle AC_1B_1$, $\angle BC_1A_1$, and $\angle ACB$ are all equal. Let M, N, and P be the circumcentres of $\triangle AC_1B_1$, $\triangle BA_1C_1$, and $\triangle CB_1A_1$, respectively. Prove that AM, BN, and CP are concurrent if and only if AA_1, BB_1, and CC_1 are altitudes of $\triangle ABC$.

3349. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$6\prod_{\text{cyclic}} \frac{a^3+1}{a^2+1} \;\geq\; \max\left\{\sum_{\text{cyclic}} \frac{a(1+bc)(a^2+1)}{a^3+1}\,,\ \sum_{\text{cyclic}} \frac{ab(1+c)(a^2b^2+1)}{a^3b^3+1}\right\}\,.$$

3350. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let x, y, and z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{yz}{1+x}+\frac{zx}{1+y}+\frac{xy}{1+z} \;\leq\; \frac{1}{4}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

KLAMKIN–05. [2005 : 328] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \;\geq\; n(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_{n+k})$$

(where the subscripts are taken modulo n) in the following cases:

(a) $n = 2k$; (b) $n = 4k$; (c)★ $2 < \frac{n}{k} < 4$.

Solution to part (c)★ *by the proposer.*

Solutions to parts (a) and (b) appeared in [2006 : 315]. Let $n = 4k - j$, where j is an integer such that $1 \leq j \leq 2k - 1$, and let $i = \left\lfloor \frac{j}{2} \right\rfloor$. Then $0 \leq i \leq k - 1$. Let a be a real number such that $a_{2k-i} \leq a \leq a_{2k-i+1}$, and let $S_1 = a_1 + a_2 + \cdots + a_n$ and $S_2 = a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_{n+k}$. We have to show that

$$\frac{S_1^2}{4k - j} \geq S_2\,.$$

We consider two cases: $2k < n \leq 3k$ and $3k \leq n < 4k$.

Case 1. Let $2k < n \leq 3k$. Then $k \leq j \leq 2k - 1$, and hence $\lfloor \frac{k}{2} \rfloor \leq i \leq k - 1$. Let

$$S \;=\; \left(\sum_{m=k-i+1}^{3k-i} a_m \right) \quad \text{and} \quad P \;=\; \left(\sum_{m=k-i+1}^{2k-i} a_m a_{k+m} \right)\,.$$

Applying the original inequality in part (b) to the non-decreasing sequence of $4k$ real numbers $a_1, \ldots, a_{2k-i}, \underbrace{a, \ldots, a}_{j \text{ times}}, a_{2k-i+1}, \ldots, a_n$, we obtain

$$\begin{aligned}\frac{(S_1 + ja)^2}{4k} \;&\geq\; (j - k)a^2 \;+\; Sa \;+\; \left(\sum_{m=1}^{k-i} a_m a_{k+m} \right) \\ &\quad + \left(\sum_{m=2k-i+1}^{n} a_m a_{k+m} \right)\,,\end{aligned}$$

or equivalently,

$$\frac{(S_1+ja)^2}{4k} \;\geq\; (j-k)a^2 \;+\; Sa \;+\; S_2 \;-\; P\,.$$

The inequality to be proved follows by adding this inequality to

$$\frac{S_1^2}{4k-j} \;+\; (j-k)a^2 \;+\; Sa - P \;\geq\; \frac{(S_1+ja)^2}{4k}\,.$$

The last inequality is true, because

$$\frac{S_1^2}{4k-j} \;+\; ja^2 \;-\; \frac{(S_1+ja)^2}{4k} \;=\; \frac{j[S_1-(4k-j)a]^2}{4k(4k-j)} \;\geq\; 0$$

and

$$-ka^2 \;+\; Sa \;-\; P \;=\; \sum_{m=k-i+1}^{2k-i} (a-a_m)(a_{k+m}-a) \;\geq\; 0\,.$$

Case 2. Let $3k \leq n < 4k$. Then $1 \leq j \leq k$, and hence $0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$. Let

$$S \;=\; \left(\sum_{m=k-i+1}^{k-i+j} a_m\right) \;+\; \left(\sum_{m=3k-i-j+1}^{3k-i} a_m\right),$$

$$P \;=\; \left(\sum_{m=k-i+1}^{2k-i} a_m a_{k+m}\right), \quad \text{and} \quad Q \;=\; \left(\sum_{m=k-i+j+1}^{2k-i} a_m a_{k-j+m}\right).$$

