SKOLIAD No. 108

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 August, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Nos questions proviennent ce mois-ci du Concours de l'Association Mathématique du Québec 2006 (niveau secondaire). Nous remercions Véronique Hussin, Université de Montréal, qui s'occupe des concours de l'AMQ du secondaire.

Concours de l'Association Mathématique du Québec
(niveau secondaire) 9 février 2006

1. **Un carré magique particulier**. Il est bien connu qu'un carré magique est obtenu en mettant des nombres dans un carré de telle sorte que la somme de chaque ligne, colonne et diagonale soit la même, comme par exemple,

8	1	6
3	5	7
4	9	2

Imaginons maintenant que nous décidions d'inventer une nouvelle forme de tel carré, en remplaçant la somme par le produit. On demande de trouver un tel carré en remplaçant les astérisques, $*$, par des nombres naturels, non nécessairement distincts ou consécutifs, dans le carré suivant :

$*$	1	$*$
4	$*$	$*$
$*$	$*$	2

2. **La promenade de Clovis**. Clovis aime se promener parmi les entiers naturels. Chaque jour, il commence avec un nombre naturel de son choix, le plus grand possible. Puis, durant la journée, il passe de nombre en nombre selon la règle suivante. En supposant que sa suite en est au nombre n :

(1) Si n est divisible par 3 sans reste, alors le nombre suivant sera $n/3$.

(2) Si le reste de la division de n par 3 est 1, alors le nombre suivant est $2n + 1$.

(3) Si le reste de la division de n par 3 est 2, alors le nombre suivant est $2n - 1$.

(4) Si $n = 1$, la suite s'arrête.

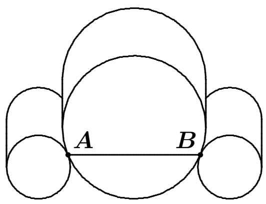
Depuis des années qu'il joue à ce jeu, il a constaté que, quel que soit le nombre de départ, la série aboutissait toujours à 1. Il se demande s'il existe une série qui croît indéfiniment, avec en moyenne des nombres de plus en plus grands, ou alors une série cyclique qui ne contient pas le nombre 1.

Dites si une telle série est possible, et donnez-en un exemple, ou alors montrez qu'une telle série n'existe pas, et pour cela démontrez que toutes les suites construites de cette façon meneront inévitablement au nombre 1.

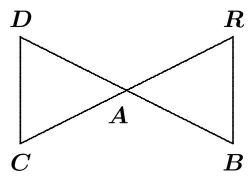
Voici un exemple d'une telle série : en commençant avec 55, on obtient 111, 37, 75, 25, 51, 17, 33, 11, 21, 7, 15, 5, 9, 3 et 1, ce qui termine la série.

3. **Huit boules dans deux urnes**. On vous confie deux urnes semblables, quatre boules blanches et quatre boules noires. Vous devez répartir les boules entre les deux urnes (pas nécessairement le même nombre dans chaque urne). On rendra ensuite les deux urnes indiscernables. Quelle répartition devez-vous choisir pour maximiser vos chances, en tirant une boule au hasard, d'en obtenir une blanche ?

4. **Les trois tonneaux attachés**. Trois gros tonneaux cylindriques, couchés parallélement sur le sol, sont attachés par un câble d'acier à leurs points de contact, A et B, de façon à ce qu'ils demeurent bien en place. Sachant que les deux plus petits ont chacun un rayon de 4 mètres et que le plus gros, au centre, a un rayon de 9 mètres, quelle est la longueur du câble d'acier ?



5. **Les mots magiques**. Un illusionniste est à la recherche de mots magiques pour accompagner ses divers tours de magie. Il décide de construire ses mots magiques en partant du diagramme à la droite. Il parcourt un chemin dans le diagramme et note les lettres rencontrées. Chaque mot magique doit contenir exactement 11 lettres et doit débuter et se terminer par la lettre A. Deux lettres consécutives ne doivent jamais être identiques. Combien y a-t'il de tels mots magiques ?



Note : Voici deux mots magiques possibles : $ABRACADABRA$ et $ARADCABARBA$.

6. **Tous les dix chiffres**. Trouver le plus petit entier positif N tel que, dans la notation décimale, N et $2N$ utilisent ensemble tous les dix chiffres : 0, 1, 2, . . . , 8, 9.

7. **Les garnitures de pizza**. À la pizzeria de Julio, toutes les pizza comportent du fromage et de la sauce tomate. Le choix de garnitures se limite aux olives noires, aux anchois et au saucisson. Sur les 200 clients que Julio a reçus hier, 40 ont pris des anchois, 80 des olives noires et 120 du saucisson, 60 ont pris à la fois des olives noires et du saucisson, mais aucun n'a pris à la fois des anchois et des olives noires, ni à la fois des anchois et du saucisson. Combien de clients n'ont pris aucune des trois garnitures ?

Mathematics Association of Quebec Contest
(Secondary level) February 9, 2006

1. **A particular magic square**. It is well known that a magic square is obtained by putting numbers in a square such that the sum of each row, column, and diagonal is the same, as for example,

8	1	6
3	5	7
4	9	2

Imagine now that we decide to invent a new form of such squares by replacing the sum by a product. We ask you to find such a square by replacing the asterisks, $*$, by natural numbers, not necessarily distinct or consecutive, in the following square:

$*$	1	$*$
4	$*$	$*$
$*$	$*$	2

2. **Clovis' outing**. Clovis likes to take an outing in the natural numbers. Each day, he starts with a natural number of his choice, the biggest possible. Then, during his day, he passes from number to number using the following rules. Suppose that the sequence of numbers is currently at n.

(1) If n is divisible by 3 without remainder, then the next number is $n/3$.

(2) If the remainder after dividing n by 3 is 1, then the next number is $2n+1$.

(3) If the remainder after dividing n by 3 is 2, then the next number is $2n-1$.

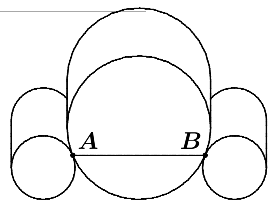
(4) If $n=1$, then the sequence stops.

Over the years that he has played this game, he noticed that, whatever the starting number, the sequence always ended up with the number 1. However, he wonders if there is a sequence that increases indefinitely, with larger and larger numbers on average, or such that it ends up in a loop of numbers that does not contain 1. Determine if such a sequence is possible and give an example, or show that such a sequence does not exist by showing that all sequences using the above rules inevitably end up at the number 1.

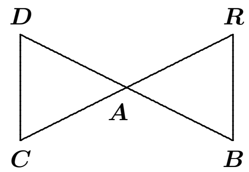
Here is an example of such a sequence: Starting with 55, we get 111, 37, 75, 25, 51, 17, 33, 11, 21, 7, 15, 5, 9, 3 and 1, which ends the sequence.

3. **Eight balls in two urns**. We give you two similar urns, four white balls, and four black balls. You must separate the balls amongst the two urns (not necessarily the same number in each urn), after which both urns will be made indistinguishable. How should the balls be distributed to maximize the chances that, if you draw a ball randomly from a randomly chosen urn, you will obtain a white ball?

4. **The three attached barrels**. Three big cylindrical barrels, lying parallel to the earth, are attached by a steel cable at their contact points, A and B, such that they stay fixed in place. Knowing that the two smaller ones each have a radius of 4 meters and the biggest one has a radius of 9 meters, what is the length of the steel cable?



5. **The magic words**. An illusionist is searching for magic words to accompany his many magic tricks. He decides to construct his magic words starting with the diagram on the right. He takes a path through the diagram and jots downs the letters he finds on it. Each magic word must have exactly 11 letters and must start and end with the letter A. Two consecutive letters must never be identical. How many magic words are there?



Note: Here are two possible magic words: $ABRACADABRA$ and $ARADCABARBA$.

6. **All ten digits**. Find the smallest positive natural number N such that, in the decimal notation, N and $2N$ together use all ten digits: $0, 1, 2, \ldots, 9$.

7. **The pizza toppings**. At the Julio pizzeria, all the pizzas have cheese and tomato sauce on them. The choice of toppings is limited to black olives, anchovies, and sausage. Of the 200 clients Julio had yesterday, 40 took anchovies, 80 took black olives, 120 took sausage, 60 took at the same time black olives and sausage, but none took at the same time anchovies and black olives or anchovies and sausage. How many clients took none of the three toppings?

We are not featuring any solutions in the Skoliad this issue, since the solutions that would normally appear here would have to be prepared before the deadline date for submissions. Solutions will continue as normal in the next issue.

That brings us to the end of another issue. Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), and Larry Rice (University of Waterloo).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier juin 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M332. *Proposé par Dionne Bailey, Elsie Campbell, et Charles R. Diminnie, Angelo State University, San Angelo, TX, É-U.*

Le rayon et la longueur d'un cylindre circulaire droit fermé sont mesurés par des entiers. La valeur de son volume est quatre fois celle de sa surface totale (extrémités comprises). Trouver le plus petit volume possible pour ce cylindre.

M333. *Proposé par l'Équipe de Mayhem.*

Anne et Berthe jouent un jeu avec un tas de n allumettes. Elles jouent à tour de rôle et c'est Anne qui commence. Chacune doit enlever soit une, trois ou six allumettes. Celle qui enlève la dernière allumette a gagné. Pour quelles valeurs de n, entre 36 et 40, Berthe a-t-elle une stratégie gagnante ?

M334. *Proposé par l'Équipe de Mayhem.*

(a) Trouver tous les entiers x pour lesquels $\dfrac{x-3}{3x-2}$ est un entier.

(b) Trouver tous les entiers y pour lesquels $\dfrac{3y^3+3}{3y^2+y-2}$ est un entier.

M335. *Proposé par l'Équipe de Mayhem.*

Dans une suite de quatre nombres, le second est le double du premier. On a aussi que la somme du premier et du quatrième nombre est 9, la somme du deuxième et du troisième est 7, et la somme des carrés des quatre nombres est 78. Trouver toutes les suites ayant ces propriétés.

M336. *Proposé par l'Équipe de Mayhem.*

Un point réseau est un point (x, y) du plan dont les coordonnées x et y sont des entiers. Soit n un entier positif. Trouver le nombre de points réseau à l'intérieur de la région $|x| + |y| \leq n$.

M337. *Proposé par l'Équipe de Mayhem.*

Sur les côtés AB et CD d'un rectangle $ABCD$ avec $AD < AB$, on choisit les points F et E de sorte que $AFCE$ soit un losange.

(a) Si $AB = 16$ et $BC = 12$, déterminer EF.

(b) Si $AB = x$ et $BC = y$, déterminer EF en fonction de x et y.

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M332. *Proposed by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

A closed right circular cylinder has an integer radius and an integer height. The numerical value of the volume is four times the numerical value of its total surface area (including its top and bottom). Determine the smallest possible volume for the cylinder.

M333. *Proposed by the Mayhem Staff.*

Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 6 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 36 to 40 inclusive does Brenda have a winning strategy?

M334. *Proposed by the Mayhem Staff.*

(a) Determine all integers x for which $\dfrac{x-3}{3x-2}$ is an integer.

(b) Determine all integers y for which $\dfrac{3y^3+3}{3y^2+y-2}$ is an integer.

M335. *Proposed by the Mayhem staff.*

In a sequence of four numbers, the second number is twice the first number. Also, the sum of the first and fourth numbers is 9, the sum of the second and third is 7, and the sum of the squares of the four numbers is 78. Determine all such sequences.

M336. *Proposed by the Mayhem Staff.*

A lattice point is a point (x, y) in the coordinate plane with each of x and y an integer. Suppose that n is a positive integer. Determine the number of lattice points inside the region $|x| + |y| \leq n$.

M337. *Proposed by the Mayhem Staff.*

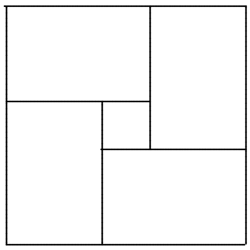
On sides AB and CD of rectangle $ABCD$ with $AD < AB$, points F and E are chosen so that $AFCE$ is a rhombus.

(a) If $AB = 16$ and $BC = 12$, determine EF.

(b) If $AB = x$ and $BC = y$, determine EF in terms of x and y.

Mayhem Solutions

M282. *Proposed by J. Walter Lynch, Athens, GA, USA.*



Four rectangles are arranged in a square pattern so that they enclose a smaller square. Let S be the area of the outer square and Q the area of the inner square. If $S/Q = 9 + 4\sqrt{5}$, determine the ratio of the sides of the rectangles.

Combination of solutions by Mihály Bencze, Brasov, Romania; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Denise Cornwell, student, Angelo State University, San Angelo, TX, USA; Hasan Denker, Istanbul, Turkey; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dragoljub Milošević, Pranjani, Serbia; Billy Suandito, Palembang, Indonesia; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comănești, Romania.

Let x and y represent the sides of one of the rectangles such that $x > y$. Then the outer square has side length $x + y$ and the inner square has side length $x - y$. The given ratio $S/Q = 9 + 4\sqrt{5}$ can then be represented as

$$\frac{(x+y)^2}{(x-y)^2} = \left(2 + \sqrt{5}\,\right)^2 .$$

Since $x > y$, we successively obtain the equivalent equations

$$\begin{aligned} \frac{x+y}{x-y} &= 2 + \sqrt{5}\,, \\ x + y &= \left(2+\sqrt{5}\,\right)x - \left(2+\sqrt{5}\,\right)y\,, \end{aligned}$$

and $x+\sqrt{5}x = 3y+\sqrt{5}y$. The ratio of the sides of the rectangle then yields

$$\frac{x}{y} = \frac{(3+\sqrt{5})(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{\sqrt{5}+1}{2},$$

which is the Golden Ratio! We can also compute $\frac{y}{x} = \dfrac{\sqrt{5}-1}{2}$.

There was one incorrect solution submitted.

M283. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the relationship between x and y if

$$x^2 + y\cos^2\alpha = x\sin\alpha\cos\alpha \qquad \text{and} \qquad x\cos 2\alpha + y\sin 2\alpha = 0\,.$$

(Assume that both x and y are non-zero.)

Essentially the same solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and Titu Zvonaru, Comăneşti, Romania.

Using the identities $2\cos^2\alpha = 1+\cos 2\alpha$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$, the first of the two given equations is successively equivalent to

$$\begin{aligned} x^2 + \tfrac{1}{2}y(1+\cos 2\alpha) &= \tfrac{1}{2}x\sin 2\alpha\,, \\ \text{and} \qquad x\sin 2\alpha - y\cos 2\alpha &= 2x^2 + y\,. \end{aligned}$$

Thus, the two given equations yield the following equivalent system of equations:

$$\begin{aligned} x\sin 2\alpha - y\cos 2\alpha &= 2x^2 + y\,, & (1) \\ x\cos 2\alpha + y\sin 2\alpha &= 0\,. & (2) \end{aligned}$$

Solving this system of equations for $\sin 2\alpha$ and $\cos 2\alpha$, we obtain

$$\sin 2\alpha = \frac{x(2x^2+y)}{x^2+y^2} \quad \text{and} \quad \cos 2\alpha = -\frac{y(2x^2+y)}{x^2+y^2}\,,$$

since x and y are non-zero. Squaring both equations and applying the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, leads to

$$\frac{x^2(2x^2+y)^2}{(x^2+y^2)^2} + \frac{y^2(2x^2+y)^2}{(x^2+y^2)^2} = 1\,,$$

which simplifies to

$$\frac{(2x^2+y)^2}{x^2+y^2} = 1\,.$$

Hence, $4x^2+4y-1=0$, where we have again used the fact that $x \neq 0$.

[*Ed:* We could have squared equations (1) and (2) to get

$$\begin{aligned} x^2\sin^2 2\alpha - 2xy\sin 2\alpha\cos 2\alpha + y^2\cos^2 2\alpha &= (2x^2+y)^2\,, \\ x^2\cos^2 2\alpha + 2xy\cos 2\alpha\sin 2\alpha + y^2\sin^2 2\alpha &= 0\,, \end{aligned}$$

and then add to get

$$x^2(\sin^2 2\alpha + \cos^2 2\alpha) + y^2(\sin^2 2\alpha + \cos^2 2\alpha) \;=\; (2x^2+y)^2\,,$$

or

$$x^2 + y^2 \;=\; (2x^2+y)^2\,,$$

which gives $4x^2+4y-1=0$, as above.]

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and BILLY SUANDITO, Palembang, Indonesia. There were four incorrect solutions submitted.

M284. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{13}\right) \;=\; \frac{\pi}{4}\,.$$

Essentially the same solution by Mihály Bencze, Brasov, Romania; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Hasan Denker, Istanbul, Turkey; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; John G. Heuver, Grande Prairie, AB; Taichi Maekawa, Takatsuki City, Osaka, Japan; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Dragoljub Milošević, Pranjani, Serbia; Billy Suandito, Palembang, Indonesia; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Titu Zvonaru, Comăneşti, Romania.

Setting $a=\tan^{-1}\left(\frac{1}{2}\right)$, $b=\tan^{-1}\left(\frac{1}{4}\right)$, and $c=\tan^{-1}\left(\frac{1}{13}\right)$, we obtain $\tan a=\frac{1}{2}$, $\tan b=\frac{1}{4}$, and $\tan c=\frac{1}{13}$ with a, b, $c\in\left(0,\frac{\pi}{4}\right)$. Applying the identity $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$ twice, we obtain

$$\begin{aligned}\tan\bigl((a+b)+c\bigr) \;&=\; \frac{\tan(a+b)+\tan c}{1-\tan(a+b)\tan c} \;=\; \frac{\dfrac{\tan a+\tan b}{1-\tan a\tan b}+\tan c}{1-\dfrac{\tan a+\tan b}{1-\tan a\tan b}\tan c}\\ &=\; \frac{\tan a+\tan b+\tan c-\tan a\tan b\tan c}{1-\tan a\tan b-\tan b\tan c-\tan a\tan c}\\ &=\; \frac{\frac{1}{2}+\frac{1}{4}+\frac{1}{13}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{13}}{1-\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{4}\cdot\frac{1}{13}-\frac{1}{2}\cdot\frac{1}{13}} \;=\; 1\,.\end{aligned}$$

Hence, $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{13}\right)=a+b+c=\tan^{-1}(1)=\frac{\pi}{4}$.

Also solved by MIHÁLY BENCZE, Brasov, Romania (second solution); JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comăneşti, Romania (second solution).

