SKOLIAD No. 109

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 September, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our questions this month come from the 2006 British Columbia Secondary School Mathematics Contest. We thank Clint Lee of Okanagan College for sending us this material.

British Columbia Secondary School Mathematics Contest 2006

Junior Final Round, Part B, Friday, May 5, 2006

1. Equilateral triangles I, II, III, and IV are such that the altitude of triangle I is the side of triangle II, the altitude of triangle II is the side of triangle III, and the altitude of triangle III is the side of triangle IV. If the area of triangle I is 2, find the area of triangle IV.

2. A square has an area of 3 square units, and a cube has a volume of 5 cubic units. Which is larger, the edge length of the square or the edge length of the cube? Justify your answer using the exact values of the two quantities.

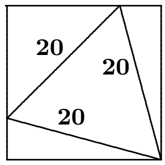
3. A certain positive integer has "6" as its last (rightmost) digit. This number is transformed into a new number by moving the "6" to the beginning of the number (leftmost position). For example, the number 1236 would be transformed to 6123, while 51476 becomes 65147. What is the smallest such positive integer for which this transformation increases the value of the number by a factor of 4?

4. The members of a committee sit at a circular table so that each committee member has two neighbours. Each member of the committee has a certain number of dollars in his or her wallet. The chairperson of the committee has one more dollar than the vice chairperson, who sits on his right and has one more dollar than the member on her right, who has one more dollar than the person on his right, and so on, until the member on the chair's left is reached. The chairperson now gives one dollar to the vice chair, who gives two dollars to the member on her right, who gives three dollars to the member on his right, and so on, until the member on the chair's left is reached. There are then two neighbours, one of whom has four times as much as the other.

(a) What is the smallest possible number of members of the committee? In this case, how much did the poorest member of the committee have at first?

(b) If there are at least 12 members of the committee, what is the smallest possible number of members of the committee? In this case, how much did the poorest member of the committee have at first?

5. An equilateral triangle, 20 centimetres on a side, is inscribed in a square, as shown in the diagram. Find the length of the side of the square.



Concours 2006 de Mathématique du secondaire de Colombie Britannique

Ronde Finale Junior partie B, vendredi 5 mai 2006

1. Les triangles équilatéraux I, II, III et IV sont tels que la hauteur du triangle I soit le côté du triangle II, la hauteur du triangle II soit le côté du triangle III, et que la hauteur du triangle III soit le côté du triangle IV. Si l'aire du triangle I est 2, trouver l'aire du triangle IV.

2. Un carré a une aire de 3 unités carrées, et un cube a un volume de 5 unités cubiques. Quel est le plus large, le côté du carré ou le côté du cube ? Justifier votre réponse en utilisant la valeur exacte des deux quantités.

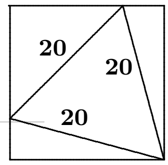
3. Un certain entier positif a «6» comme dernière unité (à droite). Ce nombre est transformé en un nouveau nombre en déplaçant le «6» au début du nombre (à gauche). Par exemple, 1236 est transformé en 6123, tandis que 51476 devient 65147. Quel est le plus petit de ces entiers pour lequel cette transformation augmente la valeur du nombre par un facteur 4 ?

4. Les membres d'un comité s'assoient autour d'une table circulaire de telle sorte que chaque membre du comité a deux voisins. Chaque membre du comité a un certain nombre de dollar dans son portefeuille. Le président a un dollar de plus que le vice-président qui s'assoit à sa droite qui lui, a aussi un dollar de plus que le membre à sa droite, et ainsi de suite jusqu'à l'atteinte du membre à la gauche du président. Maintenant, le président donne un dollar au v-p qui donne deux dollars au membre à sa droite qui donne trois dollars au suivant, et ainsi de suite jusqu'à l'atteinte du membre à la gauche du président. Il y a deux voisins tels que l'un a quatre fois l'argent de l'autre.

(a) Quel est le plus petit nombre possible de membres du comité ? Dans ce cas, spécifier combien le plus pauvre des membres du comité avait au début.

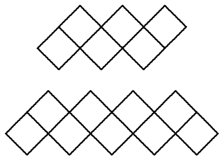
(b) S'il y a au moins 12 membres dans le comité, quel est le plus petit nombre possible de membres pour ce comité ? Dans ce cas, spécifier combien le plus pauvre des membres du comité avait au début.

5. Un triangle équilatéral de 20 cm de côté, est inscrit dans un carré, comme l'indique le diagramme. Trouver la longueur du côté du carré.



Next we give the solutions to the 6th Annual CNU Regional High School Mathematics Contest [2007 : 257–259].

1. To the right are two zigzag shapes made from identical little squares 1 cm on a side. The first shape has 6 squares and a perimeter of 14 cm. The second has 9 squares and a perimeter of 20 cm. What is the perimeter of the zigzag shape with 35 squares?



Solution by the editor.

Each time we add a square, we add replace 1 exterior side by 3 new exterior sides. Hence, if we call P_k the perimeter of the zigzag with k squares, we have $P_k = P_{k-1}+3-1 = P_{k-1}+2$. But we also have $P_1 = 4$. Therefore, we get $P_k = 2 + 2k$, and, in particular $P_{35} = 72$.

2. Three cards each have one of the digits from 1 through 9 written on them. When the three cards are arranged in some order, they make a three digit number. The largest number that can be made plus the second largest number that can be made is 1233. What is the largest number that can be made?

Solution by the editor.

Clearly, the intent of the problem is to assume that the selected digits are distinct. However, nothing in the problem statement indicates that. We must at least consider the possibility that a digit may be repeated.

Let $100a + 10b + c$ be the largest number we can make with the cards. This implies that $a \geq b \geq c$. If $b > c$, the second largest number will be $100a + 10c + b$ (for example, with 983, we can make 938). Their sum is $200a + 11(b + c) = 1233$. But we want the largest such number; thus, we must extract the largest a possible, namely $a = 6$ and $b + c = 3$, or $b = 2$ and $c = 1$, since they are in decreasing order. The largest number was 621.

Next we suppose that $b = c$. Then the largest number we can form is $100a + 10b + b = 100a + 11b$, while the second largest number is $100b + 10a + b = 101b + 10a$. Their sum is $110a + 112b = 1233$. However, this has no solution since 1233 is odd while both 110 and 112 are even.

Therefore, the largest number had to be 621.

3. You begin counting on your left hand starting with the thumb, then the index finger, the middle finger, the ring finger, then the little finger, and back to the thumb, and so on. What is the 2005^{th} finger you count?

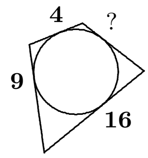
Solution by the editor.

There are two possible interpretations here. It could mean that you go directly from the little finger to the thumb, or it could mean that you are supposed to go through all the other fingers in going back to the thumb.

If we interpret it the first way, we see that each multiple of 5 lands on the little finger. Since 2005 is a multiple of 5, the 2005^{th} finger is the little finger.

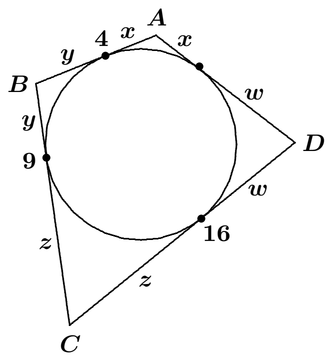
Let us use the second interpretation. In this case we get a cycle of length 8. Since 2005 leaves a remainder of 5 on division by 8, we are interested in the 5^{th} finger counted using this method. This is clearly the little finger (again).

4. A quadrilateral circumscribes a circle. Three of its sides have length 4, 9, and 16 cm, as shown. What is the length in cm of the fourth side?



Solution by the editor.

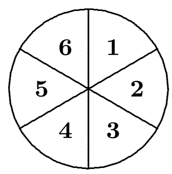
Let us label the quadrilateral $ABCD$, as in the diagram at the right. Note that the two tangents from point A to the circle have the same length, say x. We make similar observations about the lengths of the tangents to the circle from points B, C, and D, and label those lengths y, z, and w, respectively, as in the diagram. Then we see that



$$\begin{aligned} x+y &= AB = 4\,, \\ y+z &= BC = 9\,, \\ z+w &= CD = 16\,, \\ w+x &= AD\,. \end{aligned}$$

Thus, $AD = w + x = (x + y) + (z + w) - (y + z) = 4 + 16 - 9 = 11$.

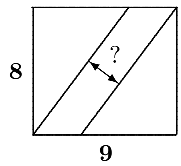
5. A pizza is cut into six pie-shaped pieces. Trung can choose any piece to eat first, but after that, each piece he chooses must have been next to a piece that has already been eaten (to make it easy to get the piece out of the pan). In how many different orders could he eat the six pieces?



Solution by the editor.

There are six different pieces where he can start. After the first piece is removed and eaten, the number of ways to remove the remaining five pieces is the same. Unless there is only one piece remaining, there are two choices for the next piece to be removed and eaten (one on either edge of the gap formed by the pieces already eaten). Hence, after the removal of the first piece, we have 2^4 ways to remove and eat the remaining 5 pieces, for a total of $6 \times 2^4 = 96$.

6. The picture shows an 8×9 rectangle cut into three pieces by two parallel slanted lines. The three pieces all have the same area. How far apart are the slanted lines?



Solution by the editor.

Let x be the length of the short side of the parallelogram in the diagram. The area of the parallelogram is then $8x$, and the two triangles are congruent and each has area $\frac{1}{2}\,8(9-x) = 36 - 4x$. Since the three areas are equal, we have $36 - 4x = 8x$, or $x = 3$.

By the Theorem of Pythagoras, the hypotenuse of the two congruent triangles has length $\sqrt{8^2 + (9-3)^2} = 10$. The distance between the two parallel lines is the height of the parallelogram which has base 10 and area $8x = 8 \times 3 = 24$; thus, the distance is 2.4.

7. Find a positive integer N so that there are exactly 25 integers x satisfying $2 \leq \dfrac{N}{x} \leq 5$.

Solution by the editor.

We can rearrange the left inequality to get $x \leq N/2$. Similarly the right inequality can be rewritten as $x \geq N/5$. Thus, we must find the smallest value of N such that there are 25 integers x such that $\frac{1}{5}N \leq x \leq \frac{1}{2}N$. Therefore, we see that

$$\frac{N}{2} - \frac{N}{5} \geq 24\,.$$

(The value on the right is 24, since it is possible that both $N/5$ and $N/2$ are integers.) On clearing the fractions, this inequality becomes $3N \geq 240$, or $N \geq 80$. We may try values of N beginning with $N = 80$ until we find the smallest one satisfying the conditions. When $N = 80$, we get $N/5 = 16$ and $N/2 = 40$, and there are exactly 25 integers x satisfying $16 \leq x \leq 40$. Therefore, the smallest value of N for this problem is $N = 80$.

8. Amy, Bart, and Carol ate some carrot sticks. Amy ate half the number that Bart ate, plus one-third the number that Carol ate, plus one. Bart ate half the number that Carol ate, plus one-third the number that Amy ate, plus two. Carol ate half the number that Amy ate, plus one-third the number that Bart ate, plus three. How many carrot sticks did they eat altogether?

Solution by the editor.

Let a, b, and c be the number of carrot sticks eaten by Amy, Bart, and Carol, respectively. The text then translates to the following system of equations:

$$\begin{aligned} a &= \tfrac{1}{2}b + \tfrac{1}{3}c + 1\,, \\ b &= \tfrac{1}{2}c + \tfrac{1}{3}a + 2\,, \\ c &= \tfrac{1}{2}a + \tfrac{1}{3}b + 3\,. \end{aligned}$$

When we clear the fractions, we get the equivalent system:

$$\begin{aligned} 6a &= 3b + 2c + 6\,, \\ 6b &= 3c + 2a + 12\,, \\ 6c &= 3a + 2b + 18\,. \end{aligned}$$

Summing all three equations yields $6(a+b+c) = 5(a+b+c) + 36$, or $a+b+c = 36$.

9. A motorized column is advancing over flat country at the rate of 15 kilometres per hour. The column is 1 kilometre long. A dispatch rider is sent from the rear to the front on a motorcycle travelling at a constant speed. He returns immediately at the same speed and his total time is 3 minutes. How fast is he going?

Solution by the editor.

Let t_1 be the time in hours that the rider takes to go from the back to the front of the column, let d_1 be the distance in km that the column travels during that time, and let v be the constant speed of the rider. Then we have

$$t_1 \;=\; \frac{d_1}{15} \;=\; \frac{1+d_1}{v}\,.$$

Solving the right equation for d_1 yields $d_1 = 15/(v-15)$. Using this in the left equation gives us $t_1 = 1/(v-15)$. Defining t_2 and d_2 in a similar manner for the trip from the front of the column to the rear, we get

$$t_2 \;=\; \frac{d_2}{15} \;=\; \frac{1-d_2}{v}\,,$$

which yields $t_2 = 1/(v+15)$. Hence,

$$t_1 + t_2 \;=\; \frac{1}{v-15} + \frac{1}{v+15} \;=\; \frac{3}{60}\,.$$

On clearing the fractions, we have

$$60(v+15) + 60(v-15) \;=\; 3(v^2 - 225)\,,$$

which resolves to

$$v^2 - 40v - 225 \;=\; 0\,.$$

Thus, $(v-45)(v+5)=0$, implying that $v=45$ or $v=-5$. We exclude the extraneous negative solution to see that the rider's speed is 45 km/h.

10. Find the remainder when the polynomial $x+x^3+x^9+x^{27}+x^{81}+x^{243}$ is divided by x^2-1.

Solution by the editor.

Note first that the exponent of each term of the dividend is odd. Next we observe that $x^{2k+1} = (x^2-1)x^{2k-1} + x^{2k-1}$, which tells us that the remainder when x^{2k+1} is divided by x^2-1 is the same as when x^{2k-1} is divided by x^2-1. This can be continued $k-1$ more times to see that the remainder is the same as when x^1 is divided by x^2-1, and that remainder is x. Thus, each of the six terms in the dividend yields a remainder of x when divided by x^2-1, which implies that the remainder is $6x$.

11. Determine the perimeter of a right triangle with hypotenuse H and area A.

Solution by the editor.

Let a and b be the lengths of the legs of the right triangle. Then we have $A=\frac{1}{2}ab$, and $H^2=a^2+b^2$ (by the Theorem of Pythagoras). Now $(a+b)^2 = a^2+2ab+b^2 = H^2+2A$, which implies that $a+b=\sqrt{H^2+4A}$. Since the perimeter of the triangle is $a+b+H$, we see that the perimeter is $H+\sqrt{H^2+4A}$.

12. When a positive integer n is divided by 3, the remainder is 1. When $n+1$ is divided by 2, the remainder is 1. What is the remainder when $n-1$ is divided by 6?

Solution by the editor.

From the second sentence of the problem statement, we see that $n+1$ is odd. Hence, the number n is even. From the first sentence of the problem statement, the number n is 1 greater than a multiple of 3. Let $n=6k+a$ for some integers k and a, $0\le a\le 5$. Then a must be even since n is even, and $a\ne 0$ and $a\ne 2$ since n is 1 greater than a multiple of 3; thus, n has the form $n=6k+4$ for some integer k. Therefore, $n-1=6k+3$ and $\dfrac{n-1}{6}=k+\dfrac{3}{6}$, and the remainder we seek is 3.

That brings us to the end of another issue. Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by **15 June 2008**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M338. *Proposed by the Mayhem Staff.*

Two students miscopy the quadratic equation $x^2 + bx + c = 0$ that their teacher writes on the board. Jim copies b correctly but miscopies c; his equation has roots 5 and 4. Vazz copies c correctly, but miscopies b; his equation has roots 2 and 4. What are the roots of the original equation?

M339. *Proposed by the Mayhem Staff.*

(a) Determine the number of integers between 100 and 199, inclusive, which contain *exactly* two equal digits.

(b) An integer between 1 and 999 is chosen at random, with each integer being equally likely to be chosen. What is the probability that the integer has *exactly* two equal digits?

M340. *Proposed by the Mayhem Staff.*

Let ABC be an isosceles triangle with $AB = AC$, and let M be the mid-point of BC. Let P be any point on BM. A perpendicular is drawn to BC at P, meeting BA at K and CA extended at T. Prove that $PK + PT$ is independent of the position of P (that is, the value of $PK + PT$ is always the same, no matter where P is placed).

M341. *Proposed by the Mayhem Staff.*

Let ABC be a right triangle with right angle at B. Sides BA and BC are in the ratio $3 : 2$. Altitude BD divides CA into two parts that differ in length by 10. What is the length of CA?

M342. *Proposed by the Mayhem Staff.*

Quincy and Celine have to move 10 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	1 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 am. What is the earliest time at which they can be finished moving all of the boxes?

M343. *Proposed by the Mayhem Staff.*

The Fibonacci numbers are defined by $f_1 = f_2 = 1$ and, for $n \geq 2$, by $f_{n+1} = f_n + f_{n-1}$. The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Find the sum of the first 100 even Fibonacci numbers.

....

M338. *Proposé par l'Équipe de Mayhem.*

Deux étudiants font une erreur en recopiant l'équation quadratique $x^2 + bx + c = 0$ que leur professeur écrit au tableau. Jean copie b correctement, mais pas c ; son équation possède alors les racines 5 et 4. Victor copie c correctement, mais pas b ; son équation possède les racines 2 et 4. Quelles sont les racines de l'équation originale ?

M339. *Proposé par l'Équipe de Mayhem.*

(a) Déterminer le nombre d'entiers entre 100 et 199, bornes comprises, contenant *exactement* deux chiffres égaux.

(b) Un entier entre 1 et 999 est choisi au hasard, chaque entier ayant la même chance d'être choisi. Quelle est la probabilité pour que cet entier ait *exactement* deux chiffres égaux ?

M340. *Proposé par l'Équipe de Mayhem.*

Soit ABC un triangle isocèle avec $AB = AC$, et soit M le point milieu de BC. Soit P un point quelconque sur BM. Par P, on dessine une perpendiculaire à BC, coupant BA en K et la droite CA en T. Montrer que $PK + PT$ est indépendant de la position de P (c'est-à-dire, la valeur de $PK + PT$ est toujours la même, peu importe la position de P).

M341. *Proposé par l'Équipe de Mayhem.*

Soit ABC un triangle rectangle, d'angle droit en B. Ses côtés BA et BC sont dans le rapport $3 : 2$. La hauteur BD divise CA en deux parties dont la différence des longueurs est 10. Quelle est la longueur de CA ?