Applying the original inequality in part (b) to the non-decreasing sequence of $4k$ real numbers $a_1, \ldots, a_{2k-i}, \underbrace{a, \ldots, a}_{j \text{ times}}, a_{2k-i+1}, \ldots, a_n$, we obtain

$$\begin{aligned} \frac{(S_1+ja)^2}{4k} \;\geq\; & \left(\sum_{m=1}^{k-i} a_m a_{k+m}\right) \;+\; a\left(\sum_{m=k-i+1}^{k-i+j} a_m\right) \\ & +\; \left(\sum_{m=k-i+j+1}^{2k-i} a_m a_{k-j+m}\right) \;+\; a\left(\sum_{m=3k-i-j+1}^{3k-i} a_m\right) \\ & +\; \left(\sum_{m=2k-i+1}^{n} a_m a_{k+m}\right), \end{aligned}$$

or equivalently,

$$\frac{(S_1+ja)^2}{4k} \;\geq\; S_2 - P + Q + Sa\,.$$

The required inequality follows by adding this inequality to

$$\frac{S_1^2}{4k-j} - P + Q + Sa \geq \frac{(S_1+ja)^2}{4k},$$

which is true, because

$$\frac{S_1^2}{4k-j} - \frac{(S_1+ja)^2}{4k} + ja^2 = \frac{j[S_1-(4k-j)a]^2}{4k(4k-j)} \geq 0$$

and

$$\begin{aligned} -ja^2 - P + Q + Sa &= \sum_{m=k-i+1}^{2k-i-j} (a_{j+m}-a_m)(a_{k+m}-a) \\ &\quad + \sum_{m=2k-i-j+1}^{2k-i} (a-a_m)(a_{k+m}-a) \geq 0\,. \end{aligned}$$

There were no other solutions submitted.

3239. [2007 : 237, 239] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer. If $\alpha = 1 + \frac{1}{12(n+1)}$, prove that

$$e < \left(\frac{(n+1)^{2n+1}}{(n!)^2}\right)^{\frac{1}{2n}} < e^{\alpha}\,.$$

Solution by Kee-Wai Lau, Hong Kong, China, modified by the editor.

Taking logarithms, the given inequalities are successively equivalent to

$$\begin{aligned} 1 &< \frac{1}{2n}\big[(2n+1)\ln(n+1) - 2\ln(n!)\big] < \alpha\,, \\ 2n &< (2n+1)\ln(n+1) - 2\ln(n!) < 2n + \frac{n}{6(n+1)}\,. \end{aligned}$$

Let $f(n) = (2n+1)\ln(n+1) - 2\ln(n!) - 2n$, and $g(n) = f(n) - \frac{n}{6(n+1)}$. We are to show that $f(n) > 0$ and $g(n) < 0$. For $0 < x < 1$, it is well known that $\ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. Hence,

$$\begin{aligned} &f(n+1) - f(n) \\ &\quad = (2n+3)\ln(n+2) - 2\ln\big((n+1)!\big) - 2(n+1) \\ &\qquad\quad - (2n+1)\ln(n+1) + 2\ln(n!) + 2n \\ &\quad = (2n+3)\ln\left(\frac{n+2}{n+1}\right) - 2 = (2n+3)\ln\left(1+\frac{1}{n+1}\right) - 2 \\ &\quad > (2n+3)\left(\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} - \frac{1}{4(n+1)^4}\right) - 2 \\ &\quad = A_n\,. \end{aligned}$$

By tedious but straightforward computations, we find that

$$A_n = \frac{2n^2+2n-3}{12(n+1)^4} > 0\,.$$

Hence, $f(n) > f(1) = 3\ln 2 - 2 > 0$.