M285. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be strictly positive numbers such that $a+b+c \geq 3abc$. Prove that $a^2+b^2+c^2 \geq 2abc$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

From the Arithmetic Mean–Geometric Mean Inequality, we have $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$, or $(a+b+c)^3 \geq 27abc$; hence,

$$(a+b+c)^4 \;=\; (a+b+c)^3(a+b+c) \;\geq\; (27abc)(3abc) \;=\; 81a^2b^2c^2\,.$$

Thus, taking square roots, we get $(a+b+c)^2 \geq 9abc$, since a, b, and c are all positive. Next,

$$\begin{aligned} a^2+b^2+c^2-\tfrac{1}{3}(a+b+c)^2 & \\ &= a^2+b^2+c^2-\tfrac{1}{3}\big(a^2+b^2+c^2+2ab+2bc+2ac\big) \\ &= \tfrac{2}{3}\big(a^2+b^2+c^2-ab-bc-ac\big) \\ &= \tfrac{2}{3}\Big(\tfrac{1}{2}\big((a-b)^2+(b-c)^2+(a-c)^2\big)\Big) \;\geq\; 0\,. \end{aligned}$$

Therefore, $a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2 \geq \frac{1}{3}(9abc) = 3abc > 2abc$.

Also solved by ARKADY ALT, San Jose, CA, USA; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and VEDULA N. MURTY, Dover, PA, USA. Three incomplete solutions were also submitted.

Three of the solvers actually proved the stronger inequality attained in the solution above.

M286. *Proposed by K.R.S. Sastry, Bangalore, India.*

If $xy+yz+zx=1$, show that

(a) $$\left|\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\right| \;=\; \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}\,;$$

(b) $$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} \;=\; \frac{2}{x+y+z-xyz}\,.$$

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

If $xy+yz+zx=1$, then $1+x^2 = xy+yz+zx+x^2 = (x+y)(x+z)$. Similarly, $1+y^2 = (y+x)(y+z)$ and $1+z^2 = (z+x)(z+y)$. This yields $\sqrt{(1+x^2)(1+y^2)(1+z^2)} = \pm(x+y)(y+z)(z+x)$. We also note that

$$\begin{aligned} (x+y)(y+z)(z+x) &= (1+x^2)(y+z) \\ &= y+z+x(xy+xz) \\ &= y+z+x(1-yz)\,; \end{aligned}$$

hence, $(x+y)(x+z)(y+z) = x+y+z-xyz$. Therefore,

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} = \frac{x}{(x+y)(x+z)}+\frac{y}{(y+x)(y+z)}+\frac{z}{(z+x)(z+y)}\,.$$

The right side of the above equation is equal to

$$\frac{x(y+z)+y(x+z)+z(x+y)}{(x+y)(y+z)(z+x)} = \frac{2}{(x+y)(y+z)(z+x)}\,.$$

Since $(x+y)(y+z)(z+x) = x+y+z-xyz$ from above, we have proved part (b).

Since $\sqrt{(1+x^2)(1+y^2)(1+z^2)} = \pm(x+y)(y+z)(z+x)$, we see that part (a) also holds.

Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.M. MILOŠEVIĆ, Pranjani, Serbia; BILLY SUANDITO, Palembang, Indonesia (part (b) *only*)*; TITU ZVONARU, Comănești, Romania; and the proposer.*

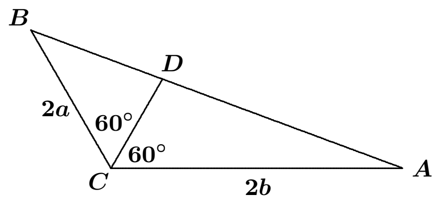
Part (a) *originally appeared without the absolute value signs on the left side. Malikić and Gómez Moreno both provided a counterexample to the equation as it originally appeared.*

M287. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given two positive real numbers a and b, construct their harmonic mean with straightedge and compass.

Solution by Taichi Maekawa, Takatsuki City, Osaka, Japan.

Construction: As shown in the diagram, draw a triangle whose sides AB and AC have length $2a$ and $2b$, respectively, in such a way that $\angle ACB = 120^\circ$ [*Ed.:* this is well known to be constructible with straightedge and compass]. Let D be the point of intersection of AB and the internal angle bisector of $\angle ACB$. Then the length of CD is the harmonic mean of a and b.



Proof: Since the area of $\triangle ACD$ plus the area of $\triangle BCD$ is equal to the area of $\triangle ABC$, we see that

$$\tfrac{1}{2}2a\cdot CD\cdot\sin 60^\circ+\tfrac{1}{2}2b\cdot CD\cdot\sin 60^\circ = \tfrac{1}{2}2a\cdot 2b\cdot\sin 120^\circ\,.$$

Therefore, $CD = \dfrac{2ab}{a+b}$; hence, CD is the harmonic mean of a and b.

Also solved by MIHÁLY BENCZE, Brasov, Romania; HASAN DENKER, Istanbul, Turkey; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Cománeşti, Romania. One incorrect solution was also submitted.

Interestingly enough, the proposer had an article in the issue previous to the one in which this proposal appeared [2007 : 17–18] *which showed a way to construct harmonic means. The method indicated there was similar to the above solution.*

Problem of the Month

Ian VanderBurgh

I hope that you've been doing some strengthening exercises recently—we're in for some heavy (and not-so-heavy) lifting this month.

Problem (2006 Cayley Contest)

Quincy and Celine have to move 16 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	2 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 a.m. What is the earliest time at which they can be finished moving all of the boxes?

(A) 9:41 a.m. (B) 9:42 a.m. (C) 9:43 a.m. (D) 9:44 a.m. (E) 9:45 a.m.

First, some ground rules. No throwing boxes. No stacking boxes. You are not allowed to put one box inside another. (For some reason, when I do this type of problem with a group of students, they always want to try these sneaky things!)

While it makes sense to get out a piece of scrap paper and do a bit of investigation, let's look before we leap. Since Celine is faster on smaller boxes and Quincy is faster on large boxes, let's try having Celine move all of the small boxes (taking $16 \times 2 = 32$ minutes) and having Quincy move all of the large boxes (taking $10 \times 5 = 50$ minutes). Thus, they would finish at 9:50 a.m. Can you see why this minimizes the *total working time*? Interestingly, this does not necessarily mean that it gives the earliest end time using the given rules.

While this division of labour is fast in one sense, this does not seem optimal in others. Celine moves boxes for 32 minutes and then sits around for 18 more minutes while Quincy keeps moving boxes. (Does this remind you of what happens sometimes when you're doing chores at home?)

How could we improve on this time? What if Celine pitches in by also moving 1 large box? Quincy's time is reduced to $9 \times 5 = 45$ minutes and Celine's time increases to $16 \times 2 + 6 = 38$ minutes. We are now at 9:45 a.m. as the finishing time. This is definitely better. But is it the best possible?

Take a few minutes to see if you can do better than 9:45 a.m. While you're doing this, I'll try to distract you a bit by pointing out that there might be some unrealistic things in this problem. At the same time, though, this problem does seem to be more of a "real life" problem than some of the ones that we encounter. (I will say more about this at the end.)

Did you do better than 9:45 a.m.? In fact, 9:43 a.m. is possible. Suppose that Celine moves 15 small boxes and 2 large boxes. This takes her $15 \times 2 + 2 \times 6 = 42$ minutes. Quincy will thus move 1 small box and 8 large boxes, which will take $1 \times 3 + 8 \times 5 = 43$ minutes. Overall, they will finish at 9:43 a.m. (and Celine won't even have time to get much of a break in her 1 minute of spare time at the end).

After some more trial and error, you will probably get frustrated like I did when you can't get a shorter time than 43 minutes. So it seems that 9:43 a.m. is the earliest possible finish time. Let's prove this.

Solution: If Celine moves 15 small boxes and 2 large boxes, it takes her 42 minutes. If Quincy moves 8 large boxes and 1 small box, it takes him 43 minutes. In this configuration, the boxes are fully moved at 9:43 a.m.

Suppose in general that Celine moves x small and y large boxes. Thus, Quincy moves $16 - x$ and $10 - y$ small boxes. In this case, Celine takes $2x + 6y$ minutes and Quincy takes $3(16 - x) + 5(10 - y) = 98 - 3x - 5y$ minutes.

Suppose now that they finish the job at 9:42 a.m. or earlier. (We'll show that this cannot actually happen.) In this case, each works for at most 42 minutes, so the total time that they work would have to be at most 84 minutes. From the information above, their total time is

$$(2x + 6y) + (98 - 3x - 5y) \;=\; 98 + y - x\,.$$

If their total time is at most 84 minutes, then $98 + y - x \leq 84$; that is, $x - y \geq 14$. But x and y are non-negative integers with $0 \leq x \leq 16$ and $0 \leq y \leq 10$. Therefore, $x - y \leq 16$ (because x is at most 16). Hence, we have $14 \leq x - y \leq 16$.

The possible pairs (x, y) that satisfy these restrictions are $(16, 0)$, $(16, 1)$, $(16, 2)$, $(15, 0)$, $(15, 1)$, and $(14, 0)$. Let's make a table of the length of time that each of Celine and Quincy takes for these values of x and y:

x	y	$2x + 6y$	$98 - 3x - 5y$	Finish at
16	0	32	50	9:50 a.m.
16	1	38	45	9:45 a.m.
16	2	44	40	9:44 a.m.
15	0	30	53	9:53 a.m.
15	1	36	48	9:48 a.m.
14	0	28	56	9:56 a.m.

Since these are all of the possible configurations in which Celine and Quincy could possibly finish at or before 9:42 a.m., and in none of them do they actually finish at 9:42 a.m. or before, we conclude that they cannot finish at or before 9:42 a.m.

Summarizing what we have seen, Celine and Quincy can finish at 9:43 a.m., and cannot finish any earlier. Hence, 9:43 a.m. is the earliest possible finishing time.

Looking back, we solved this problem by looking at the combined time and showing that we couldn't make the pair of eager workers finish earlier than 9:43 a.m.

We could have tried looking at the conditions $2x + 6y \leq 42$ and $98 - 3x - 5y \leq 42$ (or $3x + 5y \geq 56$) to show that they cannot be satisfied at the same time. We could approach this algebraically or even graphically (showing that there isn't a point in this region with non-negative integer coordinates).

This problem can actually be solved in "easier" ways using more advanced techniques from "linear programming". However, it is important from time to time to try to solve these problems using more elementary techniques.

I want to return briefly to the problem and the "real life" aspect of it. Suppose that we change the problem to the following:

> The Dunkley Piano Company makes small pianos and large pianos in their factories in Caracas and Quito. The chart below indicates the time that each factory takes to make each type of piano.
>
> | | Caracas | Quito |
> |---|---|---|
> | small piano | 2 days | 3 days |
> | large piano | 6 days | 5 days |
>
> They start making the pianos on March 1. What is the earliest date on which they can be finished making all of the pianos?

This is exactly the same problem, and so may not appear to be that interesting. But you can quite easily see how applicable this type of problem could be in business and industry. So math (and even math contest problems) can be useful!

THE OLYMPIAD CORNER

No. 268

R.E. Woodrow

We begin this number of the *Corner* with the six problems of the Estonian IMO Team Selection Contest 2004/2005. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

ESTONIAN IMO TEAM SELECTION CONTEST
2004–2005
First Day

1. In a plane, a line l and two circles c_1 and c_2 of different radii are given such that l touches both circles at point P. Point $M \neq P$ on l is chosen so that the angle Q_1MQ_2 is as large as possible, where Q_1 and Q_2 are the tangency points of the tangent lines drawn from M to c_1 and c_2, respectively, differing from l. Find $\angle PMQ_1 + \angle PMQ_2$.

2. The planet Automory has infinitely many inhabitants. Each Automorian loves exactly one Automorian and honours exactly one Automorian. Suppose that

(a) each Automorian is loved by some Automorian;

(b) if Automorian A loves Automorian B, then all Automorians honouring A love B;

(c) if Automorian A honours Automorian B, then all Automorians loving A honour B.

Does each Automorian then honour and love the same Automorian?

3. Find all pairs (x, y) of positive integers satisfying $(x + y)^x = x^y$.

Second Day

4. Find all pairs (a, b) of real numbers such that all roots of the polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are non-negative real numbers.

5. On a horizontal line, 2005 points are marked, each of which is either white or black. For each point, one finds the sum of the number of white points on the right of it and the number of black points on the left of it. Among the 2005 sums, exactly one number occurs an odd number of times. Find all possible values of this number.

6. In a plane, a line l and a circle c do not intersect, and the diameter AB of c is perpendicular to l, with B nearer to l than A. Let C be a point on c different from A and B. Line AC intersects l at point D, and E is the point of tangency of a line drawn from D to c such that E lies on the same side of AC as B. Line EB intersects l at point F, and line FA intersects c a second time at point G. Prove that the reflection of G in AB lies on FC.

Next, we give the four problems of the Trentième Olympiad Mathématique Belge Maxi Finale 2005. Thanks again to to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

TRENTIÈME OLYMPIAD MATHÉMATIQUE BELGE
Maxi Finale
Mercredi 20 Avril 2005

1. Dans l'expression

$$(x_1 + x_2 + \cdots + x_n)^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + 2x_1x_2 + 2x_1x_3 + \cdots + 2x_1x_n + 2x_2x_3 + \cdots + 2x_{n-1}x_n$$

les nombres réels non nuls x_1, x_2, x_3, $\ldots$, x_n ne sont pas tous positifs. Existe-t-il des valeurs de ces nombres réels qui rendent le nombre de doubles produits positifs égal au nombre de doubles produits négatifs

(a) si $n = 4$? (b) si $n = 2005$?

Donner une condition nécessaire et suffisante sur n pour que le nombre de doubles produits positifs soit égal au nombre de doubles produits négatifs.

2. Dans l'espace de dimension 3, existe-t-il deux points P et Q à coordonnées rationnelles tels que $|PQ| = \sqrt{7}$?

3. Dans le triangle ABC, les droites AE et CD sont les bissectrices intérieures des angles $\angle BAC$ et $\angle ACB$ respectivement; E appartient à BC et D appartient à AB. Pour quelles amplitudes de l'angle $\angle ABC$ a-t-on certainement

(a) $|AD| + |EC| = |AC|$? (b) $|AD| + |EC| > |AC|$?
(c) $|AD| + |EC| < |AC|$?

4. La suite infinie 1, 2, 3, 4, 0, 9, 6, 9, 4, 8, 7, 8, $\ldots$, ne comprend que des nombres appartenant à l'ensemble $\{0, 1, 2, \ldots, 9\}$ et est construite de la manière suivant: après le quatrième nombre, chaque nouveau nombre est formé du chiffre des unités de la somme des quatre nombres précédents.

(a) Les nombres 2, 0, 0, 5 apparaissent-ils de manière consécutive dans cette suite?

(b) Les nombres 1, 2, 3, 4 apparaissent-ils une deuxième fois de manière consécutive dans cette suite?

Next, we give the six problems of the 2005 Vietnam Mathematical Olympiad. Thanks again go to Felix Recio for collecting them for our use.

2005 VIETNAM MATHEMATICAL OLYMPIAD

Day 1 (Time: 3 hours)

1. Find the smallest and largest values of the expression $P = x + y$, where x and y are real numbers satisfying $x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$.

2. In a plane, let Γ be a circle with centre O and radius R, and let A and B be points on Γ such that AB is not a diameter. Let C be a point on Γ distinct from A and B. Construct the circle Γ_1 through A and tangent to BC at C, and construct the circle Γ_2 through B and tangent to AC at C. Let the circles Γ_1 and Γ_2 intersect again at D, distinct from C.

Prove that $CD \leq R$, and that the line CD passes through a fixed point when C moves on Γ in such a way that C does not coincide with A and B.

3. Let A_i, $1 \leq i \leq 8$, be the vertices of an octagon in a plane, such that no three of its diagonals are concurrent. Any point of intersection of any two diagonals of the octagon is called a cross. A subquadrilateral of the octagon is any convex quadrilateral whose vertices are also vertices of the octagon.

Given a colouring of a subset of the crosses, let $s(i,k)$, $i \neq k$, be the number of subquadrilaterals having A_i and A_k as vertices and having a coloured cross as the point of intersection of their diagonals. Find the least positive integer n such that one can colour n crosses so that the values $s(i,k)$ are all equal.

Day 2 (Time: 3 hours)

4. Find all real-valued functions f defined on $\mathbb{R}$ that satisfy the identity $f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy$.

5. Find all triples of non-negative integers (x, y, m) such that $\dfrac{x! + y!}{n!} = 3^n$ (with the convention $0! = 1$).

6. Let the sequence x_1, x_2, x_3, $\ldots$, be defined by $x_1 = a$, where a is a real number, and the recursion $x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n$ for $n \geq 1$.

Find all values of a for which the sequence has a finite limit as n tends to infinity, and find this limit.

To round out your problem solving pleasure, here are the six problems of the German Mathematical Olympiad, Final Round, Grades 12–13. Thanks again go to Felix Recio for collecting these for our use.

2005 GERMAN MATHEMATICAL OLYMPIAD

Final Round, Grades 12–13

Saarbrücken, May 10–12, 2005

1. Determine all pairs (x, y) of reals, which satisfy the system of equations

$$\begin{aligned} x^3 + 1 - xy^2 - y^2 &= 0\,, \\ y^3 + 1 - x^2y - x^2 &= 0\,. \end{aligned}$$

2. Let A, B, and C be three distinct points on the circle k. Let the lines h and g each be perpendicular to BC with h passing through B and g passing through C. The perpendicular bisector of AB meets h in F and the perpendicular bisector of AC meets g in G. Prove that the product $|BF| \cdot |CG|$ is independent of the choice of A, whenever B and C are fixed.

3. A lamp is placed at each lattice point (x, y) in the plane (that is, x and y are both integers). At time $t = 0$ exactly one lamp is switched on. At any integer time $t \geq 1$, exactly those lamps are switched on which are at a distance of 2005 from some lamp which is already switched on. Prove that every lamp will be switched on at some time.

4. Let $Q(n)$ denote the sum of the digits of the positive integer n. Prove that $Q\Big(Q\big(Q(2005^{2005})\big)\Big) = 7$.

5. Let r be the radius of the inscribed sphere of a tetrahedron $ABCD$ and let r_1, r_2, r_3, and r_4 be the radii of the other four spheres each of which is tangent externally to one of the faces of $ABCD$ and also tangent to the planes containing the other three faces. Prove that

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\,.$$

6. A sequence x_0, x_1, x_2, $\ldots$, of real numbers is periodic with period p, if $x_{n+p} = x_n$ for all non-negative integers n and $p > 0$.

(a) Prove that there exists a sequence with period 2, which satisfies

$$x_{n+1} = x_n - \frac{1}{x_n}\,, \qquad n = 0, 1, 2, \ldots\,.$$

(b) Prove that for any integer $p > 2$, there is a sequence satisfying the condition in part (a) and having p as smallest period.