M342. *Proposé par l'Équipe de Mayhem.*

Sophie et Céline doivent déplacer des boîtes, 10 grandes et 10 petites. Le tableau ci-dessous indique les temps requis pour ce faire, dans chaque cas et pour chaque personne.

	Céline	Sophie
petite boîte	1 min.	3 min.
grande boîte	6 min.	5 min.

Leur travail commence à 9 heures du matin. Trouver à quelle heure, au plus tôt, elles pourraient finir leur déménagement ?

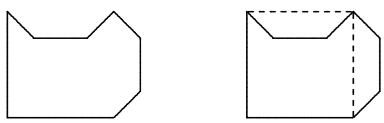
M343. *Proposé par l'Équipe de Mayhem.*

Les nombres de Fibonacci sont définis par $f_1 = f_2 = 1$ et, pour $n \geq 2$, par $f_{n+1} = f_n + f_{n-1}$. Voici donc le début de la liste des nombres de Fibonacci : $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Trouver la somme des 100 premiers nombres pairs de cette liste.

Mayhem Solutions

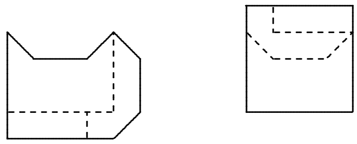
M288. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The following figure can be cut into two pieces and reassembled into a square, by simply cutting off the 'tab' and placing it in the cutaway at the top, as shown in the second image.



Determine a method to cut the given figure into three pieces which can be reassembled to form a square. (Find a method which is essentially different from cutting it into two pieces; for example, cutting the tab into two pieces would not be considered different from the two-piece dissection.)

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.



Also solved by PETER HURTHIG, Columbia College, Vancouver, BC; OWEN REN, student, Magee Secondary School, Vancouver, BC; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC. There was also a correct solution that was not essentially different than the two-piece dissection.

No two solutions to this problem were the same.

M289. *Proposed by K.R.S. Sastry, Bangalore, India.*

Solve the following equation for real x:

$$\log\left(x+\sqrt{5x-\tfrac{13}{4}}\right) \;=\; -\log\left(x-\sqrt{5x-\tfrac{13}{4}}\right)\,.$$

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

First we begin by bringing everything to one side to get

$$\log\left(x+\sqrt{5x-\tfrac{13}{4}}\right)+\log\left(x-\sqrt{5x-\tfrac{13}{4}}\right) \;=\; 0\,.$$

By the properties of logarithms, we can rewrite the equation as

$$\log\left[\left(x+\sqrt{5x-\tfrac{13}{4}}\right)\left(x-\sqrt{5x-\tfrac{13}{4}}\right)\right] \;=\; 0\,.$$

We can now see that the product inside the logarithm must be 1, because $\log y = 0$ if and only if $y = 1$. Therefore, we successively obtain

$$\begin{aligned}\left(x+\sqrt{5x-\tfrac{13}{4}}\right)\left(x-\sqrt{5x-\tfrac{13}{4}}\right) &= 1\,,\\ x^2-5x+\tfrac{13}{4} &= 1\,,\\ x^2-5x+\tfrac{9}{4} &= 0\,,\\ \left(x-\tfrac{1}{2}\right)\left(x-\tfrac{9}{2}\right) &= 0\,.\end{aligned}$$

Thus, $x=\frac{1}{2}$ or $x=\frac{9}{2}$.

We now must check for extraneous solutions. When $x=\frac{1}{2}$, we get $x+\sqrt{5x-\frac{13}{4}} = \frac{1}{2}+\sqrt{5(\frac{1}{2})-\frac{13}{4}} = \frac{1}{2}+\sqrt{\frac{-3}{4}}$. Since we are looking for real values x, we can stop here and say that $x=\frac{1}{2}$ is not a solution.

When $x=\frac{9}{2}$, we have

$$x+\sqrt{5x-\tfrac{13}{4}} \;=\; \tfrac{9}{2}+\sqrt{\tfrac{45}{2}-\tfrac{13}{4}} \;=\; \tfrac{9+\sqrt{77}}{2}$$

and

$$x - \sqrt{5x - \frac{13}{4}} = \frac{9}{2} - \sqrt{\frac{45}{2} - \frac{13}{4}} = \frac{9-\sqrt{77}}{2} = \frac{(9-\sqrt{77})(9+\sqrt{77})}{2(9+\sqrt{77})} = \frac{2}{9+\sqrt{77}}\,.$$

When we substitute these back into the logarithm equation, we can see that $x = \frac{9}{2}$ is a valid solution.

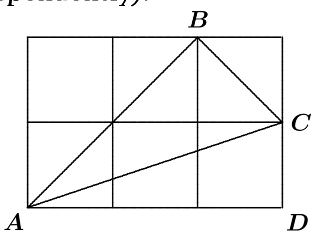
Also solved by HASAN DENKER, Istanbul, Turkey; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OWEN REN, student, Magee Secondary School, Vancouver, BC; NICK WILSON, student, Valley Catholic School, Beaverton, OR, USA; and the proposer. There were 3 incorrect solutions submitted.

M290. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Give a purely geometric proof that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

Solution by Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comănești, Romania (independently).

In the diagram, each of the six squares is a unit square. Note that $\angle CAD = \tan^{-1}(\frac{1}{3})$, since $CD = 1$ and $AD = 3$, and that $\angle BAC = \tan^{-1}(\frac{1}{2})$, since AB and BC are perpendicular, $BC = \sqrt{2}$, and $AB = 2\sqrt{2}$. Then $\tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{2}) = \angle BAD = \frac{\pi}{4}$.



Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; OWEN REN, student, Magee Secondary School, Vancouver, BC; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; D.J. SMEENK, Zaltbommel, the Netherlands; J. SUCK, Essen, Germany; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan.

M291. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

The right triangle having sides 3, $\sqrt{7}$, and 4, has the strange property that the two integer lengths sum to the value under the square root sign for the length of the third side.

1. Find another such triangle.
2. Prove that there are infinitely many such triangles, and show how to construct them.
3. Does the formula work only for integers?

Adapted from the solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

(a) Another triangle with the same property is the right-angled triangle with legs 5 and $\sqrt{11}$ and hypotenuse 6.

(b) To find the formula to construct infinitely many such triangles, we let the integer sides be called a and c. Then the side under the square root is $a+c$. We try to find an infinite family of triangles which are right-angled with legs of lengths a and $\sqrt{a+c}$ and hypotenuse of length c. For this to happen, we must have

$$\begin{aligned}
a^2+(\sqrt{a+c})^2 &= c^2\,, \\
a^2+(a+c) &= c^2\,, \\
a^2+a &= c^2-c\,, \\
a^2+a+\tfrac{1}{4} &= c^2-c+\tfrac{1}{4}\,, \\
\left(a+\tfrac{1}{2}\right)^2 &= \left(c-\tfrac{1}{2}\right)^2\,, \\
a+\tfrac{1}{2} &= c-\tfrac{1}{2} \qquad \text{(since } a \text{ and } c \text{ are positive)}, \\
c &= a+1\,.
\end{aligned}$$

Therefore, we may construct infinitely many of these right-angled triangles by letting the legs of triangle be a and $\sqrt{a+(a+1)}=\sqrt{2a+1}$ and letting the hypotenuse be $a+1$.

(c) The formula will work for any real number $a>0$ (to ensure the triangle does not have a negative side).

Also solved by HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and TITU ZVONARU, Comăneşti, Romania.

There was some confusion about whether the word "construct" meant to demonstrate explicitly the infinite family or show how these triangles can be created using compass and straightedge.

M292. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x be a positive number. Prove that $\sqrt{\dfrac{[x]}{x+\{x\}}}+\sqrt{\dfrac{\{x\}}{x+[x]}}>1$, where $[x]$ and $\{x\}$ represent the integer part and the fractional part of x, respectively.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

If x is an integer, then $\{x\}=0$ and $x=[x]$, and we have

$$\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}} \;=\; 1+0 \;=\; 1\,.$$

Similarly, if x is from interval $(0,1)$, then $[x] = 0$ and $x = \{x\}$, and

$$\sqrt{\frac{[x]}{x+\{x\}}} + \sqrt{\frac{\{x\}}{x+[x]}} \;=\; 0+1 \;=\; 1\,.$$

(This means that in the problem "$>$" should be replaced by "$\geq$" because, as we see, equality can be achieved when x is an integer or $x \in (0,1)$.)

Next, let $[x]$ be denoted by a and $\{x\}$ by b. Then $x = a+b$. Note that $0 \leq b < 1$. We may assume that x is not integer and that $x \notin (0,1)$. Thus, both a and b are greater than 0. Since a is an integer, we see that $a \geq 1 > b$; thus, $a \neq b$. We rewrite the given inequality as $\sqrt{\dfrac{a}{a+2b}} + \sqrt{\dfrac{b}{2a+b}} > 1$, which is equivalent to

$$\sqrt{a(2a+b)} + \sqrt{b(a+2b)} \;>\; \sqrt{(a+2b)(2a+b)}\,.$$

Since both sides are positive, after squaring this is equivalent to

$$2a^2 + 2b^2 + 2ab + 2\sqrt{a(2a+b)}\sqrt{b(a+2b)} \;>\; 2a^2 + 2b^2 + 5ab\,,$$

which is equivalent to

$$2\sqrt{a(2a+b)}\sqrt{b(a+2b)} \;>\; 3ab\,.$$

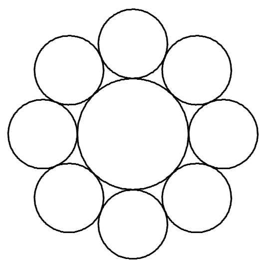
Since both sides are positive again, we can square this to obtain the equivalent inequality

$$4(ab)(2a+b)(2b+a) \;>\; 9(ab)(ab)\,,$$

and, after dividing by $ab > 0$, we get $4(5ab + 2a^2 + 2b^2) > 9ab$, which is equivalent to $11ab + 8a^2 + 8b^2 > 0$, which is true because $11ab \geq 0$, $8a^2 \geq 8$, and $8b^2 \geq 0$. Therefore, the result follows.

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÔMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; J. SUCK, Essen, Germany; and TITU ZVONARU, Comăneşti, Romania. There was also one incorrect solution submitted.

M293. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



Eight equal circles are mutually tangent in pairs and tangent externally to a unit circle. Determine the common radius of the eight smaller circles.

I. *Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let O_i ($i = 1, 2, \ldots, 8$) be the centre of the i^{th} small circle and let O be the centre of the circle of radius 1. Let r be the common radius of the eight smaller circles. We need to determine r. Let $\angle O_1OO_2 = \alpha$. By symmetry, $\angle O_1OO_2 = \angle O_2OO_3 = \cdots = \angle O_8OO_1 = \alpha$. We have $8\alpha = 360°$; whence, $\alpha = 45°$. Applying the Cosine Law to $\triangle O_1OO_2$, and noting that $OO_1 = OO_2 = 1 + r$ and $O_1O_2 = 2r$, we have

$$\begin{aligned}
O_1O_2^2 &= OO_1^2 + OO_2^2 - 2OO_1 \cdot OO_2 \cos\angle O_1OO_2\,,\\
(2r)^2 &= (1+r)^2 + (1+r)^2 - 2(1+r)(1+r)\tfrac{1}{\sqrt{2}}\,,\\
4r^2 &= (2-\sqrt{2})(1+r)^2\,,\\
2r &= (1+r)\sqrt{2-\sqrt{2}} \qquad \text{(since } r > 0\text{)},\\
2r - r\sqrt{2-\sqrt{2}} &= \sqrt{2-\sqrt{2}}\,,\\
r &= \frac{\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}\,.
\end{aligned}$$

II. *Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.*

Let O denote the centre of the large circle and A the centre of one of the small circles. The line OA passes through C, the point of tangency between the large and small circle. Draw the tangent OB to this small circle. Then OB is perpendicular to AB. By symmetry, $\angle BOA = \frac{1}{2} \times \frac{1}{8} \times 360° = 22.5°$. Applying the Sine Law in $\triangle OBA$ and using the fact that the radius of the large circle is 1, we get

$$\begin{aligned}
\frac{AB}{\sin(22.5°)} &= \frac{OA}{\sin(90°)}\\
\text{or} \quad \frac{r}{\sin(22.5°)} &= \frac{1+r}{1}\,.
\end{aligned}$$

Thus, $\sin(22.5°) = r\big(1 - \sin(22.5°)\big)$. Hence, $r = \dfrac{\sin(22.5°)}{1-\sin(22.5°)}$. Now, $\frac{\sqrt{2}}{2} = \cos(45°) = \cos\big(2(22.5°)\big) = 1 - 2\sin^2(22.5°)$, which implies that $\sin^2(22.5°) = \frac{2-\sqrt{2}}{4}$. Therefore, $\sin(22.5°) = \sqrt{\frac{2-\sqrt{2}}{4}}$, and we get

$$r = \frac{\sqrt{\frac{2-\sqrt{2}}{4}}}{1-\sqrt{\frac{2-\sqrt{2}}{4}}} = \frac{\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}\,.$$

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; OWEN REN, student, Magee Secondary School, Vancouver, BC; J. SUCK, Essen, Germany; NICK WILSON, student, Valley Catholic School, Beaverton, OR, USA; and TITU ZVONARU, Comănești, Romania.

Problem of the Month

Ian VanderBurgh

This month, we look at two similar problems involving exponents and numbers of digits.

Problem 1 (2005 Senior Australian Mathematics Competition)
The number of digits in the decimal expansion of 2^{2005} is closest to

(A) 400 (B) 500 (C) 600 (D) 700 (E) 800

Problems involving exponents always present interesting challenges. Exponents and logarithms tend to mystify many students, who seem to enjoy creating their own exponent and logarithm rules to try to solve this sort of problem. I promise that we will not make up any rules while trying to solve this one.

We can deduce from the problem above that calculators are not likely allowed on the AMC; otherwise, I'm pretty sure that we could solve this in some snazzy way using a calculator. Unfortunately, I can't find mine right now, so we'll try to do this without one.

The general strategy to solve such a problem is to try to estimate the size of the given number (2^{2005}) compared to powers of 10.

Let's do some preliminary work before looking at the solution. We first look at how we might find an estimate for a large power of 10 that is less than 2^{2005}. We will use the fact that if a and b are positive with $a > b$ and n is a positive integer, then $a^n > b^n$.

Since $2^4 = 16 > 10^1$, then $2^{2005} = 2^1 2^{2004} = 2^1(2^4)^{501} > 2 \cdot 10^{501}$.

Since $2^5 = 32 > 10^1$, then $2^{2005} = (2^5)^{401} > 10^{401}$.

Since $2^6 = 64 > 10^1$, then $2^{2005} = 2^1 2^{2004} = 2^1(2^6)^{334} > 2 \cdot 10^{334}$.

Of these three attempts, the first inequality gives us the best estimate (that is, the largest lower bound). In general, it will be the first power of 2 larger than a given power of 10 (or the last power of 2 smaller than a given power of 10) that will give us the best estimates. Let's use this principle to solve the problem.

Let's also write out the first several powers of 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, $16\,384$. We'll use this list to determine the largest power of 2 smaller than a given power of 10 and the smallest power of 2 larger than a given power of 10.

Solution to Problem 1: We look at the smallest powers of 2 larger than the first several powers of 10.

Since $2^4 = 16 > 10^1$, then $2^{2005} = 2^1 2^{2004} = 2^1(2^4)^{501} > 2 \cdot 10^{501}$ (as above).

Since $2^7 = 128 > 10^2$, then $2^{2005} = 2^3 2^{2002} = 2^3(2^7)^{286} > 8 \cdot 10^{572}$.

Since $2^{10} = 1024 > 10^3$, then $2^{2005} = 2^5(2^{10})^{200} > 32 \cdot 10^{600}$.

Since $2^{14} = 16\,384 > 10^4$, then $2^{2005} = 2^3(2^{14})^{143} > 8 \cdot 10^{572}$.

Okay! We got larger lower bounds for a little while, then it got worse. So let's collect our thoughts and try to find an upper bound, noting that the best we know now is that $2^{2005} > 32 \cdot 10^{600}$.

We look at the largest powers of 2 which are smaller than the first few powers of 10.

Since $2^3 = 8 < 10^1$, then $2^{2005} = 2^1(2^3)^{668} < 2 \cdot 10^{668}$.

Thus, we definitely know that $32 \cdot 10^{600} < 2^{2005} < 2 \cdot 10^{668}$. Can you translate this into a range for the number of digits for 2^{2005}? Try this before reading on.

From the last inequality, 2^{2005} has between 602 and 669 digits. Hence, the answer is either (C) or (D). In order to answer the question, we need to refine our estimate to determine if the number of digits is between 600 and 649 or between 650 and 699. Let's keep going.

Since $2^6 = 64 < 10^2$, then $2^{2005} = 2^1(2^6)^{334} < 2 \cdot 10^{668}$. (That didn't help much.)

Since $2^9 = 512 < 10^3$, then $2^{2005} = 2^7(2^9)^{222} < 128 \cdot 10^{666}$, which we can express as $1.28(10^{668})$. That's a slightly better bound than $2 \cdot 10^{668}$, but it doesn't actually reduce the number of digits!

Since $2^{13} = 8192 < 10^4$, then $2^{2005} = 2^3(2^{13})^{154} < 8 \cdot 10^{616}$.

Aha! We can combine this with our lower bound to conclude that $32 \cdot 10^{600} < 2^{2005} < 8 \cdot 10^{616}$. Therefore, 2^{2005} has between 602 and 617 digits, so (C) is the answer.

When you stop to think about it, we have actually done a pretty good job of narrowing down the range. Hold on a second! I found my calculator. Using the calculator, we get $\log_{10}(2^{2005}) \approx 603.57$, which tells us that in fact $10^{603} < 2^{2005} < 10^{604}$, so that 2^{2005} has exactly 604 digits.

In our second problem, we'll try to determine the exact number of digits of a power of 2 using the method above.

Problem 2 (1995 Special K Competition)

Determine the exact number of digits in the decimal expansion of 2^{100}.

Let's use our method above and try to narrow the range as much as possible.

Solution to Problem 2: We look at the smallest powers of 2 which are larger than the first several powers of 10.

Since $2^4 = 16 > 10^1$, then $2^{100} = (2^4)^{25} > 10^{25}$.

Since $2^7 = 128 > 10^2$, then $2^{100} = 2^2(2^7)^{14} > 2^2(10^2)^{14} = 4 \cdot 10^{28}$.

Since $2^{10} = 1024 > 10^3$, then $2^{100} = (2^{10})^{10} > (10^3)^{10} = 10^{30}$.