Similarly, for $0 < x < 1$, it is well known that

$$\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\,.$$

Hence,

$$\begin{aligned} & g(n+1) - g(n) \\ &= f(n+1) - f(n) - \frac{n+1}{6(n+2)} + \frac{n}{6(n+1)} \\ &= (2n+3)\ln\left(1 + \frac{1}{n+1}\right) - 2 - \frac{1}{6(n+1)(n+2)} \\ &< A_n + \frac{2n+3}{5(n+1)^5} - \frac{1}{6(n+1)(n+2)} \\ &= B_n\,. \end{aligned}$$

Again, by tedious but straightforward computations, we find that

$$B_n = \frac{-n^2+19n+32}{60(n+1)^5(n+2)} = \frac{-\big(n(n-19)-32\big)}{60(n+1)^5(n+2)} < 0$$

for $n \geq 21$. On the other hand, using a calculator, we can readily check that $g(n) < 0$ for $n = 1, 2, \ldots, 21$. It follows that $g(n) < 0$ for each positive integer n, and our proof is complete.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. A partial solution was submitted by SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

3240. [2007 : 237, 239] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer. Prove that

$$\left\lfloor \sqrt{n} + \sqrt{n + 2\sqrt[3]{n} + 1} \right\rfloor = \left\lfloor \sqrt{4n + 4\sqrt[3]{n} + 2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

The asserted equality is false in general.

For example, when $n = 45$, we have

$$\left\lfloor \sqrt{n} + \sqrt{n + 2\sqrt[3]{n} + 1} \right\rfloor = \lfloor 13.996\cdots \rfloor = 13\,,$$

while

$$\left\lfloor\sqrt{4n+4\sqrt[3]{n}+2}\right\rfloor \;=\; \lfloor 14.008\cdots\rfloor = 14\,.$$

It also fails for $n = 95$ and 616.

Hawkins and Stone also gave the following comment: It is straightforward to show that the function $\sqrt{n}+\sqrt{n+2\sqrt[3]{n}+1}$ *is less than* $\sqrt{4n+4\sqrt[3]{n}+2}$*, and the gap between them narrows as* n *increases. But sometimes the values "straddle" an integer, as shown above, so the desired equality fails. We suspect that happens infinitely often (the next instance is* 1249*).*

SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina, also gave the counterexample $n = 45$*. There were three incorrect "proofs" for the assertion.*

3241. [2007 : 237, 239] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, c be any real numbers such that $a^2+b^2+c^2 = 9$. Prove that

$$3\cdot\min\{a,\,b,\,c\} \;\le\; 1+abc\,.$$

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

Without loss of generality, we may assume $a \le b \le c$. We want to prove the inequality $abc + 1 \ge 3a$.

If $a \le 0$, then using the inequality $bc \le \frac{1}{2}(b^2+c^2)$, we obtain

$$\begin{aligned} abc+1-3a &\ge a|bc| - 3a + 1 \;\ge\; \tfrac{1}{2}a(b^2+c^2) - 3a + 1 \\ &= \tfrac{1}{2}a(9-a^2) - 3a + 1 \;=\; \tfrac{1}{2}(a+1)^2(2-a) \;\ge\; 0\,, \end{aligned}$$

with equality if and only if $a = -1$ and $b = c = 2$.

Now, let $a > 0$. Since $a \le b \le c$, we get $9 = a^2+b^2+c^2 \ge 3a^2$; whence, $a \le \sqrt{3}$. We have the obvious inequality $(c^2-a^2)(b^2-a^2) \ge 0$, which yields

$$bc \;\ge\; a\sqrt{c^2+b^2-a^2} \;=\; a\sqrt{9-2a^2}\,.$$

Hence, it suffices to prove the (stronger) inequality

$$a\bigl(a\sqrt{9-2a^2}\,\bigr)+1 \;\ge\; 3a\,.$$

If $0 < a \le 1$, then $\sqrt{9-2a^2} \ge \sqrt{7} > \frac{9}{4}$; thus, $4a^2\sqrt{9-2a^2} > 9a^2$. Using the inequality $(t+1)^2 \ge 4t$, we obtain

$$\bigl(a^2\sqrt{9-2a^2}+1\bigr)^2 \;\ge\; 4a^2\sqrt{9-2a^2} \;>\; 9a^2\,,$$

which yields

$$a\bigl(a\sqrt{9-2a^2}\,\bigr)+1 \;\ge\; 3a\,.$$

If $1 < a \le \sqrt{3}$, then we can prove an equivalent form of our inequality, namely, the one obtained by squaring both sides of