Next, we turn to solutions from our readers to problems given in the April 2007 number of the *Corner*, starting with the XX Olimpiadi Italiane Della Matematica, Cesenatico, 7 May 2004, given at [2007 : 149–150].

1. Reading the temperatures in Cesenatico for the months of December and January, Stefano notices an odd feature: on each day in that period, except for the first and the last, the lowest temperature was the sum of the lowest temperatures on the day before and the day after.

The lowest temperature was 5°C on December 3 and 2°C on January 31. Find the lowest temperature on December 25.

Solved by Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's write-up.

Let t_1, t_2, …, t_{62} be the lowest temperatures for the 62 days in the months of December and January.

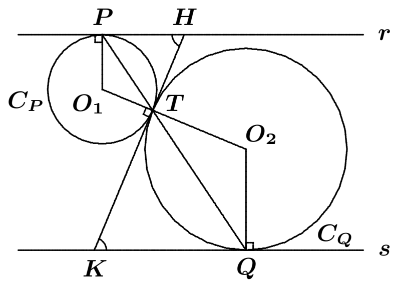
We know that $t_3 = 5$ and $t_{62} = 2$. For $i = 1, 2, \ldots, 59$ we have $t_{i+1} = t_i + t_{i+2}$ and $t_{i+2} = t_{i+1} + t_{i+3}$, hence $t_i = -t_{i+3}$. It follows that for $i = 1, 2, \ldots, 56$, we have $t_i = t_{i+6}$. Thus,

$$t_{26} = t_{32} = t_{38} = t_{44} = t_{50} = t_{56} = t_{62} = 2,$$
$$t_{24} = t_{18} = t_{12} = t_6 = -t_3 = -5.$$

Hence, the lowest temperature on December 25 is $t_{25} = t_{24} + t_{26} = -3$°C.

2. Let r and s be two parallel lines in the plane, and P and Q two points such that $P \in r$ and $Q \in s$. Consider circles C_P and C_Q such that C_P is tangent to r at P, C_Q is tangent to s at Q, and C_P and C_Q are tangent externally to each other at some point, say T. Find the locus of T when (C_P, C_Q) varies over all pairs of circles with the given properties.

Solution by Ioannis Katsikis, Athens, Greece.



Let KH be the common tangent of the circles C_P and C_Q. It is obvious that quadrilaterals PO_1TH and QO_2TK are cyclic, and the fact that $\angle K = \angle H$ means that $\angle PO_1T = \angle TO_2Q$.

Thus, $\triangle PO_1T \sim \triangle TO_2Q$ (similar triangles), and the fact that O_1TO_2 is a straight line tells us that PTQ is also a straight line.

Consequently, T belongs to the constant line PQ. Also, we get from the above similarity of triangles that $PT = PQ \cdot \dfrac{R}{R+r}$.

3. (a) Determine whether the number 2005^{2004} can be written as the sum of the squares of two positive integers.

(b) Determine whether the number 2004^{2005} can be written as the sum of the squares of two positive integers.

Combined solution to (a) *and* (b) *by R. Laumen, Deurne, Belgium.*

A natural number n is the sum of two squares if and only if the prime factorization of n does not contain any prime of the form $4k + 3$ to an odd power [see W. Sierpiński, *Elementary Theory of Numbers*, Hafner, NY 1964, p. 351].

We have the prime power factorizations $2005^{2004} = 5^{2004} \cdot 401^{2004}$ and $2004^{2005} = 2^{4010} \cdot 3^{2005} \cdot 167^{2005}$, so we conclude that 2005^{2004} is the sum of two squares, and that 2004^{2005} is not the sum of two squares.

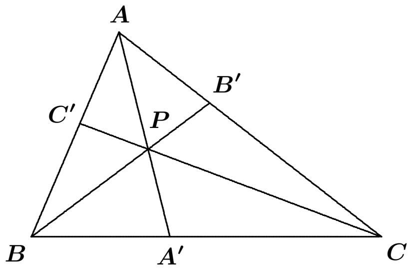
6. Let P be a point inside the triangle ABC. Say that the lines AP, BP, and CP meet the sides of ABC at A', B', and C', respectively. Let

$$x = \frac{AP}{PA'}, \quad y = \frac{BP}{PB'}, \quad z = \frac{CP}{PC'}.$$

Prove that $xyz = x + y + z + 2$.

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We first give the solution of Andrieux.

Dans la suite, on notera $[UVW]$ l'aire d'un triangle UVW. On a



$$\frac{PA}{PA'} = \frac{[PAB]}{[PA'B]} = \frac{[PAC]}{[PA'C]} = \frac{[PAB]+[PAC]}{[PA'B]+[PA'C]} = \frac{[PAB]+[PAC]}{[PBC]}.$$

D'où, en posant $S_A = [PBC]$, $S_b = [PCA]$ et $S_c = [PAB]$:

$$x = \frac{PA}{PA'} = \frac{S_C + S_B}{S_A}, \quad y = \frac{PB}{PB'} = \frac{S_A + S_C}{S_B},$$
$$\text{et} \quad z = \frac{PC}{PC'} = \frac{S_B + S_A}{S_C}.$$

On a alors :

$$\begin{aligned} xyz &= \frac{S_C + S_B}{S_A} \times \frac{S_A + S_C}{S_B} \times \frac{S_B + S_A}{S_C} \\ &= \frac{2S_AS_BS_C + S_A^2S_C + S_C^2S_A + S_C^2S_B + S_B^2S_C + S_B^2S_A + S_A^2S_B}{S_AS_BS_C} \\ &= 2 + \frac{S_A + S_C}{S_B} + \frac{S_C + S_B}{S_A} + \frac{S_B + S_A}{S_C} = x + y + z + 2. \end{aligned}$$

Next, we give the argument of Bataille.

Define positive real numbers α, β, and γ by $\alpha+\beta+\gamma=1$ and $P=\alpha A+\beta B+\gamma C$. Then $P-\alpha A=\beta B+\gamma C=(1-\alpha)A'$; hence, $\alpha\overrightarrow{PA}+(1-\alpha)\overrightarrow{PA'}=\overrightarrow{0}$. It follows that

$$\frac{AP}{PA'} = x = \frac{1-\alpha}{\alpha}.$$

Similarly, $y=\frac{1-\beta}{\beta}$ and $z=\frac{1-\gamma}{\gamma}$; hence,

$$xyz = \frac{(1-\alpha)(1-\beta)(1-\gamma)}{\alpha\beta\gamma}$$

and

$$x+y+z+2 = \frac{1-\alpha}{\alpha}+\frac{1-\beta}{\beta}+\frac{1-\gamma}{\gamma}+2\,.$$

Thus, it is sufficient to show that

$$(1-\alpha)(1-\beta)(1-\gamma) = \beta\gamma(1-\alpha)+\gamma\alpha(1-\beta)+\alpha\beta(1-\gamma)+2\alpha\beta\gamma\,.$$

Recalling that $\alpha+\beta+\gamma=1$, this identity is easily checked.

Next, we turn to solutions to problems of the 17^{th} Irish Mathematical Olympiad, First Paper, given at [2007 : 150].

1. (a) For which positive integers n does $2n$ divide the sum of the first n positive integers?

(b) Determine, with proof, those positive integers n (if any) which have the property that $2n+1$ divides the sum of the first n positive integers.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's solution.

Let $S_n=1+2+\cdots+n=\frac{n(n+1)}{2}$, and let P be the set of positive integers.

(a) We have $2n\mid S_n$ if and only if $2n\mid\frac{n(n+1)}{2}$, which holds if and only if $n(n+1)=4np$ for some $p\in P$, and this is true if and only if $n+1=4p$ for some $p\in P$.

(b) We claim that $(2n+1)\nmid S_n$ for any $n\in P$. To see this, suppose (to the contrary) that $(2n+1)\mid S_n$; that is, suppose that $\frac{n(n+1)}{2}=(2n+1)p$ for some $p\in P$. Then $n(n+1)=4np+2p$; whence, $2p=np_1$ where $p_1=n+1-4p\in P$, since $4p=2n(n+1)/(2n+1)<n+1$. Consequently, $n+1=4p+p_1=2np_1+p_1\geq 2n+1$, a contradiction.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C, and C beat A.

Solution by Titu Zvonaru, Comăneşti, Romania.

Let P_1, P_2, $\ldots$, P_n be the players, and let b_i be the number of players P_i beat. We know that $b_i \geq 1$. We can assume (by relabelling if necessary) that b_1 is minimum and that P_1 beat P_2, P_3, $\ldots$, P_{b_1+1}. Since b_1 is minimum, there is a player P_k such that P_2 beat P_k and $k > b_1 + 1$ (otherwise $b_2 < b_1$). Then P_k beat P_1, and we can take $A = P_1$, $B = P_2$, and $C = P_k$.

3. Let AB be a chord of length 6 of a circle of radius 5 centred at O. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB, and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

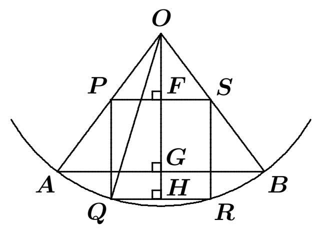
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Construct points F, G, and H, as in the diagram. Since $OA = 5$ and $AG = 3$, we see that $OG = 4$, from the Theorem of Pythagoras. Since $\triangle OPF \sim \triangle OAG$, it follows that $PF = 3t$ and $OF = 4t$ for some t. Then

$$OH \;=\; OF + FH \;=\; 4t + 6t \;=\; 10t$$

and $QH = 3t$. It now follows from the Theorem of Pythagoras (in $\triangle OQH$) that $(10t)^2 + (3t)^2 = 25$. Hence, $t^2 = \frac{25}{109}$ and

$$[PQRS] \;=\; (6t)^2 = \frac{900}{109}\,.$$



4. Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; 720\,.$$

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Smeenk.

Setting $y = x^2 - 7x$, the given equation is successively equivalent to

$$\begin{aligned}
(x^2-7x+6)(x^2-7x+10)(x^2-7x+12) &= 720,\\
(y+6)(y+10)(y+12) &= 720,\\
y^3 + 28y^2 + 252y &= 0,\\
y\left((y+14)^2 + 56\right) &= 0.
\end{aligned}$$

Thus, $y = 0$ and $x \in \{0,\, 7\}$.

5. Let a, $b \geq 0$. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}\right) \;\leq\; 3\left(a^2+b^2\right),$$

with equality if and only if $a = b$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comănești, Romania. We present the approach of Bataille.

Using the Cauchy-Schwarz Inequality

$$\begin{aligned}\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2} &= (a+b)\cdot\sqrt{a^2+ab} + \sqrt{a^2+b^2}\cdot b \\ &\leq \sqrt{2a^2+2b^2+2ab}\cdot\sqrt{a^2+ab+b^2}\,;\end{aligned}$$

hence, since $2ab \leq a^2+b^2$,

$$\begin{aligned}\sqrt{2}\left(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}\right) &\leq 2(a^2+ab+b^2) \\ &= 2(a^2+b^2) + 2ab \;\leq\; 3(a^2+b^2)\,.\end{aligned}$$

Equality calls for $a = b$ (otherwise, $2ab < a^2+b^2$ and the inequality is strict) and conversely, equality holds if $a = b$, as is readily checked. The proof is complete.

Next, we turn to solutions from our readers to problems of the 17th Irish Mathematical Olympiad 2004, Second Paper, given at [2007 : 151].

1. Determine all pairs of prime numbers (p, q), with $2 \leq p,\ q < 100$, such that $p+6$, $p+10$, $q+4$, $q+10$, and $p+q+1$ are all prime numbers.

Solved by Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comănești, Romania. We give Katsikis' write-up.

The only pairs are $(p,q) \in \{(7,3),\ (13,3),\ (37,3),\ (97,3)\}$.

Every prime different from 2 or 3 is of the form $6k+1$ or $6k+5$, where k is some positive integer. Obviously $p \neq 2$ and $p \neq 3$, since $p+6$ is prime. Since $p+10$ is a prime, $p \neq 6k+5$; thus, $p = 6k+1$ for some positive integer $k \geq 1$. If $q > 6$, the fact that $q+10$ is prime tells us that $q = 6\lambda+1$ for some positive integer $\lambda \geq 1$. However, we would then have

$$p+q+1 \;=\; 6k+1+6\lambda+1+1 \;=\; 3(2k+2\lambda+1)\,,$$

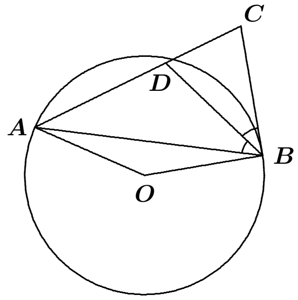
which is not prime. Thus, $q < 6$, and obviously $q \neq 2$ (since $q+4$ is prime). Therefore, we must have $q = 3$. Now we must find prime numbers $p = 6k+1$ for some integer k, $1 \leq k \leq 16$, such that $p+6$, $p+10$, and $p+4$ are all primes.

By direct checking, $k \in \{1,\ 2,\ 6,\ 16\}$; that is, $p \in \{7,\ 13,\ 37,\ 97\}$.

2. Let A and B be distinct points on a circle T. Let C be a point distinct from B such that $|AB| = |AC|$ and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72^\circ$.

Solution by Titu Zvonaru, Comănești, Romania.

Let O be the centre of T, let $AB = AC = b$ and $BC = a$, and let $\angle ABC = \alpha$. We then have $\cos\alpha = \frac{a}{2b}$, $\angle ABO = 90^\circ - \alpha$, and $\angle OBD = 90^\circ - \frac{\alpha}{2}$. In the isosceles triangle AOB, we have



$$OB = \frac{AB}{2\cos\angle ABO} = \frac{b}{2\sin\alpha}.$$

It is known that $BD = \dfrac{2ab\cos\frac{\alpha}{2}}{a+b}$ and that $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$. The Law of Cosines gives

$$OD^2 = OB^2 + BD^2 - 2\,OB\cdot BD\cdot\cos\left(90^\circ - \frac{\alpha}{2}\right).$$

Thus, $OD^2 < OB^2$ if and only if $BD^2 < 2\cdot OB\cdot BD\cdot\sin\frac{\alpha}{2}$; that is, D is inside T if and only if

$$\frac{2ab\cos\frac{\alpha}{2}}{a+b} < 2\cdot\frac{b}{2\sin\alpha}\cdot\sin\frac{\alpha}{2}.$$

If we cancel $2b$, write $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$, and $a = 2b\cos\alpha$, we get

$$\frac{2\cos\alpha\cos\frac{\alpha}{2}}{2\cos\alpha+1} < \frac{1}{4\cos\frac{\alpha}{2}}.$$

This gives $4\cos\alpha\cdot 2\cos^2\frac{\alpha}{2} - (2\cos\alpha+1) < 0$, which, by employing the identity $2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$, is equivalent to $4\cos^2\alpha + 2\cos\alpha - 1 < 0$. Therefore, $\cos\alpha \in \left(\frac{-1-\sqrt{5}}{4}, \frac{-1+\sqrt{5}}{4}\right)$; thus,

$$\cos\alpha < \frac{-1+\sqrt{5}}{4} = \sin 18^\circ.$$

However, $\sin 18^\circ = \cos 72^\circ$; hence, $\alpha > 72^\circ$ since $\alpha < 90^\circ$.

Comment. A proof that $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$ follows. We have

$$\begin{aligned}\sin 18^\circ \cos 36^\circ &= \frac{2\sin 18^\circ \cos 18^\circ \cos 36^\circ}{2\cos 18^\circ} = \frac{\sin 36^\circ \cos 36^\circ}{2\cos 18^\circ}\\ &= \frac{\sin 72^\circ}{4\cos 18^\circ} = \frac{\cos 18^\circ}{4\cos 18^\circ} = \frac{1}{4}\,. \qquad (1)\end{aligned}$$

We also see that

$$\begin{aligned}\sin 18^\circ - \cos 36^\circ &= \cos 72^\circ - \cos 36^\circ\\ &= -2\sin\frac{72^\circ+36^\circ}{2}\sin\frac{72^\circ-36^\circ}{2}\\ &= -2\sin 54^\circ \sin 18^\circ = -2\sin 18^\circ \cos 36^\circ\\ &= -\frac{1}{2}\,. \qquad (2)\end{aligned}$$

From (2), we get $\cos 36^\circ = \sin 18^\circ + \frac{1}{2}$. Then, using (1), we see that

$$\sin 18^\circ\left(\sin 18^\circ + \frac{1}{2}\right) = \frac{1}{4}\,.$$

Since $\sin 18^\circ > 0$, we obtain $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$.

3. Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number $\sqrt[3]{n^3+2n^2+n}$.

Solved by Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comănești, Romania. We give Katsikis' solution.

We prove that the first digit after the decimal point is 6.

First of all, we observe that

$$n = \sqrt[3]{n^3} < \sqrt[3]{n^3+2n^2+n} < \sqrt[3]{n^3+3n^2+3n+1} = n+1\,.$$

It is sufficient to show that

$$n+\frac{6}{10} < \sqrt[3]{n^3+2n^2+n} < n+\frac{7}{10}\,. \qquad (1)$$

The first inequality of (1) holds for $n \geq 2$, since

$$(5n+3)^3 < 125(n^3+2n^2+n) \iff 5n(5n-2) > 27\,.$$

The second inequality of (1) holds for $n \geq 2$, since

$$(10n+7)^3 > 1000(n^3+2n^2+n) \iff 100n^2+470n+343 > 0\,.$$

Thus, (1) holds for $n \geq 2$, and the first decimal digit is 6.

4. Define the function m of the three real variables x, y, and z by

$$m(x,y,z) = \max\{x^2, y^2, z^2\}\,.$$

Determine, with proof, the minimum value of m if x, y, and z vary in $\mathbb{R}$ subject to the restrictions $x+y+z=0$ and $x^2+y^2+z^2=1$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give Bataille's write-up.

We show that the required minimum of m is $\frac{1}{2}$.

Let x, y, z be such that $x + y + z = 0$ and $x^2 + y^2 + z^2 = 1$. Clearly, x, y, and z are neither all positive nor all negative; hence, up to a change of order or sign, we may suppose that $x \leq 0$ and y, $z \geq 0$. Then

$$2x^2 = x^2 + (-y - z)^2 = 1 + 2yz \geq 1 .$$

Thus, $m(x^2, y^2, z^2) \geq \frac{1}{2}$.