Since $2^{14} = 16\,384 > 10^4$, then $2^{100} = 2^2(2^{14})^7 > 2^2(10^4)^7 = 4{\cdot}10^{28}$.

Again, this has stopped getting better so let's switch directions noting that our best estimate here is $2^{100} > 10^{30}$.

Next, we look at the largest powers of 2 which are smaller than the first few powers of 10.

Since $2^3 = 8 < 10^1$, then $2^{100} = 2^1(2^3)^{33} < 2^1 10^{33} = 2 \cdot 10^{33}$.

Since $2^6 = 64 < 10^2$, then $2^{100} = 2^4(2^6)^{16} < 2^4(10^2)^{16} = 16 \cdot 10^{32}$. (That's only slightly better.)

Since $2^9 = 512 < 10^3$, then $2^{100} = 2^1(2^9)^{11} < 2^1(10^3)^{11} = 2 \cdot 10^{33}$. (That's not any better.)

Since $2^{13} = 8192 < 10^4$, then $2^{100} = 2^9(2^{13})^7 < 2^9(10^4)^7$, which is $512 \cdot 10^{28}$.

Does that help? This tells us that $2^{100} < 5.12(10^{30})$. Aha! If we combine this with our earlier findings, we see that $10^{30} < 2^{100} < 5.12(10^{30})$. This range is narrow enough to conclude that 2^{100} has exactly 31 digits, since both the lower bound and the upper bound are integers with 31 digits.

So we used the same technique to, in the first case, bound the number of digits and, in the second case, determine the exact number of digits. In theory, we should be able to use this method to determine the exact number of digits of 2^{2005}, but we might need an enormous number of estimates to get this to work.

Next month, we'll have a problem that will literally and figuratively make you dizzy.

Note from the Mayhem Editor

Greetings from your friendly neighbourhood Mathematical Mayhem Editor! I am very excited about joining the ***CRUX with MAYHEM*** team in a more significant way. You may have noticed already a slightly different flavour to the problems through the first few issues in 2008. We are going to make a real effort to keep the Mayhem problems at a more accessible level as we move forward. In addition, we are also going to move our timelines for submission up to try to get solutions published 6 issues after the problems are printed rather than 8 issues later. To facilitate this, you will notice that we have already moved up the submission deadline for the problems in this issue and you will see a plethora of Mayhem solutions in the first couple of issues this coming Fall. Happy problem solving!

THE OLYMPIAD CORNER

No. 269

R.E. Woodrow

In this number we begin with the Hungarian contests for 2004–2005. First we give the Hungarian National Olympiad, Competition for Specialized Classes. Thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

HUNGARIAN NATIONAL OLYMPIAD 2004–2005
Specialized Mathematical Classes
First Round

1. The quadrilateral $ABCD$ is cyclic. Prove that

$$\frac{AC}{BD} = \frac{DA \cdot AB \,+\, BC \cdot CD}{AB \cdot BC \,+\, CD \cdot DA}\,.$$

2. How many real numbers x are there in the interval $0 < x < 2004$ such that $x + \lfloor x^2 \rfloor = x^2 + \lfloor x \rfloor$? (Here $\lfloor c \rfloor$ denotes the greatest integer k such that $k \leq c$.)

3. Let $s(n)$ be the sum of those positive divisors of n that are less than n. A triple of three integers, (a,b,c), is a *friendly* triple if $1 < a \leq b \leq c$ and $s(a)+s(b) = c$, $s(b)+s(c) = a$, and $s(c)+s(a) = b$. Determine all friendly triples (a,b,c) where c is even.

4. The set A of positive integers has k elements. If the positive integers x and y are not in A, then $2x$, $2y$, and $x+y$ are also not in A. The sum of the elements in A is s. Find the maximum possible value of s.

5. Let $ABCDE$ be a pyramid, where $ABCD$ is a cyclic quadrilateral. The perpendicular projection of E onto the plane $ABCD$ is F. Prove that the perpendicular projections of F onto AE, BE, CE, and DE all lie on a circle.

Final Round

1. Let $ABCD$ be a trapezoid with parallel sides AB and CD. Let E be a point on the side AB such that EC and AD are parallel. Further, let the area of the triangle determined by the lines AC, BD, and DE be t, and the area of ABC be T. Determine the ratio $AB : CD$, if $t : T$ is maximal.

2. Find the greatest integer k which has the following property: For all integers x and y, whenever $xy+1$ is divisible by k, then $x+y$ is also divisible by k.

3. Haydn and Beethoven celebrate the birthday of Mozart with a game. They take numbers alternately according to the following rules. First Haydn takes the number 2. The next player can take the sum or the product of any two numbers which were taken earlier (it is possible to choose just one number twice, thus taking the square of it). The numbers which are taken must be distinct and smaller than 1757. The winner is the player who takes the number 1756. Which player has a winning strategy?

Next we give the Hungarian National Olympiad for 2004–2005. Thanks again go to Felix Recio for collecting them for our use.

HUNGARIAN NATIONAL OLYMPIAD 2004–2005

Grades 11–12

Second Round

1. Find all real solutions to the following system of equations:

$$\begin{aligned} \sqrt{x+y}+\sqrt{x-y} &= 10\,, \\ x^2-y^2-z^2 &= 476\,, \\ 2^{(\log|y|-\log z)} &= 1\,. \end{aligned}$$

2. In triangle ABC, the points B_1 and C_1 are on BC, point B_2 is on AB, and point C_2 is on AC such that the segment B_1B_2 is parallel to AC and the segment C_1C_2 is parallel to AB. Let the lines B_1B_2 and C_1C_2 meet at D. Denote the areas of triangles BB_1B_2 and CC_1C_2 by b and c, respectively.

(a) Prove that if $b=c$, then the centroid of ABC is on the line AD.

(b) Find the ratio $b:c$ if D is the incentre of ABC and $AB=4$, $BC=5$, and $CA=6$.

3. At each vertex of a pentagon there is a real number. On each side and on each diagonal, the sum of the numbers at the end-points is written. Of these ten numbers, at least seven are integers. Prove that each of the ten numbers is an integer.

4. The divisors of n are $d_1<d_2<\cdots<d_8$, where $d_1=1$ and $d_8=n$. It is known that $20\le d_6\le 25$. Find all possible values of n.

Final Round

1. A positive integer n is *charming* if there are integers a_1, a_2, …, a_n (not necessarily distinct) such that $a_1+a_2+\cdots+a_n=a_1a_2\cdots a_n=n$. Find all charming integers.

2. Let a, b, and c be positive real numbers.

(a) Prove that

$$\sqrt{\frac{a^2+b^2}{2}} + \frac{2}{\frac{1}{a}+\frac{1}{b}} \geq \frac{a+b}{2} + \sqrt{ab}\,.$$

(b) Is it true always that

$$\sqrt{\frac{a^2+b^2+c^2}{3}} + \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \frac{a+b+c}{3} + \sqrt[3]{abc}\,?$$

3. Triangle ABC is acute angled, $\angle BAC = 60^\circ$, $AB = c$, and $AC = b$ with $b > c$. The orthocentre and the circumcentre of ABC are M and O, respectively. The line OM intersects AB and CA at X and Y, respectively.

(a) Prove that the perimeter of triangle AXY is $b + c$.

(b) Prove that $OM = b - c$.

The next group of problems for your puzzling pleasure are those used to select the Indian Team to the IMO 2002. Thanks again go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for us.

INDIAN TEAM SELECTION TEST TO IMO 2002

1. Let A, B, and C be three points on a line with B between A and C. Let Γ_1, Γ_2, and Γ_3 be semicircles, all on the same side of AC, and with AC, AB, and BC as diameters, respectively. Let l be the line perpendicular to AC through B. Let Γ be the circle which is tangent to the line l, tangent to Γ_1 internally, and tangent to Γ_3 externally. Let D be the point of contact of Γ and Γ_3. The diameter of Γ through D meets l in E. Show that $AB = DE$.

2. Show that there is a set of 2002 consecutive positive integers containing exactly 150 primes. (You may use the fact that there are 168 primes less than 1000.)

3. Let $X = \{2^m 3^n \mid 0 \leq m, n \leq 9\}$. How many quadratics are there of the form $ax^2 + 2bx + c$, with equal roots, and such that a, b, and c are distinct elements of X?

4. Let ABC be an acute triangle with orthocentre H and circumcentre O. Show that there are points D, E, and F on BC, CA, and AB, respectively, such that AD, BE, and CF are concurrent and

$$DO + DH = EO + EH = FO + FH\,.$$

5. Let a, b, and c be positive real numbers such that $a^2 + b^2 + c^2 = 3abc$. Prove that

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}.$$

6. Determine the number of n-tuples of integers $(x_1, x_2, \ldots, x_n)$ such that $|x_i| \leq 10$ for each $1 \leq i \leq n$ and $|x_i - x_j| \leq 10$ for $1 \leq i, j \leq n$.

7. Given two distinct circles touching each other internally, show how to construct a triangle with the inner circle as its incircle and the outer circle as its nine-point circle.

8. Let $\sigma(n) = \sum\limits_{d|n} d$, the sum of the positive divisors of an integer $n > 0$.

(a) Show that $\sigma(mn) = \sigma(m)\sigma(n)$ for positive integers m and n with $\gcd(m,n) = 1$.

(b) Find all positive integers n such that $\sigma(n)$ is a power of 2.

9. On each day of their tour of the West Indies, Sourav and Srinath have either an apple or an orange for breakfast. Sourav has oranges for the first m days, apples for the next m days, followed by oranges for the next m days, and so on. Srinath has oranges for the first n days, apples for the next n days, followed by oranges for the next n days, and so on.

If $\gcd(m,n) = 1$ and the tour lasted for mn days, on how many days did they eat the same kind of fruit?

10. Let T be the set of all ordered triples (p,q,r) of non-negative integers. Determine all functions $f : T \to \mathbb{R}$ such that if $pqr = 0$, then $f(p,q,r) = 0$, and if $pqr \neq 0$, then

$$\begin{aligned} f(p,q,r) \;=\; & 1 + \tfrac{1}{6}\big[f(p+1,q-1,r) + f(p-1,q+1,r) \\ & + f(p-1,q,r+1) + f(p+1,q,r-1) \\ & + f(p,q+1,r-1) + f(p,q-1,r+1)\big]. \end{aligned}$$

11. Let ABC be a triangle and let P be an exterior point in the plane of the triangle. Let AP, BP, and CP meet the (possibly extended) sides BC, CA, and AB in D, E, and F, respectively. If the areas of the triangles PBD, PCE, and PAF are all equal, prove that their common area is equal to the area of the triangle ABC.

12. Let a and b be integers with $0 < a < b$. A set $\{x, y, z\}$ of non-negative integers is *olympic* if $x < y < z$ and if $\{z-y, y-x\} = \{a, b\}$. Show that the set of all non-negative integers is the union of pairwise disjoint olympic sets.

13. Let ABC and PQR be two triangles such that

(a) P is the mid-point of BC and A is the mid-point of QR, and

(b) QR bisects $\angle BAC$ and BC bisects $\angle QPR$.

Prove that $AB + AC = PQ + PR$.

14. Let p be an odd prime and let a be an integer not divisible by p. Show that there are $p^2 + 1$ triples of integers (x, y, z) with $0 \le x,\ y,\ z < p$ and such that $(x + y + z)^2 \equiv axyz \pmod{p}$.

15. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2} < \sqrt{n}\,.$$

16. Is there a set of positive integers, $\{a_1, a_2, \ldots, a_{100} : a_i \le 25000\}$, with the property that the sums $a_i + a_j$, $1 \le i < j \le 100$, are all distinct?

17. Let n be a positive integer, and let $(1 + iT)^n = f(T) + ig(T)$, where i is the square root of -1, and f and g are polynomials with real coefficients. Show that for any real number k the equation $f(T) + kg(T) = 0$ has only real roots.

18. Consider the square grid with $A = (0, 0)$ and $C = (n, n)$ at its diagonal ends. Paths from A to C are composed of moves one unit to the right or one unit up. Let C_n be the number of paths from A to C which stay on or below the diagonal AC (C_n is the n^{th} Catalan Number). Show that the number of paths from A to C which cross AC from below at most twice is equal to $C_{n+2} - 2C_{n+1} + C_n$.

19. Let PQR be an acute triangle. Let SRP, TPQ, and UQR be isosceles triangles exterior to PQR, with $SP = SR$, $TP = TQ$, and $UQ = UR$, such that $\angle PSR = 2\angle QPR$, $\angle QTP = 2\angle RQP$, and $\angle RUQ = 2\angle PRQ$. Let S', T', and U' be the points of intersection of SQ and TU, TR and US, and UP and ST, respectively. Determine the value of

$$\frac{SQ}{SS'} + \frac{TR}{TT'} + \frac{UP}{UU'}\,.$$

20. Let a, b, and c be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}\,.$$

21. Given a prime p, show that there is a positive integer n such that the decimal representation of p^n has a block of 2002 consecutive zeros.

The next set of problems are from the 2004 Kürschák Competition, also collected for us by Felix Recio, Canadian Team Leader to the IMO in Mexico.

2004 KÜRSCHÁK COMPETITION

1. The circle k and the circumcircle of the triangle ABC are touching externally. The circle k also touches the rays AB and AC at the points P and Q, respectively. Prove that the mid-point of the segment PQ is the centre of the excircle touching the side BC of the triangle ABC.

2. Find the smallest positive integer n, different from 2004, with the property that there exists a polynomial $f(x)$ with integer coefficients such that the equation $f(x) = 2004$ has at least one integer solution and the equation $f(x) = n$ has at least 2004 distinct integer solutions.

3. Some red points and some blue points are on the circumference of a circle. The following operations can be performed:

(a) A new red point can be inserted somewhere and the colours of each of its two neighbours changed to the opposite colour.

(b) If there are at least three points, at least one of which is red, then a red point can be deleted and the colours of each of its two neighbours changed to the opposite colour.

At the start, there are exactly two points on the circle, both blue. Can these two blue points be changed into (exactly) two red points by a sequence of the two operations?

Next we turn to our file of solutions to problems given in the May number of the *Corner*. First is a solution to a problem from the final round of the 18th Korean Mathematical Olympiad given at [2007 : 214–215].

2. Show that no pair of positive integers x and y satisfies $3y^2 = x^4 + x$.

Solved by Ioannis Katsikis, Athens, Greece; and Andrea Munaro, student, University of Trento, Trento, Italy. We give the solution of Munaro.

We have

$$3y^2 \;=\; x^4 + x \;=\; x(x+1)(x^2 - x + 1)\,. \tag{1}$$

Let $\gcd(a, b)$ be the greatest common divisor of the integers a and b. Then

$$\begin{aligned} \gcd(x, x+1) &= 1, \\ \gcd(x^2 - x + 1, x) &= \gcd(x^2 + 1, x) = 1, \\ \gcd(x^2 - x + 1, x + 1) &= \gcd\left((x+1)^2 - 3x, x + 1\right) = \gcd(3x, x + 1)\,, \end{aligned}$$

where the very last greatest common divisor is either 1 or 3. This leads to two cases.

Case 1. $\gcd(x^2 - x + 1, x + 1) = 1$.

The three factors on the right side of (1) are coprime in pairs; thus, one factor is of the form $3a^2$ and the other two factors are perfect squares.

If $x^2 - x + 1 = 3a^2$ for some integer a, then both x and $x + 1$ are perfect squares, a contradiction.

If $x + 1 = 3a^2$ or $x = 3a^2$ for some integer a, then $x^2 - x + 1$ must be a perfect square. Noting that $x^2 - 2x + 1 = (x - 1)^2$ is a perfect square and that $(x - 1)^2 < x^2 - x + 1 \leq x^2$, we must have $x^2 - x + 1 = x^2$, yielding $x = 1$, which is not a solution to the original equation.

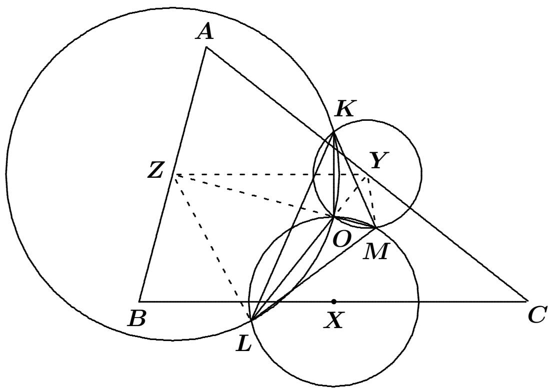
Case 2. $\gcd(x^2 - x + 1, x + 1) = 3$.

Then $x + 1 = 3a$ and $x^2 - x + 1 = 3b$ with $\gcd(a, b) = 1$, and x is a perfect square. Equation (1) becomes $3y^2 = x \cdot 3a \cdot 3b$ or $y^2 = x \cdot a \cdot 3b$, with $\gcd(x, a) = \gcd(x, b) = \gcd(a, b) = 1$. Hence, a or b is divisible by 3. Simple calculations show that $x^2 - x + 1$ is never divisible by 9; thus, $3 \nmid b$. This means that $3 \mid a$, implying that $9 \mid (x + 1)$. However, this implies that x and $\frac{1}{9}(x + 1)$ are perfect squares, hence x and $x + 1$ are perfect squares, a contradiction.

We now present a solution from our readers to a problem from the 21st Balkan Mathematical Olympiad 2004, given at [2007 : 215].

3. Let O be the circumcentre of the acute triangle ABC. The circles centred at the mid-points of the triangle's sides and passing through O intersect one another at the points K, L, and M. Prove that O is the incentre of triangle KLM.

Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's write-up.



Let X, Y, and Z be the mid-points of BC, CA, and AB, respectively. Since OK is perpendicular to the line through the centres Y and Z, and $YZ \parallel BC$, we see that KO is perpendicular to BC and it follows that O lies on the line KX. Similarly, O is on LY and MZ. Note that O is the orthocentre of $\triangle XYZ$ and that K, L, and M are the reflections of O in the sides YZ, ZX, and XY, respectively. It follows that K, L, and M are on the circumcircle Γ of $\triangle XYZ$. Since O is interior to $\triangle XYZ$ (an acute-angled triangle, as it is similar to $\triangle ABC$), K is on the arc YZ of Γ which does not contain X. Analogous observations can be made for L and M, so that X, M, Y, K, Z, L, and X occur in this order on Γ. In addition, ZX is clearly the internal bisector of $\angle LZO$, so that X is the mid-point of the arc LM of Γ (recall that Z, O, and M are collinear). Thus, KX is the internal bisector of $\angle MKL$. The result follows.