$$a\bigl(a\sqrt{9-2a^2}\,\bigr) \;\ge\; 3a-1\,;$$

that is,

$$2a^6 - 9a^4 + 9a^2 - 6a + 1 \leq 0 .$$

We have

$$\begin{aligned} 2a^6 - 9a^4 + 9a^2 - 6a + 1 &< 2a^6 - 9a^4 + 9a^2 - 5 \\ &= (2a^2 - 1)(a^2 - 1)(a^2 - 3) - (a^2 + 2) \\ &< 0 , \end{aligned}$$

which completes the proof.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

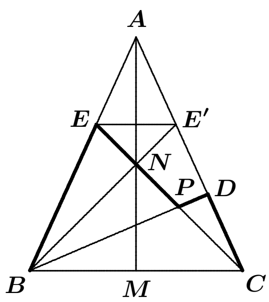
3243. [2007 : 237, 239] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be an isosceles triangle with $AB = AC$, and let P be an interior point. Let the lines BP and CP intersect the opposite sides at the points D and E, respectively. Find the locus of P if

$$PD + DC = PE + EB .$$

Solution by Titu Zvonaru, Cománeşti, Romania.

The locus consists of the points inside the triangle on the altitude from A.



Note that since the triangle is isosceles, the foot, M, of the altitude is the mid-point of BC, and the triangle is symmetric about AM. Without loss of generality, label the figure so that $BE \geq CD$ and let E' be the point between A and C for which $CE' = BE$, and let BE' and CE intersect at N. Because $\triangle ABC$ is isosceles, $BCE'E$ is an isosceles trapezoid that is symmetric about AM; whence, N belongs to AM and $NE = NE'$. As a consequence, the following statements are equivalent:

$$\begin{aligned} PD + DC &= PE + EB = PN + NE + E'C \\ &= PN + NE' + E'D + DC , \\ PD &= PN + NE' + E'D . \end{aligned}$$

The last equality says that the quadrilateral $PDE'N$ is degenerate; that is, D coincides with E' and P with N, so that P lies on AM. The locus of P is therefore the open line segment AM, as claimed.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3244. [2007 : 238, 239] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be an isosceles triangle with $AB = AC$, and let P be an interior point. Let the lines BP and CP intersect the opposite sides at the points D and E, respectively. Find the locus of P if

$$BD + DC \;=\; BE + EC\,.$$

Composite of similar solutions by Václav Konečný, Big Rapids, MI, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

As with problem 3243, the locus of P is the portion of the altitude AM inside $\triangle ABC$.

The condition $BD + DC = BE + EC$ means that points D and E lie on one ellipse of the family of ellipses with foci B and C, centre M, and axes along the lines BC and AM. Since the ellipse is symmetric about AM and intersects each of the line segments AB and AC in exactly one point, BD must intersect CE at P on AM. Conversely, if P lies on AM, then the diagram is symmetric about AM, implying that $BD = CE$ and $DC = EB$, so that $BD + DC = BE + EC$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comănești, Romania; and the proposer.

Some of the submitted solutions were designed to solve simultaneously this problem and problem 3243 above. For example, a straightforward modification of the featured solution to 3243 will likewise provide a solution to this problem.

3245. [2007 : 238, 240] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that the centre of the nine-point circle of a triangle lies on the incircle of the triangle. Show that its antipodal point is the Feuerbach Point; that is, the point where the nine-point circle and the incircle are tangent to each other.

I. *Composite of similar solutions by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; the Skidmore College Problem Solving Group, Skidmore College, Saratoga Springs, NY, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA.*

The result holds for tangent circles in general: the line joining the centres of two tangent circles passes through the point of tangency. If, moreover, the centre of one lies on the circumference of the other, the centre of the first circle is diametrically opposed in the second circle to the common tangency point.

II. *Composite of similar solutions by Michel Bataille, Rouen, France; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

Let I be the incentre, let N be the nine-point centre, and let r and R be the inradius and circumradius. Many standard references prove that the incircle is internally tangent to the nine-point circle, whose radius is $\frac{1}{2}R$; hence, $NI = \frac{1}{2}R - r$. Moreover, if N is on the incircle, we also have $NI = r$ and, therefore, $\frac{1}{2}R = 2r$. The antipodal point of N on the incircle, say N', satisfies $NN' = 2r$, so that $NN' = \frac{1}{2}R$. Thus, not only is N' on the incircle, but it is also on the nine-point circle. As a result, N' coincides with the unique common point to these two circles, namely, the Feuerbach Point.