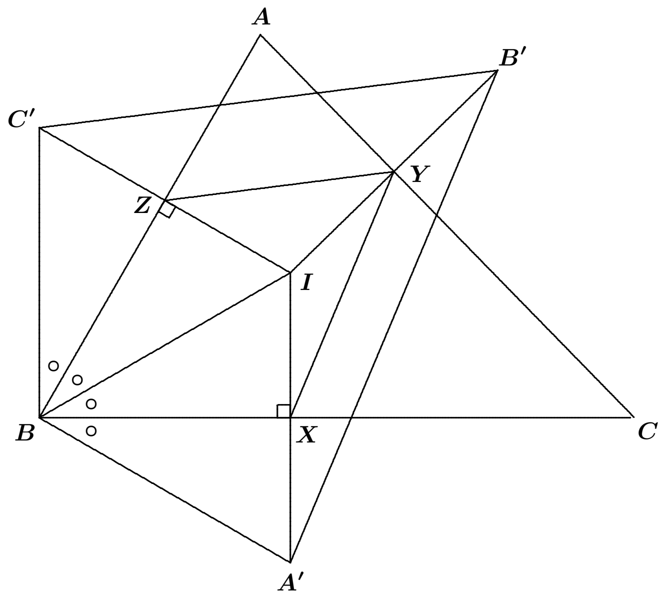
To complete the proof, we just observe that for $x = -\frac{\sqrt{2}}{2}$, $y = 0$, $z = \frac{\sqrt{2}}{2}$, we have $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$ and $m(x^2, y^2, z^2) = \frac{1}{2}$.

Next, we give solutions to problems of the New Zealand Mathematical Olympiad, IMO Squad Selection Problems 2004 given at [2007 : 151–152].

1. Let I be the incentre of triangle ABC, and let A', B', and C' be the reflections of I in BC, CA, and AB, respectively. The circle through A', B', and C' passes also through B. Find the angle $\angle ABC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comǎneşti, Romania. We give the write-up of Amengual Covas.

Let X, Y, and Z be the feet of the perpendiculars from I to the sides BC, CA, and AB , respectively.



Then X, Y, Z are the mid-points of segments IA', IB', IC' respectively. Therefore XY is parallel to $A'B'$ and YZ is parallel to $B'C'$, so we have

$$\begin{aligned} \angle A'B'C' = \angle XYZ &= \tfrac{1}{2}\angle XIZ = \tfrac{1}{2}(180^\circ - \angle ZBX) \\ &= \tfrac{1}{2}(180^\circ - \angle ABC) = 90^\circ - \tfrac{1}{2}\angle ABC \end{aligned}$$

On the other hand, since BI bisects $\angle ABC$, we have

$$\angle C'BZ = \angle ZBI = \tfrac{1}{2}\angle ABC = \angle IBX = \angle XBA' ;$$

whence,

$$\begin{aligned} \angle C'BA' &= \angle C'BZ + \angle ZBI + \angle IBX + \angle XBA' \\ &= 4 \cdot \tfrac{1}{2}\angle ABC = 2\angle ABC . \end{aligned}$$

Since A', B', C', and B are concyclic, we have $\angle C'BA' + \angle A'B'C' = 180^\circ$, which, on substitution, gives $2\cdot\angle ABC + \left(90^\circ - \frac{1}{2}\angle ABC\right) = 180^\circ$. Therefore, $\angle ABC = 60^\circ$.

3. For positive x_1, x_2, y_1, y_2, prove the inequality

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} \geq \frac{(x_1+x_2)^2}{y_1+y_2} .$$

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and Titu Zvonaru, Comăneşti, Romania. We give Alt's solutions and comment.

Solution 1. We have

$$\begin{aligned} (y_1+y_2)\left(\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2}\right) &= x_1^2 + x_2^2 + \frac{x_1^2 y_2}{y_1} + \frac{x_2^2 y_1}{y_2} \\ &\geq x_1^2 + x_2^2 + 2\sqrt{\frac{x_1^2 y_2}{y_1}\cdot\frac{x_2^2 y_1}{y_2}} \\ &= x_1^2 + x_2^2 + 2x_1x_2 = (x_1+x_2)^2 . \end{aligned}$$

Solution 2. Let $a = \dfrac{x_1}{x_1+x_2}$ and $b = \dfrac{y_1}{y_1+y_2}$. Then $1 - a = \dfrac{x_2}{x_1+x_2}$ and $1 - b = \dfrac{y_2}{y_1+y_2}$, and the original inequality can be rewritten in the form

$$\frac{a^2}{b} + \frac{(1-a)^2}{1-b} \geq 1 ,$$

where a, $b \in (0,1)$. This is successively equivalent to

$$\begin{aligned} a^2(1-b) + b(1-a)^2 &\geq b(1-b) , \\ a^2 - a^2b + b - 2ab + a^2b &\geq b - b^2 , \end{aligned}$$

or $a^2 + b^2 \geq 2ab$, which is true.

Solution 3. Since $\frac{x^2}{y} \geq 2x - y$ for $y > 0$, we apply this twice to obtain

$$\frac{a^2}{b} + \frac{(1-a)^2}{1-b} \;\geq\; (2a - b) + \big(2(1-a) - (1-b)\big) \;=\; 1\,,$$

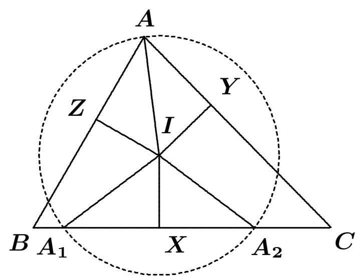
where a and b are defined as in Solution 2.

Comment. The original inequality is very simple relative to the high level of math olympiads, but it is a good occasion to perform different elementary techniques at an introductory level and, as well, for generalizations obtained by applying the Cauchy-Schwarz Inequality to $(\sqrt{y_1}, \sqrt{y_2}, \ldots, \sqrt{y_n})$ and $\left(\frac{x_1}{\sqrt{y_1}}, \frac{x_2}{\sqrt{y_2}}, \ldots, \frac{x_n}{\sqrt{y_n}}\right)$, where each x_i and y_i is a positive number:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \;\geq\; \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}\,.$$

5. Let I be the incentre of triangle ABC. Let points $A_1 \neq A_2$ lie on the line BC, points $B_1 \neq B_2$ lie on the line AC, and points $C_1 \neq C_2$ lie on the line AB so that $AI = A_1I = A_2I$, $BI = B_1I = B_2I$, $CI = C_1I = C_2I$. Prove that $A_1A_2 + B_1B_2 + C_1C_2 = P$, where P is the perimeter of $\triangle ABC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Amengual Covas.



Let X, Y, Z be the feet of the perpendiculars from I to the sides BC, CA, AB respectively. Then $IX = IY = IZ$. Right triangles IAZ, IA_1X, IA_2X, IYA are congruent, and since X is the mid-point of segment A_1A_2, we have

$$A_1A_2 \;=\; A_1X + XA_2 \;=\; AZ + YA\,.$$

Similarly, we obtain

$$\begin{aligned} B_1B_2 &= B_1Y + YB_2 = BX + ZB \\ \text{and} \quad C_1C_2 &= C_1Z + ZC_2 = CY + XC\,. \end{aligned}$$

Therefore,

$$\begin{aligned} A_1A_2 + B_1B_2 + C_1C_2 &= (AZ + YA) + (BX + ZB) + (CY + XC) \\ &= (AZ + ZB) + (BX + XC) + (CY + YA) \\ &= AB + BC + CA \;=\; P\,. \end{aligned}$$

7. A function $f(x)$ is defined on the interval $[0,1]$, so that $f(0)=f(1)=0$ and

$$f\left(\frac{a+b}{2}\right) \;\leq\; f(a)+f(b)\,.$$

for all a and b from $[0,1]$.

(a) Show that the equation $f(x)=0$ has infinitely many solutions on $[0,1]$.

(b) Are there functions on $[0,1]$ which satisfy the above conditions but are not identically zero?

Solved by Ioannis Katsikis, Athens, Greece; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give the solution of Katsikis.

(a) For $a=b\in[0,1]$, we have that $f(a)\leq 2f(a)$, hence $f(a)\geq 0$, for all $a\in[0,1]$. Let $a_n=\left(\frac{1}{2}\right)^n$, $n\geq 1$. We will show by induction that $f(a_n)=0$ for each n.

Taking $a=0$ and $b=1$, we have $f\left(\frac{0+1}{2}\right)\leq f(0)+f(1)$, which gives $f\left(\frac{1}{2}\right)\leq 0$; whence, $f(a_1)=f\left(\frac{1}{2}\right)=0$. Next, we show that if $f(a_n)=0$, then $f(a_{n+1})=0$. Taking $a=0$ and $b=a_n$, the basic inequality yields $f\left(\frac{0+a_n}{2}\right)\leq f(0)+f(a_n)=0$. Since $f\left(\frac{a_n}{2}\right)\leq 0$, we have $f(a_{n+1})=0$.

(b) Let $c>0$. Define the function

$$f(x) \;=\; \begin{cases} 0\,, & \text{if } x\in[0,1]\cap\mathbb{Q}, \\ c\,, & \text{if } x\in[0,1]\cap(\mathbb{R}\setminus\mathbb{Q}). \end{cases}$$

We will prove that f satisfies the condition of the hypothesis.

Case 1. Let x_1 and x_2 be rational numbers.

Then we have

$$f\left(\frac{x_1+x_2}{2}\right) \;=\; 0 \;\leq\; 0+0 \;=\; f(x_1)+f(x_2)\,.$$

Case 2. Let x_1 be a rational number and x_2 an irrational number.

Then we have

$$f\left(\frac{x_1+x_2}{2}\right) \;=\; c \;\leq\; 0+c \;=\; f(x_1)+f(x_2)\,.$$

Case 3. Let x_1 and x_2 be irrational numbers.

Then we have

$$f\left(\frac{x_1+x_2}{2}\right) \;\leq\; c \;<\; c+c \;=\; f(x_1)+f(x_2)\,.$$

Thus, there exist infinitely many functions on $[0, 1]$ that are not identically zero, and such that the conditions of the hypothesis hold.

8. Prove that any prime number $2^{2^n}+1$ cannot be represented as a difference of two fifth powers of integers.

Solution by Ioannis Katsikis, Athens, Greece.

For $n = 1$, we have $2^{2^n} + 1 = 5$. For integers x and y such that

$$5 = x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4),$$

we have $x - y = 1$, or $x = y + 1$. Generally, if $a \equiv U \pmod 5$, where $1 \leq U \leq 4$, then $a^5 \equiv U \pmod 5$. Thus, from $5 = x^5 - y^5$, we get $x \equiv y \pmod 5$, which contradicts the fact that $x = y + 1$. Consequently, 5 is not the difference of two fifth powers of integers.

For $n > 1$, the equation $2^{2^n} + 1 = x^5 - y^5$ again implies $x = y + 1$, since $2^{2^n} + 1$ is prime. Thus,

$$\begin{aligned} 2^{2^n} + 1 &= (y + 1)^4 + (y + 1)^3y + (y + 1)^2y^2 + (y + 1)y^3 + y^4 \\ &= 5y^4 + 10y^3 + 10y^2 + 5y + 1, \end{aligned}$$

which is a contradiction, since 2^{2^n} cannot be a multiple of 5.

That completes the material on file for this number of the *Corner*. Send me your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

Math Through the Ages: A Gentle History for Teachers and Others (*Expanded Edition*)
By William P. Berlinghoff and Fernando Q. Gouvêa, published by the Mathematical Association of America and Oxton House Publishers, 2004
ISBN 0-88385-736-7, hardcover, 273+xii pages, US$47.95
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB*

This book is not typical of those received for review in ***CRUX with MAYHEM***. As the book review editor, I must set some books aside as they clearly do not fall within the mandate of the journal. Of course, this is a problem-solving journal. Usually problems tend to appear like those in the various collections seen throughout the journal. However, problem-solving from an historical perspective takes on other images. It is hard to appreciate the challenges of doing mathematics without a standard notation for operational symbols or in the absence of a symbol to represent zero. *Math Through the Ages* offers insight into such problems through "sketches" including *Reading and Writing Arithmetic: Where the Symbols Came From* and *Nothing Becomes a Number: The Story of Zero*.

Math Through the Ages is divided into two core sections: The History of Mathematics in a Large Nutshell (approximately sixty pages in length) and Sketches (twenty-five pieces of about six to eight pages each, intended to "open up a deeper understanding of both the mathematics and the historical context of each topic covered"). Any teacher of mathematics will appreciate the gentleness in that one can open the book at any sketch to learn more about mathematics. Each sketch concludes with a collection of questions and project ideas that seem designed more to focus discussion in a class that may use the book as a text. However, there are mathematical problems sprinkled within the questions. These take the form of justifications, explanations, calculations, analysis of situations (as in the discussion of probability), and derivations (as with the Pythagorean Theorem).

The original publisher, Oxton House (Farmington, Maine), provided a very inexpensive book that was used as a text in various colleges. Subsequently the Mathematical Association of America (MAA) adopted the book for publication. The hard cover along with extensions of the questions and projects were the noteworthy changes in the "expanded edition". In fact, the MAA can easily reach a larger audience; however, the approximate doubling of the price reinforces the fact that those of us who obtained the original green paperback version have a good book in a very accessible form. *Math Through the Ages* would be a valuable acquisition as a library addition or teaching resource. The breadth of the book's content blends with the stories and its alternate presentation to provide insight to any reader wishing to learn more about topics that they commonly encounter in teaching or studies.

The book concludes with an extensive listing of suggested resources and references. Those interested in learning more about the history of mathematics may avail themselves of a compact disc entitled *Historical Modules for the Teaching and Learning of Mathematics* edited by Victor J. Katz and Karen Dee Michalowicz. The authors recommend it as a forthcoming resource that was published later in 2004 by the MAA. The eleven modules are designed as teaching units for mainly secondary school topics including statistics, combinatorics, Archimedes, polynomials, and functions.

Minnesota Math League XXV 1980–2005
By A. Wayne Roberts, Beaver's Pond Press, 2005
ISBN 978-1-59298-111-3, softcover, 242+ix pages, US$25.
Reviewed by **Robert L. Crane**, *Sydney Academy, Sydney, NS*

Having taught high school mathematics for the past 34 years and overseeing our school's active participation in the Nova Scotia Math League (a high school mathematics competition in Nova Scotia) since its inception five years ago, my interest was piqued by the title. For any high school which has not yet participated in such mathematics competitions, this book should be considered as a primary resource to illuminate the path toward successful participation.

One of the accomplishments of *Minnesota Math League* is to document in a coherent manner what its back cover promises: ". . . a marshalling of evidence that should lay to rest the myth that mathematical talent is an inborn gift that will develop with or without special programs." To mathematics teachers who love their craft, this book preaches to the converted. However, it does show that the importance of teachers in the process can never be underestimated.

The author reveals that good things just do not happen on their own. To go from zero to a well-organized and involved series of mathematics competitions takes first the recognition of a need and then the commitment to see it through. The pride of what has been accomplished in Minnesota rings true throughout the book.

Thirty pages are dedicated to questions used in Minnesota High School Mathematics League: the individual and team events. Considering that coaching a team takes time to prepare, these can be used as a valuable resource in a pinch. For schools not yet involved in mathematics competitions, the format of this competition should also be of interest.

About forty percent of the book recognizes the work and accomplishments of coaches, students, and teams over the first twenty-five year history of the Minnesota Math League. Whether this would be of great interest to non-Minnesotans I will leave for each individual reader to judge.

I am grateful for the chance to read and review the book; however, I would have reservations about purchasing it.

Sharpening the Hadwiger-Finsler Inequality

Cezar Lupu and Cosmin Pohoaţă

In memory of Alexandru Lupaş

1 Introduction and Preliminaries.

The Hadwiger–Finsler Inequality is known in the literature as a generalization of the following:

Theorem 1 In any triangle ABC with side lengths a, b, c, and area S, the following inequality holds:

$$a^2+b^2+c^2 \ \geq\ 4S\sqrt{3}\,.$$

This inequality is due to Weitzenböck [1] but also appeared in the International Mathematical Olympiad in 1961. In [5], one can find eleven proofs. In fact, in any triangle ABC the following sequence of inequalities is valid:

$$\begin{aligned} a^2+b^2+c^2 &\geq ab+bc+ca \ \geq\ a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \\ &\geq 3\sqrt[3]{a^2b^2c^2} \ \geq\ 4S\sqrt{3}\,. \end{aligned}$$

In 1937, Finsler and Hadwiger found a stronger version [2]:

Theorem 2 In any triangle ABC with side lengths a, b, c, and area S, the following inequality holds:

$$a^2+b^2+c^2 \ \geq\ 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2\,.$$

Now, we give an algebraic inequality due to Schur (See [3], for example), namely

Theorem 3 For non-negative x, y, z, and positive t, we have

$$x^t(x-y)(x-z)+y^t(y-x)(y-z)+z^t(z-y)(z-x) \ \geq\ 0\,.$$

When $t=1$ (the most common case), this is successively equivalent to:

$$\begin{aligned} x^3+y^3+z^3+3xyz &\geq xy(x+y)+yz(y+z)+zx(z+x)\,, \\ x^3+y^3+z^3+6xyz &\geq (x+y+z)(xy+yz+zx)\,. \end{aligned}$$

Since

$$x^3+y^3+z^3-3xyz \ =\ (x+y+z)(x^2+y^2+z^2-xy-yz-zx)\,,$$

one can easily deduce that

$$x^2+y^2+z^2+\frac{9xyz}{x+y+z} \ \geq\ 2(xy+yz+zx)\,.$$

We let $m = 1/x$, $n = 1/y$, and $p = 1/z$ to get the equivalent form below:

Theorem 4 For any positive reals m, n, and p, we have

$$\frac{mn}{p} + \frac{np}{m} + \frac{mp}{n} + \frac{9mnp}{mn+np+mp} \;\geq\; 2(m+n+p)\,.$$

2 Main result.

Our refinement of the Hadwiger–Finsler Inequality is as follows.