Next we examine the solutions in our files to problems posed in the 14th Japanese Mathematical Olympiad given at [2007 : 215–216].

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y, $f(xf(x) + f(y)) = \big(f(x)\big)^2 + y$.

Solved by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give Zhou's write-up.

Clearly $f(x) = x$ and $f(x) = -x$ are solutions. We show that they are the only solutions.

We obtain $f\Big(f\big(-f(0)^2\big)\Big) = 0$ by taking $x = 0$ and $y = -f(0)^2$ in the identity. Then setting $x = f\big(-f(0)^2\big)$, we obtain $f\big(f(y)\big) = y$, for all y. Putting $x = f(z)$ in the identity we now obtain

$$f\big(f(z)z + f(y)\big) \;=\; z^2 + y\,,$$

for all y and z, and hence $f(z)^2 = z^2$ for all z, as the left side of the above is also equal to $f(z)^2 + y$.

If $f(1) = 1$, then

$$\begin{aligned} 1 + 2x + x^2 &= f(1+x)^2 \;=\; f\Big(1 \cdot f(1) + f\big(f(x)\big)\Big)^2 \\ &= \big(f(1)^2 + f(x)\big)^2 \;=\; 1 + 2f(x) + x^2\,, \end{aligned}$$

and thus $f(x) = x$ for all x. Similarly, if $f(1) = -1$, then

$$\begin{aligned} 1 - 2x + x^2 &= f(-1+x)^2 \;=\; f\Big(1 \cdot f(1) + f\big(f(x)\big)\Big)^2 \\ &= \big(f(1)^2 + f(x)\big)^2 \;=\; 1 + 2f(x) + x^2\,, \end{aligned}$$

and thus $f(x) = -x$ for all x.

4. For positive real numbers a, b, and c with $a + b + c = 1$, show that

$$\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} \;\le\; 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)\,.$$

You need not state when equality holds.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Babis Stergiou, Chalkida, Greece. We give Stergiou's account.

Since $a+b+c=1$, we can write

$$\frac{1+a}{1-a} = \frac{a+b+c+a}{a+b+c-a} = \frac{2a+b+c}{b+c} = \frac{2a}{b+c}+1\,.$$

Thus, the given inequality is successively equivalent to

$$\begin{aligned}\left(\frac{2a}{b+c}+1\right)+\left(\frac{2b}{c+a}+1\right)+\left(\frac{2c}{a+b}+1\right) &\leq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\,;\\ \left(\frac{a}{c}-\frac{a}{b+c}\right)+\left(\frac{b}{a}-\frac{b}{c+a}\right)+\left(\frac{c}{b}-\frac{c}{a+b}\right) &\geq \frac{3}{2}\,;\\ \frac{ab}{c(b+c)}+\frac{bc}{a(c+a)}+\frac{ca}{b(a+b)} &\geq \frac{3}{2}\,.\end{aligned}$$

To prove the last inequality, we use a consequence of the Cauchy-Schwartz Inequality,

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}\,,$$

where x, y, and z are positive integers. Using this inequality, we obtain

$$\begin{aligned}&\frac{ab}{c(b+c)}+\frac{bc}{a(c+a)}+\frac{ca}{b(a+b)}\\ &= \frac{(ab)^2}{abc(b+c)}+\frac{(bc)^2}{abc(c+a)}+\frac{(ca)^2}{abc(a+b)}\\ &\geq \frac{(ab+bc+ca)^2}{abc(b+c)+abc(c+a)+abc(a+b)}\\ &= \frac{(ab+bc+ca)^2}{2abc(a+b+c)}\,.\end{aligned}$$

Thus, it suffices to prove that

$$\frac{(ab+bc+ca)^2}{2abc(a+b+c)} \geq \frac{3}{2}\,,$$

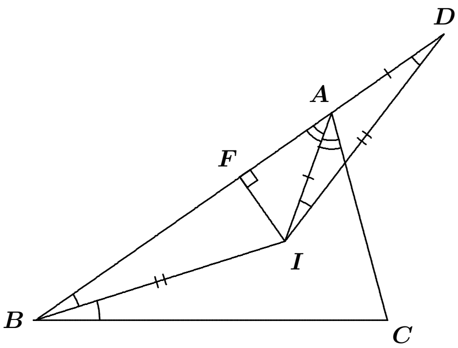
which is the same as proving $(ab+bc+ca)^2 \geq 3abc(a+b+c)$, which follows from the basic inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$. In the last inequality we have equality only if $x=y=z$, so in the given inequality we have equality only if $a=b=c=\frac{1}{3}$.

Next we turn to problems in the September 2007 *Corner*. We present solutions to selected problems of the Thai Mathematical Olympiad 2003, given at [2007 : 277–278].

1. Triangle ABC has $\angle A = 70^\circ$ and $CA+AI = BC$, where I is the incentre of triangle ABC. Find $\angle B$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's version.

First note that IA bisects $\angle BAC$, and IB bisects $\angle ABC$. Extend BA to a point D such that $AD = AI = a - b$. Let F be the foot of the perpendicular from I onto AB; F is the point of contact of the incircle with AB. It is well-known that $BF = s - b$ and $AF = s - a$. Then



$$DF = (a-b)+(s-a) = s-b = BF,$$

so $IB = ID$. Consequently,
$\angle IBF = \angle IDA = \angle AID = \frac{1}{2}\angle BAI$; hence, $\angle ABC = \angle BAI = \frac{1}{2}\angle BAC$. We were given that $\angle BAC = 70°$, so $\angle ABC = 35°$.

2. Let $f : \mathbb{Q} \to \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers, be such that

$$f(x+y) \;=\; f(x) + f(y) + 2547\,,$$

for all x, $y \in \mathbb{Q}$ and $f(2004) = 2547$. Find $f(2547)$.

Solved by Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Titu Zvonaru, Comăneşti, Romania. We first give Bataille's write-up.

We show that $f(2547) = \frac{2547\cdot 515}{334}$. Let $g : \mathbb{Q} \to \mathbb{Q}$ be defined by

$$g(x) \;=\; f(x) + 2547\,.$$

Then,

$$g(x+y) \;=\; f(x+y)+2547 \;=\; f(x)+2547+f(y)+2547 \;=\; g(x)+g(y)$$

for all rational numbers x and y. It follows that for some rational number r, we have $g(x) = rx$ for all $x \in \mathbb{Q}$ (this is a well-known result about additive functions on the rational numbers).

Since $f(2004) = 2547$, we get $g(2004) = 2 \cdot 2547$ and, observing that $g(2004) = 2004r$ as well, we obtain

$$r \;=\; \frac{2\cdot 2547}{2004} \;=\; \frac{849}{334}\,.$$

Finally, $f(2547) = g(2547) - 2547 = 2547 \cdot \frac{849}{334} - 2547 = 2547 \cdot \frac{515}{334}$.

Next we give Tsai's write-up.

Let $x \in \mathbb{Q}$ and let n be a positive integer. We prove by induction on n that

$$f(nx) \;=\; nf(x) + 2547(n-1)\,.$$

For $n = 1$, this is trivial. Assume the above equation holds for some positive integer n and for all $x \in \mathbb{Q}$. The induction step is completed by the calculation

$$\begin{aligned} f\big((n+1)x\big) &= f(nx+x) \;=\; f(nx) + f(x) + 2547 \\ &= nf(x) + 2547(n-1) + f(x) + 2547 \\ &= (n+1)f(x) + 2547n\,. \end{aligned}$$

Now, we have $f(2004 \cdot 2547) = 2004f(2547) + 2547 \cdot 2003$ and also $f(2547 \cdot 2004) = 2547f(2004) + 2547 \cdot 2546$; thus,

$$\begin{aligned} f(2547) &= \frac{2547f(2004) + 2546 \cdot 2547 - 2547 \cdot 2003}{2004} \\ &= \frac{2547^2 + 2546 \cdot 2547 - 2547 \cdot 2003}{2004} \;=\; \frac{1311705}{334}\,. \end{aligned}$$

Remark. Let $C \in \mathbb{Q}$ and let $f : \mathbb{Q} \to \mathbb{Q}$. Then $f(x+y) = f(x) + f(y) + C$ for all $x, y \in \mathbb{Q}$ if and only if $f(nx) = nf(x) + C(n-1)$ for all $x \in \mathbb{Q}$ and all $n \in \mathbb{N}$.

3. Let a, b, and c be positive real numbers such that $a+b+c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $a^3 + b^3 + c^3 \geq a + b + c$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; Vedula N. Murty, Dover, PA, USA; George Tsapakidis, Agrinio, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give Alt's generalization.

By Jensen's Inequality, we have

$$\frac{a^3+b^3+c^3}{3} \;\geq\; \left(\frac{a+b+c}{3}\right)^3.$$

We also have

$$(a+b+c)^2 \;\geq\; (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \;\geq\; 9\,.$$

We conclude that

$$a^3+b^3+c^3 \;\geq\; \frac{(a+b+c)(a+b+c)^2}{9} \;\geq\; a+b+c\,.$$

Generalization. Let n be a non-negative integer. With the same hypotheses, we have

$$a^{n+1} + b^{n+1} + c^{n+1} \;\geq\; a^{n-1} + b^{n-1} + c^{n-1}\,.$$

Proof. For non-negative integers n and m, we have

$$a^{n+m} + b^{n+m} + c^{n+m} \;\geq\; \frac{(a^n + b^n + c^n)(a^m + b^m + c^m)}{3}\,.$$

Indeed,

$$\begin{aligned} & 3(a^{n+m} + b^{n+m} + c^{n+m}) - (a^n + b^n + c^n)(a^m + b^m + c^m) \\ &\quad = \sum_{\text{cyclic}} (a^{n+m} + b^{n+m} - a^n b^m - a^m b^n) \\ &\quad = \sum_{\text{cyclic}} (a^n - b^n)(a^m - b^m) \;\geq\; 0\,. \end{aligned}$$

Using this inequality and

$$a^2 + b^2 + c^2 \;\geq\; \frac{(a+b+c)^2}{3} \;\geq\; \frac{(a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})}{3} \;\geq\; \frac{9}{3} \;=\; 3\,,$$

we immediately obtain

$$\begin{aligned} a^{n+1} + b^{n+1} + c^{n+1} &\;\geq\; \left(a^{n-1} + b^{n-1} + c^{n-1}\right)\left(\frac{a^2 + b^2 + c^2}{3}\right) \\ &\;\geq\; a^{n-1} + b^{n-1} + c^{n-1}\,. \end{aligned}$$

6. Let $ABCD$ be a convex quadrilateral. Prove that

$$[ABCD] \;\leq\; \tfrac{1}{4}\left(AB^2 + BC^2 + CD^2 + DA^2\right).$$

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; George Tsapakidis, Agrinio, Greece; and Titu Zvonaru, Cománeşti, Romania. We give the solution by Zvonaru.

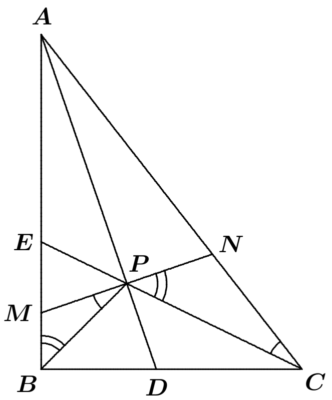
We have

$$\begin{aligned} [ABCD] &= [ABC] \;+\; [CDA] \\ &= \tfrac{1}{2} \cdot AB \cdot BC \cdot \sin\angle ABC \;+\; \tfrac{1}{2} CD \cdot DA \cdot \sin\angle CDA \\ &\leq \tfrac{1}{2} AB \cdot BC \;+\; \tfrac{1}{2} CD \cdot DA \\ &\leq \tfrac{1}{2}\left(\tfrac{1}{2}(AB^2 + BC^2) + \tfrac{1}{2}(CD^2 + DA^2)\right) \\ &= \tfrac{1}{4}(AB^2 + BC^2 + CD^2 + DA^2)\,. \end{aligned}$$

Equality holds if and only if $ABCD$ is a square.

9. Given a right triangle ABC with $\angle B = 90°$, let P be a point on the angle bisector of $\angle A$ inside ABC and let M be a point on the side AB (with $A \neq M \neq B$). Lines AP, CP, and MP intersect BC, AB, and AC at D, E, and N, respectively. Suppose that $\angle MPB = \angle PCN$ and $\angle NPC = \angle MBP$. Find $[APC]/[ACDE]$.

Solved by Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.



Let $a = BC$, $b = CA$, and $c = AB$, and let $c_a = AD$, the bisector of $\angle A$. We have

$$c_a = \frac{2bc\cos\frac{A}{2}}{b+c}.$$

By the Bisector Theorem, we deduce that

$$DC = \frac{ab}{b+c}, \qquad BD = \frac{ac}{b+c}.$$

We have $\angle BMP = \angle CNP$; thus, $\triangle AMN$ is isosceles. Hence , $AM = AN$ and P is the mid-point of MN. Let x be the common length of AM and AN. It follows that

$$MP = NP = x\sin\frac{A}{2}, \qquad AP = x\cos\frac{A}{2}.$$

Since $\triangle MBP$ and $\triangle CPN$ are similar, we have $\dfrac{BM}{PN} = \dfrac{PM}{NC}$, which yields successively

$$\begin{aligned}
(b-x)(c-x) &= x^2\sin^2\frac{A}{2};\\
x^2\left(1-\sin^2\frac{A}{2}\right) - (b+c)x + bc &= 0;\\
x^2\cdot\frac{1+\cos A}{2} - (b+c)x + bc &= 0;\\
(b+c)x^2 - 2b(b+c)x + 2b^2c &= 0.
\end{aligned}$$

Solving this equation we obtain

$$\begin{aligned}
x &= \frac{b(b+c) \pm \sqrt{b^2(b+c)^2 - 2b^2c(b+c)}}{b+c}\\
&= \frac{b(b+c) \pm b\sqrt{(b+c)(b-c)}}{b+c}\\
&= \frac{b(b+c\pm a)}{b+c},
\end{aligned}$$

and since $x < b$, we take $x = \dfrac{b(b+c-a)}{b+c}$.

We now have

$$\begin{aligned}\frac{AP}{PD} &= \frac{x\cos\frac{A}{2}}{c_a - x\cos\frac{A}{2}} = \frac{(b+c)x\cos\frac{A}{2}}{(b+c)c_a - (b+c)x\cos\frac{A}{2}} \\ &= \frac{b(b+c-a)\cos\frac{A}{2}}{2bc\cos\frac{A}{2} - b(b+c-a)\cos\frac{A}{2}} \\ &= \frac{b(b+c-a)}{2bc - b(b+c-a)} = \frac{b+c-a}{a+c-b}\,.\end{aligned}$$

Menelaus' Theorem applied to $\triangle ABD$ gives $\dfrac{CD}{CB}\cdot\dfrac{EB}{EA}\cdot\dfrac{PA}{PD} = 1$; hence, $\dfrac{EB}{EA} = \dfrac{CB}{CD}\cdot\dfrac{PD}{PA}$. Substituting the previously obtained expressions gives

$$\frac{EB}{EA} = \frac{(b+c)(a-b+c)}{b(b+c-a)}\,.$$

If r is the ratio $EB : EA$, then the ratio $EB : c$ is equal to $r : 1+r$. Hence,

$$EB = \frac{(b+c)(a-b+c)}{a+b+c}\,.$$

Finally, we have

$$\begin{aligned}[ACDE] &= [ABC] - [BDE] \\ &= \frac{ac}{2} - \frac{ac}{2(b+c)}\cdot\frac{(b+c)(a-b+c)}{a+b+c} \\ &= \frac{ac}{2}\left(1 - \frac{a-b+c}{a+b+c}\right) = \frac{abc}{a+b+c}\,,\end{aligned}$$

and also

$$\begin{aligned}[APC] &= \frac{AP\cdot AC\cdot\sin\frac{A}{2}}{2} = \frac{x\cos\frac{A}{2}\cdot b\sin\frac{A}{2}}{2} \\ &= \frac{bx\sin A}{4} = b\cdot\frac{b(b+c-a)}{4(b+c)}\cdot\frac{a}{b} = \frac{ab(b+c-a)}{4(b+c)}\,;\end{aligned}$$

hence,

$$\frac{[APC]}{[ACDE]} = \frac{(b+c-a)(a+b+c)}{4c(b+c)}\,.$$

That completes this number of the *Corner*. Send solutions soon!

BOOK REVIEWS

John Grant McLoughlin

The Magic Numbers of the Professor
By Owen O'Shea and Underwood Dudley, Mathematical Association of America, 2007
ISBN 978-0-88385-557-7, paper, 168 pages, US$39.95
Reviewed by **Jeff Hooper**, *Acadia University, Wolfville, NS*

This fascinating and fun book is crammed with number curiosities and coincidences, enough to keep the reader surprised and entertained for a considerable length of time.

Narrator Owen O'Shea first meets the fictional American Richard Stein in the Commodore Hotel in Cobh, Ireland, where his editor has sent him to meet the eccentric "professor". During dinner the professor delves into Irish history in his unique way. Ireland's Patron saint, Patrick, he says, first came to Ireland in 432 A.D. Mathematically, this was a very suitable date, the professor points out, since the island of Ireland contains 4 provinces and 32 counties, and since $432 = 4 \cdot 3^3 \cdot 2^2$. Not only that, but $432 + 1$ and $432 - 1$ are twin primes. Within the next few pages we are taken through numerical curiosities associated with the 1962 Cuban Missile Crisis, past connections between President Kennedy and the Apollo 11 moon landing, and hence into an array of curiosities among digits of numbers. For instance, the equation $192 + 384 = 576$ not only includes all 9 non-zero digits exactly once, but has the property that it has the form $n + 2n = 3n$ for the value $n = 192$. The professor then suggests that the reader discover the 3 other values of n for which this equation holds, before continuing with lots more curiosities, such as the equation

$$291548736 \;=\; 8 \cdot 92 \cdot 531 \cdot 746\,,$$

in which each of the 9 non-zero digits occurs on each side of the equation, and

$$335180136^2 \;=\; 112345723568978496\,,$$

a square in which each of the 9 digits occurs exactly twice. The book then follows O'Shea and Stein as they meet at various locations throughout Ireland. At each meeting, Stein continues weaving a numerical path through a variety of topics: more curiosities concerning digits, the September 11 tragedy, darts and cards, the King James Bible, the number of the beast, the US-Iraq war, Celtic Football Club, and James Joyce's *Ulysses*, to name only a few.