There were no other solutions submitted.

3246. [2007 : 238, 240] *Proposed by Marian Tetiva, Bîrlad, Romania.*

Let a, b, c, d be any positive real numbers with $d = \min\{a, b, c, d\}$. Prove that

$$\begin{aligned} & a^4+b^4+c^4+d^4-4abcd \\ & \qquad \geq \; 4d\big[(a-d)^3+(b-d)^3+(c-d)^3-3(a-d)(b-d)(c-d)\big] \,. \end{aligned}$$

Solution by Kee-Wai Lau, Hong Kong, China.

Let $x = a - d$, $y = b - d$, and $z = c - d$. Thus, $x \geq 0$, $y \geq 0$, and $z \geq 0$. It can then be checked that

$$\begin{aligned} & a^4+b^4+c^4+d^4-4abcd \\ & \qquad -4d\big[(a-d)^3+(b-d)^3+(c-d)^3-3(a-d)(b-d)(c-d)\big] \\ = \; & 2d^2\big[x^2+y^2+z^2+(x-y)^2+(y-z)^2+(z-x)^2\big] \\ & \qquad +8xyzd+x^4+y^4+z^4 \\ \geq \; & 0 \,, \end{aligned}$$

and the inequality is proved.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALEX REMOROV, *student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Janous pointed out that the inequality has the equivalent form

$$a^4+b^4+c^4+d^4-4abcd \;\geq\; 4d(a+b+c-3d)(a^2-ab-ac+b^2-bc+c^2)\,.$$

Since the three factors on the right side are non-negative, this gives a clever proof for the four-variable AM–GM Inequality.

Malikić commented that on the right side of the inequality the factor 3 *can be changed to* 1, *resulting in a stronger inequality, namely*

$$x^4+y^4+z^4+2(d^2x^2+d^2y^2+d^2z^2)+2\big[(dx-dy)^2+(dy-dz)^2+(dz-dx)^2\big] \;\geq\; 0\,.$$

Alt also provided the proof for the inequality

$$a^4+b^4+c^4+d^4-4abcd \;\geq\; 4d\big[(a-d)^3+(b-d)^3+(c-d)^3-kd(a-d)(b-d)(c-d)\big]\,,$$

where k *is in the interval* $\big[1-3/\sqrt{2},1+3/\sqrt{2}\big]$.

3247. [2007 : 238, 240] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a_1, a_2, $\ldots$, a_n be real numbers, each greater than 1. Prove that

$$\sum_{k=1}^{n}\big(1+\log_{a_k}(a_{k+1})\big)^2 \;\geq\; 4n\,,$$

where $a_{n+1}=a_1$.

Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

By the AM–GM Inequality and the change of base formula, we have

$$\begin{aligned}\sum_{k=1}^{n}\big(1+\log_{a_k}(a_{k+1})\big)^2 \;&\geq\; \sum_{k=1}^{n}\Big(2\sqrt{\log_{a_k}(a_{k+1})}\,\Big)^2\\ &=\; 4\sum_{k=1}^{n}\log_{a_k}(a_{k+1}) \;\geq\; 4n\left(\prod_{k=1}^{n}\log_{a_k}(a_{k+1})\right)^{\frac{1}{n}}\\ &=\; 4n\left(\prod_{k=1}^{n}\frac{\ln a_{k+1}}{\ln a_k}\right)^{\frac{1}{n}} \;=\; 4n\,.\end{aligned}$$