Theorem 5 In any triangle ABC with side lengths a, b, c, area S, inradius r, and circumradius R, the following inequality is valid:

$$a^2+b^2+c^2 \;\geq\; 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2 + (b-c)^2 + (c-a)^2\,.$$

Proof: In Theorem 4, we set $m = \frac{1}{2}(b+c-a)$, $n = \frac{1}{2}(c+a-b)$, and $p = \frac{1}{2}(a+b-c)$. This yields

$$\sum_{\text{cyclic}} \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{9(b+c-a)(c+a-b)(a+b-c)}{\sum_{\text{cyclic}}(b+c-a)(c+a-b)} \;\geq\; 2(a+b+c)\,.$$

Let s be the semiperimeter. Since

$$ab+bc+ca \;=\; s^2+r^2+4Rr \qquad (2)$$

$$\text{and} \quad a^2+b^2+c^2 \;=\; 2(s^2-r^2-4Rr)\,, \qquad (3)$$

we deduce that

$$\sum_{\text{cyclic}}(b+c-a)(c+a-b) \;=\; 4r(4R+r)\,.$$

On the other hand, we have $(b+c-a)(c+a-b)(a+b-c) = 8sr^2$ by Heron's Formula; hence, our inequality is successively equivalent to

$$\sum_{\text{cyclic}} \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{18sr}{4R+r} \;\geq\; 4s\,;$$

$$\sum_{\text{cyclic}} \frac{(s-a)(s-b)}{(s-c)} + \frac{9sr}{4R+r} \;\geq\; 2s\,;$$

$$\sum_{\text{cyclic}} (s-a)^2(s-b)^2 + \frac{9s^2r^3}{4R+r} \;\geq\; 2s^2r^2\,.$$

Now, according to the identity

$$\sum_{\text{cyclic}} (s-a)^2(s-b)^2 \;=\; \Big(\sum_{\text{cyclic}}(s-a)(s-b)\Big)^2 - 2s^2r^2\,,$$

we have

$$\Bigg(\sum_{\text{cyclic}} (s-a)(s-b)\Bigg)^2 - 2s^2r^2 + \frac{9s^2r^3}{4R+r} \;\geq\; 2s^2r^2 \,.$$

Since $\sum\limits_{\text{cyclic}} (s-a)(s-b) = r(4R+r)$, it follows that

$$r^2(4R+r)^2 + \frac{9s^2r^3}{4R+r} \;\geq\; 4s^2r^2 \,,$$

which can be rewritten as

$$\left(\frac{4R+r}{s}\right)^2 + \frac{9r}{4R+r} \;\geq\; 4 \,.$$

From the identities (2) and (3) we deduce that

$$\frac{4R+r}{s} \;=\; \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S} \,,$$

so our final succession of inequalities is

$$\left(\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}\right)^2 \;\geq\; 4 - \frac{9r}{4R+r} \,,$$

$$\left(\frac{(a^2+b^2+c^2)-((a-b)^2+(b-c)^2+(c-a)^2)}{4S}\right)^2 \;\geq\; 3 + \frac{4(R-2r)}{4R+r} \,,$$

$$a^2+b^2+c^2 \;\geq\; 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2+(b-c)^2+(c-a)^2 \,,$$

the last of which is the desired refinement. ∎

We remark that by using Euler's Inequality, $R \geq 2r$, we get Theorem 2.

3 Applications.

In this section, we give some basic applications of our refinement of the Hadwiger–Finsler Inequality. We begin with

Problem 1. In any triangle ABC with sides of lengths a, b, c and with exradii r_a, r_b, r_c, prove that

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \;\geq\; 2\sqrt{3+\frac{4(R-2r)}{4R+r}} \,.$$

Solution: From the well-known relation $r_a = S/(s-a)$ and its analogues, the inequality is equivalent to

$$\begin{aligned} \frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \;&=\; \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2S} \\ &\geq\; 2\sqrt{3+\frac{4(R-2r)}{4R+r}} \,, \end{aligned}$$

where the last inequality follows from Theorem 5. ∎

Problem 2. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \;\geq\; \sqrt{3+\frac{4(R-2r)}{4R+r}}\,.$$

Solution: From the cosine law we get $a^2 = b^2+c^2-2bc\cos A$. Since $S=\frac{1}{2}bc\sin A$, it follows that

$$a^2 \;=\; (b-c)^2+4S\cdot\frac{1-\cos A}{\sin A}\,.$$

On the other hand, by the trigonometric formulae $1-\cos A = 2\sin^2\frac{A}{2}$ and $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, we get

$$a^2 \;=\; (b-c)^2+4S\tan\frac{A}{2}\,.$$

Doing the same for all sides of the triangle ABC and adding up we obtain

$$\begin{aligned} a^2+b^2+c^2 \;&=\; (a-b)^2+(b-c)^2+(c-a)^2 \\ &\qquad +4S\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)\,. \end{aligned}$$

Now the inequality follows from Theorem 5. ■

The following are left as exercises:

Problem 3. In any triangle ABC with sides of lengths a, b, c and with corresponding exradii and altitudes r_a, r_b, r_c and h_a, h_b, h_c, prove that

$$\frac{1}{h_ar_a}+\frac{1}{h_br_b}+\frac{1}{h_cr_c} \;\geq\; \frac{1}{S}\sqrt{3+\frac{4(R-2r)}{4R+r}}\,.$$

Problem 4. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{c+a-b} \;\geq\; \frac{1}{2r}\sqrt{4-\frac{9r}{4R+r}}\,.$$

Problem 5. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c} \;\geq\; 3R\sqrt{4-\frac{9r}{4R+r}}\,.$$

[We note that by combining the above with Euler's Inequality, $R\geq 2r$, we obtain the weaker result

$$\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c} \;\geq\; 3R\sqrt{3}\,,$$

which represents an old proposal of Laurenţiu Panaitopol at the Romanian IMO Team Selection Test, held in 1990.]

Problem 6. In any triangle ABC with sides of lengths a, b, and c, and with corresponding exradii r_a, r_b, and r_c, prove that

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c} \geq \frac{s(5R-r)}{R(4R+r)}.$$

Problem 7. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{c+a-b} \geq \frac{1}{2r}\sqrt{4-\frac{9r}{4R+r}}.$$

Problem 8. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{a(b+c-a)}+\frac{1}{b(c+a-b)}+\frac{1}{c(a+b-c)} \geq \frac{r}{8R}\left(5-\frac{9r}{4R+r}\right).$$

Problem 9. In any triangle ABC with sides of lengths a, b, and c, prove that

$$\frac{1}{(b+c-a)^2}+\frac{1}{(c+a-b)^2}+\frac{1}{(a+b-c)^2} \geq \frac{1}{r^2}\left(\frac{1}{2}-\frac{9r}{4(4R+r)}\right).$$

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[3] G.N. Watson, Schur's Inequality, *The Mathematical Gazette*, **39** (1955), 207–208.

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[5] A. Engel, *Problem-Solving Strategies*, Springer Verlag, New York, 1998.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er septembre 2008**. *Une étoile* (⋆) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3313. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne et Pantelimon George Popescu, Bucarest, Roumanie.*

Soit $x_1, x_2, \ldots, x_n$ des nombres réels tels que $x_k > 1$ pour $1 \leq k \leq n$. Si l'on pose $x_{n+1} = x_1$, montrer que

$$\frac{1}{n}\sum_{k=1}^{n}\left(\log_{x_k} x_{k+1} + \log_{x_{k+1}} x_k\right) \leq \left(\prod_{k=1}^{n}\left(1 + \log_{x_k}^{n} x_{k+1}\right)\right)^{\frac{1}{n}}.$$

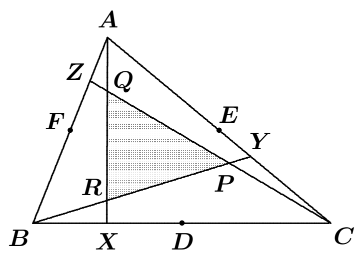
3314. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sum_{\text{cyclique}} \frac{a}{b} \geq \frac{3}{4} + \sum_{\text{cyclique}} \frac{(a+c)^2 + (a+b)c}{(b+c)(2a+b+c)}.$$

3315. *Proposé par Stanley Rabinowitz, MathPro Press, Chelmsford, MA, É-U.*

Soit D, E et F les milieux respectifs des côtés BC, CA et AB du triangle ABC. Soit respectivement X, Y et Z des points sur les segments BD, CE et AF. Les droites AX, BY et CZ bordent un triangle central (en ombragé dans la figure). Soit respectivement X', Y' et Z' les réflexions de X, Y et Z par rapport aux points D, E et F. A leur tour, les points X', Y' et Z' déterminent un autre triangle central $P'Q'R'$.



Montrer que

$$\frac{2+\sqrt{3}}{4} \leq \frac{[PQR]}{[P'Q'R']} \leq 8 - 4\sqrt{3},$$

où $[STU]$ représente l'aire du triangle STU.

3316. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sum_{\text{cyclique}} \frac{a}{b} + \Big(\sum_{\text{cyclique}} a^2\Big)^{\frac{1}{2}}\Big(\sum_{\text{cyclique}} \frac{1}{a^2}\Big)^{\frac{1}{2}} \ \geq\ \frac{2}{3}\Big(\sum_{\text{cyclique}} a\Big)\Big(\sum_{\text{cyclique}} \frac{1}{a}\Big).$$

3317. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\begin{aligned}\Big(\sum_{\text{cyclique}} \frac{a^3}{b^2-bc+c^2}\Big)\Big(\sum_{\text{cyclique}} \frac{b^2c^2}{a^3(b^2-bc+c^2)}\Big) &\\ \geq\ \Big(\sum_{\text{cyclique}} a\Big)\Big(\sum_{\text{cyclique}} \frac{1}{a}\Big) &\\ \geq\ 16\Big(\sum_{\text{cyclique}} \frac{ab}{a+b+2c}\Big)\Big(\sum_{\text{cyclique}} \frac{c}{2ab+bc+ca}\Big). &\end{aligned}$$

3318. *Proposé par D.E. Prithwijit, University College Cork, République d'Irlande.*

On suppose que les hauteurs AD, BE et CF d'un triangle ABC coupent respectivement le cercle circonscrit aux points X, Y et Z. Montrer que

$$\frac{AX}{AD}+\frac{BY}{BE}+\frac{CZ}{CF} \ =\ 4.$$

3319. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit m un nombre naturel avec $m \geq 2$, et soit r un nombre réel quelconque avec $r \geq 1/m$. Si a et b sont des nombres réels positifs satisfaisant $ab = r^2$, montrer que

$$\frac{1}{(1+a)^m}+\frac{1}{(1+b)^m} \ \geq\ \frac{2}{(1+r)^m}.$$

3320. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle d'angle droit en A et soit O le milieu de BC. Soit M un point dans le plan du triangle ABC, et désignons respectivement par M', M'', N, N' et N'' les orthocentres des triangles MAB, MAC, $AM'M''$, NAB et NAC. Si O est le point milieu de $M'M''$, montrer que O est aussi le point milieu de $N'N''$.

3321. *Proposé par Michel Bataille, Rouen, France.*

Le cercle inscrit du triangle ABC a son centre en I et touche les côtés AC et AB en E et F respectivement. Si M est un point sur le segment EF, montrer que les triangles MAB et MCA ont même aire si et seulement si $MI \perp BC$.

3322. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit a, b et c trois nombres réels non négatifs tels que $a \leq b \leq c$, et soit n un entier positif. Montrer que

$$\big(a+(n+1)b\big)\big(b+(n+2)c\big)(c+na) \;\geq\; (n+1)(n+2)(n+3)abc\,.$$

3323. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit a, b et c trois nombres réels non négatifs tels que $a^2+b^2+c^2=1$. Montrer que

$$\sum_{\text{cyclique}} (1-2a^2)(b-c)^2 \;\geq\; 0\,.$$

3324. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit a, b et c trois nombres réels non négatifs tels que $a^2+b^2+c^2=1$. Montrer que

$$3-5(ab+bc+ca)+6abc(a+b+c) \;\geq\; 0\,.$$

3325. *Proposé par Manuel Benito Muñoz, IES P.M. Sagasta, Logroño, Espagne.*

On désigne par $\sigma(n)$ la somme des diviseurs du nombre naturel n.

(a) Trouver un nombre naturel n tel que

$$\sigma(n)+500 \;=\; \sigma(n+2)\,.$$

(b)★ Quel est le nombre de solutions de la partie (a)?

. . . .

3313. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let x_1, x_2, $\ldots$, x_n be real numbers such that $x_k > 1$ for $1 \leq k \leq n$. If we set $x_{n+1} = x_1$, prove that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\log_{x_k} x_{k+1} + \log_{x_{k+1}} x_k\right) \;\leq\; \left(\prod_{k=1}^{n}\left(1+\log_{x_k}^{n} x_{k+1}\right)\right)^{\frac{1}{n}}.$$

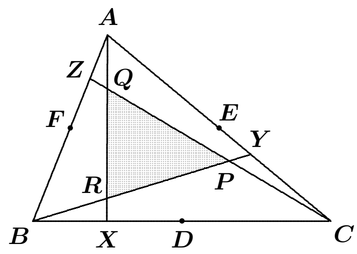
3314. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\sum_{\text{cyclic}} \frac{a}{b} \;\geq\; \frac{3}{4} + \sum_{\text{cyclic}} \frac{(a+c)^2+(a+b)c}{(b+c)(2a+b+c)}\,.$$

3315. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Let D, E, and F be the mid-points of sides BC, CA, and AB, respectively, in $\triangle ABC$. Let X, Y, and Z be points on the segments BD, CE, and AF, respectively. The lines AX, BY, and CZ bound a central triangle (shaded in the diagram). Let X', Y', and Z' be the reflections of X, Y, and Z in the points D, E, and F, respectively. The points X', Y', and Z' determine in a similar manner another central triangle $P'Q'R'$.



Prove that

$$\frac{2+\sqrt{3}}{4} \;\le\; \frac{[PQR]}{[P'Q'R']} \;\le\; 8-4\sqrt{3}\,,$$

where $[STU]$ represents the area of $\triangle STU$.

3316. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\sum_{\text{cyclic}} \frac{a}{b} + \Big(\sum_{\text{cyclic}} a^2\Big)^{\frac{1}{2}}\Big(\sum_{\text{cyclic}} \frac{1}{a^2}\Big)^{\frac{1}{2}} \;\ge\; \frac{2}{3}\Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} \frac{1}{a}\Big)\,.$$

3317. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\begin{aligned}&\Big(\sum_{\text{cyclic}} \frac{a^3}{b^2-bc+c^2}\Big)\Big(\sum_{\text{cyclic}} \frac{b^2c^2}{a^3(b^2-bc+c^2)}\Big)\\ &\quad\ge\; \Big(\sum_{\text{cyclic}} a\Big)\Big(\sum_{\text{cyclic}} \frac{1}{a}\Big) \;\ge\; 16\Big(\sum_{\text{cyclic}} \frac{ab}{a+b+2c}\Big)\Big(\sum_{\text{cyclic}} \frac{c}{2ab+bc+ca}\Big)\,.\end{aligned}$$

3318. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

The altitudes AD, BE, and CF of $\triangle ABC$ are produced to meet the circumcircle at X, Y, and Z, respectively. Prove that

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} \;=\; 4\,.$$

3319. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let m be a natural number, $m \geq 2$, and let r be any real number such that $r \geq 1/m$. If a and b are positive real numbers satisfying $ab = r^2$, prove that

$$\frac{1}{(1+a)^m} + \frac{1}{(1+b)^m} \;\geq\; \frac{2}{(1+r)^m}\,.$$

3320. *Proposed by Michel Bataille, Rouen, France.*

Let $\triangle ABC$ be right-angled at A and let O be the mid-point of BC. Let M be a point in the plane of $\triangle ABC$, and let M', M'', N, N', and N'' denote the orthocentres of $\triangle MAB$, $\triangle MAC$, $\triangle AM'M''$, $\triangle NAB$, and $\triangle NAC$, respectively. If O is the mid-point of $M'M''$, show that O is also the mid-point of $N'N''$.

3321. *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of $\triangle ABC$ have centre I and meet the sides AC and AB at E and F, respectively. For a point M on the line segment EF, show that $\triangle MAB$ and $\triangle MCA$ have the same area if and only if $MI \perp BC$.

3322. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be non-negative real numbers such that $a \leq b \leq c$, and let n be a positive integer. Prove that

$$\big(a+(n+1)b\big)\big(b+(n+2)c\big)(c+na) \;\geq\; (n+1)(n+2)(n+3)abc\,.$$

3323. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be non-negative real numbers with $a^2+b^2+c^2 = 1$. Prove that

$$\sum_{\text{cyclic}} (1-2a^2)(b-c)^2 \;\geq\; 0\,.$$

3324. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be non-negative real numbers with $a^2+b^2+c^2 = 1$. Prove that

$$3 - 5(ab+bc+ca) + 6abc(a+b+c) \;\geq\; 0\,.$$

3325. *Proposed by Manuel Benito Muñoz, IES P.M. Sagasta, Logroño, Spain.*

Let $\sigma(n)$ denote the sum of the divisors of the natural number n.

(a) Find a natural number n such that

$$\sigma(n) + 500 \;=\; \sigma(n+2)\,.$$

(b)★ How many solutions are there to part (a)?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3214. [2007 : 110, 113] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be an acute-angled triangle.

(a) Prove that $\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} > \left(\frac{6}{\pi}\right)^2$.

(b) Prove that $A\cot A + B\cot B + C\cot C < \left(\frac{\pi}{2}\right)^2$.

(c)★ Determine the best constants $c_1 \ge \left(\frac{6}{\pi}\right)^2$ and $0 < c_2 < c_3 \le \left(\frac{\pi}{2}\right)^2$ such that

$$\sum_{\text{cyclic}} \frac{\tan A}{A} \ \ge\ c_1 \qquad \text{and} \qquad c_2 \ \le\ \sum_{\text{cyclic}} A\cot A \ \le\ c_3\,.$$

Composite of solutions by Roy Barbara, Lebanese University, Fanar, Lebanon; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Dragoljub Milošević, Pranjani, Serbia.

(a) and (c) Applying the AM–GM Inequality, we obtain (for any acute-angled triangle ABC)

$$\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} \ \ge\ 3\left(\frac{\tan A\tan B\tan C}{ABC}\right)^{\frac{1}{3}}\,, \tag{1}$$

and

$$\tan A + \tan B + \tan C \ \ge\ 3(\tan A\tan B\tan C)^{\frac{1}{3}}\,. \tag{2}$$

Using (2) and the well-known identity

$$\tan A\tan B\tan C \ =\ \tan A + \tan B + \tan C\,,$$

we get

$$\tan A\tan B\tan C \ \ge 3(\tan A\tan B\tan C)^{\frac{1}{3}}\,;$$

whence,

$$\tan A\tan B\tan C \ \ge\ 3\sqrt{3}\,. \tag{3}$$

Since $ABC \le \left(\frac{1}{3}(A+B+C)\right)^3 = \left(\frac{\pi}{3}\right)^3$, inequalities (3) and (1) imply that

$$\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} \ \ge\ \frac{9\sqrt{3}}{\pi}\,,$$

with equality if and only if $A = B = C = \frac{\pi}{3}$. Thus, $c_1 = \frac{9\sqrt{3}}{\pi}$ in part (c). Since $\frac{9\sqrt{3}}{\pi} > \left(\frac{6}{\pi}\right)^2$, part (a) is also proved.