To give just a sampling, the professor describes numerous numerical connections between John Lennon and the number 9. For instance, Lennon was born on Wednesday, October 9^{th}, at 9 Newcastle Road, Penny Lane, Liverpool (and all four words and phrases *Wednesday*, *Newcastle*, *Penny Lane*, and *Liverpool* have 9 letters). The Beatles first album, *Please Please Me*, hit number 1 in the charts on February 9, 1963, Lennon met Yoko Ono for the first time on November 9, 1966, and their son Sean was born on October 9. The list of these continues even longer. There are enough such topical numerical coincidences throughout the book to satisfy even the most dedicated numerologist, even including new connections between Lincoln and Kennedy.

A word of caution may be necessary. As the authors indicate in the Introduction, this book is not meant to be read linearly, like a novel, but is better taken in small pieces, parts of a chapter at a time. Each chapter is filled with a plethora of curiosities of the sort described above, with several problems and challenges for the reader mixed in. The challenges are mainly aimed at the high school level, though there are exceptions. Solutions to these problems are given at the end of each chapter.

For instance, among the problems one encounters the following: In a standard game of darts, what is the smallest number that cannot be scored with a single dart? with two darts? While examining numerous curiosities surrounding the number 13 (including numerous 13s involved in the Apollo 13 mission), the professor serves up a number of puzzles involving Friday the 13^{th}. For example, can you see why two consecutive years can each contain exactly one occurrence of Friday the 13^{th}, yet three consecutive years can never do so? The professor even delves into probability in a few places, offering several challenges and seeming paradoxes. For instance, if, during a game of bridge, a player announces 'I have an ace', the probability that she holds a second ace is a little less than 37%; if, however, she then makes her statement a little more specific and announces 'I have the ace of spades', then the probability that she holds a second ace is now more than 56%! (Note that in bridge a hand consists of 13 cards.)

The real treasure trove here, though, is the multitude of number curiosities and patterns. Mathematical topics include triangular numbers, Mersenne primes, Lucas numbers, probability, Smith numbers, Friedman numbers, finite differences, Pythagorean triples, amicable pairs, perfect numbers, and multiply perfect numbers; still, this is only a sampling.

For the reader who has not seen them, a perfect number is one which equals the sum of its proper factors; for a multiply perfect number, the sum of its proper factors is a multiple of itself. An amicable pair is a pair of positive integers, each of which is the sum of the proper factors of the other. As the reader can verify, the smallest perfect number is 6 (since $1 + 2 + 3 = 6$), the smallest multiply perfect number which is not at the same time a perfect number is 120 (its proper factors sum to $240 = 2 \cdot 120$), and the smallest amicable pair is 220 and 284. The professor notes the remarkable coincidence that $6 = 1 \cdot 2 \cdot 3$, while $120 = 4 \cdot 5 \cdot 6$ and 504, the sum of 220 and 284, equals $7 \cdot 8 \cdot 9$. The reader is left to ponder on the possible connections between similar types of numbers and the next such product: $10 \cdot 11 \cdot 12$.

For the teacher, this is an interesting sort of reference. Not only are there a number of mathematical problems and puzzles included, but many of the coincidences and number properties can be used to lead students into further interesting topics regarding numbers, or to discovering similar coincidences of their own. For problem purists, however, this is not so useful as a straight problem resource: the number of true problems is not that large, and the problems themselves are embedded in the text, so one really needs to work through it to find them. But there are some nice ones in this book.

On the whole, though, this is still a fascinating and useful book. If you enjoy properties of numbers and numerical curiosities and coincidences of the sort mentioned above, this book is simply stuffed with them, and you will be busy for some time.

Geometric Puzzle Design
By Stewart Coffin, A K Peters, 2007
ISBN 1-56881-312-0, softcover, 204+xvi pages, US$39.00
Reviewed by ***Jim Totten****, Thompson Rivers University, Kamloops, BC*

Stewart Coffin has a reputation as a brilliant puzzle designer. I have appreciated many of his puzzle designs over the years without knowing they were his! Not only is he a great designer of puzzles, but he is an excellent writer.

Geometric Puzzle Design is a great book for the puzzle collector and enthusiast, but it is also a wonderful book for someone who is (or wishes to be) a wood-working craftsman. It is very apparent in reading the book that Coffin is all of these! On every page he conveys his infectious enthusiasm for all puzzles, but for those made of wood, he has a particular fondness.

Over the years I have amassed a small collection of puzzles, some of which I have made from scrap wood according to plans obtained from various sources. Reading this book has rekindled my interest in gaining access again to a wood-working facility so that I can make many other intriguing designs. Coffin not only discusses a lot of the geometry of the puzzles in the book, but actually provides guidance for those readers who wish to create such puzzles in their own workshops. Indeed, he rarely provides a solution to any of the puzzles in the book. Rather he focusses on the design and discusses the relative difficulty levels.

While the avid puzzle solver might find this annoying, most of them would likely agree that they would prefer to solve the puzzle on their own anyway, and this provides further impetus to their either buying or creating the puzzles in question. In many cases, the puzzles are not readily available for purchase. Hence, the focus on how to create them takes on added value.

The variety of puzzles discussed in the book is truly amazing. Coffin begins with a few chapters on two-dimensional puzzles (including dissections and sliding block puzzles). Then he begins to discuss various types of three-dimensional puzzles, from burrs to polyhedral puzzles to blocks and pins.

Throughout the book, Coffin provides guidelines on the type of jigs needed by a wood-worker in order to cut many of the pieces used to build the puzzles. In addition to these guidelines interspersed through the book, he also has a chapter at the end on woodworking techniques, in which he goes so far as to discuss the type of power tools one should have, the types of wood that should be considered for finished pieces, and the kinds of glue that he himself prefers.

I highly recommend this book to all puzzle-lovers, but especially to those who wish to try their hand at puzzle creation. I expect to derive a lot of enjoyment from it in my retirement!

The Liar Paradox and the Towers of Hanoi:
the Ten Greatest Math Puzzles of All Time
By Marcel Danesi, John Wiley and Sons, Inc., 2004
ISBN 0-471-64816-7, softcover, 248+vii pages, CDN$22.99
Reviewed by **Amar Sodhi**, *Sir Wilfred Grenfell College, Corner Brook, NL*

If one was looking for a suitable textbook for a *Mathematics is Fun* course designed for non-mathematics majors, then *The Liar Paradox and the Towers of Hanoi* would certainly be worthy of consideration. Danesi has selected his version of "The ten greatest math problems of all time" as a conduit to discuss riddles, paradoxes, and a variety of puzzles. In doing so, deductive and inductive reasoning and other proof techniques are introduced in an informal and easy to understand fashion.

Danesi starts by using *The Riddle of the Sphinx* to demonstrate how "insight thinking" is indeed a problem-solving strategy. Alcuin's *River-Crossing Puzzle* is used to introduce techniques in counting and the search for patterns is explored via Fibonacci's *Rabbit Puzzle*. Graph theory and topology come to the forefront in chapters centred on Euler's *Königsberg Bridges Puzzle* and Guthrie's *Four Colour Problem*. Problems from antiquity, such as *The Lo Shu Magic Square* and *The Cretan Labyrinth*, as well as Loyd's relatively modern *Get off the Earth Puzzle* are also featured in the book.

The book contains many historical anecdotes and even more exercises. The author's informal style allows him to deviate freely from the chapter topic and mention a score or more other puzzles and theorems which would be of interest to a novice mathematician. Thus, this book is a useful addition to the library of any educator who is looking for enrichment material to spark the interest of an impressionable teenager.

Unfortunately, the book fails to live up to the publisher's claim that "die-hard puzzle mavens to math aficionados" will be enlightened, entertained, and impressed by this volume. The author, who has established a program for students with difficulties in mathematics, assumes that readers of this book may also have difficulties in mathematics. An example of Danesi's thoroughness is when he takes pains to explain why $n^2(n^2+1)$ can be simplified to n^4+n^2, though it is unclear why this alleged simplification makes computations easier. Also, the exercises, for the most part, are of either a routine or textbook nature. The historical anecdotes are both enlightening and interesting, but a die-hard puzzle maven or math aficionado should find this book a light read and would probably be more entertained and impressed with the numerous books and articles that the author has included in his bibliography.

Industrial Grade Primes with a Money-Back Guarantee

Michael P. Abramson

Abstract

A subset of the integers is exhibited for which the converse of Fermat's Little Theorem holds. Strong evidence is given that this set contains infinitely many primes, though a proof of this is known to be very hard.

Many methods for generating large prime numbers begin with Fermat's Little Theorem:

Theorem 1 (Fermat). If n is prime and a is relatively prime to n, then

$$a^{n-1} \equiv 1 \pmod{n} . \tag{1}$$

The converse statement is:

If $a^{n-1} \equiv 1 \pmod{n}$ *for some* a *relatively prime to* n*, then* n *is prime.*

The converse is false. However, it is false so rarely that Henri Cohen jokingly coined the term *industrial grade prime* to denote any number n for which $2^{n-1} \equiv 1 \mod n$ (see [5, p. 5]). The choice of $a = 2$ is what is usually done in practice, and even though we write a everywhere for generality, we are thinking of $a = 2$. Many prime generation methods start with this converse, and then add a secondary test to eliminate the composite numbers for which (1) holds. A composite number n which satisfies (1) for some a relatively prime to n is called a *base* a *pseudo-prime*. If n is a base a pseudo-prime for all a relatively prime to n, then n is called a *Carmichael number*. Pseudo-primes and Carmichael numbers have been studied extensively [4]; in fact, it was only recently proved that there are an infinite number of Carmichael numbers [1]. Instead of studying when composite numbers fail (1), we investigate the following dual question:

Question. When does satisfying (1) guarantee primality for n?

Or, in the colloquial language of [3], when does an industrial grade prime come with a money-back guarantee that it is prime? In this note, we will describe a classic method of prime number generation, and show when its secondary test can be eliminated, thus exhibiting a set of integers for which passing (1) guarantees primality. We then give strong evidence for the conjecture that our set contains infinitely many primes.

1 A Theorem from Hardy and Wright.

We present here a slight generalization of part of Theorem 101 in [2]. Before doing so, we recall the definition of the Euler ϕ-function: $\phi(n)$ is the number of positive integers less than or equal to n which are relatively prime to n. Euler showed that $a^{\phi(n)} \equiv 1 \pmod{n}$ for all a relatively prime to n. This is a generalization of Fermat's Little Theorem, because $\phi(n) = n - 1$ for prime n.

Theorem 2. Let $n = hp + 1$, where p is prime and h is an even positive integer such that $h < 4p + 4$. If n satisfies (1) for some a relatively prime to n and if

$$a^h \not\equiv 1 \pmod{n}\ , \tag{2}$$

then n is prime.

Proof: Assume n is not prime. Let x be the order of a modulo n; that is, x is the smallest positive integer such that $a^x \equiv 1 \pmod{n}$. Then x divides every integer k for which $a^k \equiv 1 \pmod{n}$. In particular, (1) and (2) imply that x divides hp but not h. This means that x is a multiple of p or, in other words, p divides x. Similarly, $a^{\phi(n)} \equiv 1 \pmod{n}$ implies that x divides $\phi(n)$ and, by transitivity, p divides $\phi(n)$. Let $n = q_1^{e_1} q_2^{e_2} \cdots q_k^{e_k}$ be the unique prime factorization of n. Using properties of Euler's ϕ-function, one can show that

$$\phi(n) = \prod_{i=1}^{k} q_i^{e_i - 1}(q_i - 1).$$

Since p does not divide n, p cannot divide any factor q_i of n. But p does divide $\phi(n)$; hence, p divides $q_i - 1$ for some i. Let P denote this prime q_i. Then P is a factor of n such that $P \equiv 1 \pmod{p}$. Since n is not prime, we must have $n = PM$ for some $M > 1$. Now $n \equiv 1 \equiv P \pmod{p}$ implies that $M \equiv 1 \pmod{p}$, and since n and p are odd, we see that $P \equiv 1 \equiv M \pmod{2p}$. Thus, $hp + 1 = n = PM = (2pu + 1)(2pv + 1)$ for some $u, v \geq 1$, so $h = 4puv + 2u + 2v \geq 4p + 4$, contradicting our hypothesis. Therefore, n is prime. ∎

We can generate large primes by iterating Theorem 2; that is, at each step, the newly found prime n plays the role of p in the next step, and h is randomly chosen until we find a new prime n.

2 Eliminating the Secondary Test.

The next theorem gives one way of eliminating the secondary test (2).

Theorem 3. Let $n = 2pq + 1$, where p and q are odd primes satisfying

$$\tfrac{1}{2}(p - 2) \ < \ q \ < \ 2(p + 1)\,. \tag{3}$$

Then n is prime if and only if (1) holds for some a with $a^2 \not\equiv 1 \pmod{n}$.

(Note that the condition $a^2 \not\equiv 1 \pmod{n}$ is really not restrictive, especially since $a = 2$ is usually used in practice.)

Proof: If n is prime, then (1) holds by Fermat's Little Theorem. Conversely, suppose (1) holds for some a with $a^2 \not\equiv 1 \pmod{n}$. If $p \equiv \pm 1 \pmod{3}$, then $2p^2+1 = 2(3k\pm1)^2+1 = 3(6k^2\pm4k+1)$ for some $k > 1$. Then since p is prime, $2p^2+1$ is prime if and only if $p = 3$. Hence, we may assume $p \neq q$. Writing $h = 2q$ and $k = 2p$, condition (3) implies $n = hp + 1 = kq + 1$ with $h < 4p + 4$ and $k < 4q + 4$. Suppose $a^h \equiv a^k \equiv 1 \pmod{n}$. Then $a^{2p} \equiv a^{2q} \equiv 1 \pmod{n}$, so $a^2 = a^{\gcd(2p,2q)} \equiv 1 \pmod{n}$, a contradiction. Therefore, $a^h \not\equiv 1 \pmod{n}$ or $a^k \not\equiv 1 \pmod{n}$. Either way, n is prime by Theorem 2. ■

Unfortunately, the method of generating primes suggested by iterating Theorem 3 may be impractical because two primes are needed to construct one new prime at each step. Practicality aside, if we let

$$\begin{aligned} \overline{M} &= \{2pq+1 : p \text{ and } q \text{ are odd primes}\} \\ M &= \{2pq+1 \in \overline{M} : p \text{ and } q \text{ satisfy (3)}\}, \end{aligned}$$

then M has the property that a given integer n in M is prime if and only if (1) holds. In other words, M contains no base a pseudo-primes. In order for the set M to be of any real interest, M must contain infinitely many primes.

Conjecture. The sets M and $\overline{M}$ contain infinitely many primes.

3 Evidence for the Conjecture.

Let p_k represent the k^{th} odd prime. Figure 1 depicts M and $\overline{M}$, where the axes represent the indices of the odd primes p_i and p_j. M is then represented by the shaded region and $\overline{M}$ is represented by the entire (square) region inside the axes. Theorem 3 asserts that the primality of integers of the form $n = 2p_ip_j + 1$, where (i, j) lies in the shaded region of Figure 1, can be determined using only Fermat's Little Theorem.

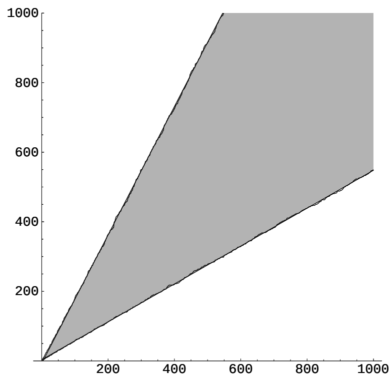


Figure 1: The region (3)

Since primes tend to be uniformly distributed in intervals, we expect that the ratio of the number of primes in M to the number of primes in $\overline{M}$ should be close to the ratio of the areas of M (the shaded region) to $\overline{M}$ (the whole square). Let $R(t)$ be this ratio of areas of M and $\overline{M}$, where t is the size of the square ($t = 1000$ in Figure 1). We approximate the boundaries of the shaded area in Figure 1 by (least-squares) lines through the origin $j = a_1(t)i$ and $j = a_2(t)i$, where a_1 and a_2 are dependent on t. Then we can approximate the desired area of the shaded region by subtracting the areas of the two triangles, $\frac{a_1(t)t^2}{2}$ and $\frac{t^2}{2a_2(t)}$, from the area t^2 of the square to obtain $t^2\left(1 - \frac{a_1(t)}{2} - \frac{1}{2a_2(t)}\right)$. Thus,

$$R(t) \approx 1 - \frac{a_1(t)}{2} - \frac{1}{2a_2(t)}.$$

We note here that although $R(t)$ is dependent on t the dependence is very small: as t increases, the slopes a_1 and a_2 will change only slightly. For $t = 1000$ as depicted in Figure 1, $a_1 = .5645$ and $a_2 = 1.788$. Thus, $R(1000) \approx .438$. Let $R_p(t)$ be the true ratio of the number of primes in M to the number of primes in $\overline{M}$. $R_p(t)$ can be easily computed for specific values of t. For example, $R_p(1000) \approx .422$. Figure 2 below shows both R and R_p as t increases up to $t = 1000$. Note that $R(t)$ looks like a horizontal line because of the very slight dependence on t.

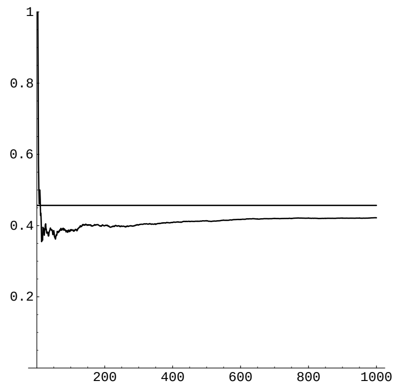


Figure 2: The functions R and R_p

Although these approximations are somewhat crude, the point of this discussion is not the actual values of the ratios, but rather that the ratios appear to be positive numbers, well away from zero. If we could prove rigorously that $\lim_{t\to\infty} R_p(t) > 0$, then this would imply that M and $\overline{M}$ both contain finitely many primes or both contain infinitely many primes.

We now introduce two functions to study the growth of these primes. Let $Q(k)$ be the number of odd primes q such that (3) holds (with $p = p_k$) and $2p_kq + 1$ is prime. We also need a function that counts all such primes in M up to p_k. To prevent over-counting, we define $\overline{Q}(k)$ to be the number of odd primes q such that $\frac{1}{2}(p_k - 2) < q < p_k$ and $2p_kq + 1$ is prime and let $\sigma(k) = \sum_{i=1}^{k} \overline{Q}(i)$. Figure 3 on the next page shows the functions $Q(k)$

and $\sigma(k)$ for $k \leq 2000$. The apparent growth of these functions leads to the stronger conjecture that $\sigma(k) \geq k$ for all $k > 0$; we have verified this statement computationally for the first 100,000 odd primes p_k.