For equality to hold, we must have $\log_{a_k}(a_{k+1})=1$ or $a_k=a_{k+1}$ for all $k=1,2,\ldots,n$. Conversely, if $a_1=a_2=\cdots=a_n$, then clearly equality holds.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, *ON; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Barbara pointed out that the inequality is a special case of the more general result that if $x_1, x_2, \ldots, x_n$ *are positive reals such that* $\prod_{k=1}^{n} x_k = 1$, *then* $\sum_{k=1}^{n} (1+x_k)^2 \geq 4n$. *Bencze obtained the generalization that if* $a_k > 1$ *for* $k = 1, 2, \ldots, n$, *and if* $P(x)$ *is any polynomial with positive coefficients, then* $\sum_{k=1}^{n} P\big(\log_{a_k}(a_{k+1})\big) \geq nP(1)$. *The current problem is the special case when* $P(x) = (1+x)^2$. *The proofs of these two generalizations are virtually the same as the one featured above.*

3248. [2007 : 238, 240] *Proposed by Titu Zvonaru, Comănești, Romania, and Bogdan Ioniță, Bucharest, Romania.*

If a, b, and c are positive real numbers, prove that

$$\frac{a^2(b+c-a)}{b+c} + \frac{b^2(c+a-b)}{c+a} + \frac{c^2(a+b-c)}{a+b} \leq \frac{ab+bc+ca}{2}\,.$$

Solution by Mohammed Aassila, Strasbourg, France.

The proof is as follows

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a^2(b+c-a)}{b+c} &= \sum_{\text{cyclic}} \frac{a}{b+c}\, a(b+c-a) \\ &\leq \sum_{\text{cyclic}} \frac{a}{b+c}\left(\frac{a+(b+c-a)}{2}\right)^2 \\ &= \sum_{\text{cyclic}} \frac{a(b+c)}{4} = \sum_{\text{cyclic}} \frac{bc}{2}\,.\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; PHI THAI THUAN, student, Tran Hung Dao High School, Phan Thiet, Vietnam; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PANOS E. TSAOUSSOGLOU, Athens, Greece; APOSTOLIS VERGOS, student, University of Patras, Patras, Greece; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Janous provided a proof of the more general inequality

$$\frac{a^{2n}(b+c-a)}{b+c} + \frac{b^{2n}(c+a-b)}{c+a} + \frac{c^{2n}(a+b-c)}{a+b} \leq \frac{(ab)^n+(bc)^n+(ca)^n}{2}$$

for any non-negative integer n. *At the same time he would like to leave the following conjecture to the readers of* ***CRUX with MAYHEM***:

$$\frac{(a^n+b^n+c^n)(abc)^n}{2} \leq \frac{a+b-c}{a+b}(ab)^{2n} + \frac{b+c-a}{b+c}(bc)^{2n} + \frac{c+a-b}{c+a}(ca)^{2n}$$

for any positive integer n.

3249. [2007 : 238, 240] *Proposed by Titu Zvonaru, Comăneşti, Romania, and Bogdan Ioniţă, Bucharest, Romania.*

Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab} \;\geq\; 6\,.$$

I. *Composite of solutions by Mohammed Aassila, Strasbourg, France; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Nguyen Thanh Liem, Tran Hung Dao High School, Phan Thiet, Vietnam; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; Phi Thai Thuan, student, Tran Hung Dao High School, Phan Thiet, Vietnam; and Panos E. Tsaoussoglou, Athens, Greece.*

We have

$$\begin{aligned}\sum_{\text{cyclic}}\left[\frac{(b+c)^2}{a^2+bc}-2\right] &= \sum_{\text{cyclic}}\frac{b^2+c^2-2a^2}{a^2+bc}\\ &= \sum_{\text{cyclic}}(a^2-b^2)\left(\frac{1}{b^2+ca}-\frac{1}{a^2+bc}\right)\\ &= \frac{\sum\limits_{\text{cyclic}}(a-b)^2(a+b)(a+b-c)(c^2+ab)}{\prod\limits_{\text{cyclic}}(c^2+ab)}\\ &\geq 0\,.\end{aligned}$$

II. *Second solution by Mohammed Aassila, Strasbourg, France.*

We prove the result for any positive real numbers a, b, and c. After a lengthy computation, the original inequality

$$\sum_{\text{cyclic}}\frac{(b+c)^2}{a^2+bc}-6 \;\geq\; 0$$

transforms to the following equivalent form

$$\frac{(b-c)^2(c-a)^2(a-b)^2}{abc}+\sum_{\text{cyclic}}\frac{(b+c)^2(b-c)^2}{a} \;\geq\; 0\,,$$

which is obviously true for any positive real numbers a, b, and c, as claimed.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (second solution); MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain (three solutions); and the proposer.