(b) and (c) Since $\lim_{x\to 0} x\cot x = 1$, we may extend $h(x) = x\cot x$ to the continuous function $f(x)$ defined as

$$f(x) = \begin{cases} x\cot x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$. Next consider a lemma:

Lemma 1. The function $f(x) = x\cot x$ is concave on $\left[0, \frac{\pi}{2}\right]$.

Proof: The second derivative of $f(x)$ is

$$f''(x) = \frac{2(x\cos x - \sin x)}{\sin^3 x} < 0,$$

since $\tan x > x$ on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x)$ is concave on $\left[0, \frac{\pi}{2}\right]$. ■

Therefore,

$$f(A) + f(B) + f(C) \leq 3f\left(\frac{A+B+C}{3}\right);$$

that is,

$$A\cot A + B\cot B + C\cot C \leq \frac{\pi\sqrt{3}}{3},$$

with equality if and only if $A = B = C = \frac{\pi}{3}$. Thus, $c_3 = \frac{\pi\sqrt{3}}{3}$ in part (c). Since $\frac{\pi\sqrt{3}}{3} < \left(\frac{\pi}{2}\right)^2$, part (b) is also proved.

(c) It only remains to find c_2. For that purpose we introduce another lemma:

Lemma 2. Let f be a concave function on some real interval I, and let a, b, u, $v \in I$ be such that $a \leq u \leq v \leq b$ and $u + v = a + b$. Then $f(u) + f(v) \geq f(a) + f(b)$.

Proof: If $a = b$, the result is obvious. Now assume that $a < b$. Let ℓ be the line containing the points $(a, f(a))$ and $(b, f(b))$. Since ℓ is not vertical, it has finite slope. Let $y = Ax + B$ be the equation of ℓ. Since f is concave, we have $f(x) \geq Ax + B$ for every $x \in [a, b]$. In particular, we have

$$f(u) \geq Au + B \quad \text{and} \quad f(v) \geq Av + B.$$

Hence,

$$\begin{aligned} f(u) + f(v) &\geq (Au + B) + (Av + B) = A(u + v) + 2B \\ &= A(a + b) + 2B = (Aa + B) + (Ab + B) \\ &= f(a) + f(b). \end{aligned}$$

■

Without loss of generality, we may assume that $0 < A \le B \le C < \frac{\pi}{2}$. Note that $A + B > \frac{\pi}{2}$; hence, $0 < \frac{\pi}{2} - A < B$. Then $\frac{\pi}{2} - A < B \le C < \frac{\pi}{2}$ all belong to the interval $\left[0, \frac{\pi}{2}\right]$ and satisfy $B + C = \left(\frac{\pi}{2} - A\right) + \frac{\pi}{2}$. By Lemma 2 and the concavity of f, we have

$$f(B) + f(C) \ \ge\ f\left(\tfrac{\pi}{2} - A\right) + f\left(\tfrac{\pi}{2}\right).$$

Since $f\left(\frac{\pi}{2}\right) = 0$, we obtain $f(B) + f(C) = f\left(\frac{\pi}{2} - A\right)$. Then

$$\lambda \ =\ f(A) + f(B) + f(C) \ \ge\ f(A) + f\left(\tfrac{\pi}{2} - A\right).$$

Let $u = \min\left\{A,\ \frac{\pi}{2} - A\right\}$ and $v = \max\left\{A,\ \frac{\pi}{2} - A\right\}$. Then $0 < u \le v < \frac{\pi}{2}$ with $u + v = 0 + \frac{\pi}{2}$. By Lemma 2 again, we get

$$f(u) + f(v) \ \ge\ f(0) + f\left(\tfrac{\pi}{2}\right) \ =\ 1 + 0\,.$$

That is, $f(A) + f\left(\frac{\pi}{2} - A\right) \ge 1$. Therefore, $c_2 \ge 1$.

If we consider an acute isosceles triangle ABC with $AB = AC = k$ (a constant), then as $\angle A \to 0$ we see that $A \cot A \to 1$, $B \cot B \to 0$, and $C \cot C \to 0$. Hence, $\lambda \to 1$. Thus, $c_2 = 1$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; VEDULA N. MURTY, Dover, PA, USA (parts (a) and (b) only); VO QUOC BA CAN, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer (parts (a) and (b) only). Not all solvers of part (c) obtained a value for c_2.

3215. [2007 : 110, 113] *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*

Given any integers k, ℓ, m, greater than 2, an integer n is called *expressible* for (k, ℓ, m) if there exist positive real numbers $a_1, a_2, \ldots, a_k$ such that $\prod\limits_{i=1}^{k} a_i = 1$ and

$$\sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} a_{i+j-1}\right)^{m} \ =\ n\,,$$

where the subscripts are taken modulo k.

Suppose that for some (k, ℓ, m) the integer 2005 is expressible while 1987 is not. Find the ordered triple (k, ℓ, m).

Solution by Joel Schlosberg, Bayside, NY, USA.

We will show that $(74, 3, 3)$ and $(16, 5, 3)$ are the only two possibilities for the triple (k, ℓ, m). This will be deduced from the fact that n is expressible for (k, ℓ, m) if and only if $n \ge k\ell^m$.

Suppose that n is expressible for (k,ℓ,m). Then by repeated use of the AM–GM Inequality, we have

$$\begin{aligned} n &= \sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} a_{i+j-1}\right)^{m} \;\geq\; \sum_{i=1}^{k}\left(\ell\left(\prod_{j=1}^{\ell} a_{i+j-1}\right)^{\frac{1}{\ell}}\right)^{m} \\ &= \ell^m \sum_{i=1}^{k}\left(\prod_{j=1}^{\ell} a_{i+j-1}\right)^{\frac{m}{\ell}} \;\geq\; k\ell^m\left(\prod_{i=1}^{k}\left(\prod_{j=1}^{\ell} a_{i+j-1}\right)^{\frac{m}{\ell}}\right)^{\frac{1}{k}} \\ &= k\ell^m\left(\prod_{j=1}^{\ell}\prod_{i=1}^{k} a_{i+j-1}\right)^{\frac{m}{k\ell}} \;=\; k\ell^m\left(\prod_{j=1}^{\ell} 1\right)^{\frac{m}{k\ell}} \;=\; k\ell^m\,. \end{aligned}$$

For $a \geq 1$, let $f(a)$ be the value of $\sum\limits_{i=1}^{k}\left(\sum\limits_{j=1}^{\ell} a_{i+j-1}\right)^{m}$ when we set $(a_1,a_2,\dots,a_n) = (a,a^{-1},1,\dots,1)$. Then f is continuous, and $f(a)$ is clearly expressible for (k,ℓ,m) for all $a \geq 1$. Now, $f(1) = k\ell^m$, and $f(a) > a^m \geq a$; hence, $f(a) \to \infty$ as $a \to \infty$. By the Intermediate Value Theorem, for any integer $n \geq k\ell^m$, there exists some $a \geq 1$ such that $f(a) = n$, and thus, n is expressible for (k,ℓ,m).

Therefore, 2005 is expressible for (k,ℓ,m), and 1987 is not expressible for (k,ℓ,m), precisely when $1987 < k\ell^m \leq 2005$. In this case, $3\ell^3 \leq k\ell^m \leq 2005$, so that $3 \leq \ell \leq \lfloor (2005/3)^{1/3} \rfloor = 8$. For fixed ℓ the inequality $1987 < k\ell^m \leq 2005$ can hold for only finitely many m, and when ℓ and m are both fixed the inequality can hold for only finitely many k. We need only consider $\ell = 3$, 4, 5, and 7, for if $\ell = ab$, $b \geq 3$ and n is (k,l,m) expressible, then n is also (ka^m,b,m) expressible. Sifting the values of k, ℓ, and m reveals that (k,ℓ,m) is either $(74,3,3)$ or $(16,5,3)$.

There were two incomplete solutions submitted, each of which found only one of the two answers.

3216. [2007 : 110, 114] *Proposed by Mihály Bencze, Brasov, Romania.*

If a, b, c, and d are positive integers, prove that

$$45\left(\frac{1}{a+b+c+d+1} - \frac{1}{(a+1)(b+1)(c+1)(d+1)}\right) \leq 4 + \sum_{\text{cyclic}}\left[\frac{1}{a+1} + \frac{1}{(a+1)(b+1)}\right].$$

Solution by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

The solver proved this inequality for any non-negative real numbers a, b, c, and d. Setting $x = \frac{1}{4}(a+b+c+d)$, and applying the AM–GM Inequality, we have

$$\begin{aligned}
\frac{1}{(a+1)(b+1)(c+1)(d+1)} &\geq \frac{256}{(a+b+c+d+4)^4} = \frac{1}{(x+1)^4}\,,\\
\sum_{\text{cyclic}} \frac{1}{a+1} &\geq \frac{4}{\sqrt[4]{(a+1)(b+1)(c+1)(d+1)}}\\
&\geq \frac{16}{a+b+c+d+4} = \frac{4}{x+1}\,,\\
\text{and} \quad \sum_{\text{cyclic}} \frac{1}{(a+1)(b+1)} &\geq \frac{4}{\sqrt{(a+1)(b+1)(c+1)(d+1)}}\\
&\geq \frac{64}{(a+b+c+d+4)^2} = \frac{4}{(x+1)^2}\,.
\end{aligned}$$

It therefore suffices to show that

$$45\left(\frac{1}{4x+1} - \frac{1}{(x+1)^4}\right) \leq 4 + \frac{4}{x+1} + \frac{4}{(x+1)^2}\,,$$

or

$$\frac{16x^5 + 39x^4 - 86x^2 + 84x + 12}{(4x+1)(x+1)^4} \geq 0\,.$$

If $x \leq 1$, then

$$\begin{aligned}
&16x^5 + 39x^4 - 86x^2 + 84x + 12\\
&\qquad = 16x^5 + 39x^4 + 84x(1-x) + 2(1-x^2) + 10 > 0\,.
\end{aligned}$$

If $x > 1$, then

$$\begin{aligned}
&16x^5 + 39x^4 - 86x^2 + 84x + 12\\
&\qquad > 55x^3 - 86x^2 + 84x + 12\\
&\qquad = 43x(x-1)^2 + 12x^3 + 41x + 12 > 0\,.
\end{aligned}$$

Therefore the inequality holds and the equality does not.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer.

3217. [2007 : 110, 114] *Proposed by Michel Bataille, Rouen, France.*

Let $\{L_n\}$ be the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$. Prove that, for all non-negative integers n, we have

$$\sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+1}{2k+1}\frac{L_{2k}}{2^n} \;=\; \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{L_k}{2^{2k}}\,.$$

Solution by the proposer, expanded by the editor.

It is well known that $L_k = \alpha_1^k + \alpha_2^k$ for $k = 0, 1, 2, \ldots$, where $\alpha_1 = \frac{1}{2}(1+\sqrt{5}\,)$ and $\alpha_2 = \frac{1}{2}(1-\sqrt{5}\,)$ are the roots of $x^2 - x - 1 = 0$. The right side of the proposed identity then suggests the introduction of the polynomials $P_n(x) = \sum\limits_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\left(\frac{x}{4}\right)^k$, for $n = 0, 1, 2, \ldots$. It is readily seen that $P_0(x) = P_1(x) = 1$.

We claim that $P_n(x)$ satisfies the following recurrence relation:

$$P_{n+1}(x) \;=\; P_n(x) + \frac{x}{4}P_{n-1}(x)\,, \tag{1}$$

for all $n \in \mathbb{N}$. To establish (1), note first that

$$\begin{aligned}\frac{x}{4}P_{n-1}(x) &= \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-k}{k}\left(\frac{x}{4}\right)^{k+1}\\ &= \sum_{k=1}^{\lfloor (n-1)/2\rfloor+1}\binom{n-k}{k-1}\left(\frac{x}{4}\right)^{k}.\end{aligned}$$

If n is even (say $n = 2\ell$), then

$$\lfloor (n-1)/2\rfloor + 1 \;=\; \lfloor (n+1)/2\rfloor \;=\; \lfloor n/2\rfloor \;=\; \ell\,.$$

Hence,

$$\begin{aligned}P_n(x) + \frac{x}{4}P_{n-1}(x) &= \sum_{k=0}^{\ell}\binom{n-k}{k}\left(\frac{x}{4}\right)^k + \sum_{k=1}^{\ell}\binom{n-k}{k-1}\left(\frac{x}{4}\right)^k\\ &= \sum_{k=1}^{\ell}\left[\binom{n-k}{k}+\binom{n-k}{k-1}\right]\left(\frac{x}{4}\right)^k + 1\\ &= \sum_{k=1}^{\ell}\binom{n+1-k}{k}\left(\frac{x}{4}\right)^k + 1\\ &= \sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+1-k}{k}\left(\frac{x}{4}\right)^k \;=\; P_{n+1}(x)\,.\end{aligned}$$

Using the same argument with minor modification regarding the upper limits of the summations, we can easily show that (1) is also true when n is odd.

Now, suppose that $x > -1$. Since the characteristic equation of the recurrence relation given in (1) is $4w^2 - 4w - x = 0$, the characteristic roots are $q_1 = \frac{1}{2}\left(1 + \sqrt{1+x}\,\right)$ and $q_2 = \frac{1}{2}\left(1 - \sqrt{1+x}\,\right)$.

From known theory, the general solution to the recurrence relation in (1) is given by $P_n(x) = c_1 q_1^{n+1} + c_2 q_2^{n+1}$, where c_1 and c_2 are to be determined. Solving $c_1 q_1 + c_2 q_2 = P_0(x) = 1$ and $c_1 q_1^2 + c_2 q_2^2 = P_1(x) = 1$, we find that $c_1 = 1/\sqrt{1+x}$ and $c_2 = -1/\sqrt{1+x}$. Let $z = \sqrt{1+x}$. We then have

$$\begin{aligned} P_n(x) &= \frac{1}{z}\left[\left(\frac{1+z}{2}\right)^{n+1} - \left(\frac{1-z}{2}\right)^{n+1}\right] \\ &= \frac{1}{2^{n+1}z}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}z^k - \sum_{k=0}^{n+1}\binom{n+1}{k}(-z)^k\right] \\ &= \frac{1}{2^n z}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n+1}{2j+1}z^{2j+1} = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}(1+x)^k\,. \end{aligned}$$

Therefore, we have established the identity

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\left(\frac{x}{4}\right)^k = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}(1+x)^k\,. \tag{2}$$

Since $1 + \alpha_i = \alpha_i^2$ for $i = 1$ and $i = 2$, by substituting α_i for x in (2), we see that

$$\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\alpha_1^{2k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{\alpha_1^k}{4^k} \tag{3}$$

and

$$\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\alpha_2^{2k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{\alpha_2^k}{4^k}\,. \tag{4}$$

The proposed identity follows by adding (3) and (4).

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and EDMUND SWYLAN, Riga, Latvia.

The proposer remarked that from the proof given above, we readily see that $\{L_n\}$ can be replaced by any sequence $\{U_n\}$ satisfying the recursion $U_{n+1} = U_n + U_{n-1}$ and in particular, by the Fibonacci sequence.

3218. [2007 : 111, 114] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let n be an integer with $n \geq 2$. In $\mathbb{R}^n$, let E be the set of points $(x_1, x_2, \ldots, x_n)$ such that $x_i \geq 0$ for all i and $0 < x_1 + x_2 + \cdots + x_n \leq 1$. Calculate the integral over E of the fractional part of $\dfrac{1}{x_1 + x_2 + \cdots + x_n}$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of the real number x, let I be integral in question, and let $0 = z_0 < z_1 < \cdots < z_s = 1$ be a partition of $[0, 1]$ with mesh size $\Delta = \max\{\Delta z_1, \Delta z_2, \ldots, \Delta z_s\}$, where $\Delta z_k = z_k - z_{k-1}$ for $k = 1, 2, \ldots, s$. By straightforward multiple integration, the volume of the simplex determined by $0 < x_1 + x_2 + \cdots + x_n \le z$ with $x_i \ge 0$ (for all i) is $V(z) = z^n/n!$. The k^{th} slice of the simplex E is determined by $z_{k-1} < x_1 + x_2 + \cdots + x_n \le z_k$, which has volume $V(z_k) - V(z_{k-1})$; this is equal to $V'(z_k^*)\Delta z_k$ for some $z_k^* \in (z_{k-1}, z_k)$, by the Mean-Value Theorem. The point $(z_k^*, 0, 0, \ldots, 0)$ is in the k^{th} slice, and the integrand evaluates to $\{1/z_k^*\}$ at this point. Thus, the limit of the sum

$$S = \sum_{k=1}^{s} \left\{\frac{1}{z_k^*}\right\} \big(V(z_k) - V(z_{k-1})\big)$$

is I as $\Delta \to 0$. On the other hand, $S = \sum\limits_{k=1}^{s} \left\{\frac{1}{z_k^*}\right\} V'(z_k^*)\Delta z_k$, and, when viewed this way, the limit of S is

$$\int_0^1 \left\{\frac{1}{z}\right\} V'(z)\,dz$$

as $\Delta \to 0$. Therefore,

$$\begin{aligned}
I &= \int_0^1 \left\{\frac{1}{z}\right\} V'(z)\,dz = \int_0^1 \left\{\frac{1}{z}\right\} \frac{z^{n-1}}{(n-1)!}\,dz \\
&= \frac{1}{(n-1)!} \sum_{j=1}^{\infty} \int_{\frac{1}{j+1}}^{\frac{1}{j}} \left(\frac{1}{z} - j\right) z^{n-1}\,dz \\
&= \frac{1}{(n-1)!} \left[\int_0^1 z^{n-2}\,dz - \sum_{j=1}^{\infty} \int_{\frac{1}{j+1}}^{\frac{1}{j}} j z^{n-1}\,dz\right] \\
&= \frac{1}{(n-1)!} \left[\frac{1}{n-1} - \frac{1}{n}\left(\frac{1}{1^n} - \frac{1}{2^n} + \frac{2}{2^n} - \frac{2}{3^n} + \frac{3}{3^n} - \frac{3}{4^n} + \cdots\right)\right] \\
&= \frac{1}{(n-1)!} \left[\frac{1}{n-1} - \frac{1}{n}\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right)\right] \\
&= \frac{1}{(n-1)!} \left[\frac{1}{n-1} - \frac{\zeta(n)}{n}\right],
\end{aligned}$$

where ζ is the Riemann zeta function.