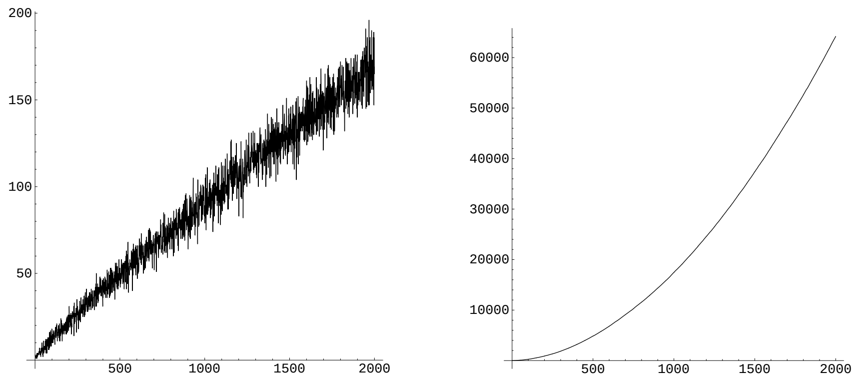


Figure 3: The functions $Q(k)$ and $\sigma(k)$

In this paper, we have exhibited a set of integers whose primality can be determined using only Fermat's Little Theorem. Although we have not proved that there are infinitely many primes in this set, we have given strong experimental evidence for it. Proving there are infinitely many primes of a form which is not linear is generally a very hard problem that requires deep results from analytic number theory.

Acknowledgments. The author thanks Robert L. Ward, who generalized Theorem 2 as stated, and Ezra Brown for kind comments on expository style.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 October 2008**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3326. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers.

(a) Show that $\prod\limits_{\text{cyclic}}(a^2+2)+4\prod\limits_{\text{cyclic}}(a^2+1) \;\geq\; 6(a+b+c)^2$.

(b)★ What is the largest constant k such that

$$\prod_{\text{cyclic}}(a^2+2)+4\prod_{\text{cyclic}}(a^2+1) \;\geq\; k(a+b+c)^2\,?$$

3327. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers.

(a) Show that $\prod\limits_{\text{cyclic}}(a^4+3a^2+2) \;\geq\; \frac{9}{4}(a+b+c)^4$.

(b)★ What is the largest constant k such that

$$\prod_{\text{cyclic}}(a^4+3a^2+2) \;\geq\; k(a+b+c)^4\,?$$

3328★. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. For $1 \leq k \leq n$, define

$$A_k \;=\; \frac{1}{k}\sum_{i=1}^{k} a_i\,, \quad G_k \;=\; \left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k}}, \quad \text{and} \quad H_k \;=\; k\left(\sum_{i=1}^{k}\frac{1}{a_i}\right)^{-1}.$$

(a) Show that $\frac{1}{n}\sum\limits_{k=1}^{n} G_k \;\leq\; \left(\prod\limits_{k=1}^{n} A_k\right)^{\frac{1}{n}}$.

(b) Show that $n\left(\sum\limits_{k=1}^{n}\frac{1}{G_k}\right)^{-1} \;\geq\; \left(\prod\limits_{k=1}^{n} H_k\right)^{\frac{1}{n}}$.

3329. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r be a real number, $0 < r \leq 1$, and let x, y, and z be positive real numbers such that $xyz = r^3$. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \;\leq\; \frac{3}{\sqrt{1+r^2}}\,.$$

3330. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let n be a natural number, let r be a real number, and let $a_1, a_2, \ldots, a_n$ be positive real numbers satisfying $\prod\limits_{k=1}^{n} a_k = r^n$; prove that

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^3} \;\geq\; \frac{n}{(1+r)^3}\,,$$

(a) for $n = 2$ if and only if $r \geq \frac{1}{3}$;

(b) for $n = 3$ if $r \geq \frac{1}{\sqrt[3]{4}}$;

(c) for $n = 4$ if $r \geq \frac{1}{\sqrt[3]{4}}$;

(d) for $n \geq 5$ if and only if $r \geq \sqrt[3]{n} - 1$.

3331. *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be the lengths of the sides of triangle ABC, and let R be its circumradius. Prove that

$$\sqrt[3]{a^2b} + \sqrt[3]{b^2c} + \sqrt[3]{c^2a} \;\leq\; 3\sqrt{3}\,R\,.$$

3332. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a_1, a_2, a_3, and a_4 be positive real numbers and let λ and μ be positive integers.

(a) Prove that

$$\frac{a_1}{\lambda a_2 + \mu a_3} + \frac{a_2}{\lambda a_3 + \mu a_1} + \frac{a_3}{\lambda a_1 + \mu a_2} \;\geq\; \frac{3}{\lambda + \mu}\,.$$

(b) Prove that

$$\frac{a_1}{\mu a_2 + \mu a_3 + \mu a_4} + \frac{a_2}{\lambda a_3 + \lambda a_4 + \lambda a_1} + \frac{a_3}{\mu a_4 + \lambda a_1 + \mu a_2} + \frac{a_4}{\lambda a_1 + \mu a_2 + \lambda a_3} \;\geq\; \frac{8}{3(\lambda + \mu)}\,.$$

3333. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

The n points P_1, P_2, $\ldots$, P_n, labelled counterclockwise about a circle, form a convex n-gon Q. Denote by P_i' the point where the interior angle bisector at P_i intersects the circle. Suppose that the points P_i' determine another convex, cyclic n-gon Q', whose interior angle bisectors intersect the circle in the vertices of Q''. In this manner, we construct a sequence of convex, cyclic n-gons $Q^{(k)}$. For which values of n can we start with an n-gon that is not equiangular and arrive in k steps at an equiangular n-gon $Q^{(k)}$?

3334. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

(a) Prove that

$$\sum_{n=0}^{\infty} \frac{\sum\limits_{k=1}^{n+1} \frac{(-1)^{k-1}\binom{n+1}{k}}{k}}{(n+1)^2} = 2\zeta(3)\,.$$

(b) Prove that

$$\sum_{n=0}^{\infty} \frac{\sum\limits_{k=1}^{n+1} \frac{(-1)^{k-1}\binom{n+1}{k}}{k^2}}{(n+1)^2} = \frac{\pi^4}{30} = 3\zeta(4)\,.$$

[The function ζ is the Riemann Zeta Function: $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$.]

3335. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let a and b be positive real numbers with $a < b$.

(a) Prove that

$$\frac{\ln b - \ln a}{b - a + 1} > \frac{b^{b-a} - a^{b-a}}{b^{b-a+1} - a^{b-a+1}}\,.$$

(b) Prove that

$$\int_a^b (x-a)^b (b-x)^a \, dx < \frac{1}{a+b+1}\left(b^{a+b+1} - a^{a+b+1}\right)\left(\frac{b-a}{b+a}\right)^{a+b}\,.$$

3336. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle, and let B_1 and B_2 be points on AC and C_1 and C_2 be points on AB such that $AB_1 = CB_2$, $AC_1 = BC_2$, and B_1C_2 intersects B_2C_1 at a point P in the interior of $\triangle ABC$. If $[KLM]$ denotes the area of $\triangle KLM$, show that

$$[PCB] > [PCA] + [PAB]\,.$$

3337. *Proposed by Michel Bataille, Rouen, France.*

In the plane of $\triangle ABC$, what is the locus of points P such that the circumradii of $\triangle PBC$, $\triangle PCA$, and $\triangle PAB$ are all equal?

. . . .

3326. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs.

(a) Montrer que $\prod_{\text{cyclique}} (a^2+2) + 4 \prod_{\text{cyclique}} (a^2+1) \ \geq\ 6(a+b+c)^2$.

(b)★ Quelle est la plus grande constante k telle que

$$\prod_{\text{cyclique}} (a^2+2) + 4 \prod_{\text{cyclique}} (a^2+1) \ \geq\ k(a+b+c)^2\,?$$

3327. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs.

(a) Montrer que $\prod_{\text{cyclique}} (a^4+3a^2+2) \ \geq\ \frac{9}{4}(a+b+c)^4$.

(b)★ Quelle est la plus grande constante k telle que

$$\prod_{\text{cyclique}} (a^4+3a^2+2) \ \geq\ k(a+b+c)^4\,?$$

3328★. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $a_1, a_2, \ldots, a_n$ des nombres réels positifs. Si $1 \leq k \leq n$, on définit

$$A_k \ =\ \frac{1}{k}\sum_{i=1}^{k} a_i\,, \quad G_k \ =\ \left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k}} \quad \text{et} \quad H_k \ =\ k\left(\sum_{i=1}^{k} \frac{1}{a_i}\right)^{-1}.$$

(a) Montrer que $\frac{1}{n}\sum_{k=1}^{n} G_k \ \leq\ \left(\prod_{k=1}^{n} A_k\right)^{\frac{1}{n}}$.

(b) Montrer que $n\left(\sum_{k=1}^{n} \frac{1}{G_k}\right)^{-1} \ \geq\ \left(\prod_{k=1}^{n} H_k\right)^{\frac{1}{n}}$.

3329. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit r un nombre réel, $0 < r \leq 1$, et soit x, y et z trois nombres réels positifs tels que $xyz = r^3$. Montrer que

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \ \leq\ \frac{3}{\sqrt{1+r^2}}\,.$$

3330. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit n un nombre naturel, r un nombre réel, et a_1, a_2, $\dots$, a_n des nombres réels positifs satisfaisant $\prod\limits_{k=1}^{n} a_k = r^n$; montrer que

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^3} \;\geq\; \frac{n}{(1+r)^3}\,,$$

(a) pour $n = 2$ si et seulement si $r \geq \frac{1}{3}$;

(b) pour $n = 3$ si $r \geq \frac{1}{\sqrt[3]{4}}$;

(c) pour $n = 4$ si $r \geq \frac{1}{\sqrt[3]{4}}$;

(d) pour $n \geq 5$ si et seulement si $r \geq \sqrt[3]{n} - 1$.

3331. *Proposé par José Gibergans-Báguena et José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b et c les longueurs des côtés du triangle ABC dont le rayon du cercle circonscrit est R. Montrer que

$$\sqrt[3]{a^2b} + \sqrt[3]{b^2c} + \sqrt[3]{c^2a} \;\leq\; 3\sqrt{3}\,R\,.$$

3332. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit a_1, a_2, a_3 et a_4 quatre nombres réels positifs, et soit λ et μ des entiers positifs.

(a) Montrer que

$$\frac{a_1}{\lambda a_2 + \mu a_3} + \frac{a_2}{\lambda a_3 + \mu a_1} + \frac{a_3}{\lambda a_1 + \mu a_2} \;\geq\; \frac{3}{\lambda + \mu}\,.$$

(b) Montrer que

$$\begin{aligned}&\frac{a_1}{\mu a_2 + \mu a_3 + \mu a_4} + \frac{a_2}{\lambda a_3 + \lambda a_4 + \lambda a_1}\\ &\qquad + \frac{a_3}{\mu a_4 + \lambda a_1 + \mu a_2} + \frac{a_4}{\lambda a_1 + \mu a_2 + \lambda a_3} \;\geq\; \frac{8}{3(\lambda + \mu)}\,.\end{aligned}$$

3333. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Les n points P_1, P_2, $\dots$, P_n, étiquetés dans le sens antihoraire sur un cercle forment un n-gone convexe Q. Désignons par P_i' le point d'intersection de la bissectrice intérieure en P_i avec le cercle. Supposons que les points P_i' forment à leur tour un n-gone circulaire convexe Q', dont les bissectrices intérieures coupent le cercle aux sommets de Q''. Ce processus donne lieu à une suite de n-gones circulaires convexes $Q^{(k)}$. Pour quelles valeurs de n peut-on commencer avec un n-gone non régulier pour aboutir en k étapes à un n-gone régulier $Q^{(k)}$?

3334. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

(a) Montrer que

$$\sum_{n=0}^{\infty} \frac{\sum_{k=1}^{n+1} \frac{(-1)^{k-1}\binom{n+1}{k}}{k}}{(n+1)^2} \;=\; 2\zeta(3)\,.$$

(b) Montrer que

$$\sum_{n=0}^{\infty} \frac{\sum_{k=1}^{n+1} \frac{(-1)^{k-1}\binom{n+1}{k}}{k^2}}{(n+1)^2} \;=\; \frac{\pi^4}{30} \;=\; 3\zeta(4)\,.$$

[La fonction ζ est la fonction zéta de Riemann : $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.]

3335. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit a et b deux nombres réels positifs tels que $a < b$.

(a) Montrer que

$$\frac{\ln b - \ln a}{b-a+1} \;>\; \frac{b^{b-a} - a^{b-a}}{b^{b-a+1} - a^{b-a+1}}\,.$$

(b) Montrer que

$$\int_a^b (x-a)^b (b-x)^a \, dx \;<\; \frac{1}{a+b+1}\left(b^{a+b+1} - a^{a+b+1}\right)\left(\frac{b-a}{b+a}\right)^{a+b}.$$

3336. *Proposé par Michel Bataille, Rouen, France.*

Dans un triangle ABC, soit B_1 et B_2 deux points sur AC, C_1 et C_2 deux points sur AB, de sorte que $AB_1 = CB_2$, $AC_1 = BC_2$ et que B_1C_2 coupe B_2C_1 en un point P à l'intérieur du triangle. Si $[KLM]$ désigne l'aire du triangle KLM, montrer que

$$[PCB] \;>\; [PCA] + [PAB]\,.$$

3337. *Proposé par Michel Bataille, Rouen, France.*

Dans le plan du triangle ABC, quel est le lieu des points P tels que les rayons des cercles circonscrits aux triangles PBC, PCA et PAB soient tous égaux ?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We have received a late solution to problem #3203 from Konstantine Zelator, University of Toledo, Toledo, OH, USA, and to problem #3221 from Pavlos Maragoudakis, Pireas, Greece.

3226. [2007 : 169, 172] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let ABC be a triangle. Let $S = \sum_{\text{cyclic}} \cos\frac{A}{2}$ and $P = \prod_{\text{cyclic}} \cos\frac{A}{2}$. Prove that

(a) $\frac{S}{P} \leq 2\sqrt{3}\max\left\{\sec\frac{A}{2},\ \sec\frac{B}{2},\ \sec\frac{C}{2}\right\}$;

(b) $\frac{S}{P} \geq 4\max\left\{\sec^2\frac{B-C}{4},\ \sec^2\frac{A-B}{4},\ \sec^2\frac{C-A}{4}\right\}$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

(a) Let α, β, and γ be defined by $A = 2\alpha$, $B = 2\beta$, and $C = 2\gamma$. Thus,

$$\frac{S}{P} = \Phi = \frac{\cos\alpha + \cos\beta + \cos\gamma}{\cos\alpha\cos\beta\cos\gamma},$$

and we must prove that

$$\Phi \leq 2\sqrt{3}\max\{\sec\alpha,\ \sec\beta,\ \sec\gamma\}. \tag{1}$$

Without loss of generality, we may assume $\alpha \geq \beta \geq \gamma$, so that $\sec\alpha \geq \sec\beta \geq \sec\gamma$. Inequality (1) is successively equivalent to

$$\begin{aligned} \cos\alpha + \cos\beta + \cos\gamma &\leq 2\sqrt{3}\cos\beta\cos\gamma, \\ \sin(\beta+\gamma) + \cos\beta + \cos\gamma &\leq 2\sqrt{3}\cos\beta\cos\gamma, \end{aligned}$$

which expands to

$$\begin{aligned} &\sin(\beta+\gamma) + 2\cos\left(\tfrac{1}{2}(\beta+\gamma)\right)\cos\left(\tfrac{1}{2}(\beta-\gamma)\right) \\ &\qquad\qquad \leq \sqrt{3}\left[\cos(\beta-\gamma) + \cos(\beta+\gamma)\right]. \end{aligned}$$

Setting $2p = \beta+\gamma$ and $2q = \beta-\gamma$, we see that this inequality is successively equivalent to

$$\begin{aligned}
\sin(2p) + 2\cos p \cos q &\le \sqrt{3}\big[\cos(2q) + \cos(2p)\big]\,, \\
2\sin p \cos p + 2\cos p \cos q &\le \sqrt{3}(2\cos^2 q + 2\cos^2 p - 2)\,, \\
\cos p(\sin p + \cos q) &\le \sqrt{3}(\cos^2 q - \sin^2 p) \\
&= \sqrt{3}(\cos q + \sin p)(\cos q - \sin p)\,, \\
\cos p &\le \sqrt{3}(\cos q - \sin p)\,, \\
\cos p + \sqrt{3}\sin p &\le \sqrt{3}\cos q\,.
\end{aligned}$$

Multiplying the last inequality by $\frac{1}{2}$, we see that we need to show

$$\sin\left(\frac{\pi}{6} + p\right) \le \frac{\sqrt{3}}{2}\cos q\,. \tag{2}$$

Since α is the largest of the three angles and $\alpha + \beta + \gamma = \pi$, we must have $\alpha \ge \frac{\pi}{6}$; hence, $\beta+\gamma \le \frac{\pi}{3}$. Therefore, $p = \frac{1}{2}(\beta+\gamma) \le \frac{\pi}{6}$. We now distinguish two cases.

Case 1. $0 \le p \le \frac{\pi}{8}$.
Then

$$\sin\left(\frac{\pi}{6} + p\right) \le \sin\left(\frac{\pi}{6} + \frac{\pi}{8}\right) = \sin\left(\frac{7\pi}{24}\right) \approx 0.793 < \frac{4}{5}\,.$$

Since $q \le p$, we have

$$\frac{\sqrt{3}}{2}\cos q \ge \frac{\sqrt{3}}{2}\cos\frac{\pi}{8} > \frac{4}{5}\,.$$

Thus, inequality (2) holds.