Janous and Ros have also shown that the inequality is true for any positive real numbers a, b, and c. Janous has proven that the following generalization holds

$$\sum_{\text{cyclic}} \frac{(b+c)^2}{a^2+\mu bc} \geq \frac{12}{\mu+1}$$

for any positive real numbers a, b, and c and $\mu \in [0,1] \cup [2,\infty)$. Malikić mentions that this is a well-known problem, originally created by Darij Grinberg and Peter Scholze; he gives the web address `http://www.mathlinks.ro/viewtopic.php?search_id=7020805&t=60128`, *where the problem can be found together with several solutions, including solutions similar to both of those featured above.*

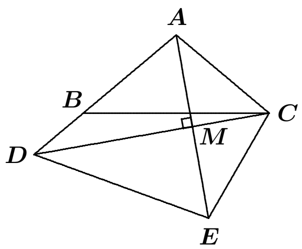
3250. [2007 : 239, 240] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let ABC be an isosceles triangle with $AB = AC$ and $\angle BAC = 100^\circ$. Let D be the point on the production of AB such that $AD = BC$. Find $\angle ADC$.

I. *Solution by Taichi Maekawa, Takatsuki City, Osaka, Japan.*

Construct the equilateral triangle ADE, as shown in the picture. Then $\triangle ABC$ and $\triangle CAE$ are congruent, since $AB = CA$, $BC = AD = AE$, and

$$\angle CAE \;=\; 100^\circ - 60^\circ \;=\; 40^\circ \;=\; \angle ABC\,.$$



Then $AC = CE$ and $AD = DE$, so that points D and C both lie on the perpendicular bisector of the segment AE. Let M be the point of intersection of DC and AE. Then $\angle AMD = 90^\circ$, and since $\angle DAE = 60^\circ$, it follows that $\angle ADC = 30^\circ$.

II. *Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $\angle ADC = \alpha$. Applying the Law of Sines to triangles ABC and ACD, we have

$$\frac{AC}{\sin 40^\circ} = \frac{BC}{\sin 100^\circ} \quad\text{and}\quad \frac{AC}{\sin\alpha} = \frac{AD}{\sin(80^\circ - \alpha)}\,,$$

and, since $AD = BC$, we get

$$\sin(80^\circ - \alpha)\sin 40^\circ \;=\; \sin\alpha\sin 100^\circ\,.$$

Now, $\sin 100^\circ = \sin 80^\circ = 2\sin 40^\circ\cos 40^\circ$, so that we obtain

$$\sin(80^\circ - \alpha) \;=\; 2\sin\alpha\cos 40^\circ\,,$$

or

$$\sin 80^\circ\cos\alpha - \sin\alpha\cos 80^\circ \;=\; 2\sin\alpha\cos 40^\circ\,.$$

The value $\alpha = 90^\circ$ is not a solution to this equation, so that we can divide both sides by $\cos\alpha$ to obtain an equation for $\tan\alpha$:

$$\begin{aligned}\tan\alpha &= \frac{\sin 80^\circ}{2\cos 40^\circ + \cos 80^\circ} = \frac{\cos 10^\circ}{2\cos(30^\circ + 10^\circ) + \sin 10^\circ} \\ &= \frac{\cos 10^\circ}{\sqrt{3}\cos 10^\circ} = \frac{1}{\sqrt{3}}.\end{aligned}$$

Therefore, $\alpha = 30^\circ$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOS ANASTASSIADES, student, Apostles Peter and Paul Lyceum, Limassol, Cyprus; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan (*second solution*)*; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JOEL SCHLOSBERG, Bayside, NY, USA; EDMUND SWYLAN, Riga, Latvia; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer* (*7 solutions*).

Two solvers mentioned that the problem is known. Covas supplied the references [1] *and* [2]*, and Lau gave the earliest reference* [3].

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[1] Ayoub B. Ayoub, "Problem 1104", Pi Mu Epsilon Journal, 12(2005); solution in 13(2005), pp. 186–187.

[2] Ion Patrascu, Probleme de Geometrie Plana, Craiova, 1996, Problem 8.12, p. 51.

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