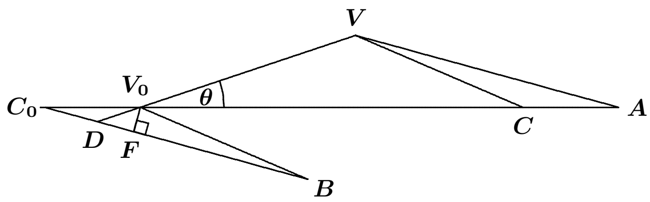
Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

Schlosberg obtains $1/(n-1)(n-1)!$ and $\zeta(n)/n!$ for the values of the integrals over E of $1/(x_1 + \cdots + x_n)$ and its integer part, respectively. The proposer passed from many variables to a single variable by using Liouville's result, that for suitable ϕ, the integral of $\phi(x_1 + \cdots + x_n)x_1^{p_1-1} \cdots x_n^{p_n-1}$ over E equals the integral of $\phi(z)z^{p_1+\cdots+p_n-1}$ over $[0, 1]$ multiplied by $\Gamma(p_1)\cdots\Gamma(p_n)/\Gamma(p_1 + \cdots + p_n)$.

3219. [2007 : 111, 114] *Proposed by Dan Vetter, Regina, SK.*

A vulture with a university education, when approached by a car while dining on the road, will always fly off in a direction chosen to maximize the distance of closest approach of the car. Show that the ratio of the speed of the car to the speed of the bird is $\sec\theta$, where θ is the angle that the vulture's flight path makes with the road.

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.



Let C and V be the positions of the car and vulture after t seconds, with C_0 and V_0 being their initial positions. The road then follows the line C_0C while the bird's flight path is along V_0V (with V_0 on C_0C). We denote $r = C_0V_0$, and the speeds of the car and the bird by v_c and v_b. With this notation, we have $\frac{C_0C}{V_0V} = \frac{v_c}{v_b}$. If A is the point on line C_0C for which $V_0A = C_0C$, we then have

$$\frac{V_0A}{V_0V} = \frac{v_c}{v_b}.$$

Next, let B be the point for which both $V_0B \parallel VC$ and $C_0B \parallel VA$, let D be the point where V_0V meets C_0B, and let F be the foot of the perpendicular from V_0 to C_0B. Since $C_0V_0 = CA$, the triangles VCA and BV_0C_0 are congruent. Consequently, $V_0B = CV$, the distance between the car and the vulture at time t. Because $\triangle V_0DC_0 \sim \triangle V_0VA$, we have $\frac{V_0D}{V_0V} = \frac{V_0C_0}{V_0A}$; that is, $V_0D = \frac{V_0V}{V_0A} \cdot C_0V_0$. Hence,

$$V_0D = \frac{v_b}{v_c} \cdot r.$$

In words, the length of V_0D is independent of the vulture's flight path. It follows that for a given flight path, the minimum distance between the car and the bird occurs when B is at F (so that V_0B and VC are perpendicular to C_0B). We therefore want to determine the flight path for which V_0F is as large as possible. But, for any flight path, we have $V_0D \geq V_0F$; that is, $\frac{v_b}{v_c} \cdot r \geq V_0F$. We interpret this to mean that $\frac{v_b}{v_c} \cdot r$ is the maximum value of the minimum distance between the car and the vulture. I believe that any well-educated vulture would choose the flight path that maximizes the distance V_0F—the direction in which V_0D coincides with V_0F (so that

$V_0D \perp C_0B$). Of course, we then have

$$\cos\theta = \cos VV_0A = \cos C_0V_0D = \frac{V_0D}{C_0V_0},$$

or

$$\sec\theta = \frac{r}{\frac{v_b}{v_c}\cdot r} = \frac{v_c}{v_b},$$

which is the desired relation.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer.

All other solvers used coordinates, an approach that produces a function to be maximized. The problem is therefore suitable as an exercise in a beginning calculus course, although the maximum can easily be obtained without derivatives. Demis added that he doubts that the vulture would resort to calculus. Indeed, it is doubtful that many vultures would exhibit the behaviour called for in our problem; however, dogs do seem to use an optimal strategy when fetching tennis balls according to the recent debate in The College Mathematics Journal, *34:3 (May 2003) 178–192, 37:1 (January 2006) 16–23, and 38:5 (November 2007) 356–361. In our problem, coordinates make clear what happens when the bird moves as fast or faster than the car. The proposer deduces that if the speeds are equal, the vulture must fly directly away from the car to maintain the minimum distance. When the bird is faster, the critical point of the distance function is irrelevant: the required minimum distance occurs when the bird takes flight, as long as it flies in a direction that has a large enough component away from the car.*

3220. [2007 : 111, 114] *Proposed by Marian Tetiva, Bîrlad, Romania.*

Let n be a positive integer. Prove that the set $\{1^2, 2^2, \ldots, n^2\}$ of the first n perfect squares can be partitioned into four subsets each having the same sum of elements if and only if $n = 8k$ or $n = 8k - 1$ for some integer $k \geq 2$.

Composite of similar solutions by Brian D. Beasley, Presbyterian College, Clinton, SC, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and the proposer.

Let $P(n)$ denote the statement that the set $\{1^2, 2^2, \ldots, n^2\}$ can be partitioned into four subsets each having the same sum of elements.

Suppose first that $P(n)$ is true. Then $4 \Bigm| \sum\limits_{m=1}^{n} m^2$, which implies that $24 \Bigm| n(n+1)(2n+1)$. Since $2n+1$ is odd and $(n, n+1) = 1$, we must have $8 \mid n$ or $8 \mid n+1$. Hence, $n = 8k$ or $n = 8k-1$ for some $k \geq 1$. However, since $\frac{1}{4}\sum\limits_{m=1}^{7} m^2 = 35 < 7^2$ and $\frac{1}{4}\sum\limits_{m=1}^{8} m^2 = 51 < 8^2$, we may eliminate $n = 7$ and $n = 8$ as possibilities. Therefore, $n = 8k$ or $n = 8k - 1$ for some $k \geq 2$.

To prove that the condition is sufficient, we first show that if $P(n)$ is true, then so is $P(n + 32)$. To see this, it suffices to show that for non-negative integers x, the set $S_x = \{x^2, (x+1)^2, \ldots, (x+31)^2\}$ can be partitioned into four subsets each having the same sum of elements. Define

$$\begin{aligned} B_1 &= \{0,\ 7,\ 11,\ 12,\ 18,\ 21,\ 25,\ 30\}\,, \\ B_2 &= \{1,\ 6,\ 8,\ 15,\ 19,\ 20,\ 26,\ 29\}\,, \\ B_3 &= \{2,\ 5,\ 9,\ 14,\ 16,\ 23,\ 27,\ 28\}\,, \\ B_4 &= \{3,\ 4,\ 10,\ 13,\ 17,\ 22,\ 24,\ 31\}\,; \end{aligned}$$

and then define $A_i = \{(x+a)^2 \mid a \in B_i\}$ for $i = 1, 2, 3, 4$.

Then by tedious but straightforward calculations, we find that the sum of the elements in A_k equals $8x^2 + 248x + 2604$ for $k = 1, 2, 3, 4$. Since $S_x = A_1 \cup A_2 \cup A_3 \cup A_4$, our claim is established.

To complete the proof, it now suffices to show that the statements $P(n)$ is true for $n \in \{15, 16, 23, 24, 31, 32, 39, 40\}$.

Setting $x = 0$ and $x = 1$ in the partition given above, we see that $P(31)$ and $P(32)$ are true. For the six remaining values of n, squaring each element below gives the required partitions.

$$\begin{aligned} &\{1, 2, \ldots, 15\} \\ &\quad = \{1, 7, 8, 14\} \cup \{2, 9, 15\} \cup \{3, 6, 11, 12\} \cup \{4, 5, 10, 13\}\,; \\ &\{1, 2, \ldots, 16\} \\ &\quad = \{1, 6, 9, 16\} \cup \{2, 4, 8, 11, 13\} \cup \{3, 5, 12, 14\} \cup \{7, 10, 15\}\,; \\ &\{1, 2, \ldots, 23\} \\ &\quad = \{1, 3, 10, 13, 19, 21\} \cup \{2, 8, 22, 23\} \cup \{4, 5, 9, 11, 15, 17, 18\} \\ &\qquad \cup \{6, 7, 12, 14, 16, 20\}\,; \\ &\{1, 2, \ldots, 24\} \\ &\quad = \{1, 5, 7, 10, 13, 15, 16, 20\} \cup \{4, 14, 22, 23\} \\ &\qquad \cup \{2, 3, 11, 17, 19, 21\} \cup \{6, 8, 9, 12, 18, 24\}\,; \\ &\{1, 2, \ldots, 39\} \\ &\quad = \{1, 8, 11, 15, 16, 19, 27, 28, 35, 37\} \cup \{2, 5, 20, 29, 30, 38, 39\} \\ &\qquad \cup \{3, 6, 7, 12, 13, 14, 17, 21, 22, 23, 26, 32, 33\} \\ &\qquad \cup \{4, 9, 10, 18, 24, 25, 31, 34, 36\}\,; \\ &\{1, 2, \ldots, 40\} \\ &\quad = \{1, 6, 18, 20, 32, 34, 35, 37\} \cup \{2, 7, 14, 15, 16, 28, 30, 39, 40\} \\ &\qquad \cup \{3, 4, 5, 9, 13, 21, 22, 27, 29, 36, 38\} \\ &\qquad \cup \{8, 10, 11, 12, 17, 19, 23, 24, 25, 26, 31, 33\}. \end{aligned}$$

This completes the proof.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA. There was one incomplete solution submitted.

It should be pointed out that the partitions of the sets S_x and $\{1^2, 2^2, \ldots, n^2\}$ for $n \in \{15, 16, 23, 24, 31, 32, 39, 40\}$ are not unique in general; for example,

$$\begin{aligned} \{1^2, 2^2, \ldots, 23^2\} &= \{1^2, 5^2, 8^2, 9^2, 15^2, 18^2, 19^2\} \cup \{2^2, 7^2, 12^2, 20^2, 22^2\} \\ &\quad \cup \{3^2, 4^2, 6^2, 11^2, 13^2, 17^2, 21^2\} \cup \{10^2, 14^2, 16^2, 23^2\} \end{aligned}$$

is an admissible partition different from the one given above.

3221. Correction. [2007 : 111, 114] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle with sides $a \geq b \geq c$ opposite the angles A, B, C, respectively. Let AH be perpendicular to the side BC with H on BC. Set $m = BH$ and $n = CH$. Prove that $a(bm + cn) - bc(b + c)$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

Essentially the same solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain; Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; Geoffrey A. Kandall, Hamden, CT, USA; Vedula N. Murty, Dover, PA, USA; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; D.J. Smeenk, Zaltbommel, the Netherlands; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Since $m = c\cos B$ and $n = b\cos C$, we have

$$a(bm + cn) \;=\; abc(\cos B + \cos C)\,.$$

Using the Law of Cosines repeatedly, we obtain

$$\begin{aligned} a(bm + cn) - bc(b + c) &= abc(\cos B + \cos C) - bc(b + c) \\ &= b(ac\cos B) + c(ab\cos C) - bc(b + c) \\ &= b\frac{a^2 + c^2 - b^2}{2} + c\frac{a^2 + b^2 - c^2}{2} - \frac{2bc(b + c)}{2} \\ &= \frac{a^2b - bc^2 - b^3 + a^2c - b^2c - c^3}{2} \\ &= \frac{b + c}{2}(a^2 - b^2 - c^2) \\ &= -bc(b + c)\cos A\,. \end{aligned}$$

Therefore, the expression $a(bm + cn) - bc(b + c)$ is positive, negative, or zero if and only if $\cos A$ is negative, positive, or zero, respectively.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; KATRINA BRICKER and NATALIE KALMINK, students, California State University, Fresno, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA (second solution); JOE HOWARD, Portales, NM, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands (second solution); EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comănești, Romania; and the proposer.

3222. [2007 : 111, 115] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Given positive real numbers a, b, c such that $a+b+c=1$, prove that

$$\frac{(1-a)(1-b)(1-c)}{\left(1-a^2\right)^2+\left(1-b^2\right)^2+\left(1-c^2\right)^2} \;\le\; \frac{1}{8}\,.$$

Solution by Michel Bataille, Rouen, France.

We prove that $L \geq 8$, where

$$L \;=\; \frac{(1-a^2)^2+(1-b^2)^2+(1-c^2)^2}{(1-a)(1-b)(1-c)} \;=\; \sum_{\text{cyclic}} \frac{(1-a)(1+a)^2}{(1-b)(1-c)}\,.$$

From

$$(1+a)^2 \;=\; (1+1-b-c)^2 \;=\; \big((1-b)+(1-c)\big)^2 \;\geq\; 4(1-b)(1-c)\,,$$

and similar inequalities for $(1+b)^2$ and $(1+c)^2$, it follows that

$$L \;\geq\; 4(1-a)+4(1-b)+4(1-c) \;=\; 8\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.M. MILOŠEVIĆ, Pranjani, Serbia; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; VO QUOC BA CAN, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3223. [2007 : 111, 115] *Proposed by Achilleas Pavlos Porfyriadis, student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece.*

Let a, b, c be positive real numbers which satisfy

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;=\; \frac{1}{abc}\,.$$

Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \;\le\; \frac{3\sqrt{3}}{4}\,.$$

Solution by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

The given condition is equivalent to $ab + bc + ca = 1$. Thus,

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a}{a^2+1} &= \sum_{\text{cyclic}} \frac{a}{a^2+ab+bc+ca} \\ &= \sum_{\text{cyclic}} \frac{a}{(a+b)(a+c)} = \frac{2}{(a+b)(b+c)(c+a)}.\end{aligned}$$

On the other hand, by the AM–GM Inequality, we have

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc \\ &\geq \tfrac{8}{9}(a+b+c)(ab+bc+ca) = \tfrac{8}{9}(a+b+c) \\ &\geq \tfrac{8}{9}\sqrt{3(ab+bc+ca)} = \tfrac{8}{9}\sqrt{3}.\end{aligned}$$

Therefore,

$$\sum_{\text{cyclic}} \frac{a}{a^2+1} \leq \frac{3\sqrt{3}}{4}.$$

Equality holds if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HASAN DENKER, Istanbul, Turkey; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

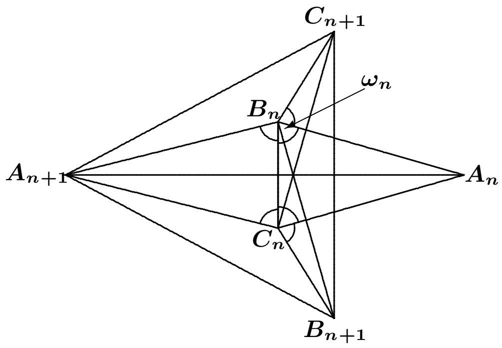
3224. [2007 : 112, 115] *Proposed by J. Chris Fisher and Harley Weston, University of Regina, Regina, SK.*

Let $A_0B_0C_0$ be an isosceles triangle whose apex angle A_0 is not 120°. We define a sequence of triangles $A_nB_nC_n$ in which $\triangle A_{i+1}B_{i+1}C_{i+1}$ is obtained from $\triangle A_iB_iC_i$ by reflecting each vertex in the opposite side (that is, B_iC_i is the perpendicular bisector of A_iA_{i+1}, and so forth). Prove that all three angles approach 60° as $n \to \infty$.

[*Ed.:* This problem is a special case of an open problem described by Judah Schwartz in "Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?", *International Journal of Computers for Mathematical Learning*, 4 (1999), pp. 99–119.]

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece, modified by the editor.

We start with the observation that the problem, as stated, is incorrect. The condition $\angle A_0 \neq 2\pi/3$ does not guarantee that the angles of $\triangle A_nB_nC_n$ approach $\pi/3$ as $n \to \infty$. In what follows, we will assume that $\angle A_0$ is such that $\angle A_n \neq 2\pi/3$ *for all* n. [*Ed.:* This immediately evokes the important question of whether such angles A_0 exist. Perhaps, our readers will share some insights on the set of "prohibited values" for $\angle A_0$; we will also say a bit more on that later. At this point, we will just assume that this set is not the entire interval $(0, \pi)$].



Let ω_n denote the angle $C_nB_nA_n$, $j_n = \cos 2\omega_n + i\sin 2\omega_n$ (then $\overline{j_n} = \cos 2\omega_n - i\sin 2\omega_n$), $x_n = \dfrac{B_nC_n}{A_nB_n}$, and a_n, b_n, and c_n be the complex numbers representing the points A_n, B_n, and C_n, respectively. An easy induction shows that $\triangle A_nB_nC_n$ is isosceles and that $0 < \omega_n < \frac{\pi}{2}$. Then $\angle C_nB_nC_{n+1} = \angle A_{n+1}B_nA_n = \angle B_{n+1}C_nB_n = 2\omega_n$, and

$$(a_n - c_n)j_n = b_n - a_n \,, \tag{1}$$
$$(c_n - b_n)j_n = c_{n+1} - b_n \,, \tag{2}$$
$$(a_{n+1} - b_n)j_n = a_n - b_n \,, \tag{3}$$
and
$$(b_{n+1} - c_n)j_n = b_n - c_n \,. \tag{4}$$

From equations (1), (4), and (2), respectively, we obtain

$$a_n = \frac{1}{j_n+1}b_n + \frac{j_n}{j_n+1}c_n \,, \tag{5}$$
$$b_{n+1} = \frac{1}{j_n}b_n + \frac{j_n-1}{j_n}c_n \,, \tag{6}$$
$$c_{n+1} = (1-j_n)b_n + j_nc_n \,, \tag{7}$$

and from (3) and (5), we get

$$a_{n+1} = \frac{j_n}{j_n+1}b_n + \frac{1}{j_n+1}c_n \,. \tag{8}$$

We have $\cos\omega_n = \dfrac{B_nC_n}{2A_nB_n}$ or $x_n = 2\cos\omega_n$. Clearly, $0 < x_n < 2$.