Case 2. $\frac{\pi}{8} < p \le \frac{\pi}{6}$.
Then

$$\begin{aligned}
\beta - \gamma &= 2\beta - 2p \le 2\alpha - 2p \\
&= 2\left(\frac{\pi}{2} - 2p\right) - 2p = \pi - 6p\,;
\end{aligned}$$

hence, $q \le \frac{1}{2}(\pi - 6p)$. Therefore,

$$\frac{\sqrt{3}}{2}\cos q \ge \frac{\sqrt{3}}{2}\cos\left(\frac{\pi - 6p}{2}\right) = \frac{\sqrt{3}}{2}\sin(3p)\,.$$

It thus suffices to show that

$$\sin\left(\frac{\pi}{6} + p\right) \le \frac{\sqrt{3}}{2}\sin(3p)\,.$$

This is successively equivalent to

$$\begin{aligned}
\frac{1}{2}\cos p + \frac{\sqrt{3}}{2}\sin p &\le \frac{\sqrt{3}}{2}(3\sin p - 4\sin^3 p)\,, \\
\cos p &\le 2\sqrt{3}\sin p - 4\sqrt{3}\sin^3 p \\
&= 2\sqrt{3}\sin p(1 - 2\sin^2 p)\,, \\
1 - \sin^2 p = \cos^2 p &\le 12\sin^2 p(1 - 2\sin^2 p)^2\,.
\end{aligned}$$

Setting $u = \sin^2 p$, this is successively equivalent to

$$\begin{aligned} 1-u &\le 12u(1-2u)^2\,,\\ 48u^3-48u^2+13u-1 &\ge 0\,,\\ (4u-1)(12u^2-9u+1) &\ge 0\,. \end{aligned}$$

For $\frac{\pi}{8} < p \le \frac{\pi}{6}$, we have $\frac{2-\sqrt{2}}{4} < u \le \frac{1}{4}$, and both factors in the last inequality above are non-positive for such u, proving inequality (2).

(b) Using the notation from part (a), we must prove that

$$\Phi \ge 4\max\left\{\sec^2\frac{\beta-\gamma}{2},\ \sec^2\frac{\alpha-\beta}{2},\ \sec^2\frac{\gamma-\alpha}{2}\right\}\,. \tag{3}$$

We may assume, without loss of generality, that

$$\max\left\{\sec^2\frac{\beta-\gamma}{2}\,,\ \sec^2\frac{\alpha-\beta}{2}\,,\ \sec^2\frac{\gamma-\alpha}{2}\right\} \;=\; \sec^2\frac{\beta-\gamma}{2}\,.$$

Thus, we see that the inequality (3) is equivalent successively to

$$\begin{aligned} \cos\alpha+\cos\beta+\cos\gamma &\ge 4\sec^2\frac{\beta-\gamma}{2}\cos\alpha\cos\beta\cos\gamma\,,\\ \sin(2p)+\cos\beta+\cos\gamma &\ge 4\sec^2 q\sin(2p)\cos\beta\cos\gamma\,,\\ \sin(2p)+2\cos p\cos q &\ge 2\sec^2 q\sin(2p)[\cos(\beta-\gamma)+\cos(\beta+\gamma)]\\ &= 2\sec^2 q\sin(2p)[\cos(2q)+\cos(2p)]\,,\\ 2\sin p\cos p+2\cos p\cos q &\ge 4\sin p\cos p\sec^2 q(2\cos^2 q-2\sin^2 p)\,,\\ \cos p(\sin p+\cos q) &\ge 4\sin p\cos p\sec^2 q(\cos^2 q-\sin^2 p)\,,\\ \cos^2 q(\sin p+\cos q) &\ge 4\sin p(\cos^2 q-\sin^2 p)\\ &= 4\sin p(\cos q+\sin p)(\cos q-\sin p)\,,\\ \cos^2 q &\ge 4\sin p(\cos q-\sin p)\,, \end{aligned}$$

which is equivalent to $(\cos q - 2\sin p)^2 \ge 0$. Since this inequality is always true, the inequality (3) holds.

Also solved by ARKADY ALT, San Jose, CA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; and the proposer. There was also one incorrect solution submitted.

3228. [2007 : 169, 172] *Proposed by Mihály Bencze, Brasov, Romania.*

For $x \in \left(0, \frac{\pi}{2}\right)$, prove that

$$\frac{(n+1)!}{2\prod\limits_{k=2}^{n}(k+\cos x)} \;\le\; \left(\frac{x}{\sin x}\right)^{n-1} \;\le\; \left(\frac{\pi}{2}\right)^{n-1}\cdot\frac{n!}{\prod\limits_{k=2}^{n}(k+\cos x)}\,.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

For $k \geq 2$, let $f_k(x) = k\sin x + \sin x\cos x - x - kx\cos x$. Then $f_k(0) = 0$, and for all $x \in (0, \pi/2)$, we have

$$\begin{aligned} f_k'(x) &= k\cos x - \sin^2 x + \cos^2 x - 1 + kx\sin x - k\cos x \\ &= kx\sin x - 2\sin^2 x \;=\; (kx - 2\sin x)\sin x \;>\; 0\,, \end{aligned}$$

since $0 < \sin x < x$ on $(0, \pi/2)$. Hence, $f_k(x) > 0$ on $(0, \pi/2)$ for all $k \geq 2$.

Next, define $g_k(x) = \dfrac{x(k+\cos x)}{\sin x}$ for $x \in (0, \pi/2]$. Then

$$\begin{aligned} g_k'(x) &= \frac{(-x\sin x + k + \cos x)\sin x - x(k+\cos x)\cos x}{\sin^2 x} \\ &= \frac{f_k(x)}{\sin^2 x} \;>\; 0\,, \end{aligned}$$

which implies that $g_k(x) \geq \lim\limits_{t\to 0+} g_k(t) = k+1$ for $x \in (0, \pi/2)$.

Therefore, for $x \in (0, \pi/2)$, and $n \geq 2$, we have

$$\prod_{k=2}^{n}(k+1) \;\leq\; \prod_{k=2}^{n}\frac{x(k+\cos x)}{\sin x} \;\leq\; \prod_{k=2}^{n}\frac{k\pi}{2}\,;$$

that is,

$$\frac{(n+1)!}{2} \;\leq\; \left(\frac{x}{\sin x}\right)^{n-1}\prod_{k=2}^{n}(k+\cos x) \;\leq\; \left(\frac{\pi}{2}\right)^{n-1} n!\,,$$

from which the result follows immediately.

Also solved by MICHEL BATAILLE, Rouen, France; and the proposer.

3229. [2007 : 170, 172] *Proposed by Mihály Bencze, Brasov, Romania.*

(a) Let x and y be positive real numbers, and let n be a positive integer. Prove that

$$(x+y)^n\sum_{k=0}^{n}\frac{1}{\binom{n}{k}x^{n-k}y^k} \;\geq\; n+1+2\sum_{i=1}^{n}\sum_{k=0}^{n-i}\frac{\binom{n}{k}}{\binom{n}{k+i}} \;\geq\; (n+1)^2.$$

(b)★ Let x_1, x_2, $\ldots$, x_k be positive real numbers, and let n be a positive integer. Determine the minimum value of

$$(x_1+x_2+\cdots+x_k)^n\sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k\geq 0}}\frac{i_1!i_2!\cdots i_k!}{n!x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}}\,.$$

Solution to (a) *by Michel Bataille, Rouen, France.*

Let L be the leftmost expression, and let $a = y/x$. Then we have

$$\begin{aligned}
L &= \sum_{k=0}^{n} \frac{(x+y)^n}{\binom{n}{k} x^{n-k} y^k} = \sum_{k=0}^{n} \frac{(1+a)^n}{\binom{n}{k} a^k} = \sum_{j=0}^{n} \sum_{k=0}^{n} \frac{\binom{n}{j}}{\binom{n}{k}} a^{j-k} \\
&= n+1+\sum_{j=1}^{n} \sum_{k=0}^{j-1} \frac{\binom{n}{j}}{\binom{n}{k}} a^{j-k} + \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} \frac{\binom{n}{j}}{\binom{n}{k}} \frac{1}{a^{k-j}} \\
&= n+1+\sum_{j=1}^{n} \sum_{k=0}^{j-1} \frac{\binom{n}{j}}{\binom{n}{k}} a^{j-k} + \sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \frac{\binom{n}{j}}{\binom{n}{j+i}} \frac{1}{a^{i}}\,.
\end{aligned}$$

We further note that

$$\begin{aligned}
\sum_{j=1}^{n} \sum_{k=0}^{j-1} \frac{\binom{n}{j}}{\binom{n}{k}} a^{j-k} &= \sum_{r=0}^{n-1} \sum_{k=0}^{n-r-1} \frac{\binom{n}{n-r}}{\binom{n}{k}} a^{n-r-k} \\
&= \sum_{r=0}^{n-1} \sum_{i=1}^{n-r} \frac{\binom{n}{n-r}}{\binom{n}{n-r-i}} a^{i} \\
&= \sum_{r=0}^{n-1} \sum_{i=1}^{n-r} \frac{\binom{n}{r}}{\binom{n}{r+i}} a^{i} = \sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \frac{\binom{n}{j}}{\binom{n}{j+i}} a^{i}\,,
\end{aligned}$$

from which it follows that

$$L = n+1+\sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \frac{\binom{n}{j}}{\binom{n}{j+i}} \left(a^i + \frac{1}{a^i}\right) \geq n+1+2\sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \frac{\binom{n}{j}}{\binom{n}{j+i}}$$

(using the well-known inequality: $\alpha + \frac{1}{\alpha} \geq 2$ for positive α). It just remains to note that, changing the order of summation, we get

$$\sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \frac{\binom{n}{j}}{\binom{n}{j+i}} = \sum_{i=1}^{n} \sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}}\,.$$

As for the right inequality, we observe that it can be rewritten as

$$\sum_{i=1}^{n} \sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \geq \frac{n(n+1)}{2}\,.$$

Since $\frac{1}{2}n(n+1) = 1+2+\cdots+n$, it suffices to show that

$$\sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \;\geq\; n+1-i\,. \tag{1}$$

Now let S denote the sum on the left in (1). Letting $j = n-i-k$, we get

$$S \;=\; \sum_{j=0}^{n-i} \frac{\binom{n}{n-i-j}}{\binom{n}{n-j}}\,.$$

Then

$$2S \;=\; \sum_{k=0}^{n-i}\left(\frac{\binom{n}{k}}{\binom{n}{k+i}} + \frac{\binom{n}{n-i-k}}{\binom{n}{n-k}}\right) \;=\; \sum_{k=0}^{n-i}\left(\frac{\binom{n}{k}}{\binom{n}{k+i}} + \frac{\binom{n}{i+k}}{\binom{n}{k}}\right)\,,$$

a sum in which each of the $n-i+1$ terms is greater than or equal to 2. Thus, $2S \geq (n-i+1)\cdot 2$ and (1) follows.

Note that the central term of the proposed inequalities is the minimum value of L attained when $a=1$; that is, when $x=y$.

Also solved by ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; and the proposer.

None of the solvers offered any insight into part (b). *Part* (b) *remains open.*

3230. [2007 : 170, 173] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, x, and y be positive real numbers. Prove that

$$\begin{aligned}\left(x^{a+1}+x+y\right)\left(y^{a+1}+y+x\right)\left(x^{a+1}+(x^a+1)y\right)\left(y^{a+1}+(y^a+1)x\right)&\\ \geq\; (xy)^a\left(x+\sqrt{xy}+y\right)^4\,.&\end{aligned}$$

I. *Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Setting $s=x^a$, $t=y^a$, $u=x+y$, $v=\sqrt{xy}$, and $w=\sqrt{st}$, the given inequality can be written as:

$$(sx+u)(ty+u)(su+y)(tu+x) \;\geq\; st(u+v)^4\,. \tag{1}$$

By the Cauchy–Schwarz Inequality, we have

$$(sx+u)(ty+u) \;\geq\; \left(\sqrt{stxy}+u\right)^2$$

and

$$(su+y)(tu+x) \;\geq\; \left(\sqrt{stu^2}+v\right)^2,$$

which imply that

$$(sx+u)(ty+u)(su+y)(tu+x) \ \geq \ \Big(\big(\sqrt{stxy}+u\big)\big(\sqrt{stu^2}+v\big)\Big)^2 .$$

Hence, to prove (1) it suffices to show that

$$\big(\sqrt{stxy}+u\big)\big(\sqrt{stu^2}+v\big) \ \geq \ \sqrt{st}(u+v)^2 . \qquad (2)$$

We now observe that (2) is equivalent, in succession, to

$$\begin{aligned} (vw+u)(uw+v) &\geq w(u+v)^2 , \\ uvw^2+u^2w+v^2w+uv &\geq u^2w+2uvw+v^2w , \\ uv(w^2-2w+1) &\geq 0 , \\ uv(w-1)^2 &\geq 0 . \end{aligned}$$

This completes the proof.

II. *Solution by the proposer.*

For any $z>0$, by applying the AM–GM Inequality, we have

$$\begin{aligned} (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) &= 3+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{z}{y}+\frac{x}{z}\right)+\left(\frac{y}{z}+\frac{z}{x}\right) \\ &\geq 3+\frac{x}{y}+\frac{y}{x}+2\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{x}} \\ &= \left(1+\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)^2 . \end{aligned}$$

Setting $z=x^{a+1}$, we then have

$$\left(x^{a+1}+x+y\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{x^{a+1}}\right) \ \geq \ \left(1+\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)^2 .$$

Multiplying both sides by $x^{a+1}y$, we obtain

$$\begin{aligned} \left(x^{a+1}+x+y\right)\left(x^{a+1}+(x^a+1)y\right) &\geq \left(\sqrt{x^{a+1}y}+\sqrt{x^{a+2}}+\sqrt{x^a}y\right)^2 \\ &= x^a\left(x+\sqrt{xy}+y\right)^2 . \qquad (1) \end{aligned}$$

By symmetry, we also have

$$\left(y^{a+1}+y+x\right)\left(y^{a+1}+(y^a+1)x\right) \ \geq \ y^a\left(y+\sqrt{yx}+x\right)^2 . \qquad (2)$$

The desired result follows by multiplying (1) and (2).

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Clearly the condition $a>0$ is not necessary. This was explicitly pointed out by Malikić.

3231★. [2007 : 170, 173] *Proposed by Ignotus, Tauramena, Casanare, Colombia.*

(a) A flea lives on the real number line at the number 1. One fine day it decides to take an n–day vacation. On the first day it jumps forward one unit landing at the number 2. Thereafter, for the remaining $n - 1$ days, it jumps forward a number of units of its choice, as long as the number of units is a proper divisor of the number it is currently visiting. A sample 11–day vacation is

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 15,\ 20,\ 30,\ 45,\ 60.$$

What is the furthest away from home the flea can get during its n–day vacation? Note that the 11–day vacation above does not get the flea as far as possible; here is one that gets the flea further:

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 81.$$

(b) Suppose the flea wishes to visit, under the same rules as in (a), a certain number n. What is the least number, $V(n)$, of vacation days it will need to get there? For example, here is a scheme to get the flea to the number 100 in 13 days:

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 72,\ 96,\ 100.$$

Comment: We have not received any satisfactory solution to this problem. Indeed, we received only one attempt at a solution; this attempt contained an admittedly unproved claim. The problem remains open.

3232. [2007 : 170, 173] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let P be a point in the interior of $\angle QOR$. Find the segment AB of minimum length which contains P with A on the ray OQ and B on the ray OR.

Solved by Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Richard I. Hess, Rancho Palos Verdes, CA, USA; Václav Konečný, Big Rapids, MI, USA; and the proposer.

This is a well-known problem. The answer is the Philo line. Bataille gives a reference to [1, p. 77], for the proof, and [1, p. 1285] for information about the impossibility of constructing the Philo line with ruler and compass. The text contains further references, including one that goes back to Newton. Kǒnecný provides the reference [2], and the moderator of this problem found it as problem 66 in [3, p. 33].

References

[1] F. Gabriel-Marie, *Exercices de géométrie*, Maison A. Mame et fils, 6e édition, 1920.

[2] Eric W. Weisstein, *Philo Line*, from MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/PhiloLine.html`

[3] D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *Geometricheskie neravenstva i zadachi na maximum i minimum*, Moskva, Nauka, 1970.

3233. [2007 : 171, 173] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2A_3$ be a triangle, and let P be an interior point. The cevian A_iP intersects the opposite side at A_i' for $1 \le i \le 3$. If $[XYZ]$ denotes the area of triangle XYZ, set $\Delta_1 = [PA_2A_1']$, $\Delta_2 = [PA_3A_2']$, $\Delta_3 = [PA_1A_3']$, and $\Delta = [A_1A_2A_3]$. Find the locus of P if $\Delta_1 + \Delta_2 + \Delta_3 = \frac{1}{2}\,\Delta$.

Solution by Joel Schlosberg, Bayside, NY, USA.

Let the barycentric coordinates of P be $x = [PA_2A_3]$, $y = [PA_3A_1]$, and $z = [PA_1A_2]$, and let us normalize them by taking Δ to be 1:

$$\Delta \;= x + y + z \;=\; 1\,.$$

Then

$$\begin{aligned} \frac{\Delta_1}{\Delta_1 + z} &= \frac{[PA_2A_1']}{[A_1A_2A_1']} = \frac{PA_1'}{A_1A_1'} = \frac{x}{x+y+z} = x\,, \\ \frac{\Delta_2}{\Delta_2 + x} &= \frac{[PA_3A_2']}{[A_2A_3A_2']} = \frac{PA_2'}{A_2A_2'} = \frac{y}{x+y+z} = y\,, \\ \text{and}\quad \frac{\Delta_3}{\Delta_3 + y} &= \frac{[PA_1A_3']}{[A_3A_1A_3']} = \frac{PA_3'}{A_3A_3'} = \frac{z}{x+y+z} = z\,. \end{aligned}$$

These three equations can be solved to yield

$$\Delta_1 \;=\; \frac{zx}{y+z}\,, \qquad \Delta_2 \;=\; \frac{xy}{z+x}\,, \qquad \text{and} \qquad \Delta_3 \;=\; \frac{yz}{x+y}\,.$$

Therefore, $\Delta_1 + \Delta_2 + \Delta_3 = \frac{1}{2}\Delta = \frac{1}{2}$ if and only if

$$\frac{zx}{y+z} + \frac{xy}{z+x} + \frac{yz}{x+y} \;=\; \frac{1}{2}\,. \tag{1}$$

By a straightforward but tedious computation, this equality is equivalent to

$$(x-y)(y-z)(z-x) \;=\; 0\,, \tag{2}$$

which holds if and only if $x = y$, $y = z$, or $z = x$. Now,

$$\frac{y}{[A_1A_1'A_3]} \;=\; \frac{A_1P}{A_1A_1'} \;=\; \frac{z}{[A_1A_2A_1']}\,,$$

so that

$$\frac{y}{z} \;=\; \frac{[A_1A_1'A_3]}{[A_1A_2A_1']} \;=\; \frac{A_1'A_3}{A_2A_1'}\,.$$

Therefore, $y = z$ if and only if A_1' is the mid-point of A_2A_3 or, equivalently, if and only if P lies on the median from vertex A_1. Similarly, $z = x$ if and only if P lies on the median from vertex A_2, and $x = y$ if and only if P lies on the median from vertex A_3. Therefore, the locus of P is the union of the medians of $\triangle A_1A_2A_3$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. One submission was incomplete.

We are grateful to our alert correspondent, Salem Malikić, for finding our problem on the USA Team Selection Test for 2003 (day 1, problem 2). That led him to the discovery that the problem previously appeared in **CRUX with MAYHEM** *as problem 2021 proposed by Toshio Seimiya* [1995 : 89; 1996 : 87–88]. *Meanwhile, Bataille found the problem as number 1612 in* Mathematics Magazine, **74**:5 *(December, 2001), page 408. The solution there used Ceva's Theorem much like solution I from 1996.*

This editor is also grateful to Demis who, in an approach quite similar to our featured solution, filled in the missing step by verifying that equations (1) *and* (2) *are equivalent. The editor agrees with Schlosberg that the computation is both straightforward (although one must use the relation $x + y + z = 1$ wisely) and tedious. Curiously, so did the team of problem solvers that submitted the second solution featured in 1996 (which is also similar to the solution above). Woo in his solution avoided the computation by observing that equation* (1) *is satisfied by each of $x = y$, $y = z$, and $z = x$. This means that $(x - y)$, $(y - z)$, and $(z - x)$ are divisors of the cubic polynomial $p(x, y)$ formed from* (1) *by clearing of fractions and setting $z = 1 - x - y$; whence, $p(x, y)$ must be a constant times the equation in* (2) *(with z replaced by $1 - x - y$).*

3234. [2007 : 171, 173] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be an equilateral triangle, and let P be an interior point. The lines AP, BP, and CP intersect the opposite sides at the points A', B', and C', respectively. Determine the position of the point P if

$$AC' + CB' + BA' \;=\; A'C + C'B + B'A\,.$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina (independently).

Without loss of generality, let us assume that the side of the equilateral triangle ABC has length 1, and let $AC' = x$, $BA' = y$, and $CB' = z$. Then $C'B = 1 - x$, $A'C = 1 - y$, $B'A = 1 - z$, and the given condition becomes $x + y + z = \frac{3}{2}$. Ceva's Theorem gives

$$\frac{x}{1-x} \cdot \frac{y}{1-y} \cdot \frac{z}{1-z} \;=\; 1\,,$$

or

$$xyz \;=\; (1-x)(1-y)(1-z)\,.$$

Using the condition $x + y + z = \frac{3}{2}$, we successively obtain

$$\begin{aligned} 2xyz - (xy + yz + zx) + (x + y + z) - 1 &= 0\,, \\ xyz - \tfrac{1}{2}(xy + yz + zx) + \tfrac{1}{2}(x + y + z) - \tfrac{1}{2} &= 0\,, \\ xyz - \tfrac{1}{2}(xy + yz + zx) + \tfrac{1}{4} &= 0\,, \\ xyz - \tfrac{1}{2}(xy + yz + zx) + \tfrac{1}{4}(x + y + z) - \tfrac{1}{8} &= 0\,, \\ \left(x - \tfrac{1}{2}\right)\left(y - \tfrac{1}{2}\right)\left(z - \tfrac{1}{2}\right) &= 0\,. \end{aligned}$$

It follows that $x = \frac{1}{2}$, $y = \frac{1}{2}$, or $z = \frac{1}{2}$. Hence, the point P is on one of the medians of $\triangle ABC$.

Conversely, if point P is on one of the medians, say, on the median through the vertex C, then $x = 1 - x = \frac{1}{2}$, and then, by Ceva's Theorem, we get $yz = (1 - y)(1 - z)$, or $y + z = 1$. But then $x + y + z = \frac{3}{2}$, so that the condition $AC' + CB' + BA' = A'C + C'B + B'A$ holds.

This shows that the locus of the point P is the union of all the interior points of the three medians of the triangle.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; D.J. SMEENK, Zaltbommel, the Netherlands; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was also one incomplete solution submitted.

3235. [2007 : 171, 174] *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

Let ABC be a triangle, and let A_1, B_1, C_1 be points on the sides BC, CA, AB, respectively, such that

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k\,.$$

Let $\alpha = AA_1$, $\beta = BB_1$, $\gamma = CC_1$, and $\lambda = \dfrac{k^2 + k + 1}{(k + 1)^2}$. Prove that

(a) $\alpha^2 + \beta^2 + \gamma^2 = \lambda(a^2 + b^2 + c^2)$;

(b) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \lambda^2\left(a^2b^2 + b^2c^2 + c^2a^2\right)$;

(c) $\alpha^4 + \beta^4 + \gamma^4 = \lambda^2\left(a^4 + b^4 + c^4\right)$.

Composite of similar solutions by Roy Barbara, Lebanese University, Fanar, Lebanon; and D.J. Smeenk, Zaltbommel, the Netherlands.

With $BA_1 = \dfrac{ka}{k + 1}$, $A_1C = \dfrac{a}{k + 1}$, and $AA_1 = \alpha$, Stewart's Theorem tells us that

$$a\left(\alpha^2 + \frac{ka \cdot a}{(k+1)^2}\right) \;=\; b^2\frac{ka}{k+1} + c^2\frac{a}{k+1}\,.$$

From this we easily obtain

$$(k+1)^2\alpha^2 \;=\; k(k+1)b^2 + (k+1)c^2 - ka^2\,. \tag{1}$$

Similarly,

$$(k+1)^2\beta^2 \;=\; k(k+1)c^2 + (k+1)a^2 - kb^2 \tag{2}$$

and

$$(k+1)^2\gamma^2 \;=\; k(k+1)a^2 + (k+1)b^2 - kc^2\,. \tag{3}$$

(a) Adding (1), (2), and (3), we get

$$(k+1)^2\big(\alpha^2+\beta^2+\gamma^2\big) \;=\; \big(k^2+k+1\big)\big(a^2+b^2+c^2\big)\,;$$

hence, $\alpha^2+\beta^2+\gamma^2 = \lambda\big(a^2+b^2+c^2\big)$, as desired.

(b) Multiply together the left sides of pairs of the three numbered equations above, add the three resulting products, and set the sum equal to the sum of the corresponding products of the right sides. After a straightforward computation, one obtains

$$\begin{aligned}(k+1)^4&\big(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\big)\\ &=\; \big(k^4+2k^3+3k^2+2k+1\big)\big(a^2b^2+b^2c^2+c^2a^2\big)\\ &=\; \big(k^2+k+1\big)^2\big(a^2b^2+b^2c^2+c^2a^2\big)\,;\end{aligned}$$

hence, $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = \lambda^2\big(a^2b^2+b^2c^2+c^2a^2\big)$.

(c) By part (a), we have

$$\big(\alpha^2+\beta^2+\gamma^2\big)^2 \;=\; \lambda^2\big(a^2+b^2+c^2\big)^2\,,$$

while from (b), we have

$$2\big(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\big) \;=\; 2\lambda^2\big(a^2b^2+b^2c^2+c^2a^2\big)\,.$$

Subtract to get $\alpha^4+\beta^4+\gamma^4 = \lambda^2(a^4+b^4+c^4)$, as required.

Also solved by MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOE HOWARD, Portales, NM, USA (part (a) *only); VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Bataille notes that this problem is related to problem 3007 [2005 : 44, 47; 2006 : 62–64], *where it is shown that α, β, and γ are the sides of a triangle T_k, and various results about this triangle are proved. In particular, the ratio $[T_k] : [ABC]$ of the areas of triangles T_k and ABC is shown to be λ, a result that can easily be deduced from* (b) *and* (c) *above.*

3236. [2007 : 171, 174] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\le\; 3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\,.$$

Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain, modified by the editor.

Since $abc = 1$, we have

$$ab+bc+ac \;=\; \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad\text{and}\quad \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \;=\; a+b+c\,,$$

and the given inequality is equivalent to

$$\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right) \;\le\; 1\,. \tag{1}$$

If $a+\dfrac{1}{b}-1 \le 0$, then $a \le 1$ and $b \ge 1$. Hence,

$$b+\frac{1}{c}-1 \;\ge\; \frac{1}{c} \;>\; 0 \qquad\text{and}\qquad c+\frac{1}{a}-1 \;\ge\; c \;>\; 0\,,$$

and the inequality (1) holds. We reach the same conclusion if either of the other factors in (1) is non-positive. If all three factors in (1) are positive, then

$$\begin{aligned}\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right) &= \frac{a}{c}-ac-\frac{1}{ac}+2\\ &= \frac{a}{c}-\left(\sqrt{ac}-\frac{1}{\sqrt{ac}}\right)^2 \;\le\; \frac{a}{c}\,.\end{aligned}$$

Similarly,

$$\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right) \;\le\; \frac{b}{a} \quad\text{and}\quad \left(c+\frac{1}{a}-1\right)\left(a+\frac{1}{b}-1\right) \;\le\; \frac{c}{b}\,.$$

The product of these inequalities yields

$$\left[\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right)\right]^2 \;\le\; \frac{a}{c}\frac{b}{a}\frac{c}{b} \;=\; 1\,,$$

with equality if and only if $a = b = c = 1$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; KEE-WAI LAU, Hong Kong, China; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; PHI THAI THUAN, student, High School Tran Hung Dao, Binh Thuan, Viet Nam; EDMUND SWYLAN, Riga, Latvia; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

*Malikić indicated that this same problem was proposed by Aaron Pixton, and it is published as number 6 on page 31 of "Olympiad Inequalities" by Thomas J. Mildorf (*Dec. 22, 2005*) available at* `www.artofproblemsolving.com/Resources/Papers/MildorfInequalities.pdf`

3237. [2007 : 171, 174] *Proposed by Michel Bataille, Rouen, France.*

Find all integers n such that

$$\frac{7n-12}{2^n}+\frac{2n-14}{3^n}+\frac{24n}{6^n} \;=\; 1\,.$$

Solution by Dionne Bailey, Elsie Campbell, Charles R. Diminnie, and Roger Zarnowski, Angelo State University, San Angelo, TX, USA.

Since the left side of the equation is negative for $n \leq 0$, we may assume that $n \geq 1$. The equation can be written as

$$(7n-12)3^n+(2n-14)2^n+24n \;=\; 6^n \;=\; 3^n2^n\,;$$

hence,

$$\begin{aligned}(2^n-7n+12)(3^n-2n+14) &= 24n+(7n-12)(2n-14)\\ &= 14(n-4)(n-3)\,.\end{aligned}$$

By Mathematical Induction, $2^n > 9n-20$ for $n \geq 5$. Hence, for $n \geq 5$,

$$3^n \;>\; 2^n \;>\; 9n-20 \;>\; 9n-35\,.$$

It follows that, for $n \geq 5$,

$$\begin{aligned}2^n-7n+12 &> 2(n-4) \;>\; 0\,,\\ 3^n-2n+14 &> 7(n-3) \;>\; 0\,,\end{aligned}$$

and therefore,

$$(2^n-7n+12)(3^n-2n+14) \;>\; 14(n-4)(n-3)\,.$$

Checking directly for $n = 1$, 2, 3, and 4, we conclude that the only solution is $n = 4$.

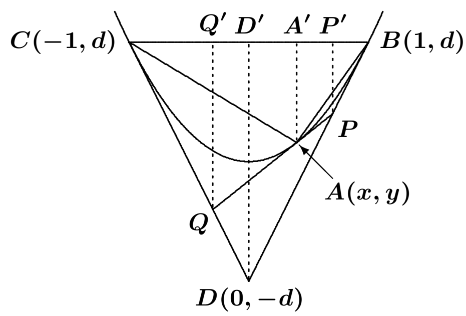
Also solved by MOHAMMED AASSILA, Strasbourg, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comănești, Romania; and the proposer. There were two incomplete submissions.

3238. [2007 : 171, 174] *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{T} = DBC$ be a triangle with $DB = DC$, and let A be a variable point in the interior of $\mathcal{T}$. The perpendiculars to BC through the mid-points of AB and AC meet DB and DC at P and Q, respectively. Find the locus of A for which P, A, and Q are collinear.

I. *Solution by Edmund Swylan, Riga, Latvia.*

With a frame of reference chosen so that $D = (0, -d)$, $B = (1, d)$, and $C = (-1, d)$ as in the diagram, the lines DB and DC have equations $y = 2dx - d$ and $y = -2dx - d$, respectively. It follows that the points on these lines with the same x-coordinate as that of the mid-points of the segments joining B and C to an arbitrary point $A = (x, y)$ will have respective coordinates



$$P = \left(\tfrac{1}{2}(x+1), dx\right) \quad \text{and} \quad Q = \left(\tfrac{1}{2}(x-1), -dx\right) .$$

The point $A = (x, y)$ lies on PQ, therefore, if and only if the slope from A to Q equals the slope from P to Q. This is equivalent, in turn, to

$$\begin{aligned} \frac{y + dx}{x - \frac{1}{2}(x-1)} &= 2dx , \\ \frac{y + dx}{x + 1} &= dx , \\ y + dx &= dx^2 + dx , \\ y &= dx^2 . \end{aligned}$$

Although there is no apparent reason to restrict the point A to the interior of triangle DBC, the required locus of A consists of the points of the parabola $y = x^2$ inside the triangle (with $-1 < x < 1$).

Remarks. The slope of DB is $2d = \tan\angle CBD$; thus, DB is tangent to the parabola at B. Similarly, DC is tangent at C and PQ at A. The parabola may therefore be described as the conic that is tangent to DB at B and to DC at C, and that passes through the mid-point of the line segment joining D to the midpoint of BC.

II. *Solution by J. Chris Fisher, University of Regina, Regina, SK.*

We first see that $BP = DQ$. Let P', A', and Q' be the feet of the perpendiculars to BC from P, A, and Q, and let D' be the mid-point of BC. Since a dilatation with centre A' and ratio $1/2$ takes B to P' and C to

Q', we must have

$$\frac{BA'}{A'C} = \frac{P'A'}{A'Q'} = \frac{PA}{AQ}. \tag{1}$$

Similarly, the dilatation with centre B and ratio $1/2$ takes A' to P' and C to D', so that

$$\frac{BA'}{A'C} = \frac{BP'}{P'D'} = \frac{BP}{PD}, \tag{2}$$

while the dilatation with centre C and ratio $1/2$ takes B to D' and A' to Q', so that

$$\frac{BA'}{A'C} = \frac{D'Q'}{Q'C} = \frac{DQ}{QC}. \tag{3}$$

Equations (2) and (3) imply that $\frac{BP}{PD} = \frac{DQ}{QC}$. Because $BD = DC$, we conclude that $BP = DQ$, as claimed. It follows that as P moves along the line BD, the lines PQ are tangent to the parabola which is tangent to BD at B and to DC at C. Indeed, $BD = DC$ is not required to obtain the envelope of a parabola (see the article "A Parabola Is Not an Hyperbola" by Dan Pedoe in this journal [1979 : 122–124], or see any standard reference on parabolas or on string art).

To finish our problem, we note that the locus of A will be the points of the parabola—the points where the lines of the envelope touch the curve. This property of the point A is guaranteed by a theorem attributed to Apollonius: *Two tangents of a parabola* (here DB and DC) *are divided into segments of like proportion* (namely $BP : PD = DQ : QC$) *by a third tangent* (namely PQ), *and this third tangent is divided in the same proportion by its point of tangency* (which is A by equation (1)). Alternatively, we can invoke a theorem attributed to Archimedes: *If the tangents to a parabola from the point P touch it at A and B, then the line through P parallel to the parabola's axis* (namely DD') *bisects the chord AB.*

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Most of the submitted solutions used coordinates in some way. The exceptions were from Bataille and Demis, who each used a synthetic argument to determine the focus and directrix, then proved that the distance to the focus of any point A of the locus had to equal the distance to the directrix.

3242. [2007 : 237, 239] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $\mathcal{A} = \left\{ z \in \mathbb{C}^* : \left| z + \frac{1}{z} \right| \leq 2 \right\}$. Let $n \geq 2$ be an integer. Prove that, if $\alpha^n \in \mathcal{A}$, then $\alpha \in \mathcal{A}$.

Solution by Kee-Wai Lau, Hong Kong, China.

Let $\alpha = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 \le \theta < 2\pi$. If $\alpha^n \in \mathcal{A}$, then $|\alpha^n + \alpha^{-n}| \le 2$; hence, we successively obtain

$$\begin{aligned}
\left|(r^n + r^{-n})\cos n\theta + i(r^n - r^{-n})\sin n\theta\right| &\le 2\,,\\
(r^n + r^{-n})^2\cos^2 n\theta + (r^n - r^{-n})^2\sin^2 n\theta &\le 4\,,\\
(r^{2n} + r^{-2n}) + 2(\cos^2 n\theta - \sin^2 n\theta) - 4 &\le 0\,,\\
(r^n - r^{-n})^2 - 4\sin^2 n\theta &\le 0\,,\\
|r^n - r^{-n}| &\le 2|\sin n\theta|\,. \qquad (1)
\end{aligned}$$

It can be proved by induction that $|\sin k\theta| \le k|\sin\theta|$ for any positive integer k. Thus, from (1), we have

$$\begin{aligned}
2n|\sin\theta| &\ge \left|r^n - r^{-n}\right|\\
&= \left|r - r^{-1}\right|\left(r^{n-1} + r^{n-2}r^{-1} + \cdots + rr^{-(n-2)} + r^{-(n-1)}\right).
\end{aligned}$$

Since $x + x^{-1} \ge 2$ for $x > 0$, by considering n odd and n even separately, we conclude that

$$r^{n-1} + r^{n-2}r^{-1} + \cdots + rr^{-(n-2)} + r^{-(n-1)} \ \ge\ n\,.$$

From these last two statements, we obtain $|r - r^{-1}| \le 2|\sin\theta|$, which, by (1), is equivalent to $|\alpha + \alpha^{-1}| \le 2$. That is, $\alpha \in \mathcal{A}$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and the proposer.

WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, did not provide a solution of the problem but presented the following conjecture. Let $n > 1$ be fixed. Then, for all real numbers x such that $\cos x \le 0$ and $\cos nx \le 0$, we have

$$\cosh^{-1}(1 - \cos nx) \ \le\ n\cosh^{-1}(1 - \cos x)\,.$$

He showed how this inequality implies the inequality of our problem.