Using $j_n\overline{j_n} = 1$ and equations (6), (7), and (8), we obtain a recurrence for the sequence $\{x_n\}$:

$$\begin{aligned} x_{n+1} &= \frac{B_{n+1}C_{n+1}}{A_{n+1}B_{n+1}} = \frac{|c_{n+1}-b_{n+1}|}{|b_{n+1}-a_{n+1}|} = \frac{\left|\dfrac{j_n^2-j_n+1}{j_n}(c_n-b_n)\right|}{\left|\dfrac{j_n^2-j_n-1}{j_n(j_n+1)}(c_n-b_n)\right|} \\ &= \frac{|(j_n+1)(j_n^2-j_n+1)|}{|j_n^2-j_n-1|} = \frac{|j_n+1|\cdot|(j_n^2-j_n+1)\overline{j_n}|}{|(j_n^2-j_n-1)\overline{j_n}|} \\ &= \frac{|j_n+1|\cdot|(j_n+\overline{j_n}-1)|}{|(j_n-\overline{j_n}-1)|} \\ &= \frac{|\cos 2\omega_n+1+i\sin 2\omega_n|\cdot|2\cos 2\omega_n-1|}{|-1+2i\sin 2\omega_n|} \\ &= \frac{\sqrt{2+2\cos 2\omega_n}\cdot|2\cos 2\omega_n-1|}{\sqrt{1+4\sin^2 2\omega_n}} \\ &= \frac{\sqrt{2(1+\cos 2\omega_n)}\cdot|2(2\cos^2\omega_n-1)-1|}{\sqrt{1+16\sin^2\omega_n\cos^2\omega_n}} \\ &= \frac{2\cos\omega_n\cdot|4\cos^2\omega_n-3|}{\sqrt{1+16\cos^2\omega_n-16\cos^4\omega_n}} = \frac{x_n\cdot|x_n^2-3|}{\sqrt{1+4x_n^2-x_n^4}}. \end{aligned}$$

Define the functions $f : [0,2] \to [0,2]$ as $f(x) = \dfrac{x\cdot|x^2-3|}{\sqrt{1+4x^2-x^4}}$ and $g : [0,2] \to [0,4]$ as $g(x) = \big(f(x)\big)^2 = \dfrac{x^2(x^2-3)^2}{1+4x^2-x^4}$. Then $x_{n+1} = f(x_n)$. Observe that if the sequence $\{x_n\}$ has a limit L, then $L = \dfrac{L\cdot|L^2-3|}{\sqrt{1+4L^2-L^4}}$, which gives 0, 1, and 2 as possible values for L.

We can now determine the "prohibited values" for x_0. Clearly, the equation $f(x) = \sqrt{3}$ has a solution x_0 in $(0,2)$ (by the Intermediate Value Theorem, since $f(0) = 0$, $f(2) = 2$, and f is continuous on $[0,2]$). Then $x_1 = f(x_0) = \sqrt{3}$, and the geometry problem does not make sense. Similarly, it does not make sense if x_0 is a solution of $f^n(x) = \sqrt{3}$, where $f^n(x)$ denotes $\underbrace{f\big(f(\ldots f(x))\big)}_{n \text{ times},\ n\geq 1}$. Hence, the set of "prohibited values" for x_0 is $\{x \mid f^n(x) = \sqrt{3}\} \cap (0,2)$.

Further, we have

$$g'(x) = \frac{-2x(x^2-3)(x^2+1)\left(x^2-(3-\sqrt{6})\right)\left(x^2-(3+\sqrt{6})\right)}{(1+4x^2-x^4)^2},$$

so that both functions g and f are strictly increasing on $\left[0,\sqrt{3-\sqrt{6}}\right]$ and $\left[\sqrt{3},2\right]$, and strictly decreasing on $\left[\sqrt{3-\sqrt{6}},\sqrt{3}\right]$.

Now consider the difference $g(x) - x^2 = \dfrac{2x^2(x^2-1)(x^2-4)}{1+4x^2-x^4}$. Since $1+4x^2-x^4 = 5-(x^2-2)^2 > 0$ for $x \in [0,2]$, we have $g(x) - x^2 > 0$ for every $x \in (0,1)$ and $g(x) - x^2 < 0$ for every $x \in (1,2)$. Hence, $f(x) > x$ for every $x \in (0,1)$, $f(x) < x$ for every $x \in (1,2)$, and $f(x) = x$ if and only if $x = 0$, $x = 1$, or $x = 2$.

Let $a = \sqrt{3-\sqrt{6}}$. Since $a < 1$, we have $a < f(a)$. Let $I = \big[a, f(a)\big]$. It is easy to check that $1 < f(a) < \sqrt{3}$. Thus, we have $f\big(f(a)\big) < f(a)$, since $f(a) > 1$. Since f is strictly decreasing and continuous on I and $I \subseteq \left[\sqrt{3-\sqrt{6}}, \sqrt{3}\right]$, we see that $f(I) = \big[f\big(f(a)\big), f(a)\big]$, which implies that $f(I) \subseteq I$, since $f\big(f(a)\big) = \sqrt{\frac{5346}{1345} - \frac{1674}{1345}\sqrt{6}} > a$. Therefore, $a < f\big(f(a)\big) < 1 < f(a) < \sqrt{3}$.

We are now going to show that, for each $x_0 \in (0,2)$ which is not "prohibited", there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in I$. Then it will follow that $x_n \in I$ for every $n \in \mathbb{N}$ with $n \geq n_0$, since $f(I) \subseteq I$.

Suppose, for the purpose of contradiction, that the sequence $\{x_n\}$ has the following property:

$x_n \notin I$ for every $n \in \mathbb{N}$.

Let us denote this property by $\mathcal{P}$. We consider several cases.

Case 1. $x_0 < a$.

Then (by induction) we obtain $x_n < a$ for every $n \in \mathbb{N}$, since f is strictly increasing on $[0,a]$. (If $x_k < a$ for some $k \in \mathbb{N}$, then $f(x_k) < f(a)$, or $x_{k+1} < f(a)$. By property $\mathcal{P}$, we get $x_{k+1} < a$.) But then we have $x_n < x_{n+1}$ for every $n \in \mathbb{N}$, since $x < f(x)$ for every $x \in (0,1)$. Thus, the sequence $\{x_n\}$ is increasing and bounded above, and therefore it has a limit, L. Recall that the possible values of L are 0, 1, and 2. Since $x_n \in (0,a)$ and $a < 1$, then L must be 0. This is a contradiction, because an increasing sequence with terms in $(0,a)$ cannot have limit 0.

Case 2. $f(a) < x_0 < \sqrt{3}$.

Then $f(x_0) < f\big(f(a)\big)$, or $x_1 < f\big(f(a)\big)$, since the function f is strictly decreasing on $\left[a, \sqrt{3}\right]$. By property $\mathcal{P}$, we get $x_1 < a$. But in this case, by induction, we get $x_n < a$ for every $n \in \mathbb{N}$ with $n \geq 1$, which leads us to a contradiction, similar to the one obtained in Case 1.

Case 3. $\sqrt{3} \leq x_0$ and there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} < \sqrt{3}$.

Then $x_n < a$ for every $n \in \mathbb{N}$ with $n \geq n_0 + 1$, and we again obtain a contradiction, similar to those obtained in Cases 1 and 2.

Case 4. $\sqrt{3} \leq x_n$ for every $n \in \mathbb{N}$.

Then we have $f(x_n) < x_n$, or $x_{n+1} < x_n$ for every $n \in \mathbb{N}$, since $f(x) < x$ for every $x \in (1,2)$. Hence, the sequence $\{x_n\}$ is strictly decreasing and therefore, there exists $\lim x_n = L$, with $L \in \left[\sqrt{3}, 2\right]$. The only possible value for L is 2, but this is a contradiction, because a decreasing sequence with terms in $[\sqrt{3}, 2)$ cannot have limit 2.

Consequently, the assumption that the sequence $\{x_n\}$ has property $\mathcal{P}$ leads to a contradiction. Therefore, it is true that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in I$ and then $x_n \in I$ for every $n \in \mathbb{N}$ with $n \geq n_0$, since $f(I) \subseteq I$. Now define the sequence $\{y_n\}$ as $y_n = x_{n+n_0}$ for every $n \in \mathbb{N}$. Then $y_n \in I$ for every $n \in \mathbb{N}$. Define the function $h : I \to \mathbb{R}$ with $h(x) = f\big(f(x)\big)$. The function h is strictly increasing on I, since f is strictly decreasing on I. We have $y_{2n+2} = h(y_{2n})$ and $y_{2n+3} = h(y_{2n+1})$ for every $n \in \mathbb{N}$. The sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are monotone. (For example, if $y_0 \leq y_2$, then $y_2 = h(y_0) \leq h(y_2) = y_4$, etc.; an easy induction completes the proof that $\{y_{2n}\}$ is increasing.) Both sequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are bounded, because $y_n \in I$. Therefore, there exist L_1, $L_2 \in I$ such that $\lim y_{2n} = L_1$ and $\lim y_{2n+1} = L_2$. Observe that

$$L_1 \;=\; \lim y_{2n+2} \;=\; \lim h(y_{2n}) \;=\; h(\lim y_{2n}) \;=\; h(L_1)\,.$$

Similarly, $L_2 = h(L_2)$. In fact, $L_1 = L_2 = 1$, as $x = 1$ is the only root in I of $h(x) = x$. To see this, let $b \in I \setminus \{1\}$ satisfy $h(b) = b$. By the Mean-Value Theorem, $b - f(b) = f(f(b)) - f(b) = f'(x)(f(b) - b)$ for some x between b and $f(b)$. Since $b \neq f(b)$ (as f has no fixed points in $I \setminus \{1\}$), we have $f'(x) = -1$ and $x \in I$. Then $f'(x) = -1$ reduces to $\ell(x) = r(x)$, where $\ell(x) = x^6 + 3 + \big(5 - (x^2-2)^2\big)^2$ and $r(x) = x^2(5x^2+3)$. Now $\ell(x)$ and $r(x)$ are strictly increasing on I, $\ell(a) > r(1)$ (a long calculation), and $\ell(1) > r(f(a))$, and therefore $\ell(x) > r(x)$ on I, a contradiction. Thus $x = 1$ is the only root in I of $h(x) = x$.

Finally, we have

$$\begin{aligned} \lim y_n \;=\; 1 \qquad &\Longleftrightarrow \qquad \lim x_n \;=\; 1 \\ &\Longleftrightarrow \qquad \lim \cos\omega_n \;=\; \tfrac{1}{2} \\ &\Longleftrightarrow \qquad \lim \omega_n \;=\; \tfrac{\pi}{3}\,, \end{aligned}$$

which completes the proof.

There were also two incomplete solutions submitted.

3225★. [2007 : 112, 115, 297] *Proposed by George Tsapakidis, Agrinio, Greece.*

The sides ℓ and m of an acute angle α with vertex A intersect the sides of a fixed acute angle β with vertex B in four distinct points P, Q, R, and S, labelled so that P lies between A and Q and also between B and S.

(a) If the measure of $\angle\alpha$ is fixed, can A and ℓ be chosen so that

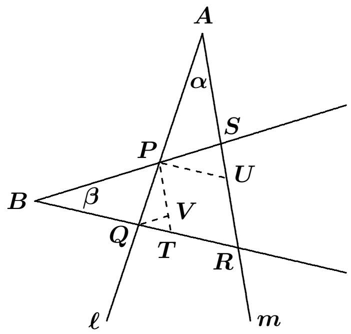
$$[PBQ] + [APS] \;=\; [PQRS]\,,$$

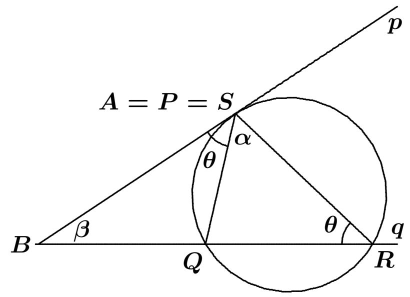
where $[XYZ]$ denotes the area of polygon XYZ?

(b) When are the lines ℓ and m constructible with Euclidean tools to satisfy the condition in part (a) for a given fixed value of α?

Solution to part (a) by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

We shall investigate the degenerate case in which the points A, P, and S coincide on one side of the angle whose vertex is B. Since $[APS] = 0$ for the degenerate configuration, the area condition becomes $[ABQ] = [AQR]$, and Q is necessarily the mid-point of BR.





Claim 1. To any non-degenerate configuration that satisfies the area condition there corresponds a degenerate configuration that likewise satisfies the area condition.

Proof: Let the parallel to QR through P meet SR at U, and the parallel to SR at P meet QR at T; thus, $PTRU$ is a parallelogram inside $PQRS$. Moreover, let V be the point on PT where the parallel to PS through Q meets PT. Then,

$$\frac{AS}{SU} = \frac{PV}{VT} = \frac{BQ}{QT};$$

the first equality follows because the triangles APU and PQT (as well as their corresponding cevians PS and QV) are homothetic, and the second because VQ is parallel to the base BP of $\triangle BTP$. It follows that $TQ \leq BQ$ and $US \leq SA$; otherwise, should $TQ > BQ$ and $US > SA$, we would have $[PQRS] > [PQT] + [PUS] > [PBQ] + [APS]$, contrary to the area condition. Note that $\angle QPT = \alpha$. Rotate this angle about P in either direction (sliding Q and T along the line BR so that $\angle QPT$ remains equal to α): in one direction the distance between Q and B shrinks to zero, while in the other the distance between Q and T grows without bound. Either way there is a position in which Q is the mid-point of BT. In those positions, the points P, B, Q, and T play the roles of A, B, Q, and R in a degenerate configuration with angle α at A and β at B, in which the area condition $[ABQ] = [AQR]$ is satisfied. ∎

Claim 2. There exists a degenerate configuration that satisfies the area condition if and only if

$$\alpha \leq 2\tan^{-1}\left[\left(3 - 2\sqrt{2}\right)\cot\frac{\beta}{2}\right].$$

It is constructible with Euclidean tools.

Proof: Denote the rays that bound the given angle β by p and q. The point R can be arbitrarily chosen on q, while Q is defined to be the mid-point of BR. Construct the circular arc QXR for which $\angle QXR = \alpha$ (on the same side of BR as p). This circle will intersect p in two, one, or zero points. In the first case, either of those points can be labeled A to obtain the desired configuration; in the last case, no such configuration can exist. The second case, where the circle is tangent to p, therefore provides the maximum permissible value of α for the given β. We define A to be the point where the circle QXR is tangent to p. It follows that $BA^2 = BQ \cdot BR = 2BQ^2$; whence,

$$\frac{BR}{BA} = \frac{BA}{BQ} = \sqrt{2}\,.$$

Moreover, $\angle BAQ = \angle ARQ$ equals some value θ, say. Then, in $\triangle BRA$, we have $2\theta = \pi - \alpha - \beta$. Also, the Sine Law applied to that triangle gives

$$\frac{\sin(\alpha+\theta)}{\sin\theta} = \frac{BR}{BA} = \sqrt{2}\,.$$

Replace θ in this last equation by $(\pi - \alpha - \beta)/2$ to get

$$\cos\frac{\alpha-\beta}{2} = \sqrt{2}\cos\frac{\alpha+\beta}{2}\,,$$

or

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{2}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\,.$$

Dividing both sides by the cosines yields

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2}\,,$$

which provides the promised extreme value of α. ■

[*Editor's comments*. The above argument provides a technically sound solution to the problem. We note that there exists a procedure, however, that turns a degenerate configuration into a non-degenerate one having the same α and β:

Claim 3. If $\alpha < 2\tan^{-1}\left[(3-2\sqrt{2})\cot\frac{\beta}{2}\right]$, then there is a non-degenerate configuration that satisfies the area condition.

Proof: Because of the strict inequality for α, the circle QXR in the proof of Claim 2 intersects the ray p in two points. Define P to be any point of p between those two points. Define a new position of Q on BR such that $\angle QPR = \alpha$. Since now we have $BQ > QR$, we also have $[PBQ] > [PQR]$. It remains to find a new position for R so that the line through it that makes an angle of α with PQ will intersect PQ in A, and p in S, in such a way that the area condition is satisfied. Such a position for R will exist because the area $[PQRS]$ grows faster than $[APS]$ as R moves away from B on the line BQ: at its initial position $[PBQ]+[APS]=[PBQ]>[PQR]=[PQRS]$, while the inequality is reversed when R is sufficiently removed from Q. ■

Claim 1 tells us that the condition on α is necessary, so our construction provides a solution to part (a). It provides no information, however, about the constructability of the configuration; therefore, part (b) remains open in the non-degenerate case.]

No other solutions were received; in particular, there was no completely satisfactory solution to part (b) *submitted.*

3227. [2007 : 169, 172] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let $\alpha \in [0,1]$ and define

$$x_n = \left(\frac{\zeta(2)+\cdots+\zeta(n+1)}{n}\right)^{n^\alpha},$$

where ζ is the Riemann Zeta Function, defined by $\zeta(k)=\sum\limits_{p=1}^{\infty}\frac{1}{p^k}$. Prove that

$$\lim_{n\to\infty} x_n = \begin{cases} 1, & \text{if } \alpha\in[0,1),\\ e, & \text{if } \alpha = 1.\end{cases}$$

Solution by Michel Bataille, Rouen, France.

Note first that

$$\begin{aligned}
&\zeta(2)+\zeta(3)\cdots+\zeta(n+1)\\
&\quad= \sum_{p=1}^{\infty}\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots+\frac{1}{p^{n+1}}\right) = \sum_{p=1}^{\infty}\frac{1}{p^2}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}\right)\\
&\quad= n+\sum_{p=2}^{\infty}\frac{1}{p^2}\cdot\frac{1-\frac{1}{p^n}}{1-\frac{1}{p}} = n+\sum_{p=2}^{\infty}\frac{p^n-1}{p^{n+1}(p-1)}\,. \qquad (1)
\end{aligned}$$

Since $\dfrac{p^n-1}{p^{n+1}(p-1)} = \dfrac{1}{p(p-1)} - \dfrac{1}{p^{n+1}(p-1)}$ and

$$\sum_{p=2}^{\infty}\frac{1}{p(p-1)} = \sum_{p=2}^{\infty}\left(\frac{1}{p-1}-\frac{1}{p}\right) = 1\,,$$

we obtain

$$\sum_{p=2}^{\infty}\frac{p^n-1}{p^{n+1}(p-1)} = 1-\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{n+1}}\,. \qquad (2)$$

From (1) and (2), we have $\ln(x_n) = n^\alpha \ln\left(1 + \frac{1}{n} - \frac{1}{n}\sum_{p=1}^{\infty} \frac{1}{p(p+1)^{n+1}}\right)$.

Since $f(x) = 1/x^{n+1}$ is concave up on $[1, \infty)$, we have

$$0 \leq \sum_{p=1}^{\infty} \frac{1}{p(p+1)^{n+1}} \leq \sum_{p=1}^{\infty} \frac{1}{(p+1)^{n+1}} \leq \int_1^{\infty} \frac{dx}{x^{n+1}} = \frac{1}{n}.$$

Hence, $\ln(x_n) = n^\alpha \ln\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right) \sim n^{\alpha-1}$ as $n \to \infty$.

[*Ed.:* Following the usual convention, $f = o(g)$ means $f/g \to 0$ as $n \to \infty$ and $f \sim g$ means $f/g \to 1$ as $n \to \infty$. See, for example, page 89 of the book: *A Course of Pure Mathematics* by G.H. Hardy.]

This yields

$$\lim_{n\to\infty} \ln(x_n) = \begin{cases} 1, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha \in [0, 1), \end{cases}$$

and the result follows immediately.

Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.